Section 4

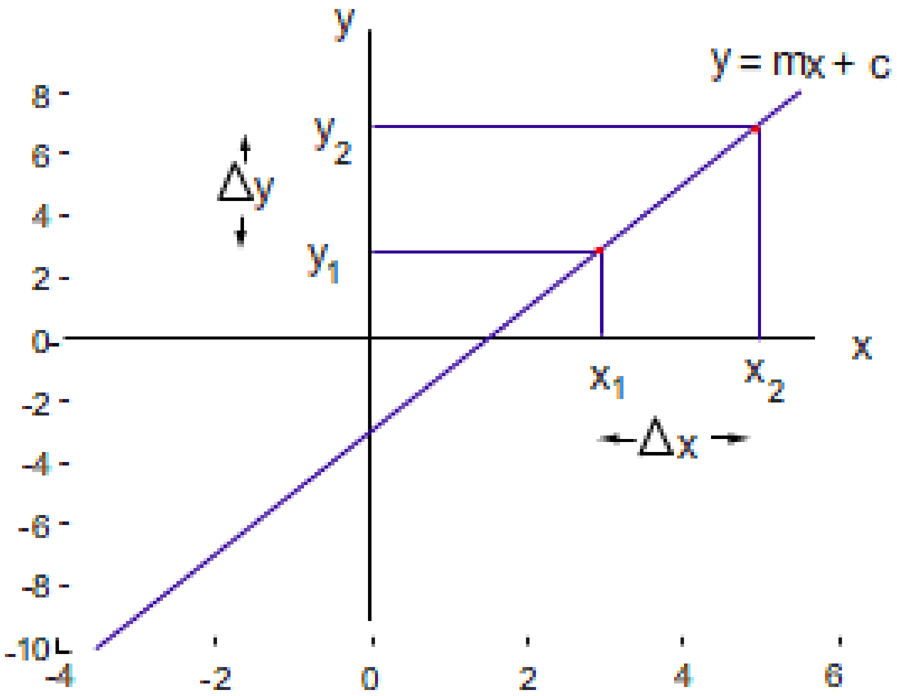
Mathematics 1

Slopes , tangent Lines and derivatives:

- **Slope definition:**
  It is the change in the dependent variable (y) between two points divided by the relative change in the independent variable *x*

- Differentiation is all about calculating the slope or slope of a curve y(x), at a given point, x.

$$slope = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



- For a straight-line graph of equation *y*(*x*) = *mx* + *c*, the slope is given simply by the value of *m*.

Example 1:

What is the slope of $y(x) = x^2 - 4x - 1$ at the point $x = 4$

Solution

$$slope = \frac{dy}{dx} = 2x - 4$$

So when $x = 4$

$$\frac{dy}{dx} = (2 \times 4) - 4 = 4$$

Example 2:

What is the slope of $y = 3x^2 + 2x + 7$ at the point x = 2

Solution

$$\frac{dy}{dx} = 6x + 4$$

So when $x = 2$

$$\frac{dy}{dx} = (6 \times 2) + 4 = 16$$

Example 3:

What is the slope of $y = 6x^2 - 5x + 3$ at the point x = 2

Solution

$$\frac{dy}{dx} = 12x - 5$$

So when $x = 2$

$$\frac{dy}{dx} = (12 \times 2) - 5 = 19$$

Example 4:

What is the slope of $y = \sqrt{2x} + 2\sqrt{x}$ at the point $x = 1$

Solution

$$\frac{dy}{dx} = \frac{1 + \sqrt{2}}{\sqrt{2x}}$$

So when $x = 1$

$$\frac{dy}{dx} = \frac{1 + \sqrt{2}}{\sqrt{2 \times 1}} = \frac{2 + \sqrt{2}}{2} = 1.707$$

Example 5:

What is the slope of $y = \frac{3x+2}{2x+3}$ at the point $x = 1$

Solution

$$\frac{dy}{dx} = \frac{5}{(2x+3)^2}$$

So when $x = 1$

$$\frac{dy}{dx} = \frac{5}{(2 \times 1 + 3)^2} = \frac{1}{5}$$

Given that $f(x) = \frac{x^2+x-2}{x^3+6}$, find f\`(x)

