
بسم الله الرحمن الرحيم

EELU
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MATH – 1

BY

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CHAPTER 2

Limits and Continuity

(Continued)

LECTURE 4.

The notion of continuity

Aims and Objectives:

(1) Understand the notion of continuity.
(2) Apply condition of continuity at a given point.
(3) Use the methods of evaluating limits.
(4) Have a strong intuitive feeling for these important concepts.

Continuous Function:

Some of the functions are "continuous" in the sense that there are no breaks in the graphs, while others have breaks or "discontinuities".

In the definition we distinguish between continuity at endpoint (which involves a one side limit) and the continuity at an interior point (which involves a two sided limit).

Continuity:

A function $y = f(x)$ is continuous if it is continuous at each point of its domain.

The Continuity Test:

A function $y = f(x)$ is continuous at $x = c$ if and only if all three of the following statements are true:

- $f(c)$ exists ("c" is in the domains of f)
- $lim\ f(x)$ exists (f has a limit as x c)
- $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Definition :

Interior Point :

A function f is continuous at an interior point c of its domain if the following conditions are satisfied :

(i) $f(c)$ is defined .

(ii) $\lim_{x \to c} f(x)$ exists .

(iii) $\lim_{x \to c} f(x) = f(c)$.

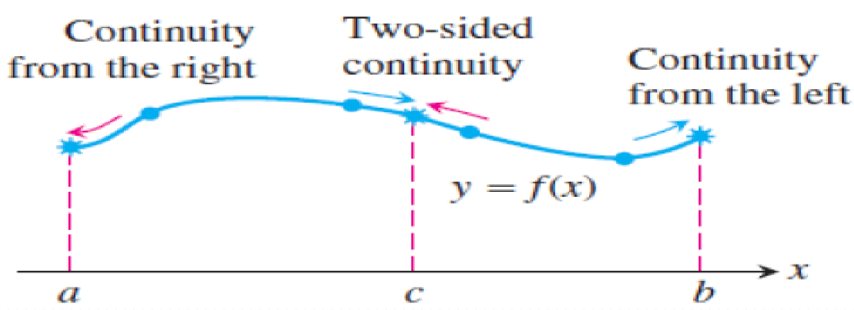
Endpoint :

(i) *A function f is continuous from right at a left end point a of its domain if*

$$\lim_{x \to a^+} f(x) = f(a)$$

(ii) *A function f is continuous from left at a right end point b of its domain if*

$$\lim_{x \to b^-} f(x) = f(b)$$

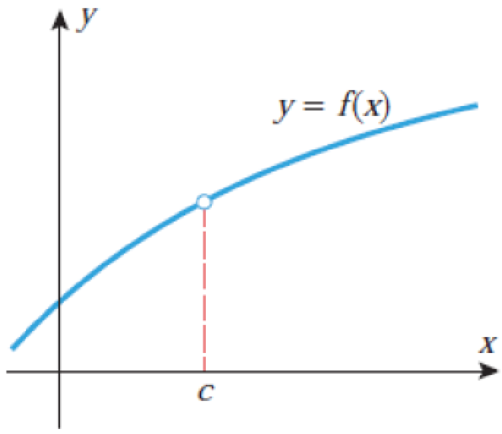


Types of Discontinuity :

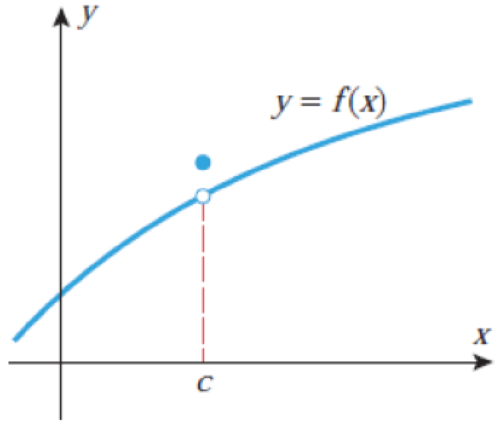
In the following figures the function is discontinuous at $x = c$.

