Objectives:

After completing this topic, you will be able to:

- Understand the notion of slopes, tangent lines and derivatives.
- Use the relation between limits and derivatives.
- Have a strong intuitive feeling for these important concepts.

In the following we introduce notion and definition of the slope.

Slope definition:

It is the change in the dependent variable (y) between two points divided by the relative change in the independent variable x

Differentiation is all about calculating the slope or slope of a curve y(x), at a given point, x.

The slope is the $\frac{increase\ \ in\ \ y}{increase\ \ in\ \ x} = \frac{height\ \ moved}{length\ moved}$

Notation:

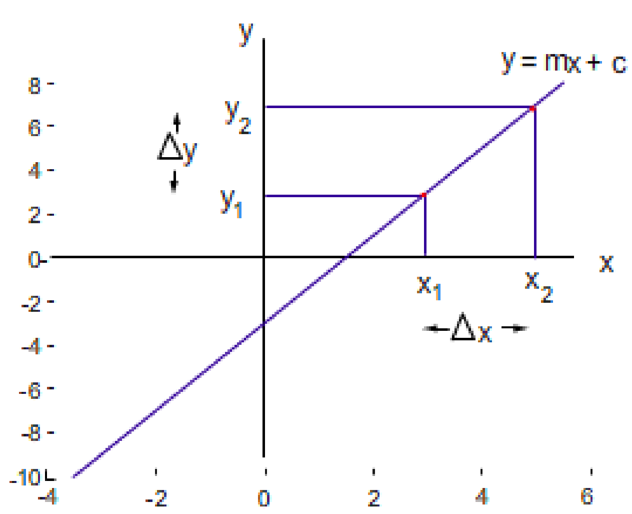
We use the symbol, Δ delta, to mean a (large) change in the value of a variable.

If say x changes from a value of x_1, to a new value, x_2, then $\Delta x = x_2 - x_1$,

So the slope of a curve $y(x)$ can be written as:

slope = $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

Linear Equations



For a straight-line graph of equation $y(x) = mx + c$, the slope is given simply by the value of m.

Here you find some examples of straight line equations and how we find its slope

Example 1:

- $y = 3x + 6$, slope $= 3$
- $y = 5x - 3$, slope $= 5$
- $y = -2x + 1$, slope $= -2$
- $y = 6 - 3x$, slope $= -3$

Measuring slopes:

If we do not know the equation of the straight line, we can work out the slope by tabulating the values of y vs. x and plotting the graph.

Here you find some examples of straight line equations and how we find its slope.

Example 2:

Values of y and x are given below, what is the slope?

x	-3	-2	-1	0	1	2	3
y	-11	-8	-5	-2	1	4	7

i) Plotting the graph

ii) Choose any 2 points along the line (x_1, y_1) and (x_2, y_2)

iii) Draw the triangles (as in the diagram), or just calculate x and y.

iv) Calculate slope from: slope = $\frac{\Delta y}{\Delta x}$

Numerical Method:

Choose any two points we have values for, say, $(-2, -8)$ and $(1,1)$.

We now have:

slope $= \frac{\Delta y}{\Delta x} = \frac{1-(-8)}{1-(-2)} = 3$

Since the intercept is at $y=-2$, we know that the equation of this line must be $y(x)=3x-2$.

Examples including negative powers of variables:

In the following you find additional solved examples showing different cases of differentiation including negative powers of given variables

Example 3:

1- $y = \frac{5}{x} = 5x^{-1}$, $\frac{dy}{dx} = -5x^{-2} = -\frac{5}{x^2}$

2- $y = \frac{3}{x^2} - \frac{1}{2x^3} = 3x^{-2} - \frac{x^{-3}}{2}$, $\frac{dy}{dx} = -6x^{-3} + \frac{3x^{-4}}{2} = \frac{-6}{x^3} + \frac{3}{2x^4}$

3- $V(r) = -\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e^2}{r}$

(= the potential between 2 electrons at a separation of r)

$$\frac{dV(r)}{dr} = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e^2}{r^2}$$

(= the force of repulsion between them - the famous inverse square law)

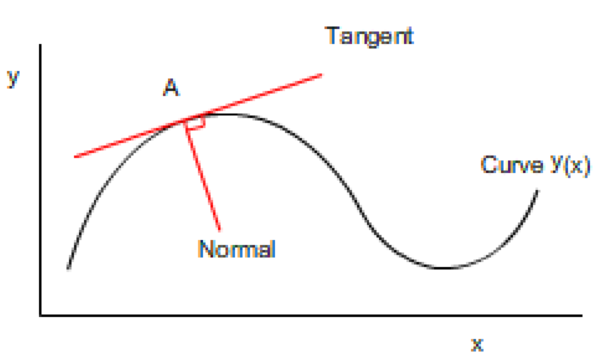
Slope of acurve at any given point:

In the following we discuss how to find the slope of a curve at an arbitrary point of continuity.

Finding the slope of a general function:

Linear curves are simple, but how do we find the slope of any curve, $y(x)$ at the point x ?

The slope of the curve at point $\mathcal{A}$ is the same as that of the tangent at point $\mathcal{A}$.



So, all we need to do is construct the tangent and measure its slope $\frac{\Delta y}{\Delta x}$.

Analytical Differentiation:

Drawing tangents is a rather cumbersome method of obtaining slopes. Is there an analytic method?

The answer is differentiation.

A simplified derivation of this is given in the handout, but we only really need to learn the 'magic formula'.

Notation:

The slope, or slope, or differential, or derivative can be written in many equivalent ways:

$$y' = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

For other variable names and functions, there is the equivalent notation.

e.g. for $s(t)$, we have $\frac{ds}{dt}$,

for $\mathcal{E}(v)$, we have $\frac{dE(v)}{dv}$

for $\phi(\lambda)$, we have $\frac{d\varphi}{d\lambda}$

Differentiation 'magic formula' (for standard polynomials)

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

To differentiate a polynomial function, multiply together the leading factor, a, and the exponent (power), n, then subtract one from the exponent.

