Section 9

By Eng. Ahmad Alaa Aziz

Nonhomogeneous Differential Equations

It's now time to start thinking about how to solve nonhomogeneous differential equations. A second order, linear nonhomogeneous differential equation is

$$y'' + p(t)\,y' + q(t)\,y = g(t) \tag{1}$$

where $g(t)$ is a non-zero function.

The general solution to a differential equation can then be written as.

$$y(t) = y_c(t) + Y_P(t)$$

$g(t)$	$Y_P(t)$ **guess**
$a\mathbf{e}^{\beta t}$	$A\mathbf{e}^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n^{th} degree polynomial	$A_n t^n + A_{n-1}t^{n-1} + \cdots A_1 t + A_0$

$r(x)$	Initial guess for $y_p(x)$
k (a constant)	A (a constant)
$ax+b$	$Ax+B$ (*Note*: The guess must include both terms even if $b=0$.)
ax^2+bx+c	Ax^2+Bx+C (*Note*: The guess must include all three terms even if b or c are zero.)
Higher-order polynomials	Polynomial of the same order as $r(x)$
$ae^{\lambda x}$	$Ae^{\lambda x}$
$a\cos\beta x+b\sin\beta x$	$A\cos\beta x+B\sin\beta x$ (*Note*: The guess must include both terms even if either $a=0$ or $b=0$.)
$ae^{\alpha x}\cos\beta x+be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x+Be^{\alpha x}\sin\beta x$
$(ax^2+bx+c)e^{\lambda x}$	$(Ax^2+Bx+C)e^{\lambda x}$
$(a_2x^2+a_1x+a0)\cos\beta x$ $+(b_2x^2+b_1x+b_0)\sin\beta x$	$(A_2x^2+A_1x+A_0)\cos\beta x$ $+(B_2x^2+B_1x+B_0)\sin\beta x$
$(a_2x^2+a_1x+a_0)e^{\alpha x}\cos\beta x$ $+(b_2x^2+b_1x+b_0)e^{\alpha x}\sin\beta x$	$(A_2x^2+A_1x+A_0)e^{\alpha x}\cos\beta x$ $+(B_2x^2+B_1x+B_0)e^{\alpha x}\sin\beta x$

Example 1

Solve the following D.E :-

$$y'' + 2y' + y = 4e^{-2x}$$

$$y(x) = C_1e^{-x} + C_2xe^{-x} + 4e^{-2x}$$

SOLUTION

First we solve the related homogeneous D.E :-

$$y'' + 2y' + y = 0$$

the roots of the characteristic equation are:-

$$r^2 + 2r + 1 = 0$$

$$(r + 1)(r + 1) = 0$$

$$\therefore r_{1,2} = -1$$

Hence the general solution of the homogeneous equation is given by :

$$y_o(x) = C_1 e^{-x} + C_2 x e^{-x}$$

Based on the form "$g(x) = 4e^{-2x}$" the particular solution would be in the form of : "$y_p = Ae^{-2x}$"

The derivatives are given by :-

$$y_p{}' = -2Ae^{-2x}$$

$$y_p'' = 4Ae^{-2x}$$

$$\because y_p{}'' + 2y_p' + y_p = 4e^{-2x}$$

$$\therefore 4Ae^{-2x} - 4Ae^{-2x} + Ae^{-2x} = 4e^{-2x}$$

$$\therefore Ae^{-2x} = 4e^{-2x}$$

$$\therefore A = 4$$

$$y_p = 4e^{-2x}$$

Hence the general solution of the homogeneous equation is given by :

$$y(x) = C_1e^{-x} + C_2xe^{-x} + 4e^{-2x}$$

Example 2

Solve the differential equation $y'' - 5y' + 4y = e^{4x}$

SOLUTION

First we solve the related homogeneous equation $y'' - 5y' + 4y = 0$. The roots of the characteristic equation are

$$k^2 - 5k + 4 = 0, \quad \Rightarrow D = 25 - 4 \cdot 4 = 9, \quad \Rightarrow k_{1,2} = \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2}$$

$$= 4, 1.$$

Hence, the general solution of the homogeneous equation is given by

$$y_0\left(x\right) = C_1 e^{4x} + C_2 e^{x},$$

where C_1, C_2 are constant numbers.

