Section 7

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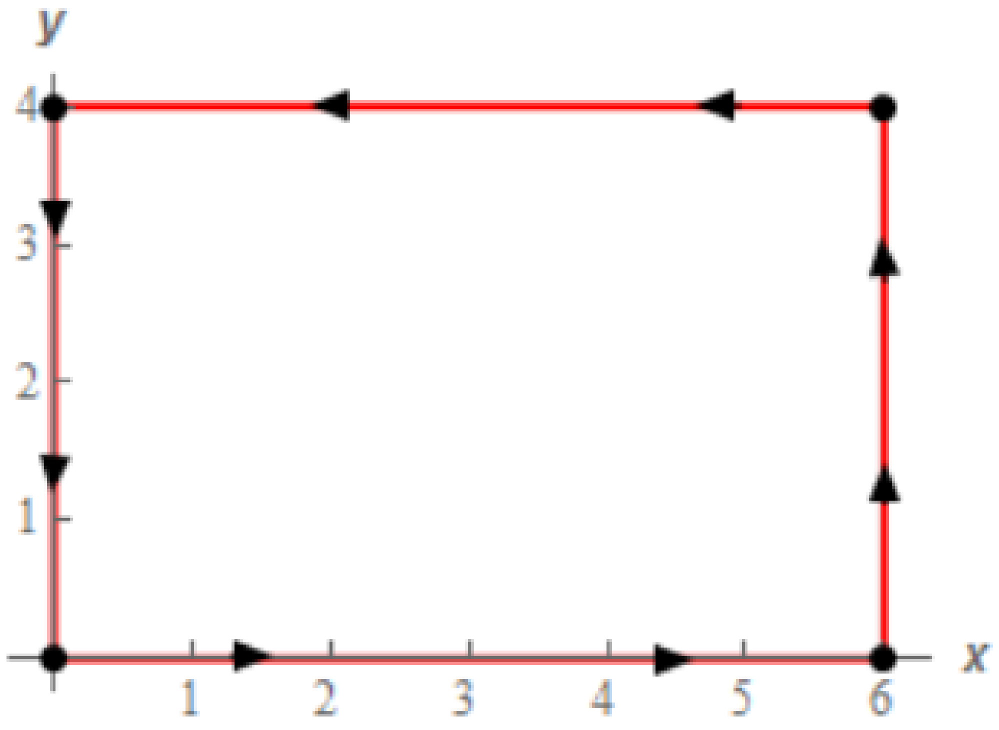
Ex.1 (Green's theorem)

$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Use Green's Theorem to evaluate $\int_C (y^4 - 2y)\,dx - (6x - 4xy^3)\,dy$ where C is shown below.



Solution:

$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$\int_C \left(y^4 - 2y\right)\,dx - \left(6x - 4xy^3\right)\,dy$$

$$\boldsymbol{M} = y^4 - 2y \qquad \boldsymbol{N} = -\left(6x - 4xy^3\right) = 4xy^3 - 6x$$

Using Green's Theorem the line integral becomes,

$$\int_C \left(y^4 - 2y\right)\,dx - \left(4xy^3 - 6x\right)\,dy = \iint_D 4y^3 - 6 - \left(4y^3 - 2\right)\,dA = \iint_D -4\,dA = -4\iint_D dA$$

D is the region enclosed by the curve.

$$= -4\,(\text{Area of } D) = -4\,[(6)\,(4)] = \boxed{-96}$$

Ex.2 (Green's theorem)

$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Using Green's formula, evaluate the integral $\oint\limits_C (x+y)\,dx - (x-y)\,dy$, where the curve C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Figure 2).

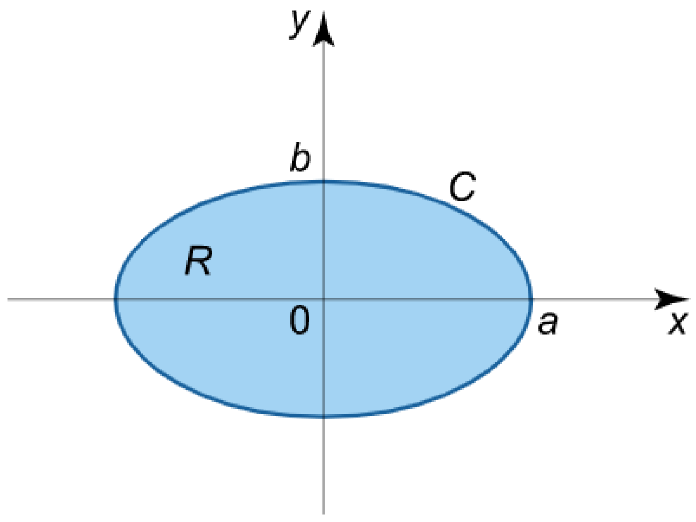


Figure 2.

Solution:

$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$\oint_C (x+y)\,dx - (x-y)\,dy,$$

$$\boldsymbol{M} = x + y, \quad \boldsymbol{N} = -(x-y)$$

$$I = \oint_C (x+y)\,dx - (x-y)\,dy = \iint_R (-1-1)\,dxdy = -2\iint_R dxdy.$$

As the double integral $\iint_R dxdy$ is equal to the area of the ellipse πab, the integral is

$$I = -2\iint_R dxdy = -2\pi ab.$$

Ex.3 (Green's theorem)

$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Using Green's theorem, calculate the integral $\oint\limits_C x^2y\,dx - xy^2\,dy$. The curve C is the circle $x^2 + y^2 = a^2$ (Figure 1), traversed in the counterclockwise direction.

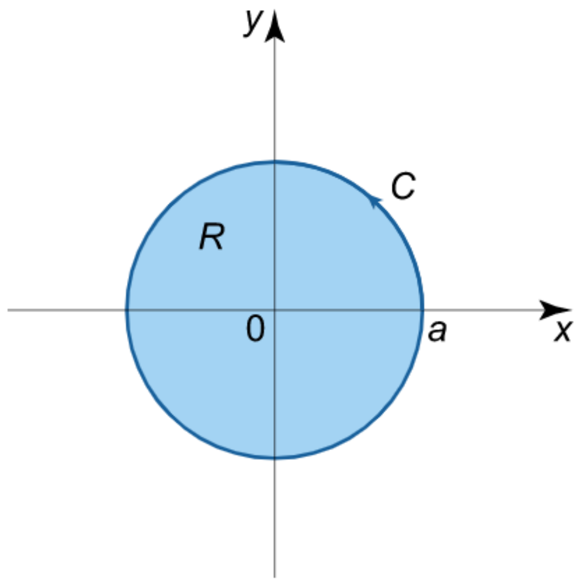


Figure 1.

Solution:

$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$\oint_C x^2 y\,dx - xy^2\,dy.$$

$$\boldsymbol{M} = x^2 y, \qquad \boldsymbol{N} = -xy^2$$

$$I = \oint_C x^2 y\,dx - xy^2\,dy = \iint_R \left(-y^2 - x^2\right) dx\,dy = -\iint_R \left(x^2 + y^2\right) dx\,dy,$$

where R is the circle with radius a centered at the origin. Transforming to polar coordinates, we obtain

$$I = -\iint\limits_R \left(x^2 + y^2\right) dxdy = -\int\limits_0^{2\pi} d\theta \int\limits_0^a \left(r^2\cos^2\theta + r^2\sin^2\theta\right) rdr = -\int\limits_0^{2\pi} d\theta \int\limits_0^a r^3 dr$$

$$= -2\pi \cdot \left[\left.\left(\frac{r^4}{4}\right)\right|_0^a\right] = -\frac{\pi a^4}{2}.$$

Remember section 6

The divergence of a vector field

If $\vec{F} = M\vec{i} + N\vec{j} + R\vec{k}$ is a vector field on and $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial y}, \frac{\partial R}{\partial z}$ exist, the divergence of $\vec{F}$ is the function of three variables defined by :

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial R}{\partial z}$$

In terms of the vector differential operator ∇ ("del" or "nabla")

$$\nabla = \left(\frac{\partial}{\partial x}\right)\vec{i} + \left(\frac{\partial}{\partial y}\right)\vec{j} + \left(\frac{\partial}{\partial z}\right)\vec{k}$$

the divergence of $\vec{F}$ can be written symbolically as the dot product of ∇ and $\vec{F}$.