Examples:

1- $y = x^2$, $dy/dx = 2x$

2- $y = 2x^3$, $dy/dx = 6x^2$

3- $y = 9x^{27}$, $dy/dx = 243x^{26}$

4- $u = 3m^6$, $du/dm = 18m^5$

5- $\phi = 7\lambda$, $d\phi/d\lambda = 7$

6- $\Psi = x3/12$,$d\Psi/dx = x^2/4$

7- $p = -5q2$, $dp/dq = -10q$

8- $y = 5$, $dy/dx = 0$

The differential of a constant is always zero, i.e. its slope is zero, as we expect.

Addition Rule:

Differentiate all adding terms sequentially, and add the results together.

Examples :

1. $y = 3x^2 + 2x + 7$, $dy/dx = 6x + 2$
2. $p = 9m - 2m^3$, $dy/dm = 9 - 6m^2$
   $dp/dm = 9 - 6m^2$
3. $y = mx + c$, $dy/dx = m$ (as we saw before).
4. $\phi = 14\lambda^2 - \lambda^{10}/5 + \lambda^3 + 3$,
   $d\phi/d\lambda = 28\lambda - 2\lambda^9 + 3\lambda^2$
5. $p(T) = nRT/V$ (at constant n, R, V),
   $dP/dT = nR/V$

Derivatives of roots:

Roots - Fractional powers of x

Square roots, cube roots, fourth roots, etc., can all be represented as fractional powers of x

e.g. $\sqrt{x} = x^{1/2}$, $\sqrt[3]{x} = x^{1/3}$

$\sqrt[3]{x^2} = x^{2/3}$, $1/\sqrt{x} = x^{-1/2}$

$\sqrt{x+1} = (x+1)^{1/2}$, $1/\sqrt[3]{x^2+3} = (x^2+3)^{-1/3}$

These too obey the magic formula.

How to convert roots to fractional powers:

In the following you find some examples illustrating how to convert roots to fractional powers and then using the given rules to find the derivative.

Example 4:

1- $y = \sqrt{x} = x^{1/2}$, $\dfrac{dy}{dx} = \frac{1}{2} x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

2- $y = \sqrt[3]{x^2} = x^{2/3}$, $\dfrac{dy}{dx} = \dfrac{2}{3} x^{-\frac{1}{3}} = \dfrac{2}{3\sqrt[3]{x}}$

3- $y = 1/\sqrt{x} = x^{-1/2}$, $\dfrac{dy}{dx} = -\frac{1}{2} x^{-3/2} = -\dfrac{1}{2\sqrt{x^3}}$

4- $\varphi(\lambda) = \sqrt{\lambda} - \dfrac{3}{\sqrt{\lambda^3}} = \lambda^{1/2} - 3\lambda^{-3/2}$, $\dfrac{d\varphi}{d\lambda} = \dfrac{1}{2\sqrt{\lambda}} + \dfrac{9}{2\sqrt{\lambda^5}}$

Derivative of functions including roots:

Using Differentiation to calculate slopes:

Now we have a method to calculate the value of a slope at any point along a curve,
without having to draw the graph and construct the tangent.

Example 5:

1- What is the slope of $y(x) = x^2 - 4x - 1$ at the point $x = 4$

Note: this is the same function we solved graphically, earlier.

Slope $= \dfrac{dy}{dx} = 2x - 4$

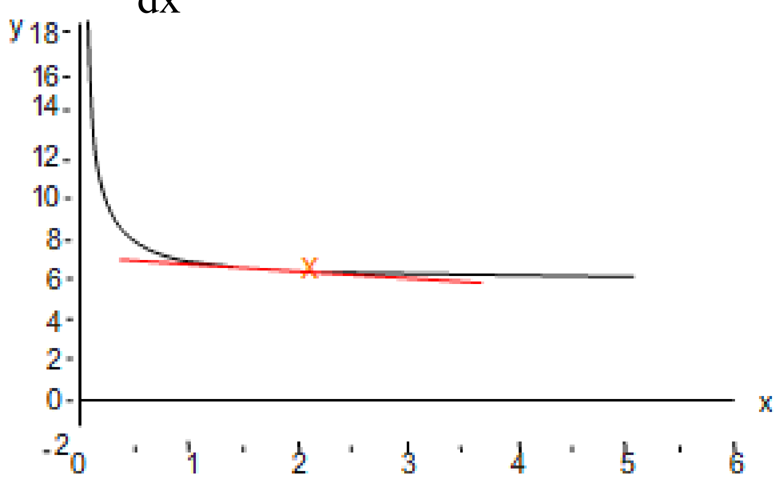
so, when $x = 4$

$\dfrac{dy}{dx} = (2 \times 4) - 4 = 4$ (as we found before)

2- What is the slope of $y(x) = \frac{1}{x} + 6$

at the point $x = 2$ $\frac{dy}{dx} = -\frac{1}{x^2}$

so, when $x = 2$, $\frac{dy}{dx} = -\frac{1}{4}$



3- What is the slope of $p(q) = \frac{q^3}{3} - 2q^2$ at $q = 3$

$\frac{dp(q)}{dq} = q^2 - 4q$, so when $q = \frac{dp(q)}{dq} = -3$

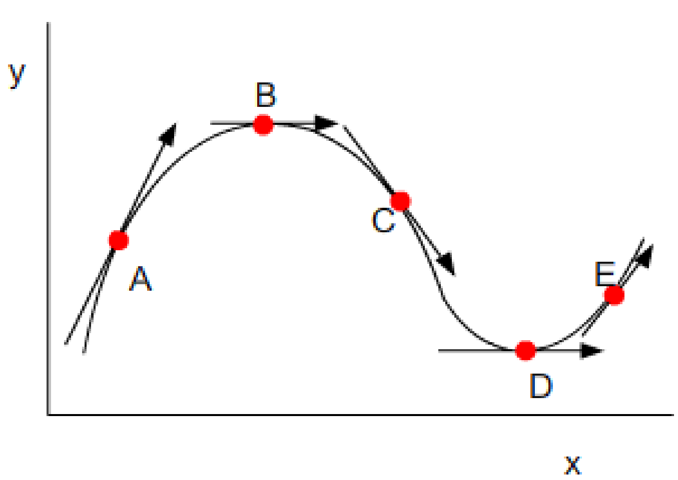
Maxima and minima:

In the following you find the notion of Zero slopes, turning points, and maxima and minima.

Zero slopes Turning points, maxima and minima.

