Objectives:

After completing this topic, you will be able to:

- Understand the notion of the area.
- Use the concepts of integration.
- Have strong intuitive feeling for these important concepts.

Notion of the Area:

Calculus consists of two main parts, differential calculus and integral calculus. Differential calculus is based upon the derivative. In this chapter you will learn the concept which is the basis for integral calculus: the definite integral and related topics.

Consider the area of a region in a plane:

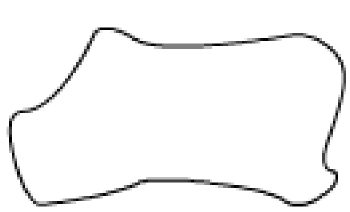
It is easy to calculate the area of a plan region bounded by straight lines.

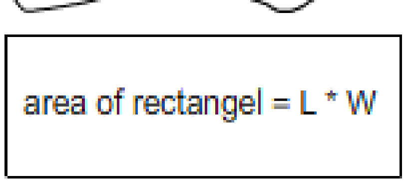
For example:

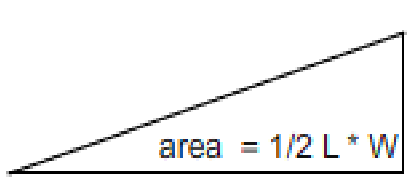
the area of a rectangle is the product of its length and width.

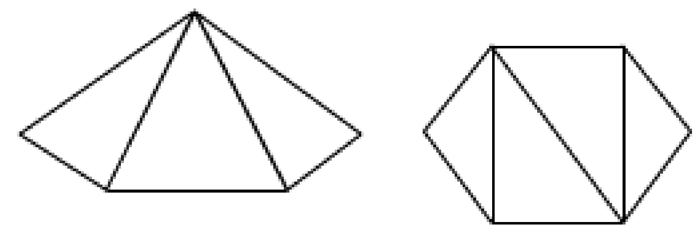
The area of a triangle is one –half the product of the altitude and base.

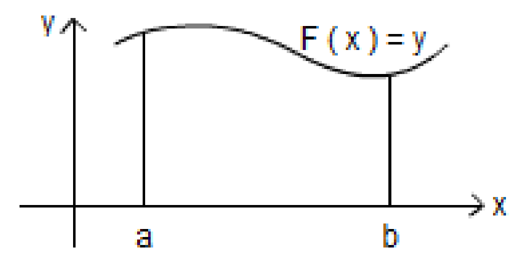
The area of any polygon can be found by subdividing boundaries involve graphs of functions, it's necessary to introduce a limiting process and then use methods of calculus.











Illustrating graph:

Given a region ς in a coordinate plane, bounded by vertical lines with χ-intercepts a and b, by the χ-axis, and by the graph of a function f, which is continuous and non negative on the closed interval $[a, b]$.

The question is how to find the given area? In the given graph you see the area bounded by the two vertical lines from a and b, the graph $f(\chi)$ and the χ axis.

In the given graph you see the area bounded by the two vertical lines from a and b, the graph $f(\chi)$ and the χ axis.

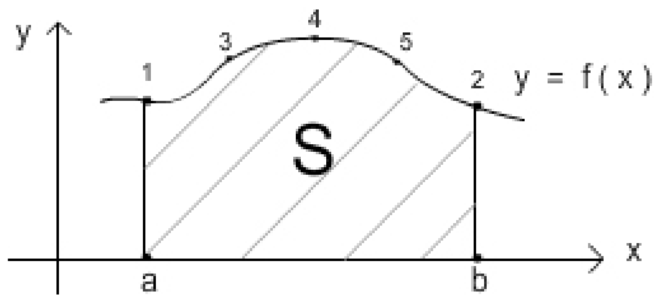
In particular, let us consider a region S in a coordinate plane, bounded by vertical lines with x-intercepts a and b, by the x-axis, and by the graph of a function f, which is continuous and non negative on the closed interval $[a, b]$.

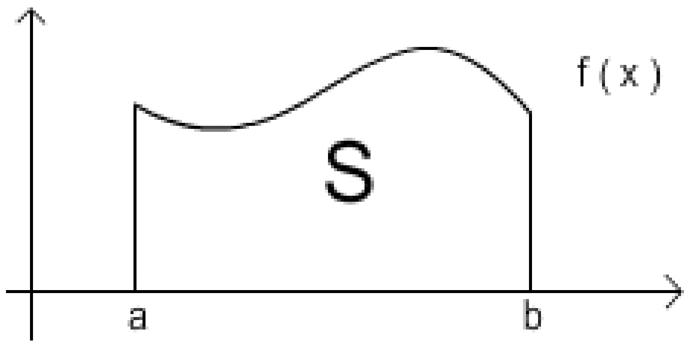
A region of this type is illustrated in figure 4.1.

