# Mathematics (2)

## Section (2)

**Parametric equations and polar coordinates**

# Parametric equations

**Definition:** If $f$ and $g$ are continuous functions of $t$ on an interval $I$, then the equations

$$x = f(t), \qquad y = g(t)$$

are called ***parametric equations*** and $t$ is called the ***parameter***.

The set of points $(x, y)$ obtained as $t$ varies on the interval $I$, is called the ***graph*** of the parametric equations. Taken together, the parametric equations and the graph are called a ***parameterized plane curve.***

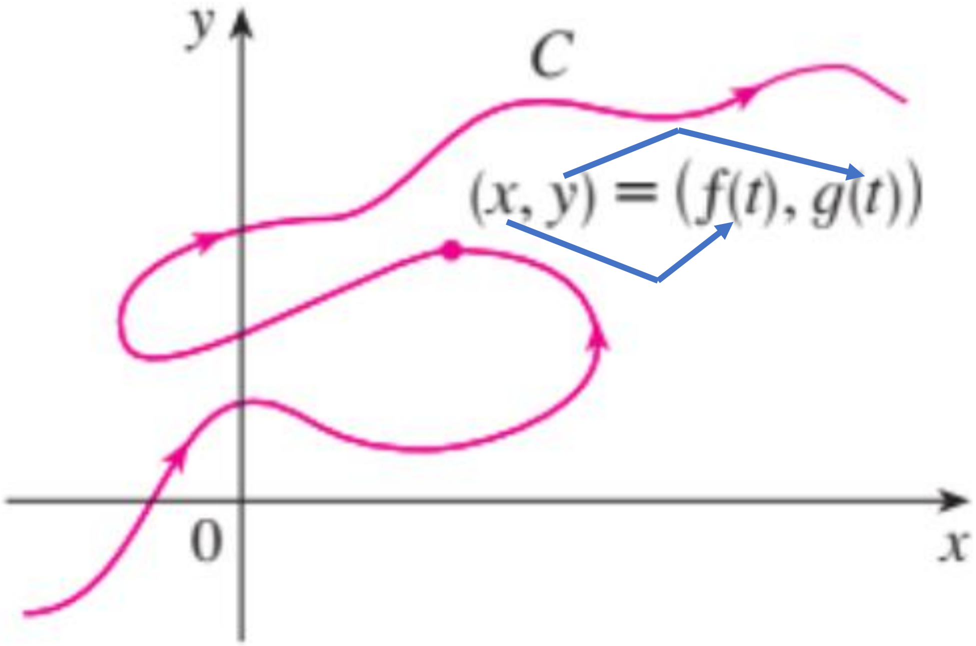
***Example***

***Cartesian equation***

$$x^2 + y^2 = 4$$

***parametric equations***

$$x = 2\cos(t), y = 2\sin(t)$$



# Examples

## Example 1

**Sketch the curve $C$ described by the parametric equations**

## Solution

$$x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad \text{for } -2 \le t \le 3.$$

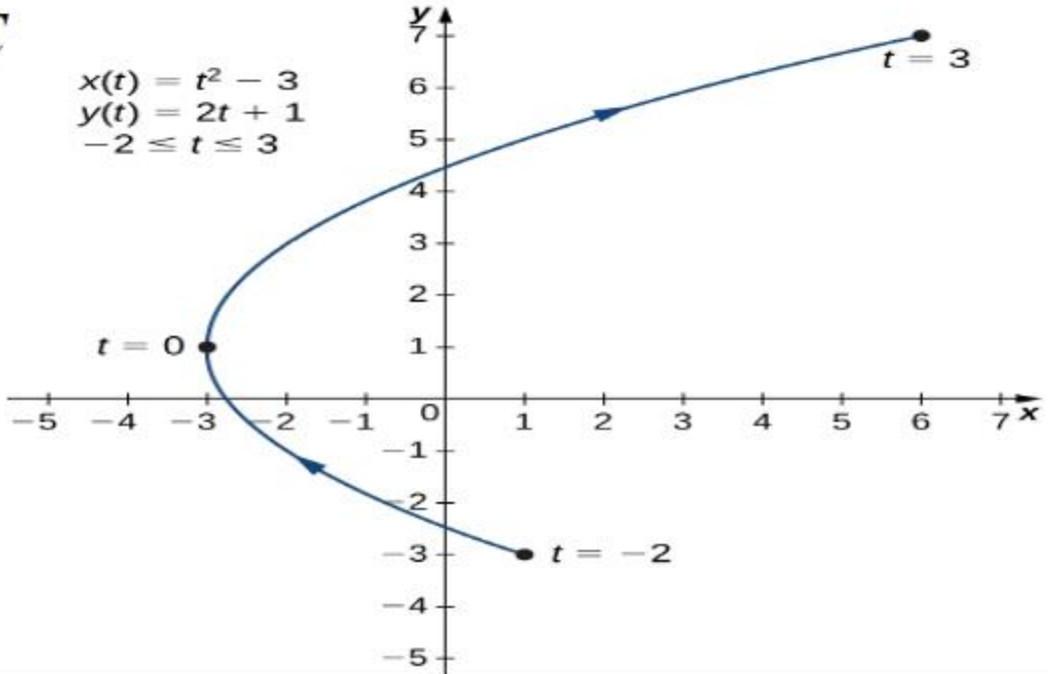
For values of $t$ on the given interval, the parametric equations yield the points $(x, y)$ shown in the following table.

**Input** → $t$

**Output** ← $x$, $y$

| $t$ | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $x$ | 1 | -2 | -3 | -2 | 1 | 6 |
| $y$ | -3 | -1 | 1 | 3 | 5 | 7 |

By plotting these points, we obtain the curve $C$



**Example 2** **Eliminate the parameter for the plane curve defined by the following parametric equations and describe the resulting graph**

**Solution**

$$x(t) = \frac{1}{\sqrt{t+1}}, \quad y(t) = \frac{t}{t+1} \qquad \text{for } t > -1.$$

