Mathematics (2)

Section (5)

Line integrals and the gradient of a function

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Line integrals

Up to this point, you have studied various types of integrals. For a single integral

$$\int_a^b f(x)dx$$

Integrate over interval $[a, b]$.

you integrated over the interval $[a, b]$. Similarly, for a double integral

$$\iint_R f(x, y)dA$$

Integrate over region R.

you integrated over the region R in the plane. In this section, you will study a new type of integral called a line integral

$$\int_C f(x, y)ds$$

Integrate over curve C.

for which you integrate over a piecewise smooth curve C. (The terminology is somewhat unfortunate- this type of integral might be better described as a "curve integral.")

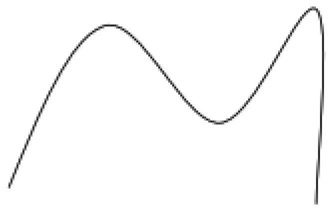
Terminology

C is a parametric curve $x = h(t), y = g(t), \qquad a \le t \le b$

C is SMOOTH

$$h, g \quad \text{continous}$$

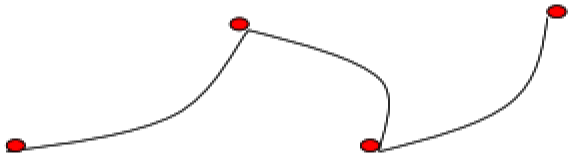
$$h'(t) \neq 0 \text{ or } g'(t) \neq 0 \quad \text{for all } t \in [a,b]$$



C is piecewise smooth

$$C = C_1 \cup C_2 \cdots \cup C_n$$

$$C_1, C_2 \cdots, C_n \quad \text{smooth}$$

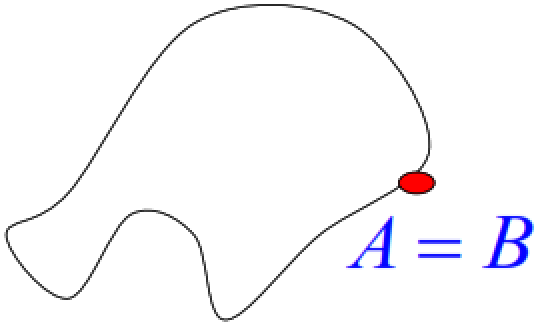


C is a parametric curve $x = h(t), y = g(t), \qquad a \le t \le b$

C is closed curve:

$$A = B$$

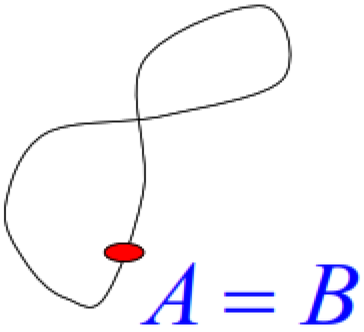
$A = \left(h(a), g(a)\right)$ and $B = \left(h(b), g(b)\right)$



C is simple closed curve:

$$A = B$$

Does not intersect itself



Evaluation of a line integral as a definite integral

- Let C be a smooth plane curve given by

$$x = h(t), y = g(t), \qquad a \le t \le b$$

- If f is defined on the curve C, then the **line integral of f along C**

$$\int_C f(x,y)\,ds = \int_a^b f\big(h(t), g(t)\big)\,ds,$$

where,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

- Let C be a smooth space curve given by

$$x = h(t), y = g(t),\ z = k(t), \qquad a \le t \le b$$

- If f is defined on the curve C, then the **line integral of f along C**

$$\int_C f(x,y,z)\,ds = \int_a^b f\big(h(t), g(t), k(t)\big)\,ds,$$

where,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

- Note that if $f = 1$, the line integral gives the arc length of the curve C. That is,

$$\int_C 1\,ds = \text{length of curve C.}$$

Examples

Example 1 **Evaluate the following line integral $\int_C y e^x \, ds$, where C is the line segment joining (1, 2) to (4, 8)**

Solution

To find the line integral, we first need to find the parameterization for C.