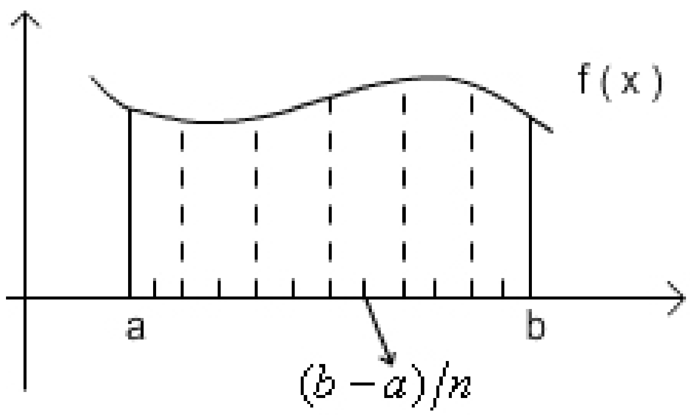
Since $f(x) \geq 0$ for every x in $[a, b]$, no part of the graph lies below the x-axis. For convenience we shall refer to S as the region under the graph of f from a to b.

Our objective is to define the area of S.

If n is any positive integer, let us begin by dividing the interval $[a, b]$ into n subintervals, all having the same length $(b-a)/n$.







This can be accomplished by choosing numbers

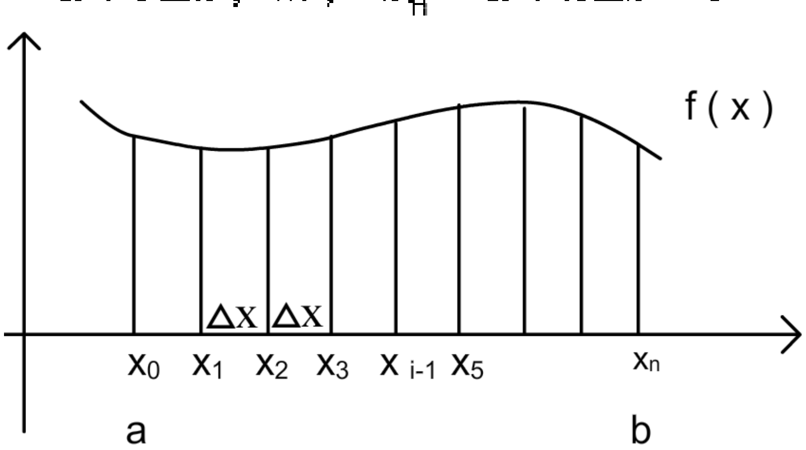
$$x_i - x_{i-1} = \frac{b-a}{n} \quad \text{where} \quad a = x_0, b = x_n$$

and $x_0, x_1, x_2, \dots, x_n$

For $i = 1,2,\dots,n$ if the length $(b-a)/n$ of each subinterval is denoted by Δx then for each I we have $\Delta x = x_i - x_{i-1}$, and $x_i = x_{i-1} + \Delta x$

Note that

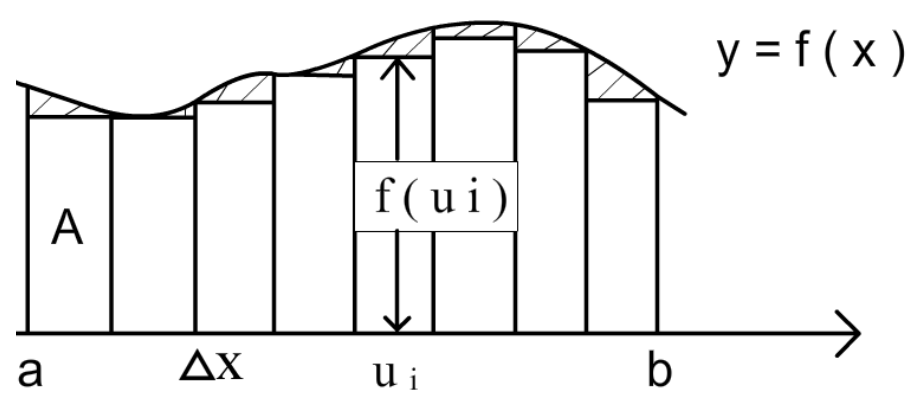
$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots$$
$$x_i = a + i\Delta x, \quad \dots, \quad x_n = a + n\Delta x = b$$



Calculus and Area:

Since f is continuous on each subinterval $[\chi_{i-1}, \chi_i]$ it follow that f takes on a minimum value at some number u_i in $[\chi_{i-1}, \chi_i]$.

For each i, let us construct a rectangle of width $\Delta = \chi_i - \chi_{i-1}$ and length equal to the minimum distance $f(u_i)$ from the χ-axis to the graph of f as shown in figure.



The area of the ith rectangle is $f(u_i)\Delta x$.

The boundary of the region formed by the totality of these rectangles is called the inscribed rectangle polygon associated with the subdivision of $[a, b]$ into n subintervals.

