
بسم الله الرحمن الرحيم

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MATH – 1

BY

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CHAPTER 2

Limits and Continuity

LECTURE 3.

Limits and their properties

Aims and Objectives:

(1) Understand the notion of limits and their properties.

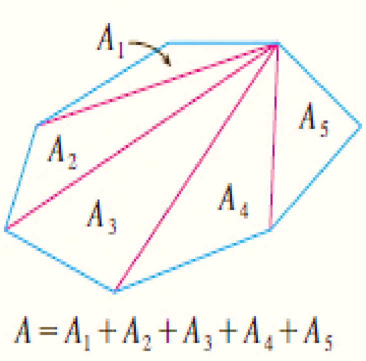
(2) Understand the methods of evaluating limits.

(3) Use the concepts of right limit and left limits.

(4) Investigate limits involving infinity.

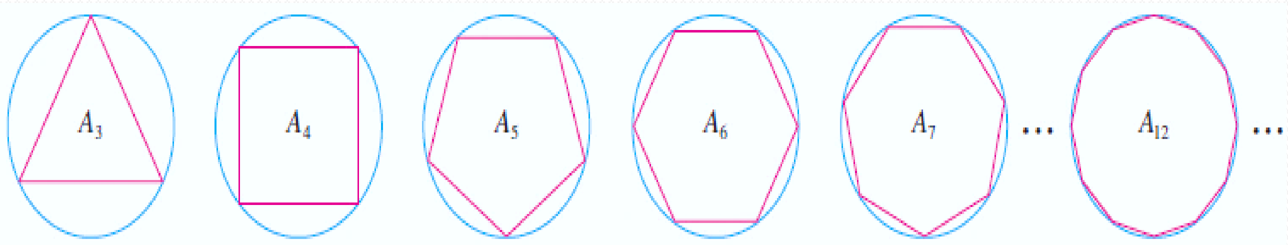
(5) Have a strong intuitive feeling for these important concepts.

- **LIMITS**



In order to find the area of any polygon by dividing it into triangles as in the Figure and adding the areas of these triangles.

To find the area of a curved figure, let An be the area of the inscribed polygon with n sides. As n increases, it appears that An becomes closer and closer to the area of the circle.

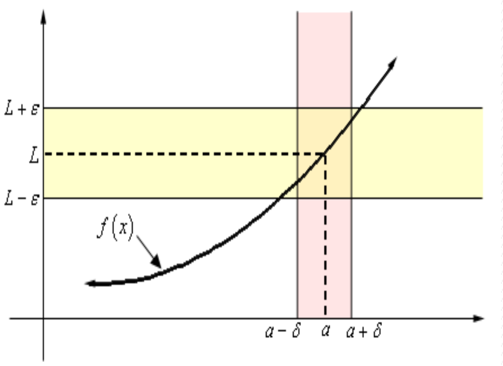


We say that the area of the circle is the limit of the areas of the inscribed polygons, and we write

$$A = \lim_{n\to\infty} A_n$$

Let f(x) be a function defined on an interval that contains x=a, except possibly at x=a. Then we say that,

$$\lim_{x\to a} f(x) = L$$



if for every number ε>0 there is some number δ>0 such that |f(x)−L|<ε whenever 0<|x−a|<δ

If we take any x in the pink region,

i.e. between a+δ and a−δ, then this x will be closer to a than either of a+δ and a−δ. Or,

$$|x-a|<\delta$$

If we now identify the point on the graph that our choice of x gives f(x), then this point on the graph will lie in the intersection of the pink and yellow region. This means that this function value f(x) will be closer to L than either of L+ε and L−ε. Or,

$$|f(x)-L|<\varepsilon$$

If we take any value of x in the pink region then the graph for those values of x will lie in the yellow region.

Understanding the Meaning of Limits:

Limits are used to describe how a function $f(x)$ behaves as the independent variable "x" moves toward a certain value "a" .

Suppose that $f(x)$ becomes arbitrarily close to the number "L" as x approaches "a". We then say that the limit of $f(x)$ as x approaches to "a" is "L", and we write $\lim\limits_{x \to a} f(x) = L.$

It is important to realize that $f(x)$ must be arbitrarily close to the number "L" for all x that are sufficiently close to "a" but different from "a" (not equal to "a").

EXAMPLE:

Find $\lim\limits_{x \to 5} (x^2 - 4x + 5)$

SOLUTION:

In this example we find $L = 10.$ It mean $f(x) = (x^2 - 4x + 5)$ that can be made arbitrarily close to 10 by requiring x to be close to 5, that is

$|(x^2 - 4x + 5) - 10| < \omega_1$ as $0 < |x - 5| < \omega_2$

where ω_1 and ω_2 are any two arbitrarily values instead of ε and δ.

