Mathematics (1)

Revision

Function and it's application

What is The domain of the following functions ?

(1) $f\left(x\right) = \frac{5-x}{x-5}$ **?**

The function is be undefined when x-5 =0 , x=-5

The domain = R-{5}

(2)$f\,(x) = \sqrt{x^2 - 4}$

The domain consist of all x such that $x \geq 2$ or $x \leq -2$ since we must have $x^2 \geq 4$

The domain is [-∞,-2]U[2, ∞]

(3) f(x)= $\frac{\sqrt{\boldsymbol{x+2}}}{\boldsymbol{x^2-9}}$

the function is undefined when $\boldsymbol{x^2} - \boldsymbol{9}$=0 , $\boldsymbol{x^2}$=9 ,x=3 ,x=-3 and

$\sqrt{\boldsymbol{x+2}}$ is defined when x+2≥ 0 , x $\geq$ -2

So domain = [-2, ∞)-{-3,3}

(4)$f(\dot{x}) = \frac{|x|}{x}$

Domain = R-{0}

(5)$f(x) = \frac{1}{\sqrt{1-x^2}}$

$x^2 < 1$, x < 1 , x > -1

So The domain= (-1,1)

(6)What is the range of $f(x) = \sin x$?

The range = [-1,1]

(7)What is the range of $y = x^2 + 3x + 4$?

to get the rang of this equation we can get the domain of x=f(y) and that be the rang of y=f(x)

$$y - 4 = x^2 - 3x$$

$$y - 4 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$$

$$y - \frac{7}{4} = \left(x - \frac{3}{2}\right)^2$$

$$x = \sqrt{y - \frac{7}{4}} + \frac{3}{2}$$

The domain of this function when $y \geq \frac{7}{4}$ is [$\frac{7}{4}$,∞]

So that the range of our function is [$\frac{7}{4}$,∞]

(10) Determine of this function is even or odd or Neither even or odd

$$g(x) = \left(\frac{x^2+1}{x-1}\right)$$

$$g(-x) = \left(\frac{(-x)^2+1}{-x-1}\right) = \left(\frac{x^2+1}{-x-1}\right) \neq g(x)$$

so this function is Neither even or odd

(11)What is the distance between (3,2),(7,8)?

d=$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

d= $\sqrt{(7-3)^2 + (8-2)^2}$ = $\sqrt{(4)^2 + (6)^2}$ = $\sqrt{16+36}$

d = $\sqrt{52}$ =7.21

(12) find the radius of circle $4(x+7)^2+4\,(y-3)^2=100$

by divide by 4

$$(x+7)^2+(y-3)^2=25$$

$$r^2=25$$

$$r=5$$

the radius of circle is 5

(13)what is the center and the radius of the circle that is given by equation

$$x^2 + y^2 - 4x - 4y = 56$$

$$x^2 - 4x + y^2 - 4y = 56$$

$$(x-2)^2 - 2^2 + (y-2)^2 - 2^2 = 56$$

$$(x-2)^2 + (y-2)^2 - 8 = 56$$

$$(x-2)^2 + (y-2)^2 = 64$$

the center is (2,2) , r^2=64 ,radius r=8

(14) find the focus of the equation of parabola x^2=8y

$(\boldsymbol{x}-\boldsymbol{h})^2$**=4p(y-k)**

$(\boldsymbol{x}-\mathbf{0})^2$**=8(y-0)**

4p=8

P=2

The focus point =(h,k+p)

=(0,2)

(15) State the vertex ,the focus and the directrix of parabola having the equation

$$y^2 - 6y + 4x - 3 = 0$$
$$y^2 - 6y = -4x + 3$$
$$(y-3)^2 - 3^2 = -4x + 3$$
$$(y-3)^2 = -4x + 12$$
$$(y-3)^2 = -4(x - 3)$$

The axis symmetry of parabola is parallel to x-axis and open to left

4p=4 ,p=1

Vertex=(3,3)