Consider a function which gives a curve like that shown.

If we measure the slope at different points we get different answers:



at points $\mathcal{A}$ and $\mathcal{E}$ slope is +ve

at point C slope is -ve

So at some points in between, $\mathcal{B}$ and $\mathcal{D}$, the function exhibits a stationary value, and this can either be a local maximum ($\mathcal{B}$) or local minimum ($\mathcal{D}$).

The points at which a curve exhibits a maximum or minimum are very important in chemistry since this often indicates when the behavior of a system changes, shows where an equilibrium position lies, or shows where something is most (or least) probable.

How do we calculate maxima and minima positions?

We know that at a local max or min, the slope = 0, i.e. $dy/dx = 0$. So, given a function, $y(x)$, all we need to do is differentiate it, and put the derivative equal to zero, then solve for x.

Example 6:

$y(x) = x3 - 3x + 1$

$\frac{dy}{dx} = 3x^2 - 3$

When $\frac{dy}{dx} = 0$, then $3x^2 - 3 = 0$

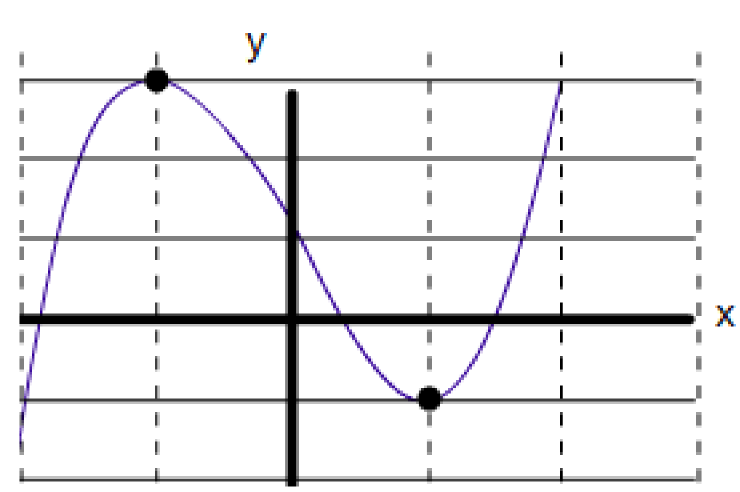
$3x^2 = 3$

$x^2 = 1$ $\qquad x = \pm 1$

$when\ x = +1, y = -1$ $\qquad when\ x = -1, y = 3$

Stationary points are (1 , -1) and (-1 , 3)

Note: later we'll show how we tell which is a max and min.



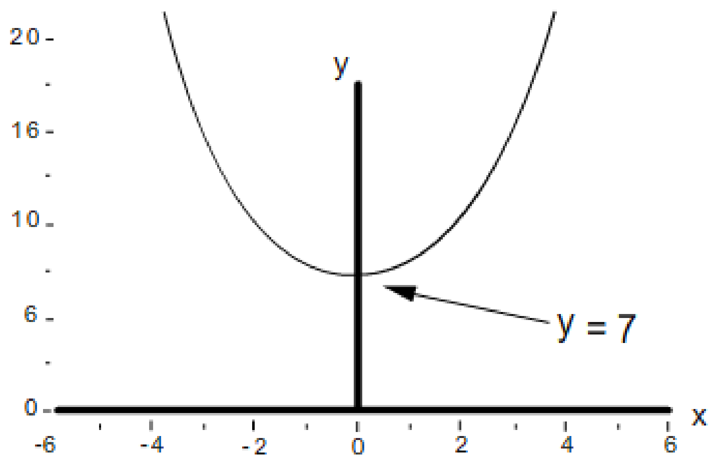
Example 7:

$y = x^2 + 7$

$\frac{dy}{dx} = 2x$

At stationary point, $\frac{dy}{dx} = 0$,

so $2x = 0$, i.e. $x = 0$ and $y = 7$.



Example 8:

3. The potential V of a diatomic molecule (e.g. Cl_2) can be approximated by a quadratic function of the bond length, r, of the form:

$V(r) = k(r - b)^2$ [k and b are constants]

What is the equilibrium bond length?

Answer:

Multiplying out: $V(r) = kr^2 - 2kbr + kb^2$

The equilibrium position will occur when the potential changes from attractive to repulsive, i.e. when the slope changes from +ve to -ve, e.g. at the minimum value of V.

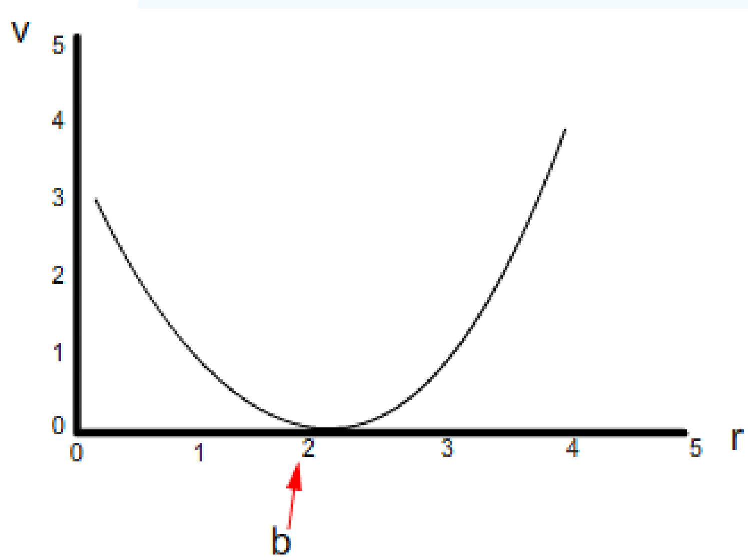
So we need $\frac{dV}{dr} = 0$,

$\frac{dV}{dr} = 2kr - 2kb = 0$

$r = b$

So the constant b is actually the equilibrium bond length.

The value of V at which this occurs is $V(r) = 0$.



Objectives:

After completing this topic, you will be able to:

- To understand how to locate the stationary points and to determine its type .
- Provide students with a strong intuitive feeling for these important concepts

DiscussTypes of stationary points:

Types of Stationary Points:

If x_{sp} is the stationary point, then if we consider points either side of x_{sp}, there are 4 types of behaviors of the slope.