EXAMPLE:

Find $\lim_{x \to 4}(x + 2)$

SOLUTION:

In this case $f(x) = x + 2$ can be made arbitrarily close to 6 by requiring x to be sufficiently close to 4 but not equal to 4.

For example, $x + 2$ can be made to be within 1/1000 of 6 that is $|(x + 2) - 6| < 1/1000$ by requiring that x be within 1/1000 of 4 but not equal to 4

$0 < |(x - 4)| < 1/1000$

We get $\lim_{x \to 4}(x + 2) = 6$.

EXAMPLE:

Let $f(x) = x + 2$ *then as we have seen*

$$\lim_{x \to 4} f(x) = \lim_{x \to 4}(x+2) = 6$$

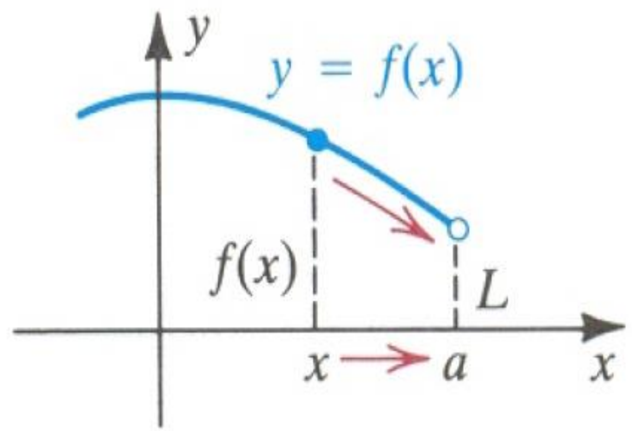
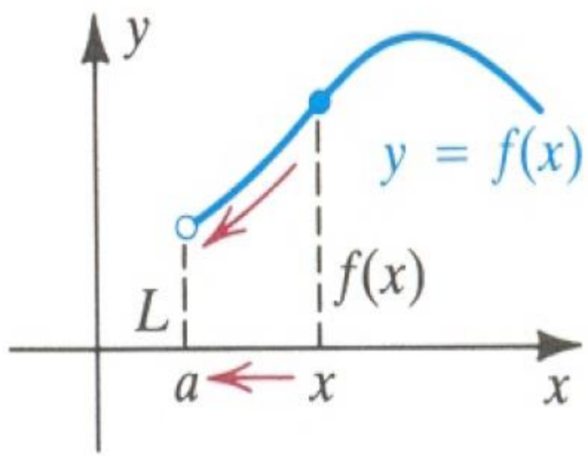
SOLUTION:

ω_1	$4 \pm \omega_1$		$f(x) = (x+2)$		ω_2
	$4+\omega_1$	$4-\omega_1$	$\lim_{x \to 4+\omega_1} f(x)$	$\lim_{x \to 4-\omega_1} f(x)$	
0 .1	4 .1	3 .9	6.1	5.9	+.1
0 .01	4 .01	3 .99	6 .01	5 .99	+ ·01
0 .001	4 .001	3 .999	6 .001	5 .999	+ .001
0 .0001	4 .0001	3 .9999	6 .0001	5 .9999	+ .0001
0 .00001	4 .00001	3 .99999	6 .00001	5 .99999	+ .00001
0 .00000	4 .00000	4 .00000	6 .00000.	6 .00000.	+ .00000

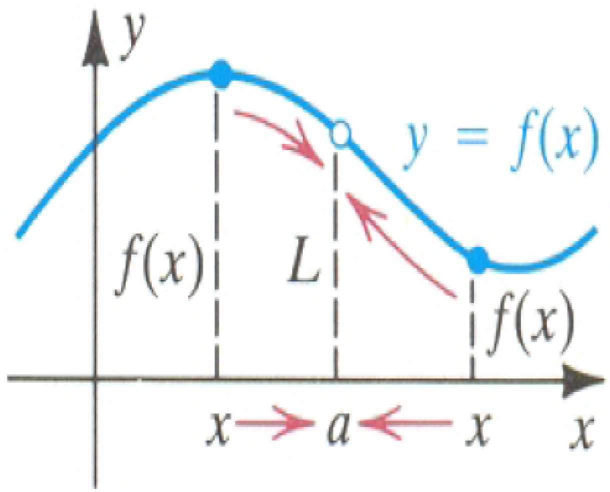
From the previous illustration we can notice since means that $-\omega_1 < (x-4) < +\omega_1$ *means that* $|(x-4)| < \omega_1$.

