Objectives:

After completing this topic, you will be able to:

- Understand the notion of limits and their properties.
- Use the concept of right limit and left limits.
- Have a strong intuitive feeling for these important concepts.

Understanding the Meaning of Limits:

Limits are used to describe how a function $f(x)$ behaves as the independent variable "x" moves toward a certain value "a"

We start this lesson by giving a general idea about the meaning of a limit.

Limits are used to describe how a function $f(x)$ behaves as the independent variable "x" moves toward a certain value "a"

Understanding the Meaning of Limits

Suppose that $f(x)$ becomes arbitrarily close to the number "L" as x approaches "a". We then say that the limit of $f(x)$ as x approaches to "a" is "L", and we write $\lim_{x\to a} f(x) = L$

We shall restate the meaning of limit in a somewhat more explicit form.

We say that $\lim_{x\to a} f(x) = L$

If $f(x)$ can be made arbitrarily close to "L" by requiring x to be sufficiently close to "a" but not equal to "a".

It is important to realize that $f(x)$ must be arbitrarily close to the number "L" for all x that are sufficiently close to "a" but different from "a".

Here we illustrate the notion of the limit by a given example.

Example 1:

Find $\lim_{x\to 4}(x+2)$

Solution:

In this case $f(x) = x + 2$ can be made arbitrarily close to 6 by requiring x to be sufficiently close to 4 but not equal to 4.

For example, $x + 2$ can be made to be within 1/1000 of 6 that is $|(x + 2) - 6| < 1/1000$

by requiring that x be within 1/1000 of 4 but not equal to 4

$0 < |(x - 4)| < 1/1000$

We get $\lim_{x\to 4} (x + 2) = 6$

Example 2:

Find $\lim_{x\to 5} (x^2 - 4x + 5)$

Solution:

In this example we find $L = 10$

It mean $f(x) = (x^2 - 4x + 5)$ that can be made arbitrarily close to 10 by requiring x to be close to 5, that is $|(x^2 - 4x + 5) - 10| < \omega_1$ as $0 < |x - 5| < \omega_2$

Where ω_1 and ω_2 are any two arbitrarily values.

We observe that the limit of $f(x)$ is a value L when x tends to a value a means that if $f(x)$ is arbitrary close to L then we will find that x is also sufficiently close to a.

Example 3:

Let $f(x) = (x + 2)$ then as we have seen

$$\lim_{x\to 4} f(x) = \lim_{x\to 4}(x+2) = 6$$

ω_1	$4 \pm \omega_1$		$f(x) = (x+2)$		ω_2
	$4+\omega_1$	$4-\omega_1$	$\lim_{x\to 4+\omega_1} f(x)$	$\lim_{x\to 4-\omega_1} f(x)$	
0 .1	4 .1	3 .9	6.1	5.9	+.1
0 .01	4 .01	3 .99	6 .01	5 .99	+ ·01
0 .001	4 .001	3 .999	6 .001	5 .999	+ .001
0 .0001	4 .0001	3 .9999	6 .0001	5 .9999	+ .0001
0 .00001	4 .00001	3 .99999	6 .00001	5 .99999	+ . 00001
0 .00000	4 .00000	4 .00000	6 .00000	6 .00000.	+ .00000

From the previous illustration we can notice since

means that $-\omega_1 < (x-4) < +\omega_1$

means that $|(x-4)| < \omega_1$

There exists ω_2 satisfying $-\omega_2 < (x+2) - 6 < \omega_2$

means that $|(x+2)-6| < \omega_2$

Objectives:

After completing this topic, you will be able to:

- Understand the methods of evaluating limits.
- Investigate limits involving infinity.
- Have a strong intuitive feeling for these important concepts.

Limits from the Right and the Left:

If the value of $f(x)$ approaches the number "$\mathcal{L}_1$" as x approaches "a" from the right side,

we write $\lim\limits_{x\to a^+} f(x) = \mathcal{L}_1$

And it is read "the limit of as x approaches "a" from the right equal "$\mathcal{L}_1$".

