Mathematics (2)

Section (6)

Vector calculus and Green's Theorem

Done by Waleed Khaled

Vector Fields and Line Integrals: Work, Circulation, and Flux

A vector field is a function that assigns a vector to each point in its domain.

A vector field on a solid region V in space might have a formula like

$$\vec{\mathrm{F}} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$$

The vector field is **continuous** if the component functions M, N, and P are **continuous**; it is **differentiable** if each of the component functions is **differentiable**.

The formula for a vector field in plane could look like

$$\vec{\mathrm{F}} = M(x,y)\vec{i} + N(x,y)\vec{j}$$

Let $\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$ be a continuous vector field defined along a smooth plane curve C parametrized by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j};\quad a \le t \le b$. Then the line integral of $\vec{F}$ along C is

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\frac{d\vec{r}}{ds}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C M dx + N dy \qquad (1)$$

where $\vec{T} = \frac{d\vec{r}}{ds}$ is a unit vector tangent to the curve C.

Remarks:

- If C is a closed smooth curve, then the line integral of $\vec{F}$ along C, given in (1), is denoted by $\oint_C \vec{F}\cdot d\vec{r}$ and is called the *circulation* of $\vec{F}$ around the curve c.

- If the vector field $\vec{F}$ is a continuous force field, then the line integral given in (1) represents the *total work* done in moving an object from the point $A = (x(a), y(a))$ to the point $B = (x(b), y(b))$ along C.

Example 1 Find the work done by the force field $\vec{F} = x^2\vec{i} - xy\vec{j}$ in moving particle along the quarter-circle $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}, \quad 0 \le t \le \pi/2$

Solution

Because $x = \cos t$ and $y = \sin t$, you have $dx = -\sin t dt$ and $dy = \cos t dt$. Therefore, the work done is

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r}} = \boxed{\int_C M dx + N dy} = \int_C x^2 dx - xy dy$$

$$= \int_0^{\frac{\pi}{2}} \cos^2(t).-\sin(t)dt - \cos(t)\sin(t)\cos(t)dt = \int_0^{\frac{\pi}{2}} \left(\cos^2(t).(-\sin(t)) - \cos^2(t)\sin(t)\right)dt$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^2(t)(-\sin(t))dt = 2\left[\frac{\cos^3(t)}{3}\right]_0^{\frac{\pi}{2}} = -\frac{2}{3}$$

The work done is negative because the field impedes movement along the curve

Example 2 Evaluate the circulation of $\vec{F} = y^3\vec{i} + \left(x^3 + 3xy^2\right)\vec{j}$ around the curve the circle C of radius 3 given by $\vec{r}(t) = 3\cos(t)\vec{i} + 3\sin(t)\vec{j}, \quad 0 \le t \le 2\pi$

Solution

Because $x = 3\cos t$ and $y = 3\sin t$, you have $dx = -3\sin t dt$ and $dy = 3\cos t dt$. So, the circulation of the vector field is

$$\oint_C M dx + N dy = \oint_C y^3 dx + \left(x^3 + 3xy^2\right) dy$$

$$= \int_0^{2\pi} \left[\left(27\sin^3 t\right)(-3\sin t) + \left(27\cos^3 t + 81\cos t \sin^2 t\right)(3\cos t)\right] dt$$

$$= 81\int_0^{2\pi} \left(\cos^4 t - \sin^4 t + 3\cos^2 t \sin^2 t\right) dt = 81\int_0^{2\pi} \left(\cos^2 t - \sin^2 t + \frac{3}{4}\sin^2 2t\right) dt$$

$$= 81\int_0^{2\pi} \left[\cos 2t + \frac{3}{4}\left(\frac{1-\cos 4t}{2}\right)\right] dt = 81\left[\frac{\sin 2t}{2} + \frac{3}{8}t - \frac{3\sin 4t}{32}\right]_0^{2\pi} = \frac{243\pi}{4}$$

Example 3 Evaluate the line integral $\int_C x\,dy - y\,dx$ along the curve C defined by the equation $y = x^3$ from the origin $(0,0)$ to $(2,8)$.