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

Surface Integrals

- In the previous chapter we looked at evaluating integrals of functions or vector fields where the points came from a curve in two- or three-dimensional space.

  We now want to extend this idea and integrate functions and vector fields where the points come from a surface in three-dimensional space.

  These integrals are called surface integrals.

Some Important shapes

1 – the plane

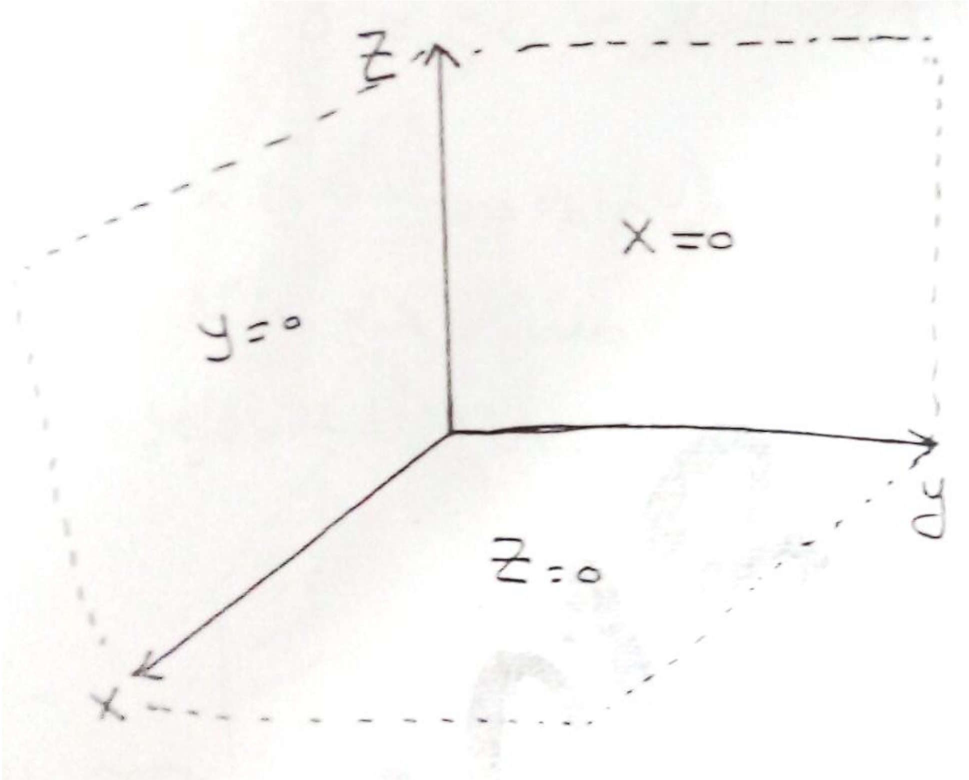
$ax + by + cz + d = 0$, the powers of (x, y, z) = 1

Examples :
(x-y) plane ------ z=0
(x-z) plane ------ y=0
(y-z) plane ------ x=0

Note:
if z=constant for example z=2
then this plane is parallel to the (x-y) plane

