Section 8

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Linear D.E

A linear differential equation involves the dependent variable “y” and it’s derivatives by themselves. (مش مضروبين في بعض) and the power of each term is one

نلاحظ وجود
"y"
مرة واحدة فقط

Linear Differential Equations
A **linear differential equation** is any differential equation that can be written in the following form.

$$a_n(t)\,y^{(n)}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \cdots + a_1(t)\,y'(t) + a_0(t)\,y(t) = g(t) \qquad (11)$$

The important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power. Also note that neither the function or its derivatives are “inside” another function, for example, $\sqrt{y'}$ or $\mathbf{e}^y$.

The coefficients $a_0(t)\,,\ \ldots\ ,a_n(t)$ and $g(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, $y(t)$, and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, (11) then it is called a **non-linear** differential equation.

In (5) - (7) above only (6) is non-linear, the other two are linear differential equations. We can’t classify (3) and (4) since we do not know what form the function F has. These could be either linear or non-linear depending on F.

Examples:

1) $\frac{dy}{dx} + 2y = 0$ $\quad$ *linear*

2) $\frac{d\omega}{dt} + 2\omega^2 = 0$ $\quad$ *not linear*

3) $\frac{dy}{dx} + x^2 = \sin x$ $\quad$ *linear*

4) $y\frac{dy}{dx} + 4y^2 = 0$ $\quad$ *non linear*

5) $\frac{dy}{dx} + 3x\cos y = 5x^2$

non linear terms

General form of D.E

1) $$\frac{dy}{dx} + P(x)y = Q(x)$$

$$OR$$

2) $$\frac{dx}{dy} + P(y)x = Q(y)$$

Steps of solution:-

1) Put the D.E in the form :-

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2) Determine $P(x)$, $Q(x)$

3) Get the integrating factor μ

$$\mu = e^{\int P(x)dx}$$

4) Evaluate y from

$$\mu y = \int \mu Q(x)dx + C$$

Example 1

$$\frac{dy}{dx} + 2xy - 8x = 0$$

SOLUTION

$p(x) = 2x$ $\qquad$ $Q(x) = 8x$

$$\mu = e^{\int 2xdx} = e^{x^2}$$

$$e^{x^2}y = \int e^{x^2}8xdx + C$$

$$e^{x^2}y = 4\int e^{x^2}dx^2 + C$$

$$e^{x^2}y = 4e^{x^2} + C$$

Example 2

$$x\frac{dy}{dx} + 2(y - 4x^2) = 0$$

SOLUTION

$$\frac{dy}{dx} + \frac{2}{x}y = 8x$$

$$p(x) = \frac{2}{x} \qquad Q(x) = 8x$$

$$\mu = e^{\int \frac{2}{x}dx} = e^{2lnx} = e^{lnx^2} = x^2$$

$$x^2y = \int 8x(x^2)dx + C$$

$$x^2y = \frac{8}{4}x^4 + C$$

Example 3

$$y' + 4\,cot(x)y = cot(x)\,cosec(x)$$

SOLUTION

$$\frac{dy}{dx} + 4\,cot(x)y = cot(x)\,cosec(x)$$

$p(x) = 4\,cot(x)$ $\qquad$ $Q(x) = cot(x)\,cosec(x)$

$$\mu = e^{4\int cot(x)dx} = e^{4lnsinx} = sin^4 x$$

$$y\,sin^4 x = \int sin^4 x \frac{cot(x)}{sin(x)} dx + C$$

$$y\,sin^4 x = \int sin^3 x\,cot(x)\,dx + C$$

Example 3

$$y\,sin^4 x = \int sin^4 x \frac{cot(x)}{sin(x)} dx + C$$

$$y\,sin^4 x = \int sin^3 x\, cot(x)\, dx + C$$

$$y\,sin^4 x = \int sin^3 x \frac{cos(x)}{sin(x)} dx + C$$