Solution

The curve C: $y = x^3$ can be parametrized as $x = t$ and $y = t^3$, $0 \le t \le 2$, you have $dx = dt$ and $dy = 3t^2 dt$. So, the line integral is

$$\int_C x\,dy - y\,dx = \int_0^2 t.3t^2 dt - t^3 dt = \int_0^2 2t^3 dt = 2\left[\frac{t^4}{4}\right]_0^2 = 8$$

~~Another solution.~~

Substituting $y = x^3$ and $dy = 3x^2 dx$ in the integrand, we obtain

$$\int_C x\,dy - y\,dx = \int_0^2 x \cdot 3x^2 dx - x^3 dx = \int_0^2 2x^3 dx = 2\left[\left(\frac{x^4}{4}\right)\Big|_0^2\right] = 8$$

Flux of the vector field

The flux of a vector field $\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$ across a smooth closed plane curve C is defined by the integral

$$\boxed{\oint_C M\,dy - N\,dx}$$

Example 4 **Find the flux of $\vec{F} = (x-y)\vec{i} + x\vec{j}$ across the circle $x^2 + y^2 = 1$ in the xy-plane.**

Solution

We can use the parametrization $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j},\quad 0 \le t \le 2\pi$. Therefore, we have

$$M = x - y = \cos t - \sin t, \qquad dy = d(\sin t) = \cos t\,dt$$

$$N = x = \cos t, \qquad dx = d(\cos t) = -\sin t\,dt$$

We find

$$\text{Flux} = \oint_C M\,dy - N\,dx = \int_0^{2\pi} \left(\cos^2 t - \sin t\cos t + \cos t\sin t\right)dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\int_0^{2\pi} (1+\cos 2t)\,dt = \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi$$

The flux of $\mathbf{F}$ across the circle is π.

The divergence of a vector field

If $\vec{F} = M\vec{i} + N\vec{j} + R\vec{k}$ is a vector field on and $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial y}, \frac{\partial R}{\partial z}$ exist, the divergence of $\vec{\mathrm{F}}$ is the function of three variables defined by :

$$\text{div } \vec{\mathrm{F}} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial R}{\partial z}$$

In terms of the vector differential operator ∇ ("del" or "nabla")

$$\nabla = \left(\frac{\partial}{\partial x}\right)\vec{i} + \left(\frac{\partial}{\partial y}\right)\vec{j} + \left(\frac{\partial}{\partial z}\right)\vec{k}$$

the divergence of $\vec{\mathrm{F}}$ can be written symbolically as the dot product of ∇ and $\vec{\mathrm{F}}$.

$$\text{div } \vec{\mathrm{F}} = \nabla \cdot \vec{\mathrm{F}}$$

The curl of a vector field

If $\vec{\mathrm{F}} = M\vec{i} + N\vec{j} + R\vec{k}$ is a vector field in space, and the partial derivatives of M, N, and R all exist. The *curl* of $\vec{\mathrm{F}}$ is the vector field defined by

$$\operatorname{curl}\vec{\mathrm{F}} = \left(\frac{\partial R}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & R \end{vmatrix} = \nabla \times \vec{\mathrm{F}}$$

Example 5 Find the divergence and curl of the vector field

$$\vec{F} = xz\vec{i} + xyz\,\vec{j} - y^2\vec{k}$$

Solution

The divergence of $\vec{F}$ is

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}\left(-y^2\right) = z + xz$$

and the curl of $\vec{F}$ is

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(-y^2\right) - \frac{\partial}{\partial z}(xyz)\right]\vec{i} - \left[\frac{\partial}{\partial x}\left(-y^2\right) - \frac{\partial}{\partial z}(xz)\right]\vec{j} + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\vec{k}$$

$$= (-2y - xy)\vec{i} - (0 - x)\vec{j} + (yz - 0)\vec{k} = -y(2 + x)\vec{i} + x\vec{j} + yz\vec{k}.$$