if y=constant for example y=2
then this plane is parallel to the (x-z) plane

if y=constant for example x=2
then this plane is parallel to the (y-z) plane



Examples of planes

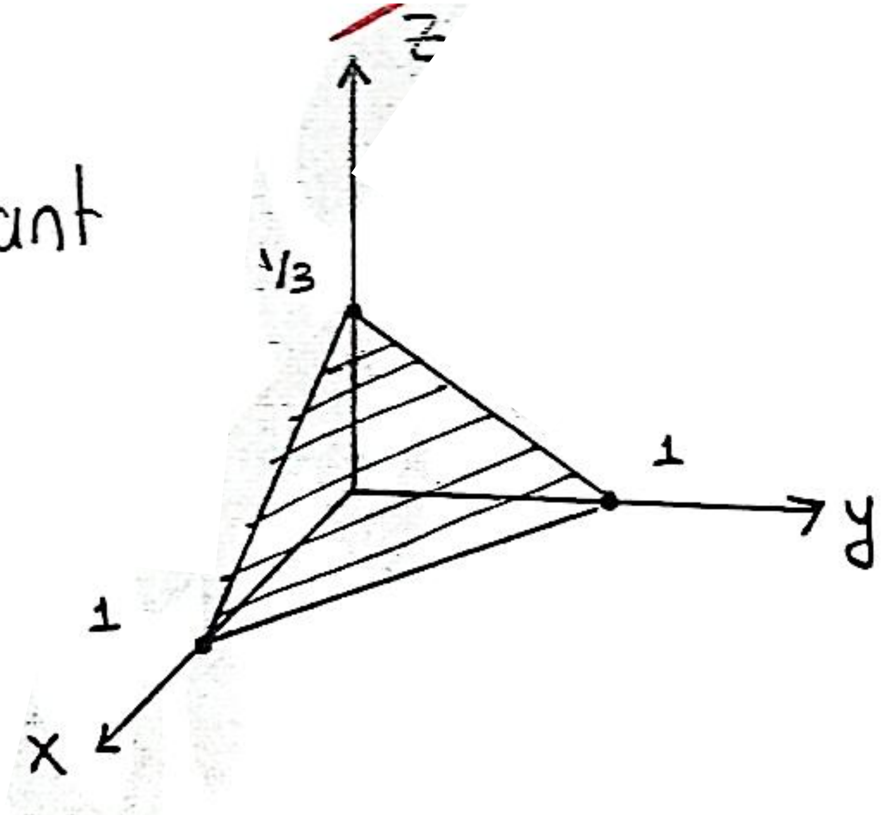
Example 1

① $x + y + 3z = 1$, 1^{st} octant

For $x = 0$ $\quad y + 3z = 1$

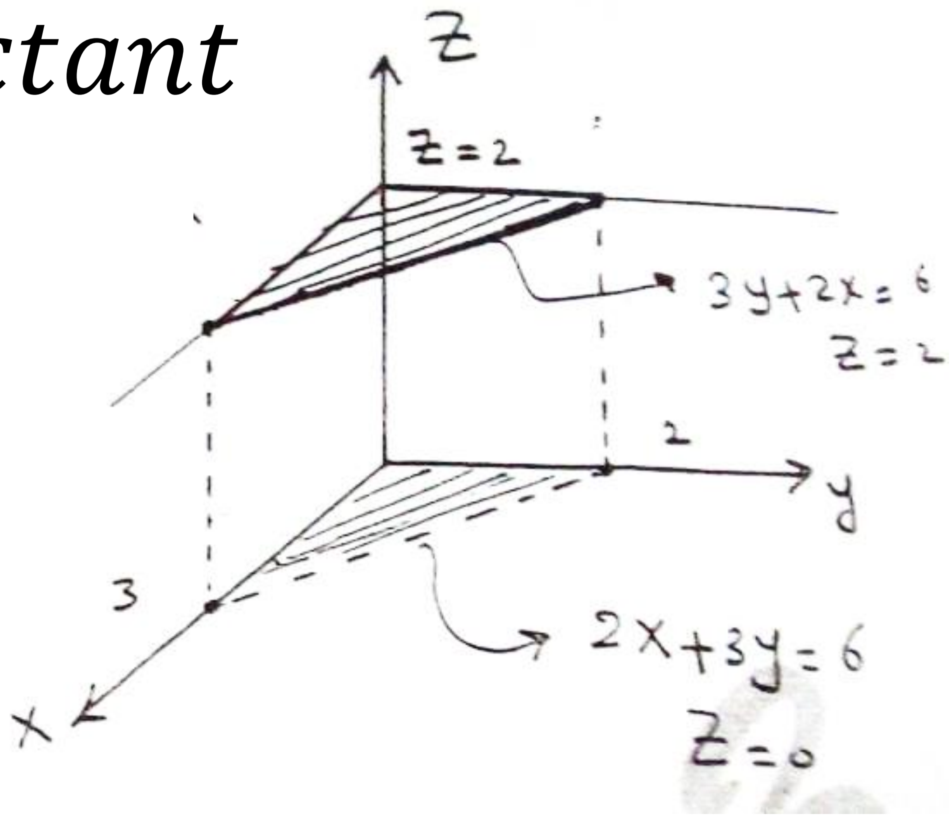
$y = 0$ $\quad x + 3z = 1$

$z = 0$ $\quad x + y = 1$



Example 2

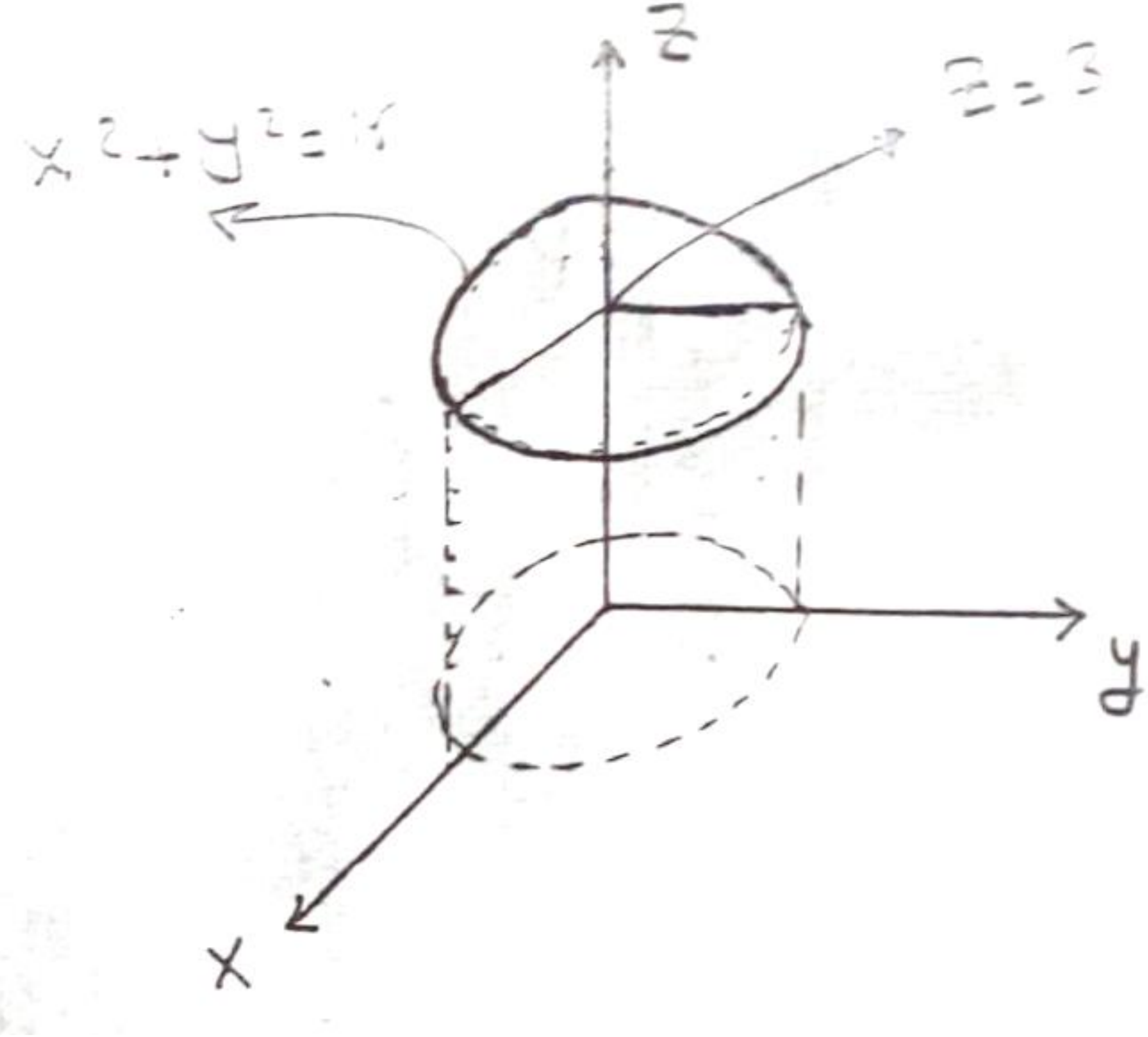
$$2x + 3y = 6\,, z = 2\,, 1^{st} octant$$