$$y\,sin^4 x = \int sin^2 x\, cos(x)\, dx + C$$

$$y\,sin^4 x = \int sin^2 x\, dsinx + C$$

$$y\,sin^4 x = \frac{1}{3} sin^3 x + C$$

Example 4

$$e^{2x}dy + 2(ye^{2x} - x)dx = 0$$

SOLUTION

$$\frac{dy}{dx} + 2y - 2xe^{-2x} = 0$$

$$\frac{dy}{dx} + 2y = 2xe^{-2x}$$

$p(x) = 2$ $\quad$ $Q(x) = 2xe^{-2x}$

$$\mu = e^{\int 2dx} = e^{2x}$$

$$e^{2x}y = \int e^{2x}2xe^{-2x}dx + C$$

$$ye^{2x} = x^2 + C$$

Example 5

Find the solution to the following IVP.

$$t\,y' - 2y = t^5 \sin(2t) - t^3 + 4t^4 \qquad y(\pi) = \frac{3}{2}\pi^4$$

First, divide through by t to get the differential equation in the correct form.

$$y' - \frac{2}{t}y = t^4 \sin(2t) - t^2 + 4t^3$$

Now that we have done this we can find the integrating factor, $\mu(t)$.

Do not forget that the "-" is part of $p(t)$. Forgetting this minus sign can take a problem that is very easy to do and turn it into a very difficult, if not impossible problem so be careful!

Now, we just need to simplify this as we did in the previous example.

$$\mu(t) = \mathbf{e}^{-2\ln|t|} = \mathbf{e}^{\ln|t|^{-2}} = |t|^{-2} = t^{-2}$$

Again, we can drop the absolute value bars since we are squaring the term.

Now multiply the differential equation by the integrating factor (again, make sure it's the rewritten one and not the original differential equation).

$$\left(t^{-2}y\right)' = t^2 \sin(2t) - 1 + 4t$$

Integrate both sides and solve for the solution.

$$\begin{aligned} t^{-2}y(t) &= \int t^2 \sin(2t)\,dt + \int -1 + 4t\,dt \\ t^{-2}y(t) &= -\frac{1}{2}t^2\cos(2t) + \frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t) - t + 2t^2 + c \\ y(t) &= -\frac{1}{2}t^4\cos(2t) + \frac{1}{2}t^3\sin(2t) + \frac{1}{4}t^2\cos(2t) - t^3 + 2t^4 + ct^2 \end{aligned}$$

Apply the initial condition to find the value of c.

$$\frac{3}{2}\pi^4 = y(\pi) = -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^2 - \pi^3 + 2\pi^4 + c\pi^2 = \frac{3}{2}\pi^4 - \pi^3 + \frac{1}{4}\pi^2 + c\pi^2$$

$$\pi^3 - \frac{1}{4}\pi^2 = c\pi^2$$

$$c = \pi - \frac{1}{4}$$

Example 6

Solve the initial value problem.

1) $y' = x + y$, $y(0) = 2$

$x = 0$
$y = 2$

$y' - y = x$

$P(x)$ $Q(x)$

$\mu(x) = e^{\int -1\, dx} = e^{-x}$

$\mu(x) y = \int \mu(x) Q(x)\, dx$

$e^{-x} y = \int e^{-x} x\, dx$

$u = x \qquad dv = e^{-x} dx$

$du = dx \qquad v = -e^{-x}$

$e^{-x} y = -x e^{-x} + \int e^{-x} dx$

$e^{-x} y = -x e^{-x} - e^{-x} + c$

we have I.C. $y(0) = 2$

$x = 0$
$y = 2$

$e^{0}(2) = 0 - e^{0} + c$

$2 = -1 + c \longrightarrow \boxed{c = 3}$

$\therefore \boxed{e^{-x} y = -x e^{-x} - e^{-x} + 3}$

$y = -x - 1 + 3e^{x}$

Remember from section 7 what is meant by the order of the D.E????

		$\boxed{\text{degree}}$
$(y')^1$	1st order	1
$(y'')^1$	2nd order	1
$(y'')^2$	" "	2

Higher Order D.E.

1- Higher Order

2- Linear D.E. with constants coefficients.

3- Homogeneous.

