Example 5 Find an equation of the plane through the point (2, 4, –1) with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

Solution

The standard equations of a plane in space is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

In Equation (1), putting

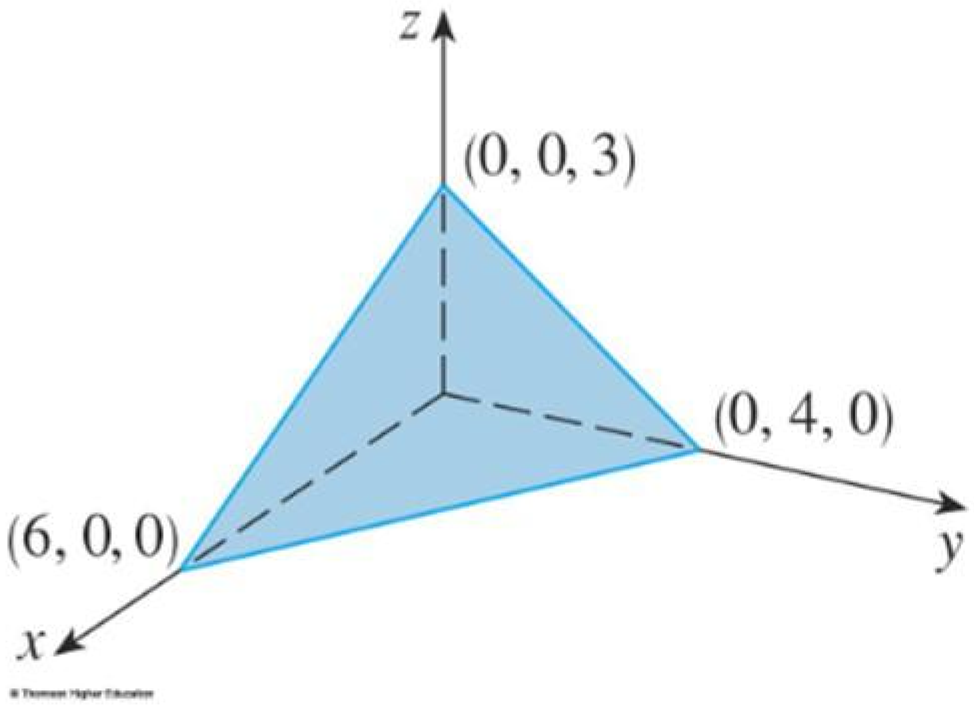
$$a = 2, b = 3, c = 4, x_0 = 2, y_0 = 4, z_0 = -1$$

we see that an equation of the plane is:

$$2(x-2) + 3(y-4) + 4(z+1) = 0$$

or $2x + 3y + 4z = 12$

- To find the *x*-intercept, we set y = z = 0 in the plane equation and obtain *x* = 6.
- To find the *y*-intercept, we set x = z = 0 in the plane equation and obtain *y* = 4.
- To find the *y*-intercept, we set x = z = 0 in the plane equation and obtain *y* = 4.

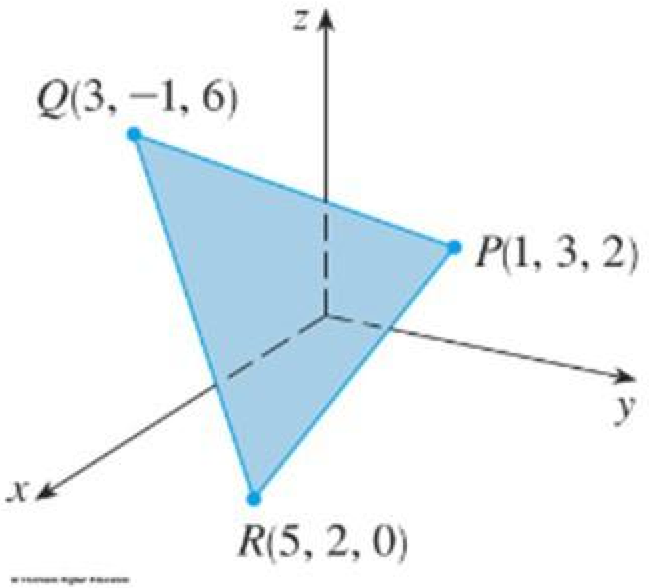


Example 6 Find an equation of the plane that passes through the points

$P(1, 3, 2)$, $Q(3, -1, 6)$, $R(5, 2, 0)$

Solution

The vectors $\vec{a}$ and $\vec{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are:



$$\vec{a} = \overrightarrow{PQ} = \langle 3-1, -1-3, 6-2 \rangle = \langle 2, -4, 4 \rangle,$$

$$\vec{b} = \overrightarrow{PR} = \langle 5-1, 2-3, 0-2 \rangle = \langle 4, -1, -2 \rangle.$$

