Mathematics (2)

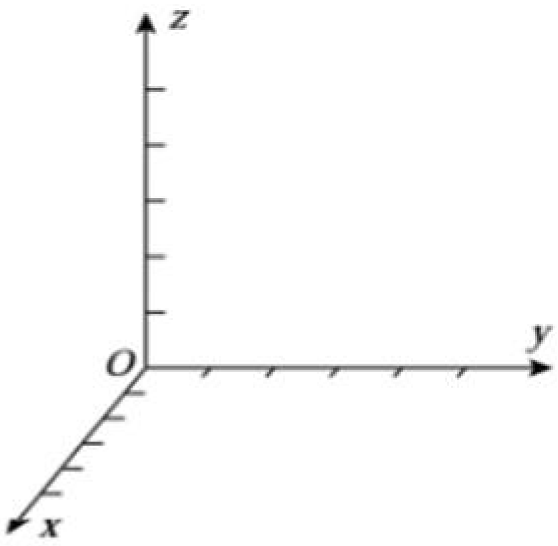
Section (4)

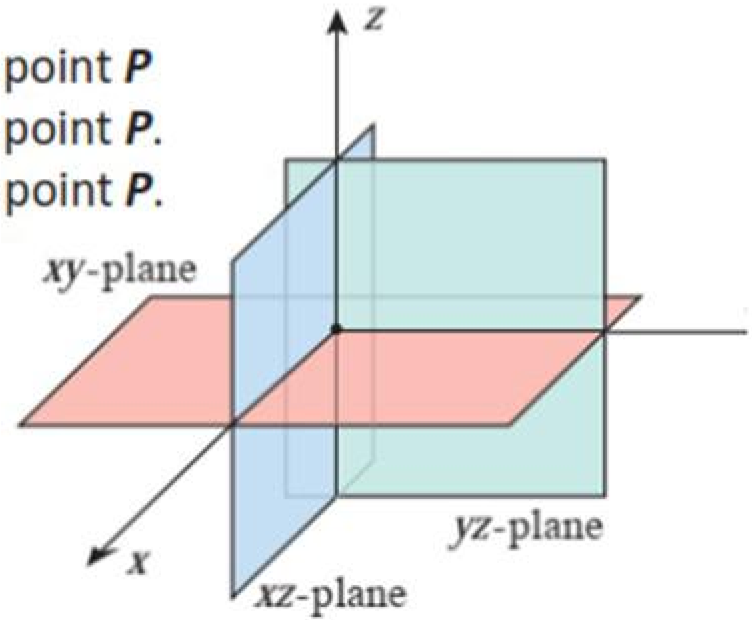
Vectors and the Geometry of Space

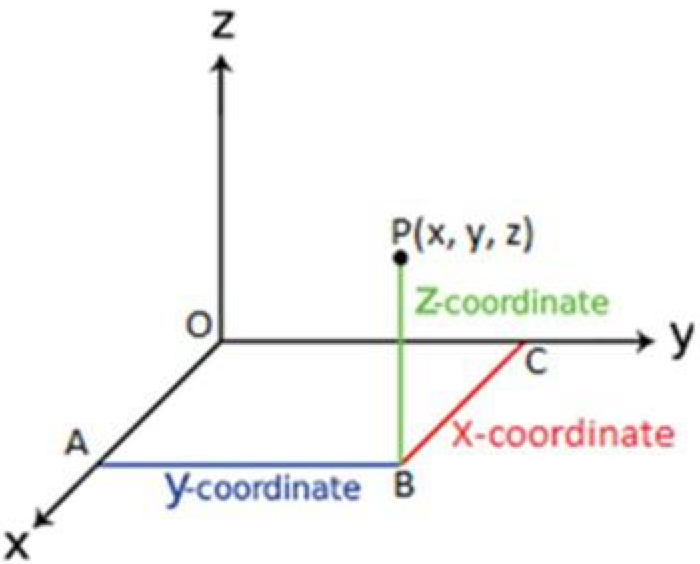
Cartesian Coordinate System

You can construct the Cartesian coordinate system by passing z axis perpendicular to both the x and y axes at the origin. The coordinate axes, taken in pairs, determine three ***coordinate planes***: the ***xy-plane***, the ***xz-plane***, and the ***yz-plane***. In this three-dimensional system, a point ***P*** in space is determined by an ordered triple (x, y, z), where x, y and z are as follows:

x = directed distance from **yz-plane** to the point ***P***
y = directed distance from **xz-plane** to the point ***P***.
z = directed distance from **xy-plane** tot he point ***P***.







