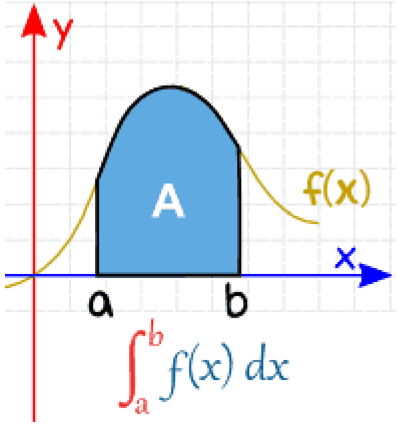
Definite Integration

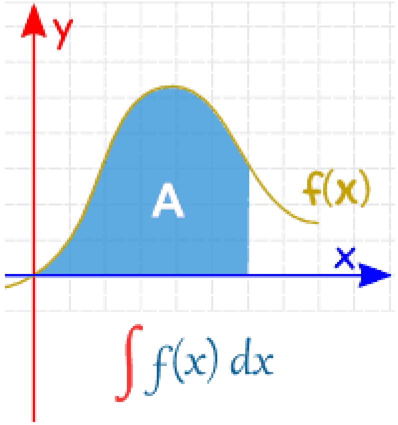
SECTION 7 MATH 1

Definite integration definition:

- A **Definite Integral** has start and end values: in other words there is an **interval** [a, b]. a and b (called limits, bounds or boundaries) are put at the bottom and top of the "S", like this:



Definite Integral
(from **a** to **b**)



Indefinite Integral
(no specific values)

We find the Definite Integral by calculating the *Indefinite* Integral at **a**, and at **b**, then subtracting

Ex: find the integral for each of the following:

- (a) $\int_1^2 2x\,dx$

Solution: First we need to find the infinite integral which is by

$$\int 2x\,dx = x^2 + C$$

Now calculate that at 1, and 2:

- At x=1: $\int 2x\,dx =$ **$1^2 + C$**
- At x=2: $\int 2x\,dx =$ **$2^2 + C$**

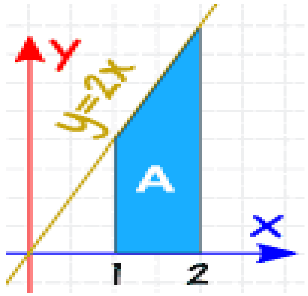
Then by Subtracting:

$(2^2 + C) - (1^2 + C)$

$2^2 + C - 1^2 - C$

$4 - 1 + C - C = 3$

Then we substitute in the original equation to get:

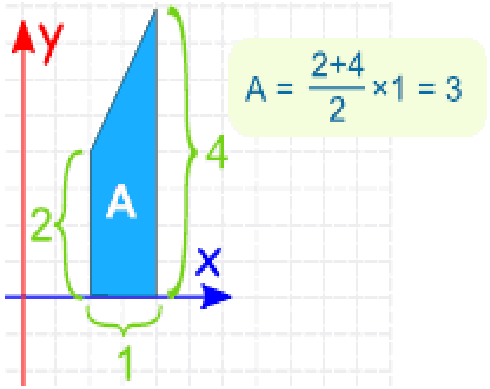


$$\int_1^2 2x\,dx = 3$$

This example can be simply solved by different ways as well :

- By finding the area under the curve or the drawing:

$$A = \frac{2+4}{2} \times 1 = 3$$



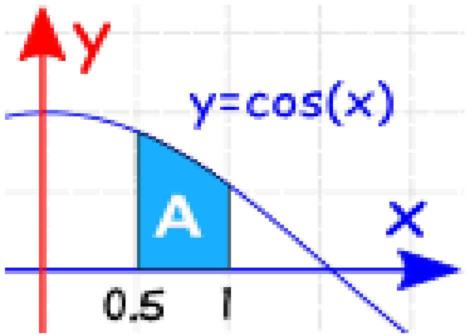
- By substituting with the limits of the integral in the variables included in the integral :

$$\int_1^2 2x\,dx = [x^2]_1^2 = [2^2 - 1^2] = [4 - 1] = 3$$

Let’s try it for another example:

