MALLIAVIN CALCULUS & APPLICATIONS TO MATHEMATICAL FINANCE

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ABSTRACT

The Malliavin calculus, also referred to as stochastic calculus of variations, allows to establish integration by parts formulas on the Wiener space that write : for some smooth function f, $E[f(X_T)G]' = E[f(X_T)H(X_T,G))]$ or $\partial_x E[f(X_T)G] = E[f(X_T)H(X_T,G)]$ for some explicit weight $H(X_T,G)$, where X_T is the solution taken at time T of some non-degenerate stochastic differential equation. It has many applications, notably in mathematical finance for the computation of Greeks for Delta hedging purpose. The aims of this project are :

- to understand the basic principle of Malliavin calculus,
- to implement the method in some simple examples related to the computation of Greeks of financial derivatives.

***K*eywords** Monte Carlo Methods · Malliavin Calculus · Quantitative Finance

1 Introduction[1]

A classical approach for option pricing in Mathematical Finance is to replicate the option payoff at maturity with a self-financing portfolio. In a no-arbitrage framework, practitioners often use delta hedging as a replication strategy. This requires computing the *sensitivity* of the the instrument's price with respect to the asset's initial one. Hence, the importance of numerical methods regarding computing *greeks*, *i.e* sensitivities of the price with respect to a model's parameter. These quantities play an essential role in risk management. In simulations, we usually use finite difference method to approach greeks. However, this method is highly unstable in the case of irregular payoffs. Here comes the role of *Malliavin Calculus*.

This report is organized as follows :

- Elemenets in Malliavin Calculus
- Applications in finance : computing greeks
- Numerical simulations
- Conclusion

[1]This work is the final report of Malliavin Calculus project in the M2MO course : Monte-Carlo methods in Finance (*Prof. Noufel FRIKHA*)

2 Rudiments in Malliavin Calculus[2]

2.1 Notations

In this whole section, we consider the following notations :

1. $(W_t)_{t\geq 0}$ is a one dimensional brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ provided with its natural filtration.
2. We consider the following Hilbert spaces provided with their canonical scalar product :
   - $L^2(\Omega) = \{F \text{ a random variable }, \mathbf{E}[|F|^2] < \infty\}$
   - $L^2([0,T]) = \{f : [0,T] \mapsto \mathbf{R}, \int_{[0,T]} |f|^2 < \infty\}$.
   - $L^2(\Omega \times [0,T]) = \{f : \Omega \times [0,T] \mapsto \mathbf{R}, \mathbf{E}[\int_{[0,T]} |f|^2] < \infty\}$.
3. Furthermore, we denote $W(h) := \int_0^T h(t)dW_t$ for every $h \in L^2([0,T])$.
4. $C_p^\infty(\mathbf{R}^n)$ set of infinitely continuously differentiable functions $f : \mathbf{R}^n \to \mathbf{R}$ with partial derivatives of polynomial growth.
5. $\mathcal{S} = \{F := f(W(h_1), \ldots, W(h_n)) | f \in C_p^\infty(\mathbf{R}^n)\ , h_i \in L^2([0,T])$ and n arbitrary $\}$ which is a dense subspace of $L^2(\Omega)$.

2.2 Derivative operator

Definition 2.1. (*Malliavin derivative of F*)

For any $F \in \mathcal{S}$, the malliavin calculus is a stochastic process given by :

$$DF = \sum_i^n \partial_i f(W(h_1), \ldots, W(h_n))h_i$$

$D : \mathcal{S} \subset L^2(\Omega) \to L^2(\Omega \times [0,T])$ is a linear operator. We will extend it by considering the norm :

$$\|F\|_{1,2} = \|F\|_{L^2(\Omega)} + \|DF\|_{L^2(\Omega\times[0,T])}$$

We define $\mathbf{D}^{1,2}$ as the closure of $\mathcal{S}$ in the norm $\|.\|_{1,2}$. Then,

$$D : \mathbf{D}^{1,2} \subset L^2(\Omega) \to L^2(\Omega \times [0,T])$$

is a closed **unbounded** operator with a dense domain $\mathbf{D}^{1,2}$.

Remark. (Interpretation)

D F can be interpreted as a directional Derivative for any $h \in L^2([0,T])$ with notation:

$$\underbrace{\langle \mathrm{D}F, h\rangle_H}_{=:\mathrm{D}_h F} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(f\left(W\left(h_1\right) + \varepsilon\left\langle h_1, h\right\rangle_H, \cdots, W\left(h_n\right) + \varepsilon\left\langle h_n, h\right\rangle_H\right) - F\right)$$

Properties.