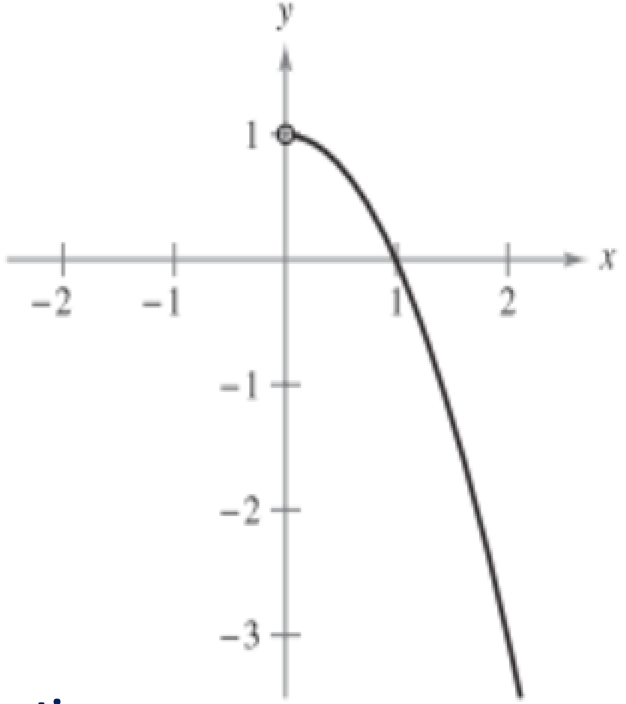
Start by solving one of the parametric equations for $t$

$x = \frac{1}{\sqrt{t+1}},$ Parametric equation for x

$x^2 = \frac{1}{t+1},$ Square both sides

$\frac{1}{x^2} = t + 1,$ Take reciprocal of both sides

$t = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2},$ Solve for t




Now, substituting into the parametric equation for $y$ yields:

$$y = \frac{\frac{1-x^2}{x^2}}{\frac{1-x^2}{x^2}+1} = \frac{\frac{1-x^2}{x^2}}{\frac{1-x^2+x^2}{x^2}} \Rightarrow \boxed{y = 1 - x^2}.$$ ← Cartesian Equation

This is a parabola with a vertex $(0,1)$. It opens to the down and it is symmetric about $y$-axis. The cartesian equation $y = 1 - x^2$ is defined for all values of $x$, but from the parametric equation of $x$, the curve is defined only if $t > -1$. This implies that we should restrict the domain of $x$ to positive values, i.e., $x > 0$.

$$y = 1 - x^2$$

$$x^2 = -y + 1$$

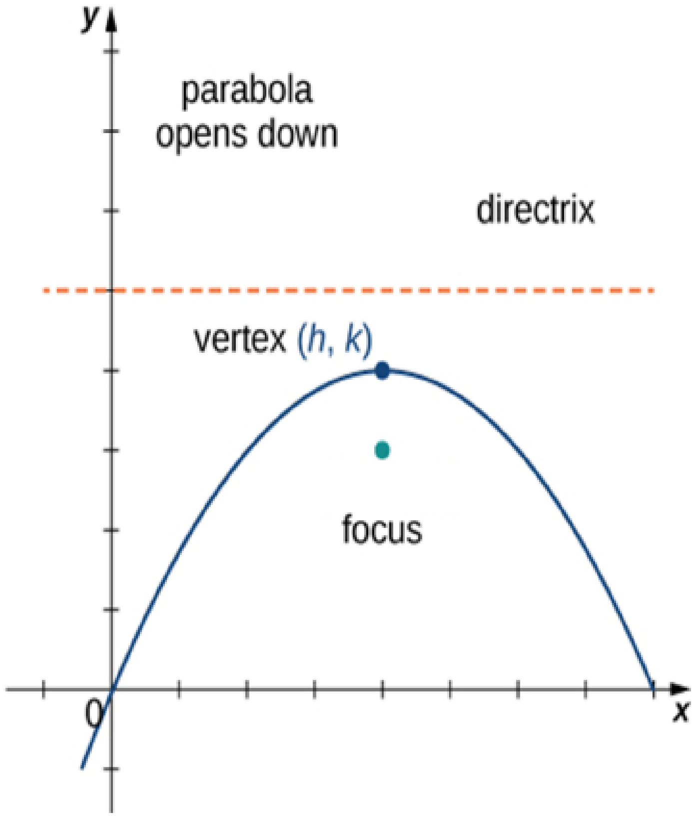
$$x^2 = -(y - 1)$$

$$(x - 0)^2 = -4\left(\frac{1}{4}\right)(y - 1)$$

$$Vertex = (0,1)$$

Case 2

| Horizontal axis of symmetry $(x - h)^2 = -4a(y-k)$ | |
|---|---|
| **Vertex** | (h, k) |
| **Focus** | (h, k-a) |
| **Directrix** | y = k+a |
| **Axis of symmetry** | x=h |




$$Axis\ of\ symmetry\ equation\ x = 0\ which\ is\ the\ equation\ of\ the\ y - axis$$

The cartesian equation $y = 1 - x^2$ is defined for all values of $x$, but from the parametric equation of $x$, the curve is defined only if $t > -1$. This implies that we should restrict the domain of $x$ to positive values, i.e., $x > 0$.