(i) Removable discontinuity

$f(c)$ is undefined



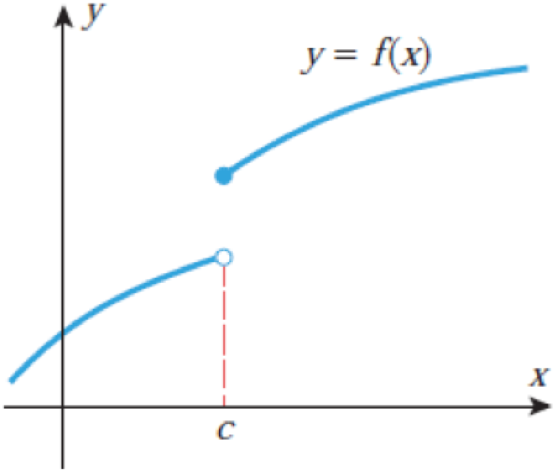
$$\lim_{x \to c} f(x) \neq f(c)$$



Types of Discontinuity :

(ii) Jump discontinuity

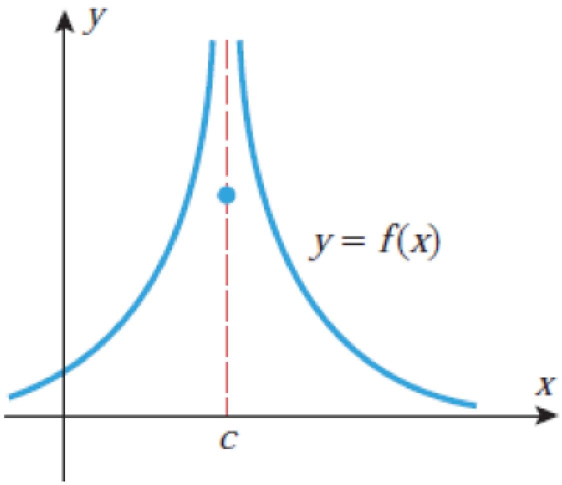
$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$$



(iii) Infinite discontinuity

$$\lim_{x \to c} f(x) = \infty \quad ,$$

$x = c$ is a *vertical asymptote*



EXAMPLE:

Discuss the continuity of the following function as:

at $x = 1$ $f(x) = \begin{cases} 7x - 2 & x < 1 \\ 5 & x = 1 \\ 5x^2 & x > 1 \end{cases}$

SOLUTION:

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} f(5x^2) = 5$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} f(7x-2) = 5$ the function has a limit

$f(1) = 5$ the limit equals the function value

Then the function is continuous at $x = 1$

EXAMPLE:

Investigate the continuity of: $f(x) = \dfrac{x^2 - 1}{x^3 + x^2 - 2x}$

SOLUTION:

The function on the form $\frac{P_1(x)}{P_2(x)}$ *is continuous except at* $P_2(x) = 0$,

i.e. at $x = 0$, $x = 1$ *and* $x = -2$.

EXAMPLES:

The function $f(x)=\dfrac{1}{x}$ is discontinuous at $x=0$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{1}{x}=+\infty$$

$\displaystyle\lim_{x\to 0-} f(x)=\lim_{x\to 0^-}\frac{1}{x}=-\infty$ the function does not have a limit at $x=0$

$f(0)$ the function is not define at $x=0$

It is easy to show that this function is continuous for any value $x\neq 0$

Definition :

A function f is continuous on an interval I if it is continuous at every number in the interval I .

Theorem :

If the function f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. Sums $\quad f + g$
2. Differences $\quad f - g$
3. Constant multiples $\quad kf$, for any number k
4. Products $\quad f \cdot g$
5. Quotients $\quad f / g$, provided $g(c) \neq 0$
6. Powers $\quad f^n$, n is positive integer
7. Roots $\quad \sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Theorem :

(i) Any polynomial is continuous everywhere ; it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(ii) Any rational function is continuous wherever it is defined ; that is , it is continuous on its domain .