If the value of x approaches the number "$\mathcal{L}_2$" as x approaches "a" from the left side,

we write $\lim\limits_{x\to a^-} f(x) = \mathcal{L}_2$

And it is read "the limit of as x approaches "a" from the left equal "$\mathcal{L}_2$ ".

If limit from the left side is the same as the limit from the right side, say $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = \mathcal{L}$

then we can say the limit exists and equal to $\mathcal{L}$ and write $\lim\limits_{x\to a} f(x) = \mathcal{L}$ where, $\mathcal{L} = \mathcal{L}_1 = \mathcal{L}_2$

Example 4:

Let $f(x) = \begin{cases} 3x & \text{if} \quad x \quad \text{is} \quad < 5 \\ 4x & \text{if} \quad x \quad \text{is} \quad \geq 5 \end{cases}$

Find $\lim\limits_{x\to 5} f(x)$

Solution:

Since $\mathcal{L}_1 = \lim\limits_{x\to 5^-} f(x) = 15$ and $\mathcal{L}_2 = \lim\limits_{x\to 5^+} f(x) = 20$

then, $\mathcal{L}_1 \neq \mathcal{L}_2$

so the limit does not exist.

Example 5:

Let $f(x) = \begin{cases} 2x & \text{if} \quad x \leq 1 \\ 2 & \text{if} \quad x > 1 \end{cases}$

Find $\lim\limits_{x\to 1} f(x)$

Solution:

Since $\mathcal{L}_1 = \lim\limits_{x\to 1^-} f(x) = 2$ and $\mathcal{L}_2 = \lim\limits_{x\to 1^+} f(x) = 2$ then

then $\mathcal{L}_1 = \mathcal{L}_2$ the limit exists and equal 1 i.e. $\mathcal{L} = 2$.

Example 6:

Find $\lim\limits_{x\to 3} \dfrac{(x^2 - x - 6)}{(x - 3)}$

Using the fact that

$$\frac{(x^2 - x - 6)}{(x-3)} = \frac{(x-3)(x+2)}{(x-3)} = (x+2)$$

Solution:

The function $f(x)$ can be written as

$$f(x) = \begin{cases} (x+2) & \text{if} \quad x \neq 3 \\ \text{No Value} & \text{if} \quad x = 3 \end{cases}$$

then $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3} f(x) = 5$

Example 7:

Discuss the existence of the limits of the following function $f(x)$ at x approaches the 1 and 2

where, $f(x) = \begin{cases} x & \text{if} \quad x \leq 1 \\ 1 & \text{if} \quad 1 < x \leq 2 \\ 3 & \text{if} \quad x > 2 \end{cases}$

Solution:

$\lim\limits_{x\to 1^-} f(x) = 1$ also $\lim\limits_{x\to 1^+} f(x) = 1$

from this results

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 1$$

There exists a limit as x approaches 1.

Second the limit of f(x) as x tends to 2

$$\lim_{x\to 2^-} f(x) = 1 \qquad \text{also} \qquad \lim_{x\to 2^+} f(x) = 3$$

from this results

$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$$

The limit does not exist as x tends to 2.

As x approach to " a " becomes more and more negative or positive without bounds and consequently approaches no fixed value

thus the limit of the function f(x) fails to exist.

In this case we would write $\lim\limits_{x\to a} f(x) = \pm\infty$

Example 8:

If $\lim\limits_{x\to a} f(x) = \pm\infty$

Find the limits: $\lim\limits_{x\to 0^+} \frac{1}{x}$, $\lim\limits_{x\to 0^-} \frac{1}{x}$ and $\lim\limits_{x\to 0} \frac{1}{x}$

As x approaches to 0 from the right, the value of $1/x$ gets larger and larger without bounds.

$$x = 1,\ \frac{1}{10},\ \frac{1}{100},\ \frac{1}{1000},\ \frac{1}{10{,}000},\ \frac{1}{100{,}000},\dots$$

$$\frac{1}{x} = 1,\ 10,\ 100,\ 1000,\ 10{,}000,\ 100{,}000,\dots$$

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

As x approaches to 0 from the left, the value of $1/x$ gets more and more negative without bounds.