1. *Linearity : $D(aF + G) = aDF + DG\ \ \forall a,\ \forall F, G \in \mathbf{D}^{1,2}$*
2. *Chain rule : $D(f(F)) = \sum_i \frac{\partial f}{\partial x_i}(F)DF_i\ \ \forall f \in C_p^\infty(\mathbf{R}^n),\ \forall F \in \mathbf{D}^{1,2}$*
3. *Product rule : $D(FG) = F.DG + G.DF\ \ \forall F, G \in \mathbf{D}^{1,2}$ given that $FG \in \mathbf{D}^{1,2}$*

Examples. *Here are some simple examples :*

- $D_t(\int_0^T h(t)dW_t) = h(t)$
- $D_t W_s = \mathbf{1}_{s\leq t}$
- $D_t f(W_s) = f'(W_s)\mathbf{1}_{s\leq t}$

[2]This section presents main theoretical results & tools that we will need henceforth, an appendix to it is provided at the end under the title *Wiener Chaos* introducing some preliminary results necessary for the proofs. These elements have been highly inspired by *Lecture notes* of Max Goldowky on the subject of Malliavin Calculus. Only a selection of proofs is provided in this report.

2.3 Skorohod Integral

Definition 2.2. *(Skorohod Integral)*

The **Skorohod Integral** δ is the adjoint of the operator D, that is, δ is an unbounded operator from $L^2(\Omega \times [0,T]) \to L^2(\Omega)$, such that :

- The domain of δ, denoted by $dom(\delta)$, the set of random variables, $u \in L^2(\Omega \times [0,T])$ such that :

$$\forall F \in \mathbf{D}^{1,2}\ |\mathbf{E}(\langle DF, u\rangle_{L^2([0,T])})| \leq C_u ||F||_{L^2(\Omega)}$$

- If $u \in dom(\delta)$, then $\delta(u) \in L^2(\Omega)$ and is caracterized by the **duality formula** :

$$\mathbf{E}(F\delta(u)) = \mathbf{E}(\langle DF, u\rangle_{L^2([0,T])})$$

Remark. 1. $u \in dom(\delta) \implies \mathbf{E}(\delta(u)) = 0$

2. δ linear bounded closed operator on $dom(\delta)$

Theorem 2.1. *Let $u \in dom(\delta)$, $F \in \mathbf{D}^{1,2}$ such that $Fu \in L^2(\Omega \times [0,T])$. Then,*

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F u_t dt$$

Proof. Let $G = g(W(G_1), \dots, W(G_n))$ arbitrary (eventually respecting regularity conditions allowing following equalities) :

$$\begin{aligned} E[GF\delta(u)] &= E\left[\int_0^T D_t(GF)u_t dt\right] \\ &= E\left[G\int_0^T D_t F u_t dt\right] + E\left[\int_0^T (D_t G)\, F u_t dt\right] \\ &= E\left[G\int_0^T D_t F u_t dt\right] + E[G\delta(Fu)] \end{aligned}$$

G is arbitrary, hence the proof completed !

□

Remark. 1. The theorem suggests clearly that Fu is integrable in the sense of Skorohod if and only if the right hand side belongs to $L^2(\Omega)$.

2. This theorem is particularly useful for computing δ of processes given by a single random variable.

Examples. *For $t_0 \in [0,T]$, we have :*

$$\begin{aligned} \delta\left(W\left(t_0\right)\right) &= W\left(t_0\right)\delta(1) - \int_0^T D_t\left(W\left(t_0\right)\right)dt \\ &= W\left(t_0\right)W(T) - \int_0^T 1_{\{t \leq t_0\}}dt \\ &= W\left(t_0\right)W(T) - t_0 \end{aligned}$$

$$\begin{aligned} \delta\left(W^2\left(t_0\right)\right) &= W\left(t_0\right)\delta\left(W\left(t_0\right)\right) - \int_0^T D_t\left(W\left(t_0\right)\right)W\left(t_0\right)dt \\ &= W\left(t_0\right)\left[W\left(t_0\right)W(T) - t_0\right] - \int_0^T 1_{\{t \leq t_0\}}W\left(t_0\right)dt \\ &= W^2\left(t_0\right)W(T) - 2t_0 W\left(t_0\right) \end{aligned}$$