Find a particular solution of the nonhomogeneous differential equation. Notice that the power of the exponential function on the right coincides with the root $k_1 = 4$ of the auxiliary characteristic equation. Therefore we will look for a particular solution of the form

$$y_1 = Axe^{4x}.$$

The derivatives are given by

$$y_1' = \left(Axe^{4x}\right)' = Ae^{4x} + 4Axe^{4x} = (A + 4Ax)\, e^{4x};$$

$$y_1'' = \left[(A + 4Ax)\, e^{4x}\right]' = 4Ae^{4x} + (4A + 16Ax)\, e^{4x} = (8A + 16Ax)\, e^{4x}.$$

Substituting the function y_1 and its derivatives in the differential equation yields:

$$(8A + 16Ax)\, e^{4x} - 5\,(A + 4Ax)\, e^{4x} + 4Axe^{4x} = e^{4x},$$

$$\Rightarrow 8A + \cancel{16Ax} - 5A - \cancel{20Ax} + \cancel{4Ax} = 1, \quad \Rightarrow 3A = 1, \quad \Rightarrow A = \frac{1}{3}.$$

Thus, the particular solution to the differential equation can be written in the form:

$$y_1 = \frac{x}{3} e^{4x}.$$

Now we can write the full solution of the nonhomogeneous equation:

$$y = y_0 + y_1 = C_1 e^{4x} + C_2 e^{x} + \frac{x}{3} e^{4x}.$$

Example 3

Find the general solution of the equation $y'' + 9y = 2x^2 - 5$.

SOLUTION

First we determine the general solution of the related homogeneous equation. Solve the auxiliary characteristic equation:

$$k^2 + 9 = 0, \quad \Rightarrow k^2 = -9, \quad \Rightarrow k_{1,2} = \pm 3i.$$

The solution is written in the form:

$$y_0(x) = C_1 \cos 3x + C_2 \sin 3x.$$

Now we construct a particular solution. The right-hand side of the given equation is a quadratic function. So we can guess on a particular solution of the same form:

$$y_1 = Ax^2 + Bx + C,$$

where the numbers A, B, C can be determined by the method of undetermined coefficients. Hence, we can write:

$$y_1' = 2Ax,\;\; y_1'' = 2A.$$

Substituting this into the original nonhomogeneous differential equation, we have

$$2A + 9\left(Ax^2 + Bx + C\right) = 2x^2 - 5,\;\; \Rightarrow 2A + 9Ax^2 + 9Bx + 9C = 2x^2 - 5.$$

By equating the coefficients of like powers of x, we obtain:

$$\begin{cases} 9A = 2 \\ 9B = 0 \\ 2A + 9C = -5 \end{cases},\;\; \Rightarrow \begin{cases} A = \frac{2}{9} \\ B = 0 \\ C = -\frac{49}{81} \end{cases}.$$

Example 4

Solve the following D.E :-

$$y'' + 2y' = 24x + e^{-2x}$$

SOLUTION

First we solve the related homogeneous D.E :-

$$y'' + 2y' = 24x + e^{-2x}$$

the roots of the characteristic equation are:-

$$r^2 + 2r = 0$$
$$r(r + 2) = 0$$
$$\therefore r_1 = 0 \quad and \quad r_2 = -2$$

Hence the general solution of the homogeneous equation is given by :

$$y_o(x) = C_1 + C_2 e^{-2x}$$

Based on the form "$g(x) = 24x + e^{-2x}$" the particular solution would be in the form of : "$y_p = Ax^2 + Bx + Cxe^{-2x}$"

The derivatives are given by :-

$$\because y_p' = 2Ax + B + C(e^{-2x} - 2xe^{-2x})$$

$$\therefore y_p{}' = 2Ax + B + Ce^{-2x} - 2Cxe^{-2x}$$

$$\because y_p'' = 2A - 2Ce^{-2x} - 2C(e^{-2x} - 2xe^{-2x})$$

$$\therefore y_p'' = 2A - 2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x}$$

$$\because y'' + 2y' = 24x + e^{-2x}$$

$$\therefore 2A - 2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x} + 4Ax + 2B + 2Ce^{-2x} - 4Cxe^{-2x} = 24x + e^{-2x}$$

