Objectives:

After completing this topic, you will be able to:

- Introduce the notion of the area.
- Understand methods of finding area.

Notion of the Area:

Applications of Definite Integrals:

The definite integral is useful for solving a large variety of applied problems. In this chapter we shall discuss area, volume, and lengths of curves.

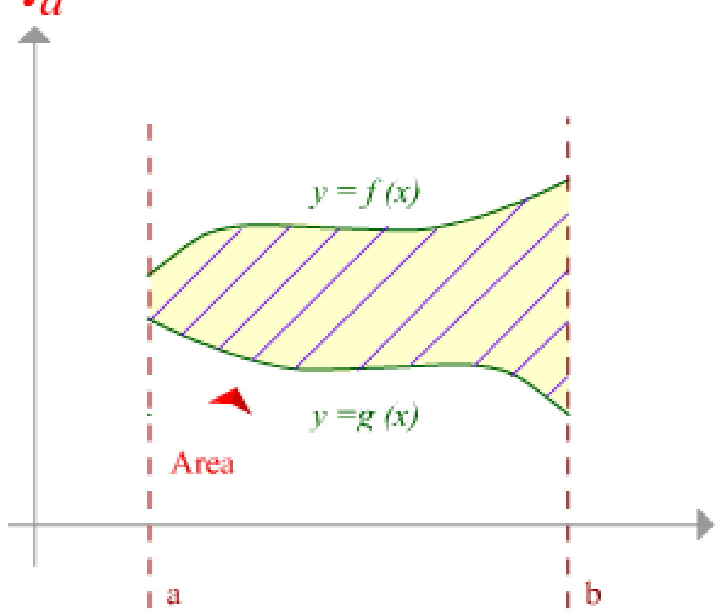
Here you find a brief introduction to applications of definite integrals and area between curves. We introduce first the notion of area bounded by the curve of the function, the χ axis, the lines $\chi = a$ and $\chi = b$.

This is mathematical and graphical illustration of area between curves.

Theorem:

If f and g are continuous and $f(\chi) \geq g(\chi)$ for all χ in $[a, b]$, then the area $\mathcal{A}$ of the region bounded by the graphs of $f, \chi, g = a$ and $\chi = b$

is $A = \int_a^b [f(x) - g(x)]dx$



$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

This formula for A can be extended to the case in which f or g is negative for some x in $[a,b]$.

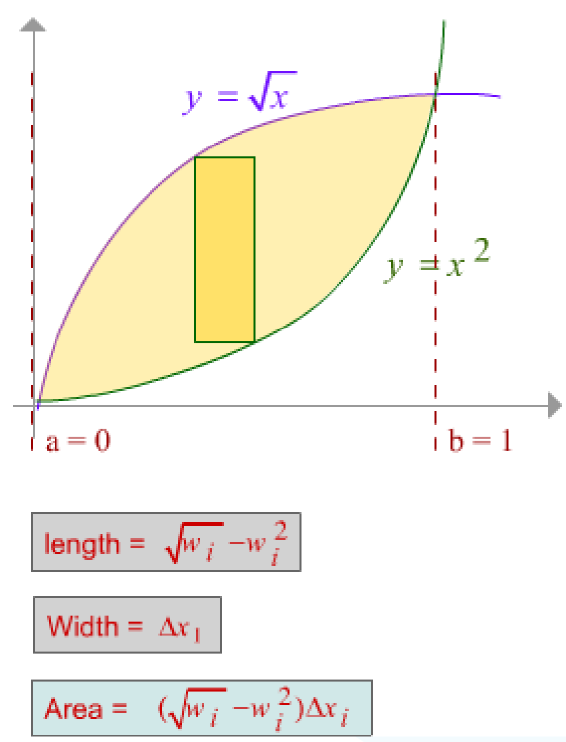
Example 1:

Find the area of the region bounded by the graphs of the equations $y = x^2$ and $y = \sqrt{x}$.

Solution:

We shall employ the Riemann sum approach.

The region and a typical rectangle are sketched in the following figure.



As indicated in the figure, the length of typical rectangle is $\sqrt{w_i} - w_i^2$ and its area is $(\sqrt{w_i} - w_i^2)\Delta x_i$.

Using the theorem with $a = 0$ and $b = 1$ we obtain

$$A = \lim_{\|p\| \to 0} \sum_i (\sqrt{w_i} - w_i^2)\Delta x_i \qquad = \int_0^1 (\sqrt{x} - x^2)dx$$

$$= [\frac{2}{3}x^{3/2} - \frac{1}{3}x^3]_0^1 \quad = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