The area of this inscribed polygon is the sum of the areas of the rectangles, that is,

$$f(u_1)\Delta x + f(u_2)\Delta x + \dots + f(u_n)\Delta x.$$

Using the summation notation we may write:

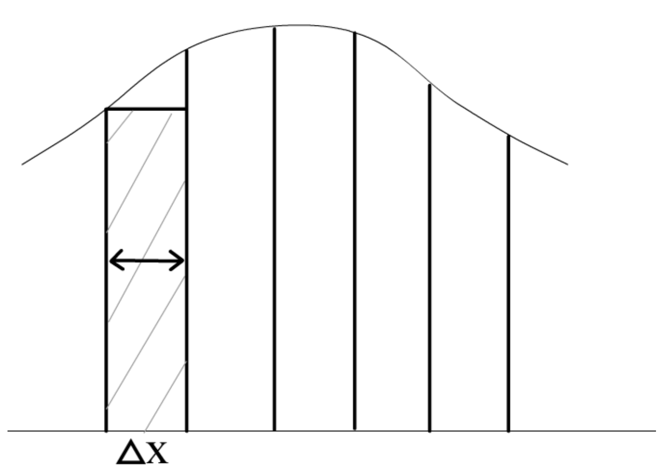
Area of inscribed rectangular polygon

$$= \sum_{i=1}^{n} f(u_i)\Delta x$$

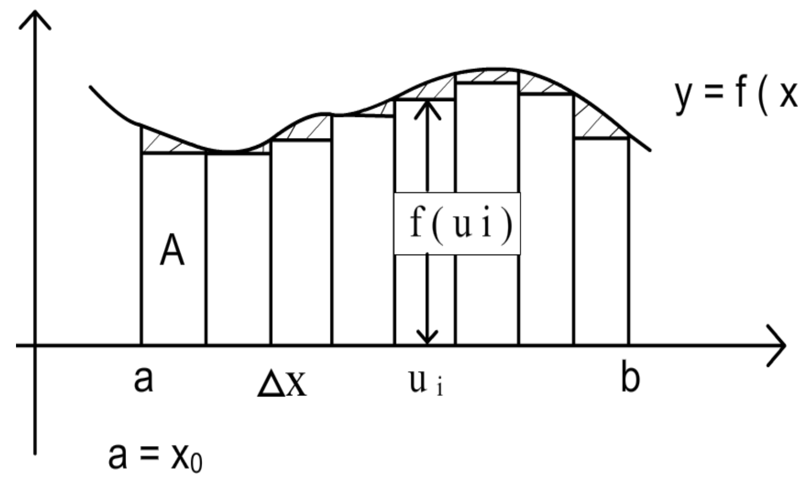
Where $f(u_i)$ is the minimum value of f on $[\chi_{i-1}, \chi_i]$. Referring to figure 4.2, we see that if n is very large or, equivalently, if $\Delta\chi$ is very small, then the sum of the rectangular areas appears to be close to what we wish to consider as the area of the region ς.

Indeed, reasoning intuitively, if there exist a number $\mathcal{A}$ that has the property that the sum = $\sum_{i=1}^{n} f(u_i)\Delta x$ gets closer and closer to $\Delta\chi \to 0$ as gets closer and closer to (but $\Delta\chi \neq 0$), then we shall call the area of ς and write $A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(u_i)\Delta x$.

The meaning of this "limit of a sum" is not the same as that for limit of a function. To eliminate the hazy phrase "closer and closer" and arrive at a satisfactory definition of $\mathcal{A}$, let us take a slightly different of view. If $\mathcal{A}$ denotes the area of ς, then the difference $A - \sum_{i=1}^{n} f(u_i)\Delta x$. it Is the area of shaded portion in figure 4.2 that lies under the graph of f and over the inscribed rectangular polygon.



This number may be thought of as the error involved in using the area of the inscribed polygon to approximate A. If we have the proper notion of area, then we should be able to make this error arbitrarily small by choosing the width Δx of the rectangles sufficiently small. This is the motivation for the following definition of the area A of S.



Definition (4.1):

Let f be continuous and nonnegative on $[a, b]$.

Let be a real number and let $f(u_i)$ be the minimum value of f on $[\chi_{i-1}, \chi_i]$.

The statement $A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(u_i)\Delta x.$ means that for every $\varepsilon > 0$ there corresponds a $\gamma > 0$ such that if $0 < \Delta\chi < \gamma$, then $A - \sum_{i=1}^{n} f(u_i)\Delta x < \varepsilon.$

you can observe that as getting smaller, the value of the summation converges to the true value of the area.

Using the summation notation:

As you see by the given graph, If the base of each rectangle getting smaller, the error of representing the area by the summation of the areas of these rectangles also getting smaller.

Here we explain again how to approximate an area by a sum of small areas.

As you observe, it is possible to make the difference between the area ant its approximation by a sum as small as desired only by choosing sufficiently small.

