
بسم الله الرحمن الرحيم

EELU

الجامعة المصرية للتعلم الإلكتروني

Egyptian E-Learning University

MATH – 1

BY

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CHAPTER 4

Integration

LECTURE 7.

Calculus and Area

Aims and Objectives:

(1) Understand the notion of the area.

(2) Use the concepts of integration.

(3) Understand the concepts of finite sums.

(4) Learn the definite integral as a limit of a sum.

(5) Gain experience in evaluating finite sums.

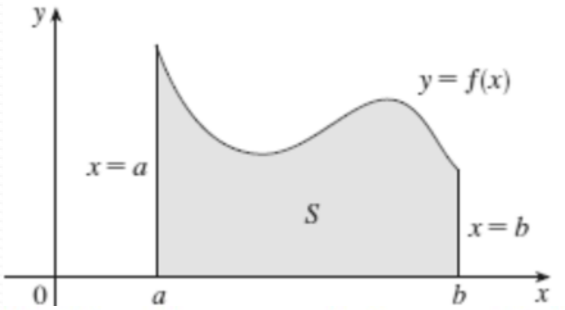
(6) Have a strong intuitive feeling for these important concepts.

- **CALCULUS IN COMPUTER SCIENCE**

Integral calculus

In this Lecture you will learn the concept which is the basis for ***integral calculus***:

the definite integral and related topics.



$$S=\{(x,y)\,|\,a\leq x\leq b,\,0\leq y\leq f(x)\}$$

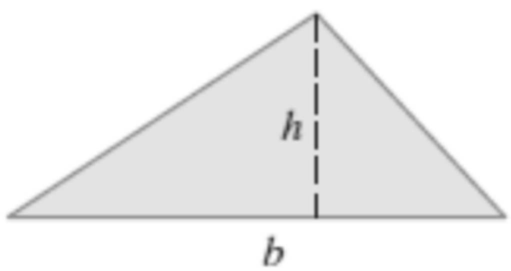
Consider the area of a region in a plane:

The area of the region S lies under the curve *y=f(x)* from *a* to *b*, which is bounded by the graph of a continuous function $f(x)\geq 0$, the vertical lines *x=a* and *x=b*, and the *x*-axis.

1. The area of a region with straight sides

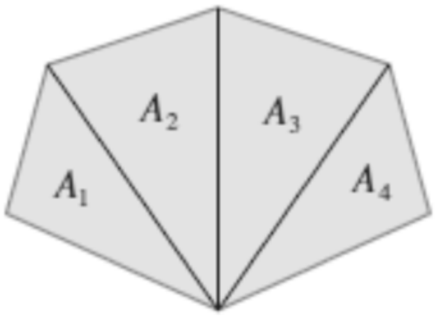
$$A=lw$$

The area of a rectangle is defined as the product of the length and the width.



$$A=\frac{1}{2}bh$$

The area of a triangle is half the base times the height.

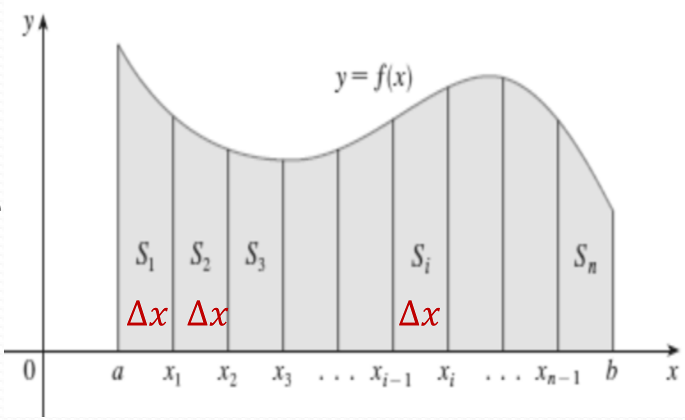


$$A=A_1+A_2+A_3+A_4$$

The area of a polygon is the sum of the triangles' areas.

2. The area of a region with curved sides

We first approximate the region *S* by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.