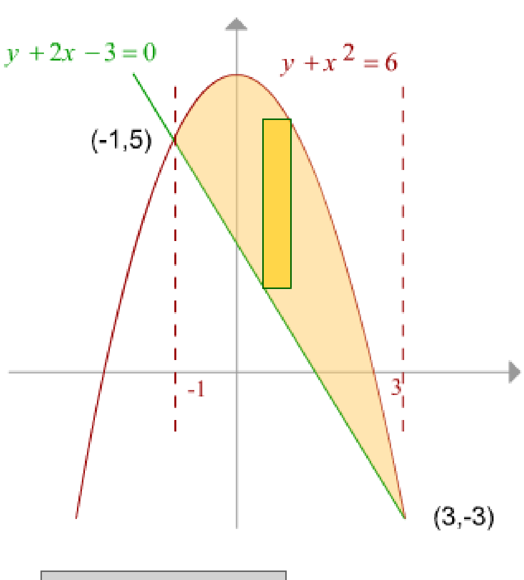
The area can be found by direct substitution in the theorem with $f(x) = \sqrt{x}$ and $g(x) = x^2$

Example 2:

Find the area of the region bounded by the graphs of $y+x^2=6$ and $y+2x-3=0$

Solution:

The region and a typical rectangle are sketched in the figure.



The points of intersection $(-1,5)$ and $(3,-3)$ of the two graphs may be found by solving the two given equations simultaneously.

It is necessary to solve each equation for y terms of x, obtaining $y=6-x^2$ and $y=3-2x$

The function $f(x)=6-x^2$ and $g(x)=3-2x$

As shown in the figure the length of a typical rectangle is $(6-w_i^2)-(3-2w_i)$

Where is some number in the subinterval of a partition $\mathcal{P}$ of $[-1,3]$ the area of this rectangle is

$$A=\lim_{\|p\|\to 0}\sum_i[(6-w_i^2)-(3-2w_i)]\Delta x_i$$

$$=\int_{-1}^{3}[(6-x^2)-(3-2x)]dx$$

Then $$=\int_{-1}^{3}(3-x^2+2x)dx$$

$$=[3x-\frac{x^3}{3}+x^2]_{-1}^{3}$$

$$=[9-\frac{27}{3}+9]-[-3-(-\frac{1}{3})+1]=\frac{32}{3}$$

Example 3:

Find the area of the region bounded by the graphs of the equations $2y^2=x+4$ and $x=y^2$

Solution:

One of two sketches of the region can be used to find the area, we use the integration with respect to y to find the area with only one integration.

Letting $f(y)=y^2,\quad g(y)=2y^2-4$, the length $f(w_i)-g(w_i)$ of a horizontal rectangle is $w_i^2-(2w_i^2-4)$ since the width is Δy the area of the rectangle is

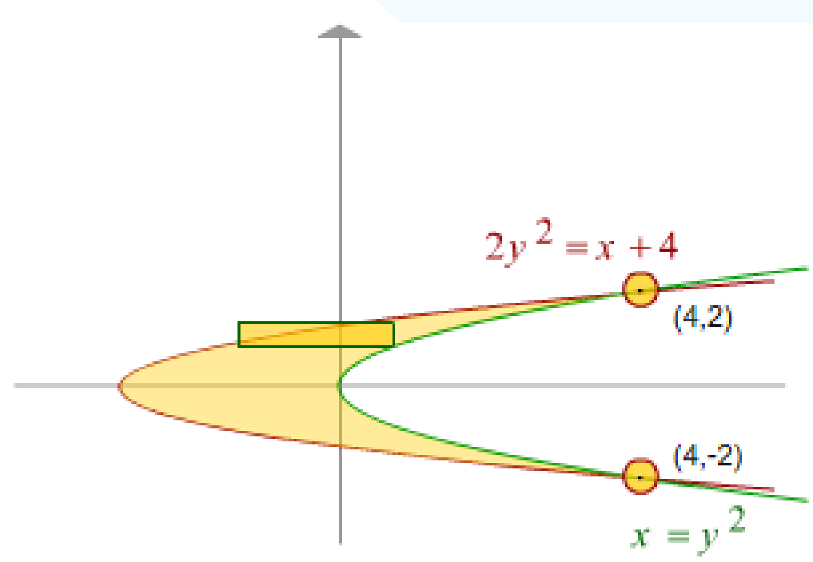
Hence, the area of $\mathcal{R}$ is $[w_i^2-(2w_i^2-4)]\Delta y_i$

$$A=\lim_{\nabla y\to 0}\sum_i[w_i^2-(2w_i^2-4)\Delta y_i$$

$$=\int_{-2}^{2}[y^2-(2y^2-4)]dy$$

$$=\int_{-2}^{2}(4-y^2)dy$$

$$=[4y-\frac{y^3}{3}]_{-2}^{2}=[8-\frac{8}{3}]-[-8-(-\frac{8}{3})]=\frac{32}{3}$$



Length $=w_i^2-(2w_i^2-4)$

Width $=\Delta y_i$

Objectives:

After completing this topic, you will be able to:

- Explain the concepts of volume of a solid.
- Show how the volume of the solid can be generated.
- Evaluate volumes of solid of revolution.