REGION	DESCRIPTION
xy-plane	Consists of all points of the form $(x, y, 0)$
xz-plane	Consists of all points of the form $(x, 0, z)$
yz-plane	Consists of all points of the form $(0, y, z)$
x-axis	Consists of all points of the form $(x, 0, 0)$
y-axis	Consists of all points of the form $(0, y, 0)$
z-axis	Consists of all points of the form $(0, 0, z)$

LINES DETERMINED BY A POINT AND A VECTOR

Consider a line L in 3-space that passes through the point $P_0 = (x_0, y_0, z_0)$ and is parallel to the nonzero vector $\vec{v} = \langle a,b,c \rangle$ Vector parallel للخط و نقطة واقعة علي الخط

The vector $\vec{v}$ is a **direction vector** for the line L, and a, b, and c are **direction numbers (ratios).**

Then L consists precisely of those points $P = (x, y, z)$ for which the vector $\overrightarrow{P_0P}$ is parallel to $\vec{v}$.
In other words, the point $P = (x, y, z)$ is on L if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\vec{v}$, say

$$\boxed{\overrightarrow{P_0P} \mathbin{//} \vec{v} \Longleftrightarrow \overrightarrow{P_0P} = t\vec{v}}$$

This equation can be written as

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a,b,c \rangle = \langle ta,tb,tc \rangle$$

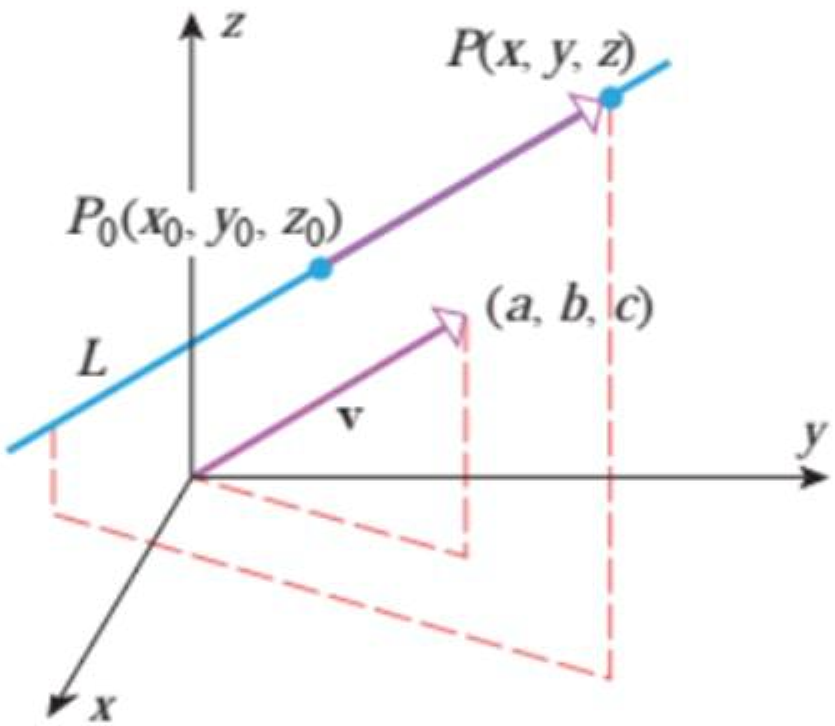
which implies that

$$x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc \qquad (1)$$

Thus, L can be described by the ***parametric equations***.

If we solve these equations for t, we get the following ***symmetric equations***.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$



Vector equations of lines

Because two vectors are equal if and only if their components are equal, the equations

$$x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc,$$

can be written in vector form as

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle,$$

or, equivalently, as

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle. \qquad (2)$$

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We define the vectors $\vec{r}$, $\vec{r}_0$, and $\vec{v}$ as

$$\vec{r} = \langle x, y, z\rangle,\ \ \vec{r}_0 = \langle x_0, y_0, z_0\rangle,\ \ \vec{v} = \langle a, b, c\rangle \qquad (3)$$

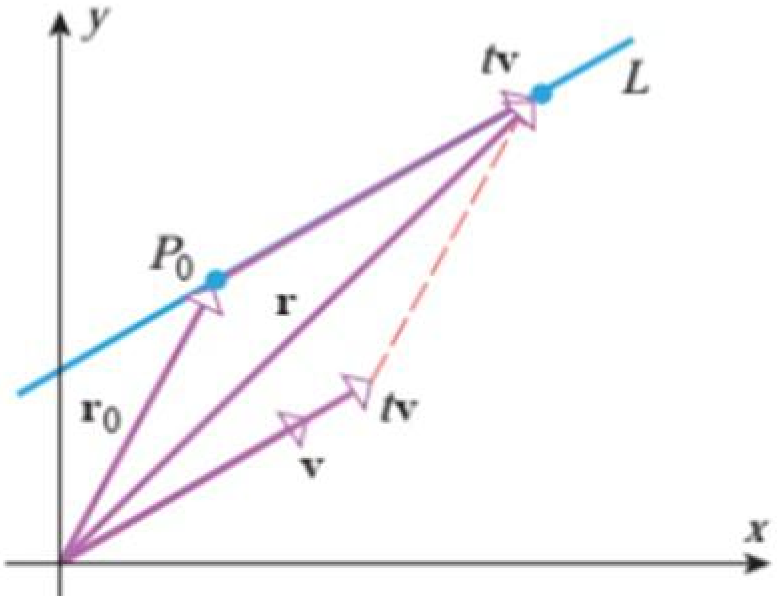
Where $\vec{r}$ is the position vector for any point (x, y, z) on the line L.