Cylinder

Example 1

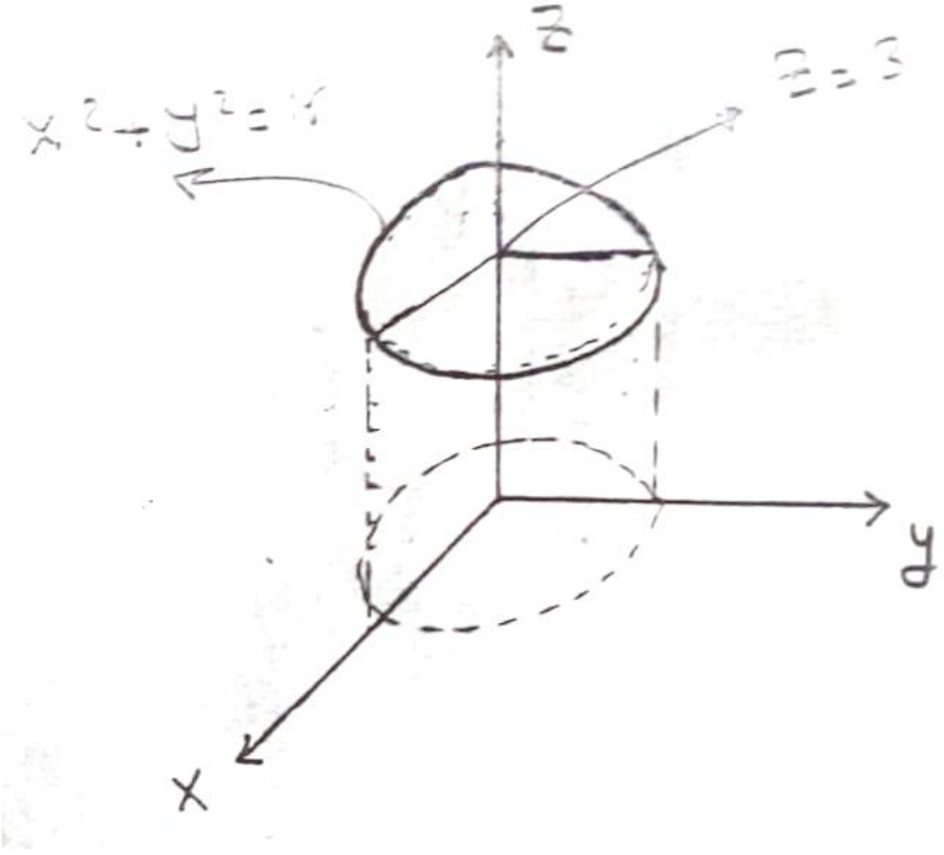
$$x^2 + y^2 = 16\,, z = 3$$



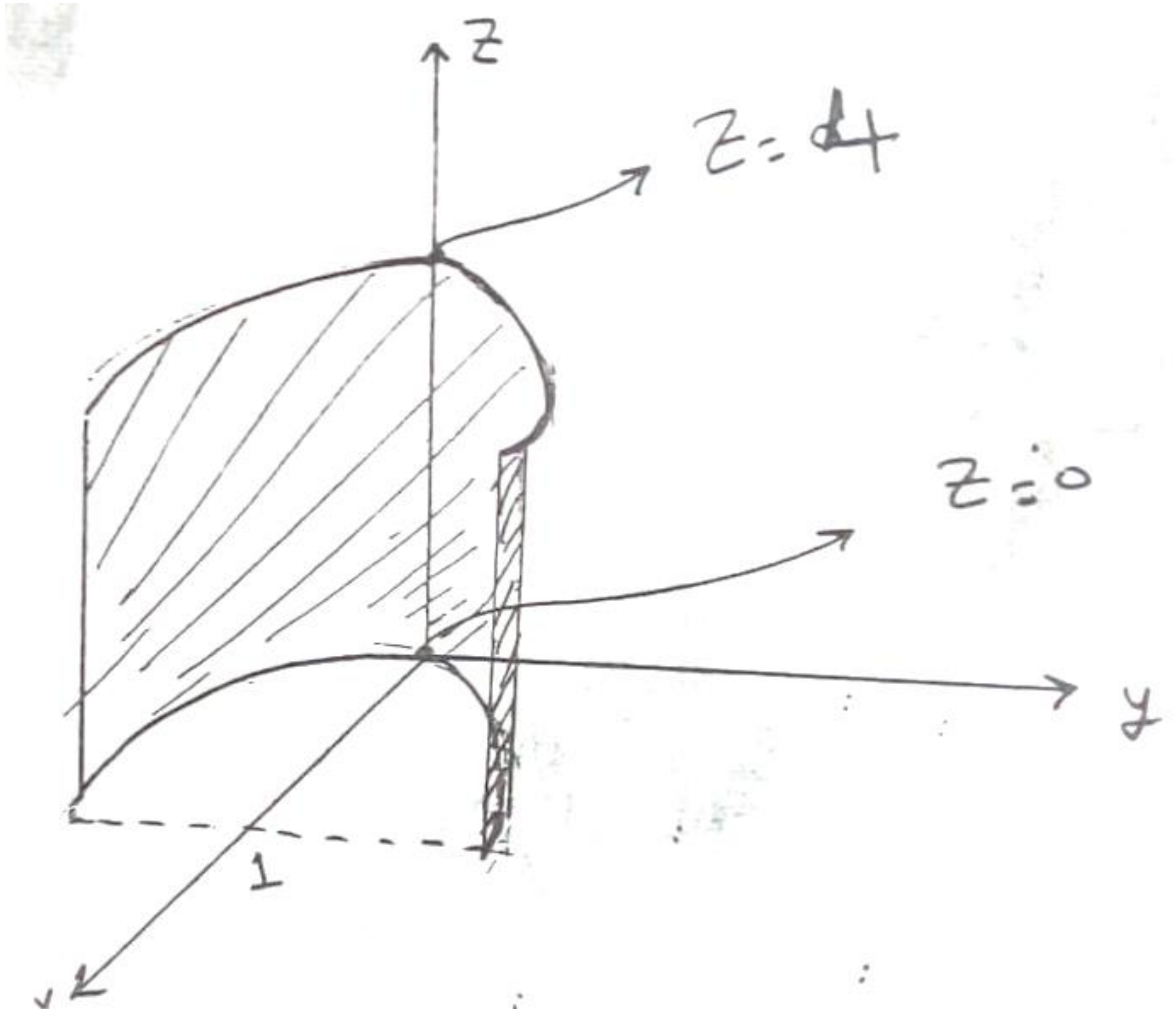
Cylinder

Example 2

$$x^2 + y^2 = 16\,, -2 \leq z \leq 2$$

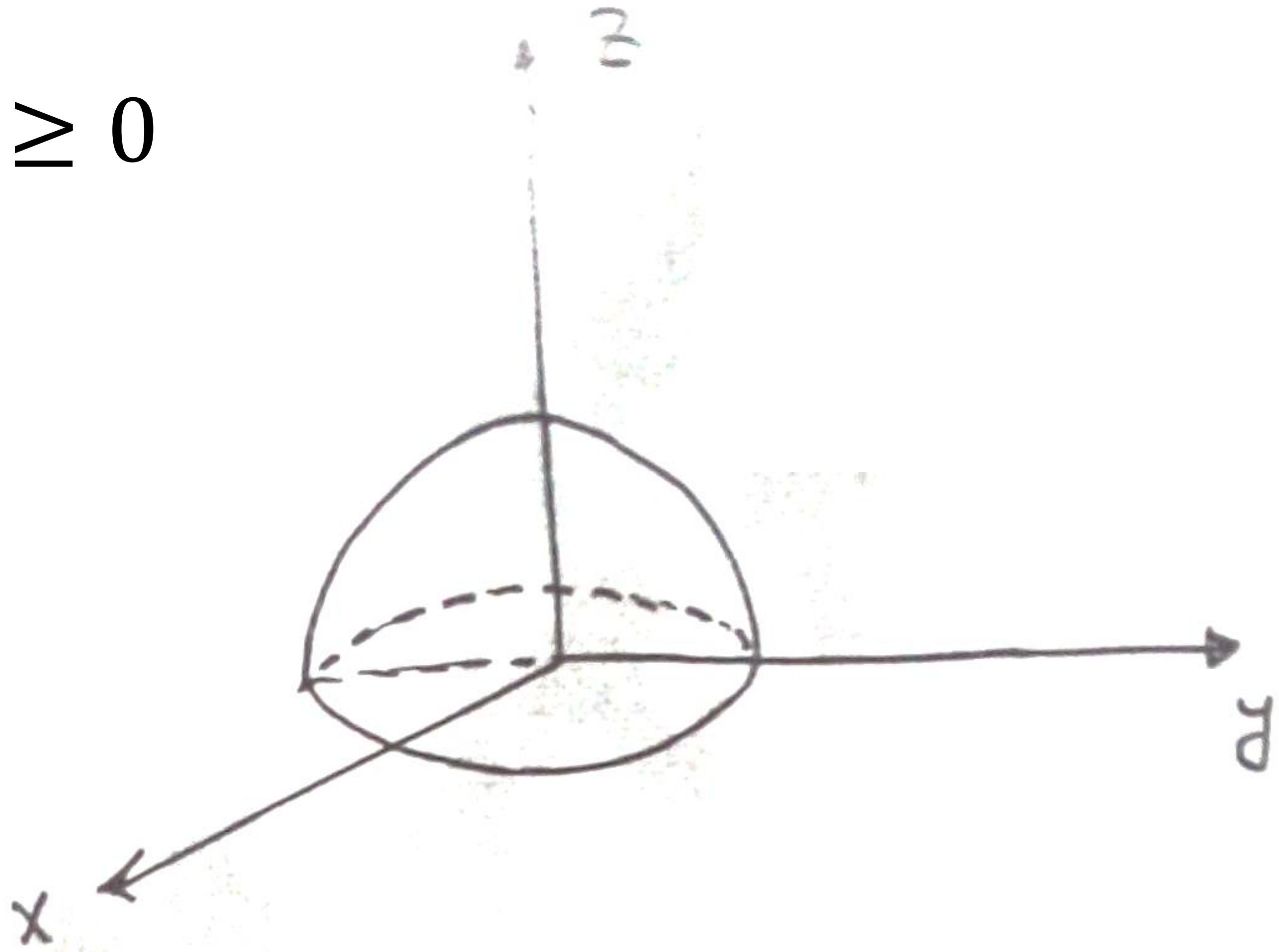


$$y^2 = x \quad , 0 \le x \le 10, \qquad 0 \le z \le 4$$



the sphere

$$x^2 + y^2 + z^2 = a^2, z \geq 0$$



the paraboloid

$$x^2 + y^2 = cz, c > 0$$

Divergence Theorem

We are going to relate surface integrals to triple integrals.
We will do this with the Divergence Theorem.

Let E be a simple solid region and S is the boundary surface of E with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous first order partial derivatives. Then,

$$\iint\limits_{S} \vec{F} \cdot d\vec{S} = \iiint\limits_{E} \operatorname{div} \vec{F} \, dV$$

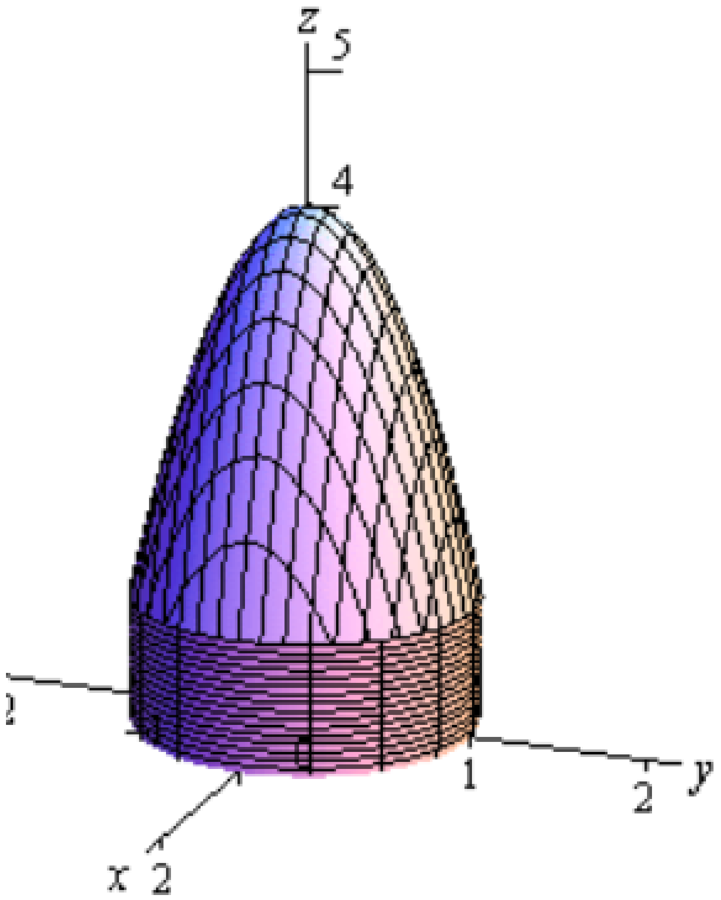
Example 1

Example 1 Use the divergence theorem to evaluate $\iint\limits_S \vec{F} \centerdot d\vec{S}$ where $\vec{F} = xy\,\vec{i} - \frac{1}{2}y^2\,\vec{j} + z\,\vec{k}$ and the surface consists of the three surfaces, $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top, $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides and $z = 0$ on the bottom.

Solution:

Let's start this off with a sketch of the surface.

The region E for the triple integral is then the region enclosed by these surfaces.
Note that cylindrical coordinates would be a perfect coordinate system for this region.
If we do that here are the limits for the ranges.



$$
\begin{gathered}
0 \leq z \leq 4 - 3r^2 \\
0 \leq r \leq 1 \\
0 \leq \theta \leq 2\pi
\end{gathered}
$$

We'll also need the divergence of the vector field so let's get that.

$$\operatorname{div}\vec{F} = y - y + 1 = 1$$

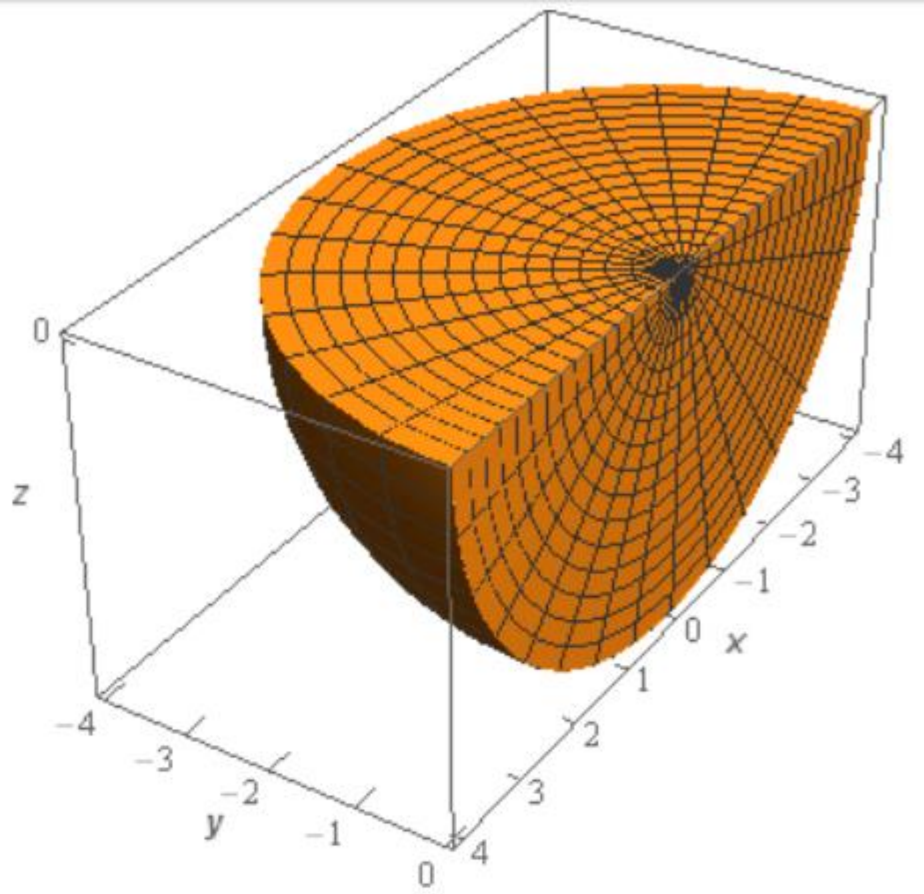
The integral is then,

$$\begin{aligned}
\iint\limits_S \vec{F} \centerdot d\vec{S} &= \iiint\limits_E \operatorname{div}\vec{F}\,dV \\
&= \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^1 4r - 3r^3\,dr\,d\theta \\
&= \int_0^{2\pi} \left.\left(2r^2 - \frac{3}{4}r^4\right)\right|_0^1 d\theta \\
&= \int_0^{2\pi} \frac{5}{4}\,d\theta \\
&= \frac{5}{2}\pi
\end{aligned}$$

Example 2

1. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = yx^2\,\vec{i} + \left(xy^2 - 3z^4\right)\,\vec{j} + \left(x^3 + y^2\right)\,\vec{k}$ and S is the surface of the sphere of radius 4 with $z \le 0$ and $y \le 0$. Note that all three surfaces of this solid are included in S.