The D.E. $\boxed{ay'' + by' + cy = 0}$

The auxiliary equation

$$\boxed{ar^2 + br + c = 0}$$

Find its roots r_1, and r_2

$$\boxed{r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

We have Three cases:-

$$r_1 \neq r_2$$

$$y_1 = e^{r_1 x}, \qquad y_2 = e^{r_2 x}$$

$$r_1 = r_2 = r$$

$$y_1 = e^{rx}, \qquad y_2 = xe^{rx}$$

$$r_{\substack{1\\2}} = \alpha \pm \beta i$$

$$y_1 = e^{\alpha x}\cos(\beta x), \qquad y_2 = e^{\alpha x}\sin(\beta x)$$

$$\boxed{y_{G.S.} = c_1 y_1 + c_2 y_2}$$

Example 1

Solve the differential equation.

1) $y'' + 9y' = 0$

$r^2 + 9r = 0$

$r(r+9) = 0$

$r_1 = 0 \quad | \quad r_2 = -9$

$y_1 = e^{0x} \quad | \quad y_2 = e^{-9x}$

$y_{G.S.} = c_1 y_1 + c_2 y_2$

$y_{G.S.} = c_1 + c_2 e^{-9x}$

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Example 2

Solve the following IVP

$$4y'' - 5y' = 0 \qquad y(-2) = 0 \qquad y'(-2) = 7$$

SOLUTION

$$\begin{aligned} 4r^2 - 5r &= 0 \\ r(4r - 5) &= 0 \end{aligned}$$

The roots of this equation are $r_1 = 0$ and $r_2 = \frac{5}{4}$. Here is the general solution as well as its derivative.

$$\begin{aligned} y(t) &= c_1\mathbf{e}^0 + c_2\mathbf{e}^{\frac{5t}{4}} = c_1 + c_2\mathbf{e}^{\frac{5t}{4}} \\ y'(t) &= \frac{5}{4}c_2\mathbf{e}^{\frac{5t}{4}} \end{aligned}$$

Up to this point all of the initial conditions have been at $t = 0$ and this one isn't. Don't get too locked into initial conditions always being at $t = 0$ and you just automatically use that instead of the actual value for a given problem.

So, plugging in the initial conditions gives the following system of equations to solve.

$$\begin{aligned} 0 &= y(-2) = c_1 + c_2\mathbf{e}^{-\frac{5}{2}} \\ 7 &= y'(-2) = \frac{5}{4}c_2\mathbf{e}^{-\frac{5}{2}} \end{aligned}$$

Solving this gives.

$$c_1 = -\frac{28}{5} \qquad c_2 = \frac{28}{5}\mathbf{e}^{\frac{5}{2}}$$

The solution to the differential equation is then.

$$y(t) = -\frac{28}{5} + \frac{28}{5}\mathbf{e}^{\frac{5}{2}}\mathbf{e}^{\frac{5t}{4}} = -\frac{28}{5} + \frac{28}{5}\mathbf{e}^{\frac{5t}{4}+\frac{5}{2}}$$

Example 3

2) $2y'' - y' - y = 0$

$$2r^2 - r - 1 = 0$$

$2r \quad +1$

$r \quad -1$

$$(2r+1)(r-1) = 0$$

$r_1 = \frac{-1}{2}$ | $r_2 = 1$

$y_1 = e^{-\frac{1}{2}x}$ | $y_2 = e^{x}$

$$y_{G.S.} = c_1 e^{-\frac{1}{2}x} + c_2 e^{x}$$

Example 4

Solve the following IVP.

$$3y'' + 2y' - 8y = 0 \qquad y(0) = -6 \qquad y'(0) = -18$$

SOLUTION

The characteristic equation is

$$\begin{aligned} 3r^2 + 2r - 8 &= 0 \\ (3r - 4)(r + 2) &= 0 \end{aligned}$$

Its roots are $r_1 = \frac{4}{3}$ and $r_2 = -2$ and so the general solution and its derivative is.