Substituting (3) in (2), respectively, yields the equation

$$\vec{r} = \vec{r}_0 + t\,\vec{v} \qquad (4)$$

This is called the *vector equation* of the line L .

Remark: Although it is not stated explicitly, it is understood in Eq.(1) and (4) that $-\infty < t < \infty$ which reflects the fact that lines extend indefinitely. Each value of the parameter t gives a point (x, y, z) on L



Example 1 Find the vector, parametric and symmetric equations of the line passing through (1, 2, -3) and parallel to **v = 4i + 5j - 7k**

Solution

Vector parallel للخط و نقطة واقعة علي الخط

Here,

$$\vec{r}=\langle x,y,z\rangle,\ \ \vec{r}_0=\langle 1,2,-3\rangle,\ \ \vec{v}=\langle 4,5,-7\rangle \qquad \boxed{\vec{r}=\langle x,y,z\rangle,\ \vec{r}_0=\langle x_0,y_0,z_0\rangle,\ \vec{v}=\langle a,b,c\rangle}$$

So, the vector Equation (4) becomes:

$$\vec{r}=\langle 1,2,-3\rangle+t\langle 4,5,-7\rangle \qquad \boxed{\vec{r}=\langle x_0,y_0,z_0\rangle+t\langle a,b,c\rangle}$$

or

$$\begin{aligned}\vec{r}&=\langle 1+4t,2+5t,-3-7t\rangle\\&=(1+4t)\vec{i}+(2+5t)\vec{j}+(-3-7t)\vec{k}\end{aligned}$$

Parametric equations are: $\boxed{x=x_0+ta,\ y=y_0+tb,\ z=z_0+tc}$

$$x=1+4t,\ y=2+5t,\ z=-3-7t.$$

Symmetric equations are:

$$\frac{x-1}{4}=\frac{y-2}{5}=\frac{z+3}{-7}. \qquad \boxed{\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}}$$

Example 2 (a) Find parametric equations of the line L passing through the points

Vector parallel للخط و نقطة واقعة علي الخط $P_1 = (2, 4, -1)$ and $P_2 = (5, 0, 7)$

(b) In what points does this line intersect the coordinate planes ?

Solution

(a) The vector $\overrightarrow{P_1P_2} = \langle 3, -4, 8 \rangle$ is parallel to L and the point $P_1 = (2, 4, -1)$ lies on L, so it follows from Eq.(1) that L has parametric equations $\boxed{x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc}$

$$x = 2 + 3t,\ y = 4 - 4t,\ z = -1 + 8t. \qquad (I)$$

Had we used P_2 as the point on L rather than P_1, we would have obtained the equations

$$x = 5 + 3t,\ y = -4t,\ z = 7 + 8t.$$

Although these equations look different from those obtained using P_1, the two sets of equations are equivalent in that both generate L as t varies from $-\infty$ to ∞.

REGION	DESCRIPTION
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(b)

- It follows from (I) in part (a) that the line intersects **the *xy*-plane** at the point where $z=-1+8t=0$ that is, when $t=\dfrac{1}{8}$. Substituting this value of t in (I) yields the point of intersection (x, y, z) = (19/8 , 7/2, 0).

$$x=2+3t,\ y=4-4t,\ z=-1+8t.$$

REGION	DESCRIPTION
xy-plane	Consists of all points of the form $(x, y, 0)$
xz-plane	Consists of all points of the form $(x, 0, z)$
yz-plane	Consists of all points of the form $(0, y, z)$
x-axis	Consists of all points of the form $(x, 0, 0)$
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z-axis	Consists of all points of the form $(0, 0, z)$

- The line intersects **the *xz*-plane** at the point where $y = 4 - 4t = 0$ that is, when $t = 1$.

