Section 9

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Nonhomogeneous Differential Equations

It's now time to start thinking about how to solve nonhomogeneous differential equations. A second order, linear nonhomogeneous differential equation is

$$y'' + p(t)y' + q(t)y = g(t)$$
 (1)

where g(t) is a non-zero function.

The general solution to a differential equation can then be written as.

$$y\left(t\right)=y_{c}\left(t\right)+Y_{P}\left(t\right)$$

g(t)	$Y_P(t)$ guess		
$a\mathbf{e}^{eta t}$	$A\mathbf{e}^{eta t}$		
$a\cos(eta t)$	$A\cos(eta t) + B\sin(eta t)$		
$b\sin(eta t)$	$A\cos(eta t) + B\sin(eta t)$		
$a\cos(eta t) + b\sin(eta t)$	$A\cos(eta t) + B\sin(eta t)$		
$n^{ m th}$ degree polynomial	$A_nt^n+A_{n-1}t^{n-1}+\cdots A_1t+A_0$		

r(x)	Initial guess for $y_p(x)$			
k (a constant)	A (a constant)			
ax + b	Ax+B (<i>Note</i> : The guess must include both terms even if $b=0$.)			
ax^2+bx+c	Ax^2+Bx+C (Note: The guess must include all three terms even if b or c are zero.)			
Higher-order polynomials	Polynomial of the same order as $r(x)$			
$ae^{\lambda x}$	$Ae^{\lambda x}$			
$a\cos eta x + b\sin eta x$	$A\cos eta x + B\sin eta x$ (<i>Note</i> : The guess must include both terms even if either $a=0$ or $b=0$.)			
$ae^{\alpha x}\cos\beta x + be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos eta x+Be^{lpha x}\sin eta x$			
$(ax^2+bx+c)e^{\lambda x}$	$(Ax^2+Bx+C)e^{\lambda x}$			
$(a_2x^2+a_1x+a_0)\cos eta x \ +(b_2x^2+b_1x+b_0)\sin eta x$	$(A_2x^2 + A_1x + A_0)\cos\beta x \ + (B_2x^2 + B_1x + B_0)\sin\beta x$			
$(a_2 x^2 + a_1 x + a_0) e^{\alpha x} \cos \beta x \ + (b_2 x^2 + b_1 x + b_0) e^{\alpha x} \sin \beta x$	$(A_2 x^2 + A_1 x + A_0) e^{\alpha x} \cos \beta x \ + (B_2 x^2 + B_1 x + B_0) e^{\alpha x} \sin \beta x$			

Solve the following D.E:-

$$y'' + 2y' + y = 4e^{-2x}$$

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + 4e^{-2x}$$

First we solve the related homogeneous D.E:-

$$y^{\prime\prime} + 2y^{\prime} + y = 0$$

the roots of the characteristic equation are:-

$$r^{2} + 2r + 1 = 0$$

 $(r + 1)(r + 1) = 0$
 $\therefore r_{1,2} = -1$

Hence the general solution of the homogeneous equation is given by :

$$y_o(x) = C_1 e^{-x} + C_2 x e^{-x}$$

Based on the form " $g(x) = 4e^{-2x}$ " the particular solution would be in the form of : " $y_p = Ae^{-2x}$ "

The derivatives are given by :-

$$y_p' = -2Ae^{-2x}$$

 $y_p'' = 4Ae^{-2x}$
 $y_p'' + 2y_p' + y_p = 4e^{-2x}$
 $Ae^{-2x} - 4Ae^{-2x} + Ae^{-2x} = 4e^{-2x}$
 $Ae^{-2x} = 4e^{-2x}$
 $Ae^{-2x} = 4e^{-2x}$
 $Ae^{-2x} = 4e^{-2x}$
 $Ae^{-2x} = 4e^{-2x}$

Hence the general solution of the homogeneous equation is given by :

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + 4e^{-2x}$$

Solve the differential equation $y'' - 5y' + 4y = e^{4x}$

First we solve the related homogeneous equation y'' - 5y' + 4y = 0. The roots of the characteristic equation are

$$k^2 - 5k + 4 = 0, \ \Rightarrow D = 25 - 4 \cdot 4 = 9, \ \Rightarrow k_{1,2} = \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2} = 4, 1.$$

Hence, the general solution of the homogeneous equation is given by

$$y_{0}\left(x
ight) =C_{1}e^{4x}+C_{2}e^{x},$$

where C_1 , C_2 are constant numbers.

