chapter(2): Basic Structures

- Sets.
- Functions.
- Sequences, Sums.

1 Sets :-

- A set is an unordered collection of objects.
- The objects in a set are called the elements or members.

- $S = \{a, b, c, d\}$

  $a \in S$. $e \notin S$

- The set A of odd integers positive integers less than 10.

  $A = \{1, 3, 5, 7, 9\}$

- The set B of positive integers less than 1000.

  $B = \{1, 2, 3, 4, \boxed{\cdots}, 999\}$

  ellipses (...)

*** Set builder notation :-

Ex: The set A of odd positive integers less than 10.

- $A = \{x \mid x$ is an odd positive integers less than $10\}$.
- $A = \{x \in Z^+ \mid x$ is odd and $x < 10\}$.

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- $\underline{\underline{N}} = \{0, 1, 2, 3, \ldots\}$ the set of all natural numbers. الأعداد الطبيعية
- $\underline{\underline{Z}} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all integers. الأعداد الصحيحة
- $\underline{\underline{Z}}^+ = \{1, 2, 3, \ldots\}$, the set of all positive integers. الأعداد الصحيحة الموجبة
- $\underline{\underline{Q}} = \{p/q \mid p \in Z, q \in Z$, and $q \neq 0\}$, the set of all rational numbers. الأعداد النسبية.
- $\underline{\underline{R}}$, the set of all real numbers. الأعداد الحقيقية
- $\underline{\underline{R}}^+$, the set of all positive real numbers. الأعداد الحقيقية الموجبة
- $\underline{\underline{C}}$, the set of all complex numbers. الأعداد المركبة

  $a + bi$

* Interval notation :- الفترات

- closed interval $[a, b]$.
- open interval (a, b). $]a, b[$

$[a, b] = \{x \mid a \leq x \leq b\}$

$[a, b) = \{x \mid a \leq x < b\}$

$(a, b] = \{x \mid a < x \leq b\}$

$(a, b) = \{x \mid a < x < b\}$

* If A and B are sets, then A and B are equal if and only if $\forall x (x \in A \leftrightarrow x \in B)$. We write $A = B$, if and only if A and B are equal sets.

* **Ex:** $A = \{1, 2, 3\}$, $B = \{3, 2, 1\}$

$\therefore$ A and B are equal.

* **Ex(2):** $A = \{1, 3, 5\}$

$B = \{1, 3, 3, 5, 5, 5\}$

$\therefore$ A and B are equal.

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* **Empty set (null set):**
  - Is a special set that has no elements.
  - denoted by : Φ , $\{\}$

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* **Cardinality :-** عدد العناصر الموجودة فى المجموعة
  - Is the number of distinct elements in S.
  - denoted by $|S|$.

* **Ex(1):**
  - $S = \{a, b, c, d\}$ $\therefore |S| = 4$.
  - $S = \{1, 2, 3, 7, 9\}$ $\therefore |S| = 5$.
  - $\Phi = \{\}$ $\therefore |\Phi| = \text{Zero}$.

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* **Ex(2):**
  - $S = \{a, b, c, d, \{2\}\}$

    $\therefore |S| = 5$.
  - $A = \{1, 2, 3, \{2, 3\}, 9\}$

    $\therefore |A| = 5$.
  - $\{\Phi\} = \{\{\}\}$

    $\therefore |\{\Phi\}| = |\{\{\}\}| = 1$.

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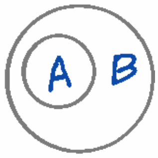
* **Infinite :-** لانهائية
  - A set is said to be infinite if it is not finite.

* **Subset :- " $\subseteq$ "**

- $\boxed{A \subseteq B \leftrightarrow \forall x (x \in A \rightarrow x \in B)}$



- $\boxed{A \subseteq B \equiv B \supseteq A}$

- For every set S:

  (i) $\Phi \subseteq S$ (ii) $S \subseteq S$.

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* **Proper subset : " $\subset$ "**
  - The set A is a subset of the set B but that $A \neq B$.

