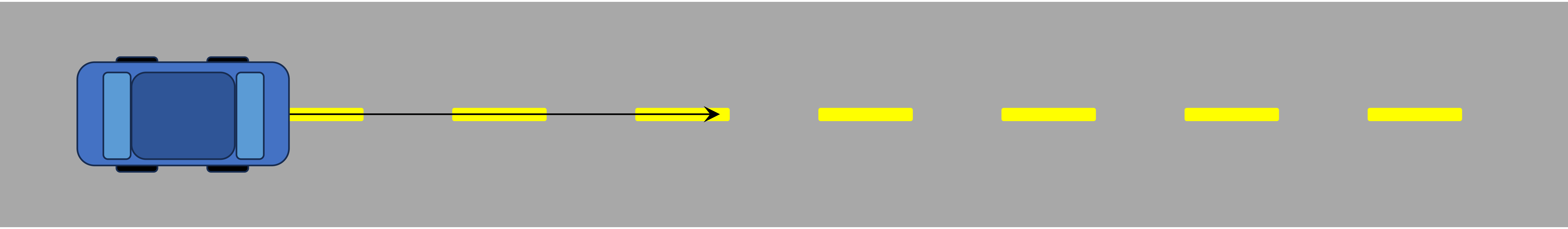
Lecture 3

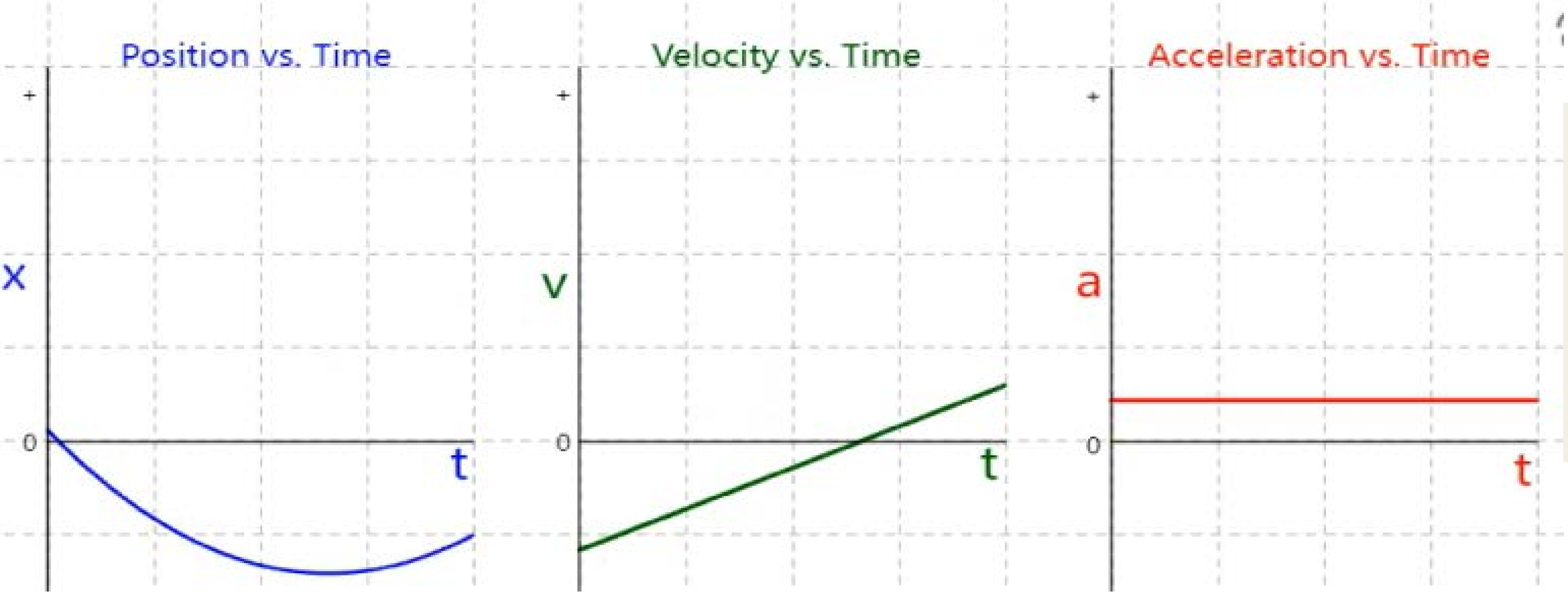
Kinematics in 3D

Date: 2/27/2025

Course Instructor:
Jingtian Hu (胡竞天)