$$x = -1,\ -\frac{1}{10},\ -\frac{1}{100},\ -\frac{1}{1000},\ -\frac{1}{10{,}000},\ -\frac{1}{100{,}000},\dots$$

$$\frac{1}{x} = -1,\ -10,\ -100,\ -1000,\ -10{,}000,\ -100{,}000,\dots$$

$\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$. It follows that : $\lim\limits_{x\to 0} \frac{1}{x}$ does not exist

Example 9:

Find the limits: $\lim\limits_{x\to +\infty} \frac{1}{x}$, $\lim\limits_{x\to -\infty} \frac{1}{x}$

Following the previous example

$$\lim_{x\to +\infty} \frac{1}{x} = \lim_{x\to -\infty} \frac{1}{x} = 0$$

Objectives:

After completing this topic, you will be able to:

- Understand the methods of evaluating limits.
- Apply limits involving infinity.
- Have a strong intuitive feeling for these important concepts.

The Limit Theorem:

Calculation of limits directly from the definition can be difficult, but the limit theorem will make the computation of many limits a straight forward task.

Rule 1:

The limit of constant function is the same constant

If $f(x) = k$ then the limit $\lim_{x \to a} f(x) = k$ where k is a constant everywhere.

Example 10:

Find the limit for the following functions as x tends to 10

$f(x) = 3$ $\quad \lim_{x \to 10} f(3) = 3$

$f(xc) = 3c$ $\quad \lim_{x \to 10} f(3) = 3c$

Rule:

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and equal to L_1 and L_2 respectively then

$$\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L_1 \cdot L_2$$

Example 11:

Find the limits for the following:

$$\lim_{x \to 1}\left(\frac{1}{x} + x\right) = \lim_{x \to 1}\left(\frac{1}{x}\right) + \lim_{x \to 1}(x) = 1 + 1 = 2$$

$$\lim_{x \to \infty}\left(\frac{1}{x} + x\right) = \lim_{x \to \infty}\left(\frac{1}{x}\right) + \lim_{x \to \infty}(X) = 0 + \infty = \infty$$

$$\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L_1 \cdot L_2$$

$$\lim_{x \to 4}\left(x\sqrt{x}\right) = \lim_{x \to 4}(x) \cdot \lim_{x \to 4}\left(\sqrt{x}\right) = (4)(2) = 8$$

$$\lim_{x \to 1}[(2x+3)(x+1)] = \lim_{x \to 1}(2x+3) \cdot \lim_{x \to 1}(x+1) = (5)(2) = 10$$

$$\lim_{x \to 1}(x-1)(X^2+2x+1) = \lim_{x \to 1}(x-1) \cdot \lim_{x \to 1}(x^2+2x+1) = (0)(4) = 0$$

$$\lim_{x \to 2}(2x+4x^2) = \lim_{x \to 2}(2x) + \lim_{x \to 2}(3x^2) = 2\lim_{x \to 2}(x) + 4\lim_{x \to 2}(x^2) = 16$$

$$\lim_{x \to 2}\left(\frac{x+2}{x}\right) = \frac{\lim_{x \to 2}(x+2)}{\lim_{x \to 2}(x)} = \frac{4}{2} = 2$$

$$\lim_{x \to 8}\left(\frac{\sqrt{2x}}{\sqrt[3]{x}}\right) = \frac{\lim_{x \to 8}\left(\sqrt{2x}\right)}{\lim_{x \to 8}\left(\sqrt[3]{x}\right)} = \frac{4}{2} = 2$$