Definition of a volume of solid of revolutions:

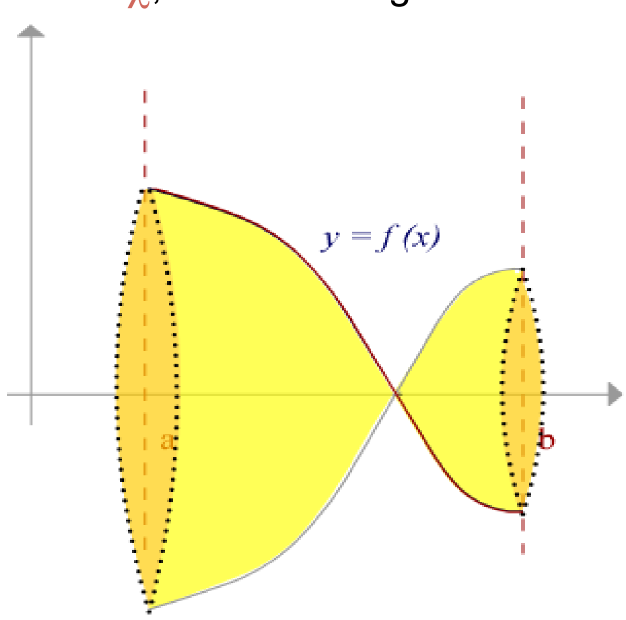
Definition:

Let f be continuous on $[a, b]$. The volume V of the solid of revolution generated by revolving the region bounded by the graphs of $f, \chi = a, \chi = b$ and the χ-axis is $V = \lim_{\|p\|\to 0} \sum_i \pi [f(w_i)]^2 \Delta x_i = \int_a^b \pi [f(x)]^2 dx$

In fact that the limit of the sum in the definition equals $\int_a^b \pi [f(x)]^2 dx$ follows from the definition of the definite integral.

The requirement that $f(\chi) \geq 0$ for all χ in $[a, b]$, was omitted in the definition.

If f is negative for some χ, and if the region bounded by the graphs of $f, \chi = a \;\; \chi = b$, and the χ-axis figure (i), a solid of the type shown in the figure (ii) is obtained.

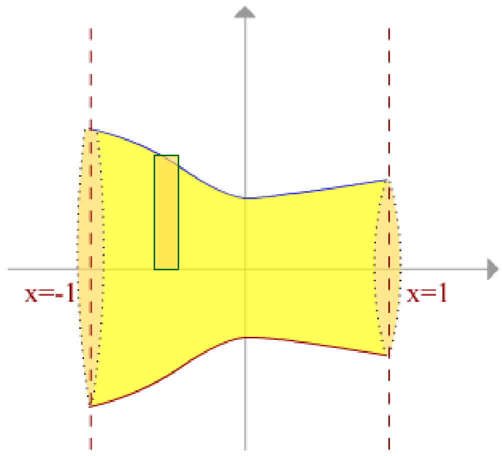


Example 4:

If $f(\chi) = \chi^2 + 1$, find the volume of the solid generated by revolving the region under the graph of f from -1 to 1 about the χ-axis.

Solution:

The solid is illustrated in the following figure included in the sketch is a typical rectangle and the disk that it generates.



Since the radius of the disc that is $w_i^2 + 1$, its volume is $\pi (w_i^2 + 1) \Delta\chi_i$

and $V = \lim_{\|p\|\to 0} \sum_i \pi (w_i^2 + 1)^2 \Delta x_i$

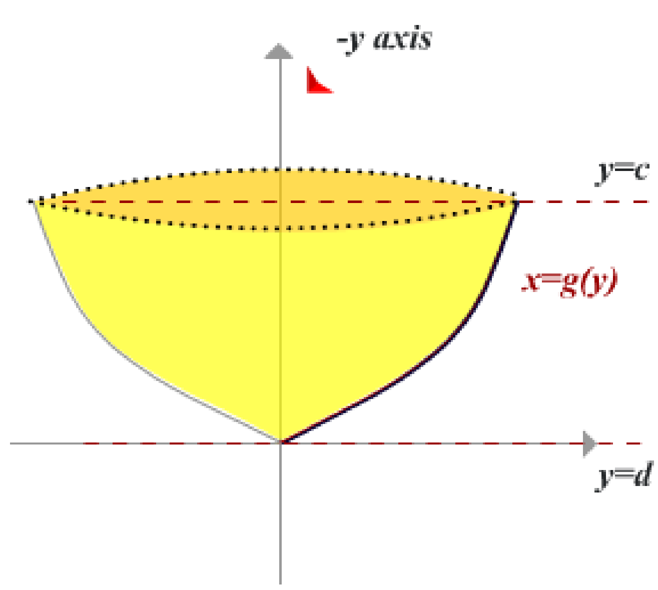
$$= \int_{-1}^{1} \pi (x^2 + 1)^2 dx = \pi \int_{-1}^{1} (x^4 + 2x^2 + 1) dx$$

$$= \pi [\frac{1}{5}x^5 + \frac{2}{3}x^3 + x]_{-1}^{1}$$

$$= \pi [(\frac{1}{5} + \frac{2}{3} + 1) - (-\frac{1}{5} - \frac{2}{3} - 1)] = \frac{56}{15}\pi$$

Definition:

Let g be continuous $[a, b]$. The volume V of t revolution generated by revolving the region bounded by the graphs of $\chi = g(y), y = c, y = d$ and the y-axis is $V = \lim_{\|p\|\to 0} \sum_i \pi [g(w_i)]^2 \Delta y_i = \int_c^d \pi [g(y)]^2 dy$



Example 5:

The region bounded by the y-axis, the graph of $y = \chi^3$, $y = 1$ and $y = 8$ is revolved about the y-axis. Find the volume of the resulting solid.