Theorem 2.2. *Let (X_t) be a stochastic process such that :*

- $\mathbf{E}(\int_0^T X_t^2) < \infty$
- (X_t) *is* $(\mathcal{F}_t)-$ *adapted*

Then,

- $X \in dom(\delta)$
- *And,*

$$\delta(X) = \int_0^T X_t dW_t$$

Proof. A classical approach consists of proving the equality for a handy class of processes, then conclude.
- Let X be an elementary $\mathcal{F}$-adapted process:

$$X_t = \sum_{j=1}^{n} \xi_j \mathbf{1}_{]t_{j-1},t_j]}$$

with $0 = t_0 \leq \cdots \leq t_n = T$

$$\begin{aligned}\delta(\xi_j \mathbf{1}_{]t_{j-1},t_j]}) &= \xi_j \delta(\mathbf{1}_{]t_{j-1},t_j]}) - \int_0^T D_t \xi_j \mathbf{1}_{]t_{j-1},t_j]} dt \\ &= \xi_j (W_{t_j} - W_{t_{j-1}})\end{aligned}$$

δ is linear, it follows :

$$\begin{aligned}\delta(X_t) &= \sum_{j}^{n} \xi_j (W_{t_j} - W_{t_{j-1}}) \\ &= \int_0^T X_t dW_t\end{aligned}$$

- Let X a L^2- adapted process, $\exists X^{(n)}$ a sequence of elementary $\mathcal{F}$- adapted processes such that $X^{(n)} \xrightarrow[n\to\infty]{\text{L2}} X$. By Ito's Isometry :

$$\delta(X^{(n)}) = \int_0^T X_t^{(n)} dW_t \xrightarrow[n\to\infty]{\text{L2}} \int_0^T X_t dW_t$$

We conclude (δ closable):

$$\delta(X) = \int_0^T X_t dW_t$$

□

2.4 Malliavin derivative of a Skorohod integral

Theorem 2.3. *Let X be a stochastic process such that :*

- $X \in L^2(\Omega \times [0,T])$
- $X_t \in \mathbf{D}^{1,2}\ \forall t \in [0,T]$

- $D_tX \in dom(\delta)$ *such that* $\delta(D_tX) \in L^2(\Omega \times [0,T])$

Then,

$$\delta(X) \in \mathbf{D}^{1,2} \text{ and } D_t(\delta(X)) = X_t + \delta\left(D_tX\right)$$

Proof. We use the results of **Appendix A**. We use the following decomposition :

$$X_t = \sum_{n=0}^{\infty} I_n\left(f_n(\,.,t)\right)$$

$$\begin{aligned}
D_t(\delta(X_t)) &= D_t(\delta(\sum_{n=0}^{\infty} I_n\left(f_n(\,.,t)\right))) \\
&= D_t(\sum_{n=0}^{\infty}(n+1)I_n(\tilde{f}_n)) \\
&= \sum_{n=0}^{\infty}(n+1)I_n(\tilde{f}_n(.,t))) \\
&= X_t + \sum_{n=0}^{\infty} I_n(\sum_{i=1}^{n} f_n(t_1,\ldots,t_{i-1},t,t_{i+1},\ldots,t_n)) \\
&= X_t + \sum_{n=0}^{\infty} nI_n(\psi_n(.,t,.))
\end{aligned}$$

Where $\psi_n(\cdot,t,\cdot)$ is a symmetrization of

$$(t_1,\cdots,t_n) \mapsto f_n\left(t_1,\cdots,t_{n-1},t,t_n\right)$$

On the other hand,

$$\begin{aligned}
\delta(D_t(X)) &= \delta(\sum_{n=0}^{\infty} nI_{n-1}f_n(.,t,.)) \\
&= \sum_{n=0}^{\infty} nI_n(\psi_n(.,t,.))
\end{aligned}$$

□

2.5 Integration by parts formula

Theorem 2.4. *Integration by parts formula*
Let F,G be Random Variables, such that $F \in \mathbb{D}^{1,2}$ and let u be an $L^2(\Omega)$ -Valued Random Variable such that $\langle \mathrm{D}f,u\rangle_H \neq 0$ a.s. and $\frac{Gu}{\langle \mathrm{D}F,u\rangle_H} \in \operatorname{dom}\delta$. Then for any $f \in C^1$ with bounded derivatives

$$E\left[f'(F)G\right] = E\left[f(F)\delta\left(\frac{Gu}{\langle \mathrm{D}F,u\rangle_{L^2(L^2([0,T])}}\right)\right]$$

Remark. Generally, $f \notin \mathcal{C}^1$. We can prove that the theorem holds for continuous functions with jump discontinuities and linear growth using an approximation argument.