Example 6 **Find the divergence and curl of the vector field**

$$\vec{F} = xy^2z^2\vec{i} + x^2yz^2\,\vec{j} + x^2y^2z\vec{k}$$

Solution

The divergence of $\vec{F}$ is

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(xy^2z^2) + \frac{\partial}{\partial y}(x^2yz^2) + \frac{\partial}{\partial z}\left(x^2y^2z\right) = y^2z^2 + x^2z^2 + x^2y^2$$

and the curl of $\vec{F}$ is

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(x^2y^2z\right) - \frac{\partial}{\partial z}(x^2yz^2)\right]\vec{i} - \left[\frac{\partial}{\partial x}\left(x^2y^2z\right) - \frac{\partial}{\partial z}(xy^2z^2)\right]\vec{j} + \left[\frac{\partial}{\partial x}(x^2yz^2) - \frac{\partial}{\partial y}(xy^2z^2)\right]\vec{k}$$

$$= (2x^2yz - 2x^2yz)\vec{i} - (2xy^2z - 2xy^2z)\vec{j} + (2xyz^2 - 2xyz^2)\vec{k} = 0\vec{i} + 0\vec{j} + 0\vec{k}.$$

Example 7 If $\vec{F} = (y^4 - x^2z^2)i + (x^2 + y^2)\vec{j} - x^2yz\vec{k}$, determine curl $\vec{F}$ at $(1,3,-2)$.

Solution

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^4 - x^2z^2 & x^2 + y^2 & -x^2yz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-x^2yz) - \frac{\partial}{\partial z}(x^2 + y^2)\right)\vec{i} - \left(\frac{\partial}{\partial x}(-x^2yz) - \frac{\partial}{\partial z}(y^4 - x^2z^2)\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y^4 - x^2z^2)\right)\vec{k}$$

$$= -x^2z\vec{i} - (-2xyz + 2x^2z)\vec{j} + (2x - 4y^3)\vec{k}.$$

At (1,3,-2),

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = -(1)^2(-2)\vec{i} - (-2(1)(3)(-2) + 2(1)^2(-2))\vec{j}$$

$$+ (2(1) - 4(3)^3)\vec{k}$$

$$= 2\vec{i} - 8\vec{j} - 106\vec{k}.$$

Exercise 1

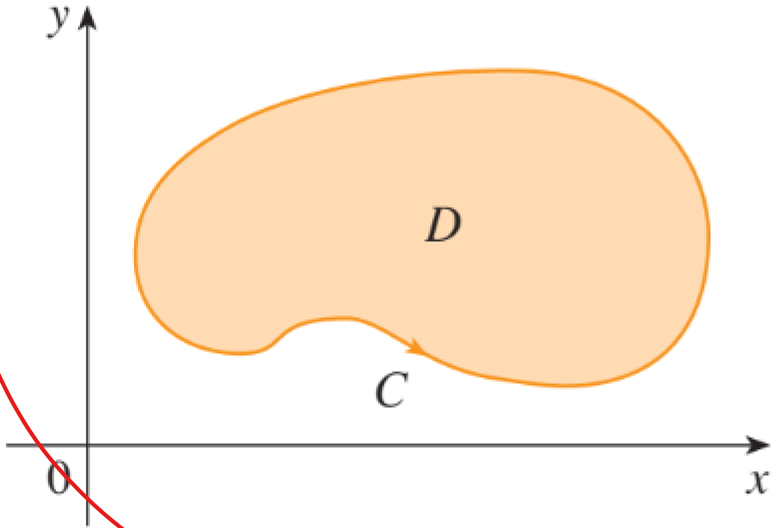
If $\vec{F} = (xy^3 - y^2z^2)\vec{i} + (x^2 + z^2)\vec{j} - x^2yz^2\vec{k}$, determine $\text{curl}\,\vec{F}$ at point $(1,2,3)$.

GREEN'S THEOREM

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C.

- If M and N have continuous partial derivatives on an open region that contains D, then

$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$



Example 8 Evaluate

$$\int_C x^4\,dx + xy\,dy$$

where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, from $(0, 1)$ to $(0, 0)$