If f is continuous on $[a, b]$ then , as it is shown in more advanced texts, a number A satisfying definition 4.1 actually exist.

We shall call A the area under the graph of f from a to b.

The area my also be obtained by means of circumscribed rectangular polygons of the type illustrated in figure 4.3.

in this case we select the number v_i in each interval $[x_{i-1}, x_i]$

such that $f(v_i)$ is the maximum value of f on $[x_{i-1}, x_i]$.

We may then write:

Area of circumscribed rectangular polygon $= \sum_{i=1}^{n} f(v_i)\Delta x$.

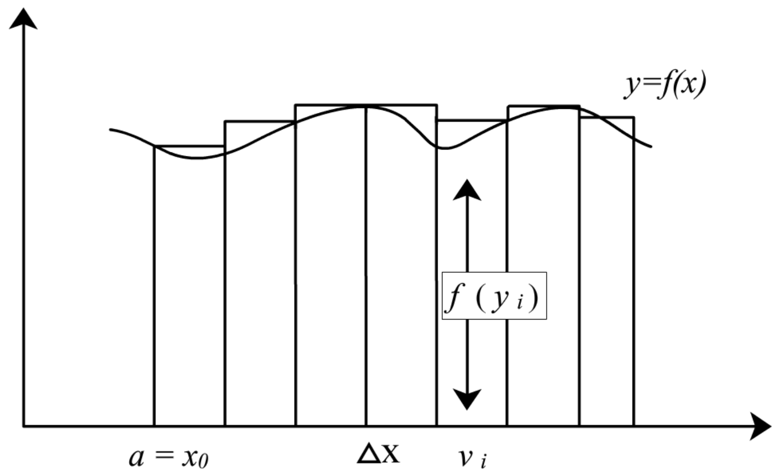
The limit of this sum as is defined as in (4.1), where the only change is that we use $\sum_{i=1}^{n} f(v_i)\Delta x - A < \varepsilon$.

In the definition, since we want this difference to be nonnegative.

It can be proved that the same number A is obtained using either or circumscribed rectangles.

In our development of the definite integral we shall employ sums of many numbers.

To express such sums compactly,



Objectives:

After completing this topic, you will be able to:

- Understand the concepts of finite sums.
- Learn the definite integral as a limit of a sum.
- Gain experience in evaluating finite sums.

Finite Sum Concept:

In the second topic we introduce the concept of finite sum. Given a finite number of elements, their sum is known as the finite sum.

It is convenient to use summation notation, to illustrate, given a collection of numbers $\{1, 2, \ldots\ldots, a_n\}$, the symbol $\Sigma^{n}_{i=1} a_i$ represent their sum,

that is $\Sigma^{n}_{i=1} a_i = a_1 + a_2 + \ldots\ldots + a_n$

Where the Greek capital letter Σ indicates a sum, and the symbol a_i represent the ith term. The letter i is called the index of summation or the summation variable, and the numbers 1 and n indicates the extreme values of the summation variable.

A theorem concerning finite sums:

Theorem 4.1:

If n is any positive integer and $\{a_1, a_2, \ldots\ldots, a_n\}$, $\{b_1, b_2, \ldots\ldots, b_n\}$ are sets of numbers, then

$$(i)\sum_{i=1}^{n}(a_i+b_i)=\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}b_i$$

$$ii)\sum_{i=1}^{n}ca_i=c(\sum_{i=1}^{n}a_i),\ \text{for any number } c;$$

$$(iii)\sum_{i=1}^{n}(a_i-b_i)=\sum_{i=1}^{n}a_i-\sum_{i=1}^{n}b_i$$

Now, the following definition will be useful in some illustrations.

$$(i)\sum_{i=1}^{n} i = 1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

$$(ii)\sum_{i=1}^{n} i^2 = 1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$(iii)\sum_{i=1}^{n} i^3 = 1^3+2^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Examples using finite sum formula:

This screen illustrates some important properties of the sum operation such as multiplying by a constant or the sum of a difference, applying the sum rules given before we can evaluate some given sums as illustrated.

Here you find some solved example illustrating the properties of the sum operator and how to evaluate the given sum.

Example 1:

Find $\Sigma^{4}_{i=1} i^2(i-3)$

Solution:

we merely substitute, in succession, the integers 1,2,3, and 4 for i and add the resulting terms. thus,

$$\Sigma^{4}_{i=1} i^2(i-3) = 1^2(1-3)+2^2(2-3)+3^2(3-3)+4^2(4-3)$$
$$= (-2) + (-4) + (0) + (16) = 10$$

Example 2:

Find $\sum_{i=0}^{3}\frac{2^i}{(i+1)}$

Solution:

$$\sum_{i=0}^{3}\frac{2^i}{(i+1)}=\frac{2^0}{(0+1)}+\frac{2^1}{(1+1)}+\frac{2^2}{(2+1)}+\frac{2^3}{(3+1)}$$

$$=1+1+\frac{4}{3}+2=\frac{16}{3}$$

Example 3:

If $f(\chi) = 16 - \chi^2$ find the area of the region under the graph of f from 0 to 1.

