
بسم الله الرحمن الرحيم

EELU

الجامعة المصرية للتعلم الإلكترونى

Egyptian E-Learning University

MATH – 1

BY

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CHAPTER 5

Applications of Definite Integrals

LECTURE 11.

Volumes and Lengths

Aims and Objectives:

(1) Explain the concepts of volume of a solid.

(2) Show how the volume of the solid can be generated.

(3) Evaluate volumes of solid of revolution.

(4) Introduce the concepts of length of curves.

(5) Calculate length of curves.

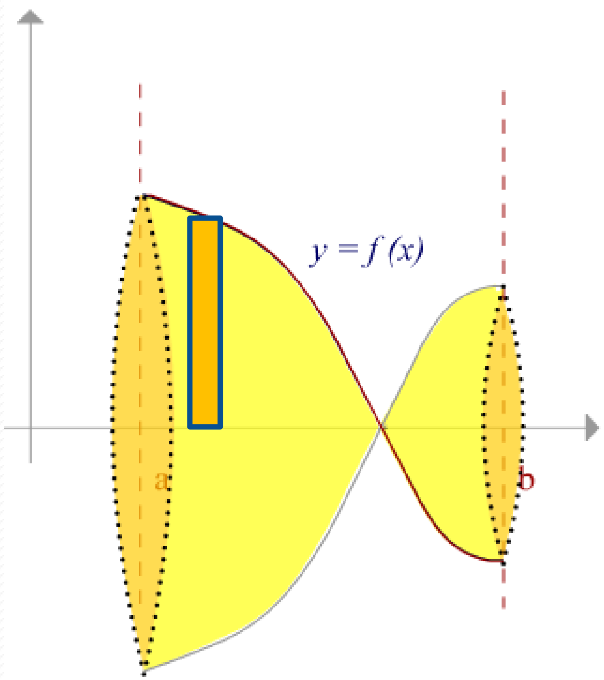
(6) Have a strong intuitive feeling for these important concepts.

Definition of a volume of solid of revolutions:

Definition:

Let f be continuous on $[a\ ,\ b\ ]$. The volume V of the solid of revolution generated by revolving the region bounded by the graphs of f, $x = a$, $x = b$ and the x-axis is

$$V = \lim_{\|p\|\to 0} \sum_{i} \pi [f\ (w_i\ )]^2 \Delta x_i = \int_a^b \pi\ [f\ (x\ )]^2\ dx$$



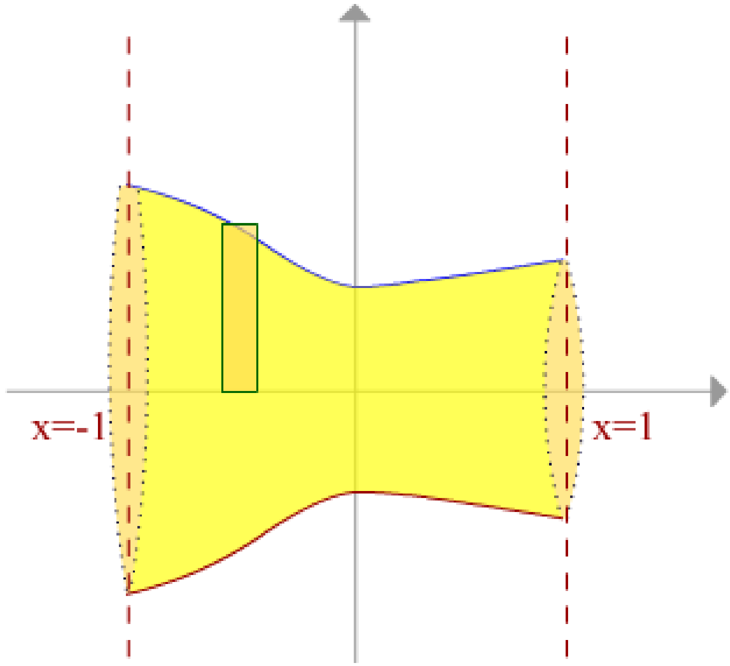
The requirement that $f(x) \geq 0$ for all x in $[a\ ,\ b\ ]$, was omitted in the definition.