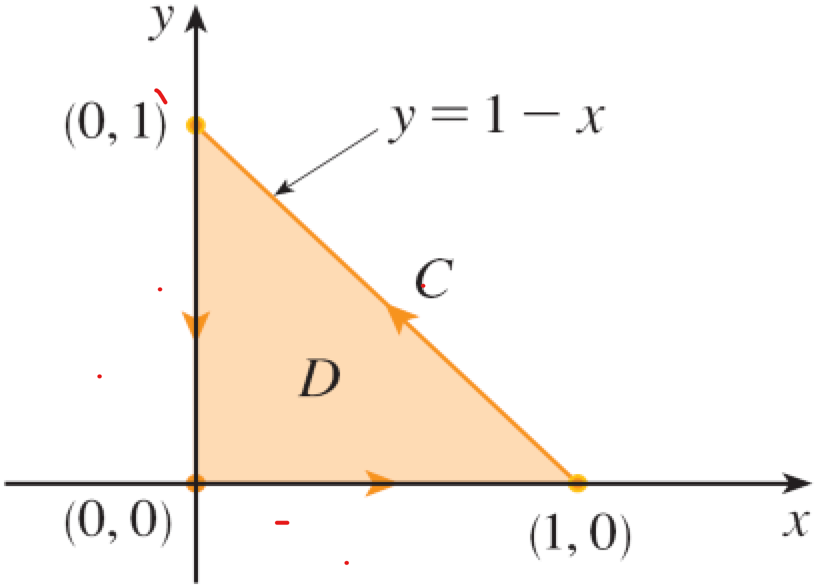
Solution

Notice that the region D enclosed by C is simple and C has positive orientation

Equation of a line passing through $(1,0)$ and $(0,1)$ is

$$\frac{y-0}{x-1} = \frac{1-0}{0-1} = -1$$

$$\Rightarrow y = -1(x-1) = 1-x$$



If we let $M(x, y) = x^4$ and $N(x, y) = xy$, then

$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx$$

$$= \int_0^1 [\tfrac{1}{2} y^2]_{y=0}^{y=1-x}\,dx$$

$$= \tfrac{1}{2} \int_0^1 (1-x)^2\,dx$$

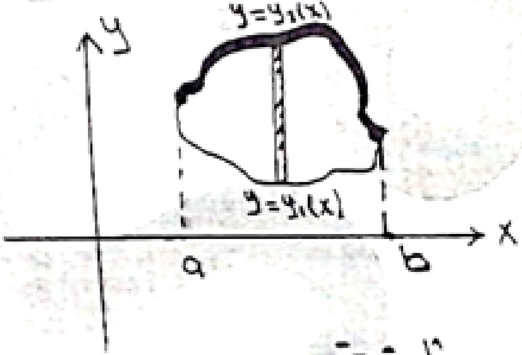
$$= -\tfrac{1}{6}(1-x)^3 \Big]_0^1 = \tfrac{1}{6}$$

Evaluation of The Double Integral

$$\iint_R f(x,y)\,dA$$

1 Evaluation of Double Integrals in Cartesian Coordinates:

1) vertical strips:



x ← ثابتة

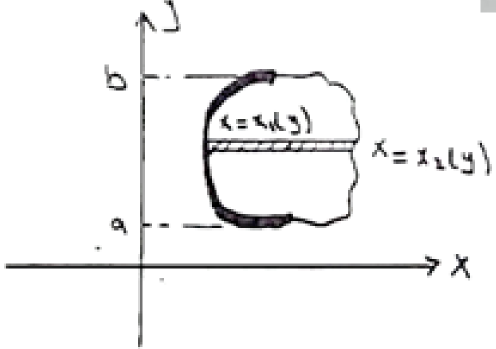
y ← متغيرة

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$$

الحد المتغير

الحد الثابت

2) Horizontal strips:



y ← ثابتة

x ← متغيرة

$$\iint_R f(x,y)\,dA = \int_{y=a}^{b}\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$$

الحد المتغير

الحد الثابت

Example 9 Use Green's Theorem to evaluate the line integral

$$\oint_C y^3 dx + \left(x^3 + 3xy^2\right) dy$$

$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$.

Solution

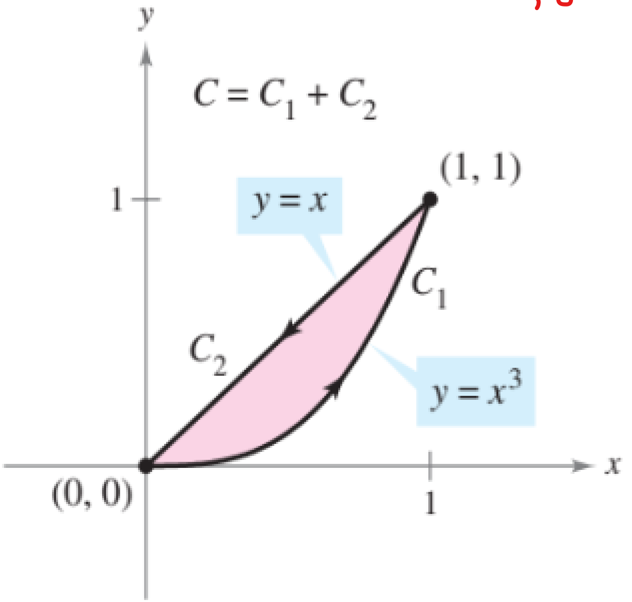
Because $M = y^3$ and $N = x^3 + 3xy^2$, it follows that $\frac{\partial N}{\partial x} = 3x^2 + 3y^2$ and $\frac{\partial M}{\partial y} = 3y^2$.

