

Section 9

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Nonhomogeneous Differential Equations

It's now time to start thinking about how to solve nonhomogeneous differential equations. A second order, linear nonhomogeneous differential equation is

$$y'' + p(t)y' + q(t)y = g(t) \quad (1)$$

where $g(t)$ is a non-zero function.

The general solution to a differential equation can then be written as.

$$y(t) = y_c(t) + Y_P(t)$$

$$2x^2 - 5 \rightarrow Ax^2 + Bx + C$$

$g(t)$	$Y_P(t)$ guess
$ae^{\beta t}$	$Ae^{\beta t}$
$a \cos(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$
$b \sin(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$
$a \cos(\beta t) + b \sin(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$
n^{th} degree polynomial	$A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0$

$r(x)$	Initial guess for $y_p(x)$
k (a constant)	A (a constant)
$ax + b$	$Ax + B$ (Note: The guess must include both terms even if $b = 0$.)
$ax^2 + bx + c$	$Ax^2 + Bx + C$ (Note: The guess must include all three terms even if b or c are zero.)
Higher-order polynomials	Polynomial of the same order as $r(x)$
$ae^{\lambda x}$	$Ae^{\lambda x}$
$a \cos \beta x + b \sin \beta x$	$A \cos \beta x + B \sin \beta x$ (Note: The guess must include both terms even if either $a = 0$ or $b = 0$.)
$ae^{\alpha x} \cos \beta x + be^{\alpha x} \sin \beta x$	$Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$
$(ax^2 + bx + c)e^{\lambda x}$	$(Ax^2 + Bx + C)e^{\lambda x}$
$(a_2x^2 + a_1x + a_0) \cos \beta x$ $+ (b_2x^2 + b_1x + b_0) \sin \beta x$	$(A_2x^2 + A_1x + A_0) \cos \beta x$ $+ (B_2x^2 + B_1x + B_0) \sin \beta x$
$(a_2x^2 + a_1x + a_0)e^{\alpha x} \cos \beta x$ $+ (b_2x^2 + b_1x + b_0)e^{\alpha x} \sin \beta x$	$(A_2x^2 + A_1x + A_0)e^{\alpha x} \cos \beta x$ $+ (B_2x^2 + B_1x + B_0)e^{\alpha x} \sin \beta x$

Example 1

Solve the following D.E :-

$$y'' + 2y' + y = 4e^{-2x}$$

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + 4e^{-2x}$$

SOLUTION

First we solve the related homogeneous D.E :-

$$y'' + 2y' + y = 0$$

the roots of the characteristic equation are:-

$$r^2 + 2r + 1 = 0$$

$$(r + 1)(r + 1) = 0$$

$$\therefore r_{1,2} = -1$$

Hence the general solution of the homogeneous equation is given by :

$$y_o(x) = C_1 e^{-x} + C_2 x e^{-x}$$

Based on the form " $g(x) = 4e^{-2x}$ " the particular solution would be in the form of : " $y_p = Ae^{-2x}$ "

The derivatives are given by :-

$$y_p' = -2Ae^{-2x}$$

$$y_p'' = 4Ae^{-2x}$$

$$\rightarrow \therefore y_p'' + 2y_p' + y_p = 4e^{-2x}$$

$$\therefore 4Ae^{-2x} - 4Ae^{-2x} + Ae^{-2x} = 4e^{-2x}$$

$$\therefore Ae^{-2x} = 4e^{-2x}$$

$$\therefore A = 4$$

$$y_p = 4e^{-2x}$$

Hence the general solution of the homogeneous equation is given by :

$$y(x) = C_1e^{-x} + C_2xe^{-x} + 4e^{-2x}$$

Example 2

$$y(0) = 1$$
$$y'(1) = 2$$

Solve the differential equation $y'' - 5y' + 4y = e^{4x}$

x ✓ (A) $y = \frac{y', y''}{-}$

x ✓ (b) $y = \frac{y}{-}$

x ✓ (c) $y = \frac{y}{-}$

(d) $y = \frac{y}{-}$

x $y_p = Ae^{4x}$

$$y_p = Axe^{4x}$$

SOLUTION

First we solve the related homogeneous equation $y'' - 5y' + 4y = 0$. The roots of the characteristic equation are

$$k^2 - 5k + 4 = 0, \Rightarrow D = 25 - 4 \cdot 4 = 9, \Rightarrow k_{1,2} = \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2} \\ = 4, 1$$

Hence, the general solution of the homogeneous equation is given by

$$y_0(x) = C_1 e^{4x} + C_2 e^x,$$

where C_1, C_2 are constant numbers.

