
بسم الله الرحمن الرحيم

EELU

الجامعة المصرية للتعلم الإلكترونى

Egyptian E-Learning University

MATH – 1

BY

DR. ADEL MORAD

CHAPTER 3

Derivatives and their Applications (Continued)

LECTURE 6.

Applications on Derivatives

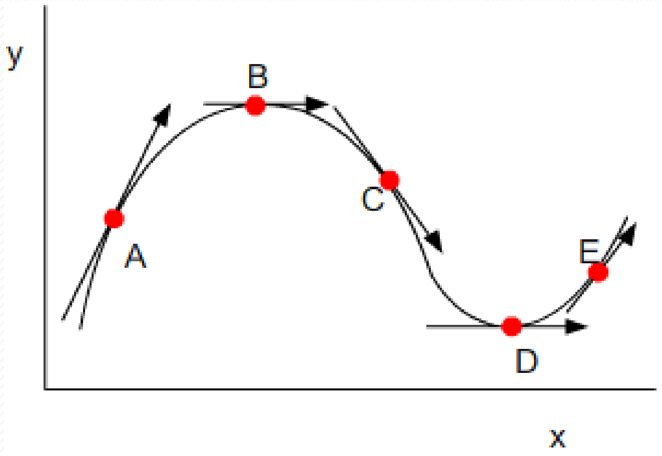
Aims and Objectives:

(1) To find the stationary points.
(2) To understand how to locate the stationary points.
(3) To determine the type of each stationary point.
(4) To define the rules of differentiation.
(5) Provide students with a strong intuitive feeling for these important concepts.

Maxima and Minima

If we measure the slope at different points we get different answers: at points $\mathcal{A}$ and $\mathcal{E}$ slope is +ve at point $\mathcal{C}$ slope is –ve.

So at some points in between, $\mathcal{B}$ and $\mathcal{D}$, the function exhibits a stationary value, and this can either be a local maximum ($\mathcal{B}$) or local minimum ($\mathcal{D}$).



How do we calculate maxima and minima positions?

We know that at a local max or min, the slope = 0, i.e. $dy/dx = 0$.

So, given a function, $y(x)$, all we need to do is differentiate it, and put the derivative equal to zero, then solve for x.

Example 6: Calculate maxima and minima positions of the function $y(x) = x^3 - 3x + 1$?

$$\frac{dy}{dx} = 3x^2 - 3$$

When $\frac{dy}{dx} = 0$, then $3x^2 - 3 = 0$

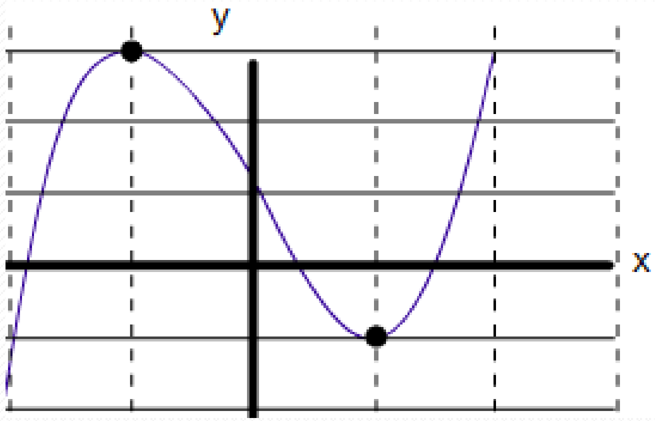
$3x^2 = 3$

$x^2 = 1$ $\qquad x = \pm 1$

$when\ x = +1,\ y = -1$ $\qquad when\ x = -1,\ y = 3$

Stationary points are (1 , -1) and (-1 , 3)

Note: later we'll show how we tell which is a max and min.



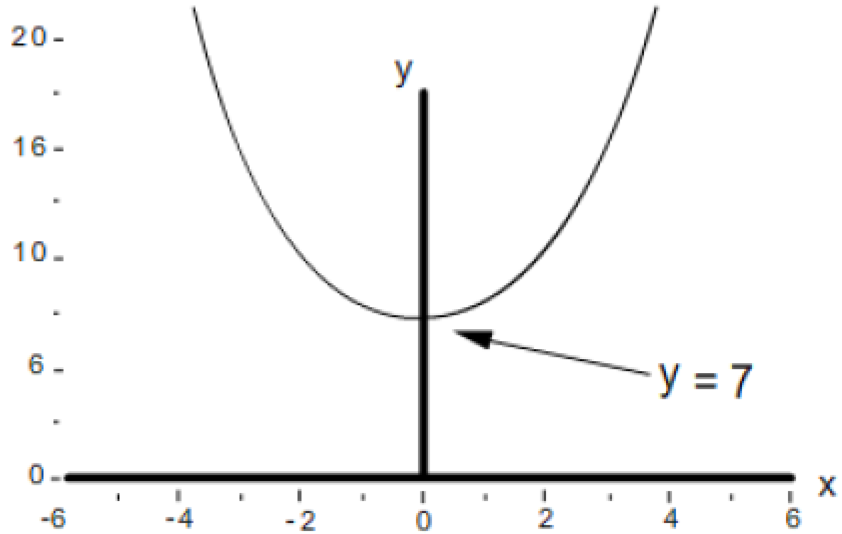
Example 7:

$$y = x^2 + 7$$

$$\frac{dy}{dx} = 2x$$

At stationary point, $\frac{dy}{dx} = 0$,

so $2x = 0$, i.e. $x = 0$ and $y = 7$.



Example 8:

The potential V of a diatomic molecule (e.g. Cl_2) can be approximated by a quadratic function of the bond length, r, of the form:

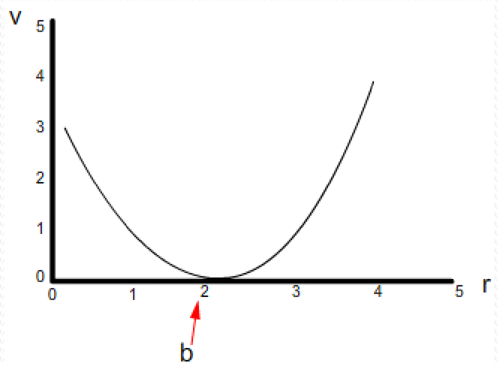
$V(r) = k(r - b)^2$ [k and b are constants]

What is the equilibrium bond length?

Answer:

Multiplying out: $V(r) = kr^2 - 2kbr + kb^2$

The equilibrium position will occur when the potential changes from attractive to repulsive, i.e. when the slope changes from +ve to -ve, e.g. at the minimum value of V.



So we need $dv/dr = 0$,

$dv/dr = 2kr - 2kb = 0$

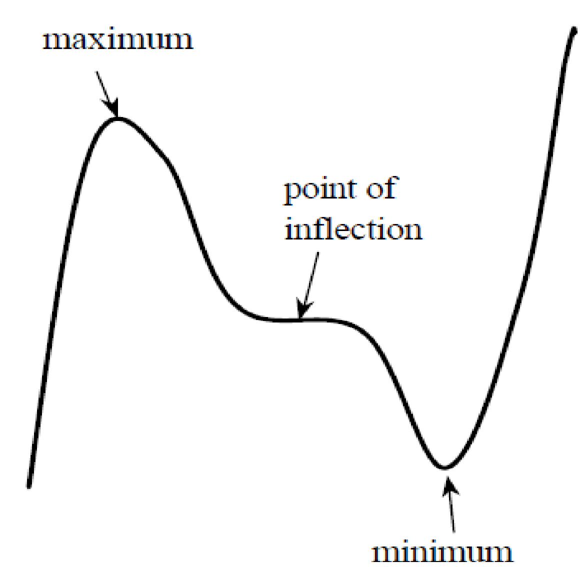
then $r = b$. So the constant b is actually the equilibrium bond length. The value of V at which this occurs is $V(r) = 0$, SP is $(b,0)$.

How to determine if a stationary point is a max, min or point of inflection?

The rate of change of the slope either side of a turning point reveals its type. But a rate of change is a differential, So all we need to do is differentiate the slope, $\frac{dy}{dx}$, with respect to x.

In other words we need the 2nd differential,

or $\frac{d}{dx}(\frac{dy}{dx})$ more usually called $\frac{d^2y}{dx^2}$



Example 9: Find the second derivative of:

1- $y(x) = 9x^2 - 2$

$\frac{dy}{dx} = 18x$ and $\frac{d^2y}{dx^2} = 18$

1- $p = 3q^3 - 4q^2 + 6$

$\frac{dp}{dq} = 9q^2 - 8q$ $\frac{d^2p}{dq^2} = 18q - 8$

Rules for stationary points:

i) At a local maximum $\frac{d^2y}{dx^2} = -ve$

ii) At a local minimum $\frac{d^2y}{dx^2} = +ve$

iii) At a point of inflection $\frac{d^2y}{dx^2} = 0$ and we must examine the slope either side of the turning point to find out if the curve is a *+ve* or *-ve* point of inflection.

Examples on Maxima and Minima of a function:

For each of the curves whose equations are given next:

- Find each stationary point and what type it is.
- Find the coordinates of the point where the curve meets the y-axis.
- Sketch the curve.
- Check by sketching the curve on your graphic calculator.