Proof. By chain rule: $\langle \mathrm{D}(f(F)),u\rangle_{L^2(\Omega)} = f'(F)\langle \mathrm{D}F,u\rangle_{L^2(\Omega)}$ then

$$\begin{aligned} E\left[f'(F)G\right] &= E\left[\frac{\langle \mathrm{D}(f(F)), u\rangle_{L^2([0,T]}}{\langle \mathrm{D}F, u\rangle_{L^2([0,T])}} G\right] \\ &= E\left[\left\langle \mathrm{D}(f(F)), \frac{Gu}{\langle \mathrm{D}F, u\rangle_{L^2([0,T]}} \right\rangle_{L^2([0,T])}\right] \\ &\overset{\text{duality}}{=} E\left[f(F)\delta\left(\frac{Gu}{\langle \mathrm{D}F, u\rangle_{L^2([0,T])}}\right)\right] \end{aligned}$$

□

3 Applications to Mathematical Finance : Greeks

3.1 Black & Scholes model[3]

The risk-neutral form of the Black and Scholes, Merton (1973) model postulates the following diffusion for S, driven by a standard $\mathcal{Q}$-Brownian motion W:

$$dS_t = S_t(\kappa dt + \sigma dW_t)$$

Where :

- $\kappa = r - q$
- r is the interest rate
- q is the dividend rate
- $\sigma > 0$ constant volatility parameter

More explicitly, it follows from applying Ito's formula on $ln(S)$:

$$S_t = e^{bt+\sigma W_t}$$

with $b = \kappa - \sigma^2/2$.

3.2 Sensitivities

Sensitivities, or more commonly called by practitioners *Greeks*, are the derivatives of the price of an option with respect to its parameters. They quantify the stability under parameters' variations.

Definition 3.1. *(Greeks)*

Let denote Π_0 the price of a european option generating a final flow $\phi(S_T)$.

- **Delta** : Δ is defined as the measure of the rate of change of the options calculated value,Π_0, with respect to the change of the underlying assets' price,S_0.

$$\Delta = \frac{\partial \Pi_0}{\partial S_0}$$

- **Gamma** : Γ is defined as the measure of the rate of change of the options calculated value,, with respect to the change of the underlying assets' price,S.

$$\Gamma = \frac{\partial^2 \Pi_0}{\partial S_0^2}$$

- **Vega** : ν is defined as the measure of the rate of change of the options calculated value,Π_0, with respect to the change of the underlying assets' price,σ.

$$\nu = \frac{\partial \Pi_0}{\partial \sigma}$$

[3] Henceforth, we will work in the Black & Scholes setup with no dividends

Remark. For the exact formulas for vanilla calls' / puts' greeks, we refer you to **Appendix B**.

3.3 Application of Malliavin Calculus : computing greeks

Proposition 1. *(European claim option Delta using Malliavin Calculus) We consider a european style option with maturity T and a payoff function ϕ. Then,*

$$\Delta = \frac{e^{-rT}}{S_0} E\left[\phi\left(S_T\right) \frac{W_T}{\sigma T}\right]$$

Proof. *(Formal calculations of Delta using Malliavin Calculus)*

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial S_0} E\left[e^{-rT} \phi\left(S_T\right)\right] \\
&= E\left[e^{-rT} \phi'\left(S_T\right) \frac{\partial S_T}{\partial S_0}\right] \\
&= \frac{e^{-rT}}{S_0} E\left[\phi'\left(S_T\right) S_T\right] \\
&= \frac{e^{-rT}}{S_0} E\left[\phi\left(S_T\right) \delta\left(\frac{S_T}{\int_0^T D_s S_T ds}\right)\right] \\
&= \frac{e^{-rT}}{S_0} E\left[\phi\left(S_T\right) \delta\left(\frac{S_T}{\int_0^T \sigma S_T 1_{\{s \leq T\}} ds}\right)\right] \\
&= \frac{e^{-rT}}{S_0} E\left[\phi\left(S_T\right) \delta\left(\frac{1}{\sigma T}\right)\right] \\
&= \frac{e^{-rT}}{S_0} E\left[\phi\left(S_T\right) \frac{W_T}{\sigma T}\right]
\end{aligned}$$