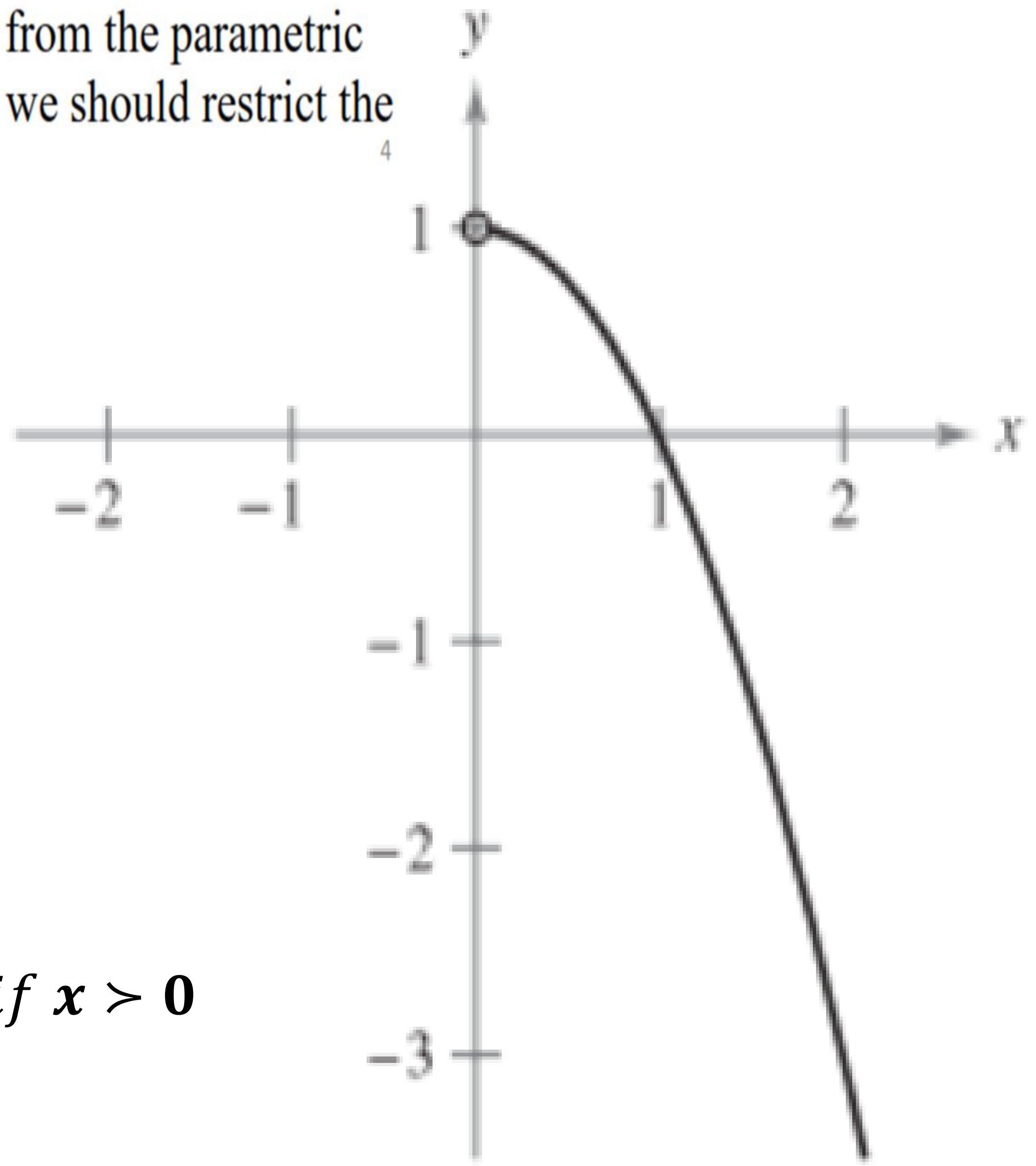
$$\because t = \frac{1}{x^2} - 1$$

$$\because t \succ -1$$

$$\therefore \frac{1}{x^2} - 1 \succ -1$$

$$\therefore \frac{1}{x^2} \succ 0$$

$$\therefore \frac{1}{x} \succ 0 \qquad \textit{which is only valid if and only if } \boldsymbol{x \succ 0}$$



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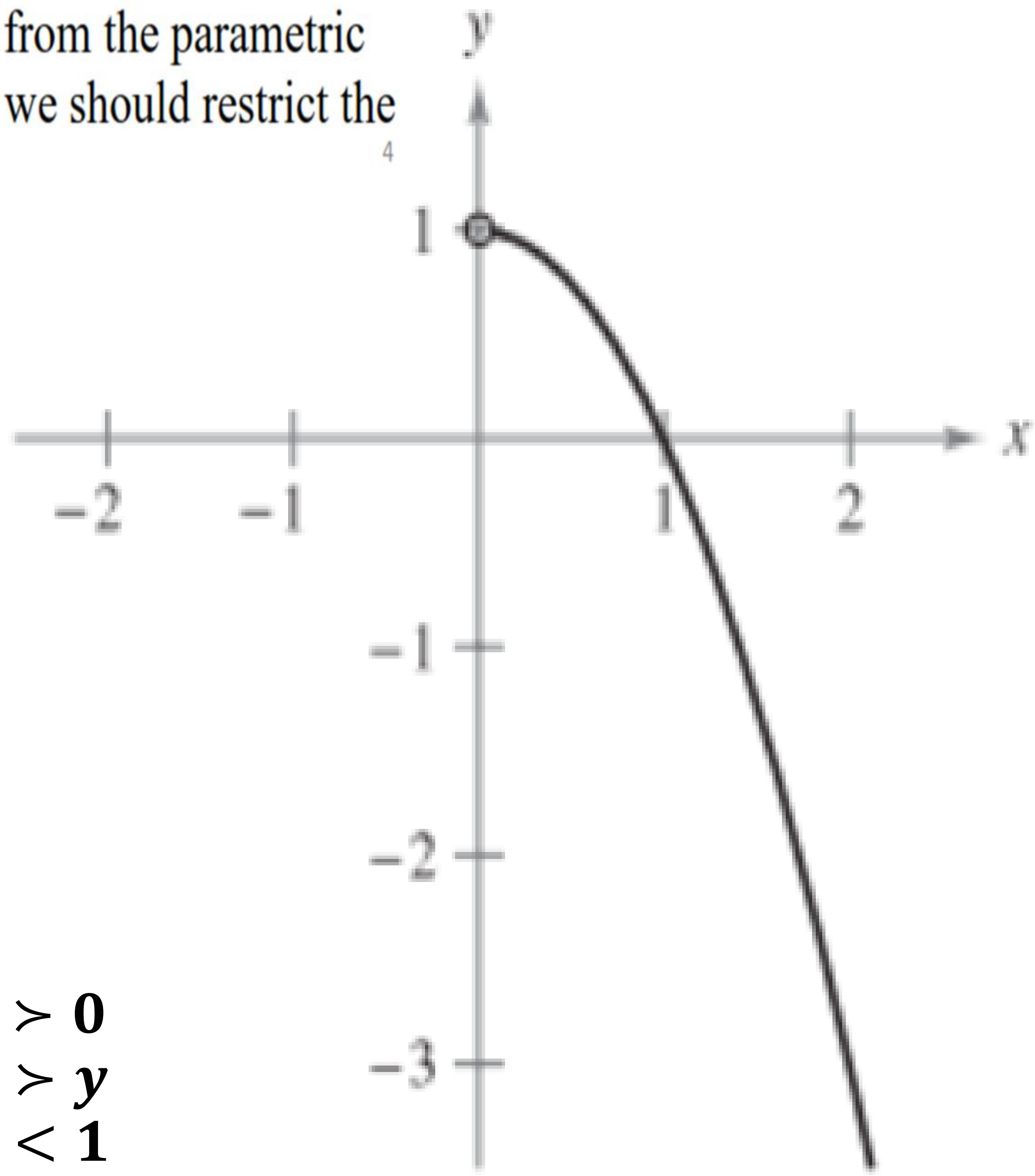



$$\because t = \frac{1}{1-y} - 1$$

$$\because t \succ -1$$

$$\therefore \frac{1}{1-y} - 1 \succ -1$$

$$\therefore \frac{1}{1-y} \succ 0$$

$$\textit{which is only valid if and only if}: \quad 1 - y \succ \mathbf{0}$$

$$1 \succ \boldsymbol{y}$$

$$y < \mathbf{1}$$

**Example 3** **Eliminate the parameter and describe the resulting graph of the parametric curve**

$$x(t) = \sin(t), \quad y(t) = -4+3\cos(t) \quad \textbf{for } 0 \le t \le 2\pi.$$