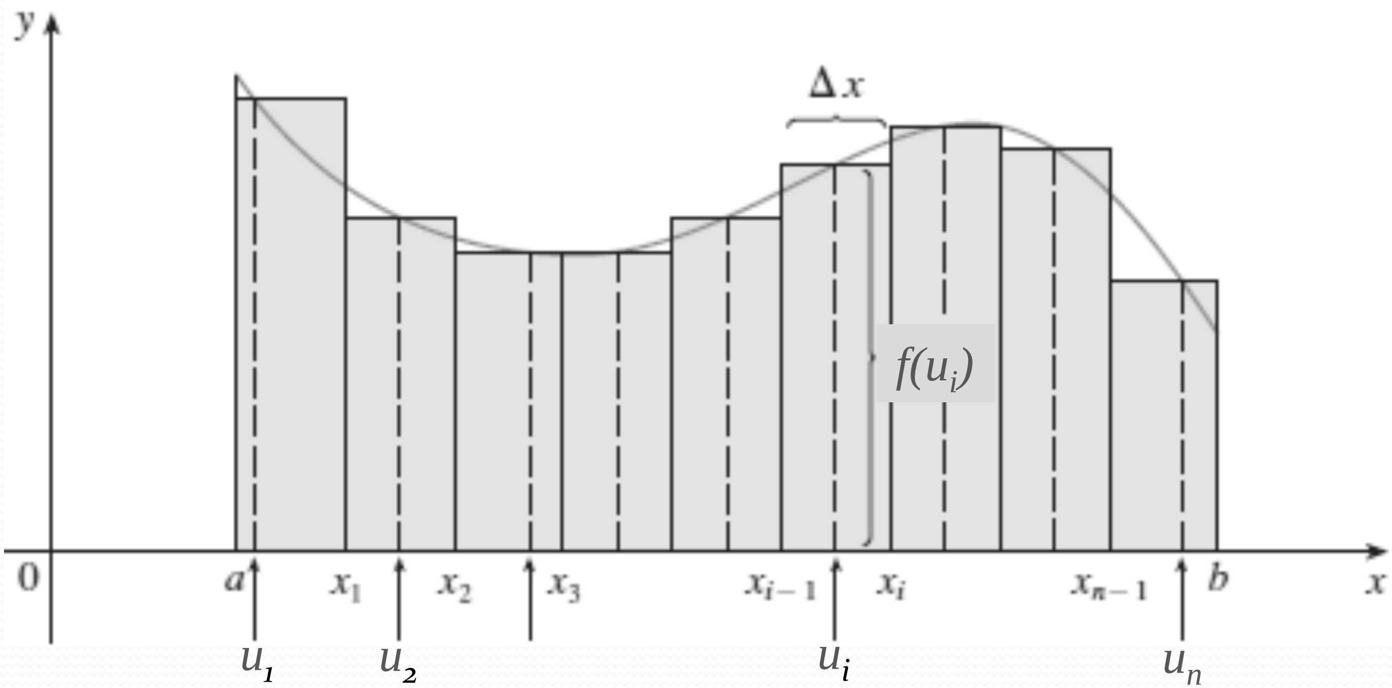
We start by subdividing into n strips $S_1, S_2, \ldots\ldots, S_n$ of equal width

$$x_i - x_{i-1} = \Delta x = \frac{b-a}{n},$$

such that the width of the interval [a, b] is b-a. These strips divide the interval [a, b] into n subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n],$$

where $x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots\ldots, x_i = a + i\Delta x, \ldots\ldots, x_n = a + n\Delta x = b$.



The area of the i^{th} rectangle is $f(u_i)\Delta x$.The boundary of the region formed by the totality of these rectangles is called the inscribed rectangle polygon associated with the subdivision of $[a, b]$ into n subintervals.