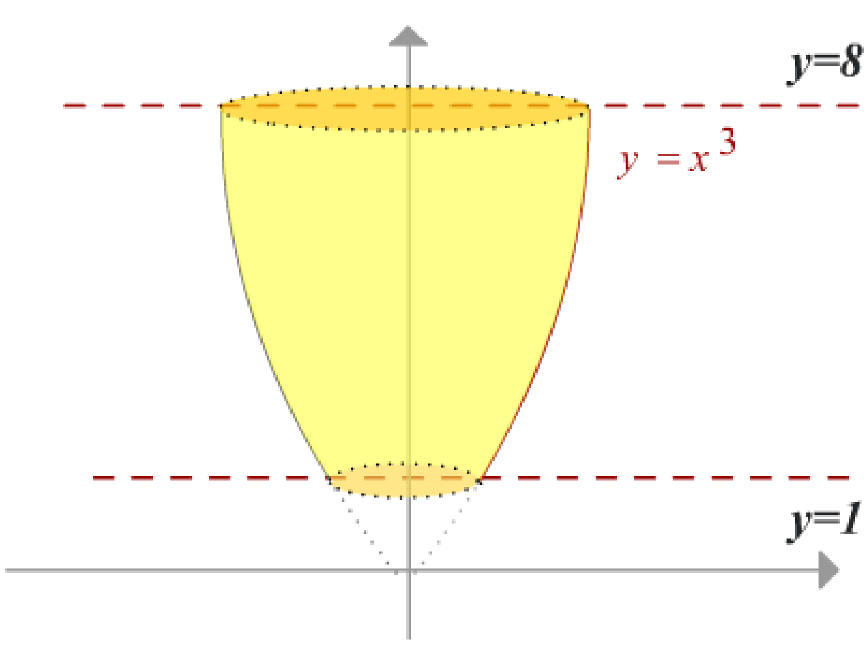
Solution:

The solid is sketched together with a disc generated by a typical rectangle. Since we plan to integrate with respect to y, we solve the equation $y = \chi^3$ for χ in terms of y ,obtaining $\chi = y\ 1/3$, and we let $\chi = g(y) = y\ 1/3$, then as shown in the figure, the radius of a typical disc is $g(w_i) = w_i^{1/3}$ and its volume is $\pi(w_i^{1/3})^2 \Delta y_i$ applying the definition with $g(y) = y\ 1/3$ gives us

$$V = \lim_{\|P\| \to 0} \sum_i \pi (w_i^{1/3})^2 \Delta y_i$$

$$= \int_1^8 \pi (y^{1/3})^2 dy = \pi \int_1^8 y^{2/3} dy$$

$$= \pi(\frac{3}{5})[y\ 5/3]_1^8 \quad = \frac{3}{5}\pi[8^{5/3} - 1] = \frac{93}{5}\pi$$



Example 6:

The region bounded by the graphs of the equations

$$x^2 = y - 2, 2y - x - 2 = 0, x = 0, and\ x = 1$$

is revolved about the χ-axis. Find the volume of the resulting solid.

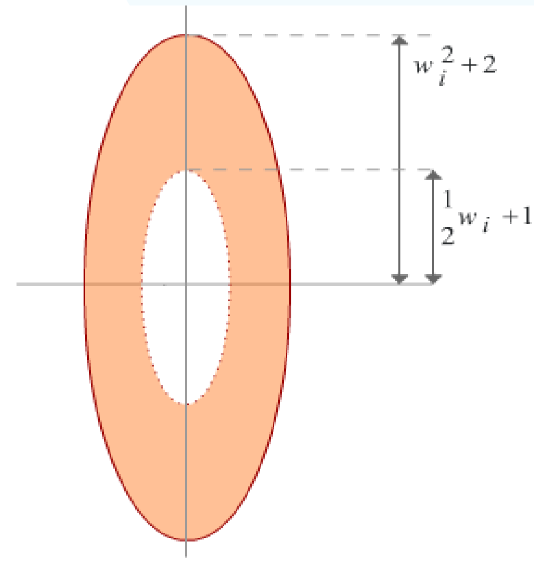
Solution:

The region and a typical rectangle are sketched in (i) then we wish to integrate with respect to χ we solve the first two equations for y in terms of χ, obtaining $y = x^2 + 2 and\ y = \frac{1}{2}x + 1.$

The generated by the rectangle in (i) is illustrated in (ii).