Example: Investigate the stationary points and examine their types of

$$y(x) = x^3 - 3x + 1$$

$$\frac{dy}{dx} = 3x^2 - 3$$

$\dfrac{dy}{dx} = 0$. This gives $3x^2=3$, then $x = \pm 1$ (at $x=-1$, $y=3$ and at $x=1$, $y=-1$)

giving stationary points at $(-1,3)$ and $(1,-1)$

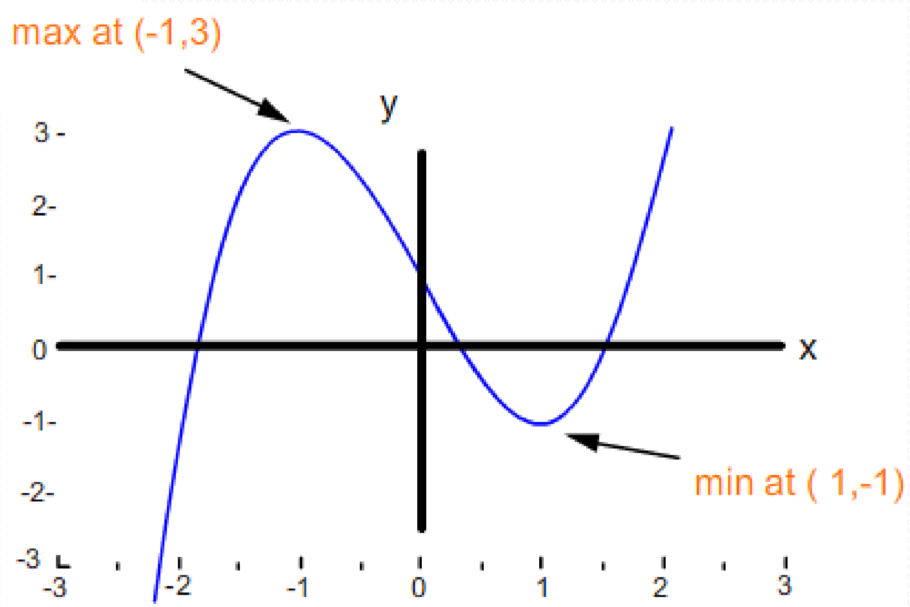
$$\frac{d^2y}{dx^2} = 6x$$

At stationary point $(-1,3)$, $x = -1$,

so $\dfrac{d^2y}{dx^2} = -6$, so it's a maximum.

At stationary point $(1,-1)$, $x = +1$,

so $\dfrac{d^2y}{dx^2} = +6$, so it's a minimum



Now, we can finally sketch the curve:

Example: Find the stationary point and what type it is?

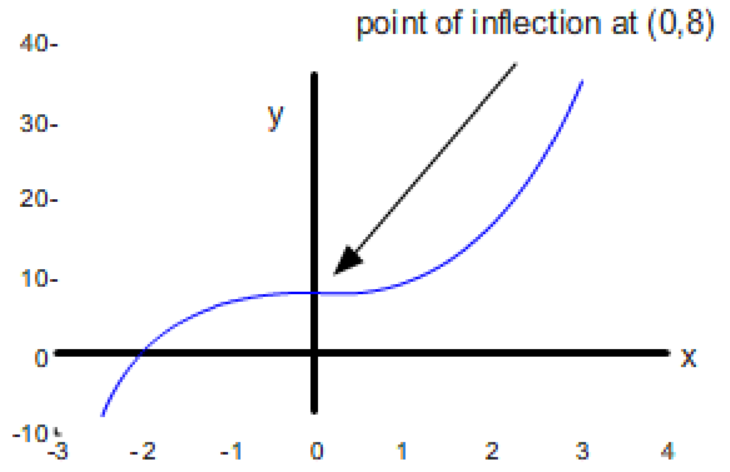
$y = x^3 + 8$ $\quad \frac{dy}{dx} = 3x^2$, which is equal to zero at the stationary point.

If $3x^2 = 0,\ x = 0$, and so $y = +8$, so the stationary point is at $(0,8)$.

$$\frac{d^2y}{dx^2} = 6x$$

So, at the stationary point $(0,8)$, $\frac{d^2y}{dx^2} = 0$, so we have a point of inflexion.

But is $\frac{dy}{dx}$ +ve either side of this point (e.g. at $x = +1,\ \frac{dy}{dx} = +3$, at $x = -1,\ \frac{dy}{dx} = +3)$ so the curve has a positive point of inflection.