Find a particular solution of the nonhomogeneous differential equation. Notice that the power of the exponential function on the right coincides with the root $k_1=4$ of the auxiliary characteristic equation. Therefore we will look for a particular solution of the form

$$y_1 = Axe^{4x}$$
.

The derivatives are given by

$$y_1' = (Axe^{4x})' = Ae^{4x} + 4Axe^{4x} = (A+4Ax)e^{4x};$$

$$y_1'' = \left[(A + 4Ax) e^{4x} \right]' = 4Ae^{4x} + (4A + 16Ax) e^{4x} = (8A + 16Ax) e^{4x}.$$

Substituting the function y_1 and its derivatives in the differential equation yields:

$$(8A + 16Ax) e^{4x} - 5(A + 4Ax) e^{4x} + 4Axe^{4x} = e^{4x},$$

 $\Rightarrow 8A + 16Ax - 5A - 20Ax + 4Ax = 1, \Rightarrow 3A = 1, \Rightarrow A = \frac{1}{3}.$

Thus, the particular solution to the differential equation can be written in the form:

$$y_1 = \frac{x}{3}e^{4x}.$$

Now we can write the full solution of the nonhomogeneous equation:

$$y = y_0 + y_1 = C_1 e^{4x} + C_2 e^x + \frac{x}{3} e^{4x}.$$

Find the general solution of the equation $y'' + 9y = 2x^2 - 5$.

First we determine the general solution of the related homogeneous equation. Solve the auxiliary characteristic equation:

$$k^2 + 9 = 0$$
, $\Rightarrow k^2 = -9$, $\Rightarrow k_{1,2} = \pm 3i$.

The solution is written in the form:

$$y_0(x) = C_1 \cos 3x + C_2 \sin 3x.$$

Now we construct a particular solution. The right-hand side of the given equation is a quadratic function. So we can guess on a particular solution of the same form:

$$y_1 = Ax^2 + Bx + C,$$

where the numbers A, B, C can be determined by the method of undetermined coefficients. Hence, we can write:

$$y_1' = 2Ax, \ y_1'' = 2A.$$

Substituting this into the original nonhomogeneous differential equation, we have

$$2A + 9(Ax^2 + Bx + C) = 2x^2 - 5, \ \Rightarrow 2A + 9Ax^2 + 9Bx + 9C = 2x^2 - 5.$$

By equating the coefficients of like powers of x, we obtain:

$$\left\{ egin{array}{ll} 9A = 2 \ 9B = 0 \ 2A + 9C = -5 \end{array}
ight. \; \Rightarrow \left\{ egin{array}{ll} A = rac{2}{9} \ B = 0 \ C = -rac{49}{81} \end{array}
ight. \;
ight.$$

Solve the following D.E:-

$$y'' + 2y' = 24x + e^{-2x}$$

First we solve the related homogeneous D.E:-

$$y'' + 2y' = 24x + e^{-2x}$$

the roots of the characteristic equation are:-

$$r^{2} + 2r = 0$$

$$r(r + 2) = 0$$

$$r_{1} = 0 \quad and \quad r_{2} = -2$$

Hence the general solution of the homogeneous equation is given by :

$$y_o(x) = C_1 + C_2 e^{-2x}$$

Based on the form " $g(x) = 24x + e^{-2x}$ " the particular solution would be in the form of : " $y_p = Ax^2 + Bx + Cxe^{-2x}$ "