□

Proposition 2. *(European claim option Gamma using Malliavin Calculus)*
We consider a european style option with maturity T and a payoff function ϕ. Then,

$$\Gamma = \frac{e^{-rT}}{S_0^2 \sigma T} E\left[\phi\left(S_T\right)\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right]$$

Proof. *(Formal calculations of Gamma using Malliavin Calculus)*

$$
\begin{aligned}
\Gamma &= \frac{\partial^2}{\partial S_0^2} E\left[e^{-rT}\phi\left(S_T\right)\right] \\
&= \frac{e^{-rT}}{S_0^2} E\left[\phi''\left(S_T\right) S_T^2\right] \\
&= \frac{e^{-rT}}{S_0^2} E\left[\phi'\left(S_T\right)\delta\left(\frac{S_T^2}{\int_0^T \sigma S_T 1_{\{s\leq T\}} ds}\right)\right] \\
&= \frac{e^{-rT}}{S_0^2} E\left[\phi'\left(S_T\right)\delta\left(\frac{S_T^2}{\sigma T S_T}\right)\right] \\
&= \frac{e^{-rT}}{S_0^2} E\left[\phi'\left(S_T\right)\delta\left(\frac{S_T}{\sigma T}\right)\right] \\
&= \frac{e^{-rT}}{S_0^2} E\left[\phi'\left(S_T\right)\left(\frac{S_T W_T}{\sigma T} - S_T\right)\right] \\
&= \frac{e^{-rT}}{S_0^2} E\left[\phi\left(S_T\right)\delta\left(\frac{\frac{W_T}{\sigma T} - 1}{\sigma T}\right)\right] \\
&= \frac{e^{-rT}}{S_0^2} E\left[\phi\left(S_T\right)\left(\delta\left(\frac{W_T}{\sigma^2 T^2}\right) - \delta\left(\frac{1}{\sigma T}\right)\right)\right] \\
&= \frac{e^{-rT}}{S_0^2} E\left[\phi\left(S_T\right)\left(\frac{W_T^2 - T}{\sigma^2 T^2} - \frac{W_T}{\sigma T}\right)\right] \\
&= \frac{e^{-rT}}{S_0^2} E\left[\phi\left(S_T\right)\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right]
\end{aligned}
$$

□

Proposition 3. *(European claim option Vega using Malliavin Calculus)*
We consider a european style option with maturity T and a payoff function ϕ. Then,

$$
\nu = e^{-rT} E\left[\phi\left(S_T\right)\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right]
$$

Proof. *(Formal calculations of Vega using Malliavin Calculus)*

$$
\begin{aligned}
\nu &= \frac{\partial}{\partial \sigma} E\left[e^{-rT}\phi\left(S_T\right)\right] \\
&= E\left[e^{-rT}\phi'\left(S_T\right)\frac{\partial S_T}{\partial \sigma}\right] \\
&= e^{-rT} E\left[\phi'\left(S_T\right)\left(W_T - \sigma T\right) S_T\right] \\
&= e^{-rT} E\left[\phi\left(S_T\right)\delta\left(\frac{\left(W_T - \sigma T\right) S_T}{\int_0^T D_s S_T ds}\right)\right] \\
&= e^{-rT} E\left[\phi\left(S_T\right)\delta\left(\frac{\left(W_T - \sigma T\right) S_T}{\int_0^T \sigma S_T 1_{\{s\leq T\}} ds}\right)\right. \\
&= e^{-rT} E\left[\phi\left(S_T\right)\delta\left(\frac{W_T - \sigma T}{\sigma T}\right)\right] \\
&= e^{-rT} E\left[\phi\left(S_T\right)\left(\frac{1}{\sigma T}\delta\left(W_T\right) - W_T\right)\right] \\
&= e^{-rT} E\left[\phi\left(S_T\right)\left(\frac{W_T^2 - T}{\sigma T} - W_T\right)\right] \\
&= e^{-rT} E\left[\phi\left(S_T\right)\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right]
\end{aligned}
$$

□

Remark. We find the classical relation : $\nu = S_0^2 \sigma T \Gamma$

4 Numerical results and conclusion

4.1 Setup

In this section, we conduct an empirical study on the Malliavin Weighted Scheme **Vs.** Finite Difference Method in the estimation of option greeks for three type of derivatives :

1. Vanilla option : The payoff function $\phi(S_T) = (S_T - K)_+$.
2. Binary option (or *digital option*) : the payoff function is given by $\phi(S_T) = \mathbf{1}_{(S_T \geq K)}$, exhibiting one discontinuity.
3. Corridor option : the payoff function is given by $\phi(S_T) = \mathbf{1}_{(K_{min} \leq S_T \leq K_{max})}$, exhibiting two discontinuities.