Example: Where are the turning point (s), and does it (or they) indicate a max or min in the function

$p(q) = 4 - 2q - 3q^2$?

$\frac{dp}{dq} = -2 - 6q$, which at the turning point = 0

So $-2 - 6q = 0,\ 6q = -2,\ q = \frac{-1}{3}$, and $p(q) = 4\frac{1}{3}$

We have one turning point at $\left(\frac{-1}{3}, 4\frac{1}{3}\right)$

$\frac{d^2p}{dq^2} = -6$, so the turning point is a maximum.

The rules of differentiation

a) Product rule:

If a function $y(x)$ can be written as the product of two other functions, say $\mathcal{U}(x)$ and $\mathcal{V}(x)$, then the differential of $y(x)$ is given by the product rule:

e.g. if $y(x) = u(x)\, v(x)$, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

b) Quotient rule:

If we have one function divided by another, such as $y(x) = \frac{u(x)}{v(x)}$, then

$$\frac{dy}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$

Examples: Find the first derivative of the functions?

1. $y(x) = (x^2 + 2)(x + 1)$

let $u = x^2 + 2$, so that $\frac{du}{dx} = 2x$

let $v = x + 1$, so that $\frac{dv}{dx} = 1$

so $\frac{dy}{dx} = u.\frac{dv}{dx} + v\frac{du}{dx} = (x^2 + 2).1 + (x + 1).2x$

$= 3x^2 + 2x + 2$

(Note: we can check this by expanding out the brackets)

$y(x) = x^3 + x^2 + 2x + 2$, $\frac{dy}{dx} = 3x^2 + 2x + 2$

2. $y(x) = x^3(3 - \frac{1}{x} + 3x^2)$

Let $u = x^3$, and $v = 3 - \frac{1}{x} + 3x^2$

$$\frac{dy}{dx} = x^3(\frac{1}{x^2} + 6x) + (3 - \frac{1}{x} + 3x^2).3x^2$$

3. $\varphi(\lambda) = (2\lambda + \lambda^2)(\frac{1}{\lambda} + \frac{1}{\lambda^2})$

$$\frac{d\varphi}{d\lambda} = (2\lambda + \lambda^2)(-\frac{1}{\lambda^2} - \frac{2}{\lambda^3}) + (\frac{1}{\lambda} + \frac{1}{\lambda^2})(2+2\lambda)$$

4. $y(x) = \frac{x^2+1}{2x+3}$ so that $u = x^2 + 1$ and $\frac{du}{dx} = 2x$

and $v = 2x + 3$ and $\frac{dv}{dx} = 2$

$$\frac{dy}{dx} = \frac{(2x+3)(2x) - (x^2+1)(2)}{(2x+3)^2} = \frac{4x^2+6x-2x^2-2}{(2x+3)^2} = \frac{2x^2+6x-2}{(2x+3)^2}$$

[Note: Alternatively we can say $\frac{u(x)}{v(x)} = uv^{-1}$ and

use the product rule and function of a function.]

c) Function of a function:

Suppose we want to differentiate $(2x-1)^3$

We could expand the bracket then differentiate term by term, but this is tedious!

We need a more direct method for expressions of this kind.

Now $(2x-1)^3$ is a cubic function of the linear function $(2x-1)$, i.e. it is a function of a function. We have 2 ways to solve this kind problems:

1- Chain rule:

If we have $y(x) = f(u)$ (complicated expression), we let $u =$ (complicated expression) then we use the chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

2- Sequential Step rule:

With this method, we start with the outermost function, and differentiate our way to the centre, multiplying everything together along the way.

Examples: Find the first derivative of the functions?

Using the first rule:

1. $y = (2 - x^3)^4$ let $u = 2 - x^3$, so that $y = u^4$

$$\frac{dy}{du} = 4u^3 \text{ and } \frac{du}{dx} = -3x^2$$

So $\dfrac{dy}{dx} = (4u^3).(-3x^2) = -12x^2(2 - x^3)^3$

2. $y(x) = \dfrac{1}{(1-x^2)}$, i.e. $y = (1 - x^2)^{-1}$

let $u = (1 - x^2)$, so that $\dfrac{du}{dx} = -2x$ and $y = u^{-1}$,

so that $\dfrac{dy}{du} = -\dfrac{1}{u^2}$

$$\frac{dy}{dx} = (-2x)\left(-\frac{1}{u^2}\right) = +\frac{2x}{(1-x^2)^2}$$

The same examples, using the second rule:

1. $y = (2 - x^3)^4$

think of this as y = (expression)^4

differentiating, $\dfrac{dy}{dx} = 4$(expression)^3

We now look at the expression in the brackets and differentiate that $(= -3x^2)$ and multiply it to our previous answer to give