- (b) The Definite Integral, from 0.5 to 1.0, of **cos(x)** dx:

$$\int_{0.5}^{1} \cos x \, dx$$



Solution:

The *Indefinite* Integral is: **∫cos(x) dx = sin(x) + C**

We can ignore C for definite integrals (as we saw above) and we get:

$$\int_{0.5}^{1} \cos x \, dx = [\,\sin x\,]_{0.5}^{1} = = \sin(1) - \sin(0.5)$$

$$= 0.841 - 0.47$$

$$= \mathbf{0.362}$$

(c) find the Definite Integral, from 0 to 1, of **sin(x)** dx:

solution:

The Definite Integral, from 0 to 1, of **sin(x)** dx:
Since we are going from 0, can we just calculate the integral at x=1?

$-\cos(1) = -0.540$

What? It is negative? But it looks positive in the graph.
Well ... we made a mistake!

Because we need to subtract the integral at x=0. We shouldn't assume that it is zero.

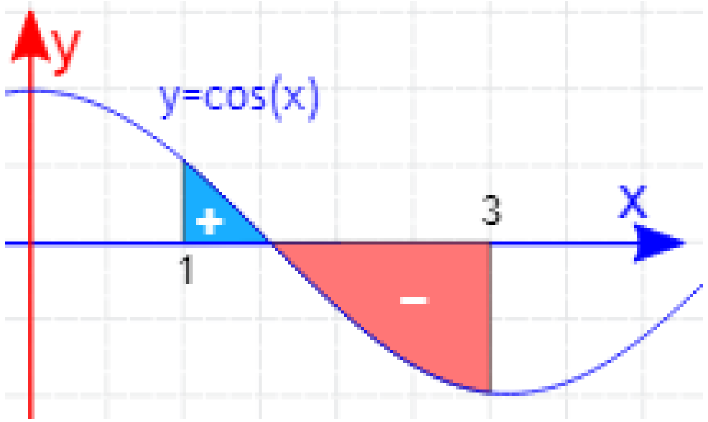
So let us do it properly, subtracting one from the other:

$$\int_0^1 \sin(x)\,dx = [-\cos x\,]_0^1 = [-\cos(1)] - [-\cos(0)] = -0.540 - (-1) = 0.46$$

(d) Find the integral $\int_1^3 \cos x \, dx$

Solution :

- Notice that some of it is positive, and some negative. The definite integral will work out the **net** value.



- $\int_1^3 \cos x \, dx = [\sin x]_1^3 = [\sin 3] - [\sin 1]$

$= 0.141 - 0.841$

$= \mathbf{-0.700}$

So important Rule to know !

Sometimes we need to have a positive integral so we have to sabstract the two areas from each other and treat each term of them separately as follows:

$$\int_a^b f(x)\,dx = (\text{Area above x axis}) - (\text{Area below x axis})$$

- Which we have already seen and checked in example (d)

Properties of the definite integration:

- 1- Adding& sabstracting Functions:

$$Ex: \int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$Ex: \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

- 2- Reversing the interval

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

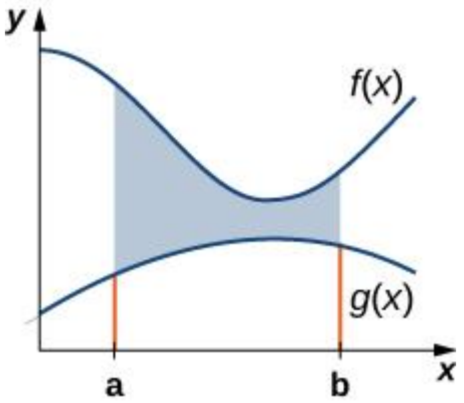
Applications of the definite integral (outlines)

