Mathematics (2)

Section (3)

Vectors and vector operations

Done by Waleed Khaled

Vectors

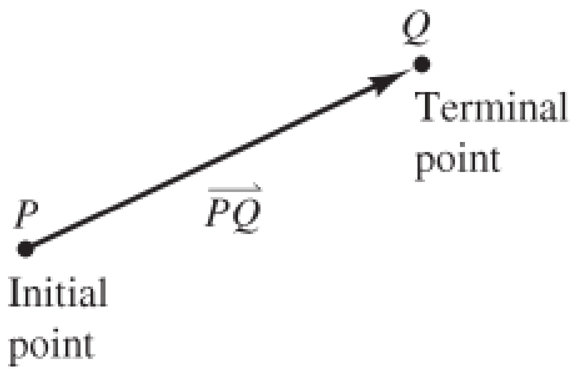
DEFINITION: A vector is a quantity that has both *magnitude* and *direction* such as force, velocity, acceleration, ….

In a graphical sense vectors are represented by directed line segments. The endpoints of the segment are called the initial point and the terminal point of the vector. The length of the line segment is the magnitude of the vector, and the direction of the line segment is the direction of the vector. A vector v is denoted by $\vec{v}$ or $\underline{v}$ or $\mathbf{v}$

If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are the initial and terminal points of a directed line segment, the component form of the vector $\mathbf{v}$ represented by $\overrightarrow{PQ}$ is $\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$. Moreover, from the Distance Formula you can see that the **length** (or **magnitude**) **of v** is

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

Length of a vector



A directed line segment

Vectors in Space

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in space and let c be a scalar.

1. *Equality of Vectors:* $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.
2. *Component Form:* If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, then
$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle.$$
3. *Length:* $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
4. *Unit Vector in the Direction of* $\mathbf{v}$: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\dfrac{1}{\|\mathbf{v}\|}\right)\langle v_1, v_2, v_3 \rangle, \quad \mathbf{v} \neq \mathbf{0}$
5. *Vector Addition:* $\mathbf{v} + \mathbf{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$
6. *Scalar Multiplication:* $c\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle$

Example 1

Give the vector for the line segment from (4,5,6) to (4,6,6). Find its magnitude and determine if the vector is a unit vector

Solution

The components of the vector are always the coordinates of the ending point minus the coordinates of the starting point. Therefore.

$$\vec{v} = \langle 4-4, 6-5, 6-6 \rangle = \langle 0,1,0 \rangle.$$

The magnitude of this vector is

$$\|\vec{v}\| = \sqrt{0^2 + 1^2 + 0^2} = 1$$

Because we can see that $\|\vec{v}\| = 1$, we know that this vector is a unit vector.

Example 2

The vector $\vec{v} = \langle 6, -4, 0 \rangle$ starts at the point $P = (-2, 5, -1)$. At what point does the vector end?

Solution

If the ending point of the vector is given by $Q = (x_2, y_2, z_2)$ then we know that the vector $\vec{v}$ can be written as,

$$\vec{v} = \overrightarrow{PQ} = \langle x_2 + 2, y_2 - 5, z_2 + 1 \rangle = \langle 6, -4, 0 \rangle$$

Now, if two vectors are equal the corresponding components must be equal. Or,

$$\begin{aligned} x_2 + 2 &= 6 \\ y_2 - 5 &= -4 \\ z_2 + 1 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_2 &= 4 \\ y_2 &= 1 \\ z_2 &= -1 \end{aligned}$$

The endpoint of the vector is then, $Q = (4, 1, -1)$

Example 3

Find a vector that points in the same direction as $\vec{v} = \langle -1, 4 \rangle$ with a magnitude of 10.

Solution

Let's determine a unit vector that points in the same direction

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle -1, 4 \rangle}{\sqrt{(-1)^2 + 4^2}} = \frac{\langle -1, 4 \rangle}{\sqrt{17}} = \left\langle -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle.$$

We know that scalar multiplication can change the magnitude of a vector. We've got a vector with magnitude of one that points in the correct direction.