Previous Lecture: Kinematics along a line





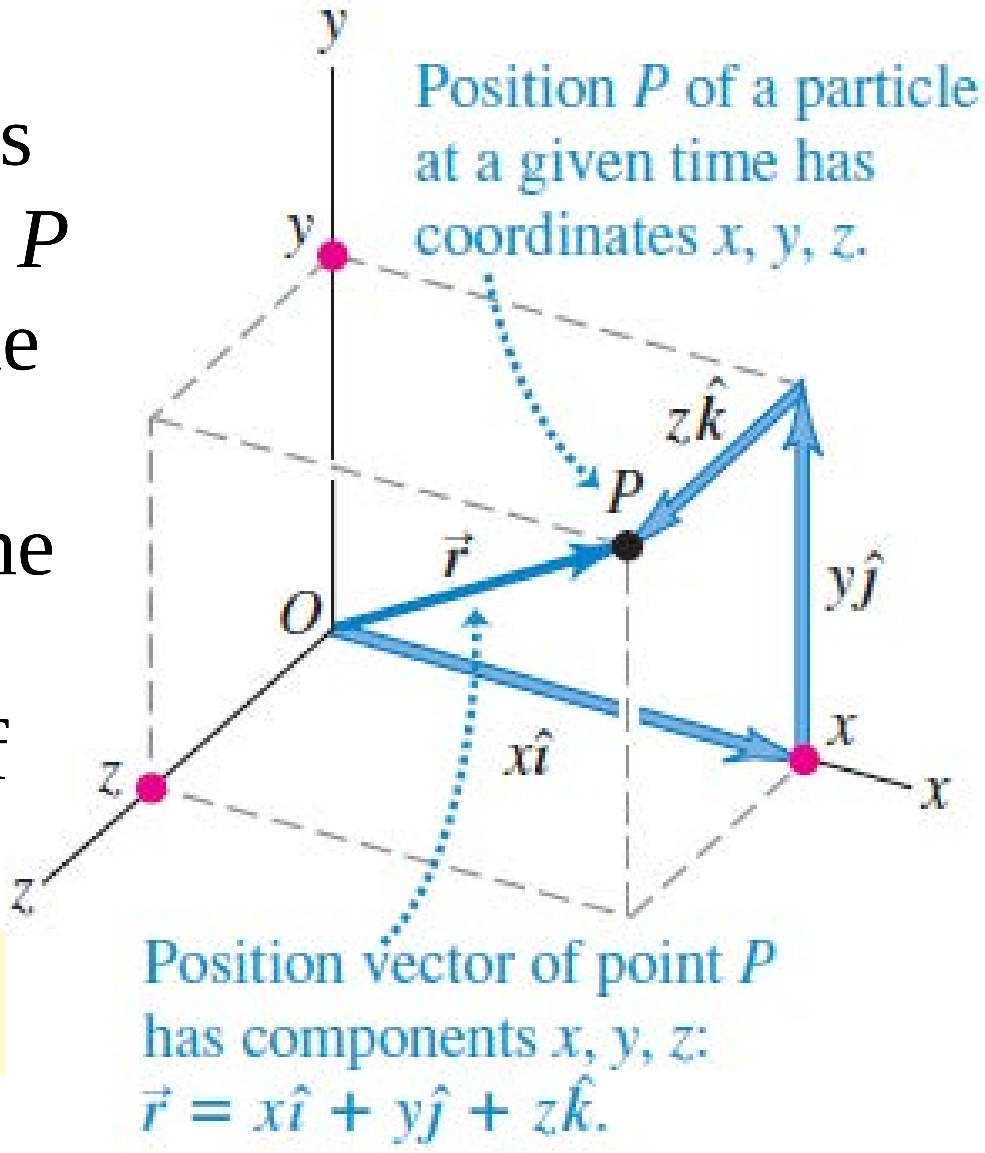
$$v_x = v_{0x} + \int_0^t a_x \, dt$$

$$x = x_0 + \int_0^t v_x \, dt$$

Position and Velocity Vectors

To describe the *motion* of a particle in space, we must first be able to describe the particle's *position*. Consider a particle that is at a point *P* at a certain instant. The **position vector** of the particle at this instant is a vector that goes from the origin of the coordinate system to the point *P* The Cartesian coordinates *x*, *y*, and *z* of point *P* are the *x*-, *y*-, and *z*-components of vector *r*

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad \text{(position vector)}$$



Position and Velocity Vectors

During a time interval Δt the particle moves from P_1, where its position vector is $\vec{r}_1$, to P_2, where its position vector is $\vec{r}_2$. The change in position (the displacement) during this interval is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$. We define the **average velocity** $\vec{v}_{av}$ during this interval in the same way we did in Chapter 2 for straight-line motion, as the displacement divided by the time interval:

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t} \qquad \text{(average velocity vector)} \qquad (3.2)$$

Recall in 1D: $\vec{r} = x\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$ (position vector)