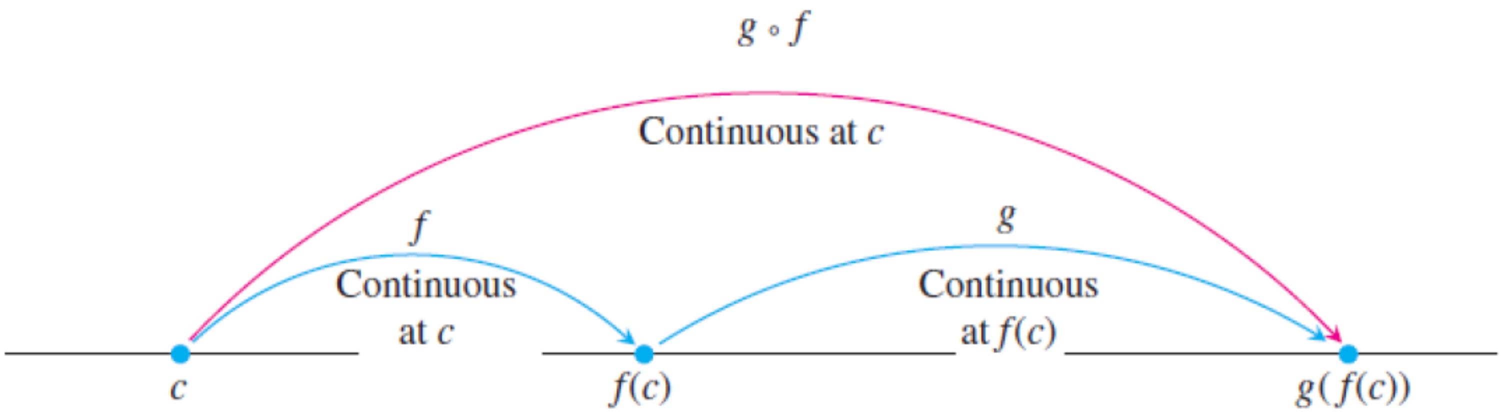
Theorem :

The following types of functions are continuous at every number in their domains :

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Composite of Continuous Functions :

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f = g(f(x))$ is continuous at c.



Composite of continuous functions are continuous

Limits of Continuous Functions :

If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g\left(\lim_{x \to c} f(x)\right).$$

Show that $f(x)=\dfrac{\sin x}{x}$, $x \neq 0$ has a continuous extension to $x=0$, and find that extension.

Solution

(i) Since $f(0)=\dfrac{0}{0}$ is undefined , then f is discontinuous at $x=0$.

(ii) Since $\lim\limits_{x \to 0} f(x)=\lim\limits_{x \to 0}\dfrac{\sin x}{x}=1$, then we can define a new function

$$F(x)=\begin{cases}\dfrac{\sin x}{x} & , \quad x \neq 0 \\ 1 & , \quad x=0\end{cases}$$

(iii) Since $\lim\limits_{x \to 0} F(x)=F(0)=1$, then F is continuous at $x=0$, and is called the continuous extension of f to $x=0$.

TECHNIQUES FOR FINDING LIMITS:

Rule:

I) *Suppose f(x) is a rational function* $f(x)=\frac{A(x)}{B(x)}=\frac{a_n x^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-1}+\ldots+a_0}{b_m x^m+b_{m-1}x^{m-1}+b_{m-2}x^{m-2}+\ldots+b_0}$

Then, $\lim_{x\to\pm\infty} f(x)=\frac{a_n}{b_m}x^{n-m}$ *and we have three cases*

1) *if* $n > m$ *then* $\lim_{x\to a} f(x)=\frac{a_n}{b_m}x^{n-m}=\pm\infty$

2) if $n = m$ *then* $\lim_{x\to a} f(x)=\frac{a_n}{b_m}x^{n-m}=\frac{a_n}{b_m}$

3) if $n < m$ *then* $\lim_{x\to a} f(x)=\frac{a_n}{b_m}x^{n-m}=0$

EXAMPLE:

$$\lim_{x\to-\infty}\frac{x^3+2x+1}{x^2-x+3}=\lim_{x\to-\infty}\frac{x^3(1+2/x^2+1/x^3)}{x^2(1-1/x+3/x^2)}$$

$$\lim_{x\to-\infty} x\frac{(1+2/x^2+1/x^3)}{(1-1/x+3/x^2)}=-\infty\frac{(1+0+0)}{(1-0+0)}=-\infty$$