4.2 Numerical results

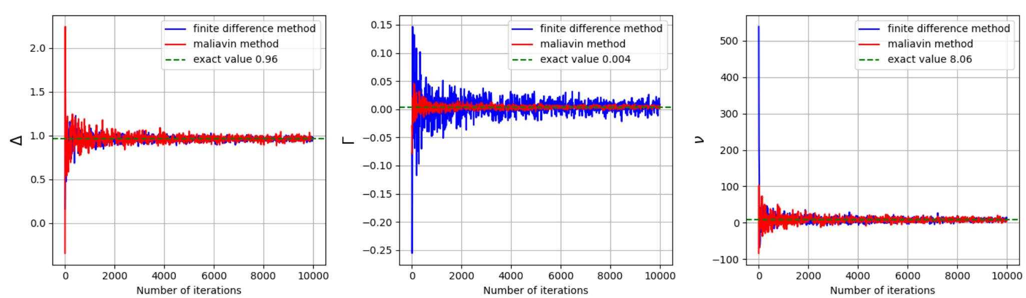

Figure 1: European call greeks - Finite Difference Vs. Malliavin

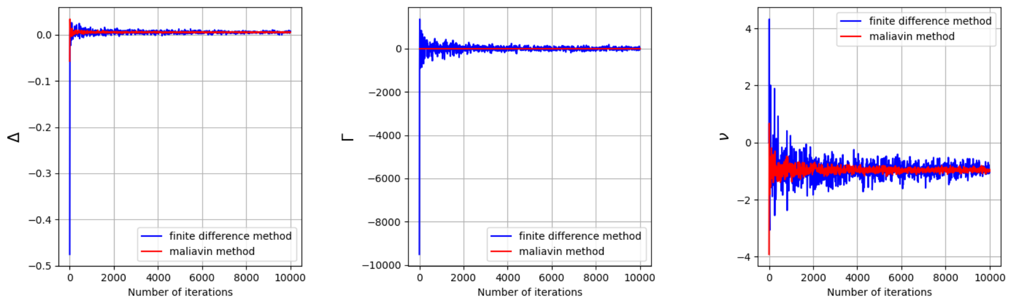

Figure 2: Digital option greeks - Finite Difference Vs. Malliavin

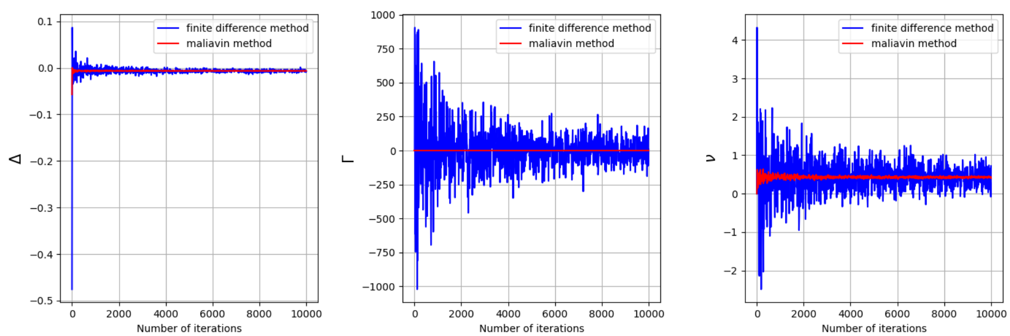

Figure 3: Corridor option greeks - Finite Difference Vs. Malliavin

$\frac{Var_{finitedifference}}{malliavin}$	Δ	Γ	ν
Vanilla option	0.40	13.21	3.93
Binary option	41.49	14740486587957.42	5.92
Corridor option	88.39	61324868176022.72	152.49

Table 1: Comparison of the Malliavin weighted scheme and the Finite difference method

4.3 Remarks and commentaries

For simulation purposes, we choose options with decreasing payoff regularity properties to infer on the effect of Malliavin Calculus. We use for the sake of arguments the figures and results of the previous section.