Example 1:

If $f(x) = x^2+1$, find the volume of the solid generated by revolving the region under the graph of f from -1 to 1 about the x-axis.

Solution:

The solid is illustrated in the following figure included in the sketch is a typical rectangle and the disk that it generates. Since the radius of the disc that is $w_i^2 + 1$, its volume is $\pi (w_i^2 +1)^2 \Delta x_i$ and



$$V = \lim_{\|p\|\to 0} \sum_i \pi (w_i^2 + 1)^2 \Delta x_i$$

$$= \int_{-1}^{1} \pi (x^2+1)^2 dx = \pi \int_{-1}^{1} (x^4 + 2x^2 + 1) dx$$

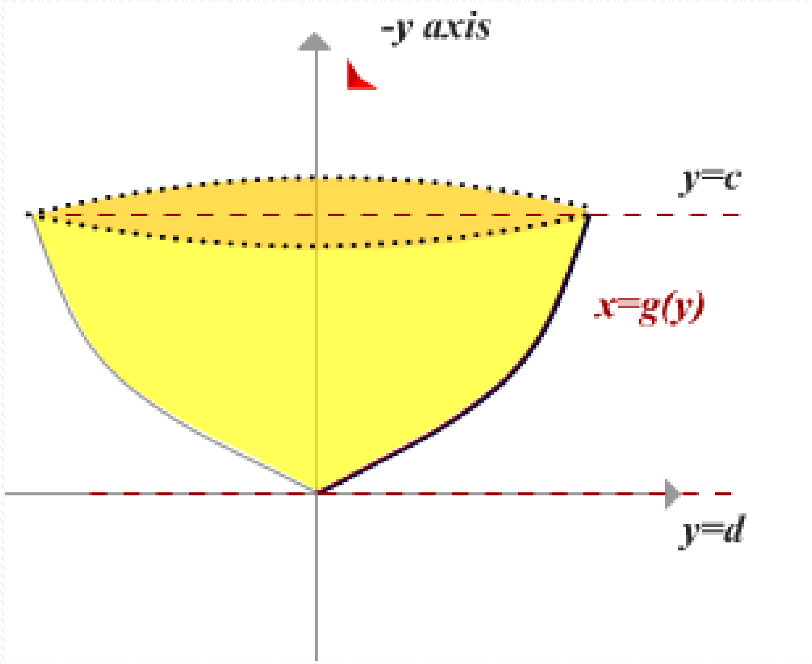
$$= \pi [\frac{1}{5} x^5 + \frac{2}{3} x^3 + x]_{-1}^{1}$$

$$= \pi [(\frac{1}{5} + \frac{2}{3} + 1) - (-\frac{1}{5} - \frac{2}{3} - 1)] = \frac{56}{15} \pi$$

Definition:

Let g be continuous $[a , b]$. The volume V of t revolution generated by revolving the region bounded by the graphs of $x = g(y)$, $y = c$, $y = d$ and the y-axis is

$$V = \lim_{\|p\| \to 0} \sum_i \pi [g(w_i)]^2 \Delta y_i = \int_c^d \pi [g(y)]^2 dy$$



Example 2:

The region bounded by the y-axis, the graph of $y = x^3$, $y = 1$ and $y = 8$ is revolved about the y-axis. Find the volume of the resulting solid.