$\dfrac{dy}{dx} = 4(2 - x^3)^3(-3x^2)$ (which is the same as before)

2. $y = \dfrac{1}{(1-x^2)} = (1 - x^2)^{-1}$

$\dfrac{dy}{dx} = -(1 - x^2)^{-2} \,.\, (-2x)$

3. $y = \sqrt{x^2 - 1} = (x^2 - 1)^{½}$

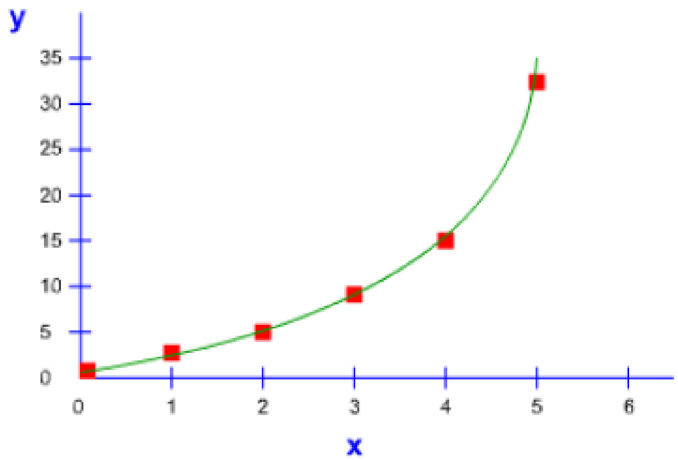
$\dfrac{dy}{dx} = ½(x^2 - 1)^{-½} \,.\, 2x$

Exponential Functions:

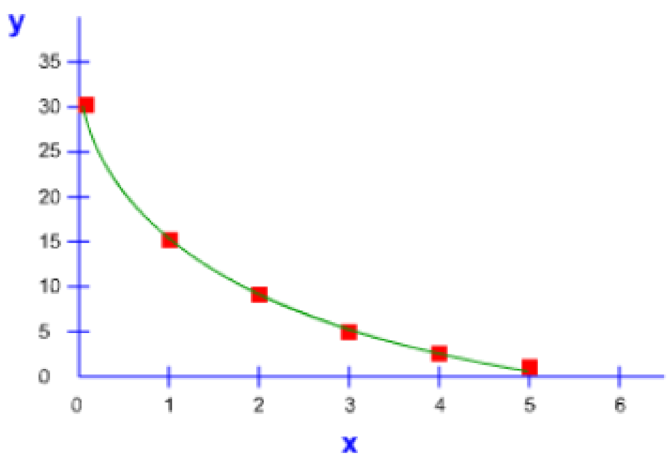
The general expression for an exponential function is

$f(x) = ka^x$, k, a = constants

a > 1



a < 1

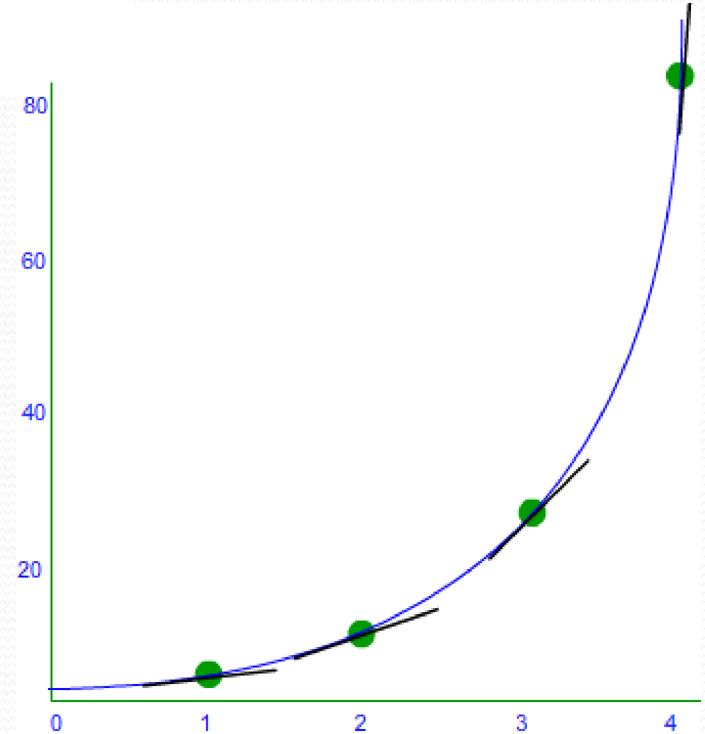


Examples on exponential Functions:

1. An example is $y = 3^x$

x	0	1	2	3	4
y	1	3	9	27	81

One of the most important properties of an exponential function is that the slope of the function at any value is proportional to the value of the function itself.