The derivatives are given by :-

$$y_p' = 2Ax + B + C(e^{-2x} - 2xe^{-2x})$$

$$y_p' = 2Ax + B + Ce^{-2x} - 2Cxe^{-2x}$$

$$y_p'' = 2A - 2Ce^{-2x} - 2C(e^{-2x} - 2xe^{-2x})$$

$$y_p'' = 2A - 2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x}$$

$$y_p'' + 2y' = 24x + e^{-2x}$$

$$2A - 2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x} + 4Ax + 2B + 2Ce^{-2x} - 4Cxe^{-2x} = 24x + e^{-2x}$$

$$(2A + 2B) + 4Ax - 2Ce^{-2x} = 24x + e^{-2x}$$

$$4A = 24, A = 6$$

$$-2C = 1, \quad C = -0.5$$

$$(2A + 2B) = 0, \quad B = -6$$

Hence the general solution of the homogeneous equation is given by:

$$y(x) = C_1 e^{-x} + C_2 e^{-x} + 6x^2 - 6x - 0.5xe^{-2x}$$

Thus, the particular solution is given by

$$y_1 = \frac{2}{9}x^2 - \frac{49}{81}$$
.

Then the general solution of the original nonhomegeneous differential equation is expressed by the formula

$$y=y_0+y_1=C_1\cos 3x+C_2\sin 3x+rac{2}{9}x^2-rac{49}{81}.$$

Solve the differential equation $y'' + 16y = 2\cos^2 x$.

First of all we solve the related homogeneous equation. The characteristic equation has roots:

$$k^2 + 16 = 0$$
, $\Rightarrow k^2 = -16$, $\Rightarrow k_{1,2} = \pm 4i$,

so the general solution has the form:

$$y_0(x) = C_1 \cos 4x + C_2 \sin 4x.$$

Now we find a particular solution for the nonhomogeneous equation. Rewrite the right-hand side as

$$2\cos^2 x = \cos 2x + 1.$$

It follows from here that the particular solution is defined by the function

$$y_1 = A\cos 2x + B\sin 2x + C,$$

where the numbers A, B, and C can be calculated using the method of undetermined coefficients. The first and second derivatives of the function y_1 are

$$y_1' = -2A\sin 2x + 2B\cos 2x,$$

$$y_1'' = -4A\cos 2x - 4B\sin 2x.$$

Substituting this back into the differential equation produces:

$$-4A\cos 2x - 4B\sin 2x + 16(A\cos 2x + B\sin 2x + C) = \cos 2x + 1$$

$$-4A\cos 2x - 4B\sin 2x + 16A\cos 2x + 16B\sin 2x + 16C = \cos 2x + 1$$

$$12A\cos 2x + 12B\sin 2x + 16C = \cos 2x + 1.$$

The latter expression is identical. Therefore we can write the following system of equations to determine the coefficients A, B, C:

$$\left\{ egin{array}{ll} 12A = 1 \ 12B = 0 \ , \end{array}
ight. \Rightarrow \left\{ egin{array}{ll} A = rac{1}{12} \ B = 0 \ . \end{array}
ight. \ C = rac{1}{16} \end{array}
ight. .$$

Thus, the particular solution has the form:

$$y_1 = \frac{1}{12}\cos 2x + \frac{1}{16}.$$

Respectively, the general solution of the original nonhomogeneous equation is written as

$$y = y_0 + y_1 = C_1 \cos 4x + C_2 \sin 4x + \frac{1}{12} \cos 2x + \frac{1}{16}$$
.