$$\begin{aligned} y(t) &= c_1 \mathbf{e}^{\frac{4t}{3}} + c_2 \mathbf{e}^{-2t} \\ y'(t) &= \frac{4}{3} c_1 \mathbf{e}^{\frac{4t}{3}} - 2c_2 \mathbf{e}^{-2t} \end{aligned}$$

Now, plug in the initial conditions to get the following system of equations.

$$\begin{aligned} -6 &= y(0) = c_1 + c_2 \\ -18 &= y'(0) = \frac{4}{3} c_1 - 2c_2 \end{aligned}$$

Solving this system gives $c_1 = -9$ and $c_2 = 3$. The actual solution to the differential equation is then.

$$y(t) = -9\mathbf{e}^{\frac{4t}{3}} + 3\mathbf{e}^{-2t}$$

Example 5

3) $4y'' - 4y' + y = 0$

$$\underbrace{\boxed{4}}_{a}r^2 \underbrace{\boxed{-4}}_{b}r \underbrace{\boxed{+1}}_{c} = 0$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16-16}}{8} = \frac{1}{2}$$

$$\begin{matrix} 2r & -1 \\ 2r & -1 \end{matrix}$$

$$(2r-1)^2 = 0$$

$$r_1 = \frac{1}{2} \quad \Big| \quad r_2 = \frac{1}{2}$$

$$y_1 = e^{\frac{1}{2}x} \quad \Big| \quad y_2 = x e^{\frac{1}{2}x}$$

$$y_{G.S.} = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$$

Example 6

Solve the following IVP

$$y'' + 14y' + 49y = 0 \qquad y(-4) = -1 \qquad y'(-4) = 5$$

SOLUTION

The characteristic equation and its roots are.

$$r^2 + 14r + 49 = (r+7)^2 = 0 \qquad r_{1,2} = -7$$

The general solution and its derivative are

$$y(t) = c_1\mathbf{e}^{-7t} + c_2 t\mathbf{e}^{-7t}$$
$$y'(t) = -7c_1\mathbf{e}^{-7t} + c_2\mathbf{e}^{-7t} - 7c_2 t\mathbf{e}^{-7t}$$

Plugging in the initial conditions gives the following system of equations.

$$-1 = y(-4) = c_1\mathbf{e}^{28} - 4c_2\mathbf{e}^{28}$$
$$5 = y'(-4) = -7c_1\mathbf{e}^{28} + c_2\mathbf{e}^{28} + 28c_2\mathbf{e}^{28} = -7c_1\mathbf{e}^{28} + 29c_2\mathbf{e}^{28}$$

Solving this system gives the following constants.

$$c_1 = -9\mathbf{e}^{-28} \qquad c_2 = -2\mathbf{e}^{-28}$$

The actual solution to the IVP is then.

$$y(t) = -9\mathbf{e}^{-28}\mathbf{e}^{-7t} - 2t\mathbf{e}^{-28}\mathbf{e}^{-7t}$$
$$y(t) = -9\mathbf{e}^{-7(t+4)} - 2t\mathbf{e}^{-7(t+4)}$$

Example 7

$$y'' + 8y' + 41y = 0$$

$\underset{a}{\boxed{1}}r^2 \underset{b}{\boxed{+8}}r \underset{c}{\boxed{+41}} = 0$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$r_{1,2} = \frac{-8 \pm \sqrt{64-164}}{2}$$

$$\boxed{\sqrt{-1} = i}$$

$$r_{1,2} = \frac{-8 \pm \sqrt{-100}}{2}$$

$$= \frac{-8 \pm 10i}{2} = \underset{\alpha}{\boxed{-4}} \pm \underset{\beta}{\boxed{5}}i$$

$$y_1 = e^{\alpha x} \cos \beta x$$

$$y_2 = e^{\alpha x} \sin \beta x$$

$$y_{G.S.} = C_1 e^{-4x} \cos 5x + C_2 e^{-4x} \sin 5x$$

Example 8

Solve the equation $y'' - 6y' + 13y = 0$.

SOLUTION

The auxiliary equation is $r^2 - 6r + 13 = 0$. By the quadratic formula, the roots are

$$r = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= 3 \pm 2i$$

the general solution of the differential equation is

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

Thank you