Solution:

the region is illustrated in figure 4.4. if the interval $[0, 3]$ is divided into n equal subintervals, then the length Δx of a typical subinterval is $3/n$.

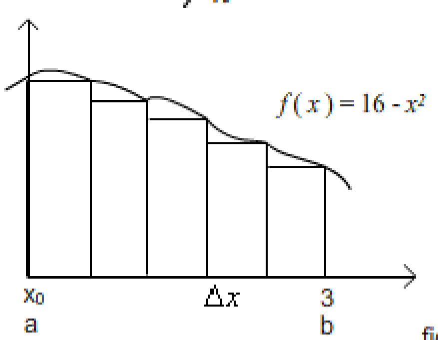


figure 4.4

employing the notation used in figure 4.2 with $a = 0$ and $b = 3$. endpoint x_i of the subinterval, that is, $u_i = x_i = 3i/n$

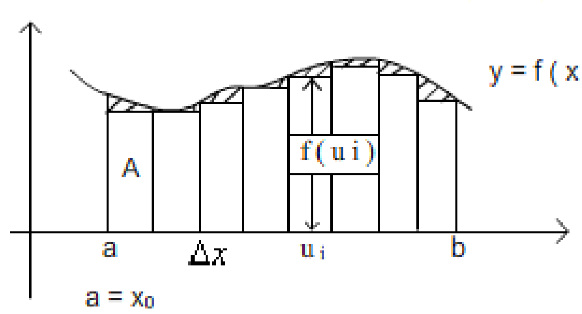


figure 4.2

Since $x_0 = 0, x_1 = \Delta x, x_2 = 2\Delta x, \ldots, x_i = i\Delta x, \ldots, x_n = n\Delta x = 3$

Using the fact that $\Delta x = 3/n$ we may write

$$x_i = i(\Delta x) = i\left(\frac{3}{n}\right) = \frac{3i}{n}$$

Since f is decreasing on $[0, 3]$, the number u_i in $[x_{i-1}, x_i]$ at which f takes on its minimum value is always at the right-hand $f(u_i) = f\left(\frac{3i}{n}\right) = 16 - \left(\frac{3i}{n}\right)^2 = 16 - \frac{9i^2}{n^2}$

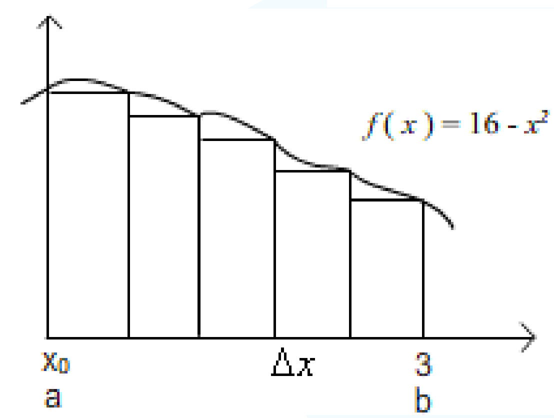
Concept of approximating an area by a finite sum:

In this screen we start introducing the concept of approximating an area by a finite sum.

Here we use the idea of finite sum to approximate an area by dividing the area into a group of rectangles each has a small area and summing the areas of these rectangles.

The summation in definition 4.1 may be written

$$\sum_{i=1}^{n} f(u_i)\Delta x = \sum_{i=1}^{n}\left(16 - \frac{9i^2}{n^2}\right)\left(\frac{3}{n}\right)$$
$$= \sum_{i=1}^{n}\left(\frac{48}{n} - \frac{27i^2}{n^3}\right)$$



The last sum may be simplified as follows:

$$\sum_{i=1}^{n}\frac{48}{n} - \sum_{i=1}^{n}\frac{27i^2}{n^3} = \left(\frac{48}{n}\right)n - \frac{27}{n^3}\sum_{i=1}^{n} i^2.$$

In order to find the area, we must now let Δx approach 0. Since $\Delta x = (b-a)/n$, this can be accomplished by letting n increase without bound. And we can replace $\Delta x \to 0$ by $n \to \infty$, we have

$$\lim_{\Delta x \to 0}\sum_{i=1}^{n} f(u_i)\Delta x = \lim_{n\to\infty}\left\{48 - \frac{9}{2n^3}\left[2n^3 + 3n^2 + n\right]\right\} = \lim_{n\to\infty} 48 - \frac{9}{2}\lim_{n\to\infty}\left[\frac{2n^3+3n^2+n}{n^3}\right]$$
$$= 48 - \frac{9}{2}\lim_{n\to\infty}\left[2 + \frac{3}{n} + \frac{1}{n^2}\right] = 48 - \frac{9}{2}[2+0+0] = 48 - 9 = 39$$

The area in the preceding example may also be found by using circumscribed rectangular polygons.