For irregular payoff functions such as digital and corridor option, the mean squared convergence of $\Pi_0(S_0+\epsilon)$ to $\Pi_0(S_0)$ is linear in ϵ [4]. As a result, for corridor and digital options, the convergence of Malliavin formula occurs very fast and with almost no oscillations per contra to the outperformed finite difference method that exhibits lengthy convergence and a pseudo-periodic regime. We note that the effect of Malliavin formula is even astounding for corridor option compared to the digital one.

For vanilla options, regular payoff function does not call for the use of the Malliavin integration by parts. In fact, the mean squared convergence of $\Pi_0(S_0 \pm \epsilon)$ to $\Pi_0(S_0)$ is quadratic in ϵ. Let's take the case of the Δ :

$$\begin{aligned}\mathbb{E}\left[|\Pi_0(S_0+\varepsilon)-\Pi_0(S_0)|^2\right] &\leq \mathbb{E}\left[|S_T(\varepsilon)-S_t|^2\right]\\ &\leq \varepsilon^2\mathbb{E}\left[|e^{(r-\mu)T+\sigma\sqrt{T}Z}|^2\right]\end{aligned}$$

Consequently :

$$\mathbb{E}\left[|\Pi_0(S_0+\varepsilon)-\Pi_0(S_0)|^2\right] = O(\varepsilon^2)$$

As a result, we observe very comparable performance for both techniques.

Another remark, Malliavin technique is not the right choice for short-term maturities as the weight function explode. Also, we need to underline the fact that the weight function is roughly polynomial of W_t. In the case of a small value payoff function, the brownian motion might take very high values; multiplying it by a polynom of W_t will lead to a low variance, which is very convenient.

5 Conclusion

In this short report, we present elements in Malliavin Calculus and some application in Mathematical Finance. We show that integration by part introduced by the theory aforenamed smoothens irregular payoff functions for computing greeks, outperforming hence finite difference method. However, this is to be used with moderation. For regular payoffs, generally there is no added value. It might be in some cases inefficient compared to finite difference method. Some extensions to the presented results is to use hybrid approaches using a local version of the Malliavin method in the dicontinuities points with a finite difference method. We can also try to conduct a similar studies on asian/american options, similar results might be drawn.

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Appendix

A Wiener Chaos

In this appendix, we denote $(\mathbf{H}, \langle\cdot,\cdot\rangle), \|.\|_H$ a separable hilbert space unless mentioned otherwise.

Definition A.1. (*Isonormal Gaussian Process*)

Let $W = \{W(h), h \in \mathbf{H}\}$ be a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ indexed by $\mathbf{H}$. We say that W is an **isonormal gaussian process** if :

- W is a family of gaussian processes
- $(\forall h \in \mathbf{H})\ \mathbf{E}[W(h)] = 0$
- $(\forall h, g \in \mathbf{H})\ \mathbf{E}[W(h)W(g)] = \langle h, g\rangle$

Remark. 1. $h \mapsto W(h)$ is an isometry of $\mathbf{H}$ into a closed subspace of $L^2(\Omega, \mathcal{F}, P)$ $\mathcal{H}_1$.

2. Let B a d-dimensional brownian motion defined on the wiener space $(\Omega = \mathcal{C}([0,\infty], \mathbf{R}^d), \mathcal{F}, P)$. In this case, $\mathbf{H} = L^2(\mathbf{R}^+, \mathbf{R}^d)$. In this case, we write Ito's integrale :

$$\forall h \in \mathbf{H},\ W(h) := \sum_{i=1}^{d} \int h_i(s) dB_s^i$$

This is the case of definition of Malliavin Calculus : *white noise derivative operator*. More general frameworks may be considered.

Definition A.2. *(Hermite Polynomial of degree n and parameter $\lambda > 0$)*

For $n \geq 1$ and $x \in \mathbf{R}$,

$$H_n(x, \lambda) := \frac{(-\lambda)^n}{n!} e^{\frac{x^2}{2\lambda}} \frac{d^n}{dx^n}\left[e^{\frac{x^2}{2\lambda}}\right]$$
$$H_0 := 1, n \geq 1, x \in \mathbb{R}$$
$$H_n(x, 1) = H_n(x)$$

Definition A.3. *(Wiener Chaos of order n)*

For $n \geq 1$, we define the Wiener Chaos of order n $\mathcal{H}$ as the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by $\{H_n(W(h)) \mid h \in H, \|h\|_H = 1\}$.

Theorem A.1. *(Wiener Chaos decomposition (I))*

Let $G = \sigma(W(h), h \in \mathbf{H})$. Then,

$$L^2(\Omega, G, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

In the rest of the appendix, we consider the white noise case.