Solution

$$f`(x) = \frac{(x^3+6)\frac{d}{dx}(x^2+x-2) - (x^2+x-2)\frac{d}{dx}(x^3+6)}{(x^3+6)^2}$$

$$= \frac{(x^3+6)(2x+1) - (x^2+x-2)\,(3x)}{(x^3+6)^2}$$

$$= \frac{(2x^4+x^3+12x+6) - (3x^4+3x^3-6x^2)}{(x^3+6)^2}$$

$$= \frac{-x^4-2x^3-6x^2+12x+6}{(x^3+6)^2}$$

Given that $f(x) = \frac{e^{-3x}}{x^2+2}$, find f`(x)

Ans: $f`(x) = \frac{-(3x^2+2x+3)e^{-3x}}{(x^2+1)^2}$

Given that $f(x) = \frac{2x+1}{x^2-3}$, find f`(x)

Ans: $f`(x) = \frac{-2(x^2+x+3)}{(x^2-3)^2}$

Determine all the critical points for the function.

1. $f\left(x\right)=6x^5+33x^4-30x^3+100$

Solution

$$\begin{aligned}f'\left(x\right)&=30x^4+132x^3-90x^2\\&=6x^2\left(5x^2+22x-15\right)\\&=6x^2\left(5x-3\right)\left(x+5\right)\end{aligned}$$

$$x=-5,\quad x=0,\quad x=\frac{3}{5}$$

2. $g(t) = \sqrt[3]{t^2}\,(2t - 1)$

Solution

$$g(t) = t^{\frac{2}{3}}(2t - 1) = 2t^{\frac{5}{3}} - t^{\frac{2}{3}}$$

$$g'(t) = \frac{10}{3}t^{\frac{2}{3}} - \frac{2}{3}t^{-\frac{1}{3}} = \frac{10t^{\frac{2}{3}}}{3} - \frac{2}{3t^{\frac{1}{3}}}$$

$$g'(t) = \frac{10t - 2}{3t^{\frac{1}{3}}}$$

$$t = 0 \qquad \text{and} \qquad t = \frac{1}{5}$$

3. $h(t) = 10t\mathbf{e}^{3-t^2}$

Solution

$$h'(t) = 10\mathbf{e}^{3-t^2} + 10t\mathbf{e}^{3-t^2}(-2t) = 10\mathbf{e}^{3-t^2} - 20t^2\mathbf{e}^{3-t^2}$$

$$h'(t) = 10\mathbf{e}^{3-t^2}\left(1 - 2t^2\right)$$

$$1 - 2t^2 = 0$$

$$1 = 2t^2$$

$$\frac{1}{2} = t^2 \qquad\qquad t = \pm\frac{1}{\sqrt{2}}$$

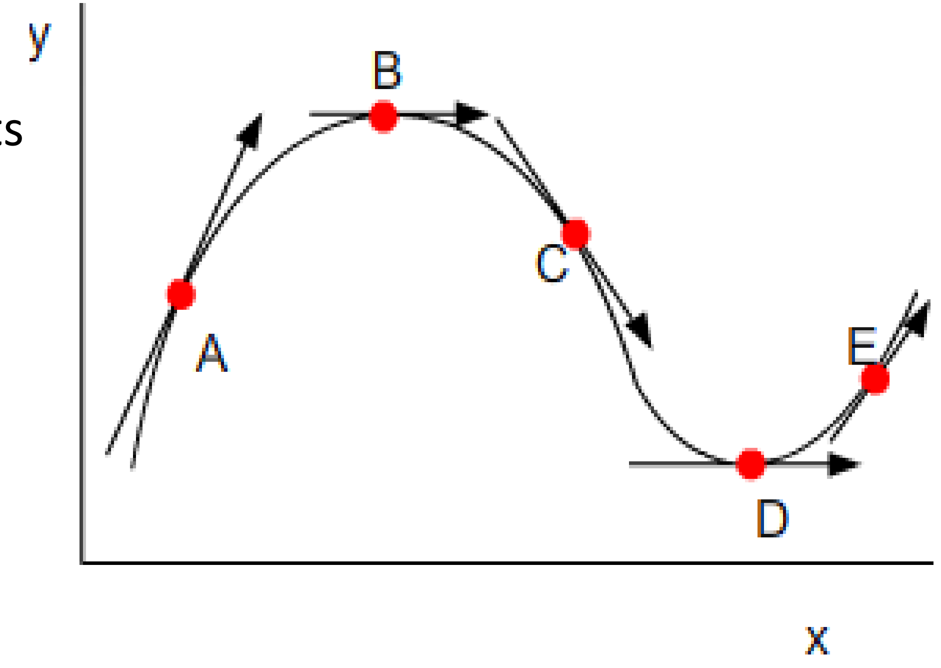
Maxima and Minima:

- In the following you find the notion of Zero slopes , turning points, and maxima and minima.
  Zero slopes Turning points, maxima and minima.
  Consider a function which gives a curve like that shown.
  If we measure the slope at different points we get different answers:
  at points *A* and *E* slope is +ve
  at point *C* slope is -ve

  So at some points in between, *B* and *D*, the function exhibits stationary value, and this can either be a local maximum (*B*) or local minimum (*D*).