Find a particular solution of the nonhomogeneous differential equation. Notice that the power of the exponential function on the right coincides with the root $k_1 = 4$ of the auxiliary characteristic equation. Therefore we will look for a particular solution of the form

$$y_1 = \underline{Axe^{4x}}.$$

$$\rightarrow y_p = \cancel{Ae^{4x}} \quad \times$$

The derivatives are given by

$$\underline{y_1' = (Axe^{4x})' = Ae^{4x} + 4Axe^{4x} = (A + 4Ax)e^{4x};}$$

$$\underline{y_1'' = [(A + 4Ax)e^{4x}]' = 4Ae^{4x} + (4A + 16Ax)e^{4x} = (8A + 16Ax)e^{4x}.}$$

Substituting the function y_1 and its derivatives in the differential equation yields:

$$(8A + 16Ax) e^{4x} - 5(A + 4Ax) e^{4x} + 4Axe^{4x} = e^{4x},$$

$$\Rightarrow 8A + \cancel{16Ax} - 5A - \cancel{20Ax} + \cancel{4Ax} = 1, \Rightarrow 3A = 1, \Rightarrow A = \frac{1}{3}.$$

Thus, the particular solution to the differential equation can be written in the form:

$$y_1 = \frac{x}{3} e^{4x}.$$

Now we can write the full solution of the nonhomogeneous equation:

$$y = y_0 + y_1 = C_1 e^{4x} + C_2 e^x + \frac{x}{3} e^{4x}.$$

y', y''

Example 3

Find the general solution of the equation $y'' + 9y = 2x^2 - 5$.

$$y'' + 9y = 0$$

$$y_{G.S} = C_1 \cos 3x + C_2 \sin 3x + \frac{2}{9}x^2 - \frac{49}{81}$$

SOLUTION

First we determine the general solution of the related homogeneous equation. Solve the auxiliary characteristic equation:

$$k^2 + 9 = 0, \Rightarrow k^2 = -9, \Rightarrow k_{1,2} = \pm 3i.$$

$k = \pm \sqrt{-9} = \pm \sqrt{-1} \sqrt{9} = \pm 3i$

The solution is written in the form:

$$y_0(x) = C_1 \cos 3x + C_2 \sin 3x.$$

Now we construct a particular solution. The right-hand side of the given equation is a quadratic function. So we can guess on a particular solution of the same form:

$$y_1 = Ax^2 + Bx + C,$$

Handwritten notes in red:

$\alpha \quad \beta$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$

$C_1 e^{2x} (\cos 5x + C_1 e^{ix} \sin 5x)$

where the numbers A, B, C can be determined by the method of undetermined coefficients. Hence, we can write:

$$y_1' = 2Ax, \quad y_1'' = 2A.$$

Substituting this into the original nonhomogeneous differential equation, we have

$$2A + 9(Ax^2 + Bx + C) = 2x^2 - 5, \Rightarrow 2A + 9Ax^2 + 9Bx + 9C = 2x^2 - 5.$$

By equating the coefficients of like powers of x , we obtain:

$$\begin{cases} 9A = 2 \\ 9B = 0 \\ 2A + 9C = -5 \end{cases}, \Rightarrow \begin{cases} A = \frac{2}{9} \\ B = 0 \\ C = -\frac{49}{81} \end{cases}.$$

Thus, the particular solution is given by

$$y_1 = \frac{2}{9}x^2 - \frac{49}{81}.$$

Then the general solution of the original nonhomogeneous differential equation is expressed by the formula

$$y = y_0 + y_1 = C_1 \cos 3x + C_2 \sin 3x + \frac{2}{9}x^2 - \frac{49}{81}.$$

Example 4

Solve the following D.E :-

$$y'' + 2y' = 24x + e^{-2x}$$

$$\underline{y_p} = \underline{\left(Ax^2 + 13x \right) + \left(Cx e^{-1x} \right)}$$