Applying Green's Theorem, you then have

$$\oint_C y^3 dx + \left(x^3 + 3xy^2\right) dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \int_0^1 \int_{x^3}^{x} \left[\left(3x^2 + 3y^2\right) - 3y^2\right] dy\,dx$$

$$\int_0^1 \int_{x^3}^{x} 3x^2\,dy\,dx = \int_0^1 \left[3x^2 y\right]_{x^3}^{x} dx = \int_0^1 \left(3x^3 - 3x^5\right) dx = \left[\frac{3x^4}{4} - \frac{x^6}{2}\right]_0^1 = \frac{1}{4}$$



Example 10 Evaluate

$$\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$$

where C is the circle $x^2 + y^2 = 9$

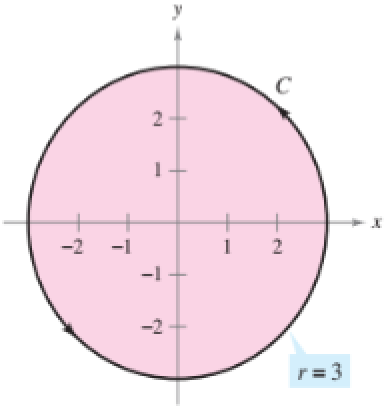
$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

Solution

- The region D bounded by C is the disk $x^2 + y^2 \le 9$. So, let's change to polar coordinates after applying Green's Theorem:

$$\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(7x + \sqrt{y^4+1}\right) - \frac{\partial}{\partial x}\left(3y - e^{\sin x}\right)\right]dA = \iint_D (7-3)\,dA == \iint_D 4\,dA$$



$$= \int_0^{2\pi}\int_0^3 4r\,dr\,d\theta = \int_0^{2\pi}\left[2r^2\right]_0^3 d\theta = 18\int_0^{2\pi} d\theta = 36\pi$$

Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4dA = 4\iint_D dA = 4\cdot\pi(3)^2 = 36\pi$$

Example 11 Find the work done an object moves in the force field

$$\vec{F} = \left(x + 2y^2\right)\vec{j}$$

once counterclockwise around the circular path $(x-2)^2 + y^2 = 1.$

Solution

Let $\vec{F} = 0\vec{i} + \left(x + 2y^2\right)\vec{j}$. Then, $M = 0$ and $N(x, y) = x + .2y^2$ Let D be the region bounded by the circle $(x-2)^2 + y^2 = 1.$

Then, by Green's theorem we have

$$W = \oint_C \vec{F} \cdot d\vec{r} = \oint_C M dx + N dy = \oint_C 0 dx + \left(x + 2y^2\right) dy$$

$$= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_D dA = \pi$$

We simply used the fact that D is a disk of radius 1.

Exercises

In Exercises 1: 6, use Green's Theorem to evaluate the line integral.

1. $\int_c 2xydx + (x + y)dy$
C: boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$
2. $\int_c y^2dx + xydy$
C: boundary of the region lying between the graphs of $y = 0$, $y = \sqrt{x}$, and $x = 9$

3. $\int_c (x^2 - y^2)dx + 2xydy$
$C: x^2 + y^2 = 4$

4. $\int_c y\,dx + \ln(x^2 + y^2)dy$
$C: x = 4 + 2\cos t, y = 4 + \sin t$

5. $\int_c \sin x \cos y dx + (xy + \cos x \sin y)dy$
C: boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$

6. $\int_c (e^{-x^2/2} - y)dx + (e^{-y^2/2} + x)dy$
C: boundary of the region lying between the graphs of the circle $x = 6\cos\theta, y = 6\sin\theta$ and the ellipse $x = 3\cos\theta, y = 2\sin\theta$