Since both $\vec{a}$ and $\vec{b}$ lie in the plane, their cross product $\vec{a} \times \vec{b}$ is orthogonal to the plane and can be taken as the normal vector.

Thus,

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\vec{i} + 20\vec{j} + 14\vec{k}$$

With the point $P(1, 2, 3)$ and the normal vector n, an equation of the plane is:

$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$ or $6x + 10y + 7z = 50$

Example 7 Find the point at which the line with parametric equations

$$x = 2 + 3t \qquad y = -4t \qquad z = 5 + t$$

intersects the plane $4x + 5y - 2z = 18$.

Solution

We substitute the expressions for x, y, and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

That simplifies to $-10t = 20$. Hence, $t = -2$.

Therefore, the point of intersection occurs when the parameter value is $t = -2$.

Then,

$$x = 2 + 3(-2) = -4, \qquad y = -4(-2) = 8, \qquad z = 5 - 2 = 3$$

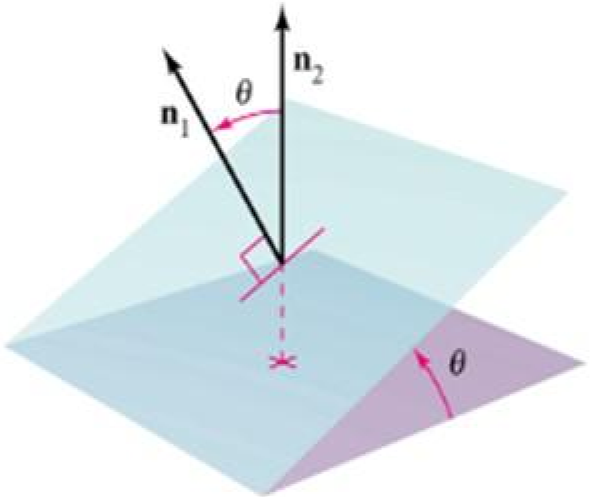
So, the point of intersection is $(-4, 8, 3)$.

Planes in Space

Two planes in space with normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ are either parallel or intersect in a line.

- They are perpendicular if and only if their normal vectors are perpendicular.
- They are parallel if and only if their normal vectors are parallel.
- The angle between two planes is equal to the angle between the normal vectors are given by

$$\cos\theta = \frac{|n_1 \cdot n_2|}{\|n_1\| \, \|n_2\|}$$



- To find the line of intersection between two planes, solve the system of two equations with three unknowns to get a point on this line.
- The line of intersection is parallel to $n_1 \times n_2$

Example 8 Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$

Solution

The normal vectors of these planes are:

$$\vec{n}_1 = \langle 1,1,1 \rangle, \quad \vec{n}_2 = \langle 1,-2,3 \rangle.$$

So, if θ is the angle between the planes,

$$\cos\theta = \frac{|n_1 \cdot n_2|}{\|n_1\|\,\|n_2\|} = \frac{|1(1)+1(-2)+1(3)|}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