General Examples:

$$\lim_{x \to 5}(x^2 - 4x + 3) = \lim_{x \to 5}(x^2) - \lim_{x \to 5}(4x) + \lim_{x \to 5}(3) = 5^2 - 4(5) + 3 = 8$$

$$\lim_{x \to 2}\left(\frac{5x^3+4}{x-3}\right) = \frac{\lim_{x \to 2}(5x^3+4)}{\lim_{x \to 2}(x-3)} = \frac{5\lim_{x \to 2}(x^3) + \lim_{x \to 2}(4)}{\lim_{x \to 2}(x) - \lim_{x \to 2}(3)} = \frac{5(2^3)+4}{2-3} = -44$$

$$\lim_{x \to 4}\frac{(2-x)}{(x-4)(x+2)} = \begin{cases} \lim_{x \to 4^+}\frac{(2-x)}{(x-4)(x+2)} = -\infty \\ \lim_{x \to 4^-}\frac{(2-x)}{(x-4)(x+2)} = +\infty \end{cases}$$

limit does not exist

Example 12:

Find $\lim_{x \to +\infty}\left(\frac{3x+5}{6x-5}\right)$

Applying the Rule 2 $\lim_{x \to +\infty}\left(\frac{3x+5}{6x-5}\right) = \frac{\infty}{\infty}$ and this undetermined value.

Divided the numerator and denominator by the highest power of x then

$$\lim_{x \to +\infty}\left(\frac{3x+5}{6x-8}\right) = \lim_{x \to +\infty}\left(\frac{3+5/x}{6-8/x}\right) = \frac{\lim_{x \to +\infty}(3+5/x)}{\lim_{x \to +\infty}(6-5/x)} =$$

$$\frac{\lim_{x \to +\infty}(3) + 5\lim_{x \to +\infty}(1/x)}{\lim_{x \to +\infty}(6) - 8\lim_{x \to +\infty}(1/x)} = \frac{3+5\cdot(0)}{6-8\cdot(0)} = \frac{1}{2}$$

Here you find an example showing how to solve problems when x tends to infinity.

Example 13:

Find $\lim_{x \to -\infty}\left(\frac{4x^2-x}{2x^3-5}\right)$

Using the previous technique then

$$\lim_{x \to -\infty}\left(\frac{4x^2-x}{2x^3-5}\right) = \lim_{x \to -\infty}\left(\frac{4/x-1/x^2}{2-5/x^3}\right) = \frac{\lim_{x \to -\infty}(4/x-1/x^2)}{\lim_{x \leftarrow -\infty}(2-5/x^3)}$$

$$\frac{4\cdot(0)-(0)}{2-5\cdot(0)} = \frac{0}{2} = 0$$

Objectives:

After completing this topic, you will be able to:

- Understand the notion of continuity.
- Apply condition of continuity at a given point.
- Have a strong intuitive feeling for these important concepts.

Continuous Function:

Some of the functions are "continuous" in the sense that there are no breaks in the graphs, while others have breaks or "discontinuities" in this section we shall formulize the concept of continuity and consider a few of major properties of continuous function.

We now come to the formal definition of continuity of a point in a function's domain. In the definition we distinguish between continuity at endpoint (which involves a one side limit) and the continuity at an interior point (which involves a two sided limit).

Continuity at an Interior Point:

A function $y=f(x)$ is continuous at an interior point "c" of its domain $[a, b]$ if $\lim\limits_{x\to c} f(x)=f(c)$

Continuity at an Endpoint:

A function $y=f(x)$ is continuous at an endpoint "a" of its domain if $\lim\limits_{x\to a^+} f(x)=f(a)$

A function $y=f(x)$ is continuous at an endpoint "b" of its domain if $\lim\limits_{x\to b^-} f(x)=f(b)$

Continuity:

A function $y=f(x)$ is continuous if it is continuous at each point of its domain.