There exists ω_2 *satisfying* $-\omega_2 < (x+2) - 6 < \omega_2$ *means that* $|(x+2) - 6| < \omega_2$

One-Sided Limits :

Notation	Meaning	Graphical Interpretation
$\lim\limits_{x \to a^-} f(x) = L$ (left–hand limit)	$f(x)$ approaches L as x approaches a, and $x < a$.	
$\lim\limits_{x \to a^+} f(x) = L$ (right–hand limit)	$f(x)$ approaches L as x approaches a, and $x > a$.	

Two-Sided Limit :

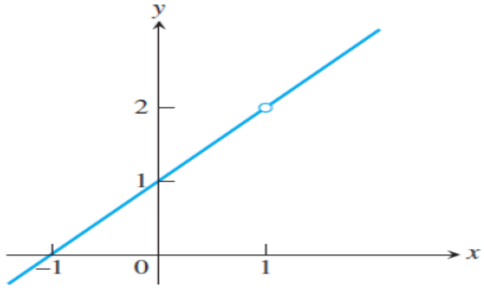
Notation	Meaning	Graphical Interpretation
$\lim_{x \to a} f(x) = L$	$f(x)$ approaches L as x approaches a.	

Theorem :

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

EXAMPLE

$$f(x)=\frac{x^2-1}{x-1}$$



* Note that 1 is not in the domain of f, since substituting $x=1$ gives us the indeterminate form $\frac{0}{0}$, $f(1)$ is undefined.

x	$f(x)$
0.9	1.9
0.99	1.99
0.999	1.999
⋮	⋮
0.999999	1.999999

$$\lim_{x\to 1^-} f(x)=2$$

(left −hand limit)

x	$f(x)$
1.1	2.1
1.01	2.01
1.001	2.001
⋮	⋮
1.000001	2.000001

$$\lim_{x\to 1^+} f(x)=2$$

(right −hand limit)

Indeterminate Forms :

$$\frac{0}{0}\ ,\ \frac{\infty}{\infty}\ ,\ 0.\infty\ ,\ \infty-\infty\ ,\ 0^0\ ,\ \infty^0\ ,\ 1^\infty$$

* Since $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = 2$ (one-sided limits) exist and are equal , then $\lim\limits_{x \to 1} f(x) = 2$ (two-sided limit or limit) exists .

It appears that $f(x)$ approaches 2 as x approaches 1 .

* Factoring the numerator and denominator and cancelling common factor (eliminating zero denominator algebraically)

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} \quad , \quad x \neq 1 \;\Rightarrow\; x - 1 \neq 0$$

$$= \lim_{x \to 1} (x+1) = \boxed{2} .$$

EXAMPLE:

Let $f(x) = \begin{cases} 3x & \text{if } x \text{ is } < 5 \\ 4x & \text{if } x \text{ is } \geq 5 \end{cases}$

Find $\lim_{x\to 5} f(x)$

SOLUTION:

Since $L_1 = \lim_{x\to 5^-} f(x) = 15$ *and* $L_2 = \lim_{x\to 5^+} f(x) = 20$

then, $L_1 \neq L_2$ *so the limit does not exist.*

EXAMPLE:

Let $f(x) = \begin{cases} 2x & \text{if } x \leq 1 \\ 2 & \text{if } x > 1 \end{cases}$

Find $\lim_{x\to 1} f(x)$

SOLUTION:

Since $L_1 = \lim_{x\to 1^-} f(x) = 2$ *and* $L_2 = \lim_{x\to 1^+} f(x) = 2$

then, $L_1 = L_2$ *the limit exists and equal 1 i.e.* $L=2$.

EXAMPLE:

Find $\lim_{x\to 3}\frac{(x^2-x-6)}{(x-3)}$

Using the fact that $\frac{(x^2-x-6)}{(x-3)}=\frac{(x-3)(x+2)}{(x-3)}=(x+2)$

SOLUTION:

The function f(x) can be written as $f(x)=\begin{cases}(x+2) & \text{if } x\neq 3\\ \text{No Value} & \text{if } x=3\end{cases}$

Then $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = 5$

EXAMPLE:

Discuss the existence of the limits of the following function f(x) at x approaches the 1 and 2 , where

$$f(x)=\begin{cases}x & \text{if } x\le 1\\ 1 & \text{if } 1<x\le 2\\ 3 & \text{if } x>2\end{cases}$$

SOLUTION:

First the limit of f(x) as x tends to 1: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 1$

There exists a limit as x approaches 1.