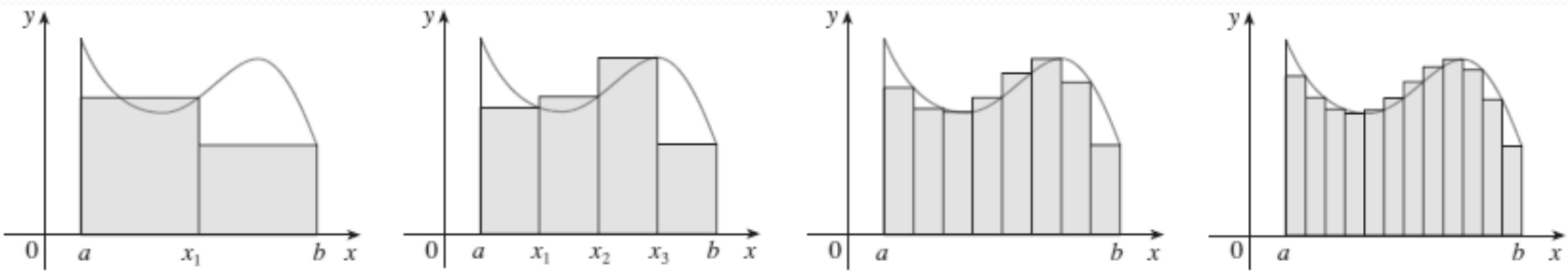
The area of this inscribed polygon is the sum of the areas of the rectangles, that is,

$$R_n = f(u_1)\Delta x + f(u_i)\Delta x + \cdots + f(u_n)\Delta x = \sum_{i=1}^{n} f(u_i)\Delta x$$

We can see that the area of S appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore we define the area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(u_1)\Delta x + f(u_i)\Delta x + \cdots + f(u_n)\Delta x]$$

$$= \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(u_i)\Delta x$$



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

The statement $A = \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(u_i)\Delta x$ means that for every $\varepsilon > 0$ there corresponds a $\gamma > 0$ such that if $0 < \Delta x < \gamma$, then $A - \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(u_i)\Delta x < \varepsilon$

we can observe that as Δx getting smaller, the value of the summation converges to the true value of the area.

Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\,\Delta x$$

Finite Sum Concept:

It is convenient to use summation notation, to illustrate, given a collection of numbers $\{1, 2, \ldots\ldots, a_n\}$,the symbol $\Sigma^n{}_{i=1}\, a_i$ represent their sum,

that is $\Sigma^n{}_{i=1}\, a_i = a_1 + a_2 + \ldots\ldots + a_n$

Where the Greek capital letter Σ indicates a sum, and the symbol a_i represent the ith term. The letter i is called the index of summation or the summation variable, and the numbers 1 and n indicates the extreme values of the summation variable.

A theorem concerning finite sums:

Theorem

If n is any positive integer and $\{a_1, a_2, \ldots\ldots, a_n\}$, $\{b_1, b_2, \ldots\ldots, b_n\}$ are sets of numbers, then

$$(i)\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$ii)\sum_{i=1}^{n} ca_i = c\Big(\sum_{i=1}^{n} a_i\Big), \text{ for any number } c;$$

$$iii)\sum_{i=1}^{n} c = nc$$

Now, the following definition will be useful in some illustrations.

$$(i)\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$(ii)\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(iii)\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Example 1:

Find $\Sigma^4_{i=1}\, i^2\,(i-3)$

Solution:

we merely substitute, in succession, the integers 1,2,3, and 4 for i and add the resulting terms. thus,

$$\Sigma^4_{i=1}\, i^2\,(i-3) = 1^2(1-3)+2^2(2-3)+3^2(3-3)+4^2(4-3)$$

$$= (-2) + (-4) + (0) + (16) = 10$$

Example 2:

Find $\displaystyle\sum_{i=0}^{3} \frac{2^i}{(i+1)}$

Solution:

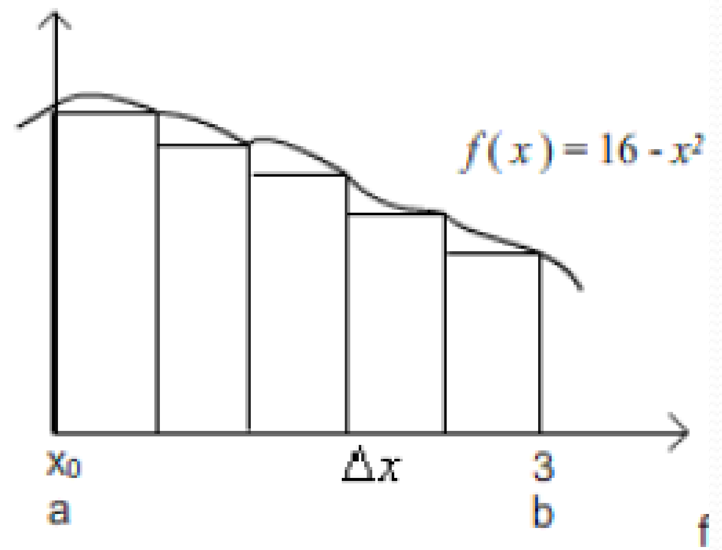
$$\sum_{i=0}^{3} \frac{2^i}{(i+1)} = \frac{2^0}{(0+1)} + \frac{2^1}{(1+1)} + \frac{2^2}{(2+1)} + \frac{2^3}{(3+1)}$$