Substituting this value of *t* in (I) yields the point of intersection (x, y, z) = (5 , 0, 7).

$$x = 2 + 3t,\ y = 4 - 4t,\ z = -1 + 8t$$

REGION	DESCRIPTION
xy-plane	Consists of all points of the form $(x, y, 0)$
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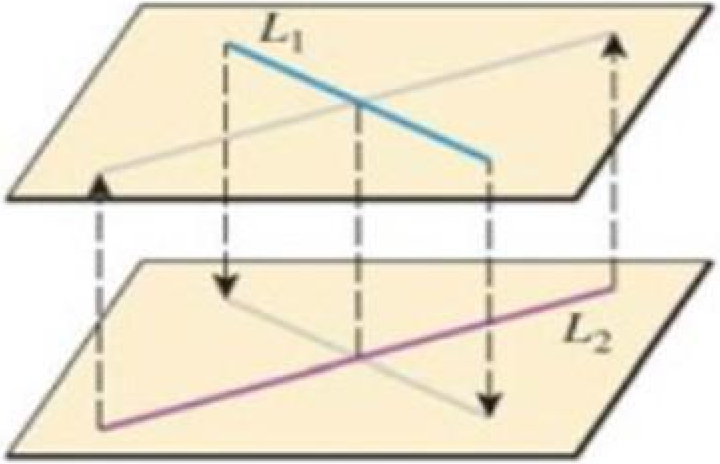
- It follows from (I) in part (a) that the line intersects **the *yz*-plane** at the point where $x = 2 + 3t$

that is, when $t = -\frac{2}{3}$. Substituting this value of t in (I) yields the point of intersection (x, y, z) = (0 , 0, 7).

$$x = 2 + 3t,\ y = 4 - 4t,\ z = -1 + 8t$$

Relationships between Lines

Given two lines in the two-dimensional plane, the lines are equal, they are parallel but not equal, or they intersect in a single point. In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew lines**



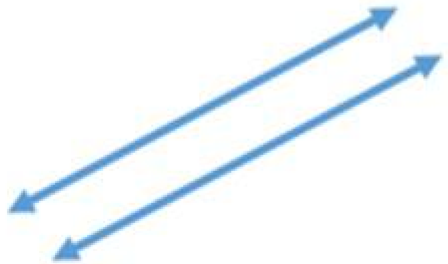
Example 3 Find the intersection point of the 2 lines

$$\begin{cases} L_1 : x = 2t, \;\; y = 3 - 4t, \;\; z = 1 + t \\ L_2 : x = 1 + s, \;\; y = -3s, \;\; z = -s \end{cases}$$

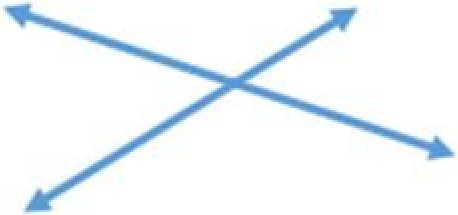
Solution

The lines intersect if there is a pair of parameters (s, t) that gives the same point on the two lines. This leads to three conditions on t and s,

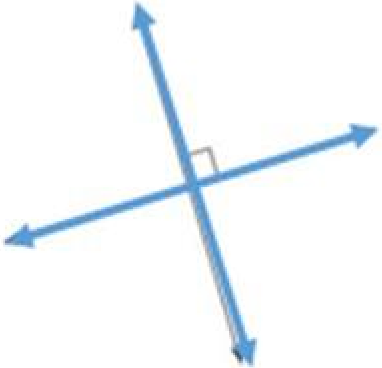
Lines



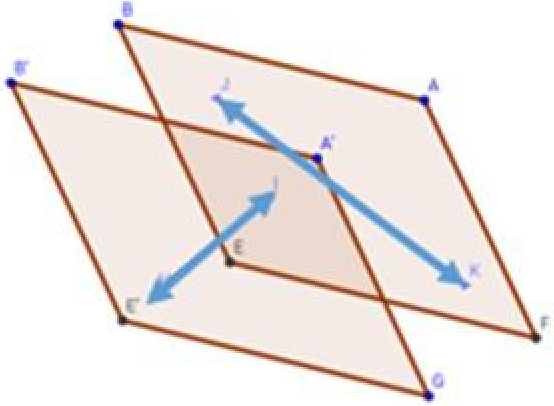
Parallel lines stay the same distance apart and never cross.



Intersecting lines cross at one point.



Perpendicular lines intersect and form right angles.



Skew lines are on different planes and do not intersect.

The lines intersect if there is a pair of parameters (s,t) that gives the same point on the two lines. This leads to three conditions on t and s,

$$\begin{aligned}2t &= 1+s,\\ 3-4t &= -3s,\\ 1+t &= -s\end{aligned}$$

Thus, the lines intersect if there are values of t and s that satisfy all three equations, and the lines do not intersect if there are no such values. We shall solve the first two equations for t and s and then check whether these values satisfy the third equation.