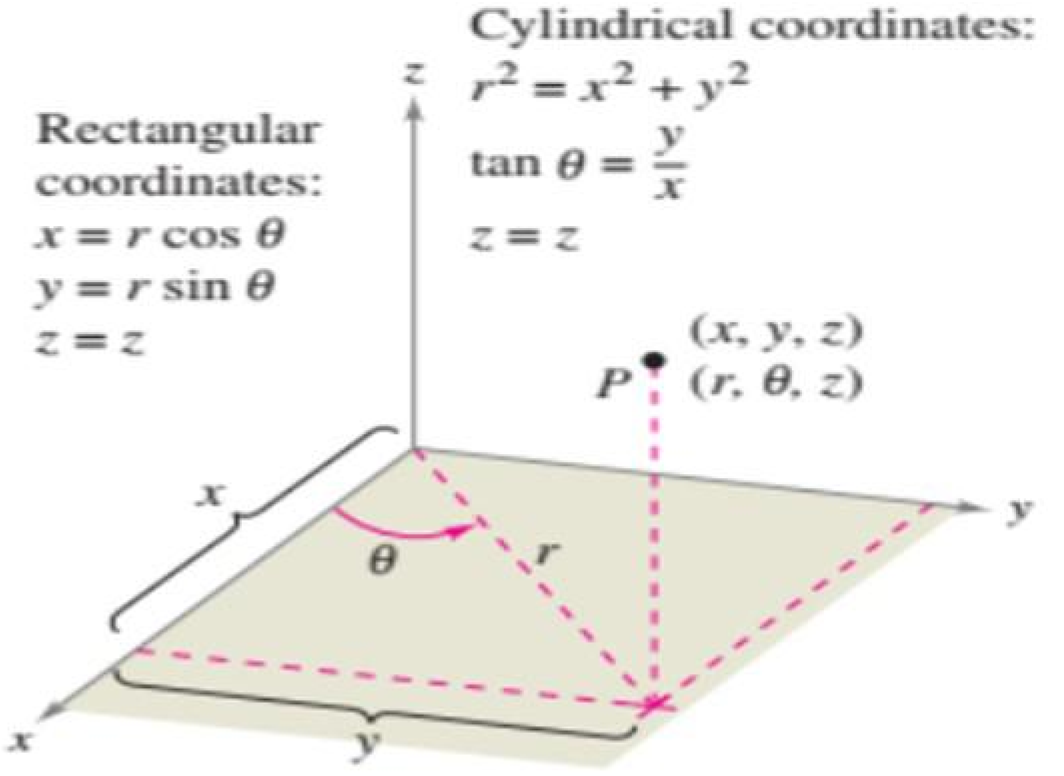
$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

Cylindrical and Spherical Coordinates

The Cylindrical Coordinate System

In a **cylindrical coordinate system,** a point P in space is represented by an ordered triple (r, θ, z).

1. (r, θ) is a polar representation of the projection of P in the xy-plane.
2. z is the directed distance from (r, θ) to P.



The Spherical Coordinate System

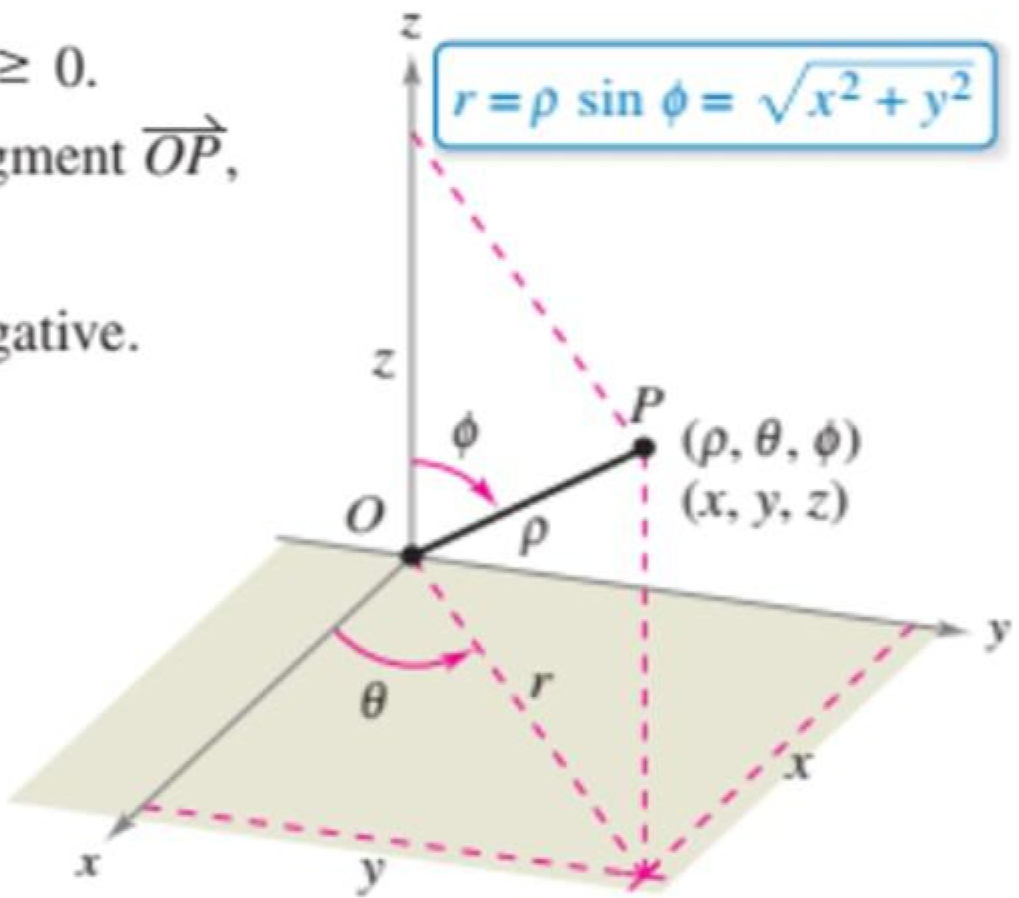
In a **spherical coordinate system,** a point P in space is represented by an ordered triple (ρ, θ, ϕ), where ρ is the lowercase Greek letter rho and ϕ is the lowercase Greek letter phi.