$$= 1 + 1 + \frac{4}{3} + 2 = \frac{16}{3}$$

Example 3:

If $f(x) = 16 - x^2$, find the area of the region under the graph of f from 0 to 3.

Solution:



For the given region, the interval $[0,3]$ is divided into n equal subintervals, then the length Δx of a typical subinterval is $\Delta x = b–a / n=3/n$.

Since $x_0 = 0$, $x_1 = \Delta x, x_2 = 2\Delta x, \ldots, x_i = i\Delta x, \ldots, x_n = n\Delta x = 3$

Using the fact that $\Delta x = 3/n$ we may write

$$x_i = i(\Delta x) = i\left(\frac{3}{n}\right) = \frac{3i}{n}$$

Since f is decreasing on $[0, 3]$, the number u_i in $[x_{i-1}, x_i]$ at which f takes on its minimum value is always at the right-hand $f(u_i) = f\left(\frac{3i}{n}\right) = 16 - \left(\frac{3i}{n}\right)^2 = 16 - \frac{9i^2}{n^2}$

Using the idea of finite sum to approximate an area by dividing the area into a group of rectangles each has a small area and summing the areas of these rectangles.

$$\sum_{i=1}^{n} f(u_i)\Delta x = \sum_{i=1}^{n}\left(16 - \frac{9i^2}{n^2}\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{48}{n} - \frac{27i^2}{n^3}\right) = \left(\frac{48}{n}\right)n - \frac{27}{n^3}\sum_{i=1}^{n} i^2.$$

Remember that $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

In order to find the area, we must now let Δx approach 0. Since $\Delta x = (b-a)/n$, this can be accomplished by letting n increase without bound. And we can replace $\Delta x \to 0$ by $n \to \infty$, we have

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(u_i)\Delta x = \lim_{n\to\infty}\left\{48 - \frac{9}{2n^3}\left[2n^3 + 3n^2 + n\right]\right\} = \lim_{n\to\infty} 48 - \frac{9}{2}\lim_{n\to\infty}\left[\frac{2n^3+3n^2+n}{n^3}\right]$$

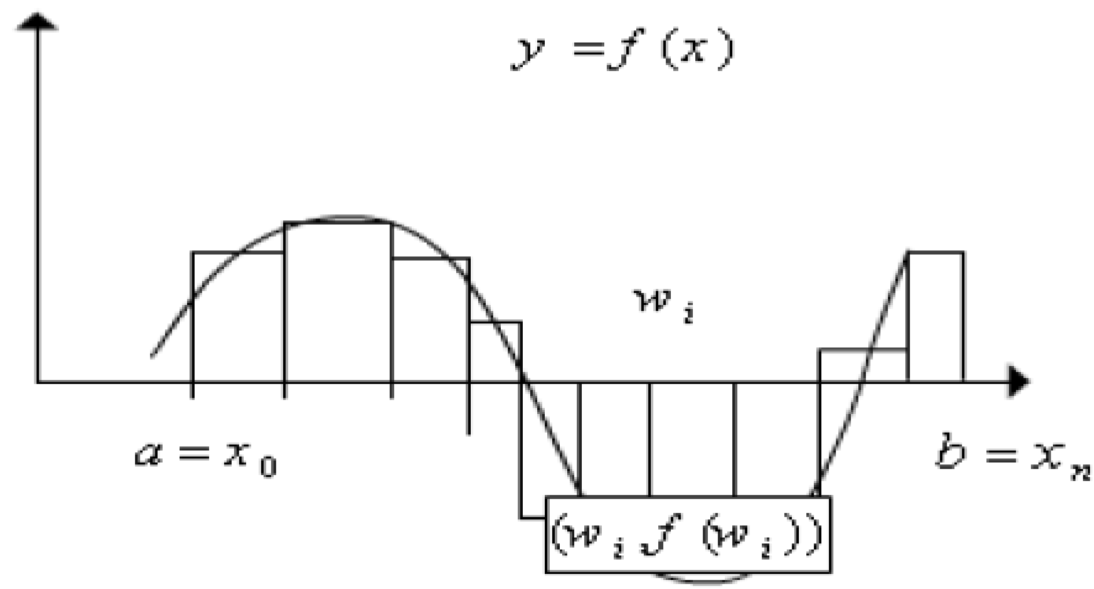
$$= 48 - \frac{9}{2}\lim_{n\to\infty}\left[2 + \frac{3}{n} + \frac{1}{n^2}\right] = 48 - \frac{9}{2}\left[2+0+0\right] = 48 - 9 = 39$$