Second the limit of f(x) as x tends to 2: $\lim_{x\to 2^-} f(x) = 1 \neq \lim_{x\to 2^+} f(x) = 3$

The limit doesn't exists as x approaches 2.

As x approaches to "a" becomes more and more positive or negative, then the limit of the function f(x) fails to exist and $\lim_{x\to a} f(x) = \pm\infty$

EXAMPLE:

If $\lim_{x \to a} f(x) = \pm\infty$

Find the limits $\lim_{x \to 0^+} \frac{1}{x}$, $\lim_{x \to 0^-} \frac{1}{x}$ and $\lim_{x \to 0} \frac{1}{x}$

SOLUTION:

As x approaches to 0 from the right, the value of 1/x gets larger and larger without bounds.

$$x = 1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10{,}000}, \frac{1}{100{,}000}, \ldots$$

$$\frac{1}{x} = 1, 10, 100, 1000, 10{,}000, 100{,}000, \ldots$$

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

As x approaches to 0 from the left, the value of 1/x gets more and more negative without bounds.

$$x = -1, -\frac{1}{10}, -\frac{1}{100}, -\frac{1}{1000}, -\frac{1}{10{,}000}, -\frac{1}{100{,}000}, \ldots$$

$$\frac{1}{x} = -1, -10, -100, -1000, -10{,}000, -100{,}000, \ldots$$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$. It follows that: $\lim_{x \to 0} \frac{1}{x}$ does not exist

EXAMPLE:

Find the limits: $\lim\limits_{x \to +\infty} \frac{1}{x}$, $\lim\limits_{x \to -\infty} \frac{1}{x}$

SOLUTION:

Following the previous example, then $\lim\limits_{x \to +\infty} \frac{1}{x} = \lim\limits_{x \to -\infty} \frac{1}{x} = 0$

Limits Laws :

*If **f** and **g** are two functions have the limit **a**, and **c** is a constant then:*

1) $\lim\limits_{x\to a} c\,f(x) = c \lim\limits_{x\to a} f(x)$

2) $\lim\limits_{x\to a}(f(x) \pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

3) $\lim\limits_{x\to a}(f(x)g(x)) = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$

4) $\lim\limits_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

5) $\lim\limits_{x\to a} c = c$, $\lim\limits_{x\to a} x = a$

6) $\lim\limits_{x\to a}(f(x))^n = \left(\lim\limits_{x\to a} f(x)\right)^n$, $n \in \mathbb{N}$

7) $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$, $n \in \mathbb{N}$

8) $\lim\limits_{x\to a} |f(x)| = \left|\lim\limits_{x\to a} f(x)\right|$

9) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

10) $\lim\limits_{x\to \infty}\left(1 + \frac{1}{x}\right)^x = \lim\limits_{y\to 0}(1+y)^{1/y} = e$, and $e \cong 2.718$

11) If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Limits of Trigonometric Functions :

Theorems :

(i) $\lim_{x \to 0} \sin x = 0$

(ii) $\lim_{x \to 0} \cos x = 1$

(iii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iv) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(v) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Corollaries :

1. $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \frac{1}{\cos x} = \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = 1$

2. $\lim_{x \to 0} \frac{\sin kx}{x} = \lim_{x \to 0} k \frac{\sin kx}{kx} = k \lim_{kx \to 0} \frac{\sin kx}{kx} = k \cdot 1 = k$

3. $\lim_{x \to 0} \frac{\sin \alpha x}{\sin \beta x} = \lim_{x \to 0} \frac{\alpha}{\beta} \frac{\frac{\sin \alpha x}{\alpha x}}{\frac{\sin \beta x}{\beta x}} = \frac{\alpha}{\beta} \frac{\lim_{x \to 0} \frac{\sin \alpha x}{\alpha x}}{\lim_{x \to 0} \frac{\sin \beta x}{\beta x}} = \frac{\alpha}{\beta} \cdot \frac{1}{1} = \frac{\alpha}{\beta}$

Theorems :

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$ ***, where e=2.71828 (Euler's constant)***

EXAMPLE:

Find the limit for the following functions

1- $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+c}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{c}=e\cdot 1=e$

2- $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{cx}=\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)^{c}=e^{c}$

3- $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^{x}=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{cy}=e^{c}$