1. ρ is the distance between P and the origin, $\rho \geq 0$.

2. θ is the same angle used in cylindrical coordinates for $r \geq 0$.

3. ϕ is the angle *between* the positive z-axis and the line segment $\overrightarrow{OP}$, $0 \leq \phi \leq \pi$.

Note that the first and third coordinates, ρ and ϕ, are nonnegative.



conversion formulas for coordinate systems

CONVERSION		FORMULAS	RESTRICTIONS
Cylindrical to rectangular Rectangular to cylindrical	$(r, \theta, z) \to (x, y, z)$ $(x, y, z) \to (r, \theta, z)$	$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$ $r = \sqrt{x^2 + y^2}, \ \tan\theta = y/x, \ z = z$	$r \geq 0, \rho \geq 0$ $0 \leq \theta < 2\pi$ $0 \leq \phi \leq \pi$
Spherical to cylindrical Cylindrical to spherical	$(\rho, \theta, \phi) \to (r, \theta, z)$ $(r, \theta, z) \to (\rho, \theta, \phi)$	$r = \rho\sin\phi, \quad \theta = \theta, \quad z = \rho\cos\phi$ $\rho = \sqrt{r^2 + z^2}, \ \theta = \theta, \ \tan\phi = r/z$	
Spherical to rectangular Rectangular to spherical	$(\rho, \theta, \phi) \to (x, y, z)$ $(x, y, z) \to (\rho, \theta, \phi)$	$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$ $\rho = \sqrt{x^2 + y^2 + z^2}, \ \tan\theta = y/x, \ \cos\phi = z/\sqrt{x^2 + y^2 + z^2}$	

Example 9 Find the rectangular coordinates of the point with cylindrical coordinates

$$(r,\theta,z)=\left(4,\frac{\pi}{3},-3\right)$$

Solution

Applying the cylindrical-to-rectangular conversion formulas, yields:

$$x=r\cos\theta=4\cos\left(\frac{\pi}{3}\right)=4\left(\frac{1}{2}\right)=2$$

$$y=r\sin\theta=4\sin\left(\frac{\pi}{3}\right)=4\left(\frac{\sqrt{3}}{2}\right)=2\sqrt{3}$$

$$z=-3.$$

Thus, the rectangular coordinates of the point are $(x,y,z)=\left(2,2\sqrt{3},-3\right)$.

Example 10 Find the rectangular coordinates of the point with spherical coordinates

$$(\rho,\theta,\phi)=\left(4,\frac{\pi}{3},\frac{\pi}{4}\right)$$

Solution

Applying the spherical -to-rectangular conversion formulas, yields:

$$x=\rho\sin\varphi\cos\theta=4\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)=4\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)=\sqrt{2},$$

$$y=\rho\sin\varphi\sin\theta=4\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)=4\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)=\sqrt{6},$$

$$z=\rho\cos\varphi=4\cos\left(\frac{\pi}{4}\right)=4\left(\frac{1}{\sqrt{2}}\right)=2\sqrt{2}.$$

Thus, the rectangular coordinates of the point are $(x,y,z)=\left(\sqrt{2},\sqrt{6},2\sqrt{2}\right)$.