Given that, C is a line segment from $(1,2)$ to $(4,8)$. Equation of a line passing through $(1,2)$ and $(4,8)$ is

$$\frac{y-2}{x-1} = \frac{8-2}{4-1} = \frac{6}{3} = 2$$

$$\Rightarrow y-2 = 2(x-1)$$

$$\Rightarrow y = 2(x-1)+2 = 2x$$

Therefore, a suitable parameterization would be

$$x = t \qquad \text{and} \qquad y = 2t$$

As t increases from 1 to 4 , the point moves from $(1,2)$ to $(4,8)$. Therefore. we will integrate from 1 to 4

Since $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2$ we have:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(1)^2 + (2)^2}\,dt = \sqrt{5}\,dt$$

The line integral becomes

$$\int_C y\,e^x\,ds = \int_1^4 2t\,e^t\sqrt{5}\,dt = 2\sqrt{5}\int_1^4 t\,e^t\,dt =$$

The integrand is the product of the algebraic function t with the exponential function e^t. So, we shall apply Integration by Parts, by letting

$$u = t \quad \text{and} \quad dv = e^t\,dt$$

so that

$$du = dt \quad \text{and} \quad v = \int e^t\,dt = e^t$$

Thus,

$$\int_1^4 te^t\,dt = \int_1^4 u\,dv = [uv]_{t=1}^{t=4} - \int_1^4 v\,du$$

$$\Rightarrow \int_1^4 te^t\,dt = \left[te^t\right]_{t=1}^{t=4} - \int_1^4 e^t\,dt = \left[4e^4 - e\right] - \left[e^t\right]_{t=1}^{t=4}$$

$$\Rightarrow \int_1^4 te^t\,dt = \left[4e^4 - e\right] - \left[e^4 - e\right] = 3e^4$$

Hence,

$$\int_C y\,e^x\,ds = 2\sqrt{5}\int_1^4 t\,e^t\,dt = 2\sqrt{5}\left(3e^4\right) = 6\sqrt{5}e^4.$$

Example 2 Evaluate the line integral $\int_C (xy + z^3)ds$ from $(1, 0, 0)$ to $(-1, 0, \pi)$ along the helix C that is represented by the parametric equations

$$x = \cos t,\ y = \sin t,\ z = t.$$

Solution

$$\int_C \left(xy + z^3\right) ds = \int_0^{\pi} \left(\cos t \sin t + t^3\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \left(\cos t \sin t + t^3\right) \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt$$

$$= \sqrt{2} \int_0^{\pi} \left(\cos t \sin t + t^3\right) dt$$

$$= \sqrt{2} \left[\frac{\sin^2 t}{2} + \frac{t^4}{4}\right]_0^{\pi} = \frac{\sqrt{2}\pi^4}{4}$$

Example 3 **Evaluate**

$$\int_C (x^2 - y + 3z)ds$$

where C is the line segment from $(0,0,0)$ to $(1,2,1)$.

Solution

Begin by writing a parametric form of the equation of a line:

$$x = t,\ y = 2t,\ \text{and}\ \ z = t,\ 0 \le t \le 1$$

Therefore, $x'(t) = 1, y'(t) = 2,$ and $z'(t) = 1,$ which implies that

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt = \sqrt{1^2 + 2^2 + 1^2}dt = \sqrt{6}dt$$

So, the line integral takes the following form.

$$\int_C (x^2 - y + 3z)ds = \int_0^1 (t^2 - 2t + 3t)\sqrt{6}dt = \sqrt{6}\int_0^1 (t^2 + t)dt = \sqrt{6}\left[\frac{t^3}{3} + \frac{t^2}{2}\right]_0^1 = \frac{5\sqrt{6}}{6}$$

Piecewise-smooth curves and line integrals

If C is a piecewise-smooth curve then C can be written as a finite union of smooth curves; that is,

$$C = C_1 \cup C_2 \ldots \cup C_n$$

The line integral of f along C is defined as the sum of the line integrals of f along each of the smooth pieces of C; that is,

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds$$

Example 4 Evaluate $\int_C x\,ds$ where C is the piecewise- smooth curve formed by the boundary region bounded by $y = x$ and $y = x^2$.