## Solution

Using the parametric equations $x = \sin(t), \ y = -4 + 3\cos(t)$, we find that

$$\sin(t) = x, \qquad \cos(t) = \frac{y+4}{3},$$

$$\boxed{\begin{array}{l} \sin^2 t = x^2 \ , \quad \cos^2 t = \dfrac{(y+4)^2}{9} \\ \sin^2 t + \cos^2 t = x^2 + \dfrac{(y+4)^2}{9} \end{array}}$$

Now, substituting into the trigonometric identity $\sin^2(t) + \cos^2(t) = 1$, yields:

$$x^2 + \frac{(y+4)^2}{9} = 1.$$

$$\boxed{\frac{(x-0)^2}{1} + \frac{(y-(-4))^2}{9} = 1}$$

This is an ellipse centered at the point (0, - 4). Its major axis is the *y*-axis and of length = 6 .

$$\frac{(x-0)^2}{1}+\frac{(y-(-4))^2}{9}=1$$

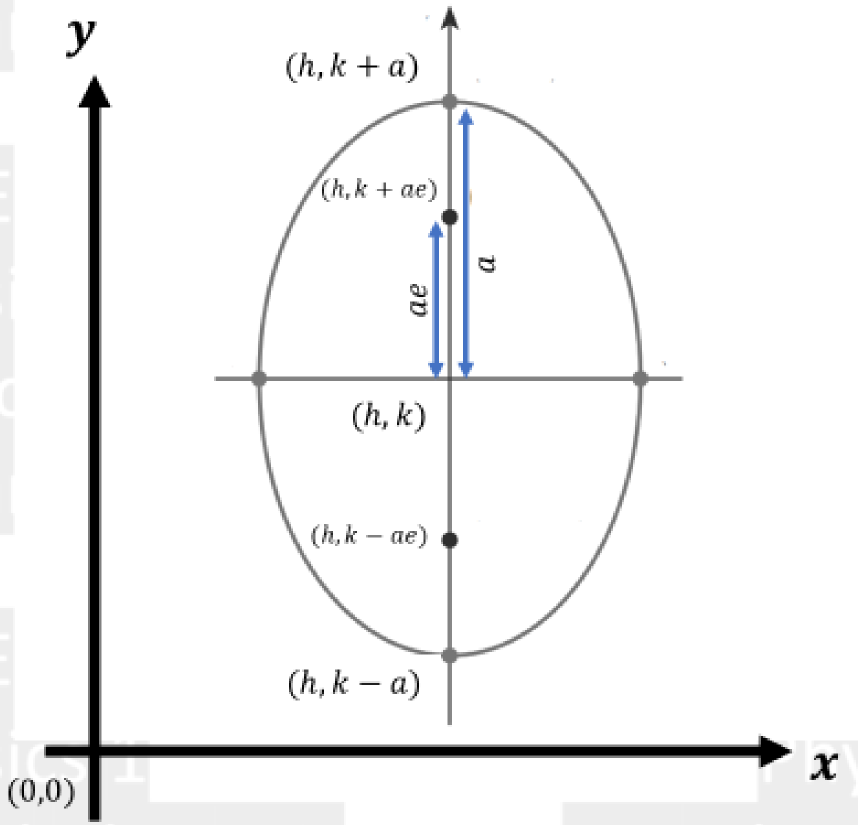
$$a^2=9$$
$$a=3$$

$$b^2=1$$
$$b=1$$

$$Vertex=(0,-4)$$

## Case 1 – Eclipse having the major axes parallel to the y-axis

| Standard Equation form $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$ | |
|---|---|
| Center | (h, k) |
| Focus points | (h, k+ae) , (h, k-ae) |
| Vertices | (h, k+a) , (h, k-a) |
| Directrix equations | $y=\frac{a}{e}+k$ , $y=-\frac{a}{e}+k$ |
| Equation of the major axis | $x=h$ |
| The length of the major axis | Length = 2a |
| Equation of the minor axis | $y=k$ |
| The length of the minor axis | Length = 2b |



# Calculus with Parametric Curves

If the parametrized curve $C$ given by $x = f(t)$ *and* $y = g(t)$ is differentiable, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad , \qquad (*)$$

provide that $\frac{dx}{dt} \neq 0$.

The slope of the tangent to the parametrized curve $C$ at the point $\mathrm{t} = t_0$, is the derivative $\frac{dy}{dx}$ evaluated at $\mathrm{t} = t_0$.

It can be seen from Equation $(*)$ that:

- the curve has a horizontal tangent when $\frac{dy}{dt} = 0$ (provided that $\frac{dx}{dt} \neq 0$) and
- it has a vertical tangent when $\frac{dx}{dt} = 0$ (provided that $\frac{dy}{dt} \neq 0$).

## Parametric Formula for $\frac{d^2y}{dx^2}$

If the equations $x = f(t), y = g(t)$ define $y$ as a twice-differentiable function of $x$, then at any point where $\frac{dx}{dt} \neq 0$, then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

**Example 4** A curve $C$ is defined by the parametric equations

$$x = t^2, \qquad y = t^3 - 3t.$$

(a) Find the tangent line to the curve at the point where $t = \sqrt{3}$ .

$$\boxed{\begin{array}{l} x = \left(\sqrt{3}\right)^2 = 3 \\ y = \left(\sqrt{3}\right)^3 - 3(\sqrt{3}) = 0 \end{array}}$$

(b) Find the points on where the tangent is horizontal or vertical.

**Solution**

(a) Since

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t},$$

the slope of the tangent at the point where $t = \sqrt{3}$ is $\mathrm{m} = \frac{dy}{dx}\Big|_{t=\sqrt{3}} = \frac{6}{2\sqrt{3}} = \sqrt{3}$.

The point on the curve, corresponding to the value $t = \sqrt{3}$ is $\boxed{(3, 0) \longrightarrow (x_1, y_1)}$

The **point- slope form** of the tangent line is

$$(y - y_1) = m(x - x_1)$$

$$(y - 0) = \sqrt{3}(x - 3) \qquad \Rightarrow. \ y = \sqrt{3}(x - 3)\,.$$

The curve $C$ has a horizontal tangent when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

$$\frac{dy}{dt} = 3t^2 - 3 = 0$$

$$3t^2 - 3 = 0$$

$$t^2 = 1$$

$$t = \pm 1$$

$\textit{The corresponding points to } (t = \pm 1) : -$

| | |
|---|---|
| $t = +1$ | $t = -1$ |
| $x = (1)^2 = 1$ | $x = (-1)^2 = 1$ |
| $y = (1)^3 - 3(1) = -2$ | $y = (-1)^3 - 3(-1) = 2$ |

$\textit{The corresponding points to } (t = \pm 1) \textit{ on the curve are : } (1, -2) \textit{ and } (1, 2)$

It has a vertical tangent when $\frac{dx}{dt} = 2t = 0$, that is, $t = 0$.