TECHNIQUES FOR FINDING LIMITS:

Rule:

II) *Suppose* $f(x)=\frac{A(x)}{B(x)}=\frac{(x-a)G(x)}{(x-a)H(x)}$

Then $\lim_{x\to a} f(x)=\lim_{x\to a}\frac{G(x)}{H(x)}$

EXAMPLE:

$$\lim_{x\to 1}\frac{x^3-3x+2}{x^4-2x^2+1}=\lim_{x\to 1}\frac{(x-1)(x^2+x-2)}{(x-1)(x^3+x^2-x-1)}$$

$$\lim_{x\to 1}\frac{x^2+x-2}{x^3+x^2-x-1}=\lim_{x\to 1}\frac{(x-1)(x+2)}{(x-1)(x+1)^2}=\lim_{x\to 1}\frac{x+2}{(x+1)^2}=3/4$$

Rule:

III) *conjugate technique*

EXAMPLE:

Find $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$

Since, $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}=\frac{0}{0}$ *, then using the conjugate* $\sqrt{x+3}+2$

$$\lim_{x\to 1}\frac{(\sqrt{x+3}-2)}{(x-1)}\frac{(\sqrt{x+3}+2)}{(\sqrt{x+3}+2)}=\lim_{x\to 1}\frac{x-1}{(x-1)\sqrt{x+3}+2}=1/4$$

LIMITS OF TRIGONOMETRIC FUNCTION:

1- $\lim_{x\to 0} \text{Sin}(x) = 0$ and $\lim_{x\to a} \text{Sin}(x) = \text{Sin}(a)$

2- $\lim_{x\to 0} \text{Cos}(x) = 1$ and $\lim_{x\to a} \text{Cos}(x) = \text{Cos}(a)$

3- $\lim_{x\to 0} \frac{\text{Sin}(x)}{x} = 1$

EXAMPLE:

Find $\lim_{x\to 0} \frac{x \sin x}{1-\cos x}$

SOLUTION:

$$\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{x\sin x\,(1+\cos x)}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0} \frac{x\,(\sin x) + x\,(\sin x)\,(\cos x)}{1-(\cos x)^2}$$

$$= \lim_{x\to 0} \frac{x\sin x + x\,(\sin x)\,(\cos x)}{(\sin x)^2} = \lim_{x\to 0} \left(\frac{x}{\sin x} + \frac{x\cos x}{\sin x} \right) =$$

$$\lim_{x\to 0}\left(\frac{x}{\sin x}\right) + \lim_{x\to 0}\left(\frac{x}{\sin x}\right).\lim_{x\to 0}(\cos x) = 1 + (1)(1) = 2$$

LIMITS OF TRIGONOMETRIC FUNCTION:

EXAMPLE:

Prove that $\lim\limits_{x \to 0} \frac{1-\cos x}{x} = 0$

SOLUTION:

$$\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x \to 0} \frac{1-(\cos x)^2}{x(1+\cos x)} = \lim_{x \to 0} \frac{1}{x} \frac{(\sin x)^2}{(1+\cos x)}$$

$$\left(\lim_{x \to 0} \frac{\sin x}{x} \right) \left(\lim_{x \to 0} \sin x \right) \left(\lim_{x \to 0} \frac{1}{1+\cos x} \right) = (1)(0)\left(\frac{1}{2}\right) = 0$$

LIMITS OF TRIGONOMETRIC FUNCTION:

EXAMPLE:

Prove that $\lim_{x \to 0} \frac{\sin(mx)}{\sin(nx)} = \frac{m}{n}$

SOLUTION:

$$\lim_{x \to 0} \frac{\sin(mx)}{\sin(nx)} = \lim_{x \to 0} \frac{m\left(\frac{\sin(mx)}{mx}\right)}{n\left(\frac{\sin(nx)}{nx}\right)} = \frac{m}{n} \frac{\lim_{x \to 0}\left(\frac{\sin(mx)}{mx}\right)}{\lim_{x \to 0}\left(\frac{\sin(nx)}{nx}\right)}$$