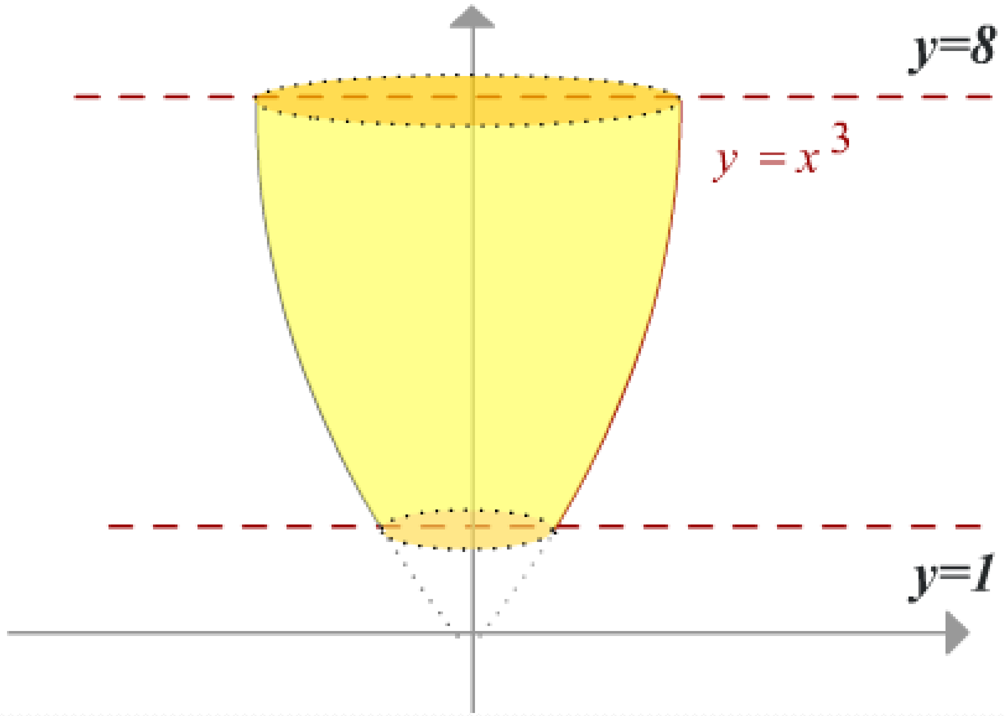
Solution:

The solid is sketched together with a disc generated by a typical rectangle. Since we plan to integrate with respect to y, we solve the equation $y = x^3$ for x in terms of y ,obtaining $x = y^{1/3}$, and we let $x = g(y) = y^{1/3}$, then as shown in the figure, the radius of a typical disc is $g(w_i) = w_i^{1/3}$ and Its volume is $:(w_i^{1/3})^2 \Delta y_i$ applying the definition with $g(y) = y^{1/3}$ gives us

$$V = \lim_{\|P\| \to 0} \sum_i \pi (w_i^{1/3})^2 \Delta y_i$$

$$= \int_1^8 \pi (y^{1/3})^2 dy = \pi \int_1^8 y^{2/3} dy$$

$$= \pi \left(\frac{3}{5}\right) [y^{5/3}]_1^8 = \frac{3}{5} \pi [8^{5/3} - 1] = \frac{93}{5} \pi$$



Example 3:

The region bounded by the graphs of the equations

$$x^2 = y - 2, 2y - x - 2 = 0, x = 0, and\ x = 1$$

is revolved about the x-axis. Find the volume of the resulting solid.

Solution:

The region and a typical rectangle are sketched. Then we wish to integrate with respect to x we solve the first two equations for y in terms of x, obtaining