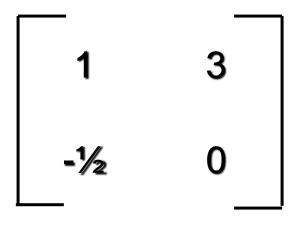
Determinants

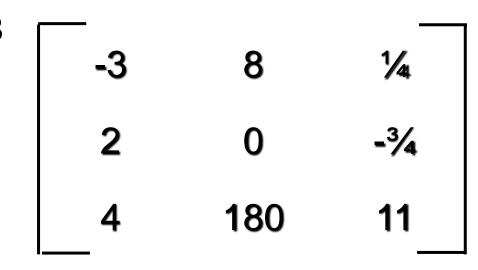
2 x 2 and 3 x 3 Matrices

Matrices

- A matrix is an array of numbers that are arranged in rows and columns.
- A matrix is "square" if it has the same number of rows as columns.
- We will consider only 2x2 and 3x3 square matrices

Note that Matrix is the singular form, matrices is the plural form!





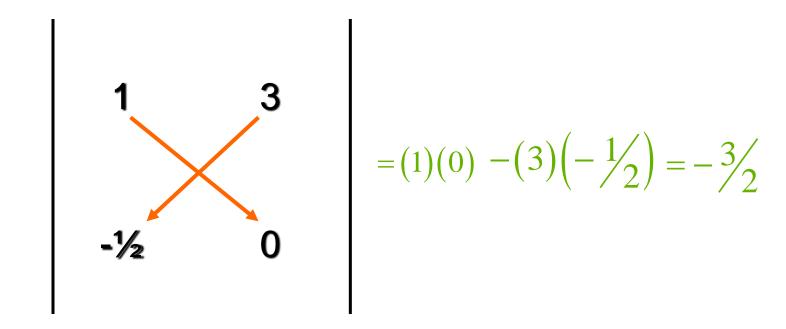
Determinants

Note the difference in the matrix and the determinant of the matrix!

- Every square matrix has a determinant.
- The determinant of a matrix is a number.
- We will consider the determinants only of 2x2 and 3x3 matrices.

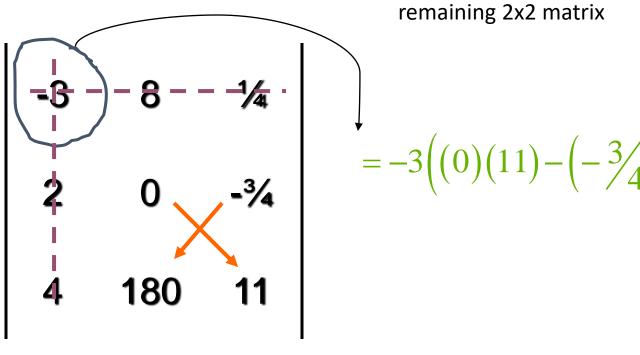
	1		3		
	-1/2		0		
ı				,	
-<	3	8		1/4	
2	2	0		-3/4	
4	ļ	180		11	

Determinant of a 2x2 matrix



Determinant of a 3x3 matrix

Imagine crossing out the first row. And the first column.

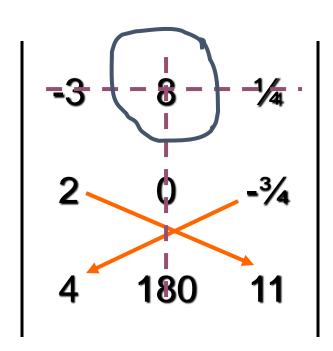


Now take the double-crossed element. . . And multiply it by the determinant of the

$$= -3\left((0)(11) - \left(-\frac{3}{4}\right)(180)\right)$$

Determinant of a 3x3 matrix

Now keep the first row crossed. Cross out the second column.

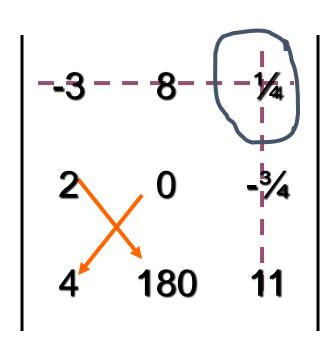


- •Now take the negative of the doublecrossed element.
- •And multiply it by the determinant of the remaining 2x2 matrix.
- Add it to the previous result.

$$=-3\Big((0)\big(11\big)-\Big(-\frac{3}{4}\Big)\big(180\big)\Big)-8\Big((2)\big(11\big)-\Big(-\frac{3}{4}\Big)\big(4\big)\Big)$$

Determinant of a 3x3 matrix

Finally, cross out first row and last column.