Idea for the final Theorem A classical approximation approach is used :

We approximate any function $f \in L^2([a, b] = T)$ by elementary functions vanishing on the diagonal. Let,

$$D := \{(t_1, \cdots, t_n) \in T^n \mid \exists i \neq j : t_i = t_j\}$$

be the Diagonal of T^n. Let $\mathcal{E}$ be the class of elementary functions T^n vanishing on D that is the set of functions of the form

$$f(t_1, \cdots, t_n) = \sum_{i_1, \cdots, i_n = 0}^{k} a_{i_1, \cdots, i_n} \mathbf{1}_{\left(T_{i_1}, T_{i_1}\right]} \times \cdots \times \mathbf{1}_{(T_{i_n}, T_{i_n}]}$$

where $a=\tau_0<\tau_1<\cdots<\tau_k=b$ and $a_{i_1,\cdots,i_n}=0$ if $i_p=i_q$ for some $p\neq q$. For $f\in\mathcal{E}_n$ define

$$I_n(f):=\sum_{i_1,\cdots,i_{n-1}}^{k} a_{i_1,\cdots,i_n}\xi_{i_1}\cdots\xi_{i_n};\xi_{i_p}=B_{\tau_{i_p}}-B_{\tau_{i_{p-1}}}$$

N.B. $*I_n(f)$ is well-defined, i.e. the definition doesn't depend on the representation of f

$*I_n(f)$ is linear

Define the symmetrization, $\tilde{f}$ of f by

$$\tilde{f}\left(t_1,\cdots,t_n\right)=\frac{1}{n!}\sum_{\sigma\in S_n} f\left(t_{\sigma(1)},\cdots,t_{\sigma(n)}\right)$$

where S_n is the set of all permutations of $[n]$. For $\sigma\in S_n$

$$\begin{gathered}\int_{T^n}\left|f\left(t_1,\cdots,t_n\right)\right|^2 dt_1\cdots dt_n=\int_{T_n}\left|f\left(t_{\sigma(1)},\cdots,t_{\sigma(n)}\right)\right|^2 dt_1\cdots dt_n\\ \Rightarrow\|\tilde{f}\|_{L^2(T^n)}\leq\frac{1}{n!}\sum_{\sigma\in S_n}\|f\|_{L^2(T^n)}=\|f\|_{L^2}\end{gathered}$$

* $I_n(f)=I_n(\tilde{f})$

Theorem A.2. *(Wiener Chaos decomposition (II))*
Any $F\in L^2(\Omega,\mathcal{G},P)$ can be expanded into

$$F=\sum_{n=0}^{\infty} I_n\left(f_n\right)$$

where $f_0=E[F], I_0$ the identity on constants) and $f_n\in L^2\left(T^n\right)$ symmetric and uniquely determined by F.

Theorem A.3. *(Wiener Chaos decomposition (III))*
Any $F\in L^2(\Omega\times[0,T])$ can be expanded into

$$F_t=\sum_{n=0}^{\infty} I_n\left(f_n(.,t)\right)$$

where $f_n\in L^2\left(T^{n+1}\right)$ symmetric and uniquely determined by F.

Lemma. *(Useful lemma for proofs)*
Under certain rigorous conditions that we will suppose fulfilled in this report we have :

- *For F in Wiener Chaos decomposition (II) : $\delta(F)=\sum_{n=0}^{\infty} I_{n+1}\left(\tilde{f}_n\right)$*
- *For F in Wiener Chaos decomposition (III) : $\mathrm{D}_t F=\sum_{n=1}^{\infty} n I_{n-1}\left(f_n(\cdot,t)\right)$*

B Black & Scholes model : Greeks

Greek	European Call	European Put
Delta	$\mathrm{e}^{-qT}\Phi(d_1)$	$\mathrm{e}^{-qT}(\Phi(d_1)-1)$
Gamma	$\frac{\Phi'(d_1)\mathrm{e}^{-qT}}{S_0\sigma\sqrt{T}}$	$\frac{\Phi'(d_1)\mathrm{e}^{-qT}}{S_0\sigma\sqrt{T}}$
Vega	$S_0\sqrt{T}\Phi'(d_1)\mathrm{e}^{-qT}$	$S_0\sqrt{T}\Phi'(d_1)\mathrm{e}^{-qT}$

Table 2: The exact formula for the Greeks for European options