$$\therefore (2A + 2B) + 4Ax - 2Ce^{-2x} = 24x + e^{-2x}$$

$$\therefore 4A = 24 \ , A = 6$$

$$-2C = 1, \qquad \mathrm{C} = -0.5$$

$$(2A + 2B) = 0 \, , \qquad \mathrm{B} = -6$$

Hence the general solution of the homogeneous equation is given by :

$$y(x) = C_1 e^{-x} + C_2 e^{-x} + 6x^2 - 6x - 0.5xe^{-2x}$$

Thus, the particular solution is given by

$$y_1 = \frac{2}{9}x^2 - \frac{49}{81}.$$

Then the general solution of the original nonhomegeneous differential equation is expressed by the formula

$$y = y_0 + y_1 = C_1 \cos 3x + C_2 \sin 3x + \frac{2}{9}x^2 - \frac{49}{81}.$$

Example 5

Solve the differential equation $y'' + 16y = 2\cos^2 x$.

SOLUTION

First of all we solve the related homogeneous equation. The characteristic equation has roots:

$$k^2 + 16 = 0, \quad \Rightarrow k^2 = -16, \quad \Rightarrow k_{1,2} = \pm 4i,$$

so the general solution has the form:

$$y_0(x) = C_1 \cos 4x + C_2 \sin 4x.$$

Now we find a particular solution for the nonhomogeneous equation. Rewrite the right-hand side as

$$2\cos^2 x = \cos 2x + 1.$$

It follows from here that the particular solution is defined by the function

$$y_1 = A\cos 2x + B\sin 2x + C,$$

where the numbers A, B, and C can be calculated using the method of undetermined coefficients. The first and second derivatives of the function y_1 are

$$y_1' = -2A\sin 2x + 2B\cos 2x,$$

$$y_1'' = -4A\cos 2x - 4B\sin 2x.$$

Substituting this back into the differential equation produces:

$$-4A\cos 2x - 4B\sin 2x + 16\left(A\cos 2x + B\sin 2x + C\right) = \cos 2x + 1,$$

$$-4A\cos 2x - 4B\sin 2x + 16A\cos 2x + 16B\sin 2x + 16C = \cos 2x + 1,$$

$$12A\cos 2x + 12B\sin 2x + 16C = \cos 2x + 1.$$

The latter expression is identical. Therefore we can write the following system of equations to determine the coefficients A, B, C :

$$\left\{ \begin{array}{l} 12A = 1 \\ 12B = 0 \\ 16C = 1 \end{array} \right., \quad \Rightarrow \left\{ \begin{array}{l} A = \frac{1}{12} \\ B = 0 \\ C = \frac{1}{16} \end{array} \right..$$

Thus, the particular solution has the form:

$$y_1 = \frac{1}{12}\cos 2x + \frac{1}{16}.$$

Respectively, the general solution of the original nonhomogeneous equation is written as

$$y = y_0 + y_1 = C_1\cos 4x + C_2\sin 4x + \frac{1}{12}\cos 2x + \frac{1}{16}.$$

Determinants

2 x 2 and 3 x 3 Matrices

Matrices

Note that Matrix is the singular form, matrices is the plural form!

- A matrix is an array of numbers that are arranged in rows and columns.
- A matrix is "square" if it has the same number of rows as columns.
- We will consider only 2x2 and 3x3 square matrices

$$\begin{bmatrix} 1 & 3 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{bmatrix}$$

Determinants

Note the difference in the matrix and the determinant of the matrix!

- Every square matrix has a determinant.
- The determinant of a matrix is a number.
- We will consider the determinants only of 2x2 and 3x3 matrices.

$$\begin{vmatrix} 1 & 3 \\ -\frac{1}{2} & 0 \end{vmatrix}$$

$$\begin{vmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{vmatrix}$$

Determinant of a 2x2 matrix

$$\begin{vmatrix} 1 & 3 \\ -\frac{1}{2} & 0 \end{vmatrix} = (1)(0) - (3)\left(-\frac{1}{2}\right) = -\frac{3}{2}$$

Determinant of a 3x3 matrix

Imagine crossing out the first row.
And the first column.