In other words $\frac{dy}{dx}$, proportional to $y(x)$, or

$$\frac{dy}{dx} = c\, y(x)$$

If $y(x) = a^u$ then $\frac{dy}{dx} = lin(a) a^u \frac{du}{dx}$

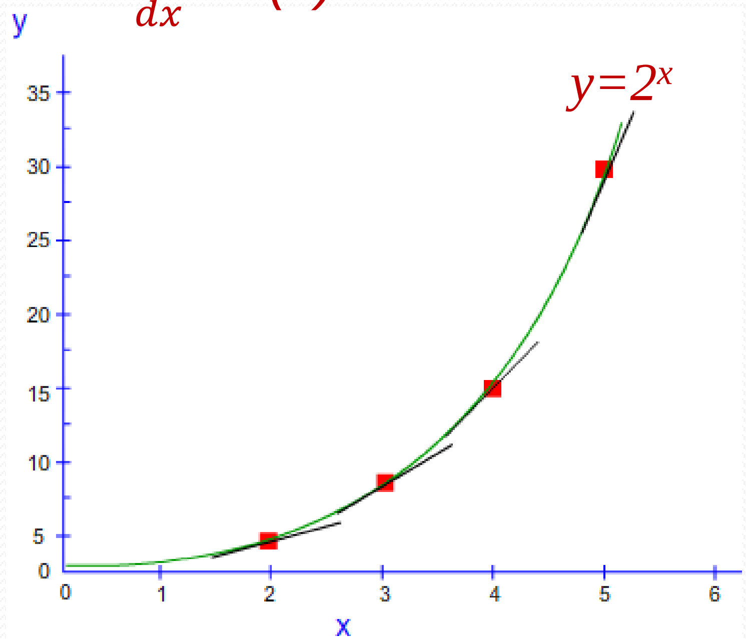
In the example $\frac{dy}{dx} = lin(3) 3^x$

2.

$y = 2^x$, plot the graph and measure the slopes at different values of x

x	y	Slope at x measured from graph	Slope/y
0	1	0.69	0.69
1	2	1.38	0.69
2	4	2.76	0.69
3	8	5.52	0.69
4	16	11.04	0.69

$\frac{dy}{dx}$=lin(2)2^x



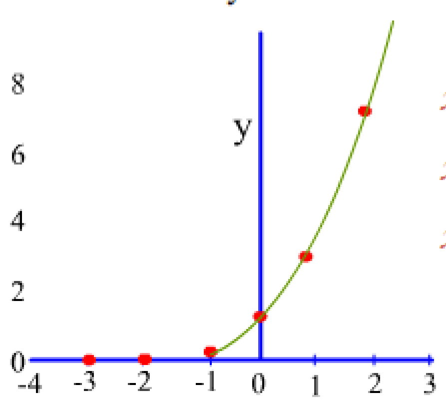
Description of the exponential function and its importance:

The function e^x or $exp(x)$ is known as the natural exponential function and is extremely important in all branches of science:

- Radioactive materials undergo exponential decay,
- World human population is increasing exponentially,
- Chemical reaction rates depend exponentially upon the temperature, etc.

x	0	1	2	3	4	5	-1	-2
e^x	1	2.72	7.39	20.1	54.6	148	0.37	0.69
e^{-x}	1	0.37	0.69	0.05	0.02	0.007	2.27	7.39

$y = e^x$

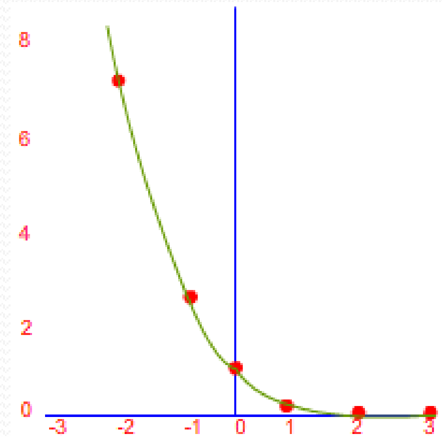


$x = 0,$ $e^x = 1$; $x = 0,$ $e^{-x} = 1$

$x = +$ infinity, $e^x = +$ infinity ; $x = +$ infinity, $e^{-x} = 0$

$x = -$ infinity, $e^x = 0$; $x = -$ infinity, $e^{-x} = +\infty$

$y=e^{-x}$



Examples: Find the first derivative of the exponential functions?