$$y = x^2 + 2\ and\ y = \frac{1}{2}x + 1.$$

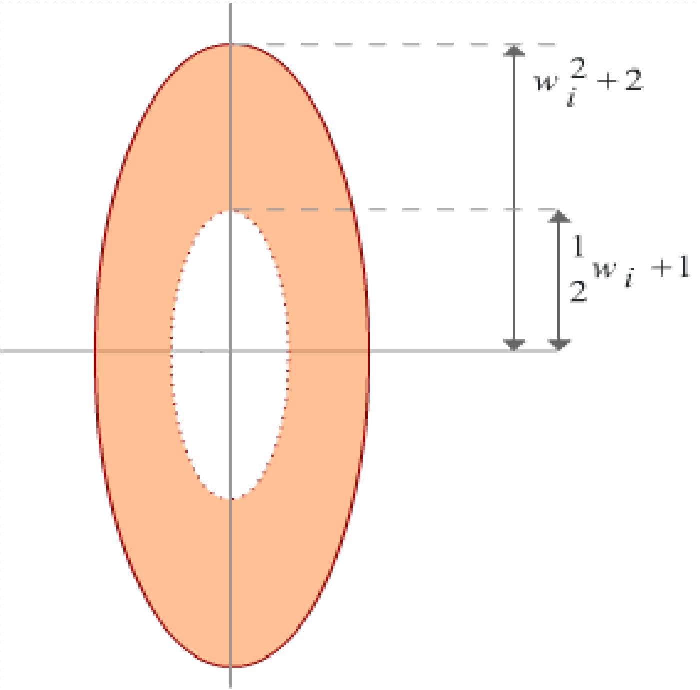
Since outer radius of the washer is $w_i^2 + 2$ and the inner radius is $1/2\ w_i + 1$, its volume is $\pi[(w_i^2 + 2)^2 - (\frac{1}{2}w_i + 1)^2]\Delta x_i$.

Taking the limits of the sum of such volumes gives us

$$V = \int_0^1 \pi[(x^2+2)^2 - (\frac{1}{2}x+1)^2]dx$$

$$= \pi \int_0^1 (x^4 + \frac{15}{4}x^2 - x + 3)dx$$

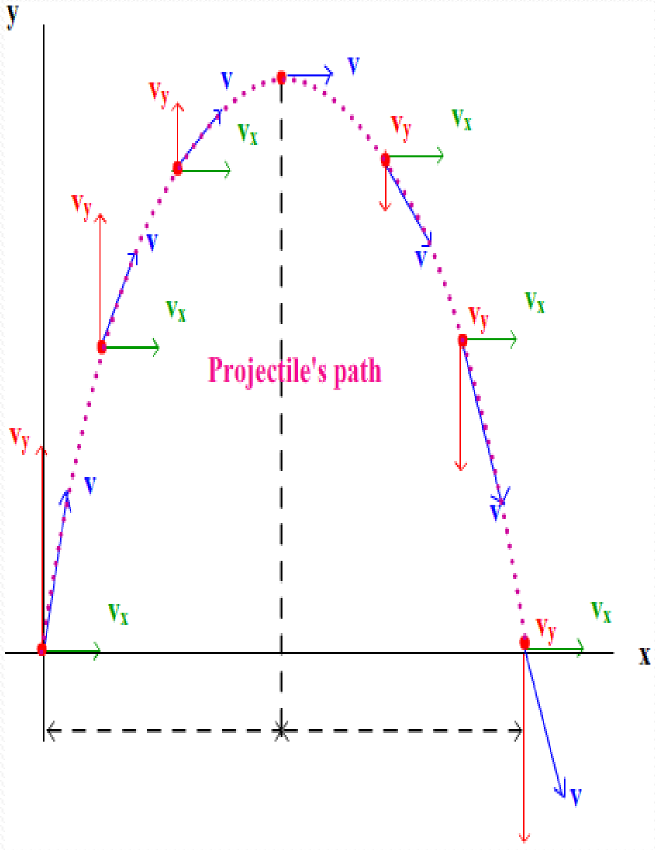
$$= \pi[\frac{1}{5}x^5 + \frac{5}{4}x^3 - \frac{1}{2}x^2 + 3x]_0^1 = \frac{79\pi}{20}$$