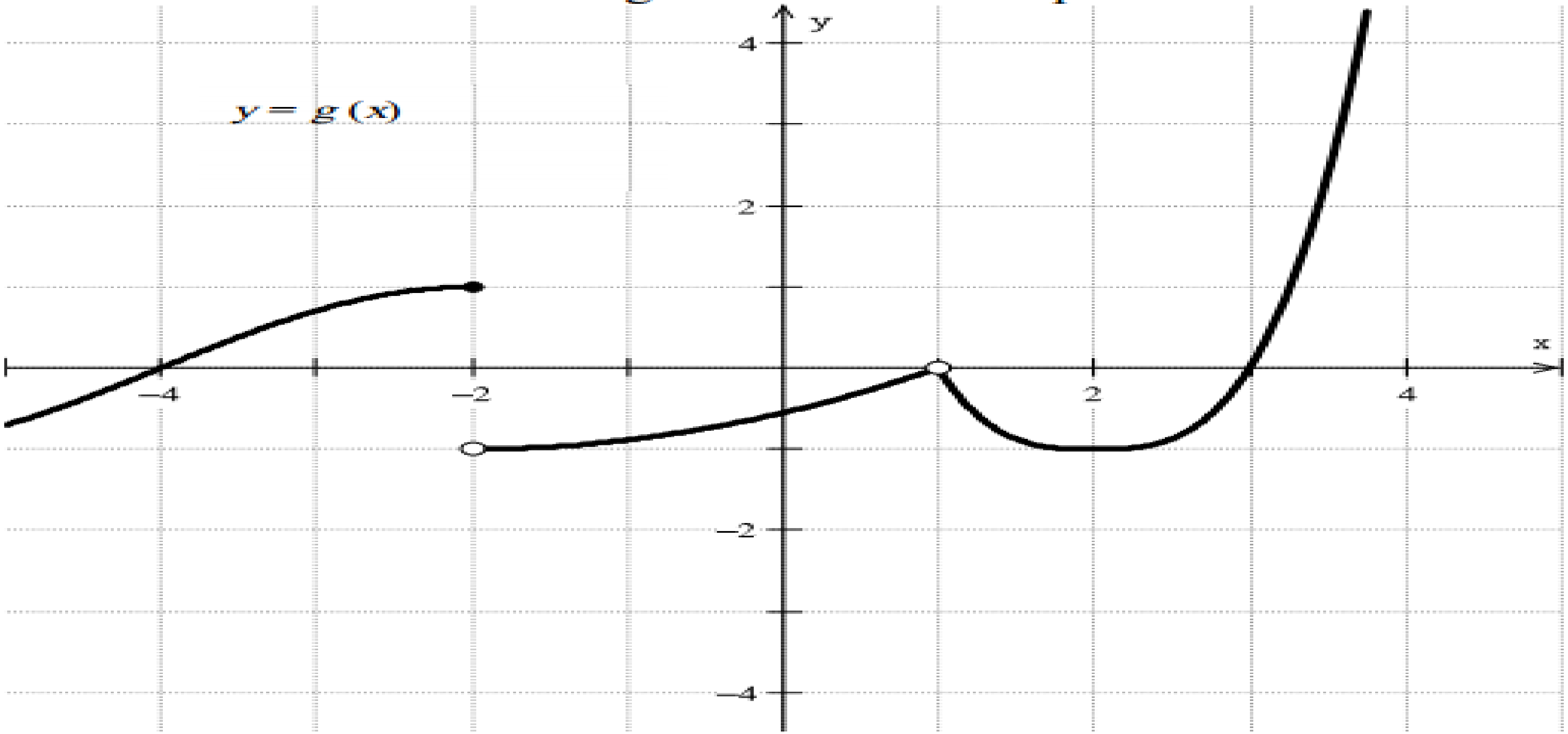
The focus point =(2,3)

The directrix x=4

The equation of the latus rectum is x=2

Limits and Continuity

Ex 2 Evaluate each of the limits given the function pictured below.



a) $\lim_{x \to -2^-} g(x)$
b) $\lim_{x \to -2^+} g(x)$
c) $\lim_{x \to -2} g(x)$
d) $g(-2)$

e) $\lim_{x \to 1^-} g(x)$
f) $\lim_{x \to 1^+} g(x)$
g) $\lim_{x \to 1} g(x)$
h) $g(1)$

i) $\lim_{x \to 2^-} g(x)$
j) $\lim_{x \to 2^+} g(x)$
k) $\lim_{x \to 2} g(x)$
l) $g(2)$

a) 1
b) -1
c) DNE
d) 1

e) 0
f) 0
g) 0
h) DNE

i) -1
j) -1
k) -1
l) -1

(2)find the limit of

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$$

Rewrite $\frac{x^2 - 3x + 2}{x - 2} = \frac{(x-1)(x-2)}{x-2} = x - 1.$

Hence $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \to 2} (x - 1) = 1.$

(3)find the limit of

$$\lim_{x\to\infty}\frac{x^3+x^2+x+1}{x^3+3x^2+5x+2}$$

$$\frac{x^3+x^2+x+1}{x^3+3x^2+5x+2}=\frac{1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{1+\frac{3}{x}+\frac{5}{x^2}+\frac{2}{x^3}}\xrightarrow[x\to\infty]{}1.$$