1-Area of a Plane Region

2-Volume

Area of a Region between Two Curves

- Let f(x) and g(x) be continuous functions over an interval [a,b] such that f(x)≥g(x) on [a,b] . We want to find the area between the graphs of the functions, as shown in Figure



The area between the graphs of two functions, f(x)f(x) and g(x)g(x), on the interval [a,b]

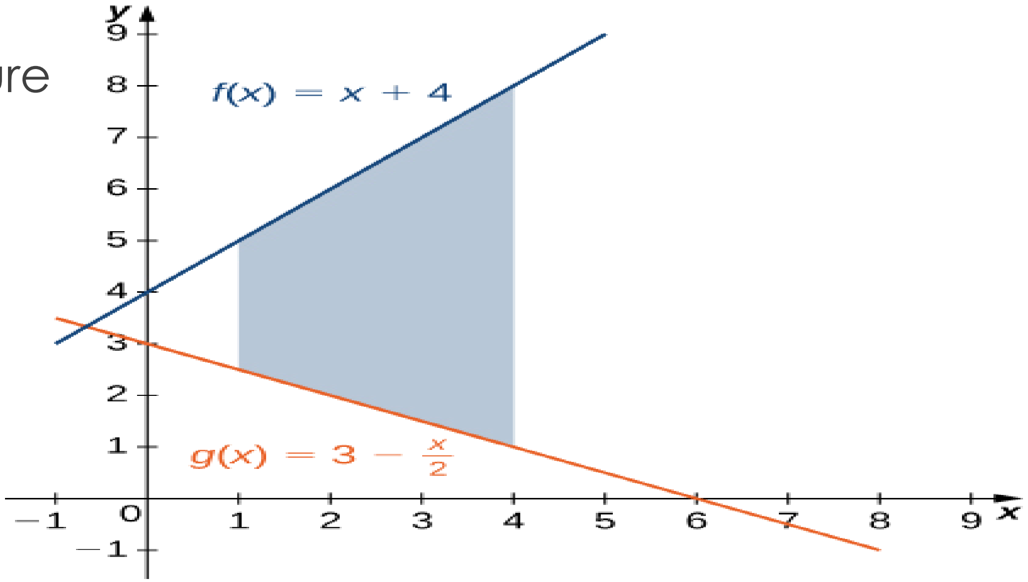
Let f(x)f(x) and g(x)g(x) be continuous functions such that f(x)≥g(x)f(x)≥g(x) over an interval [a,b]a,b]. Let R denote the region

bounded above
by the graph of f(x)f(x), below by the graph of g(x)g(x), and on the left and right by the lines x=a and x=b, respectively. Then, the area of R is given by

$$A=\int_a^b[f(x)-g(x)]dx$$

Ex 1: Finding the Area of a Region between Two Curves

- If R is the region bounded above by the graph of the function f(x)=x+4 and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval [1,4] , find the area of region R .
- Solution: The region is depicted in the following figure



$$A = \int_a^b [f(x) - g(x)]\, dx$$

$$= \int_1^4 [(x+4) - (3 - \frac{x}{2})]\, dx = \int_1^4 \left[\frac{3x}{2} + 1\right] dx$$

$$= [\frac{3x^2}{4} + x]\Big|_1^4 = (16 - \frac{7}{4}) = \frac{57}{4}.$$

Example 2

- If R is the r egion bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval [1,5] , find the area of region R .

Solution: try it yourself!