In this case we select, in each subinterval $[x_{i-1}, x_i]$, the number $v_i = (i-1)(3/n)$ at which f takes its maximum value.

Objectives:

After completing this topic, you will be able to:

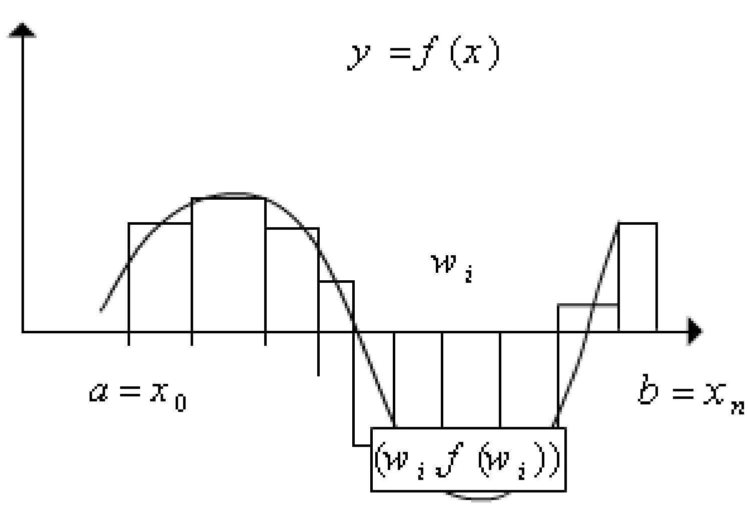
- The concepts of definite integrals.
- The importance of definite integrals.
- The real life applications of definite integrals.

Definition introducing the main concepts of definite integral:

Definition (4.2):

Let f be a function that is defined on a closed interval and let be a partition of $[a, b]$.

A Riemann sum of f for $\mathcal{P}$ is any expression of $\mathcal{R}_p$ the form $R_P = \sum_{i=1}^{n} f(w_i)\Delta x_i$



In definition 4.2, $f(w_i)$ is not necessarily a maximum or minimum value of f on $[\chi_{i-1}, \chi_i]$.

Thus if we construct a rectangle of length and width , the rectangle may be neither inscribed nor circumscribed.

Since $f(\chi)$ may be negative for some χ in $[a, b]$, some terms of $\mathcal{R}_p$ in definition 4.2 may be negative.

In this screen you find a definition introducing the main concepts of definite integral.

Here we continue introducing the Riemann's notions for finding the definite integral.

In this screen we illustrate (by a graph) how to evaluate an area With some parts above the χ axis and other parts below the χ axis.

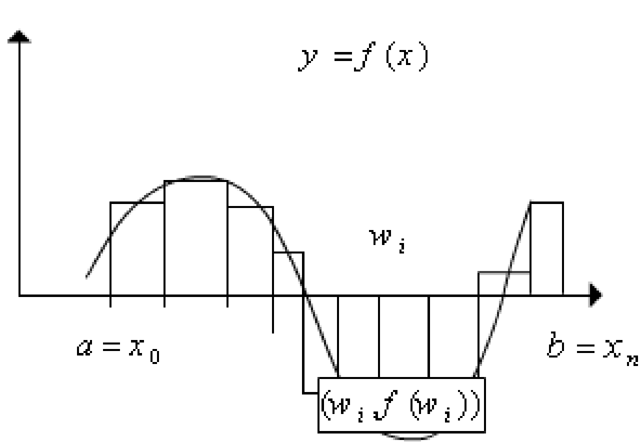
This screen is a continuation of the given Riemann's concepts.

Riemann's Concepts For Definite Integral:

Consequently, a Riemann sum does not always represent a sum of areas of rectangles. It is possible to interpret a Riemann sum geometrically as follows.

If R_p is defined as in 4.2, then for each subinterval $[x_{i-1}, x_i]$ let us construct a horizontal line segment through the point $(w_i f(w_i))$ thereby obtaining a collection of rectangles.

If $f(w_i)$ is positive the rectangle lies above the x-axis as illustrated by the shaded rectangles in figure 4.5, and the product $f(w_i)\,\Delta x_i$ is the area of this rectangle. If $f(w_i)$ is negative, then the rectangle lies below the x-axis as illustrated by the unshaded rectangles in figure 4.5.