Solution

Begin by integrating up the line $y = x$, using the following parametrization

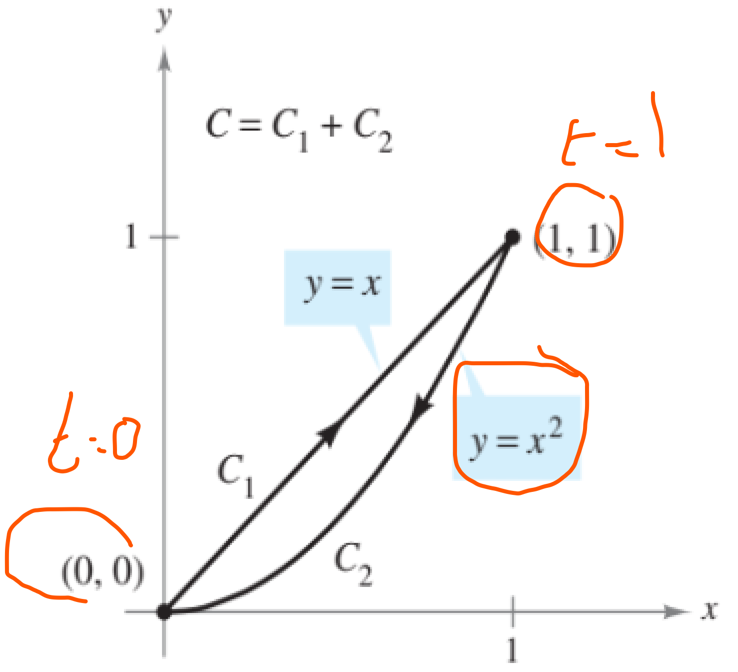
$$C_1 : x = t, y = t, 0 \le t \le 1$$

This implies that $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 1$. So,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{2}dt$$

and we have

$$\int_{C_1} x ds = \int_0^1 t\sqrt{2}dt = \left[\frac{\sqrt{2}}{2}t^2\right]_0^1 = \frac{\sqrt{2}}{2}$$



Next, integrate down the parabola $y = x^2$, using the parametrization

$$C_2 : x = 1 - t,\ y = (1-t)^2, \qquad 0 \le t \le 1$$

This implies that $\dfrac{dx}{dt} = -1$ and $\dfrac{dy}{dt} = -2(1-t)$ So,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{1 + 4(1-t)^2}\, dt$$

and we have

$$\int_{C_2} x\, ds = \int_0^1 (1-t)\sqrt{1 + 4(1-t)^2}\, dt$$

$$= -\frac{1}{8}\left[\frac{2}{3}\left[1 + 4(1-t)^2\right]^{3/2}\right]_0^1 = \frac{1}{12}\left(5^{3/2} - 1\right)$$

Consequently,

$$\int_C x\, ds = \int_{C_1} x\, ds + \int_{C_2} x\, ds = \frac{\sqrt{2}}{2} + \frac{1}{12}\left(5^{3/2} - 1\right) = 1.56$$

$$t = 1\ , \qquad x = 1$$
$$t = 2\ , \qquad x = 0$$

$$C_2 : x = 2 - t \qquad 1 \le t \le 2$$

$$y = (2-t)^2$$

$$\frac{dx}{dt} = -1 \quad , \frac{dy}{dt} = -2(2-t)$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$ds = \sqrt{(-1)^2 + \left(-2(2-t)\right)^2}\, dt$$

$$ds = \sqrt{1 + 4(t-2)^2}\, dt$$

$$I = \int_1^2 (2-t)\sqrt{1+4(t-2)^2}\, dt$$

$$I = \frac{-1}{8}\int_1^2 8(2-t)\sqrt{1+4(t-2)^2}\, dt$$

$$I = \frac{-1}{8}\int_1^2 \sqrt{1+4(t-2)^2}\, d(1+4(t-2)^2)$$

$$I = \frac{-1}{8}\left[\frac{(1+4(t-2)^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2 = \frac{1}{12}\left[5^{\frac{3}{2}} - 1\right]$$

Example ~~6~~ Evaluate the line integral $\int_C xdy - ydx$ along the curve C defined by the equation $y = x^3$ from the origin $(0,0)$ to $(2,8)$.

Solution

$t=0$ $\quad$ $t=2$

The curve C: $y = x^3$ can be parametrized as $x = t$ and $y = t^3$, $0 \le t \le 2$, you have $dx = dt$ and $dy = 3t^2dt$. So, the line integral is

$$\int_C xdy - ydx = \int_0^2 t.3t^2dt - t^3dt = \int_0^2 2t^3dt = 2\left[\frac{t^4}{4}\right]_0^2 = 8$$

Another solution.