The Continuity Test:

A function $y=f(x)$ is continuous at $x=c$ if and only if all three of the following statements are true:

- $f(c)$ exists ("c" is in the domains of f)
- $\lim f(x)$ exists (f has a limit as $x\to c$)
- $\lim\limits_{x\to c} f(x)=f(c)$ (the limit equals the function value)

In following example we introduce the notion of continuity test with an example

Example 14:

Discuss the continuity of the following function as at $x=1$ $f(x)=\begin{cases} 7x-2 & x<1 \\ 5 & x=1 \\ 5x^2 & x>1 \end{cases}$

Solution 15:

$\lim\limits_{x\to 1^+} f(x)= \lim\limits_{x\to 1^+} f(5x^2)=5$

$\lim\limits_{x\to 1^-} f(x)= \lim\limits_{x\to 1^-} f(7x-2)=5$ the function has a limit

$f(1)=5$ the limit equals the function value

Then the function is continuous at $x=1$

Example 16:

The function $f(x)=\dfrac{1}{x}$ is discontinuous at $x=0$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\dfrac{1}{x}=+\infty$

$\lim\limits_{x\to 0-} f(x)=\lim\limits_{x\to 0^-}\dfrac{1}{x}=-\infty$ the function does not have a limit at $x=0$

$f(0)$ the function is not define at $x=0$

It is easy to show that this function is continuous for any value $x\neq 0$

Properties of Continuous Functions:

In the following we give the properties of continuous functions.

Example 17:

If the function $f(x)$ and $g(x)$ are continuous at $x = a$, then all of the following combinations are continuous as $x = a$:

i) $f(x) + g(x)$

ii) $f(x) - g(x)$

iii) $f(x)\ g(x)$

iv) $kf(x)$ (k any number)

v) $f(x)\,/g(x)$ ($g(x) \neq 0$)

vi) A polynomial function is continuous at all points.

Objectives:

After completing this topic, you will be able to:

- Use the methods of evaluating limits.
- Have a strong intuitive feeling for these important concepts.

Special Limits:

The limit of the function $\frac{\sin x}{x}$ as $x \to c$

The function $\frac{\sin x}{x}$ does not define for $x \to c$ since the numerator and denominator of the function become zero.

Theorem $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

Example 18:

1. $\lim\limits_{x\to 0}\frac{\tan x}{x}=\lim\limits_{x\to 0}\frac{\sin x}{x}\frac{1}{\cos x}=\lim\limits_{x\to 0}\frac{\sin x}{x}\lim\limits_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{1}=1$

2. $\lim\limits_{x\to 0}\frac{\sin kx}{x}=\lim\limits_{x\to 0}k\frac{\sin kx}{kx}=k\lim\limits_{kx\to 0}\frac{\sin kx}{kx}=k\cdot 1=k$

3. $\lim\limits_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}=\lim\limits_{x\to 0}\frac{\alpha}{\beta}\frac{\frac{\sin\alpha x}{\alpha x}}{\frac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}\frac{\lim\limits_{x\to 0}\frac{\sin\alpha x}{\alpha x}}{\lim\limits_{x\to 0}\frac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}\cdot\frac{1}{1}=\frac{\alpha}{\beta}$

Theorem $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$ Where $e=2.71828$

Example 19:

Find the limit for the following functions

1- $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+c}=\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{c}=e\cdot 1=e$

2- $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{cx}=\left(\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)^{c}=e^{c}$

3- $\lim\limits_{x\to\infty}\left(1+\frac{c}{x}\right)^{x}=\lim\limits_{y\to\infty}\left(1+\frac{1}{y}\right)^{cy}=e^{c}$

Example 20:

Investigate the continuity of $h(x)=x^3+2x^2+x+1$

$\lim\limits_{x\to 1}(x^3+2x^2+x+1)=1+2+1+1=5$

Solution:

The rational function $\frac{P_1(x)}{P_2(x)}$ is continuous except at the points $P_2(x)=0$

Example 21:

Investigate the continuity of $f(x)=\frac{x^2-1}{x^3+x^2-2x}$

Solution:

The function on the form $\frac{P_1(x)}{P_2(x)}$ is continuous except at the $P_2(x)=0$ ($x=0$, $x=1$ and $x=-2$)

Techniques for Finding Limits:

Rule:

I) Suppose $f(x)$ is a rational function

$$f(x)=\frac{A(x)}{B(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-1}+\ldots+a_0}{b_mx^m+b_{m-1}x^{m-1}+b_{m-2}x^{m-2}+\ldots+b_0}$$