Solution:



We are going to use the Divergence Theorem in the following direction.

$$\iint\limits_{S} \vec{F} \cdot d\vec{S} = \iiint\limits_{E} \operatorname{div} \vec{F} \, dV$$

where E is just the solid shown in the sketches from Step 1.

Because E is a portion of a sphere we'll be wanting to use spherical coordinates for the integration. Here are the spherical limits we'll need to use for this region.

$$\pi \le \theta \le 2\pi$$
$$\frac{\pi}{2} \le \varphi \le \pi$$
$$0 \le \rho \le 4$$

One of the restrictions on the region in the problem statement was $y \le 0$. This means that if we look at this from above we'd see the portion of the circle of radius 4 that is below the x axis and so we need the given range of θ above to cover this region.

We were also told in the problem statement that $z \le 0$ and so we only want the portion of the sphere that is below the xy-plane. We therefore need the given range of φ to make sure we are only below the xy-plane.

We'll also need the divergence of the vector field so here is that.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(yx^2\right) + \frac{\partial}{\partial y}\left(xy^2 - 3z^4\right) + \frac{\partial}{\partial z}\left(x^3 + y^2\right) = 4xy$$

Now let's apply the Divergence Theorem to the integral and get it converted to spherical coordinates while we're at it.

$$\begin{aligned}
\iint\limits_S \vec{F} \cdot d\vec{S} &= \iiint\limits_E \operatorname{div}\vec{F}\, dV \\
&= \iiint\limits_E 4xy\, dV \\
&= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^4 4\left(\rho\sin\varphi\cos\theta\right)\left(\rho\sin\varphi\sin\theta\right)\left(\rho^2\sin\varphi\right)\, d\rho\, d\varphi\, d\theta \\
&= \int_{\pi}^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_0^4 4\rho^4\sin^3\varphi\cos\theta\sin\theta\, d\rho\, d\varphi\, d\theta
\end{aligned}$$

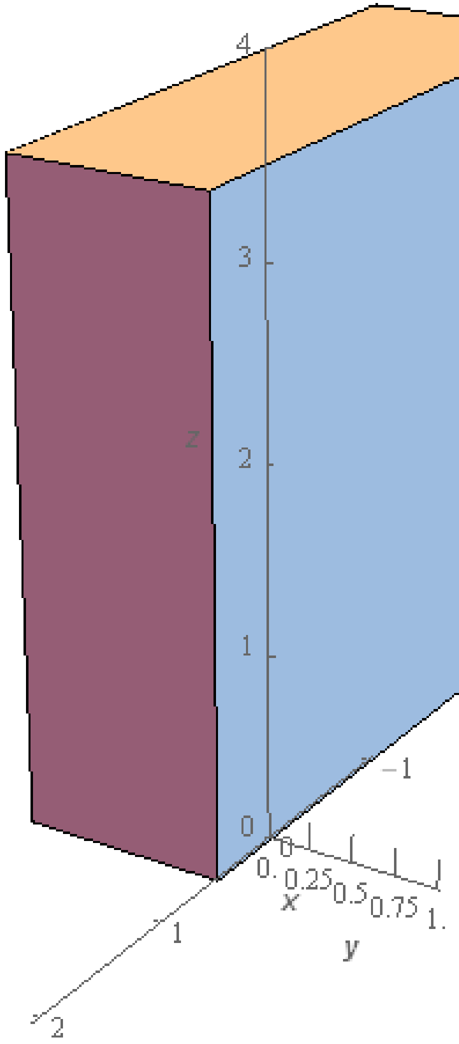
Don't forget to pick up the $\rho^2 \sin\varphi$ when converting the dV to spherical coordinates.

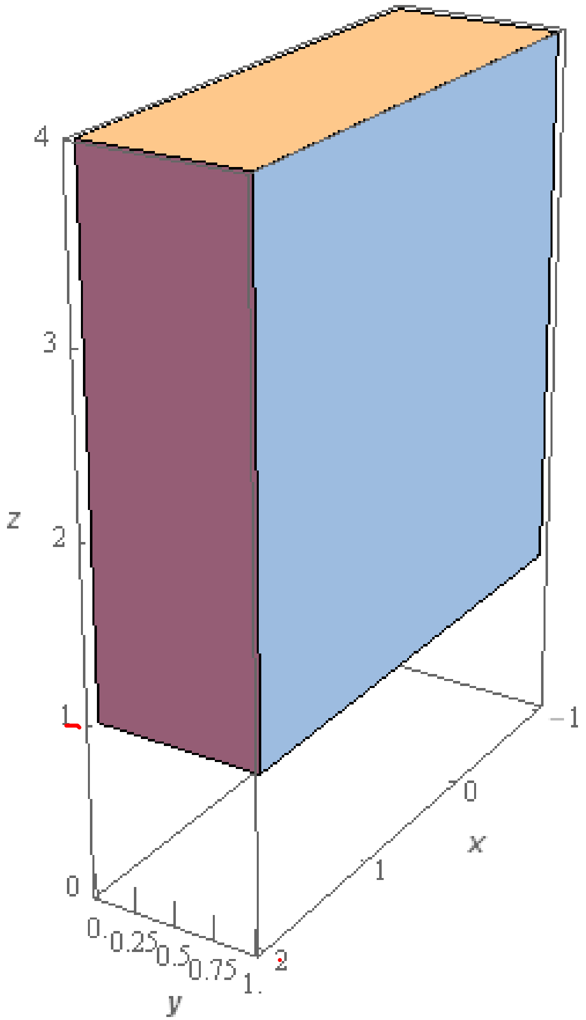
$$
\begin{aligned}
\iint\limits_S \vec{F} \cdot d\vec{S} &= \int_{\pi}^{2\pi} \int_{\frac{1}{2}\pi}^{\pi} \int_0^4 4\rho^4 \sin^3\varphi \cos\theta \sin\theta \, d\rho \, d\varphi \, d\theta \\
&= \int_{\pi}^{2\pi} \int_{\frac{1}{2}\pi}^{\pi} \left(\frac{4}{5}\rho^5 \sin^3\varphi \cos\theta \sin\theta \right) \Big|_0^4 \, d\varphi \, d\theta \\
&= \int_{\pi}^{2\pi} \int_{\frac{1}{2}\pi}^{\pi} \frac{4096}{5} \sin\varphi \left(1 - \cos^2\varphi\right) \cos\theta \sin\theta \, d\varphi \, d\theta \\
&= \int_{\pi}^{2\pi} \left(-\frac{4096}{5} \left(\cos\varphi - \frac{1}{3}\cos^3\varphi \right) \cos\theta \sin\theta \right) \Big|_{\frac{1}{2}\pi}^{\pi} \, d\theta \\
&= \int_{\pi}^{2\pi} \frac{8192}{15} \cos\theta \sin\theta \, d\theta \\
&= \int_{\pi}^{2\pi} \frac{4096}{15} \sin(2\theta) \, d\theta \\
&= -\frac{2048}{15} \cos(2\theta) \Big|_{\pi}^{2\pi} = \boxed{0}
\end{aligned}
$$

Example 3

2. Use the Divergence Theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \sin(\pi x)\,\vec{i} + zy^3\,\vec{j} + \left(z^2 + 4x\right)\,\vec{k}$ and S is the surface of the box with $-1 \le x \le 2$, $0 \le y \le 1$ and $1 \le z \le 4$. Note that all six sides of the box are included in S.