	$x < x_{sp}$	$x = x_{sp}$	$X > x_{sp}$	curve
(i)	+ve	zero	-ve	Max
(ii)	-ve	Zero	+ve	Main
(iii)	+ve	Zero	+ve	inflection
(iv)	-ve	zero	-ve	inflection

Stationary points, like (iii) and (iv), where the slope does not change sign

produce S-shaped curves, and the stationary points are called points of inflection.

How to determine if a stationary point is a max, min or point of inflection.

The rate of change of the slope either side of a turning point reveals its type.

But a rate of change is a differential, So all we need to do is differentiate the slope, $\frac{dy}{dx}$, with respect to x.

In other words we need the 2nd differential, or $\frac{d}{dx}(\frac{dy}{dx})$, more usually called $\frac{d^2y}{dx^2}$

In the following you find how to calculate the second derivative and how to use it to find the type of the stationary point.

Example 9:

1- $y(x) = 9x^2 - 2$

$\frac{dy}{dx} = 18x$ and $\frac{d^2y}{dx^2} = 18$

1- $p = 3q^3 - 4q^2 + 6$

$\frac{dp}{dq} = 9q^2 - 8q$ $\qquad \frac{d^2p}{dq^2} = 18q - 8$

Rules for stationary points:

i) At a local maximum $\frac{d^2y}{dx^2} = \text{-ve}$

ii) At a local minimum $\frac{d^2y}{dx^2} = +\text{ve}$

iii) At a point of inflexion $\frac{d^2y}{dx^2} = 0,$ and we must examine the slope either side of the turning point to find out if the curve is a +ve or -ve point of inflication.

1- Taking the same example as we used before:

$$y(x) = x^3 - 3x + 1$$
$$\frac{dy}{dx} = 3x^2 - 3$$
$$\frac{dy}{dx} = 0$$

giving stationary points at $(-1,3)$ and $(1,-1)$

$$\frac{d^2y}{dx^2} = 6x$$

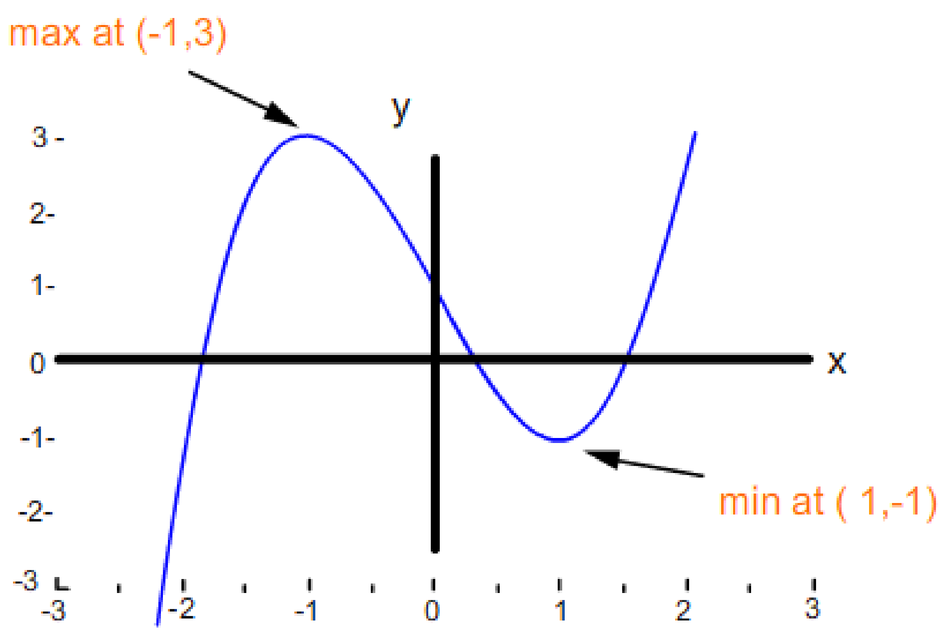
At stationary point $(-1,3)$, $x=-1$,

so $\frac{d^2y}{dx^2}=-6$, so it's a maximum.

At stationary point $(1,-1)$, $x=+1$,

so $\frac{d^2y}{dx^2}=+6$, so it's a minimum

So we can finally sketch the curve:



2-Find the stationary points and determine its type for :

$y=x^3+8$ $\quad \frac{dy}{dx}=3x^2$, which is equal to zero at the stationary point.

If $3x^2=0$, $x=0$, and so $y=+8$, so the stationary point is at $(0,8)$.

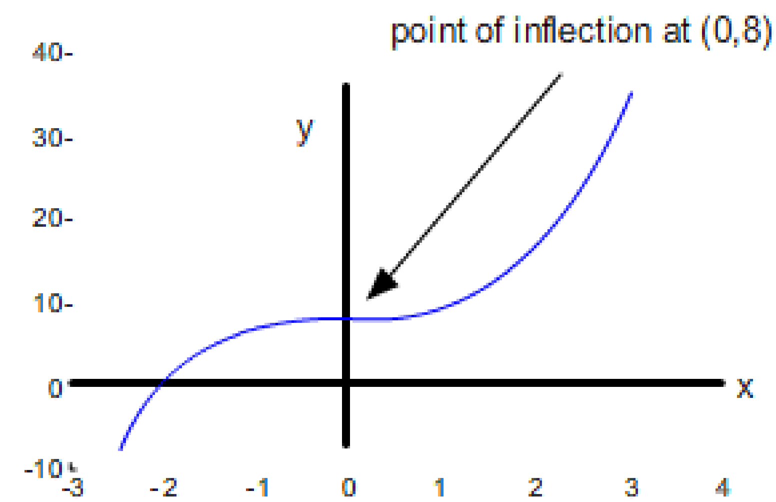
$\frac{d^2y}{dx^2}=6x$

So, at the stationary point $(0,8)$, $\frac{d^2y}{dx^2}=0$, so we have a point of inflexion.

But is $\frac{dy}{dx}$ +ve either side of this point

(e.g. at $x=+1$, $\frac{dy}{dx}=+3$, at $x=-1$, $\frac{dy}{dx}=+3$),

so the curve has a positive point of inflection.



3- Where are the turning point (s), and does it (or they) indicate a max or min in the function $p(q)=4-2q-3q^2$?