EXAMPLES:

$$\lim_{x\to 10} 3 = 3,\ \lim_{x\to 10} 3c = 3c,$$

$$\lim_{x\to 1}\left(\frac{1}{x}+x\right) = \lim_{x\to 1}\left(\frac{1}{x}\right)+\lim_{x\to 1}(x) = 1+1 = 2$$

$$\lim_{x\to\infty}\left(\frac{1}{x}+x\right) = \lim_{x\to\infty}\left(\frac{1}{x}\right)+\lim_{x\to\infty}(X) = 0+\infty = \infty$$

$$\lim_{x\to 4}\left(x\sqrt{x}\right) = \lim_{x\to 4}(x)\cdot\lim_{x\to 4}\left(\sqrt{x}\right) = (4)(2) = 8$$

$$\lim_{x\to 1}[(2x+3)(x+1)] = \lim_{x\to 1}(2x+3)\cdot\lim_{x\to 1}(x+1) = (5)(2) = 10$$

$$\lim_{x\to 1}(x-1)(X^2+2x+1) = \lim_{x\to 1}(x-1)\cdot\lim_{x\to 1}(x^2+2x+1) = (0)(4) = 0$$

EXAMPLES:

$$\lim_{x\to2}(2x+4x^2)=\lim_{x\to2}(2x)+\lim_{x\to2}(3x^2)=2\lim_{x\to2}(x)+4\lim_{x\to2}(x^2)=16$$

$$\lim_{x\to2}\left(\frac{x+2}{x}\right)=\frac{\lim\limits_{x\to2}(x+2)}{\lim\limits_{x\to2}(x)}=\frac{4}{2}=2$$

$$\lim_{x\to8}\left(\frac{\sqrt{2x}}{\sqrt[3]{x}}\right)=\frac{\lim\limits_{x\to8}\left(\sqrt{2x}\right)}{\lim\limits_{x\to8}\left(\sqrt[3]{x}\right)}=\frac{4}{2}=2$$

$$\lim_{x\to5}\left(x^2-4x+3\right)=\lim_{x\to5}\left(x^2\right)-\lim_{x\to5}(4x)+\lim_{x\to5}(3)=5^2-4(5)+3=8$$

$$\lim_{x\to2}\left(\frac{5x^3+4}{x-3}\right)=\frac{\lim\limits_{x\to2}\left(5x^3+4\right)}{\lim\limits_{x\to2}(x-3)}=\frac{5\lim\limits_{x\to2}(x^3)+\lim\limits_{x\to2}(4)}{\lim\limits_{x\to2}(x)-\lim\limits_{x\to2}(3)}=\frac{5(2^3)+4}{2-3}=-44$$

$$\lim_{x\to4}\frac{(2-x)}{(x-4)(x+2)}=\begin{cases}\lim\limits_{x\to4^+}\dfrac{(2-x)}{(x-4)(x+2)}=-\infty\\[2ex]\lim\limits_{x\to4^-}\dfrac{(2-x)}{(x-4)(x+2)}=+\infty\end{cases}$$

limit does not exist

EXAMPLE:

Find the limits: $\lim_{x\to+\infty}\left(\frac{3x+5}{6x-5}\right)$

SOLUTION:

Applying the Rule 2 $\lim_{x\to+\infty}\left(\frac{3x+5}{6x-5}\right)=\frac{\infty}{\infty}$ and this undetermined value.

Divided the numerator and denominator by the highest power of x then

$$\lim_{x\to+\infty}\left(\frac{3x+5}{6x-8}\right)=\lim_{x\to+\infty}\left(\frac{3+5/x}{6-8/x}\right)=\frac{\lim_{x\to+\infty}(3+5/x)}{\lim_{x\to+\infty}(6-5/x)}=$$

$$\frac{\lim_{x\to+\infty}(3)+5\lim_{x\to+\infty}(1/x)}{\lim_{x\to+\infty}(6)-8\lim_{x\to+\infty}(1/x)}=\frac{3+5\cdot(0)}{6-8\cdot(0)}=1/2$$

Here you find an example showing how to solve problems when x tends to infinity.

EXAMPLE:

Find the limits: $\lim_{x\to-\infty}\left(\dfrac{4x^2-x}{2x^3-5}\right)$

SOLUTION:

Divided the numerator and denominator by x^3

Using the previous technique then

$$\lim_{x\to-\infty}\left(\frac{4x^2-x}{2x^3-5}\right)=\lim_{x\to-\infty}\left(\frac{4/x-1/x^2}{2-5/x^3}\right)=\frac{\lim\limits_{x\to-\infty}\left(4/x-1/x^2\right)}{\lim\limits_{x\leftarrow-\infty}(2-5/x^3)}$$

$$\frac{4\cdot(0)-(0)}{2-5\cdot(0)}=\frac{0}{2}=0$$

THANK YOU