Integration

Integrals $\Rightarrow \int f'(x)\,dx = f(x)+c$

(1) $\int K\,dx = K\,x + c$

Examples

- $\int 4\,dx = 4x + c$

- $\int \frac{dx}{2} = \int \frac{1}{2}\,dx = \frac{1}{2}\,x + c$

(2) $\int x^n\,dn = \frac{x^{n+1}}{n+1} + c$

Examples:

- ❖ $\int x^2\ dx = \frac{x^3}{3}$ +c

- ❖ $\int \frac{1}{x^2}$ dx = $\int x^{-2}\ dx = \frac{x^{-1}}{-1}$ +c

- ❖ $\int \sqrt[3]{x}$ dx = $\int x^{\frac{1}{3}}$ dx = $\frac{3}{4}\,x^{\frac{4}{3}}$ +c

- ❖ $\int \frac{1}{\sqrt[3]{x}}$ dx = $\int x^{\frac{-3}{2}}$ dx= $\frac{x^{\frac{-1}{2}}}{\frac{-1}{2}}$ +c

(3) $\int (ax + b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1)*a}$

Examples:

- $\int (1 - 2x)^3 \, dx = \frac{(1-2x)^4}{-8} + c$

Solution :

$\int (1 - 2x)^3 \, dx = \frac{(1-2x)^4}{-8} + c$

- $\int \frac{1}{\sqrt[3]{x-1}} \, dx$

Solution:

$\int \frac{1}{\sqrt[3]{x-1}} \, dx = \int (x - 1)^{\frac{-1}{3}} \, dx = \frac{3}{2} \, (x - 1)^{\frac{2}{3}} + c$

(4) $\int k \,.\, f(x)\, dx = k \int f(x)\, dx$ where k is const

- Ex

$\int 2x^3 \, dx = 2 \int x^3 \, dx = 2 \, \frac{x^4}{4} + c$

Examples

➢ $\int x^2 + 6x - 3 \, dx$

Solution :

$= \frac{x^3}{3} + 3x^2 - 3x + c$

➢ $\int x^6 + \frac{1}{x^3} - \sqrt[5]{x^2} \, dx$

Solution

$= \frac{x^7}{7} - \frac{x^{-2}}{-3} - \frac{x^{\frac{7}{5}}}{\frac{7}{5}} + c$

- $\int x^2 \sqrt{x}$

- Solution

$\int x^2 \, x^{\frac{1}{2}}$ dx= $\int x^{\frac{5}{2}}$ dx= $\frac{2}{7} x^{\frac{7}{2}}$ +c

- $\int (1+x) \sqrt{x}$

Solution:

$\int x^{\frac{1}{2}}$ (1+x) dx= $\int x^{\frac{1}{2}}$ - $x^{\frac{3}{2}}$ dx

= $\frac{2}{3} x^{\frac{3}{2}}$ - $\frac{2}{5} x^{\frac{5}{2}}$ +c

- $\int \frac{x^3+5x^2-4}{x^2}$ dx

- Solution:

- $\int \frac{x^3+5x^2-4}{x^2}$ dx = $\int$ x+5 - $\frac{4}{x^2}$ dx =$\frac{1}{2}$ x^2 +5x+$\frac{4}{x}$ +c

- $\int \frac{x^2-4}{x-2}$ dx

Solution

= $\frac{1}{2}x^2$ +2x +c

(5) $\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$

- $\int(x^2 + x + 1)dx = \int x^2 + \int x + \int 1\,\mathrm{dx} = \frac{x^3}{3} + \frac{x^2}{2} + x + c$

- $\int(6x - 8)dx = \int 6x - \int 8\ dx = 6\,\frac{x^2}{2} - 8x + c$

- $\int \left(3x^2 - \frac{4}{x^5} + 6\right) dx = \int(3x^2 - 4x^{-5} + 6)dx$

  $= \frac{3x^3}{3} + x^{-4} + 6x + c$

Integration by Substitution

(1) $\int (x^2 + 1)^3$ (2x) dx

Solution:

Let u= $x^2 + 1$ **du= 2x dx**

$\int (x^2 + 1)^3$ **(2x) dx =** $\int u^3$ **du=** $\frac{u^4}{4}$ **+C**

= $\frac{(x^2+1)^4}{4}$ **+C**

(2) $\int Sin^3 x \cos x\, dx$

- Let u= $\sin x$ du=$\cos x$

$\int Sin^3 x \cos x\, dx = \int u^3\, du = \frac{u^4}{4} + C$

$= \frac{Sin^4 x}{4} + C$

- 3) $\int \sin 3x\, dx$

Let u = 3x du= 3dx dx= $\frac{1}{3}$ du

$\int \sin 3x\, dx = \int \sin u\, du \cdot \frac{1}{3}\, du$

$= \frac{1}{3} \int \sin u\, du = \frac{1}{3}(-\cos u + C)$

$= \frac{-1}{3} \cos 3x + C$

$$\int [f(x)]^n \, f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

- $\int \frac{\sin x}{\sqrt{\cos x}}\, dx = -\int (\cos x)^{\frac{-1}{2}} * -\sin x\, dx = -\frac{(\cos x)^{\frac{-1}{2}}}{\frac{1}{2}} + C$
- $\int (x^3+2)^5\, x^2\, dx = \int (x^3+2)^5\, 3x^2\, dx = \frac{1}{3}\frac{(x^3+2)^6}{6} + C$
- $\int \frac{lnx}{x}\, dx = \int lnx\ \frac{1}{x}\, dx = \frac{(\ln x)^2}{2} + C$