Example 11 Find the spherical coordinates of the point with rectangular coordinates

$$(x, y, z) = \left(4, -4, 4\sqrt{6}\right).$$

Solution

Applying the rectangular -to-spherical conversion formulas, yields:

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{4^2 + (-4)^2 + \left(4\sqrt{6}\right)^2} = \sqrt{128} = 8\sqrt{2},$$

$$\tan\theta = \frac{y}{x} = \frac{-4}{4} = -1,$$

$$\cos\varphi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2}$$

From the restriction $0 \le \theta < 2\pi$ and the computed value of *tan(θ)*, the possibilities for θ are $\theta = 3\pi/4$ and $\theta = 7\pi/4$. However, the given point has a negative *y*-coordinate, so we must have $\theta = 7\pi/4$. Moreover, from the restriction $0 \le \phi \le \pi$ and the computed value of $\cos\phi$, the only possibility for ϕ is $\phi = \pi/6$. Thus, the spherical coordinates of the point

$$(\rho, \theta, \phi) = \left(8\sqrt{2}, \frac{7\pi}{4}, \frac{\pi}{6}\right)$$

Example 12 Find the cylindrical coordinates of the point with rectangular coordinates

$$(x, y, z) = \left(-\sqrt{2}, \sqrt{2}, 1\right).$$

Solution

Applying the cylindrical-to-rectangular conversion formulas, yields:

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = 2$$

$$\tan(\theta) = \frac{y}{x} = \frac{\sqrt{2}}{-\sqrt{2}} = -1 \Rightarrow \theta = \frac{3\pi}{4}$$

$$z = 1.$$

Thus, the rectangular coordinates of the point are $(r, \theta, z) = \left(2, \frac{3\pi}{4}, 1\right)$

Exercises

1) Find an equation of the plane passing

   a) through the points $(3,-1,2)$, $(2,1,5)$, $(1,-2,-2)$

   b) through the points (3,2,1), (3,1,-5) and is perpendicular to $6x+7y+2z=10$

2) Show that the 2 lines are parallel

$$\begin{cases} L_1 : x = 1+t, \ \ y = 1+2t, \ \ z = 5+t \\ L_2 : x = -5-2s, \ \ y = 3-4s, \ \ z = -2s \end{cases}$$

3) Find the angle between the two planes and the line of intersection of the two plane $x-2y+z=0$ and $2x+3y-2z=0$.

4) Find parametric equations and symmetric equations of the line through the point (2,3,1) and is parallel to $x=-5t,\ y=4-2t\ ,\ z=3$

5) Convert the point from cylindrical coordinates to rectangular coordinates

1. $(5, 0, 2)$ **2.** $(4, \pi/2, -2)$

3. $(2, \pi/3, 2)$ **4.** $(6, -\pi/4, 2)$

5. $(4, 7\pi/6, 3)$ **6.** $(1, 3\pi/2, 1)$

6) Convert the point from rectangular coordinates to cylindrical coordinates

7. $(0, 5, 1)$ **8.** $(2\sqrt{2}, -2\sqrt{2}, 4)$

9. $(1, \sqrt{3}, 4)$ **10.** $(2\sqrt{3}, -2, 6)$

11. $(2, -2, -4)$ **12.** $(-3, 2, -1)$

7) Convert the point from rectangular coordinates to spherical coordinates

29. $(4, 0, 0)$ **30.** $(1, 1, 1)$

31. $(-2, 2\sqrt{3}, 4)$ **32.** $(2, 2, 4\sqrt{2})$

33. $(\sqrt{3}, 1, 2\sqrt{3})$ **34.** $(-4, 0, 0)$

8) Convert the point from cylindrical coordinates to spherical coordinates

57. $(4, \pi/4, 0)$ **58.** $(3, -\pi/4, 0)$

59. $(4, \pi/2, 4)$ **60.** $(2, 2\pi/3, -2)$

61. $(4, -\pi/6, 6)$ **62.** $(-4, \pi/3, 4)$

63. $(12, \pi, 5)$ **64.** $(4, \pi/2, 3)$