Substituting $y = x^3$ and $dy = 3x^2dx$ in the integrand, we obtain

$$\int_C xdy - ydx = \int_0^2 x \cdot 3x^2dx - x^3dx = \int_0^2 2x^3dx = 2\left[\left(\frac{x^4}{4}\right)\Bigg|_0^2\right] = 8$$

Curve	Parametric Equations
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (Ellipse)	Counter-Clockwise: $x=a\cos(t)$, $y=b\sin(t)$, $0\le t\le 2\pi$ Clockwise: $x=a\cos(t)$, $y=-b\sin(t)$, $0\le t\le 2\pi$
$x^2+y^2=r^2$ (Circle)	Counter-Clockwise: $x=r\cos(t)$, $y=r\sin(t)$, $0\le t\le 2\pi$ Clockwise: $x=r\cos(t)$, $y=-r\sin(t)$, $0\le t\le 2\pi$
$y=f(x)$	$x=t$ $y=f(t)$
$x=g(y)$	$x=g(t)$ $y=t$
Line Segment From (x_0,y_0,z_0) to (x_1,y_1,z_1)	$\vec{r}(t)=(1-t)\langle x_0,y_0,z_0\rangle+t\langle x_1,y_1,z_1\rangle \quad , \quad 0\le t\le 1$ or $x=(1-t)x_0+t\,x_1$ $y=(1-t)y_0+t\,y_1 \quad , \quad 0\le t\le 1$ $z=(1-t)z_0+t\,z_1$

Gradient of a Function

Let $z = f(x, y)$ be a function of two variables x and y such that f_x and f_y exist. Then the gradient of f, denoted by ∇f, is the vector

$$\nabla f = \left\langle f_x(x, y), f_y(x, y) \right\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

∇f is read as "del f". Another notation for the gradient is **grad** f

Let $w = f(x, y, z)$ be a function of three variables x, y and z such that f_x, f_y and f_z exist. Then the gradient of f, is the vector

$$\nabla f = \left\langle f_x(x, y), f_y(x, y), f_z(x, y) \right\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Example 1 If $f(x, y, z) = x\sin(yz)$, find the gradient of f

Solution

The gradient of f is:

$$\nabla f(x, y, z)$$

$$= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

$$= \langle \sin yz, xz\cos yz, xy\cos yz \rangle$$

Example 2 Find the gradient of $f(x,y) = y\ln x + xy^2$ at the point $(1,2)$.

Solution

Using

$$f_x(x,y) = \frac{y}{x} + y^2 \text{ and } f_y(x,y) = \ln x + 2xy$$

you have

$$\nabla f(x,y) = \left(\frac{y}{x} + y^2\right)\mathbf{i} + (\ln x + 2xy)\mathbf{j}$$

At the point (1,2), the gradient is

$$\begin{aligned}\nabla f(1,2) &= \left(\frac{2}{1} + 2^2\right)\mathbf{i} + [\ln 1 + 2(1)(2)]\mathbf{j} \\ &= 6\mathbf{i} + 4\mathbf{j}.\end{aligned}$$

Exercises

1-6 Find the gradient of f at the indicated point.

1. $f(x,y) = 5x^2 + y^4; (4,2)$

2. $f(x,y) = 5\sin x^2 + \cos 3y; (\sqrt{\pi}/2, 0)$

3. $f(x,y) = (x^2 + xy)^3; (-1,-1)$

4. $f(x,y) = (x^2 + y^2)^{-1/2}; (3,4)$

5. $f(x,y,z) = y\ln(x + y + z); (-3,4,0)$

6. $f(x,y,z) = y^2 z\tan^3 x; (\pi/4, -3, 1)$

7-14 Find ∇z or ∇w.

7. $z = \sin(7y^2 - 7xy)$

8. $z = 7\sin(6x/y)$

9. $z = \frac{6x+7y}{6x-7y}$

10. $z = \frac{6xe^{3y}}{x+8y}$

11. $w = -x^9 - y^3 + z^{12}$

12. $w = xe^{8y}\sin 6z$

13. $w = \ln\sqrt{x^2 + y^2 + z^2}$

14. $w = e^{-5x}\sec x^2 yz$