To convert this into a vector with magnitude of 10 all we need to do is multiply this new unit vector by 10 to get,

$$\vec{w} = 10\vec{u} = 10\left\langle -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle = \left\langle -\frac{10}{\sqrt{17}}, \frac{40}{\sqrt{17}} \right\rangle.$$

Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

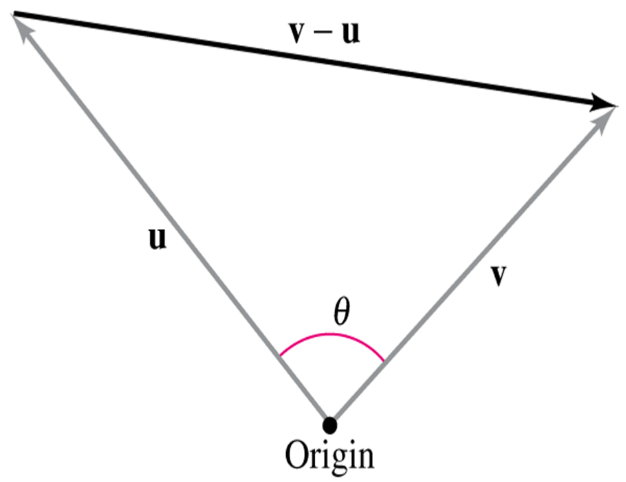
The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Angle Between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$



$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta$$

The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 4

Determine the dot product $\vec{a}\cdot\vec{b}$, if $\|\vec{a}\|=5$, $\|\vec{b}\|=\frac{3}{7}$ and the angle between the two vectors is $\theta=\frac{\pi}{12}$.

Solution

We just need to run through the formula

$$\vec{a}\cdot\vec{b}=\|\vec{a}\|\|\vec{b}\|\cos(\theta)=5\left(\frac{3}{7}\right)\cos\left(\frac{\pi}{12}\right)=2.0698$$

Example 5

Determine the value of α so that the two vectors $\vec{a}=2\vec{i}+3\vec{j}+8\vec{k}$, and $\vec{b}=4\vec{i}-4\vec{j}+\alpha\vec{k}$ are perpendicular.

Solution

Given that the two vectors $\vec{a}$ and $\vec{b}$ are perpendicular meaning that $\vec{a}\cdot\vec{b}=0.$

$$\vec{a}\cdot\vec{b}=\left(2\vec{i}+3\vec{j}+8\vec{k}\right)\cdot\left(4\vec{i}-4\vec{j}+\alpha\vec{k}\right)=2(4)+3(-4)+8(\alpha)=0$$

$$\Rightarrow -4+8\alpha=0\Rightarrow\alpha=\frac{1}{2}.$$

Example 6

Find the angle between two vectors, $\vec{a} = \vec{i} + 3\vec{j} + 5\vec{k}$ and $\vec{b} = 2\vec{i} + 6\vec{j} + 10\vec{k}$

Solution

To find the angle between two vector we use the following formula:

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} \qquad (*)$$

and solve for θ.

Given $\vec{a} = \vec{i} + 3\vec{j} + 5\vec{k}, \ \vec{b} = 2\vec{i} + 6\vec{j} + 10\vec{k}$

$$\vec{a} \cdot \vec{b} = 1 \cdot 2 + 3 \cdot 6 + 5 \cdot 10 = 70$$

$$\|\vec{a}\| = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}, \quad \|\vec{b}\| = \sqrt{2^2 + 6^2 + 10^2} = \sqrt{140}$$

Plugging these values in $(*)$, we get:

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{70}{\sqrt{35}\sqrt{140}} = \frac{70}{\sqrt{4900}} = \frac{70}{70} = 1$$

Therefore, $\theta = \cos^{-1}(1) = 0.$

Hence, two vectors $\vec{a}$ and $\vec{b}$ are parallel.

Cross Product of Two Vectors in Space

Let **u** = $< u_1, u_2, u_3 >$ and **v** = $< v_1, v_2, v_3 >$ be two vectors in space. The cross product of the vectors **u** and **v** is the vector