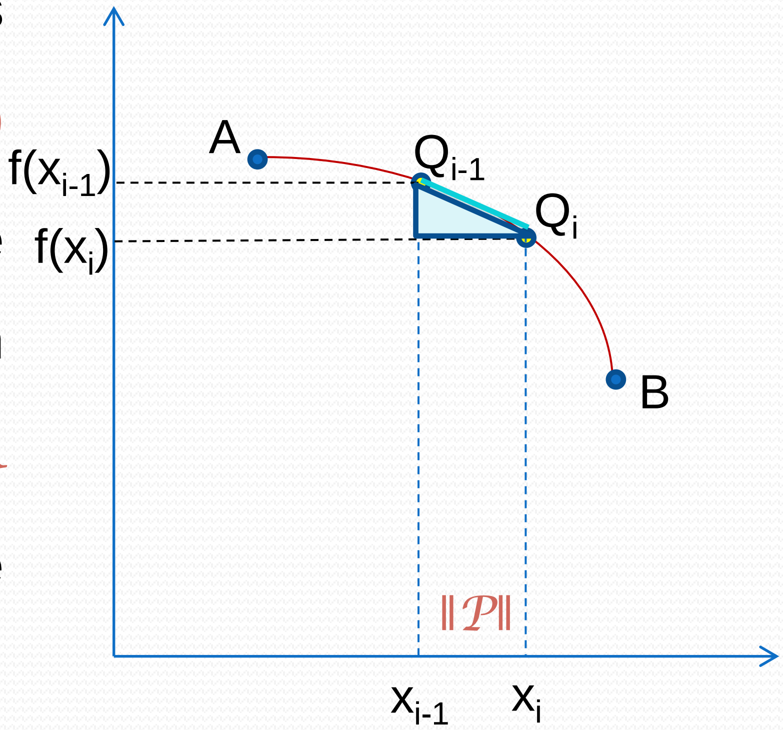
Arc Length:

To solve certain problem in the sciences it is essential to consider the length of the graph of a function.

For example, if a projectile moves along a parabolic course, we may wish to determine the distance it travels during a specified interval of time. Similarly, it may be necessary to find the length of a twisted piece of wire. We could simply straighten it and find the linear length with a ruler (or by mean of the distance formula). As we shall see, the key



If the norm $\|\mathcal{P}\|$ of the partition $\mathcal{P}$ of $[a, b]$ is small, then the distance between $Q_{i-1}(x_{i-1}, f(x_{i-1}))$ and $Q_i(x_i, f(x_i))$ for each i is very small and we expect the length $L_P = \sum_{i=1}^{n} d(Q_{i-1}, Q_i)$ to be an approximation to the length of arc between $\mathcal{A}$ and $\mathcal{B}$. This gives us a clue to suitable definition of arc length.



Specifically, we shall consider the limit of the sum $\mathcal{L}_p$ as $\|\mathcal{P}\| \to 0$ to formulate this concept precisely, and at the same time arrive at a formula for calculating arc length. By the distance formula

$$d(Q_{i-1}, Q_i) = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

Applying the mean value theorem

$$f(x_i) - f(x_{i-1}) = f'(w_i)(x_i - x_{i-1})$$

where w_i is an open interval (x_{i-1}, x_i). Substituting this into the preceding formula and letting $\Delta x_i = x_i - x_{i-1}$, we obtain

$$d(Q_{i-1}, Q_i) = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

$$d(Q_{i-1}, Q_i) = \sqrt{(\Delta x_i)^2 + [f'(w_i)\Delta x_i)]^2}$$

$$= \sqrt{1 + [f'(w_i)]^2}\,\Delta x_i$$

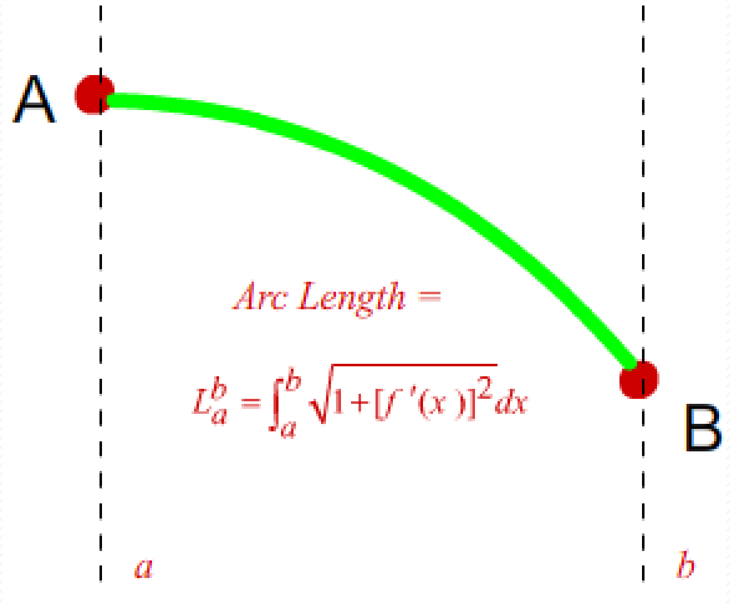
Consequently, $L_P = \sum_{i=1}^{n} \sqrt{1 + [f'(w_i)]^2}\,\Delta x_i$ $= \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