From the first equation , we have $s = 2t-1$. Substituting this value of s into the second equation yields:

$$3-4t = -3(2t-1) \Rightarrow 3-4t = -6t+3 \Rightarrow t = 0,$$

So, $s = -1$. The values $t = 0$ and $s = -1$ satisfy the third equation, so the lines intersect

To find the intersection point, substitute about the value $t = 0$ in the parametric equations of L_1 about the value $s = -1$ in the parametric equations of L_2. Hence the intersection point P is $P(0,3,1)$

Example 4 Find the relation between the lines L_1 and L_2 with parametric equations

$\boxed{x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc}$

$$x = 1 + t, \quad y = -2 + 3t, \quad z = 4 - t,$$
$$x = 2s, \quad y = 3 + s, \quad z = -3 + 4s.$$

Solution

The line L_1 is parallel to the direction vector $\langle 1, 3, -1 \rangle$, and the line L_2 is parallel to the direction vector $\langle 2, 1, 4 \rangle$. These vectors are not parallel since neither is a scalar multiple of the other (i.e., their components are not proportional). Thus, the lines are **not parallel**.

If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$1 + t = 2s$$
$$-2 + 3t = 3 + s$$
$$4 - t = -3 + 4s$$

However, if we solve the first two equations, we get:

$$t = \frac{11}{5} \quad \text{and} \quad s = \frac{8}{5}$$

These values don't satisfy the third equation. Thus, there are no values of t and s that satisfy the three equations. So, L_1 and L_2 **do not intersect**. Hence, L_1 and L_2 are **skew lines.**

Equation of a Plane

To determine a plane, we need a point $P_0 = (x_0, y_0, z_0)$ in the plane and a normal vector $\vec{n} = \langle a, b, c \rangle$ that is perpendicular to the plane.

The standard equations of a plane in space is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad (1)$$

By regrouping terms, we obtain the general form of the line

$$ax + by + cz + d = 0$$

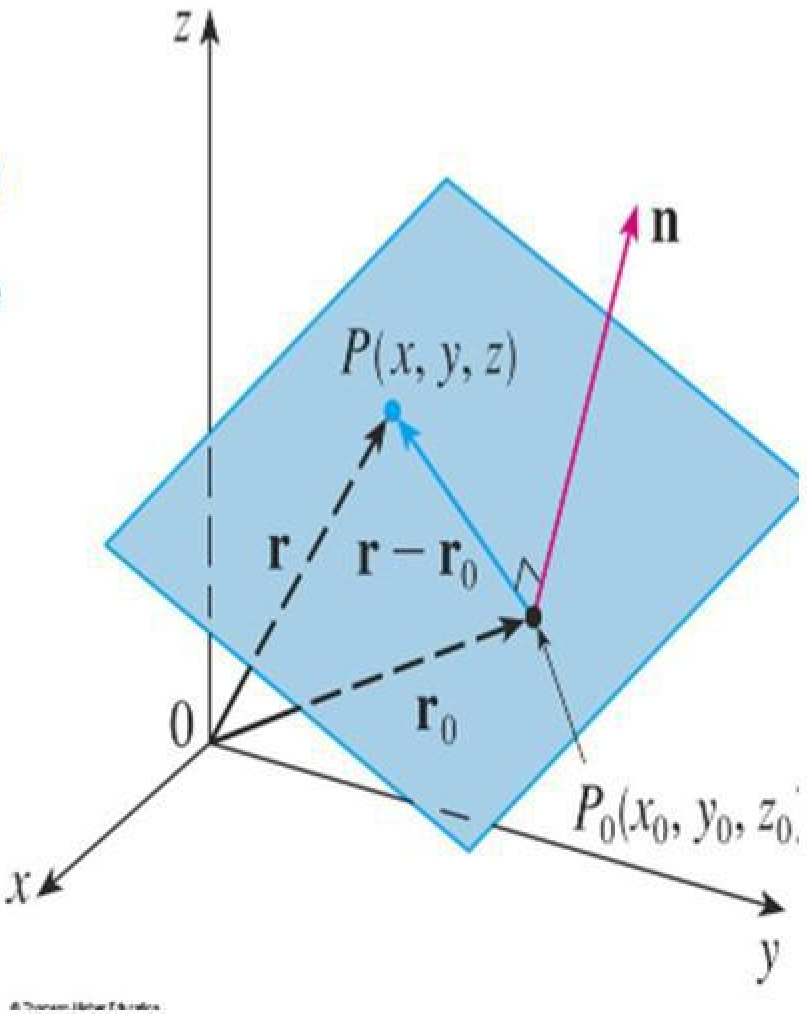
To obtain a vector equation for the plane, we write:

$$\vec{n} = \langle a, b, c \rangle, \vec{r} = \langle x, y, z \rangle, \ \vec{r}_0 = \langle x_0, y_0, z_0 \rangle$$