Since outer radius of the washer is $w_i^2 + 2$ and the inner radius is $1/2\ w_i + 1$,

its volume is $\pi[(w_i^2 + 2)^2 - (\frac{1}{2}w_i + 1)^2]\Delta x_i$



Taking the limits of the sum of such volumes gives us $V = \int_0^1 \pi[(x^2 + 2)^2 - (\frac{1}{2}x + 1)^2]dx$

$$= \pi \int_0^1 (x^4 + \frac{15}{4}x^2 - x + 3)dx$$

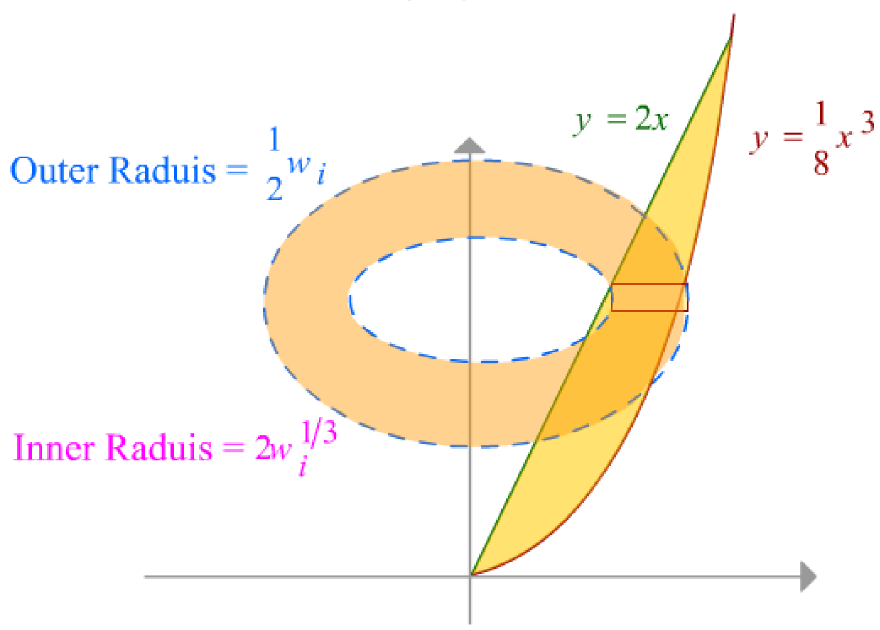
$$= \pi[\frac{1}{5}x^5 + \frac{5}{4}x^3 - \frac{1}{2}x^2 + 3x]_0^1 = \frac{79\pi}{20}$$

Example 7:

The region in the first quadrant bounded by the graphs of $y = \frac{1}{8}x^3$ and $y = 2\chi$ is revolved about the y-axis. Find the volume of the resulting solid.

Solution:

As shown in the following figure



The inner and outer radii of the washer generated by the rectangle are $\frac{1}{2}w_i$ and $2w_i^{1/3}$ respectively.

Since the thickness is Δy_i it follows that the volume of the washer is

$$\pi[(2w_i^{1/3})^2 - (\frac{1}{2}w_i)^2]\Delta y_i = \pi[4w_i^{2/3} - \frac{1}{4}w_i^2]\Delta y_i$$

Taking a limit of a sum of such terms gives us

$$V = \int_0^8 \pi[4y_i^{2/3} - \frac{1}{4}y_i^2]dy_i = \pi[\frac{12}{5}y^{5/3} - \frac{1}{12}y^3]_0^8$$

$$== \pi[\frac{12}{5}(8^{5/3}) - \frac{1}{12}(8^3)]_0^8 = \frac{512}{15}\pi$$

Objectives:

After completing this topic, you will be able to:

- Introduce the concepts of length of curves.
- Calculate length of curves.

Arc Length:

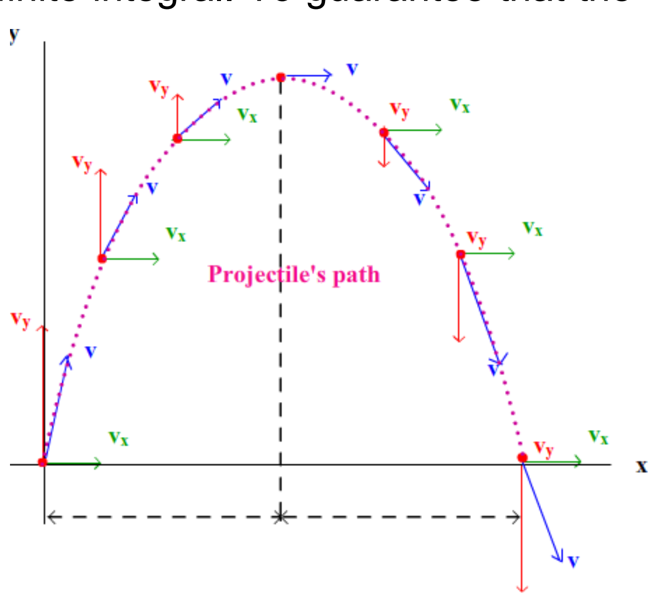
To solve certain problem in the sciences it is essential to consider the length of the graph of a function.