SOLUTION

First we solve the related homogeneous D.E :-

$$y'' + 2y' = 0$$

the roots of the characteristic equation are:-

$$r^2 + 2r = 0$$

$$r(r + 2) = 0$$

$$\therefore r_1 = 0 \text{ and } r_2 = -2$$

Hence the general solution of the homogeneous equation is given by :

$$\underline{y_o(x) = C_1 + C_2 e^{-2x}}$$

Based on the form " $g(x) = 24x + e^{-2x}$ " the particular solution would be in the form of : " $y_p = Ax^2 + Bx + Cxe^{-2x}$ "

The derivatives are given by :-

$$\therefore y_p' = 2Ax + B + C(e^{-2x} - 2xe^{-2x})$$

$$\therefore y_p' = 2Ax + B + Ce^{-2x} - 2Cxe^{-2x}$$

$$\therefore y_p'' = 2A - 2Ce^{-2x} - 2C(e^{-2x} - 2xe^{-2x})$$

$$\therefore y_p'' = 2A - 2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x}$$

$$\therefore y'' + 2y' = 24x + e^{-2x}$$

$$\therefore 2A - 2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x} + 4Ax + 2B + 2Ce^{-2x} - 4Cxe^{-2x} = 24x + e^{-2x}$$

$$\therefore (2A + 2B) + 4Ax - 2Ce^{-2x} = 24x + e^{-2x}$$

$$\therefore 4A = 24, A = 6$$

$$-2C = 1, C = -0.5$$

$$(2A + 2B) = 0, B = -6$$

Hence the general solution of the homogeneous equation is given by :

$$y(x) = C_1e^{-x} + C_2e^{-x} + 6x^2 - 6x - 0.5xe^{-2x}$$

Example 5

$$\begin{matrix} 1 \lambda^0 \\ A x^0 \end{matrix}$$

Solve the differential equation $y'' + 16y = 2\cos^2 x$.

$$y_p = \underline{(A \cos 2x + B \sin 2x)} + C$$

(Note: Red arrows in the original image point from the underlined term to the $\cos 2x$ and $\sin 2x$ terms, and from the $+1$ term to the C constant.)

SOLUTION

First of all we solve the related homogeneous equation. The characteristic equation has roots:

$$k^2 + 16 = 0, \Rightarrow \underline{k^2 = -16}, \Rightarrow k_{1,2} = \pm 4i,$$

so the general solution has the form:

$$\underline{y_0(x) = C_1 \cos 4x + C_2 \sin 4x.}$$

Now we find a particular solution for the nonhomogeneous equation. Rewrite the right-hand side as

$$\underline{2\cos^2 x = \cos 2x + 1.}$$

$$\begin{aligned} \cos 2x &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

It follows from here that the particular solution is defined by the function

$$y_1 = A \cos 2x + B \sin 2x + C,$$

where the numbers A , B , and C can be calculated using the method of undetermined coefficients. The first and second derivatives of the function y_1 are

$$y_1' = -2A \sin 2x + 2B \cos 2x,$$

$$y_1'' = -4A \cos 2x - 4B \sin 2x.$$

Substituting this back into the differential equation produces:

$$-4A \cos 2x - 4B \sin 2x + 16(A \cos 2x + B \sin 2x + C) = \cos 2x + 1,$$

$$-4A \cos 2x - 4B \sin 2x + 16A \cos 2x + 16B \sin 2x + 16C = \cos 2x + 1,$$

$$12A \cos 2x + 12B \sin 2x + 16C = \cos 2x + 1.$$

The latter expression is identical. Therefore we can write the following system of equations to determine the coefficients A, B, C :

$$\begin{cases} 12A = 1 \\ 12B = 0 \\ 16C = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{12} \\ B = 0 \\ C = \frac{1}{16} \end{cases}.$$

Thus, the particular solution has the form:

$$y_1 = \frac{1}{12} \cos 2x + \frac{1}{16}.$$

Respectively, the general solution of the original nonhomogeneous equation is written as

$$y = y_0 + y_1 = C_1 \cos 4x + C_2 \sin 4x + \frac{1}{12} \cos 2x + \frac{1}{16}.$$

Determinants

2 x 2 and 3 x 3 Matrices

Matrices

Note that Matrix is the singular form, matrices is the plural form!