Now take the double-crossed element. . .
And multiply it by the determinant of the remaining 2x2 matrix

$$\begin{vmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{vmatrix}$$

$$= -3\left((0)(11) - \left(-\tfrac{3}{4}\right)(180)\right)$$

Determinant of a 3x3 matrix

Now keep the first row crossed.
Cross out the second column.

$$\begin{vmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{vmatrix}$$

- Now take the negative of the double-crossed element.
- And multiply it by the determinant of the remaining 2x2 matrix.
- Add it to the previous result.

$$= -3\Big((0)(11) - \left(-\tfrac{3}{4}\right)(180)\Big) - 8\Big((2)(11) - \left(-\tfrac{3}{4}\right)(4)\Big)$$

Determinant of a 3x3 matrix

Finally, cross out first row and last column.

- Now take the double-crossed element.
- Multiply it by the determinant of the remaining 2x2 matrix.
- Then add it to the previous piece.

$$\begin{vmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{vmatrix}$$

$$= -3\left((0)(11)-\left(-\tfrac{3}{4}\right)(180)\right)-8\left((2)(11)-\left(-\tfrac{3}{4}\right)(4)\right)$$

$$+\left(\tfrac{1}{4}\right)\left((2)(180)-(0)(4)\right) = -515$$

Cramer's Rule for Solution of Linear Equations:

If
$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

with $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

Then $x = \dfrac{1}{\Delta}\Delta_x \quad , \quad y = \dfrac{1}{\Delta}\Delta_y \quad , \quad z = \dfrac{1}{\Delta}\Delta_z$

where $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Remember: Cramer's rule can be fruitfully applied in case of $\Delta \neq 0$.

Use Cramer's rule to solve the system of equations:

(1) $2x + 3y - z = 0, \quad x + 2z - 3y = 2, \quad y + z + 2 = 0 \qquad \{(1, -1, -1)\}$

(2) $x - 2y + 2z = 1, \quad 3x + 4z = 8, \quad 6z - y = 2 \qquad \left\{\left(2, 1, \frac{1}{2}\right)\right\}$

(3) $x - y - z = 3, \quad 3x + y = 2, \quad 2y + 3z = 1 \qquad \{(2, -4, 3)\}$

(4) $2x - y + 3z = 0, \quad y - 4x - 6z = 2, \quad 4x + 3y = 8 \qquad \left\{\left(\frac{7}{2}, -2, -3\right)\right\}$

Example 1

Use Cramer's rule to solve the system of equations:

$2x + 3y - z = 0,$ $\quad x + 2z - 3y = 2,$ $\quad y + z + 2 = 0$ $\quad \{(1, -1, -1)\}$

SOLUTION

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -3 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$= (2)(+1)(-5) + (1)(-1)(4) + 0$

$= -10 - 4$

$= -14$

$$\Delta_x = \begin{vmatrix} 0 & 3 & -1 \\ 2 & -3 & 2 \\ -2 & 1 & 1 \end{vmatrix}$$

$= 0 + (3)(-1)(6) + (-1)(+1)(-4)$

$= -18 + 4$

$= -14$

$$\Delta_y = \begin{vmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ 0 & -2 & 1 \end{vmatrix}$$

$= (2)(+1)(6) + 0 + (-1)(+1)(-2)$

$= 12 + 2$

$= 14$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 1 & -2 \end{vmatrix}$$

$= (2)(+1)(4) + (3)(-1)(-2) + 0$

$= 8 + 6$

$= 14$

$\therefore\ x = \dfrac{\Delta_x}{\Delta} = 1,$

$y = \dfrac{\Delta_y}{\Delta} = -1,$

$z = \dfrac{\Delta_z}{\Delta} = -1$

$\therefore$ S.S. $= \{(1, -1, -1)\}$

Example 2

Use Cramer's rule to solve the system of equations:

$x - 2y + 2z = 1,$ $\quad 3x + 4z = 8,$ $\quad 6z - y = 2$ $\quad \left\{\left(2, 1, \frac{1}{2}\right)\right\}$

SOLUTION

$$\Delta = \begin{vmatrix} 1 & -2 & 2 \\ 3 & 0 & 4 \\ 0 & -1 & 6 \end{vmatrix}$$

$= (1)(+1)(4) + (3)(-1)(-10) + 0$

$= 4 + 30$

$= 34$

$$\Delta_x = \begin{vmatrix} 1 & -2 & 2 \\ 8 & 0 & 4 \\ 2 & -1 & 6 \end{vmatrix}$$