1. $y = 5e^x$, $\frac{dy}{dx} = 5e^x$
2. $y = 3e^x + 2$, $\frac{dy}{dx} = 3e^x$
3. a general result is $\frac{d}{dx}(e^{bx}) = be^{bx}$
4. $y = e^{-x}$, $\frac{dy}{dx} = -e^{-x}$
5. $y = \exp(x^2 + 1)$, $\frac{dy}{dx} = \exp(x^2 + 1) \,.\, 2x$
6. $y = x.e^{-bx.}$ This requires the product rule

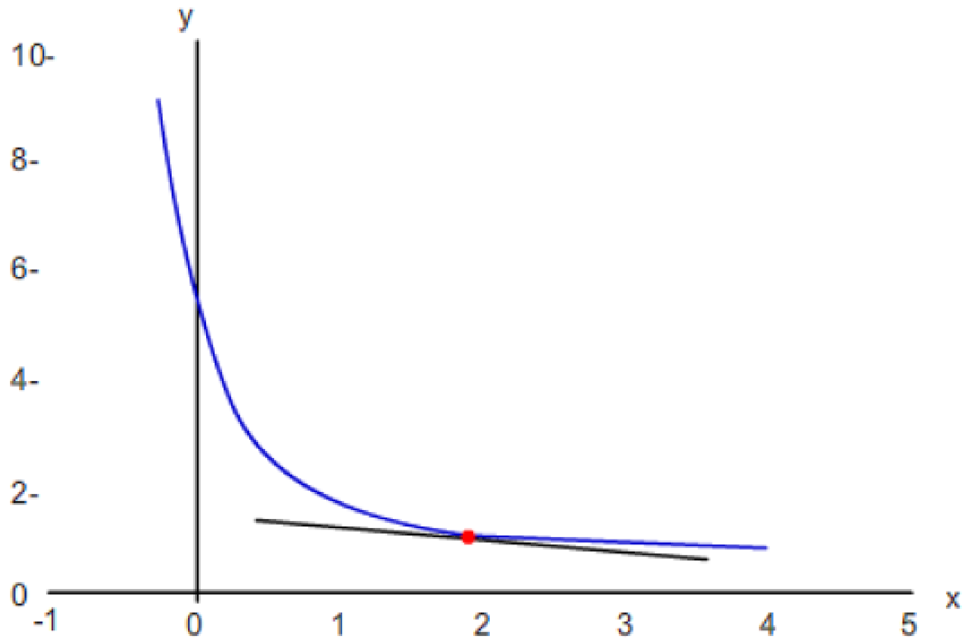
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x(-be^{-bx}) + e^{-bx}.(1) = (1 - bx)e^{-bx}$$

Example What is the slope of the curve $y = 5e^{-2x} + 1$ at the point $x = 2$?

Solution:

the slope $= y' = -10e^{-2x}$

at $x = 2$, the slope $= y' = -10e^{-4} = -0.183$



The Logarithmic Function:

If we have the relationship $y = a^x$

then there must be the inverse relationship such that

$x = f(y)$.

We call the function, $f(y)$, the logarithm to base a.

$x = \log_a(y)$, valid only for $y > o$

There are 2 types of logarithm in common use:

a) Common logs have base 10 and are written $\log_{10} x$

b) Natural logs have base e and are written $\log_e x$ or

In x So, if $y = e^x$, then In $y = x$

$y = 10^x$, then $\log_{10} y = x$

Logarithmic Functions :

1- In $\mathcal{A}$ + In $\mathcal{B}$ = In $(\mathcal{AB})$

2- In $\mathcal{A}$ − In $\mathcal{B}$ = In $(\mathcal{A} / \mathcal{B})$

3- In $\mathcal{A}^x = x$.In $\mathcal{A}$

The Differential of ln x It can be shown that, if

$y = \ln x, \qquad \frac{dy}{dx} = \frac{1}{x}$

We can now use this, together with the Product, Chain and Sequential Rules to find the Differentials of log functions.

1. $y = \ln(ax + b) \qquad \frac{dy}{dx} = \frac{1}{(ax+b)} \times a = \frac{a}{(ax+b)}$

2. $y = \ln(2\sqrt{x}) + 3x^2 = \ln(2x^{1/2}) + 3x^2$

$$\frac{dy}{dx} = \left(\frac{1}{2\sqrt{x}}\right) \cdot \frac{2}{2x^{1/2}} + 6x = \frac{1}{2x} + 6x$$

3. What are the stationary points in $y = \ln(x) - x$?

$$\frac{dy}{dx} = \frac{1}{x} - 1$$

So the slope = 0 when $\frac{1}{x} - 1 = 0$,

i.e., when $x = 1$, and $y = -1$.