$\frac{dp}{dq}=-2-6q$, which at the turning point = 0

So $-2-6q=0,\ 6q=-2,\ q=\frac{-1}{3}$, and $p(q)=4\frac{1}{3}$

We have one turning point at $\left(\frac{-1}{3},4\frac{1}{3}\right)$

$\frac{d^2p}{dq^2}=-6$, so the turning point is a maximum.

Objectives:

After completing this topic, you will be able to:

- Understand the concepts of higher-order derivatives.
- Use the notion of the chain rule.

a) Product rule:

If a function $y(x)$ can be written as the product of two other functions, say $u(x)$ and $v(x)$,

then the differential of $y(x)$ is given by the product rule:

e.g. if $y(x) = u(x)\,v(x)$

then $\frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}$

1. $y(x) = (x^2 + 2)(x + 1)$

let $u = x^2 + 2$, so that $\frac{du}{dx} = 2x$

let $v = x + 1$, so that $\frac{dv}{dx} = 1$

so $\frac{dy}{dx} = u.\frac{dv}{dx} + v\frac{du}{dx} = (x^2 + 2).1 + (x + 1).2x$

$= 3x^2 + 2x + 2$

(Note: we can check this by expanding out the brackets)

$y(x) = x^3 + x^2 + 2x + 1$, $\frac{dy}{dx} = 3x^2 + 2x + 2$

2. $y(x) = x^3(3 - \frac{1}{x} + 3x^2)$

Let $u = x^3$, and $v = 3 - \frac{1}{x} + 3x^2$

$\frac{dy}{dx} = x^3(\frac{1}{x^2} + 6x) + (3 - \frac{1}{x} + 3x^2).3x^2$

3. $\varphi(\lambda) = (2\lambda + \lambda^2)(\frac{1}{\lambda} + \frac{1}{\lambda^2})$

$\frac{d\varphi}{d\lambda} = (2\lambda + \lambda^2)(-\frac{1}{\lambda^2} - \frac{2}{\lambda^3}) + (\frac{1}{\lambda} + \frac{1}{\lambda^2})(2+2\lambda)$

b) Quotient rule:

If we have one function divided by another, such as

$y(x) = \frac{u(x)}{v(x)}$, then $\frac{dy}{dx} = \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2}$

Example 10:

$y(x) = \frac{x^2+1}{2x+3}$ so that $u = x2 + 1$ and. $\frac{du}{dx} = 2x$

and $v = 2x + 3$ and $\frac{dv}{dx} = 2$

$\frac{dy}{dx} = \frac{(2x+3)(2x) - (x^2+1)(2)}{(2x+3)^2}$

[Note: Alternatively we can say $\frac{u(x)}{v(x)} = uv^{-1}$ and use the product rule and function of a function.]

c) Function of a function:

Suppose we want to differentiate $(2x-1)^3$.

We could expand the bracket then differentiate term by term, but this is tedious!

We need a more direct method for expressions of this kind.

Now $(2x-1)^3$ is a cubic function of the linear function $(2x-1)$, i.e. it is a function of a function.

There are 2 ways to think about solving functions of a function:

1- Chain rule:

If we have $y(x) = f$ (complicated expression),

we let $u =$ (complicated expression) then work out $\frac{dy}{du}$ and $\frac{du}{dx}$.

We then use $\frac{du}{dx} = \frac{dy}{du} \cdot \frac{dy}{dx}$ The Chain Rule

Example 11:

$y = (2 - x^3)^4$ let $u = 2 - x^3$, so that $y = u^4$

$\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = -3x^2$

So $\frac{dy}{dx} = (4u^3).(-3x^2) = -12x^2(2 - x^3)^3$

2. $y(x) = \frac{1}{(1-x^2)}$, i.e. $y = (1 - x^2)^{-1}$

let $u = (1 - x^2)$, so that $\frac{du}{dx} = -2x$ and $y = u^{-1}$,

so that $\frac{dy}{du} = -\frac{1}{u^2}$

$\frac{dy}{dx} = (-2x)(-\frac{1}{u^2}) = +\frac{2x}{(1-x^2)^2}$

ii) Sequential Step Method:

With this method, we start with the outermost function, and differentiate our way to the centre, multiplying everything together along the way.

Example 12:

1- $y = (2 - x^3)^4$

think of this as y = (expression)^4

differentiating, $\frac{\mathrm{dy}}{\mathrm{dx}} = 4$(expression)^3

We now look at the expression in the brackets and differentiate that $(= -3x^2)$ and multiply it to our previous answer to give

$\frac{dy}{dx} = 4(2 - x^3)^3(-3x^2)$ (which is the same as before)

2. $y = \frac{1}{(1-x^2)} = (1 - x^2)^{-1}$

$\frac{dy}{dx} = -(1 - x^2)^{-2} . (-2x)$

3. $y = \sqrt{x^2 \quad 1} = (x^2 - 1)^{½}$

$\frac{dy}{dx} = ½(x^2 - 1)^{-½} . 2x$

Example 13:

Refer back to this later, after we've covered sin and In.

$y = \sin\{\ln(3x^2 + 2)\}$

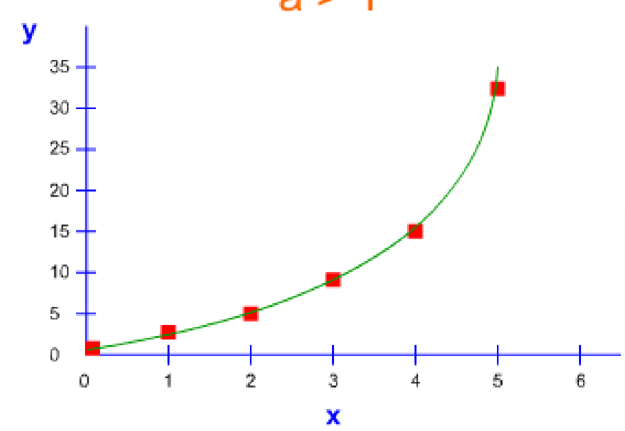
$= \cos\{\ln(3x^2 + 2)\} . \frac{1}{3x^2+2} . 6x$

Exponential Functions:

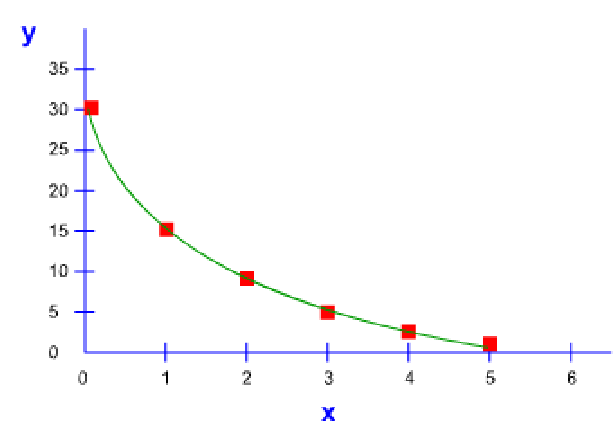
The general expression for an exponential function is

$f(x) = ka^x$, k, a = constants

a > 1



a < 1



Examples on exponential Functions:

An example is $y = 3^x$

x	0	1	2	3	4
y	1	3	9	27	81

One of the most important properties of an exponential function is that the slope of the function at any value is proportional to the value of the function itself.

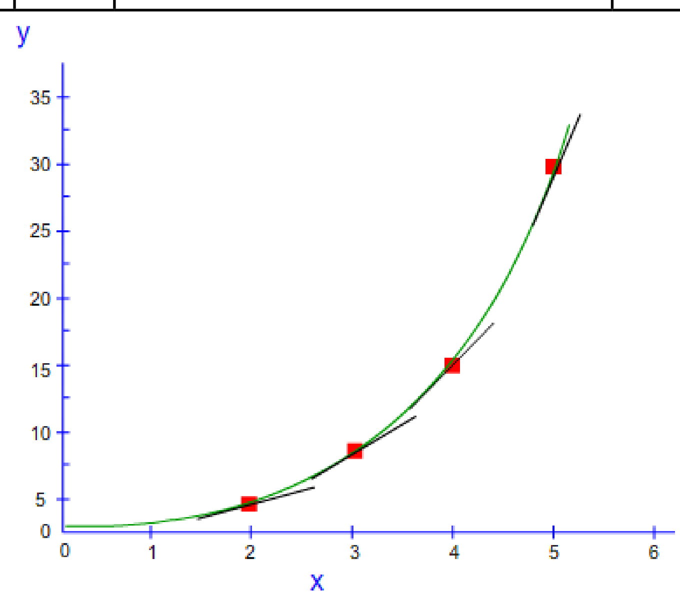
In other words $\frac{dy}{dx}$, proportional to $y(x)$, or = $\frac{dy}{dx}$, constant $y(x)$ the value of the constant depends upon the function $y(x)$.

Numerical examples:

Example 14:

$y = 2^x$, plot the graph and measure the slopes at different values of x

x	y	Slope at x measured from graph	Slope/y
0	1	0.69	0.69
1	2	1.38	0.69
2	4	2.76	0.69
3	8	5.52	0.69
4	16	11.04	0.69

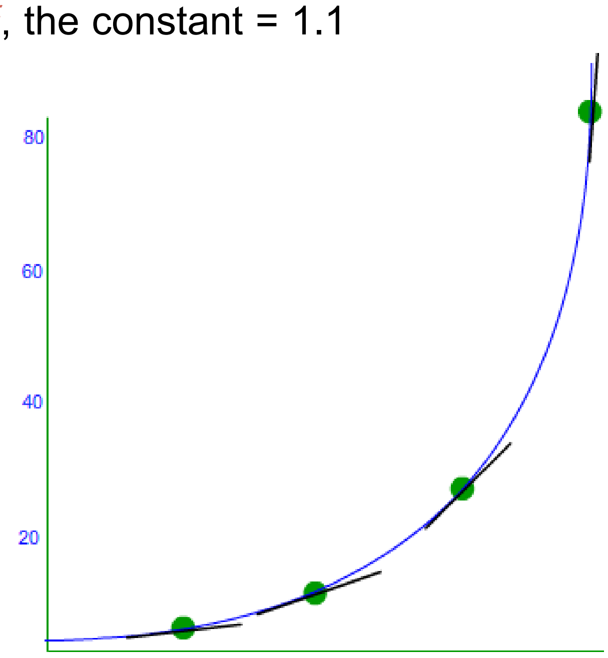


Example 15:

$y = 3^x$

x	y	Slope	Slope/y
0	1	1.1	1.1
1	2	3.3	1.1
2	4	9.9	1.1
3	8	29.7	1.1
4	16	89.0	1.1

So for $y = 3^x$, the constant = 1.1

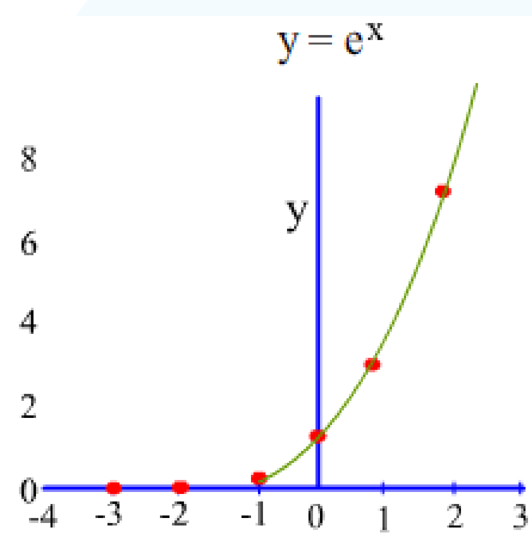


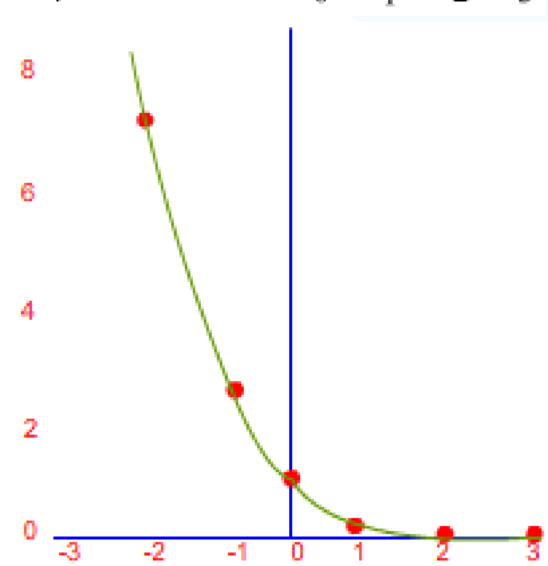
Description of the exponential function and its importance.