- A matrix is an array of numbers that are arranged in rows and columns.
- A matrix is “square” if it has the same number of rows as columns.
- We will consider only 2x2 and 3x3 square matrices

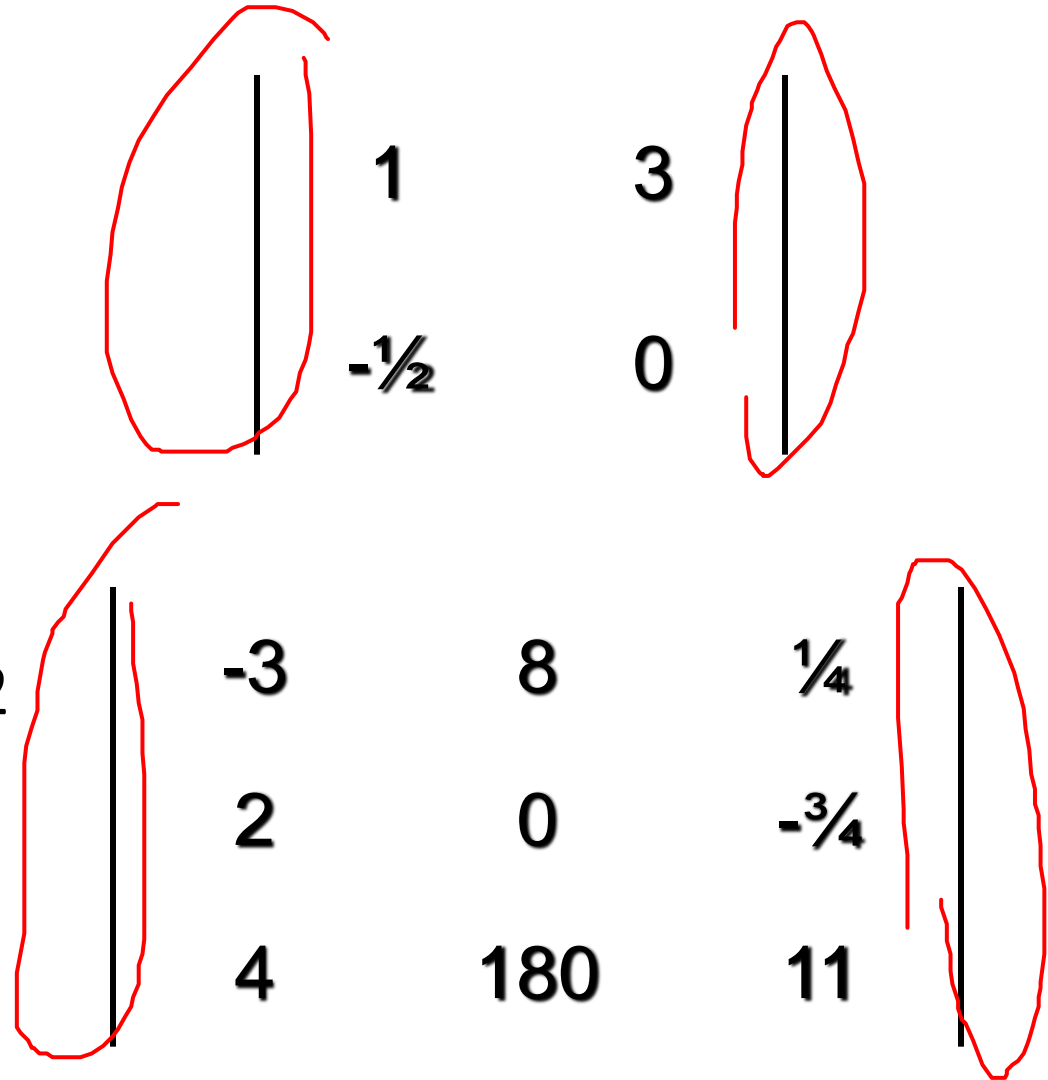
$$\begin{bmatrix} 1 & 3 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{bmatrix}$$

Determinants

Note the difference in the matrix and the determinant of the matrix!

- Every square matrix has a determinant.
- The determinant of a matrix is a number.
- We will consider the determinants only of 2x2 and 3x3 matrices.