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$, which at (1,-1) is -ve, so it is a *maximum*.

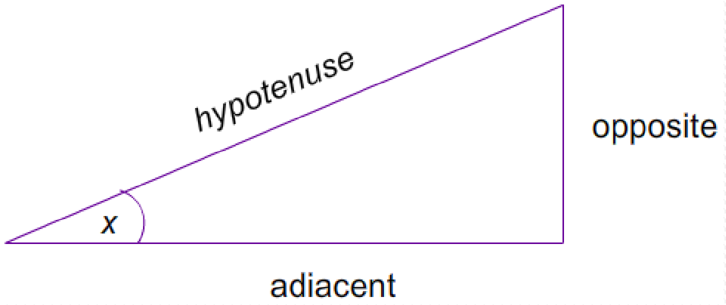
Trigonometric Functions:

The common trig. functions are defined relative to a right-angled triangle.

sin x = opposite/hypotenuse

cos x = adjacent/hypotenuse

tan x = opposite/adjacent



In calculus, we need the angles measured in radians rather than degrees, with 2π radians in a circle $= 360°$ so that 1 radian $= 360°/2\pi$.
The common points used: $0° = 0$ radians $90° = \pi/2$ radians $180° = \pi$ radians $270° = 3\pi/2$ radians $360° = 2\pi$ radians

Examples: Find the first derivative of the trigonometric functions:

1- $y = \sin(kx)$

$$\frac{dy}{dx} = \underset{\substack{\uparrow \\ \text{differential of } \sin(\ldots)}}{\cos(kx)} \times \underset{\substack{\uparrow \\ \text{differential of } kx}}{k} = k.\cos(kx) \leftarrow \text{important}$$

2- $y = 3\cos x - 4\sin(x^2)$

$$\frac{dy}{dx} = -3\sin x - 8x\cos x^2$$

3- $y = 7\sin(5x^2) + 6\ln\{\tan(5x)\}$

$$\frac{dy}{dx} = 70x\cos 5x^2 + 30\sec^2 x / (\tan 5x)$$

Speed of an object. The speed of the object is the absolute value of the velocity.

Example: The position of a particle is given by the equation of motion

$f(t) = 1/(t + 1) = (t + 1)^{-1}$

Find the velocity and speed when $t = 2$.

Solution:

$f\,'(t) = -(t + 1)^{-2}$

$f\,'(2) =- 1/9.$

The velocity at $t = 2$ is -1/9 m/s and the speed at $t = 2$ is 1/9

Example : Find the rate of change of $f(x) = x^2$, when $x = 3$.

Solution:

$f\,'(x)=2x$.

The rate of change of $f(x)$ when $x = 3$ is 6.

TABLE 1. DERIVATIVES

$f(x)$	$f'(x)$	Derivative Number
$af(x)$	$af'(x)$	D-1
$u(x)+v(x)$	$u'(x)+v'(x)$	D-2
$f(u)$	$f'(u)\dfrac{du}{dx}=\dfrac{df(u)}{du}\dfrac{du}{dx}$	D-3
a	0	D-4
$x^n \quad (n\neq 0)$	nx^{n-1}	D-5
$u^n \quad (n\neq 0)$	$nu^{n-1}\dfrac{du}{dx}$	D-6
uv	$u\dfrac{dv}{dx}+v\dfrac{du}{dx}$	D-7
$\dfrac{u}{v}$	$\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$	D-8
e^u	$e^u\dfrac{du}{dx}$	D-9

TABLE 1. DERIVATIVES (CONTINUED)

a^u	$(\ln a)a^u \dfrac{du}{dx}$	D-10
$\ln u$	$\dfrac{1}{u}\dfrac{du}{dx}$	D-11
$\log_a u$	$(\log_a e)\dfrac{1}{u}\dfrac{du}{dx}$	D-12
$\sin u$	$\cos u\left(\dfrac{du}{dx}\right)$	D-13
$\cos u$	$-\sin u \dfrac{du}{dx}$	D-14
$\tan u$	$\sec^2 u \dfrac{du}{dx}$	D-15
$\sin^{-1} u$	$\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx} \quad \left(-\dfrac{\pi}{2} \le \sin^{-1} u \le \dfrac{\pi}{2}\right)$	D-16
$\cos^{-1} u$	$\dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx} \quad \left(0 \le \cos^{-1} u \le \pi\right)$	D-17
$\tan^{-1} u$	$\dfrac{1}{1+u^2}\dfrac{du}{dx} \quad \left(-\dfrac{\pi}{2} < \tan^{-1} u < \dfrac{\pi}{2}\right)$	D-18

THANK YOU