Solution:





$$\iint\limits_S \vec{F} \centerdot d\vec{S} = \iiint\limits_E \text{div}\vec{F}\, dV$$

where E is just the solid shown in the sketches from Step 1.

E is just a box and the limits defining it where given in the problem statement. The limits for our integral will then be,

$$\begin{array}{c} -1 \le x \le 2 \\ 0 \le y \le 1 \\ 1 \le z \le 4 \end{array}$$

We'll also need the divergence of the vector field so here is that.

$$\text{div}\vec{F} = \frac{\partial}{\partial x}\left(\sin(\pi x)\right) + \frac{\partial}{\partial y}\left(zy^3\right) + \frac{\partial}{\partial z}\left(z^2 + 4x\right) = \pi\cos(\pi x) + 3zy^2 + 2z$$

Now let's apply the Divergence Theorem to the integral and get it converted to a triple integral.

$$\begin{aligned} \iint\limits_S \vec{F} \centerdot d\vec{S} &= \iiint\limits_E \text{div}\vec{F}\, dV \\ &= \iiint\limits_E \pi\cos(\pi x) + 3zy^2 + 2z\, dV \\ &= \int_{-1}^{2}\int_0^1\int_1^4 \pi\cos(\pi x) + 3zy^2 + 2z\, dz\, dy\, dx \end{aligned}$$

All we need to do then in evaluate the integral.

$$\begin{aligned}
\iint\limits_S \vec{F} \cdot d\vec{S} &= \int_{-1}^{2}\int_{0}^{1}\int_{1}^{4} \pi\cos(\pi x) + 3zy^2 + 2z\,dz\,dy\,dx \\
&= \int_{-1}^{2}\int_{0}^{1} \left(\pi z\cos(\pi x) + \frac{3}{2}z^2y^2 + z^2\right)\Bigg|_1^4 dy\,dx \\
&= \int_{-1}^{2}\int_{0}^{1} 3\pi\cos(\pi x) + \frac{45}{2}y^2 + 15\,dy\,dx \\
&= \int_{-1}^{2} \left(3y\pi\cos(\pi x) + \frac{15}{2}y^3 + 15y\right)\Bigg|_0^1 dx \\
&= \int_{-1}^{2} 3\pi\cos(\pi x) + \frac{45}{2}\,dx \\
&= \left(3\sin(\pi x) + \frac{45}{2}x\right)\Bigg|_{-1\pi}^{2} = \boxed{\frac{135}{2}}
\end{aligned}$$

Differential Equations

- The first definition that we should cover should be that of **differential equation**.
  A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.
- There is one differential equation that everybody probably knows, that is Newton's Second Law of Motion. If an object of mass mm is moving with acceleration a and being acted on with force F then Newton's Second Law tells us.

$$F = ma \tag{1}$$

To see that this is in fact a differential equation we need to rewrite it a little. First, remember that we can rewrite the acceleration, a, in one of two ways.

$$a = \frac{dv}{dt} \quad \text{OR} \quad a = \frac{d^2u}{dt^2} \tag{2}$$

Where v is the velocity of the object and u is the position function of the object at any time t. We should also remember at this point that the force, F may also be a function of time, velocity, and/or position.

So, with all these things in mind Newton's Second Law can now be written as a differential equation in terms of either the velocity, v, or the position, u, of the object as follows.

$$m\frac{dv}{dt} = F(t, v) \tag{3}$$

$$m\frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right) \tag{4}$$

So, here is our first differential equation. We will see both forms of this in later chapters.

Here are a few more examples of differential equations.

$$ay'' + by' + cy = g(t) \tag{5}$$

$$\sin(y)\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2\mathbf{e}^{-5y} \tag{6}$$

$$y^{(4)} + 10y''' - 4y' + 2y = \cos(t) \tag{7}$$

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{8}$$

$$a^2 u_{xx} = u_{tt} \tag{9}$$

$$\frac{\partial^3 u}{\partial x^2 \partial t} = 1 + \frac{\partial u}{\partial y} \tag{10}$$

Order

The **order** of a differential equation is the largest derivative present in the differential equation. In the differential equations listed above (3) is a first order differential equation, (4), (5), (6), (8), and (9) are second order differential equations, (10) is a third order differential equation and (7) is a fourth order differential equation.

Note that the order does not depend on whether or not you've got ordinary or partial derivatives in the differential equation.

Ordinary and Partial Differential Equations

A differential equation is called an **ordinary differential equation**, abbreviated by **ode,** if it has ordinary derivatives in it. Likewise, a differential equation is called a **partial differential equation**, abbreviated by **pde,** if it has partial derivatives in it. In the differential equations above (3) - (7) are ode's and (8) - (10) are pde's.

Linear Differential Equations

A **linear differential equation** is any differential equation that can be written in the following form.

$$a_n(t)\,y^{(n)}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \cdots + a_1(t)\,y'(t) + a_0(t)\,y(t) = g(t) \tag{11}$$

The important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power. Also note that neither the function or its derivatives are "inside" another function, for example, $\sqrt{y'}$ or $\mathbf{e}^y$.

The coefficients $a_0(t),\ \ldots\ ,a_n(t)$ and $g(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function,$y(t)$, and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, (11) then it is called a **non-linear** differential equation.

In (5) - (7) above only (6) is non-linear, the other two are linear differential equations. We can't classify (3) and (4) since we do not know what form the function F has. These could be either linear or non-linear depending on F.

Solution

A **solution** to a differential equation on an interval $\alpha < t < \beta$ is any function $y(t)$ which satisfies the differential equation in question on the interval $\alpha < t < \beta$. It is important to note that solutions are often accompanied by intervals and these intervals can impart some important information about the solution. Consider the following example.

Example 1 Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$.

We'll need the first and second derivative to do this.

$$y'(x) = -\frac{3}{2}x^{-\frac{5}{2}} \qquad y''(x) = \frac{15}{4}x^{-\frac{7}{2}}$$

Plug these as well as the function into the differential equation.

$$\begin{aligned} 4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right) &= 0 \\ 15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}} &= 0 \\ 0 &= 0 \end{aligned}$$

So, $y(x) = x^{-\frac{3}{2}}$ does satisfy the differential equation and hence is a solution. Why then did we include the condition that $x > 0$? We did not use this condition anywhere in the work showing that the function would satisfy the differential equation.

To see why recall that

$$y(x) = x^{-\frac{3}{2}} = \frac{1}{\sqrt{x^3}}$$

In this form it is clear that we'll need to avoid $x = 0$ at the least as this would give division by zero.

So, we saw in the last example that even though a function may symbolically satisfy a differential equation, because of certain restrictions brought about by the solution we cannot use all values of the independent variable and hence, must make a restriction on the independent variable. This will be the case with many solutions to differential equations.

In the last example, note that there are in fact many more possible solutions to the differential equation given. For instance, all of the following are also solutions

$$
\begin{aligned}
y(x) &= x^{-\frac{1}{2}} \\
y(x) &= -9x^{-\frac{3}{2}} \\
y(x) &= 7x^{-\frac{1}{2}} \\
y(x) &= -9x^{-\frac{3}{2}} + 7x^{-\frac{1}{2}}
\end{aligned}
$$

We'll leave the details to you to check that these are in fact solutions. Given these examples can you come up with any other solutions to the differential equation? There are in fact an infinite number of solutions to this differential equation.