For example, if a projectile moves along a parabolic course, we may wish to determine the distance it travels during a specified interval of time.

Similarly, it may be necessary to find the length of a twisted piece of wire. We could simply straighten it and find the linear length with a ruler (or by mean of the distance formula).

As we shall see, the key to defining the length of a graph is to divide the graph into many small pieces and then approximate each piece by means of a line segments.

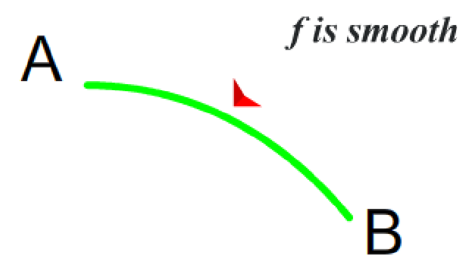
This lead to a definite integral. To guarantee that the integral exists, its necessary to place restrictions on the function, as indicated in the following discussion.

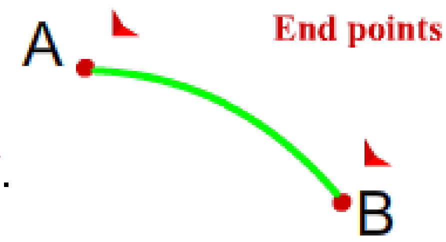


A function f is said to be smooth on an interval if it has a derivative $f^{/}$ that is continuous throughout the interval.

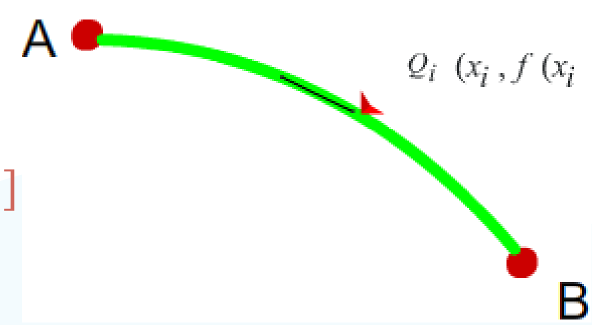
We intend to define what is meant by the length of arc between two point A and B on the graph of a smooth function.

If f is smooth on a closed interval $[a, b]$, the points $A(a, f(a))$ and $B(b, f(b))$ will be called the endpoints of the graph f.





Let us consider the partition P of $[a, b]$ determined by



$a = \chi_0, \chi_1, \chi_2, \ldots\ldots\ldots \chi_n = b$ and let Q_i denote the point with coordinates $(\chi_i, f(\chi_i))$ this gives us $n+1$ points $Q_0, Q_1, Q_2, \ldots\ldots\ldots Q_n$ on the graph of f, we connect each Q_{i-1} to Q_1 by line segment of length $d(Q_{i-1}, Q_1)$ then the length L_p of the resulting broken line is $L_P = \sum_{i=1}^{n} d(Q_{i-1}, Q_i)$

If the norm $\|P\|$ of the partition is small, then Q_{i-1} is close to Q_i for each i and we expect L_p to be an approximation to the length of arc between A and B.

This gives us a clue to suitable definition of arc length. Specifically, we shall consider the limit of the sum L_p as $\|P\| \to 0$ to formulate this concept precisely, and at the same time arrive at a formula for calculating arc length. By the Distance Formula

$$d(Q_{i-1}, Q_i) = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

Applying the mean value theorem

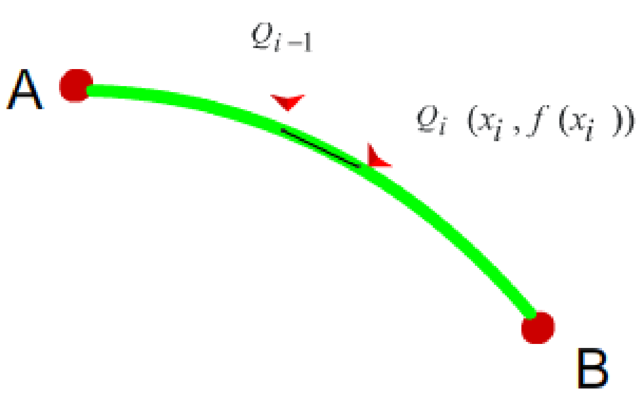
$$f(x_i)-f(x_{i-1})=f'(w_i)(x_i-x_{i-1})$$

where w_i is an open interval (x_{i-1}, x_i).