  How do we calculate maxima and minima positions?

  We know that at a local max or min, the slope = 0, i.e. $dy/dx = 0$. So, given a function, $y(x)$, all we need to do is differentiate it, and put the derivative equal to zero, then solve for x.



Find the derivative.

1) $y = x^5 + 5x^4 - 10x^2 + 6$

Ans: $\frac{dy}{dx} = 5x(x^3 + 4x^2 - 4)$

2) $y = 3x^{\frac{1}{2}} - x^{\frac{3}{2}} + 2x^{\frac{-1}{2}}$

Ans: $\frac{dy}{dx} = \frac{3}{2\sqrt{2}} - \frac{3}{2}\sqrt{x} - \frac{1}{x^{3/2}}$

3) $y = x^5 + 5x^4 - 10x^2 + 6$

Ans: $\frac{dy}{dx} = 5x(x^3 + 4x^2 - 4)$

4) $y = x^5 + 5x^4 - 10x^2 + 6$

Ans: $\frac{dy}{dx} = 5x(x^3 + 4x^2 - 4)$

5) $y = (1 - 5x)^6$

Ans: $y` = -30(1 - 5x)^5$

6) $y = (\frac{x}{1+x})^5$

Ans: $y` = \frac{5x^4}{(1+x)^6}$

7) $f(t) = \frac{2}{\sqrt{t}} + \frac{6}{\sqrt[3]{t}}$

Ans: $f`(t) = -\frac{t^{\frac{1}{2}}+2t^{\frac{2}{3}}}{t^2}$

8) $y = (x-1)\sqrt{x^2-2x+2}$

Ans: $\frac{dy}{dx} = \frac{2x^2-4x+3}{\sqrt{x^2-2x+2}}$

9) $z = \frac{\omega}{\sqrt{1-4\omega^2}}$

Ans: $\frac{dz}{d\omega} = \frac{1}{(1-4\omega^2)^{\frac{3}{2}}}$

10) $y = (x^2 + 3)^4(2x^3 - 5)^3$

Ans: $\frac{dy}{dx} = 2x(x^2 + 3)^3(2x^3 - 5)^2(17x^3 + 27x - 20)$

11) $f(x) = \sqrt{\frac{x-1}{x+1}}$

Ans: $f`(x) = \frac{1}{(x-1)\sqrt{x^2-1}}$

12) $y = (\frac{x^3-1}{2x^3+1})^4$

Ans: $y` = \frac{36x^2(x^3-1)^3}{(2x^3+1)^5}$

Determine the maximum and minimum values.

$(a)\ f(x) = x^2 + 2x - 3$

Ans: x = - 1 , min. = - 4

$(b)\ f(x) = x^3 - 6x^2 + 9x - 8$

Ans: x = 1 , max. = - 4 and x = 3 , min. = - 8

$(c)\ f(x) = (2 - x)^3$

Ans: Neither max. nor min.

(d) $f(x) = x^3 + \frac{48}{x}$

Ans: x = - 2 , max. = -32 and x = 2 , min. = 32

(e) $f(x) = (x-1)^{\frac{1}{3}}(x+2)^{\frac{2}{3}}$

Ans: x = - 2 , max. = 0 and x = 0 , min. = -$\sqrt[3]{4}$
and x = 1 neither

(f) $f(x) = (x-4)^4(x+3)^3$

Ans: x = - 0 , max. = 6912 and x = 4 , min. = 0
and x = -3 neither

(g) $y = 3x^4 - 4x^3 - 12x^2 + 2$

At x = 0 there is a maximum of y = 2.

At x = −1 there is a minimum of y = −3.

At x = 2 there is a minimum of y = −30.

(h) $f(x) = x^2 - 6x + 5$

$f'(x) = 2x - 6 = 0 \; implies \; x = 3$

Minimum (3,-4)