(4)find the limit of

$$\lim_{x\to\infty}\sqrt{x^2+1}-\sqrt{x^2-1}$$

Rewrite

$$\sqrt{x^2+1}-\sqrt{x^2-1}=\frac{\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$=\frac{\left(\sqrt{x^2+1}\right)^2-\left(\sqrt{x^2-1}\right)^2}{\sqrt{x^2+1}+\sqrt{x^2-1}}=\frac{\left(x^2+1\right)-\left(x^2-1\right)}{\sqrt{x^2+1}+\sqrt{x^2-1}}=\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

Hence $\lim\limits_{x\to\infty}\sqrt{x^2+1}-\sqrt{x^2-1}=\lim\limits_{x\to\infty}\dfrac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}=0.$

(5)find the limit of

$$\lim_{x\to 0}\frac{2x}{\sqrt{2x^2+x+1}-\sqrt{x^2-3x+1}}$$

Rewrite

$$\frac{2x}{\sqrt{2x^2+x+1}-\sqrt{x^2-3x+1}}=$$

$$\frac{2x\left(\sqrt{2x^2+x+1}+\sqrt{x^2-3x+1}\right)}{\left(\sqrt{2x^2+x+1}-\sqrt{x^2-3x+1}\right)\left(\sqrt{2x^2+x+1}+\sqrt{x^2-3x+1}\right)}$$

$$=\frac{2x\left(\sqrt{2x^2+x+1}+\sqrt{x^2-3x+1}\right)}{\left(\sqrt{2x^2+x+1}\right)^2-\left(\sqrt{x^2-3x+1}\right)^2}=\frac{2x\left(\sqrt{2x^2+x+1}+\sqrt{x^2-3x+1}\right)}{x^2+4x}$$

Next divide by *x.*

$$=\frac{2\left(\sqrt{2x^2+x+1}+\sqrt{x^2-3x+1}\right)}{x+4}\xrightarrow[x\to 0]{}1$$

(6)find the limit of

$$\lim_{x \to 0} \frac{\sin(3x)}{6x}$$

Use the fact that $\lim_{\alpha \to 0} \frac{\sin(\alpha)}{\alpha} = 1.$

Rewrite $\frac{\sin(3x)}{6x} = \frac{1}{2}\frac{\sin(3x)}{3x}$

Since $\lim_{x \to 0} \frac{\sin(3x)}{3x} = 1$, we conclude that $\lim_{x \to 0} \frac{\sin(3x)}{6x} = \frac{1}{2}.$

(7) find the limit of

$$\lim_{x \to 0} \frac{\sin(\sin(x))}{x}$$

Rewrite:

$$\frac{\sin(\sin(x))}{x} = \frac{\sin(\sin(x))}{\sin(x)} \frac{\sin(x)}{x} \xrightarrow[x \to 0]{} 1$$

since $\lim_{\alpha \to 0} \frac{\sin(\alpha)}{\alpha} = 1$. In the above, that fact

was applied first by substituting $\alpha = \sin(x)$.

Hence $\lim_{x \to 0} \frac{\sin(\sin(x))}{\sin(x)} = 1$.

(8)find the limit of

$$\lim_{x \to 0} \frac{\sin(x^2)}{x\sin(x)}$$

Rewrite:

$$\frac{\sin(x^2)}{x\sin(x)} = \frac{\sin(x^2)}{x^2}\frac{x}{\sin(x)} \xrightarrow[x \to 0]{} 1$$

CONTINUITY

One of the main topics early in Calculus is CONTINUITY. It really is a simple concept, which, like the Limit, is made complicated by its mathematical definition. Let us take a look at the formal definition of continuous:

Continuous--Defn: "A function f(x) is continuous at x = a if and only if:

i. $f(a)$ exists*,

ii. $\lim_{x \to a} f(x)$ exists*,

and iii. $\lim_{x \to a} f(x) = f(a)$."

**By "exists," we mean that it equals a real number.*

i) "$f(a)$ exists" means a must be in the domain.

ii) "$\lim_{x \to a} f(x)$ exists" means $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

iii) "$\lim_{x \to a} f(x) = f(a)$" should be self explanatory.