- $\boxed{A \subset B \leftrightarrow (\forall x (x \in A \rightarrow x \in B) \land \exists x (x \in B \land x \notin A))}$

* **Ex:** For each of the following sets, determine whether 3 is an element of that set.

• $A = \{1, 2, 3, 4\}$	$3 \in A$
• $B = \{\{1\}, \{2\}, \{3\}, \{4\}\}$	$3 \notin B$
• $C = \{1, 2, \{1, 3\}\}$	$3 \notin C$

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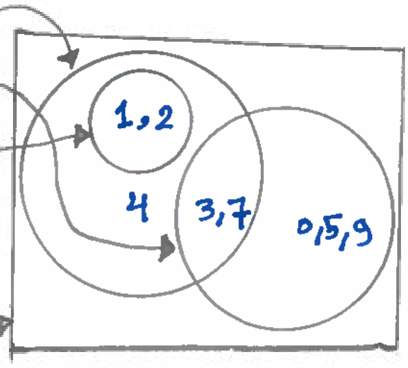
* **Venn Diagram :-**

$A = \{1, 2, 3, 4, 7\}$

$B = \{0, 3, 5, 7, 9\}$

$C = \{1, 2\}$



universal set "U"

* Power set :-

- The set of all subsets.
- denoted by $P(S)$ or 2^{S}
- The number of elements in the Power set is $2^{|S|}$.

***Ex:** $S = \{1, 2, 3\}$, Find $P(S)$, and $|P(S)|$?

Ans.

$P(S) = 2^{S} = \{\Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}.$

$|P(S)| = 2^{|S|} = 2^{3} = 8$ elements.

* Ex(2): what is the Power set of the empty set ?

Ans.

- $P(\Phi) = 2^{\Phi} = \{\Phi\}$
- $|P(\Phi)| = 2^{|\Phi|} = 2^{0} = 1$.

* Ex(3): what is the Power set of the set $\{\Phi\}$

ans.

- $P(\{\Phi\}) = \{\Phi, \{\Phi\}\}$
- $|P(\{\Phi\})| = 2^{1} = 2$.

* The ordered n-tuple :-

- $(a_1, a_2, a_3, \dots, a_n)$
- In Particular, ordered 2-tuples are called ordered Pairs .
  أزواج مرتبة

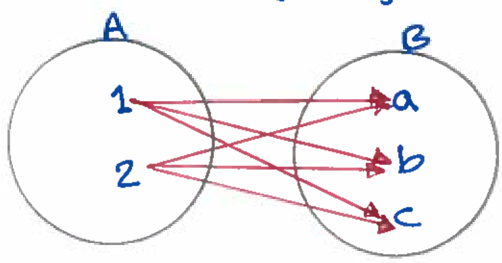
Ex. (a, b)

* Cartesian Products :-

* EX(1): Let $A = \{1, 2\}$ and $B = \{a, b, c\}$ find $A \times B$ and $|A \times B|$?

ans.

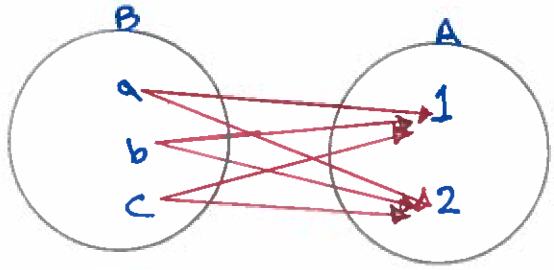
- $A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$.



- $|A \times B| = |A| * |B| = 2 * 3 = 6$.

* Ex(2): Find $B \times A$?

- $B \times A = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$.



* Ex(3):

$A = \{0, 1\}$, $B = \{1, 2\}$, $C = \{0, 1, 2\}$

Find $A \times B \times C = ?$

ans.

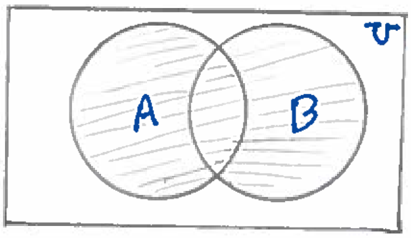
- $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$.