$\boldsymbol{The\ corresponding\ points\ to\ (t = 0) : -}$

$$t = 0$$

$$x = (0)^2 = 0$$

$$y = (0)^3 - 3(0) = 0$$

$\boldsymbol{The\ corresponding\ point\ to\ (t = 0)\ on\ the\ curve\ is :\ (0, 0)}$

**Example 5** Find $\frac{d^2y}{dx^2}$ as a function of $t$ if $x = t - t^2$ and $y = t - t^3$.

**Solution**

1. Find the first derivative $dy/dx$.

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{1-2t} = \boldsymbol{f(t)}$$

2. Find the derivative of $dy/dx$ with respect to $t$.

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{2-6t+6t^2}{(1-2t)^2}$$

Quotient Rule

3. Divide by $dx/dt$.

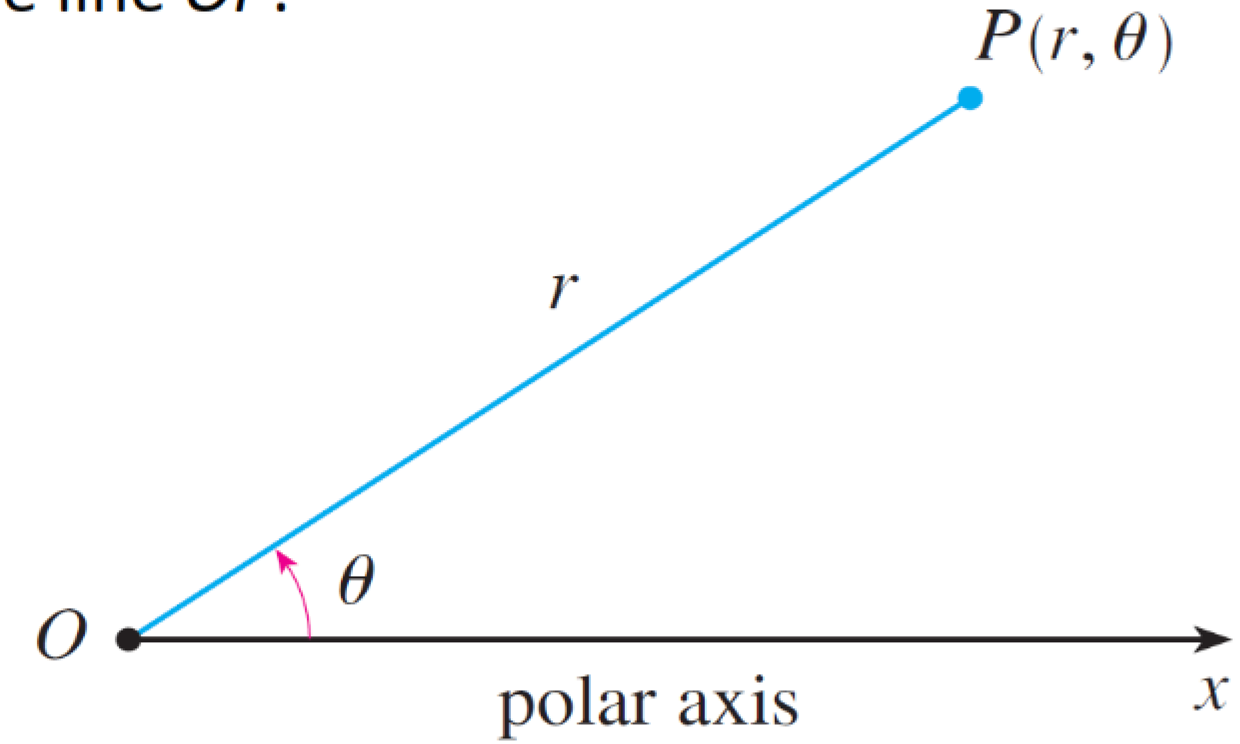
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{2-6t+6t^2}{(1-2t)^2}}{1-2t}$$

$$= \frac{2-6t+6t^2}{(1-2t)^3}$$

# Polar coordinates

We choose a point in the plane that is called the **pole** and is labeled *O*. Then we draw a ray (half-line) starting at *O* called the **polar axis (initial line)**.

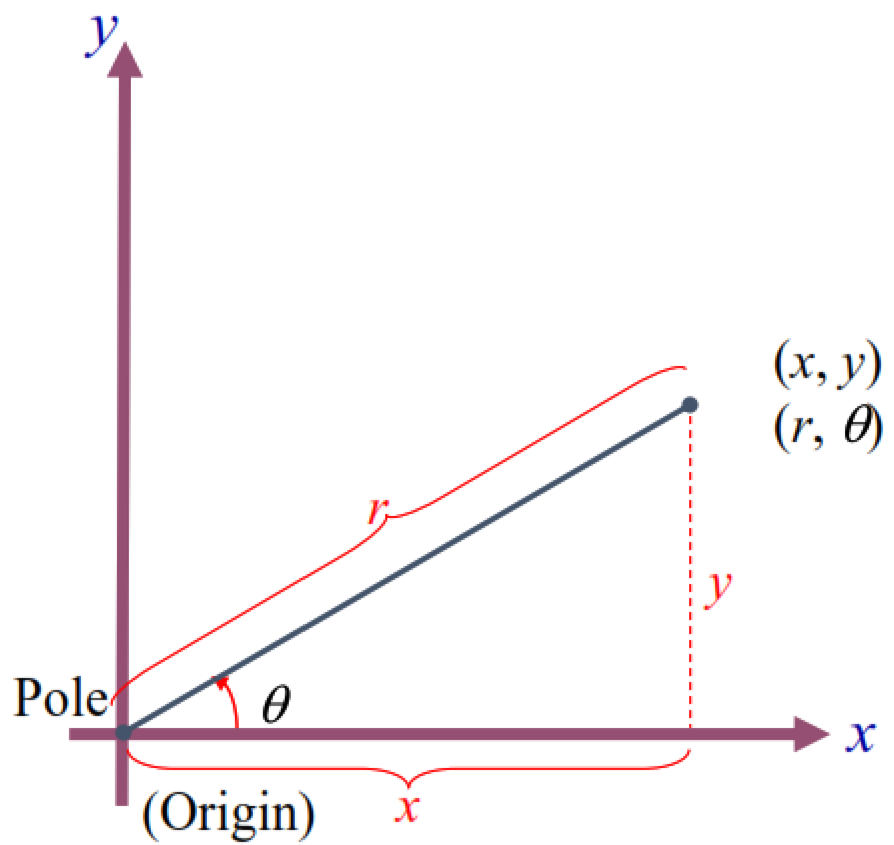
If *P* is any other point in the plane, then its location is determined by the ordered pair $(r, \theta)$ where *r* is the distance from *O* to P and $\theta$ is the angle between the polar axis and the line *OP*.