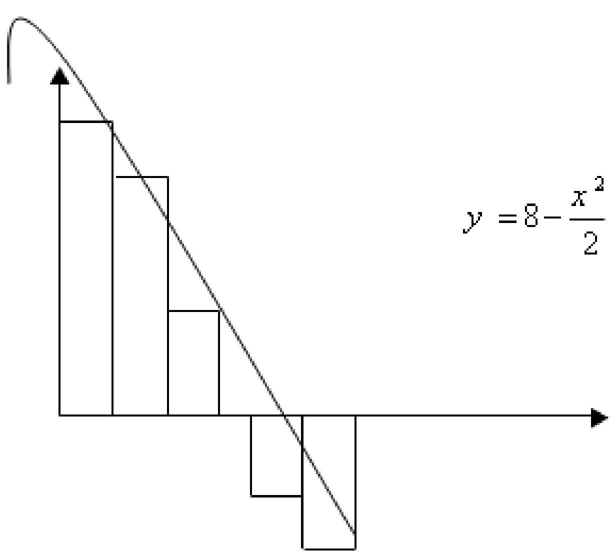
Here you find an example illustrating the idea of evaluating. An area with some parts above the χ axis and other parts below the x axis

Example 4:

Suppose $f(x)=8-(x^2/2)$ and P is the partition of $[0,6]$ into the five subintervals determined by $x_0=0, x_1=1.5, x_2=2.5, x_3=4.5, x_4=5$, and $x_5=6$

Find (a) the norm of the partition and (b) the Riemann sum

R_P if $w_1=1, w_2=2, w_3=3.5, w_4=5, w_5=5.5$.



Solution:

Solution: the graph of f is sketched in figure 4.5.

Also shown in the figure are the points on the x-axis that correspond to x_i and the rectangles of lengths $|f(w_i)|$ for $i=1,2,3,4,$ and 5.

Thus, $\Delta x_1=1.5, \Delta x_2=1, \Delta x_3=2, \Delta x_4=0.5, \Delta x_5=1$

And hence the norm $\|P\|$ of the partition is Δx_3, or 2.

By definition 4.2,

$$R_P=f(w_1)\Delta x_1+f(w_2)\Delta x_2+f(w_3)\Delta x_3+f(w_4)\Delta x_4+f(w_5)\Delta x_5$$
$$=f(1)(1.5)+f(2)(1)+f(3.5)(2)+f(5)(0.5)+f(5.5)(1)$$
$$=(7.5)(1.5)+(6)(1)+(1.875)(2)+(-4.5)(0.5)+(-7.125)(1)$$

Which reduces to $R_P=11.625$

Concepts and Conditions Of Definite Integral:

Definition (4.3):

Let f be a function that is defined on a closed interval $[a, b]$. The definite integral of f from a to b, denoted by $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx=\lim_{\|P\|\to 0}\sum_i f(w_i)\Delta x_i$$

Provided the limit exists.

If the definite integral of f from a to b exists, then f is said to be integrable on the closed interval $[a, b]$, or we say that the integral $\int_a^b f(x)dx$ exists.

The process of finding the number represented by the limit is called evaluating the integral.

The symbol $\int$ in is called an integral sign. The numbers a and b referred to as the limits of integration, a being called the lower limit and b the upper limit.

Whenever an interval $[a,b]$ is employed it is assumed that $a<b$, consequently, definition 4.3 does not take into account the cases in which the lower limit of integration greater than or equal to the upper limit.

The definition may be extended to include the case where the lower limit is greater than the upper limit

Exact Value Of Definite Integral:

Definition (4.4):

$$\text{If } c>d, \text{ then } \int_c^d f(x)dx=-\int_d^c f(x)dx$$

The case in which the lower and the upper limits of integration are equal is covered by the next definition.

Here you find another definition concerning the properties of definite integral.

Definition (4.5):

$$\text{If } f(a) \text{ exists, then } \int_a^a f(x)dx = 0$$

Not every function is integrable.

For example, if $f(\chi)$ becomes positively or negatively infinite at some numbering $[a, b]$, then the definite integral does not exist.

Definite integrals of discontinuous functions may or may not exist, depending on the nature of the discontinuities. However, according to the next theorem, continuous functions are always integrable.

Properties Of Definite Integrals:

Theorem (4.2):

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

This screen illustrates some properties of definite integral.

As an immediate illustration, we have the following important result.

Theorem(4.3):

If f is continuous and $f(\chi) \geq 0$ for all χ in $[a, b]$, Then the area of the region under the graph of f from a to b is given by $A = \int_a^b f(x)dx$

Theorem (4.4): $\int_a^b kdx = k(b-a)$

Theorem(4.5): If f is integrable on $[a, b]$ and k is any real umber, then kf is integrable on $[a, b]$ and

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

Theorem (4.6):

If f and are integrable on $[a, b]$, then $f+g$ and $f-g$ are integrable on $[a, b]$

and $i) \int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

$$ii) \int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Theorem (4.7):

If $a<c<b$, and if f is integrable on both $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Here you find some additional properties of the definite integral.