As $x \to 0$ then $mx \to 0$ and $nx \to 0$

$$= \frac{m}{n} \frac{\lim_{mx \to 0}\left(\frac{\sin(mx)}{mx}\right)}{\lim_{nx \to 0}\left(\frac{\sin(nx)}{nx}\right)} = \frac{m}{n}$$

THEOREM AND EXAMPLES OF SPECIAL CASES:

Theorem $\lim\limits_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$

EXAMPLE:

$$\lim_{x\to 3}\frac{\sqrt{x-2}-1}{x-3} = \lim_{x\to 3}\frac{(x-2)^{1/2}-1^{1/2}}{(x-2)-1} = \lim_{x-2\to 1}\frac{(x-2)^{1/2}-1^{1/2}}{(x-2)-1}$$

$$= (1/2)(1)^{-1/2} = 1/2$$

Corollary $\lim\limits_{x\to a} \frac{x^n-a^n}{x^m-a^m} = \frac{n}{m}a^{n-m}$

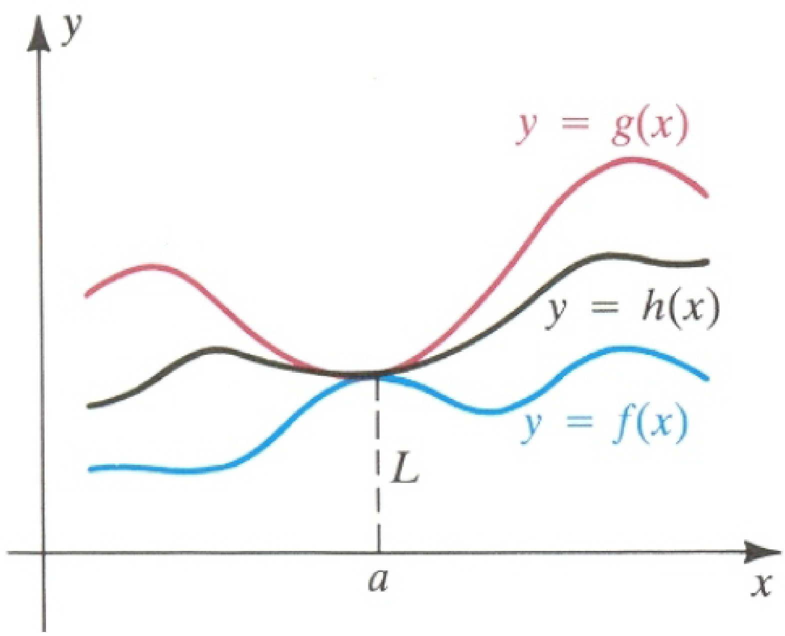
EXAMPLE:

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{x^3-2^3}{x^2-2^2} = \frac{3}{2}2^{3-2} = 3$$

The Sandwich (Squeeze) Theorem :

Suppose $f(x) \le h(x) \le g(x)$ *for every* x *in an open interval containing* a *, except possibly at* a *,*

$$\text{If} \quad \lim_{x \to a} f(x) = L = \lim_{x \to a} g(x) \text{, then} \quad \lim_{x \to a} h(x) = L \text{ .}$$



EXAMPLES:

1- $$\lim_{x\to\infty}\left(5+\frac{1}{x}\right)=\lim_{x\to\infty}5+\lim_{x\to\infty}\frac{1}{x}=5+0=5$$

2- $$\lim_{x\to 0^+}\frac{1}{x^{1/3}}=\lim_{x\to 0^+}\left(\frac{1}{x}\right)^{1/3}=\infty$$

3- $$\lim_{x\to 0^-}\frac{1}{x^{1/3}}=\lim_{x\to 0^-}\left(\frac{1}{x}\right)^{1/3}=-\infty$$

4- $$\lim_{x\to 0^+}\frac{1}{x^2}=\lim_{x\to 0^+}\left(\frac{1}{x^2}\right)=\infty$$

5- $$\lim_{x\to 1^+}\frac{1}{x-1}=\infty$$

6- $$\lim_{x\to 1^-}\frac{1}{x-1}=-\infty$$

7- $$\lim_{x\to -\infty}\frac{-15x}{7x+4}=\lim_{x\to -\infty}\frac{-15}{7+4/x}=-\frac{15}{7}$$

THANK YOU