- $|A \times B \times C| = |A| * |B| * |C|$
  $= 2 * 2 * 3 = 12$ ###

* Set operations :-

[1] Union : "$\cup$" (اتحاد)

* $\boxed{A \cup B = \{x \mid x \in A \vee x \in B\}}$



$A \cup B$ is shaded

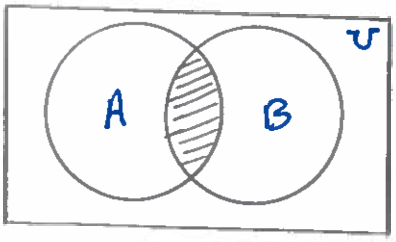
* Ex: Let set $A = \{1, 3, 5\}$ and $B = \{1, 3, 2\}$, Find $A \cup B = ?$

ans.

$A \cup B = \{1, 2, 3, 5\}$.

[2] Intersection : "$\cap$" (التقاطع)

* $\boxed{A \cap B = \{x \mid x \in A \wedge x \in B\}}$



$A \cap B$ shaded

* Ex: Let set $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$, Find $A \cap B = ?$

ans.

$A \cap B = \{1, 3\}$.

[3] Disjoint :- منفصل

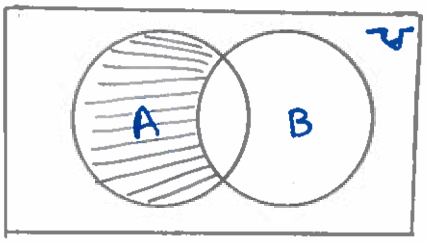
* Two sets are called disjoint if their intersection is the empty set.

$\boxed{A \cap B = \Phi}$

[4] Difference :- "$-$" (الفرق)

* $\boxed{A - B = \{x \mid x \in A \wedge x \notin B\}}$

$A - B \neq B - A$ * مش بالبدل



$A - B$ is shaded

* Ex: Let set $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$, Find $A - B = ?$

ans.

$A - B = \{5\}$.

$B - A = \{2\}$.

[5] Complement : "$\bar{A}$" المكمل

* $\boxed{\bar{A} = \{x \mid x \in U \wedge x \notin A\}}$ or

* $\boxed{\bar{A} = \{x \in U \mid x \notin A\}}$ ($\bar{A}$ $'A$ A' A^c)

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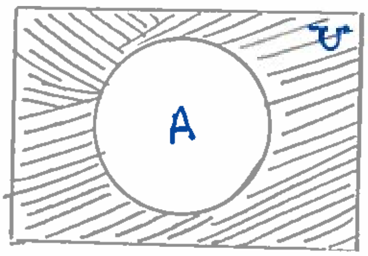


$\bar{A}$ is shaded

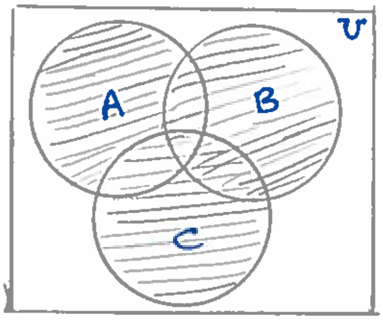
* Ex: Let $U = \{1, 2, 3, 4, 5\}$ and $A = \{1, 3\}$, find $\bar{A}$?

ans.

$\bar{A} = \{2, 4, 5\}$.

6 Generalized Unions :-

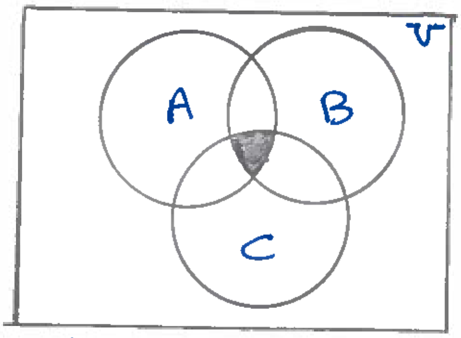
$$* \quad A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$



$A \cup B \cup C$ is shaded

7 Generalized Intersections:-

$$* \quad A_1 \cap A_2 \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$



$A \cap B \cap C$ is shaded.