- •Now take the double-crossed element.
- •Multiply it by the determinant of the remaining 2x2 matrix.
- •Then add it to the previous piece.

$$= -3((0)(11) - (-\frac{3}{4})(180)) - 8((2)(11) - (-\frac{3}{4})(4))$$
$$+ (\frac{1}{4})((2)(180) - (0)(4)) = -515$$

Cramer's Rule for Solution of Linear Equations:

If
$$a_1 x + b_1 y + c_1 z = d_1$$

 $a_2 x + b_2 y + c_2 z = d_2$
 $a_3 x + b_3 y + c_3 z = d_3$
with $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$
Then $x = \frac{1}{\Delta} \Delta_x$, $y = \frac{1}{\Delta} \Delta_y$, $z = \frac{1}{\Delta} \Delta_z$
where $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Remember: Cramer's rule can be fruitfully applied in case of $\Delta \neq 0$.

Use Cramer's rule to solve the system of equations:

(1)
$$2x + 3y - z = 0$$
,

$$x + 2z - 3y = 2$$
,

$$y+z+2=0$$

$$\{(1,-1,-1)\}$$

(2)
$$x-2y+2z=1$$
,

$$3x + 4z = 8$$
,

$$6z-y=2$$

$$\left\{ \left(2,1,\frac{1}{2}\right)\right\}$$

(3)
$$x-y-z=3$$
,

$$3x+y=2,$$

$$2y + 3z = 1$$

$$\{(2,-4,3)\}$$

(4)
$$2x-y+3z=0$$
,

$$y - 4x - 6z = 2$$

$$4x + 3y = 8$$
 $\left\{ \left(\frac{7}{2}, -2, -3 \right) \right\}$

$$\left\{ \left(\frac{7}{2}, -2, -3\right) \right\}$$

Use Cramer's rule to solve the system of equations:

$$2x + 3y - z = 0$$
, $x + 2z - 3y = 2$, $y + z + 2 = 0$

$$x + 2z - 3y = 2$$

$$y+z+2=0$$

$$\{(1,-1,-1)\}$$

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -3 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (2)(+1)(-5)+(1)(-1)(4)+0$$

$$= -10-4$$

$$= -14$$

$$\Delta_{x} = \begin{vmatrix} 0 & 3 & -1 \\ 2 & -3 & 2 \end{vmatrix}$$

$$= 0 + (3)(-1)(6) + (-1)(+1)(-4)$$

$$= -18 + 4$$

$$= -14$$

$$\Delta_{y} = \begin{vmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= (2)(+1)(6)+0+(-1)(+1)(-2)$$

$$= 12+2$$

$$= 14$$

$$\Delta_{z} = \begin{vmatrix} 2 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= (2)(+1)(4)+(3)(-1)(-2)+0$$

$$= 8+6$$

$$= 14$$

$$x = \frac{\Delta_y}{\Delta} = 1,$$

$$y = \frac{\Delta_y}{\Delta} = -1,$$

$$z = \frac{\Delta_z}{\Delta} = -1$$

 $S.S. = \{(1, -1, -1)\}$

Use Cramer's rule to solve the system of equations:

$$x-2y+2z=1$$
,

$$3x + 4z = 8$$
,

$$6z-y=2$$

$$\left\{ \left(2,1,\frac{1}{2}\right)\right\}$$

$$\Delta = \begin{vmatrix} 1 & -2 & 2 \\ 3 & 0 & 4 \\ 0 & -1 & 6 \end{vmatrix}$$

$$= (1)(+1)(4)+(3)(-1)(-10)+0$$

$$= 4+30$$

$$= 34$$

$$\Delta_x = \begin{vmatrix} 1 & -2 & 2 \\ 8 & 0 & 4 \\ 2 & -1 & 6 \end{vmatrix}$$