The function e^x is known as the exponential function (as opposed to any other exponential function) and is extremely important in all branches of science:

- Radioactive materials undergo exponential decay,
- World human population is increasing exponentially,
- Chemical reaction rates depend exponentially upon the temperature, etc.

x	0	1	2	3	4	5
e^x	1	2.72	7.39	20.1	54.6	148
e^{-x}	1	0.37	0.69	0.05	0.02	0.007





$x = 0,$	$e^x = 1$	;	$x = 0,$	$e^{-x} = 1$
$x = +$ infinity,	$e^x = +$ infinity	;	$x = +$ infinity,	$e^{-x} = 0$
$x = -$ infinity,	$e^x = 0$	;	$x = -$ infinity,	$e^{-x} = +$ infinity

The different notation for expressing the exponential function

Variants of the Exponential Function

In most applications the exponent is not simply x, but some function of x,
or the exponent is multiplied by some other function of x.

Notation: We normally write the exponential function as $y = e^x$,
but if the exponent is a function (e.g. $x^2 + 1$),
it's often easier to write it as $y = exp(x^2 + 1)$, which is equivalent to e^{x^2+1}

(**Note** - This exp is not the same EXP on your calculator, which refers to 10^x)

$y = e^{bx}$, $y = e^{-bx}$, $y = e^{\frac{-ax}{b}}$,

$y = e^{2x^2-1}$ or $y = exp(2x^2 - 1)$

$y = 5e^x$, $y = 5x.e^{-bx}$,

$y = (x2 - 1)^{e^{-x^2}}$, etc.

i.e. if $y = e^x$, $\frac{dy}{dx} = e^x$

Here you find some solved examples using the exponential function

1- $y = 5e^x$, $\frac{dy}{dx} = 5e^x$

2- $y = 3e^x + 2$, $\frac{dy}{dx} = 3e^x$

3. $y = ebx$, use chain rule with and $u = bx$, $\frac{du}{dx} = b$

so $\frac{dy}{dx} = \frac{dy}{du} \quad \frac{du}{dx} = be^{bx}$

Alternatively, using the sequential rule: $\frac{dy}{dx} = e^{bx} . b$

So a general result is $\frac{d}{dx}(e^{bx}) = be^{bx}$

4. $y = e^{-x}$, $\frac{dy}{dx} = -e^{-x}$

5. $y = exp(x^2 + 1)$, $\frac{dy}{dx} = exp(x^2 + 1) . 2x$

6. $y = x.e^{-bx.}$ This requires the product rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x(-be^{-bx}) + e^{-bx}.(1) = (1 - bx)e^{-bx}$$

In this example you find a solved example using the chain rule.

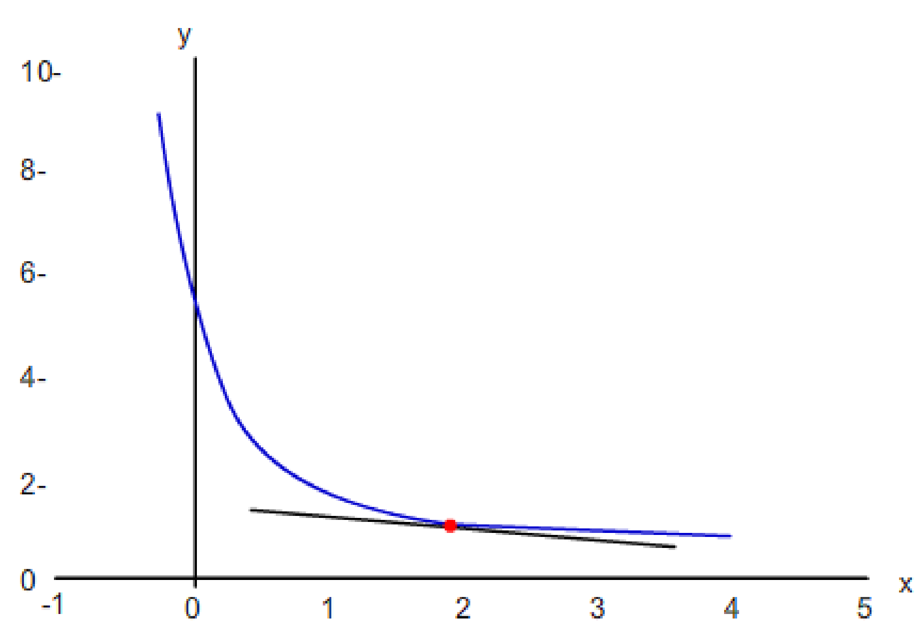
Example 16:

What is the slope of the curve $y = 5e^{-2x} + 1$ at the point $x = 2$?

Solution:

the slope $= y' = -10e^{-2x}$

at x = 2 , the slope $= y' = -10e^{4} = -0.183$



The Logarithmic Function:

If we have the relationship $y = a^x$
then there must be the inverse relationship such that $x = f(y)$.
We call the function, $f(y)$, the logarithm to base a.
$x = \log_a(y)$, valid only for $y > 0$
There are 2 types of logarithm in common use:
a) Common logs have base 10 and are written $\log_{10} x$
b) Natural logs have base e and are written $\log_e x$ or
ln x So, if $y = e^x$, then ln $y = x$
$y = 10^x$, then $\log_{10} y = x$

Logarithmic Functions :

1- $\ln \mathcal{A} + \ln \mathcal{B} = \ln(\mathcal{AB})$
2- $\ln \mathcal{A} - \ln \mathcal{B} = \ln(\mathcal{A}/\mathcal{B})$
3- $\ln \mathcal{A}^x = x.\ln \mathcal{A}$

Example 17:

1. $\ln 2 + \ln 3 = 1.792 = \ln 6$
2. $\ln x + \ln(x^2+1) = \ln\{x.(x^2+1)\} = \ln(x^3+x)$
3. $\ln 6 - \ln 3 = 0.693 = \ln 2$
4. $\ln(x+1) - \ln(3-x^2) = \ln\left(\frac{x+1}{3-x^2}\right)$
5. $\ln x + \ln(x+3) - \ln(x^2+4) = \ln\left(\frac{x(x+3)}{x^2+4}\right)$
6. $\ln(x^2+1)^3 = 3\ln(x^2+1)$
7. $\ln(x^2+3)^{x+1} = (x+1)\ln(x^2+3)$
8. $\ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln x \leftarrow$ important

Numerical Example: when $x = 2$,

9. $\ln\frac{(x+1)^2}{4x} = \ln\frac{3^2}{8} = \ln\frac{9}{8} = 0.118$

or $= 2\ln(x+1) - \ln(4x) = 2\ln 3 - \ln 8 = 0.118$

The Differential of ln x It can be shown that, if

$y = \ln x, \quad \frac{dy}{dx} = \frac{1}{x}$

We can now use this, together with the Product, Chain and Sequential Rules to find the Differentials of log functions.

1. $y = \ln(ax+b) \quad \frac{dy}{dx} = \frac{1}{(ax+b)} \times a = \frac{a}{(ax+b)}$

↑ Differential of ln (1/(...)) ↑ Diff. of (...)

2. $y = \ln(2\sqrt{x}) + 3x^2 = \ln(2x^{1/2}) + 3x^2$

$\frac{dy}{dx} = \left(\frac{1}{2\sqrt{x}}\right).\frac{2}{2x^{1/2}} + 6x = \frac{1}{2x} + 6x$

3. What are the stationary points in $y = \ln(x) - x$?

$\frac{dy}{dx} = \frac{1}{x} - 1$

So the slope = 0 when $\frac{1}{x} - 1 = 0$,

i.e., when $x = 1$, and $y = -1$.

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$, which at (1,-1) is -ve, so it is a *maximum*.

Trigonometric Functions:

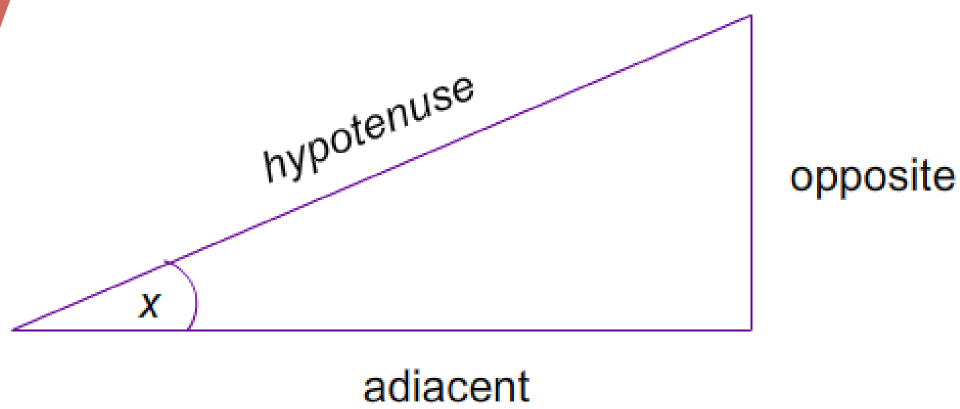
The common trig. functions are defined relative to a right-angled triangle.

sin x = opposite/hypotenuse O/H
cos x = adjacent/hypotenuse A/H
tan x = opposite/adjacent O/A

This can be remembered using sOHcAHtOA
↑ sin ↑ cos ↑ tan

In calculus, we need the angles measured in radians rather than degrees, with

2π radians in a circle $= 360°$
so that 1 radian $= 360°/2\pi$.



The common points used: $0° = 0$ radians

$90° = \pi/2$ radians $180° = \pi$ radians

$270° = 3\pi/2$ radians $360° = 2\pi$ radians

For graphs of the common trig functions, see handout 2.

Inverse Trig. Functions

If $y = \sin x$, then $x = \sin^{-1} y$ (also called arcsin y)

$y = \cos x$, then $x = \cos^{-1} y$ (also called arccos y)

$y = \tan x$, then $x = \tan^{-1} y$ (also called arctan y).

1. $\sin x = 0.32$, $x = \sin^{-1} 0.32 = 18.7°$ $= 0.33$ rads
2. $\tan x = 0.87$, $x = \tan^{-1} 0.87 = 41°$ $= 0.71$ rads

1- $y = \sin(kx)$

$$\frac{dy}{dx} = \underbrace{\cos(kx)}_{\text{differential of } \sin(\ldots)} \times \underbrace{k}_{\text{differential of } kx} = k.\cos(kx) \leftarrow \text{important}$$

2- $y = 3\cos x - 4\sin(x^2)$

$$\frac{dy}{dx} = -3 \sin x - 8x \cos x^2$$

3- $y = 7\sin(5x^2) + 6\ln\{\tan(5x)\}$

$$\frac{dy}{dx} = 70 x \cos 5x^2 + 30 \sec^2 x / (\tan 5x)$$

In the following you find definition and example on velocity and

Speed of an object.

Velocities:

The function f that describes the motion is called the position function of the object.

In the time interval from $t = a$ to $t = a + h$, the average velocity = Displacement/ time. $= [f(a+h) - f(a)]/h$. The instantaneous velocity

$$V(a) = \lim_{h \to 0} [f(a+h) - f(a)] / h = f'(a).$$

Speed:

The speed of the object is the absolute value of the velocity.

Example 18:

The position of a particle is given by the equation of motion $f(t) = 1/(t+1) = (t+1)^{-1}$.

Find the velocity and speed when $t = 2$.

Solution:

$f'(t) = -(t+1)^{-2}$

$f'(2) = -1/9$. The velocity at $t = 2$ is $-1/9\ m/s$

The speed at $t = 2$ is $1/9$

Rate of change:

The rate of change of y with respect to x at $x = x_1$ is the slope of the tangent to the curve $y = f(x)$.

Example 19:

Find the rate of change of $f(x) = x^2$, when $x = 3$.

Solution:

$f'(x) = 2x$.

The rate of change of $f(x)$ when $x = 3$ is 6.