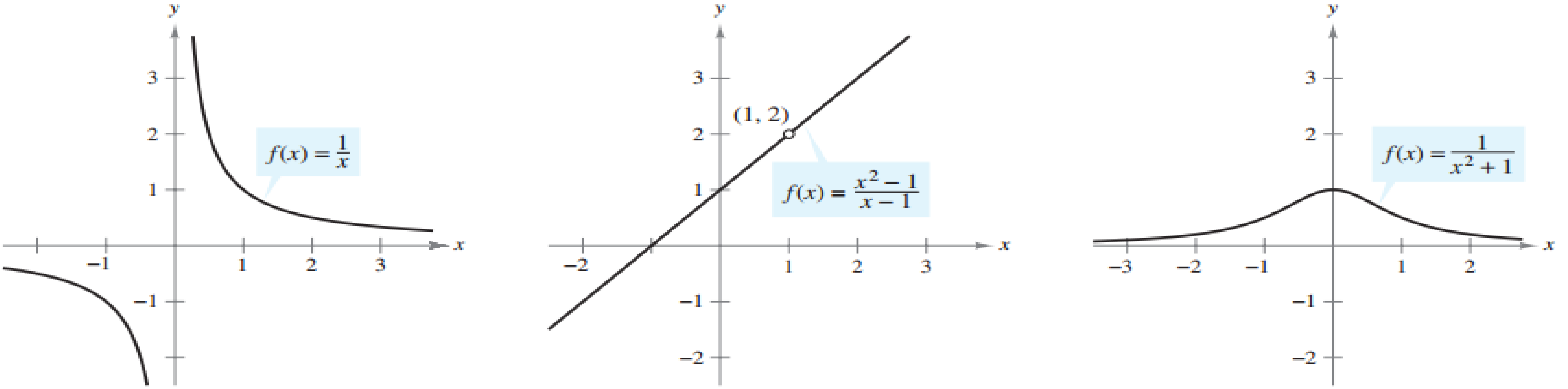
Discuss the continuity of each function.

(a) $f(x) = 1/x$ (b) $f(x) = (x^2 - 1)/(x - 1)$ (c) $f(x) = 1/(x^2 + 1)$

(a) The domain of $f(x) = 1/x$ consists of all real numbers except $x = 0$. So, this function is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$. [See Figure 1.63(a).]

(b) The domain of $f(x) = (x^2 - 1)/(x - 1)$ consists of all real numbers except $x = 1$. So, this function is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$. [See Figure 1.63(b).]

(c) The domain of $f(x) = 1/(x^2 + 1)$ consists of all real numbers. So, this function is continuous on the entire real line. [See Figure 1.63(c).]



(a) Continuous on $(-\infty, 0)$ and $(0, \infty)$ (b) Continuous on $(-\infty, 1)$ and $(1, \infty)$ (c) Continuous on $(-\infty, \infty)$

$$g(x) = \begin{cases} x^2 - 2x + 1, & if\ x > -1 \\ 2\ , & if\ x = -1 \\ 3 - x, & if\ x < -1 \end{cases}$$ Is $g(x)$ continuous at $x = -1$?

To answer this question, we must check each part of the definition.

i) Does $g(-1)$ exist? Yes, the middle line says that -1 is in the domain and it tells us that $y = 2$ if $x = -1$.

ii) Does the $\lim\limits_{x \to -1} g(x)$ exist? Yes, the two one-sided limits are equal.

iii) Does $\lim\limits_{x \to -1} g(x) = g(-1)$? No. $\lim\limits_{x \to -1} g(x) = 4$, while $g(-1) = 2$

$$\boxed{\text{So } g(x) \text{ is not continuous at } x = -1 \text{ because the limit does not equal the function.}}$$

Let $f(x) = \begin{cases} \cos(x) + 1 & , \text{ if } x \leq 0; \\ 2 - 3x & , \text{ if } x > 0. \end{cases}$ Determine if this function is continuous at $x = 0$.

1. The function is defined at $x = 0$ and the value is $f(0) = \cos(0) + 1 = 2$.
2. Since $y = \cos(x) + 1$ is continuous at $x = 0$, we have:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos(x) + 1 = \cos(0) + 1 = 2.$$

3. Since $y = 2 - 3x$ is continuous at $x = 0$, we have:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 - 3x = 2 - 3(0) = 2.$$

Since all three of these values are the same, the function is continuous at $x = 0$.

Let $f(x) = \begin{cases} e^x & , \text{ if } x < 0; \\ 9x^2 + x + 1 & , \text{ if } x \geq 0. \end{cases}$. Is f continuous at $x = 0$?

1. The function is defined at $x = 0$ and its value is $f(0) = 9(0)^2 + (0) + 1 = 1$.
2. Since $y = e^x$ is continuous at $x = 0$, we have:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^x = e^0 = 1.$$

3. Since $y = x$ is continuous at $x = 0$, we have:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 9x^2 + x + 1 = 1.$$

Since all three of these values are the same, the function is continuous at $x = 0$.