$$= (8)(-1)(-10)+0+(4)(-1)(3)$$

$$= 80-12$$

$$= 68$$

$$\Delta_{y} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 8 & 4 \\ 0 & 2 & 6 \end{vmatrix}$$

$$= (1)(+1)(40)+(3)(-1)(2)+0$$

$$= 40-6$$

$$= 34$$

$$\Delta_{z} = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 0 & 8 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= (1)(+1)(8)+(3)(-1)(-3)+0$$

$$= 8+9$$

$$= 17$$

$$x = \frac{\Delta_x}{\Delta} = 2,$$

$$y = \frac{\Delta_y}{\Delta} = 1,$$

$$z = \frac{\Delta_z}{\Delta} = \frac{1}{2}$$

$$S.S. = \left\{ \left(2, 1, \frac{1}{2} \right) \right\}$$

Use Cramer's rule to solve the system of equations:

$$x-y-z=3,$$

$$3x+y=2,$$

$$2y + 3z = 1$$

$$\{(2,-4,3)\}$$

$$\Delta = \begin{vmatrix} 1 & -1 & -1 \\ 3 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= (1)(+1)(3)+(3)(-1)(-1)+0$$

$$= 3+3$$

$$= 6$$

$$\Delta_x = \begin{vmatrix} 3 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= (-1)(+1)(3)+0+(3)(+1)(5)$$

$$= -3+15$$

$$= 12$$

$$\Delta_y = \begin{vmatrix} 1 & 3 & -1 \\ 3 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= (1)(+1)(6)+(3)(-1)(10)+0$$

$$= 6-30$$

$$= -24$$

$$\Delta_{z} = \begin{vmatrix} 1 & -1 & 3 \\ 3 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= (1)(+1)(-3)+(3)(-1)(-7)$$

$$= -3+21$$

$$= 18$$

$$\therefore x = \frac{\Delta_{x}}{\Delta} = 2,$$

$$y = \frac{\Delta_{y}}{\Delta} = -4,$$

$$S.S. = \{ (2, -4, 3) \}$$

Use Cramer's rule to solve the system of equations:

$$2x-y+3z=0$$
,

$$y - 4x - 6z = 2$$

$$4x + 3y = 8$$
 $\{(\frac{7}{2}, -2, -3)\}$

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ -4 & 1 & -6 \\ 4 & 3 & 0 \end{vmatrix}$$

$$= (3)(+1)(-16)+(-6)(-1)(10)+0$$

$$= -48+60$$

$$= 12$$

$$\Delta_{x} = \begin{vmatrix} 0 & -1 & 3 \\ 2 & 1 & -6 \\ 8 & 3 & 0 \end{vmatrix}$$

$$= 0+(-1)(-1)(48)+(3)(+1)(-2)$$

$$= 48-6$$

$$= 42$$

$$\Delta_{y} = \begin{vmatrix} 2 & 0 & 3 \\ -4 & 2 & -6 \\ 4 & 8 & 0 \end{vmatrix}$$

$$= (2)(+1)(48)+0+(3)(+1)(-40)$$

$$= -24$$

$$\Delta_{z} = \begin{vmatrix} 2 & -1 & 0 \\ -4 & 1 & 2 \\ 4 & 3 & 8 \end{vmatrix}$$

$$= (2)(+1)(2)+(-1)(-1)(-40)$$

$$= 4-40$$

$$= -36$$

$$\therefore x = \frac{\Delta_{z}}{\Delta} = \frac{7}{2},$$

$$y = \frac{\Delta_{y}}{\Delta} = -2,$$

$$z = \frac{\Delta_{z}}{\Delta} = -3$$

$$\therefore \text{ S.S.} = \left\{ \left(\frac{7}{2}, -2, -3\right) \right\}$$

Thank you