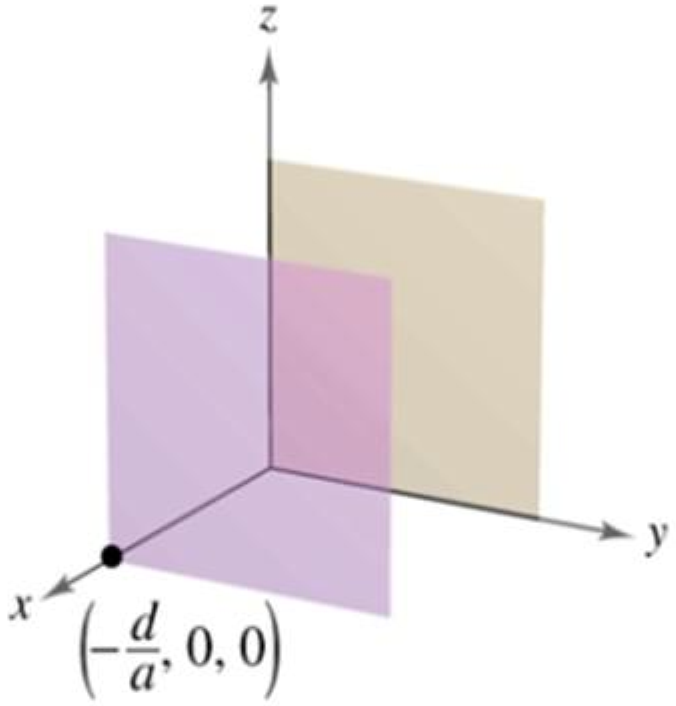
Then, the Equation (1) becomes:

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \qquad \Rightarrow \qquad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

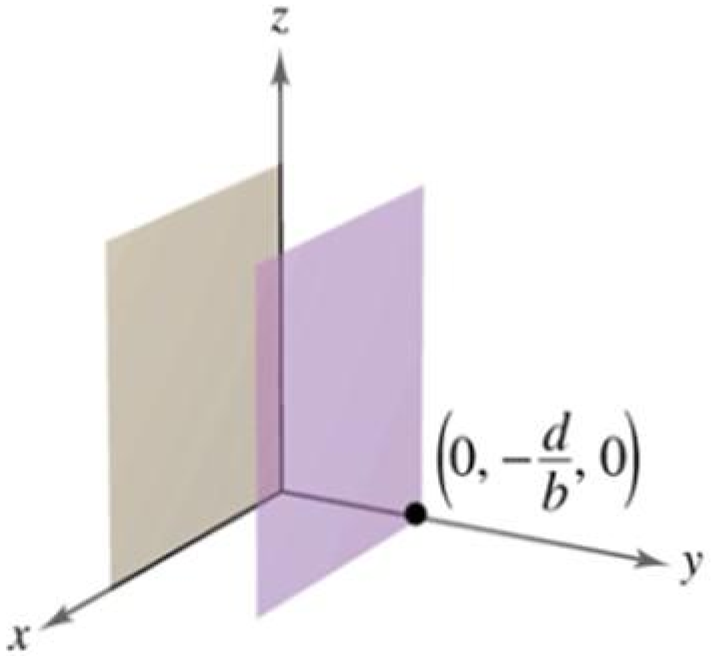
This equation is the vector equation of the plane through $P_0 = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$