# The relation between rectangular and polar coordinates.

If the point $P$ has Cartesian coordinates $(x, y)$ and polar coordinates $(r, \theta)$, then, from the figure, we have

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$




and so

$$\cos\theta = \frac{x}{r} \longrightarrow \boxed{x = r\cos\theta}$$

$$\sin\theta = \frac{y}{r} \longrightarrow \boxed{y = r\sin\theta}$$

$$\tan\theta = \frac{y}{x}, \qquad \boxed{r^2 = x^2 + y^2}$$ (Pythagorean Identity)

$(\boldsymbol{r}, \boldsymbol{\theta})$

**Example 6** Convert the point $\boxed{\left(4, -\frac{\pi}{3}\right)}$ into rectangular coordinates.

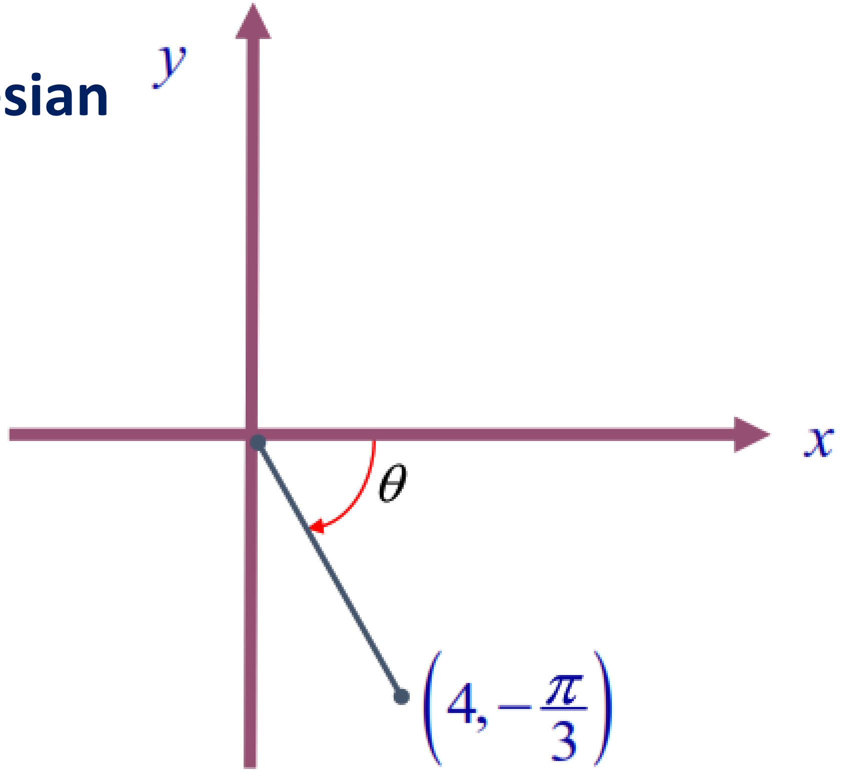
**Cartesian**

**Solution**

$$x = r\cos\theta = 4\cos\left(-\frac{\pi}{3}\right) = 4\left(\frac{1}{2}\right) = 2$$

$$y = r\sin\theta = 4\sin\left(-\frac{\pi}{3}\right) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

$$(x, y) = \left(2, -2\sqrt{3}\right)$$



**Example 7** Convert the point $(1,1)$ into polar coordinates.

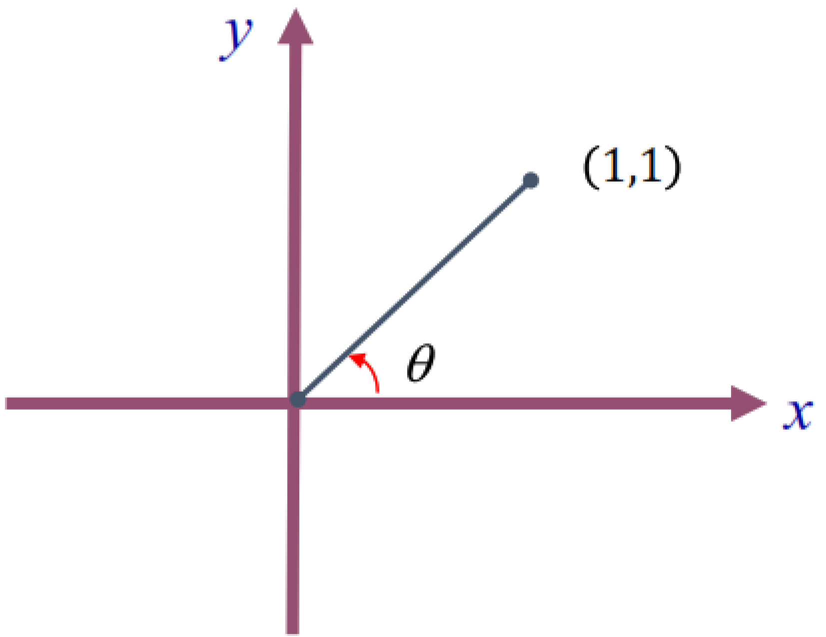
**Solution**

$$(x,y)=(1,1)$$

$$\tan\theta=\frac{y}{x}=\frac{1}{1}=1$$

$$\theta=\tan^{-1}(1)=45^{o}=45*\frac{\pi}{180}=\frac{\pi}{4}$$

$$r=\sqrt{x^2+y^2}=\sqrt{1^2+1^2}=\sqrt{2}$$




Then, the corresponding polar coordinates are $(r,\theta)=\left(\sqrt{2},\frac{\pi}{4}\right)$.