$= (8)(-1)(-10) + 0 + (4)(-1)(3)$

$= 80 - 12$

$= 68$

$$\Delta_y = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 8 & 4 \\ 0 & 2 & 6 \end{vmatrix}$$

$= (1)(+1)(40) + (3)(-1)(2) + 0$

$= 40 - 6$

$= 34$

$$\Delta_z = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 0 & 8 \\ 0 & -1 & 2 \end{vmatrix}$$

$= (1)(+1)(8) + (3)(-1)(-3) + 0$

$= 8 + 9$

$= 17$

$$\therefore\ x = \frac{\Delta_x}{\Delta} = 2,$$

$$y = \frac{\Delta_y}{\Delta} = 1,$$

$$z = \frac{\Delta_z}{\Delta} = \frac{1}{2}$$

$$\therefore\ \text{S.S.} = \left\{\left(2, 1, \frac{1}{2}\right)\right\}$$

Example 3

Use Cramer's rule to solve the system of equations:

$x - y - z = 3$, $3x + y = 2$, $2y + 3z = 1$ $\{(2, -4, 3)\}$

SOLUTION

$$\Delta = \begin{vmatrix} 1 & -1 & -1 \\ 3 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= (1)(+1)(3) + (3)(-1)(-1) + 0$$

$$= 3 + 3$$

$$= 6$$

$$\Delta_x = \begin{vmatrix} 3 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= (-1)(+1)(3) + 0 + (3)(+1)(5)$$

$$= -3 + 15$$

$$= 12$$

$$\Delta_y = \begin{vmatrix} 1 & 3 & -1 \\ 3 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= (1)(+1)(6) + (3)(-1)(10) + 0$$

$$= 6 - 30$$

$$= -24$$

$$\Delta_z = \begin{vmatrix} 1 & -1 & 3 \\ 3 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= (1)(+1)(-3) + (3)(-1)(-7)$$

$$= -3 + 21$$

$$= 18$$

$$\therefore \ x = \frac{\Delta_x}{\Delta} = 2,$$

$$y = \frac{\Delta_y}{\Delta} = -4,$$

$$z = \frac{\Delta_z}{\Delta} = 3$$

$\therefore$ S.S. $= \{(2, -4, 3)\}$

Example 4

Use Cramer's rule to solve the system of equations:

$2x - y + 3z = 0,\qquad y - 4x - 6z = 2,\qquad 4x + 3y = 8$ $\quad \left\{\left(\frac{7}{2}, -2, -3\right)\right\}$

SOLUTION

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ -4 & 1 & -6 \\ 4 & 3 & 0 \end{vmatrix}$$

$$= (3)(+1)(-16) + (-6)(-1)(10) + 0$$

$$= -48 + 60$$

$$= 12$$

$$\Delta_x = \begin{vmatrix} 0 & -1 & 3 \\ 2 & 1 & -6 \\ 8 & 3 & 0 \end{vmatrix}$$

$$= 0 + (-1)(-1)(48) + (3)(+1)(-2)$$

$$= 48 - 6$$

$$= 42$$

$$\Delta_y = \begin{vmatrix} 2 & 0 & 3 \\ -4 & 2 & -6 \\ 4 & 8 & 0 \end{vmatrix}$$

$$= (2)(+1)(48) + 0 + (3)(+1)(-40)$$

$$= -24$$

$$\Delta_z = \begin{vmatrix} 2 & -1 & 0 \\ -4 & 1 & 2 \\ 4 & 3 & 8 \end{vmatrix}$$

$$= (2)(+1)(2) + (-1)(-1)(-40)$$

$$= 4 - 40$$

$$= -36$$

$$\therefore \; x = \frac{\Delta_x}{\Delta} = \frac{7}{2},$$

$$y = \frac{\Delta_y}{\Delta} = -2,$$

$$z = \frac{\Delta_z}{\Delta} = -3$$

$$\therefore \; \text{S.S.} = \left\{\left(\frac{7}{2}, -2, -3\right)\right\}$$

Thank you