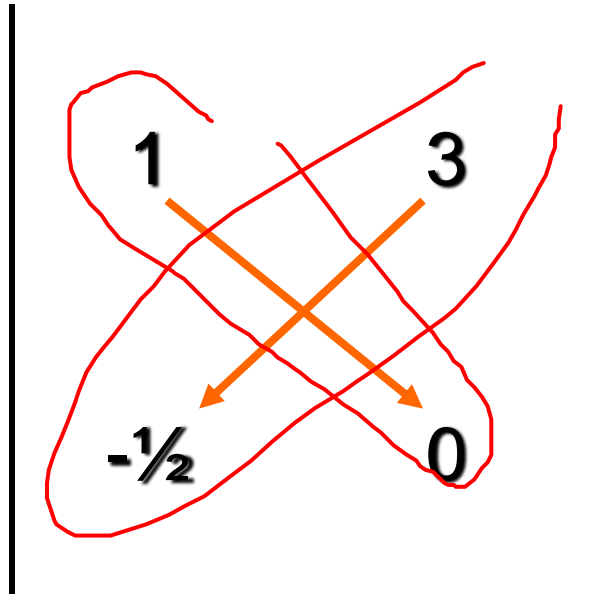


The image shows four 2x2 matrices, each enclosed in a red hand-drawn oval. The matrices are arranged in a 2x2 grid. The top-left matrix has elements 1 and 3 in the first row, and -1/2 and 0 in the second row. The top-right matrix has elements 3 and 0 in the first row, and 1 and -1/2 in the second row. The bottom-left matrix has elements -3 and 8 in the first row, and 2 and 0 in the second row. The bottom-right matrix has elements 1/4 and -3/4 in the first row, and 11 and 180 in the second row.

$\begin{vmatrix} 1 & 3 \\ -\frac{1}{2} & 0 \end{vmatrix}$	$\begin{vmatrix} 3 & 0 \\ 1 & -\frac{1}{2} \end{vmatrix}$
$\begin{vmatrix} -3 & 8 \\ 2 & 0 \end{vmatrix}$	$\begin{vmatrix} \frac{1}{4} & -\frac{3}{4} \\ 11 & 180 \end{vmatrix}$

Determinant of a 2x2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
$$= ad - bc$$



$$= (1)(0) - (3)\left(-\frac{1}{2}\right) = +\frac{3}{2}$$

Determinant of a 3x3 matrix

Imagine crossing out the first row.
And the first column.

Now take the double-crossed element. . .
And multiply it by the determinant of the
remaining 2x2 matrix

The diagram shows a 3x3 matrix with the first row and first column crossed out. The remaining 2x2 matrix is highlighted with a blue circle. The double-crossed element is $-3/4$. The determinant of the 2x2 matrix is calculated as $(0)(11) - (-3/4)(180)$. The final determinant is calculated as $0 + 405 + 176 = 581$.

$$\begin{vmatrix} -3 & 8 & -1/4 \\ 2 & 0 & -3/4 \\ 4 & 180 & 11 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 0 & 11 \\ -3/4 & 180 \end{vmatrix} + 405 + 176$$

$$= -3(0(11) - (-3/4)(180)) + 405 + 176$$

$$= -3(-135) + 405 + 176$$

$$= 405 + 176 + 405 = 986$$

Determinant of a 3x3 matrix

Now keep the first row crossed.
Cross out the second column.

A 3x3 matrix is shown within large vertical bars. The elements are: Row 1: -3, 8, 1/4; Row 2: 2, 0, -3/4; Row 3: 4, 180, 11. A blue oval highlights the element 8 in the first row, second column. A vertical dashed purple line passes through the second column. Two orange arrows cross the matrix: one from the element 2 in the second row, first column to the element 11 in the third row, third column, and another from the element 180 in the third row, second column to the element 4 in the third row, first column.

$$\begin{vmatrix} -3 & 8 & 1/4 \\ 2 & 0 & -3/4 \\ 4 & 180 & 11 \end{vmatrix}$$

- Now take the negative of the double-crossed element.
- And multiply it by the determinant of the remaining 2x2 matrix.
- Add it to the previous result.

$$= -3 \left((0)(11) - \left(-\frac{3}{4}\right)(180) \right) - 8 \left((2)(11) - \left(-\frac{3}{4}\right)(4) \right)$$

Determinant of a 3x3 matrix

Finally, cross out first row and last column.