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

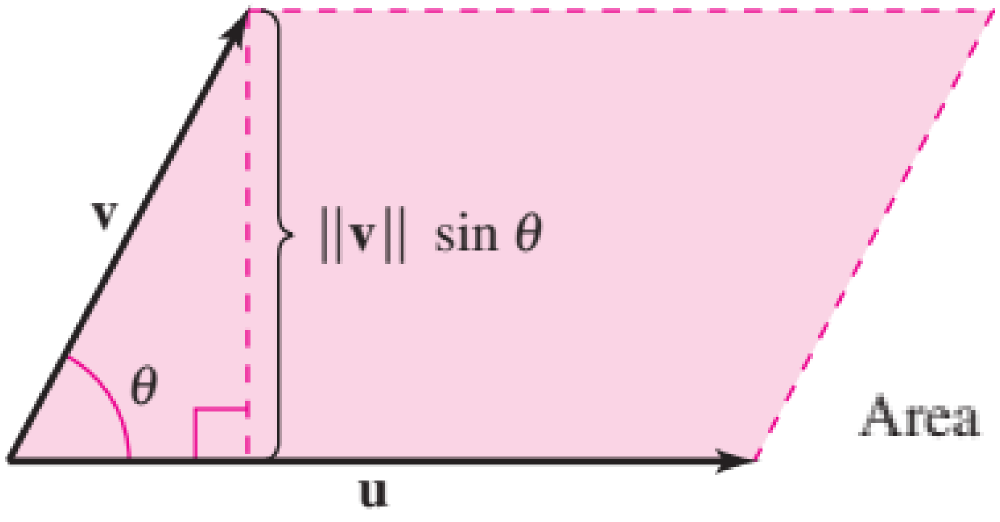
A convenient way to calculate **u x v** is to use the following *determinant form:*

$$\mathrm{u} \times \mathrm{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

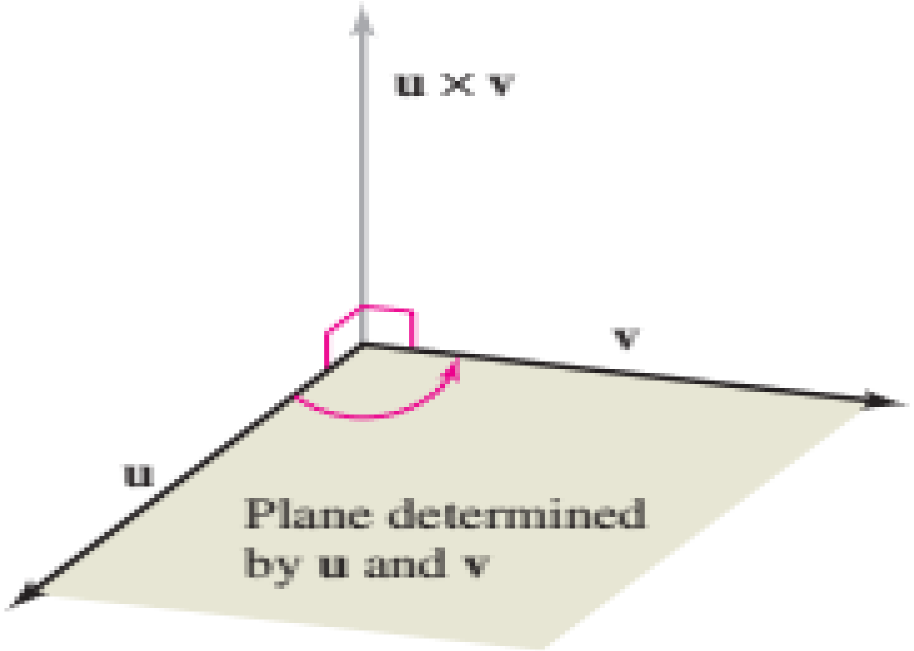
Geometric Properties of the Cross Product

Let **u** and **v** be nonzero vectors in space, and let θ be the angle between **u** and **v**.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin \theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if **u** and **v** are scalar multiples of each other(i.e., parallel to each other
4. $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having **u** and **v** as adjacent sides.



$$\begin{aligned}\text{Area} &= (\text{base})(\text{height}) \\ &= \|\mathbf{u}\| \, \|\mathbf{v}\| \sin \theta \\ &= \|\mathbf{u} \times \mathbf{v}\|.\end{aligned}$$



Example 7

Find a unit vector that is orthogonal to both $\vec{a} = \langle 1,\ 3,\ 4 \rangle$ and $\vec{b} = \langle 2,\ 7,\ -5 \rangle$.

Solution

The cross product $\vec{a} \times \vec{b}$, is orthogonal to both $\vec{a}$ and $\vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \vec{k}$$

$$= (-15 - 28)\vec{i} - (-5 - 8)\vec{j} + (7 - 6)\vec{k} = -43\vec{i} + 13\vec{j} + \vec{k}$$

Because $\left\|\vec{a} \times \vec{b}\right\| = \sqrt{(-43)^2 + 13^2 + 1^2} = \sqrt{2019},$ a unit vector orthogonal to both vectors is

$$\vec{n} = \frac{\vec{a} \times \vec{b}}{\left\|\vec{a} \times \vec{b}\right\|} = \frac{1}{\sqrt{2019}}\left(-43\vec{i} + 13\vec{j} + \vec{k}\right) = -\frac{43}{\sqrt{2019}}\vec{i} + \frac{13}{\sqrt{2019}}\vec{j} + \frac{1}{\sqrt{2019}}\vec{k}$$