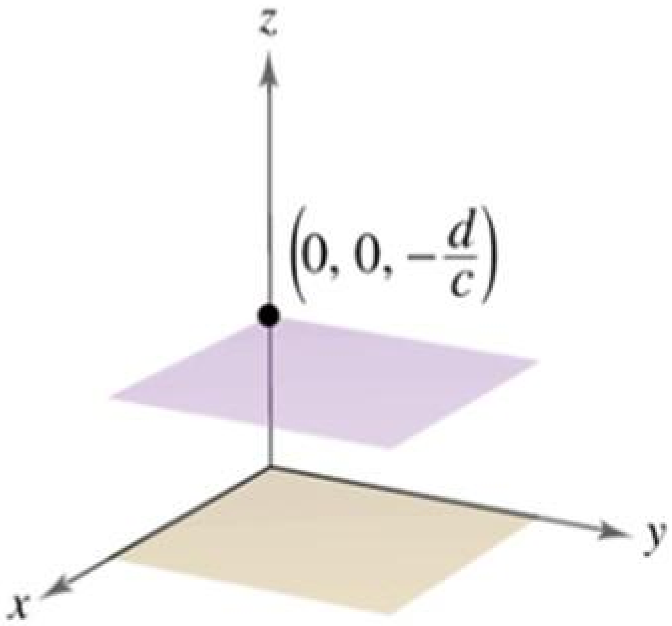
More equations of planes



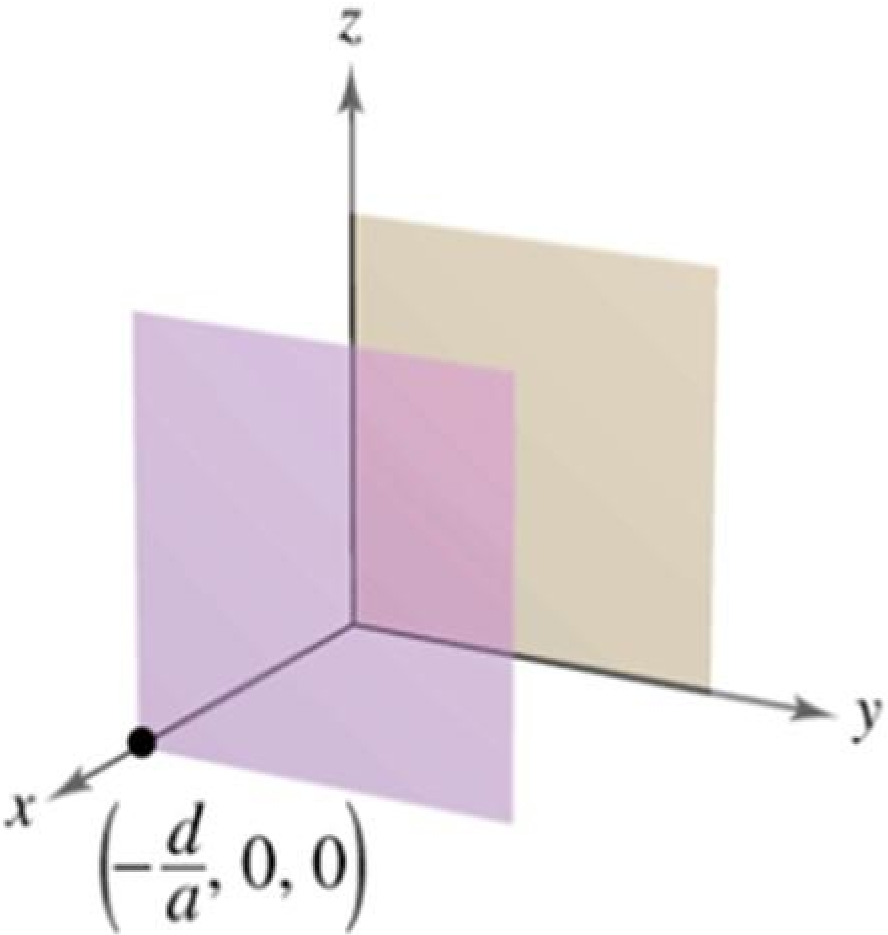
Plane $ax + d = 0$ is parallel to the yz-plane.



Plane $by + d = 0$ is parallel to the xz-plane.



Plane $cz + d = 0$ is parallel to the xy-plane.



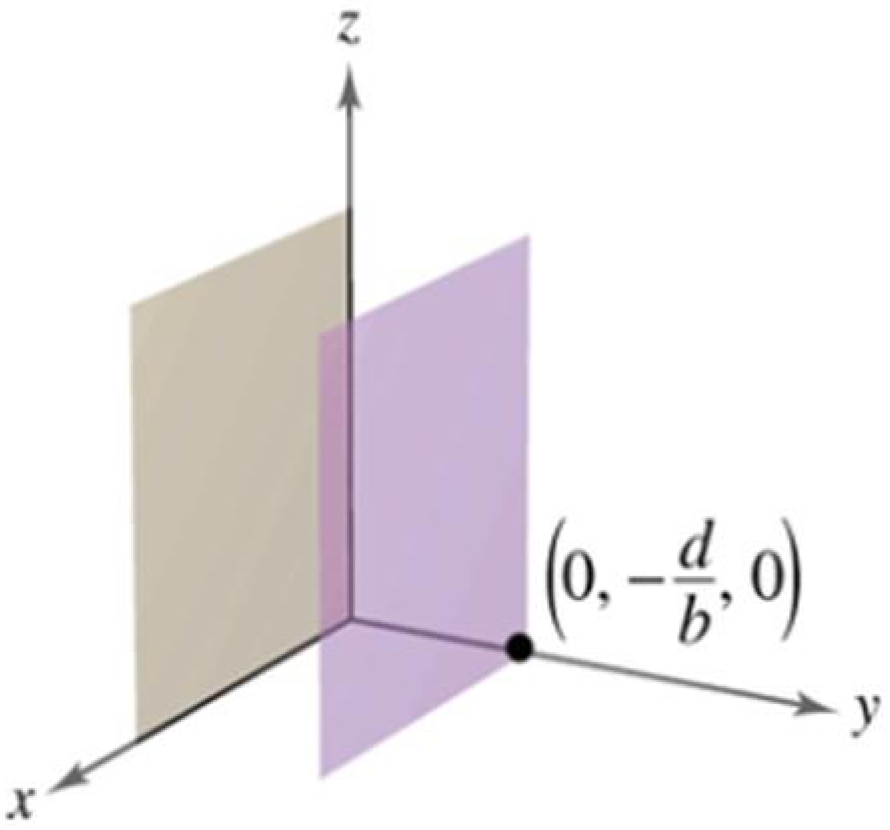
The standard equations of a plane in space is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$\vec{n} = a\hat{\imath} = < a, 0,0 >$$