Definition:

Let the function f be smooth on a closed interval $[a, b]$.

The arc length of the graph of f from $A(a, f(a))$ and $B(b, f(b))$ is given by

$$L_a^b = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$



Example 4:

If $f(x) = 3x^{2/3} - 10$, find the arc length of the graph of f from the point $A(8, 2)$ to $B(27, 17)$.

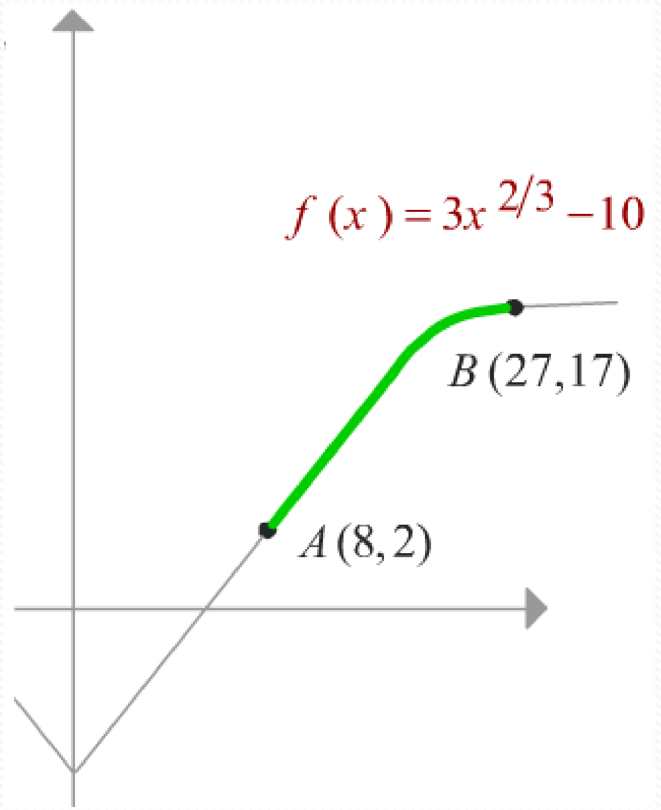
Solution:

The graph f is sketched in the opposite figure, then

$$L_a^b = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$L_8^{27} = \int_8^{27} \sqrt{1 + \left(\frac{2}{x^{1/3}}\right)^2}\, dx = \int_8^{27} \sqrt{1 + \frac{4}{x^{2/3}}}\, dx$$

$$= \int_8^{27} \frac{\sqrt{x^{2/3} + 4}}{x^{1/3}}\, dx$$



To evaluate this integral, use integration by substitution

$$L_8^{27} = \int_8^{27} \frac{\sqrt{x^{2/3}+4}}{x^{1/3}}\,dx$$

let $u = x^{2/3} + 4 \quad and \ du = \frac{2}{3}x^{-1/3}dx$

Then $L_8^{27} = \frac{3}{2}\int_8^{27} \sqrt{x^{2/3}+4}\left(\frac{2}{3x^{1/3}}\right)dx$

If $\chi = 8$ then $u = (8)^{2/3} + 4 = 8,$

whereas if $\chi = 27$ then $u = (27)^{2/3} + 4 = 13$

Making substitution and changing the limits of

integration

$$L_8^{27} = \frac{3}{2}\int_8^{13} \sqrt{u}\,du = u^{3/2}]_8^{13} = 13^{3/2} - 8^{3/2} \approx 24.2$$

THANK YOU