Theorem(4.8):

If f is integrable on a closed interval and if a, b and c are any three numbers in the interval, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Theorem(4.9):

If f is integrable on $[a, b]$, and if $f(\chi) \geq 0$ for all χ in $[a, b]$, then $\int_a^b f(x)dx \geq 0$

Corollary 4.11:

If f and g are integrable on $[a, b]$, and $f(\chi) \geq g$ for all χ in $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

Here you find some applications on the given theorems

Example 5:

1- $\int_{-2}^{3} 7dx$

Solution:

Using Theorem(4.5)

$$\int_{-2}^{3} 7dx = 7[3-(-2)] = 7(5) = 35$$

Example 6:

2- $\int_{-1}^{1} dx = 1-(-1) = 2$

Since $\int_{a}^{b} dx = b-a$

Example 7:

3- $\int_0^2 x \ \ dx = 4$ and $\int_0^2 x dx = 2$, evaluate

$$\int_0^2 (5x^3 - 3x + 6)dx$$

Solution:

$$\int_0^2 (5x^2 - 3x + 6)dx = \int_0^2 5x^2 dx - \int_0^2 3x dx + \int_0^2 6dx$$
$$= 5\int_0^2 x^2 dx - 3\int_0^2 x dx + 6\int_0^2 dx$$
$$= 5(4) - 3(2) + 12 = 26$$

The mean value theorem:

If f is continuous on a closed interval $[a, b]$, then there is a number z in the open interval (a, b) such that $\int_a^b f(x)dx = f(z)(b-a)$

Proof:

If f is a constant function, then the result follows trivially from theorem before where z is any number in (a, b) next assume that f is not a constant function and suppose that m and $\mathcal{M}$ are the minimum and maximum values of f, respectively on $[a, b]$.

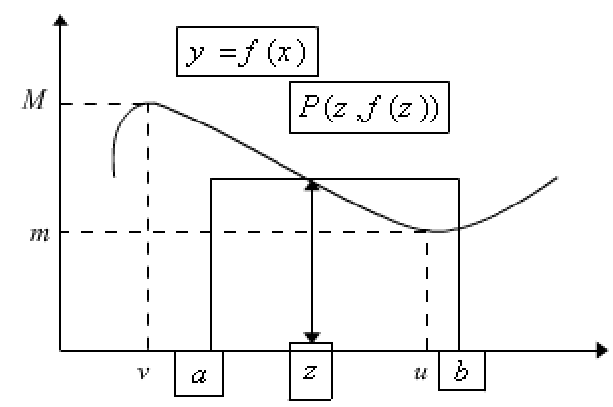
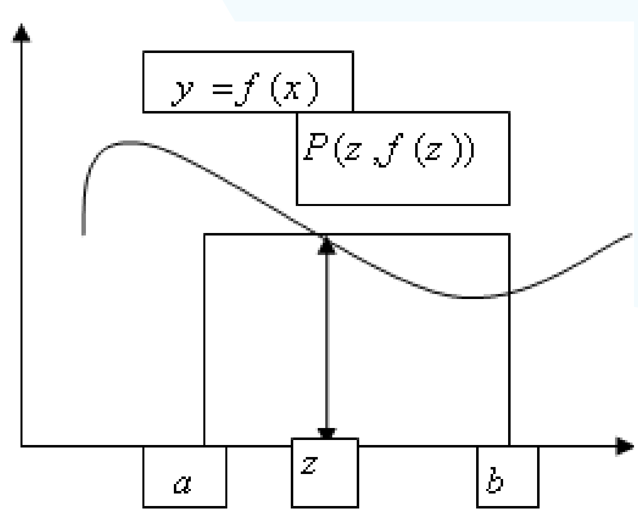


Illustration of the mean value theorem:

Let $f(u) = m$ and $f(v) = \mathcal{M}$, where u and v are in $[a, b]$ since f is not a constant function $m < f(\chi) < \mathcal{M}$, for some χ in $[a, b]$

$$\int_a^b m dx < \int_a^b f(x)dx < \int_a^b M dx$$



Employing theorem

$$m(b-a) < \int_a^b f(x)dx < M(b-a)$$

Dividing by $b-a$ and replacing m and M by $f(u)$ and $f(v)$,

it follows from the intermediate value theorem that there is a number is a number z strictly between u and v such that

$$f(z) = \frac{1}{b-a}\int_a^b f(x)dx$$

Multiplying both sides by $b-a$ gives us the conclusion of the theorem.

Example 8:

It can proved that $\int_0^3 [4-(x^2/4)]dx = \frac{39}{4}$.

Find a number that satisfies the conclusion of the mean value theorem for this integral.

Solution:

According to the mean value theorem for definite integrals, there is a number z between 0 and 3 such that $\int_0^3 (4-\frac{x^2}{4})dx = (4-\frac{z^2}{4})(3-0)$

Or, equivalently, $\frac{39}{4} = (\frac{16-z^2}{4})(3)$

Multiplying both sides of the last equation by $\frac{4}{3}$ leads to $13 = 16 - z^2$ and, therefore , $z^2 = 3$.

Consequently , $\sqrt{3}$ satisfies the condition of theorem