* Set Identities :-

Identity	Name
$A \cap U = A$ $A \cup \Phi = A$	Identity Laws
$A \cup U = U$ $A \cap \Phi = \Phi$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation Laws
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgans Laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \Phi$	Complement Laws

* Example (1) :

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

ans.

- First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.
- Second, " " " " $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

- First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$:

$x \in \overline{A \cap B}$	by assumption.
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection.
$\neg(x \in A) \vee \neg(x \in B)$	1st De. Morgan law.
$x \notin A \vee x \notin B$	defn. of negation.
$x \in \overline{A} \vee x \in \overline{B}$	defn. of Complement.
$x \in \overline{A} \cup \overline{B}$	defn. of union.

- Second, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$:

$x \in \overline{A} \cup \overline{B}$	by assumption.
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union.
$(x \notin A) \vee (x \notin B)$	defn. of complement.
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation.
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgans. Law.

$\neg(x \in A \cap B)$ defn. of intersection

$x \in \overline{A \cap B}$ defn. of complement.

* Example(2): use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

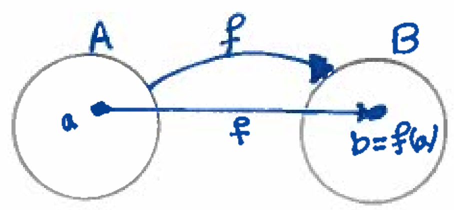
Ans.

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid \neg(x \in (A \cap B))\} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \bar{A} \vee x \in \bar{B}\} \\
&= \{x \mid x \in \bar{A} \cup \bar{B}\} \\
&= \bar{A} \cup \bar{B}.
\end{aligned}$$

2 Functions.

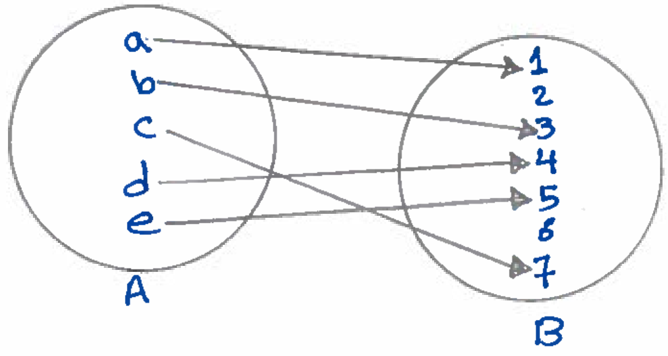
* If f is a function from A to B, we write $f: A \to B$.

*



* Domain: A
* Co-domain: B
* $f(a) = b$
  - a is a preimage of b.
  - b is the image of a.
* The range: is the set of all images of elements of A

* Ex(1):



* Domain = $\{a, b, c, d, e\}$.
* Co-Domain = $\{1, 2, 3, 4, 5, 6, 7\}$.
* Range = $\{1, 3, \cancel{4}, 5, 7\}$.

* Definition:-

$$\boxed{(f_1 + f_2)(x) = f_1(x) + f_2(x)}$$

$$\boxed{(f_1 f_2)(x) = f_1(x)\, f_2(x)}$$

* Ex: Let f_1 and f_2 be functions from R to R such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. what are the functions $f_1 + f_2$ and $f_1 f_2$?

ans.

$$\begin{aligned}
(f_1 + f_2)(x) &= f_1(x) + f_2(x) \\
&= x^2 + (x - x^2) \\
&= x^2 + x - x^2 \\
&= x.
\end{aligned}$$

$$\begin{aligned}
(f_1 f_2)(x) &= f_1(x)\, f_2(x) \\
&= x^2 (x - x^2) \\
&= x^3 - x^4.
\end{aligned}$$