So, given that there are an infinite number of solutions to the differential equation in the last example (provided you believe us when we say that anyway….) we can ask a natural question. Which is the solution that we want or does it matter which solution we use? This question leads us to the next definition in this section.

Initial Condition(s)

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s when we're feeling lazy…) are of the form,

$$y(t_0) = y_0 \quad \text{and/or } y^{(k)}(t_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. As we will see eventually, solutions to "nice enough" differential equations are unique and hence only one solution will meet the given initial conditions.

The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation as we will see.

Example 2 $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$, $y(4) = \frac{1}{8}$, and $y'(4) = -\frac{3}{64}$.

As we saw in previous example the function is a solution and we can then note that

$$y(4) = 4^{-\frac{3}{2}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{8}$$

$$y'(4) = -\frac{3}{2}4^{-\frac{5}{2}} = -\frac{3}{2}\frac{1}{\left(\sqrt{4}\right)^5} = -\frac{3}{64}$$

and so this solution also meets the initial conditions of $y(4) = \frac{1}{8}$ and $y'(4) = -\frac{3}{64}$. In fact, $y(x) = x^{-\frac{3}{2}}$ is the only solution to this differential equation that satisfies these two initial conditions.

Initial Value Problem
An **Initial Value Problem** (or **IVP**) is a differential equation along with an appropriate number of initial conditions.

Example 3 The following is an IVP.

$$4x^2y'' + 12xy' + 3y = 0 \qquad y(4) = \frac{1}{8}, \quad y'(4) = -\frac{3}{64}$$

Example 4 Here's another IVP.

$$2t\,y' + 4y = 3 \qquad y(1) = -4$$

As we noted earlier the number of initial conditions required will depend on the order of the differential equation.

General Solution

The **general solution** to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

Example 5 $y\left(t\right) = \frac{3}{4} + \frac{c}{t^2}$ is the general solution to

$$2t\,y' + 4y = 3$$

We'll leave it to you to check that this function is in fact a solution to the given differential equation. In fact, all solutions to this differential equation will be in this form. This is one of the first differential equations that you will learn how to solve and you will be able to verify this shortly for yourself.

Actual Solution

The **actual solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Example 6 What is the actual solution to the following IVP?

$$2t\,y' + 4y = 3 \qquad y(1) = -4$$

This is actually easier to do than it might at first appear. From the previous example we already know (well that is provided you believe our solution to this example…) that all solutions to the differential equation are of the form.

$$y(t) = \frac{3}{4} + \frac{c}{t^2}$$

All that we need to do is determine the value of c that will give us the solution that we're after. To find this all we need do is use our initial condition as follows.

$$-4 = y(1) = \frac{3}{4} + \frac{c}{1^2} \qquad \Rightarrow \qquad c = -4 - \frac{3}{4} = -\frac{19}{4}$$

So, the actual solution to the IVP is.

$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

First Order Differential Equations

- In this chapter we will look at solving first order differential equations.

  The most general first order differential equation can be written as,

$$\frac{dy}{dt} = f\left(y, t\right)$$

Separable Equations

We are now going to start looking at nonlinear first order differential equations.

The first type of nonlinear first order differential equations that we will look at is separable differential equations.

A separable differential equation is any differential equation that we can write in the following form:-

$$N(y)\frac{dy}{dx} = M(x)$$

$$N\left(y\right)\frac{dy}{dx} = M\left(x\right) \tag{1}$$

Note that in order for a differential equation to be separable all the y's in the differential equation must be multiplied by the derivative and all the x's in the differential equation must be on the other side of the equal sign.

To solve this differential equation we first integrate both sides with respect to x to get,

$$\int N\left(y\right)\frac{dy}{dx}\,dx = \int M\left(x\right)\,dx$$

Now, remember that y is really $y\left(x\right)$ and so we can use the following substitution,

$$u = y\left(x\right) \qquad du = y'\left(x\right)\,dx = \frac{dy}{dx}\,dx$$

Applying this substitution to the integral we get,

$$\int N\left(u\right)\,du = \int M\left(x\right)\,dx \tag{2}$$

At this point we can (hopefully) integrate both sides and then back substitute for the u on the left side. Note, that as implied in the previous sentence, it might not actually be possible to evaluate one or both of the integrals at this point. If that is the case, then there won’t be a lot we can do to proceed using this method to solve the differential equation.

Or

$$N\left(y\right)\frac{dy}{dx}=M\left(x\right) \qquad (1)$$

Note that in order for a differential equation to be separable all the y's in the differential equation must be multiplied by the derivative and all the x's in the differential equation must be on the other side of the equal sign.

To solve this differential equation we first integrate both sides with respect to x to get,

We obviously can't separate the derivative like that, but let's pretend we can for a bit and we'll see that we arrive at the answer with less work.

Now we integrate both sides of this to get,

$$\int N\left(y\right)dy=\int M\left(x\right)dx \qquad (3)$$

Example 1

Solve the following IVP and find the interval of validity of the solution.

$$y' = \mathbf{e}^{-y}\left(2x - 4\right) \qquad y\left(5\right) = 0$$

Solution:

This differential equation is easy enough to separate, so let's do that and then integrate both sides.

$$\mathbf{e}^{y}\,dy = (2x-4)\ dx$$
$$\int \mathbf{e}^{y}\,dy = \int (2x-4)\ dx$$
$$\mathbf{e}^{y} = x^{2} - 4x + c$$

Applying the initial condition gives

$$1 = 25 - 20 + c \qquad c = -4$$

This then gives an implicit solution of.

$$\mathbf{e}^{y} = x^{2} - 4x - 4$$

We can easily find the explicit solution to this differential equation by simply taking the natural log of both sides.

$$y(x) = \ln\left(x^{2} - 4x - 4\right)$$

Example 2

Solve the following IVP and find the interval of validity for the solution.

$$\frac{dr}{d\theta} = \frac{r^2}{\theta} \qquad r(1) = 2$$

Solution:

$$\frac{1}{r^2}dr = \frac{1}{\theta}d\theta$$

$$\int \frac{1}{r^2}dr = \int \frac{1}{\theta}d\theta$$

$$-\frac{1}{r} = \ln|\theta| + c$$

Now, apply the initial condition to find c.

$$-\frac{1}{2} = \ln(1) + c \qquad c = -\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

Solving for r gets us our explicit solution.

$$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

Example 3

Solve the following IVP.

$$\frac{dy}{dt} = \mathbf{e}^{y-t}\sec(y)\left(1+t^2\right) \qquad y(0) = 0$$

Solution:

$$\frac{dy}{dt} = \frac{\mathbf{e}^{y}\mathbf{e}^{-t}}{\cos(y)}\left(1+t^2\right)$$

$$\mathbf{e}^{-y}\cos(y)\,dy = \mathbf{e}^{-t}\left(1+t^2\right)dt$$

Now, with a little integration by parts on both sides we can get an implicit solution.

$$\int \mathbf{e}^{-y}\cos(y)\,dy = \int \mathbf{e}^{-t}\left(1+t^2\right)dt$$

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y)-\cos(y)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+c$$

Applying the initial condition gives.

$$\frac{1}{2}(-1) = -(3)+c \qquad c = \frac{5}{2}$$

Therefore, the implicit solution is.

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y)-\cos(y)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+\frac{5}{2}$$

Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax)$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x)$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax)$$

Thank you