A 3x3 matrix is shown within large vertical bars. The first row contains the elements -3, 8, and 1/4. The second row contains 2, 0, and -3/4. The third row contains 4, 180, and 11. A blue oval highlights the element 1/4 in the first row, third column. A dashed purple line runs horizontally through the first row, and another dashed purple line runs vertically through the third column. Two orange arrows originate from the element 2 in the second row, first column and point to the elements 180 and 4 in the third row, second and first columns respectively, illustrating the cross-out process for the first row and last column.

- Now take the double-crossed element.
- Multiply it by the determinant of the remaining 2x2 matrix.
- Then add it to the previous piece.

$$= -3 \left((0)(11) - \left(-\frac{3}{4}\right)(180) \right) - 8 \left((2)(11) - \left(-\frac{3}{4}\right)(4) \right) \\ + \left(\frac{1}{4}\right) \left((2)(180) - (0)(4) \right) = -515$$

Another method (short-cut method)

$(-3) + 405 + 176$

$(-3) + (-24) + 90$

66

-

581

= -515

Cramer's Rule for Solution of Linear Equations:

If
$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$$

with $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

Then $x = \frac{1}{\Delta} \Delta_x$, $y = \frac{1}{\Delta} \Delta_y$, $z = \frac{1}{\Delta} \Delta_z$

where $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Remember: Cramer's rule can be fruitfully applied in case of $\Delta \neq 0$.

Use Cramer's rule to solve the system of equations:

(1) $2x + 3y - z = 0,$ $x + 2z - 3y = 2,$ $y + z + 2 = 0$

(2) $x - 2y + 2z = 1,$ $3x + 4z = 8,$ $6z - y = 2$

(3) $x - y - z = 3,$ $3x + y = 2,$ $2y + 3z = 1$

(4) $2x - y + 3z = 0,$ $y - 4x - 6z = 2,$ $4x + 3y = 8$

$\{(1, -1, -1)\}$

$\left\{\left(2, 1, \frac{1}{2}\right)\right\}$

$\{(2, -4, 3)\}$

$\cdot \left\{\left(\frac{7}{2}, -2, -3\right)\right\}$

Example 1

Use Cramer's rule to solve the system of equations:

$$2x + 3y - z = 0,$$

$$x + 2z - 3y = 2,$$

$$y + z + 2 = 0$$

$$\{(1, -1, -1)\}$$

SOLUTION

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -3 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (2)(+1)(-5) + (1)(-1)(4) + 0$$

$$= -10 - 4$$

$$= -14$$

$$\Delta_x = \begin{vmatrix} 0 & 3 & -1 \\ 2 & -3 & 2 \\ -2 & 1 & 1 \end{vmatrix}$$

$$= 0 + (3)(-1)(6) + (-1)(+1)(-4)$$

$$= -18 + 4$$

$$= -14$$

$$\Delta_y = \begin{vmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= (2)(+1)(6) + 0 + (-1)(+1)(-2)$$

$$= 12 + 2$$

$$= 14$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= (2)(+1)(4) + (3)(-1)(-2) + 0$$

$$= 8 + 6$$

$$= 14$$

$$\therefore x = \frac{\Delta_x}{\Delta} = 1,$$

$$y = \frac{\Delta_y}{\Delta} = -1,$$

$$z = \frac{\Delta_z}{\Delta} = -1$$

$$\therefore \text{S.S.} = \{(1, -1, -1)\}$$

Example 2

Use Cramer's rule to solve the system of equations:

$$x - 2y + 2z = 1,$$

$$3x + 4z = 8,$$

$$6z - y = 2$$

$$\left\{ \left(2, 1, \frac{1}{2} \right) \right\}$$

SOLUTION

$$\Delta = \begin{vmatrix} 1 & -2 & 2 \\ 3 & 0 & 4 \\ 0 & -1 & 6 \end{vmatrix}$$

$$\begin{aligned} &= (1)(+1)(4) + (3)(-1)(-10) + 0 \\ &= 4 + 30 \\ &= 34 \end{aligned}$$

$$\Delta_x = \begin{vmatrix} 1 & -2 & 2 \\ 8 & 0 & 4 \\ 2 & -1 & 6 \end{vmatrix}$$