Example 8

Find a vector perpendicular to the plane that passes through the points:

$$P(1,\ 4,\ 6),\ Q(-2,\ 5,\ -1),\ R(1,\ -1,\ 1)$$

Solution

The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Therefore, it is perpendicular to the plane through P, Q, and R.

$$\overrightarrow{PQ}=(-2-1)\vec{i}+(5-4)\vec{j}+(-1-6)\vec{k}=-3\vec{i}+\vec{j}-7\vec{k}$$

$$\overrightarrow{PR}=(1-1)\vec{i}+(-1-4)\vec{j}+(1-6)\vec{k}=-5\vec{j}-5\vec{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -3 & 1 & -7\\ 0 & -5 & -5\end{vmatrix}=(-5-35)\vec{i}-(15-0)\vec{j}+(15-0)\vec{k}$$

$$=-40\vec{i}-15\vec{j}+15\vec{k}$$

Therefore, the vector $-40\vec{i}-15\vec{j}+15\vec{k}$ is perpendicular to the given plane.

Example 9

Find the area of the triangle with vertices :

$$P(1,\ 4,\ 6),\ Q(-2,\ 5,\ -1),\ R(1,\ -1,\ 1)$$

Solution

In **Example 8**, we computed that

$$\overrightarrow{PQ}\times\overrightarrow{PR}=-40\vec{i}-15\vec{j}+15\vec{k}$$

The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|=\sqrt{(-40)^2+(-15)^2+15^2}=5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is:

$$\tfrac{5}{2}\sqrt{82}.$$

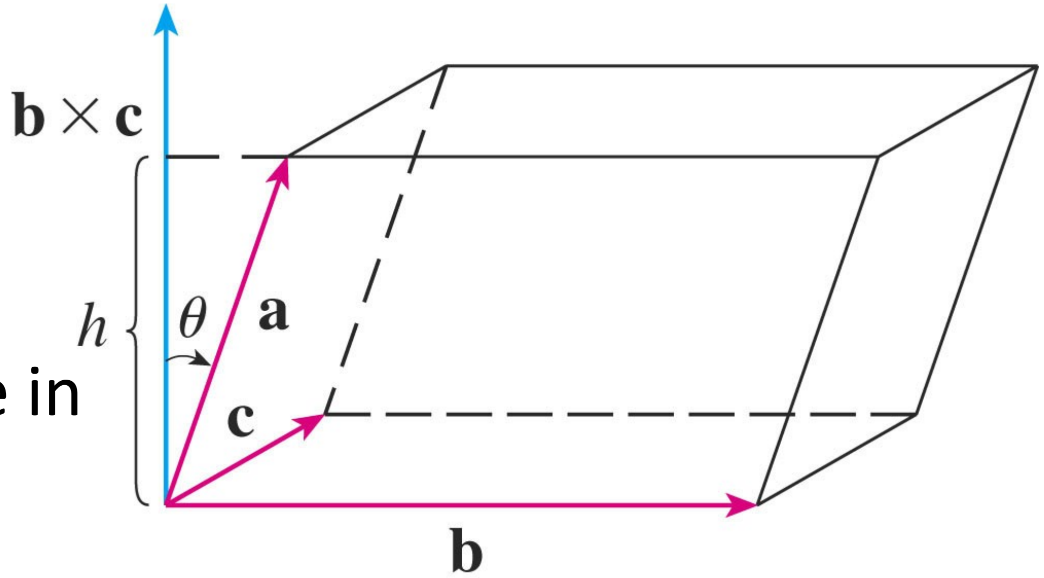
Scalar triple product

Let **a** = $\langle a_1, a_2, a_3 \rangle$, **b** = $\langle b_1, b_2, b_3 \rangle$ and **c** = $\langle c_1, c_2, c_3 \rangle$. Then. the scalar triple product of these three vectors is defined as:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$



By this formula, if $V = 0$, then the vectors must lie in the same plane. That is, they are coplanar

Example 10

Use the scalar triple product to show that the vectors **a** = <1, 4, -7>, **b** = <2, -1, 4>, **c** = <0, -9, 18> are coplanar.

Solution

The vectors **a** = <1, 4, -7>, **b** = <2, -1, 4>, **c** = <0, -9, 18> are coplanar, if their scalar triple product equals zero.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18) = 0$$