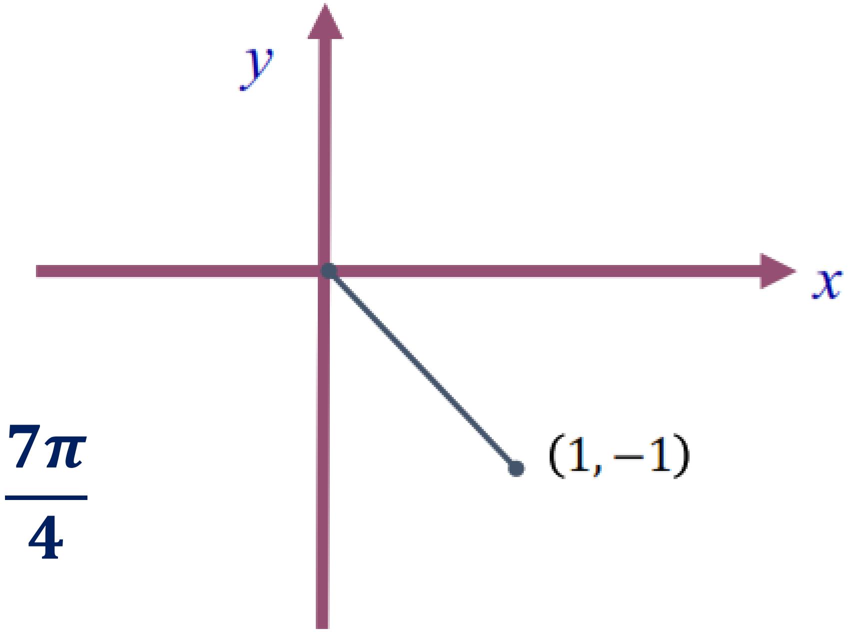
**Example 8** Convert the point (1,-1) into polar coordinates.

**Solution**

$$(x, y) = (1, -1)$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$$

$$\boldsymbol{\theta = \tan^{-1}(1) = 360 - 45^{o} = 315 = 315 * \frac{\pi}{180} = \frac{7\pi}{4}}$$




$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Then, the corresponding polar coordinates are $(r, \theta) = \left(\sqrt{2}, \frac{7\pi}{4}\right)$.

**Example 9** Convert the polar equation $r=\dfrac{4}{1+\sin\theta}$ into a rectangular equation.

**Solution**

$$r=\frac{4}{1+\sin\theta}$$ Polar form

$$r+r\sin\theta=4$$ Multiply each side by the denominator.

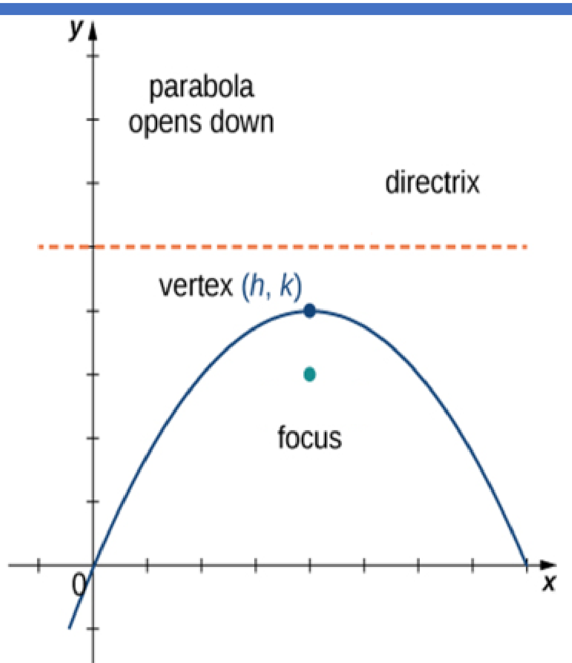
$$\sqrt{x^2+y^2}+y=4$$ Substitute rectangular coordinates $r=\sqrt{x^2+y^2}$ and $y=r\sin\theta$.

$$\Rightarrow \sqrt{x^2+y^2}=4-y$$
$$\Rightarrow x^2+y^2=(4-y)^2$$
$$\Rightarrow x^2+y^2=16-8y+y^2$$

$$x^2=-8(y-2)$$

Case 2

| Horizontal axis of symmetry $(x-h)^2=-4a(y-k)$ | |
|---|---|
| **Vertex** | (h, k) |
| **Focus** | (h, k-a) |
| **Directrix** | y = k+a |
| **Axis of symmetry** | x=h |



# Double integrals

In mathematics a multiple (iterated) integral is a definite integral of a function of several real variables, for instance , $f(x,y)$ or $f(x,y,z)$ . Integrals of a function of two variables over a region in $\mathbb{R}^2$ (the real-number plane) are called double integrals.

Suppose that $f$ is a function of two variables that is integrable on the rectangle

$$R=[a,b]\times[c,d]$$

The double integral of $f$ over the region R is defined as follows:

$$\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

Outer integral inner integral

**Example 10** Evaluate the iterated integral $\int_0^3 \int_1^2 x^2 y\,dy\,dx$

**Solution**

$$\int_0^3 \int_1^2 x^2 y\,dy\,dx = \int_0^3 \left[ x^2 \frac{y^2}{2} \right]_1^2 dx$$

$$= \int_0^3 \left[ x^2 \left( \frac{2^2}{2} \right) - x^2 \left( \frac{1^2}{2} \right) \right] dx$$

$$= \frac{3}{2} \int_0^3 x^2 dx$$

$$= \frac{3}{2} \left[ \frac{x^3}{3} \right]_0^3 = \frac{27}{2}$$

**Example 11** Evaluate the iterated integral $\iint_R (x-3y^2)\,dA$, where

$$R=\{(x,y):0\le x\le 2, 2\le y\le 3\}.$$

**Solution**

$$\iint_R (x-3y^2)\,dA=\int_0^2\int_2^3 (x-3y^2)\,dy\,dx=\int_0^3 \left[xy-y^3\right]_2^3 dx$$

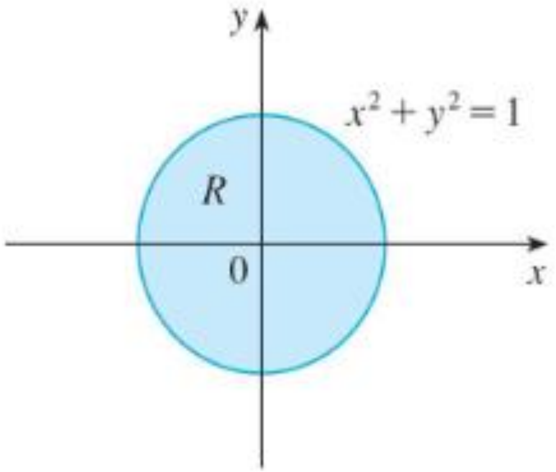
$$=\int_0^3\left[(3x-3^3)-(2x-2^3)\right]dx$$

$$=\int_0^3 (x-19)\,dx=\left[\frac{x^2}{2}-19x\right]_0^3$$

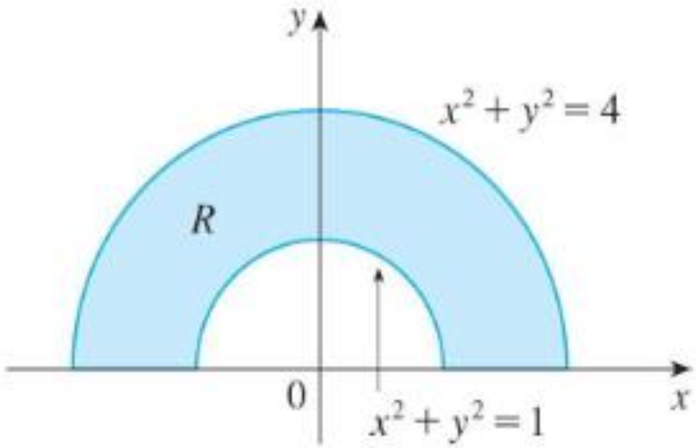
$$=\frac{9}{2}-57=-\frac{105}{2}.$$

# Double integrals in polar coordinates

- Suppose that we want to evaluate a double integral $\iint_R f(x,y)\,dA$, where $R$ is one of the regions shown in




(a) $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$




(b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

- In either case, the description of $R$ in terms of rectangular coordinates is rather complicated but $R$ is easily described by polar coordinates.