$$p = \left(-\frac{d}{a}, 0,0\right)$$

Plane $ax + d = 0$ is parallel to the yz-plane.



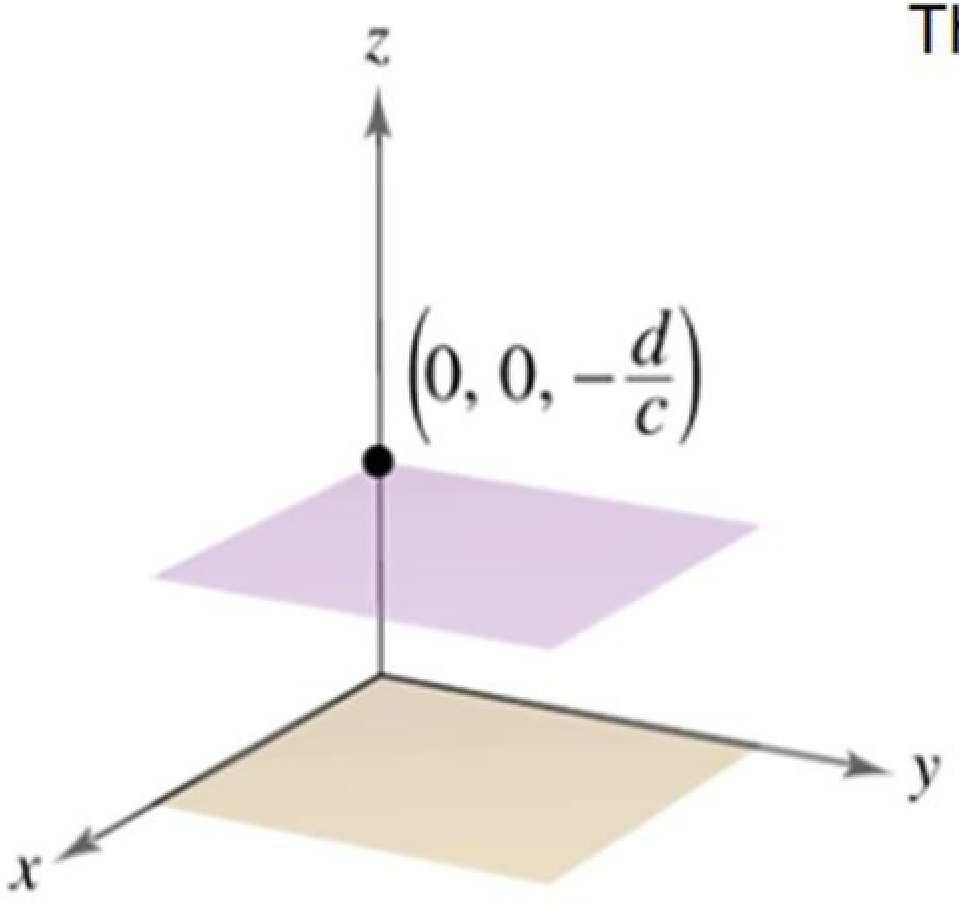
The standard equations of a plane in space is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$\vec{n} = b\hat{j} = <0, b, 0>$$

$$p = \left(0, -\frac{d}{b}, 0\right)$$

Plane $by + d = 0$ is parallel to the xz-plane.



The standard equations of a plane in space is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$\vec{n} = c\hat{k} = < 0,0,c >$$

$$p = \left(0,0,-\frac{d}{c}\right)$$

Plane $cz + d = 0$ is parallel to the xy-plane.