Substituting this into the preceding formula and letting $\Delta x_i = x_i - x_{i-1}$, we obtain

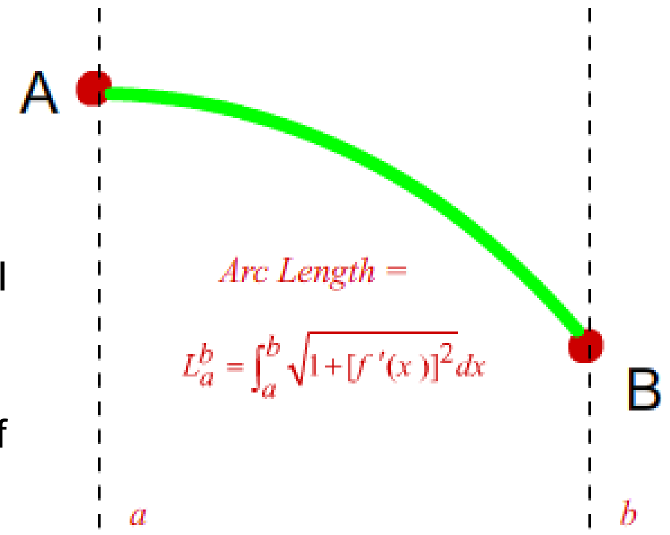
$$d(Q_{i-1},Q_i)=\sqrt{(\Delta x_i)^2+[f'(w_i)\Delta x_i]^2}$$
$$=\sqrt{1+[f'(w_i)]^2}\,\Delta x_i$$

Consequently, $L_P=\sum_{i=1}^{n}\sqrt{1+[f'(w_i)]^2}\,\Delta x_i$



Observe that L_p is a Riemann sum for the function g defined by $g(x)=\sum_{i=1}^{n}\sqrt{1+[f'(x)]^2}$ the limit of the sum is defined the arc length of the graph f from A to B. Since $g=\sqrt{1+(f')^2}$ is a continuous function, the limit exists and equals the definite integral $\int_a^b\sqrt{1+[f'(x)]^2}dx$ this arc length will be denoted by the symbol L^b_a.

Definition:



Let the function f be smooth on a closed interval $[a, b]$.

The arc length of the graph of f from $A(a, f(a))$ and $B(b, f(b))$ is given by

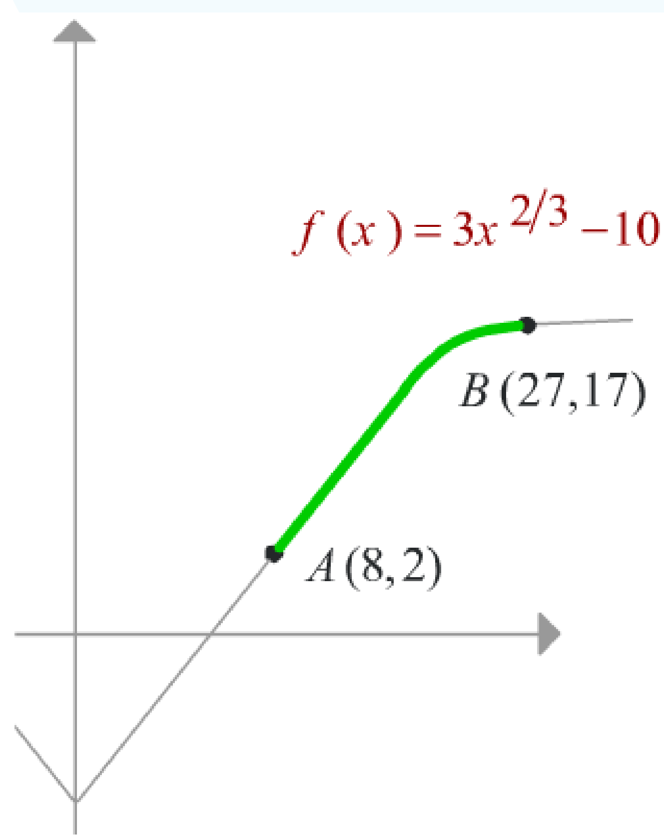
$$L_a^b=\int_a^b\sqrt{1+[f'(x)]^2}dx$$

Example 8:

If $f(x)=3x^{2/3}-10$, find the arc length of the graph of f from the point $A(8, 2)$ to $B(27, 17)$.

Solution:

The graph f is sketched in the opposite figure



$$L_8^{27}=\int_8^{27}\sqrt{1+(\frac{2}{x^{1/3}})^2}dx=\int_8^{27}\sqrt{1+\frac{4}{x^{2/3}}}dx$$
$$=\int_8^{27}\frac{\sqrt{x^{2/3}+4}}{x^{2/3}}dx$$

To evaluate this integral,

let $u = x^{2/3} + 4 \quad and \ du = \frac{2}{3} x^{-1/3} dx$

Then $L_8^{27} = \frac{3}{2} \int_8^{27} \sqrt{x^{2/3} + 4} \left(\frac{2}{3x^{1/3}}\right) dx$

If $\chi = 8$ then $u = (8)^{2/3} + 4 = 8,$

whereas if $\chi = 27$ then $u = (27)^{2/3} + 4 = 13$

Making substitution and changing the limits of

integration

$$L_8^{27} = \frac{3}{2} \int_8^{13} \sqrt{u} du = u^{3/2}]_8^{13} = 13^{3/2} - 8^{3/2} \approx 24.2$$