$$\begin{aligned} &= (8)(-1)(-10) + 0 + (4)(-1)(3) \\ &= 80 - 12 \\ &= 68 \end{aligned}$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 8 & 4 \\ 0 & 2 & 6 \end{vmatrix}$$

$$\begin{aligned} &= (1)(+1)(40) + (3)(-1)(2) + 0 \\ &= 40 - 6 \\ &= 34 \end{aligned}$$

$$\Delta_z = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 0 & 8 \\ 0 & -1 & 2 \end{vmatrix}$$

$$\begin{aligned} &= (1)(+1)(8) + (3)(-1)(-3) + 0 \\ &= 8 + 9 \\ &= 17 \end{aligned}$$

$$\therefore x = \frac{\Delta_x}{\Delta} = 2,$$

$$y = \frac{\Delta_y}{\Delta} = 1,$$

$$z = \frac{\Delta_z}{\Delta} = \frac{1}{2}$$

$$\therefore \text{S.S.} = \left\{ \left(2, 1, \frac{1}{2} \right) \right\}$$

Example 3

Use Cramer's rule to solve the system of equations:

$$\begin{array}{llll} x - y - z = 3, & 3x + y = 2, & 2y + 3z = 1 & \{(2, -4, 3)\} \end{array}$$

$$\Delta = \begin{vmatrix} 1 & -1 & -1 \\ 3 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix}$$

$$\begin{aligned} &= (1)(+1)(3) + (3)(-1)(-1) + 0 \\ &= 3 + 3 \\ &= 6 \end{aligned}$$

$$\Delta_x = \begin{vmatrix} 3 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\begin{aligned} &= (-1)(+1)(3) + 0 + (3)(+1)(5) \\ &= -3 + 15 \\ &= 12 \end{aligned}$$

$$\Delta_y = \begin{vmatrix} 1 & 3 & -1 \\ 3 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

SOLUTION

$$\begin{aligned} &= (1)(+1)(6) + (3)(-1)(10) + 0 \\ &= 6 - 30 \\ &= -24 \end{aligned}$$

$$\Delta_z = \begin{vmatrix} 1 & -1 & 3 \\ 3 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$\begin{aligned} &= (1)(+1)(-3) + (3)(-1)(-7) \\ &= -3 + 21 \\ &= 18 \end{aligned}$$

$$\therefore x = \frac{\Delta_x}{\Delta} = 2,$$

$$y = \frac{\Delta_y}{\Delta} = -4,$$

$$z = \frac{\Delta_z}{\Delta} = 3$$

$$\therefore \text{S.S.} = \{ (2, -4, 3) \}$$

Example 4

Use Cramer's rule to solve the system of equations:

$$2x - y + 3z = 0, \quad y - 4x - 6z = 2, \quad 4x + 3y = 8 \quad \cdot \quad \left\{ \left(\frac{7}{2}, -2, -3 \right) \right\}$$

SOLUTION

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ -4 & 1 & -6 \\ 4 & 3 & 0 \end{vmatrix}$$

$$= (3)(+1)(-16) + (-6)(-1)(10) + 0$$

$$= -48 + 60$$

$$= 12$$

$$\Delta_x = \begin{vmatrix} 0 & -1 & 3 \\ 2 & 1 & -6 \\ 8 & 3 & 0 \end{vmatrix}$$

$$= 0 + (-1)(-1)(48) + (3)(+1)(-2)$$

$$= 48 - 6$$

$$= 42$$

$$\Delta_y = \begin{vmatrix} 2 & 0 & 3 \\ -4 & 2 & -6 \\ 4 & 8 & 0 \end{vmatrix}$$

$$= (2)(+1)(48) + 0 + (3)(+1)(-40)$$

$$= -24$$

$$\Delta_z = \begin{vmatrix} 2 & -1 & 0 \\ -4 & 1 & 2 \\ 4 & 3 & 8 \end{vmatrix}$$

$$= (2)(+1)(2) + (-1)(-1)(-40)$$

$$= 4 - 40$$

$$= -36$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{7}{2},$$

$$y = \frac{\Delta_y}{\Delta} = -2,$$

$$z = \frac{\Delta_z}{\Delta} = -3$$

$$\therefore \text{S.S.} = \left\{ \left(\frac{7}{2}, -2, -3 \right) \right\}$$

Thank you