Definition introducing the main concepts of definite integral:

Definition (4.2):

Let f be a function that is defined on a closed interval and let be a partition of $[a, b]$.

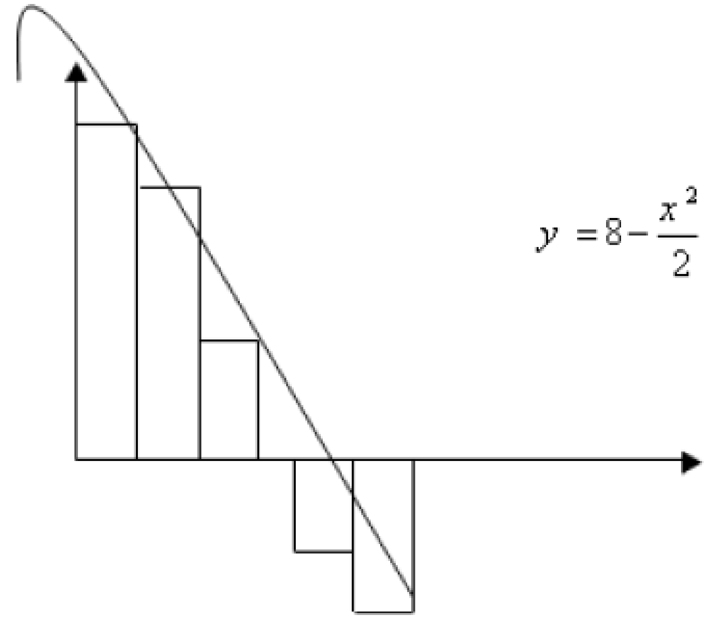
A Riemann sum of f for $\mathcal{P}$ is any expression of $\mathcal{R}_p$ the form $R_P = \sum_{i=1}^{n} f(w_i)\Delta x_i$



Example 4:

Suppose $f(x)=8-(x^2/2)$ and $\mathcal{P}$ is the partition of $[0,6]$ into the five subintervals determined by $x_0=0, x_1=1.5, x_2=2.5, x_3=4.5, x_4=5$, and $x_5=6$

Find (a) the norm of the partition and (b) the Riemann sum R_P if $w_1=1, w_2=2, w_3=3.5, w_4=5,\ w_5=5.5.$



Solution:

Solution: the graph of f is sketched in figure 4.5.

Also shown in the figure are the points on the x-axis that correspond to x_i and the rectangles of lengths $|f(w_i)|$ for $i=1,2,3,4,$ and $5.$

Thus, $\Delta x_1=1.5, \Delta x_2=1, \Delta x_3=2, \Delta x_4=0.5, \Delta x_5=1$

And hence the norm $\| P \|$ of the partition is Δx_3, or $2.$

By definition 4.2,

$$R_P=f(w_1)\Delta x_1+f(w_2)\Delta x_2+f(w_3)\Delta x_3+f(w_4)\Delta x_4+f(w_5)\Delta x_5$$

$$=f(1)(1.5)+f(2)(1)+f(3.5)(2)+f(5)(0.5)+f(5.5)(1)$$

$$=(7.5)(1.5)+(6)(1)+(1.875)(2)+(-4.5)(0.5)+(-7.125)(1)$$

Which reduces to $R_P=11.625$

THANK YOU