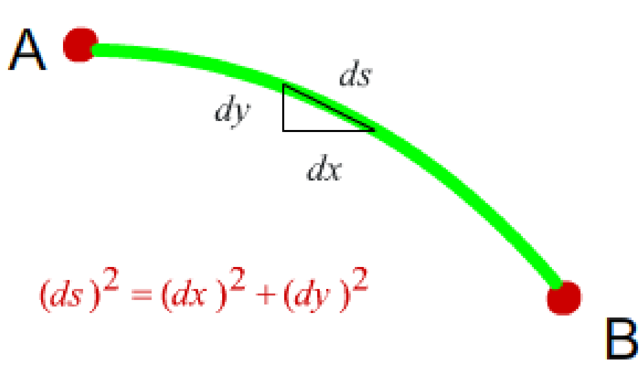
Definition:

Let the function f be smooth on a closed interval $[a, b]$.
The arc length function s for the graph of f on $[a, b]$
is given by

$$S(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt \qquad where \quad a \le x \le b$$

Theorem:

Let f be smooth $[a, b]$, and let s be the arc length for the graph of $y = f(\chi)$ on $[a, b]$.



If dx and dy are differentials of χ and y,

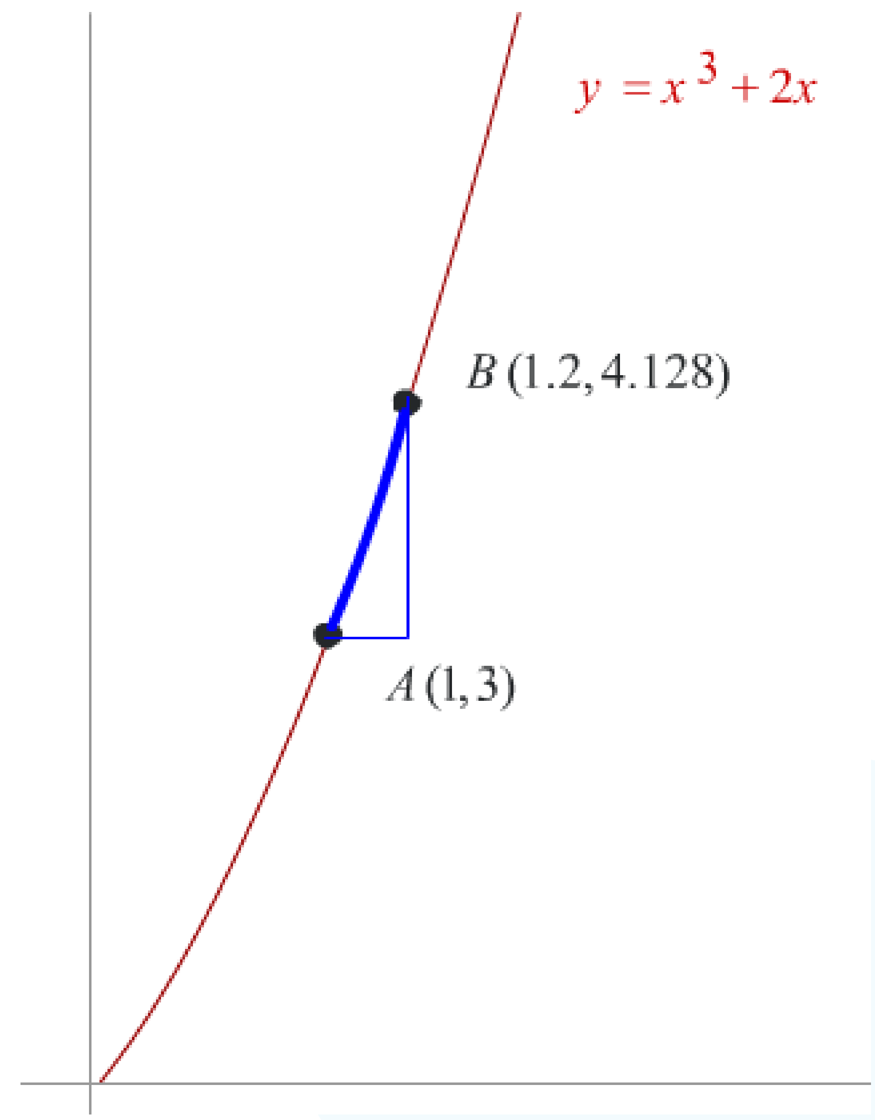
then
$$(i)\ ds = \int_a^x \sqrt{1 + [f'(x)]^2} dx$$
$$(ii)\ (ds)^2 = (dx)^2 + (dy)^2$$

Example 9:

Use differentials to approximate the arc length of

$$y = x^3 + 2x \quad from\ A(1,3)\ to\ B(1.2, 4.128).$$

Solution:



If we let $f(x) = x^3 + 2x$, then by (i) of the theorem $ds = \sqrt{1 + (3x^2 + 2)^2} dx$

An approximation may be obtained by letting $x = 1$ and $dx = 0.2$.

Thus $ds = \sqrt{1 + 5^2}(0.2) \approx 1.02$