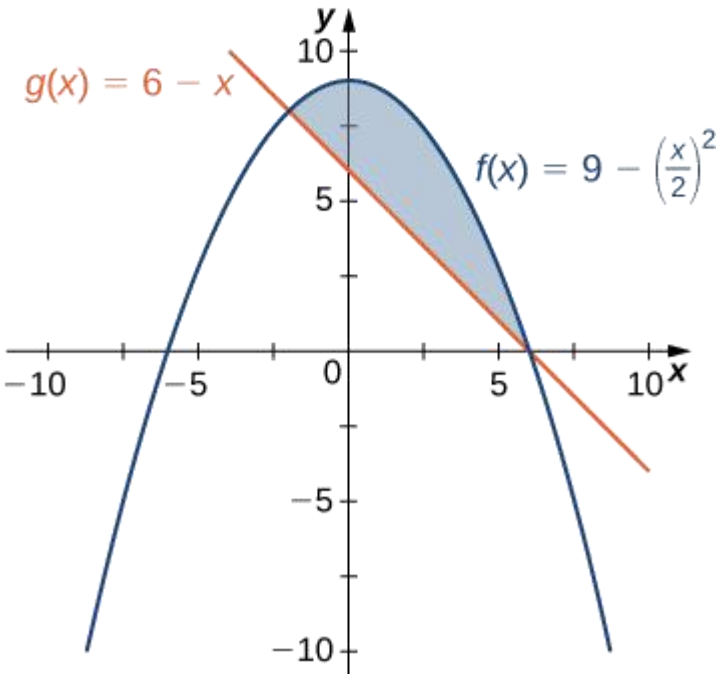
HINT :
Ans: $12\ units^2$

Example:2

- If R is the region bounded above by the graph of the function $f(x) = 9 - \left(\frac{x}{2}\right)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R .

Solution:

- The region is depicted in the following figure



We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$\begin{aligned}
f(x) &= g(x) \\
9 - \left(\frac{x}{2}\right)^2 &= 6 - x \\
9 - \frac{x^2}{4} &= 6 - x \\
36 - x^2 &= 24 - 4x \\
x^2 - 4x - 12 &= 0 \\
(x-6)(x+2) &= 0.
\end{aligned}$$

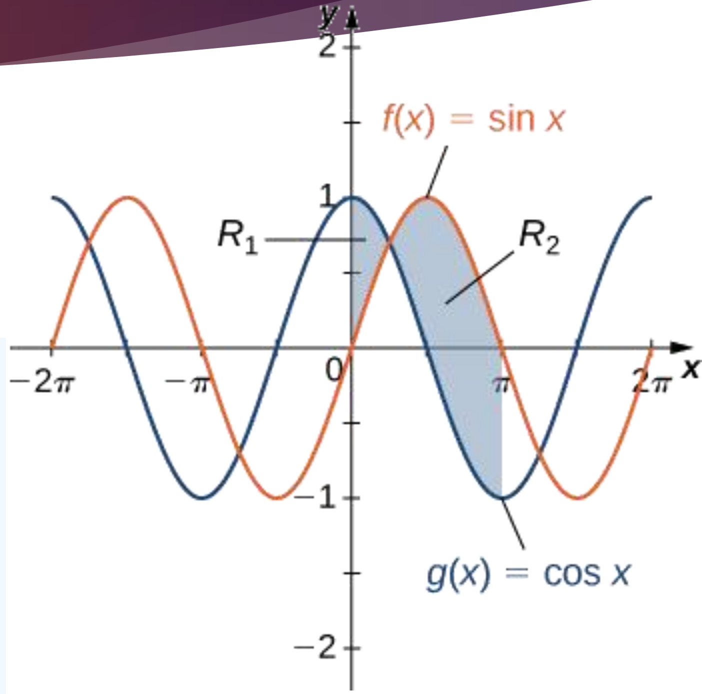
The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$\begin{aligned}
A &= \int_a^b [f(x) - g(x)]\,dx \\
&= \int_{-2}^{6} \left[9 - \left(\frac{x}{2}\right)^2 - (6 - x)\right] dx \\
&= \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right] dx \\
&= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Bigg|_{-2}^{6} = \frac{64}{3}.
\end{aligned}$$

Ex:3 If R is the region between the graphs of the functions f(x)=sinx and g(x)=cosx over the interval [0,π] , find the area of region R

- Solution:

the region is depicted in the following figure.