# Change to polar coordinates.

- To convert from rectangular to polar coordinates in a double integral by:
  - Writing $x = r\cos\theta$ and $y = r\sin\theta$
  - Using the appropriate limits of integration for $r$ and $\theta$
  - Replacing $dA$ by $r\,dr\,d\theta$

**Example 12** Evaluate $\iint_R (1 - x^2 - y^2)\, dA$

where the region is the circular disk $R$ given by $x^2 + y^2 \le 1$.

**Solution**

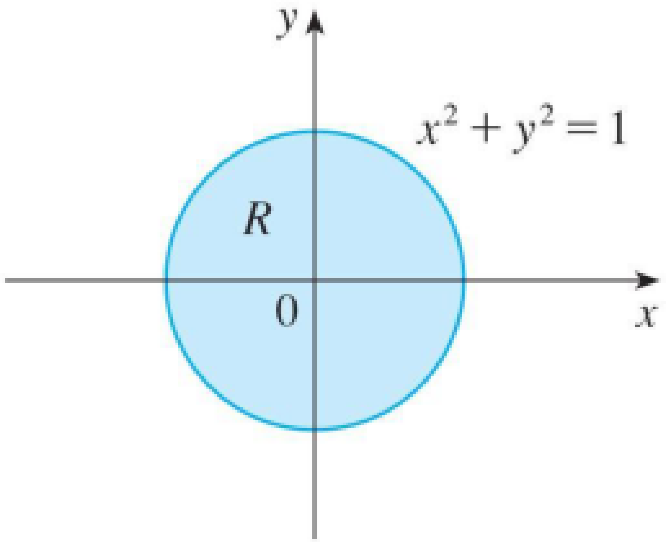
In polar coordinates, $R$ is given by $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

$$\iint_R (1-x^2-y^2)\, dA = \int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r-r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} d\theta = \frac{1}{4}(2\pi) = \frac{\pi}{2}$$




(a) $R = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

**Example 13** Evaluate $\iint_R (3x+4y^2)\,dA$

where $R$ is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

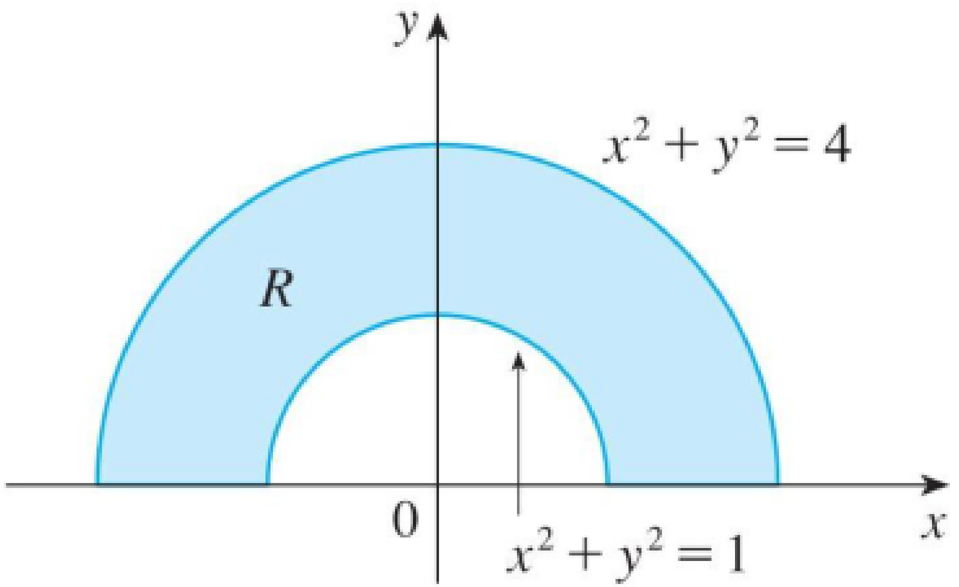
**Solution**

- The region $R$ can be described as:

$$R = \{(x, y) \mid y \geq 0,\ 1 \leq x^2 + y^2 \leq 4\}$$

- It is the half-ring shown in Opposite figure.
- In polar coordinates, it is given by:

$$1 \leq r \leq 2,\ 0 \leq \theta \leq \pi$$




(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$

Then, we have

$$\iint_R (3x+4y^2)\,dA = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{\pi} [r^3\cos\theta + r^4\sin^2\theta]_{r=1}^{r=2}\,d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta]\,d\theta$$

$$= \int_0^{\pi} [7\cos\theta + \tfrac{15}{2}(1-\cos 2\theta)]\,d\theta$$

$$= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\Big]_0^{\pi} = \frac{15\pi}{2}$$

# Exercises

1- Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter

$$(a)\ x = t^4 + 1, y = t^3 + t; \qquad t = -1.$$

$$(b)\ x = 2t^2 + 1, y = \frac{1}{3}t^3 - t; \qquad t = 3.$$

2- Find the points on the curve where the tangent is horizontal or vertical.

$$(a)\ x = 10 - t^2, y = t^3 - 12t.$$

$$(b)\ x = 2\cos(t), y = \sin(2t).$$

3-Evaluate the following double integrals

$$\iint_R x \sec^2 y \, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant \pi/4\}$$

$$\iint_R (y + xy^{-2}) \, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$$

$$\iint_R \frac{xy^2}{x^2 + 1} \, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 1, -3 \leqslant y \leqslant 3\}$$

4-Evaluate the following integral by changing to polar coordinates

(*a*) $\iint_R x^2 y dA$, where $R$ is the top half of the disck with center at the origin and radius 5

(*b*) $\iint_R e^{-x^2-y^2} dA$, where $R$ is the region bounded by the semi-circle $x = \sqrt{4 - y^2}$ and the y-axis.