Then $\lim\limits_{x\to\pm\infty}f(x)=\frac{a_n}{b_m}x^{n-m}$

We have three cases

1) if $n>m$ then $\lim\limits_{x\to\pm\infty}f(x)-\frac{a_n}{b_m}x^{n-m}-\pm\infty$

2) if $n=m$ then $\lim\limits_{x\to\pm\infty}f(x)=\frac{a_n}{b_m}x^{n-m}=\frac{a_n}{b_m}$

3) if $n<m$ then $\lim\limits_{x\to\pm\infty}f(x)=\frac{a_n}{b_m}x^{n-m}=0$

Example 22:

$\lim\limits_{x\to-\infty}\frac{x^3+2x+1}{x^2-x+3}=\lim\limits_{x\to-\infty}\frac{x^3(1+2/x^2+1/x^3)}{x^2(1-1/x+3/x^2)}$

$\lim\limits_{x\to-\infty}x\frac{(1+2/x^2+1/x^3)}{(1-1/x+3/x^2)}=-\infty\frac{(1+0+0)}{(1-0+0)}=-\infty$

Rule:

II) Suppose $f(x)=\frac{A(x)}{B(x)}=\frac{(x-a)G(x)}{(x-a)H(x)}$

then the $\lim_{x\to a} f(x)=\lim_{x\to a}\frac{G(x)}{H(x)}$

Example 23:

$$\lim_{x\to 1}\frac{x^3-3x+2}{x^4-2x^2+1}=\lim_{x\to 1}\frac{(x-1)(x^2+x-2)}{(x-1)(x^3+x^2-x-1)}$$

$$\lim_{x\to 1}\frac{x^2+x-2}{x^3+x^2-x-1}=\lim_{x\to 1}\frac{(x-1)(x+2)}{(x-1)(x+1)^2}=\lim_{x\to 1}\frac{x+2}{(x+1)^2}=3/4$$

Rule:

III) conjugate technique

Example 24:

Find $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$

Solution:

$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}=\frac{0}{0}$ using the conjugate $\sqrt{x+3}+2$

$$\lim_{x\to 1}\frac{(\sqrt{x+3}-2)}{(x-1)}\frac{(\sqrt{x+3}+2)}{(\sqrt{x+3}+2)}=\lim_{x\to 1}\frac{x-1}{(x-1)\sqrt{x+3}+2}=1/4$$

Limits of trigonometric function:

1- $\lim_{x\to 0}\text{Sin}(x)=0$ and $\lim_{x\to a}\text{Sin}(x)=\text{Sin}(a)$

2- $\lim_{x\to 0}\text{Cos}(x)=1$ and $\lim_{x\to a}\text{Cos}(x)=\text{Cos}(a)$

3- $\lim_{x\to 0}\frac{\text{Sin}(x)}{x}=1$

Example 25:

Find $\lim_{x\to 0}\frac{x\sin x}{1-\cos x}$

$$\lim_{x\to 0}\frac{x\sin x}{1-\cos x}=\lim_{x\to 0}\frac{x\sin x\,(1+\cos x)}{(1-\cos x)(1+\cos x)}=\lim_{x\to 0}\frac{x\,(\sin x)+x\,(\sin x)\,(\cos x)}{1-(\cos x)^2}$$

$$=\lim_{x\to 0}\frac{x\,\sin x+x\,(\sin x)\,(\cos x)}{(\sin x)^2}=\lim_{x\to 0}\left(\frac{x}{\sin x}+\frac{x\cos x}{\sin x}\right)=$$

$$\lim_{x\to 0}\frac{x}{\sin x}+\lim_{x\to 0}\frac{x}{\sin x}\lim_{x\to 0}\cos x=1+(1)(1)=2$$

Example 26:

Prove that $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

Solution:

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{1-\cos x}{x}\,\frac{1+\cos x}{1+\cos x}$$

$$=\lim_{x\to 0}\frac{1-(\cos x)^2}{x(1+\cos x)}=\lim_{x\to 0}\frac{1}{x}\frac{(\sin x)^2}{(1+\cos x)}$$

$$\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\sin x\right)\left(\lim_{x\to 0}\frac{1}{1+\cos x}\right)=(1)(0)\left(\tfrac{1}{2}\right)=0$$

Example 27:

Prove that $\lim_{x\to 0}\frac{\sin(mx)}{\sin(nx)}=\frac{m}{n}$

Solution:

$$\lim_{x\to 0}\frac{\sin(mx)}{\sin(nx)}=\lim_{x\to 0}\frac{m\left(\frac{\sin(mx)}{mx}\right)}{n\left(\frac{\sin(nx)}{nx}\right)}=\frac{m}{n}\,\frac{\lim_{x\to 0}\left(\frac{\sin(mx)}{mx}\right)}{\lim_{x\to 0}\left(\frac{\sin(nx)}{nx}\right)}$$

As $x\to 0$ then $mx\to 0$ and $nx\to 0$

$$=\frac{m}{n}\,\frac{\lim_{mx\to 0}\left(\frac{\sin(mx)}{mx}\right)}{\lim_{nx\to 0}\left(\frac{\sin(nx)}{nx}\right)}=\frac{m}{n}$$

Theorem and Examples:

In the following we discuss the limit of some special cases.

Theorem: $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$

Example 28:

$$\lim_{x\to3}\frac{\sqrt{x-2}-1}{x-3}=\lim_{x\to3}\frac{(x-2)^{1/2}-1^{1/2}}{(x-2)-1}=\lim_{x-2\to1}\frac{(x-2)^{1/2}-1^{1/2}}{(x-2)-1}$$
$$=(1/2)(1)^{-1/2}=1/2$$

Corollary: $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x^m-a^m}=\dfrac{n}{m}a^{n-m}$

Example 29:

$$\lim_{x\to2}\frac{x^3-8}{x^2-4}=\lim_{x\to2}\frac{x^3-2^3}{x^2-2^2}=\frac{3}{2}2^{3-2}=3$$

Example 30:

1- $\lim\limits_{x\to\infty}\left(5+\dfrac{1}{x}\right)=\lim\limits_{x\to\infty}5+\lim\limits_{x\to\infty}\dfrac{1}{x}=5+0=5$

2- $\lim\limits_{x\to0^+}\dfrac{1}{x^{1/3}}=\lim\limits_{x\to0^+}\left(\dfrac{1}{x}\right)^{1/3}=\infty$

3- $\lim\limits_{x\to0^-}\dfrac{1}{x^{1/3}}=\lim\limits_{x\to0^-}\left(\dfrac{1}{x}\right)^{1/3}=-\infty$

4- $\lim\limits_{x\to0^+}\dfrac{1}{x^2}=\lim\limits_{x\to0^+}\left(\dfrac{1}{x^2}\right)=\infty$

5- $\lim\limits_{x\to1^+}\dfrac{1}{x-1}=\infty$

6- $\lim\limits_{x\to1^-}\dfrac{1}{x-1}=-\infty$

7- $\lim\limits_{x\to-\infty}\dfrac{-15x}{7x+4}=\lim\limits_{x\to-\infty}\dfrac{-15}{7+4/x}=-\dfrac{15}{7}$

8- $\lim\limits_{x\to\infty}\dfrac{5x^2+8x-3}{3x^2+2}=\lim\limits_{x\to\infty}\dfrac{5+8/x-3/x^2}{3+2/x^2}=\dfrac{5}{3}$

9- $\lim\limits_{x\to-\infty}\dfrac{2x^2-3}{7x-4}=\lim\limits_{x\to-\infty}\dfrac{2x-3/x}{7-4/x}=-\infty$

10-
$$\lim_{x\to\infty}\frac{-4x^3+7x}{2x^2-3x-10}=\lim_{x\to\infty}\frac{-4x+7/x}{2-3/x-10/x^2}=-\infty$$