The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4]$, $\cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

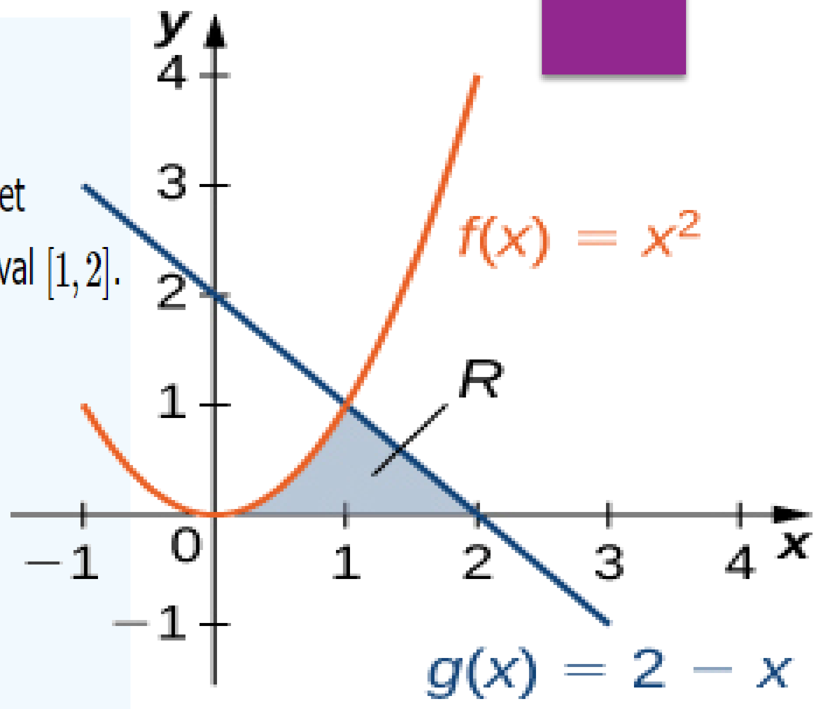
On the other hand, for $x \in [\pi/4, \pi]$, $\sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$\begin{aligned} A &= \int_a^b |f(x) - g(x)| dx \\ &= \int_0^\pi |\sin x - \cos x| dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\ &= [\sin x + \cos x]|_0^{\pi/4} + [-\cos x - \sin x]|_{\pi/4}^{\pi} \\ &= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}. \end{aligned}$$

Ex 4: Consider the region depicted in the following Figure. Find the area of R.



Solution

As with Example 6.1.3, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0,1]$ and one over the interval $[1,2]$.

Over the interval $[0,1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}.$$

Over the interval $[1,2]$, the region is bounded above by $g(x) = 2 - x$ and below by the **x-axis**, so we have

$$A_2 = \int_1^2 (2-x)dx = [2x - \frac{x^2}{2}] \Big|_1^2 = \frac{1}{2}.$$

Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

Volume application

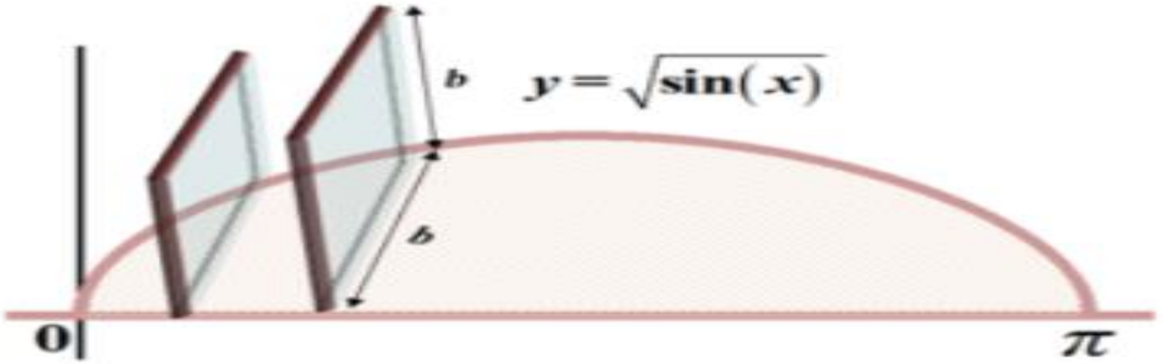
General Rule:

Given the cross sectional area A(x) in interval [[a,b], and cross sections are perpendicular to the x-axis, the volume of this solid is Volume =$\int_a^b A(x)\,dx$

Example 1:

Squares

Set up the integral to find the **volume** of solid whose base is bounded by the graph of $f(x) = \sqrt{\sin(x)}$, $x = 0$, $x = \pi$, and the x-axis, with perpendicular cross sections that are **squares**.



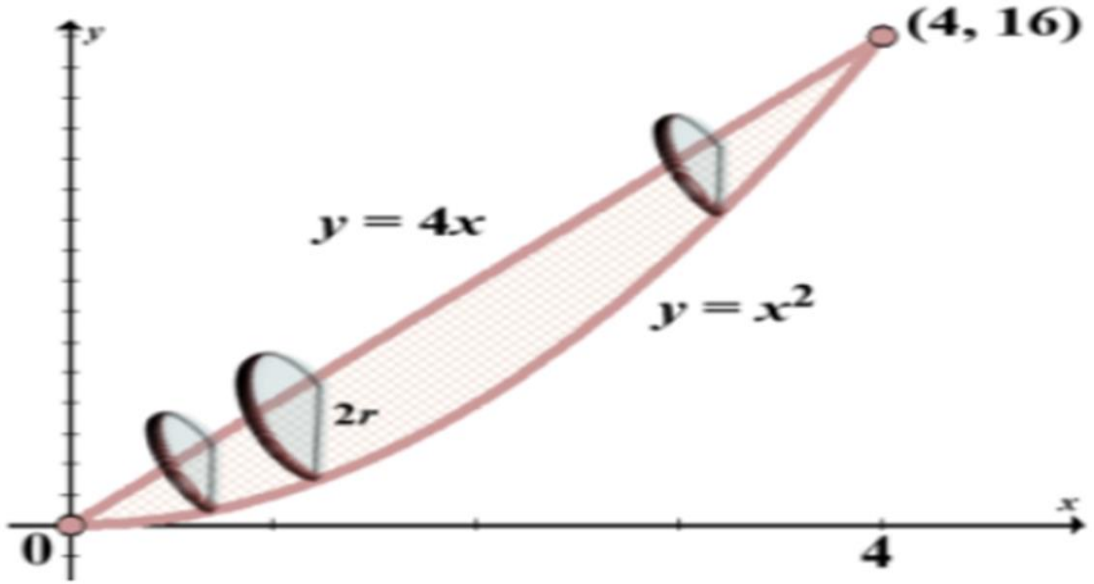
Note that the side of the square is the distance between the function and x-axis (b), and the area is b^2.

$$\text{Volume} = \int_0^{\pi} \left[\sqrt{\sin(x)} - 0\right]^2 dx = \int_0^{\pi} \sin(x)\, dx$$

Example 2:

Semicircles

Set up the integral to find the **volume** of solid whose base is bounded by graphs of $y = 4x$ and $y = x^2$, with perpendicular cross sections that are **semicircles**.



Note that the diameter ($2r$) of the semicircle is the distance between the curves, so the radius r of each semicircle is $\dfrac{4x - x^2}{2}$. Thus, the area of each semicircle is $\dfrac{\pi r^2}{2} = \dfrac{1}{2}\pi \cdot \left(\dfrac{4x - x^2}{2}\right)^2$. Thus:

$$\text{Volume} = \frac{1}{2}\pi \int_0^4 \left[\frac{(4x - x^2)}{2}\right]^2 dx = \frac{\pi}{8} \int_0^4 (4x - x^2)^2\, dx$$

Thanks a lot

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