$$\int \frac{f'(x)}{f(x)} = \ln\left(f(x)\right) + c$$

- $\int \frac{1}{x}$ dx = ln(x)+c

- $\int \frac{x}{x^2-1} = \frac{1}{2} \int \frac{2x}{x^2-1} = \frac{1}{2} \ln(x^2 - 1) + C$

- $\int \frac{dx}{4x-1} = \frac{1}{4}$ ln(4x-1) +C

- $\int \frac{sin(5x)}{2-cos(5x)} = \frac{1}{5} \int \frac{5sin(5x)}{2-cos(5x)}$ = $\frac{1}{5}$ ln(2-cos(5x)) $+ C$

- $\int \frac{(x^2+1)}{x^3+3x}$ dx= $\frac{1}{3}$ ln $(x^3 + 3x)$ +C

Integration by partial fractions

- Case (1)

g(x)= $(x - x_1)\ (x - x_2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(x - x_n)$

$$\frac{f(x)}{g(x)}=\frac{A}{(x-x_1)}+\frac{B}{(x-x_2)}+\frac{C}{(x-x_3)}+\ldots\ldots\ldots\ldots\ldots\ \frac{D}{(x-x_n)}$$

Case (2)

g(x)=$(ax+b)^n$

$$\frac{f(x)}{g(x)}=\frac{A}{((ax+b)^1)}+\frac{B}{((ax+b)^2)}+\frac{C}{((ax+b)^2)}+\ldots\ldots\ldots\ldots\ldots\ \frac{A_n}{((ax+b)^2)}$$

- Case (3)

If g(x)= $(\mathrm{a}x^2 + bx + c)(\ x - x_1)(x - x_2)$

- $\dfrac{f(x)}{g(x)} = \dfrac{Ax+B}{\mathrm{a}x^2+bx+c} + \dfrac{c}{x-x_1} + \dfrac{D}{x-x_2}$

❖ $\int \frac{dx}{x^2+x-2}$ dx

= $\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$

= $\frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

1= $A(x-1) + B(x+2)$ at x=1 A= $\frac{-1}{3}$

= $\frac{1}{x^2+x-2} = \frac{-1}{3} \cdot \frac{1}{(x+2)} + \frac{1}{3}\frac{1}{(x-1)}$ at x= -2 B= $\frac{1}{3}$

$\int \frac{1}{x^2+x-2}$ dx = $\frac{-1}{3}\int \frac{1}{(x+2)}dx + \frac{1}{3}\int \frac{1}{(x-1)}$dx

= $\frac{-1}{3}$ ln(x+2) + $\frac{1}{3}$ ln(x+1) +c

$$\int \frac{x^2+1}{x^3+2x^2+x}\,dx$$

- $$\frac{x^2+1}{x^3+2x^2+x} = \frac{x^2+1}{x(x^2+2x+1)} = \frac{x^2+1}{x(x+1)(x+1)} = \frac{x^2+1}{x(x+1)^2}$$

$$= \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2+Bx(x+1)+c(x)}{x(x+1)^2}$$

at x=0,1,-1

A= 1 , B= $\frac{-3}{4}$ **, C=** $\frac{-1}{2}$

$$\int \frac{x^2+1}{x^3+2x^2+x}\,dx = \int \frac{1}{x}\,dx - \int \frac{3}{4}\frac{1}{(x+1)}dx - \int \frac{1}{2}\frac{1}{(x+1)^2}$$

$$= \ln(x) - \frac{3}{4}\ln(x+1) + \frac{1}{2}\ln\,(x+1)^{-1} + C$$

Integration by parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ex) Evaluate $\int x \sin x \, dx$

- $\int x \sin x \, dx$

Let u =x , dv= sin x

du= dx v= -cos x

$\int x \sin x \, dx$ **= -x cosx -** $\int -\cos x \, dx$**= -x cosx +Sin x + C**

❖ $\int \ln x \, dx$

Let u= ln x **dv= dx**

du= $\frac{\mathbf{1}}{\boldsymbol{x}}$ **dx** **v= x**

$\int \mathbf{ln}\, \boldsymbol{x}\, \boldsymbol{dx}$**= x ln x -** $\int \boldsymbol{x}.\frac{\mathbf{1}}{\boldsymbol{x}}$ **dx**

= x ln x – x +C