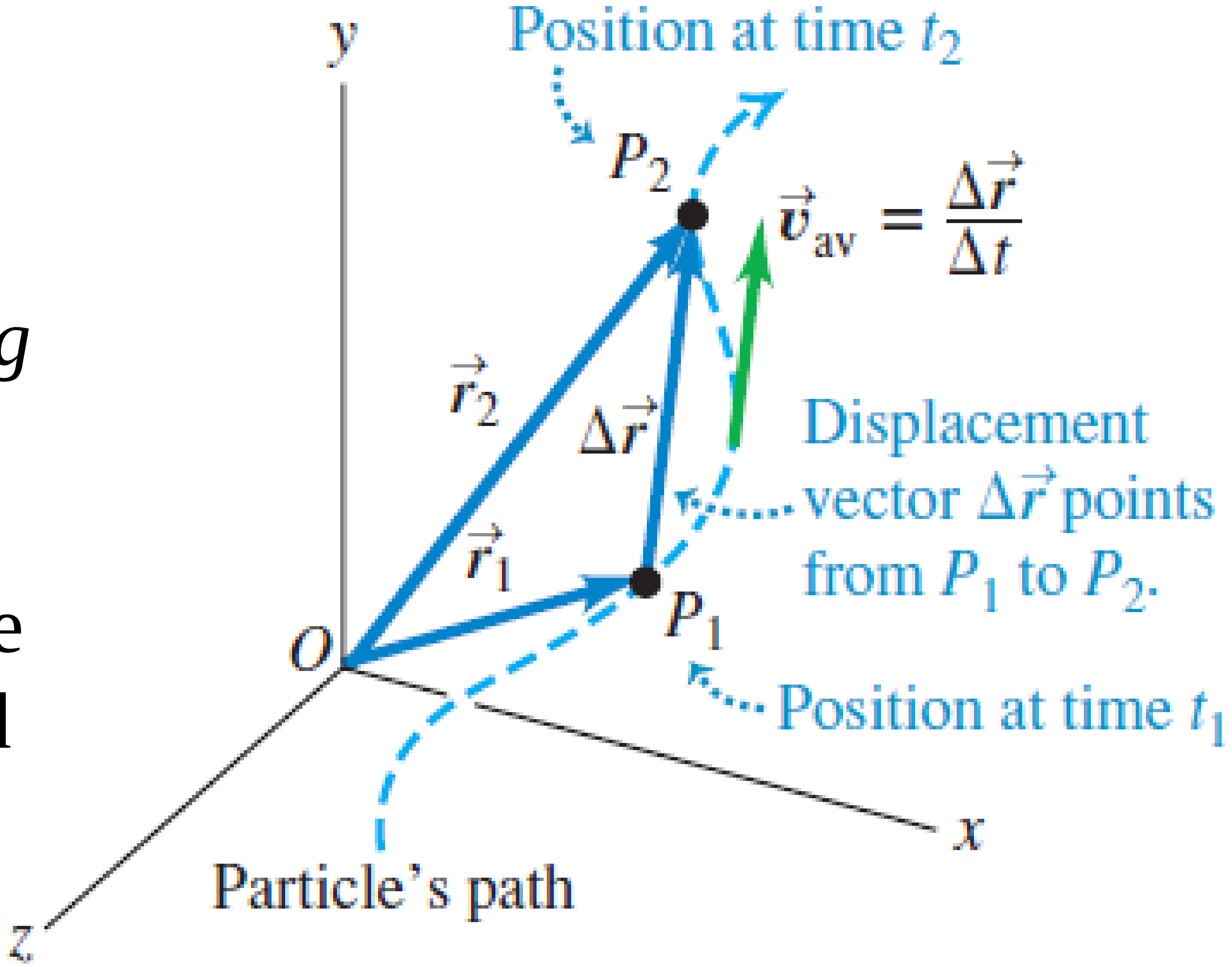
$$v_{av\text{-}x} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

Exactly the same form

Position and Velocity Vectors

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t}$$

Dividing a vector by a scalar is really a special case of *multiplying* a vector by a scalar, described in Section 1.7 of the textbook; the average velocity $\boldsymbol{v}_{av}$ is equal to the displacement vector Δr multiplied by $1/\Delta t$ the reciprocal of the time interval



Position and Velocity Vectors

We now define **instantaneous velocity** just as we did in 1D

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \qquad \text{(instantaneous velocity vector)}$$

The *magnitude* of the vector $\vec{v}$ at any instant is the *speed* v of the particle at that instant. The *direction* of $\vec{v}$ at any instant is the same as the direction in which the particle is moving at that instant.

It follows that the components v_x, v_y and v_z of the instantaneous velocity v_x are simply the time derivatives of the coordinates *x*, *y*, and *z*. That is,

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt} \qquad \text{(components of instantaneous velocity)}$$

Position and Velocity Vectors

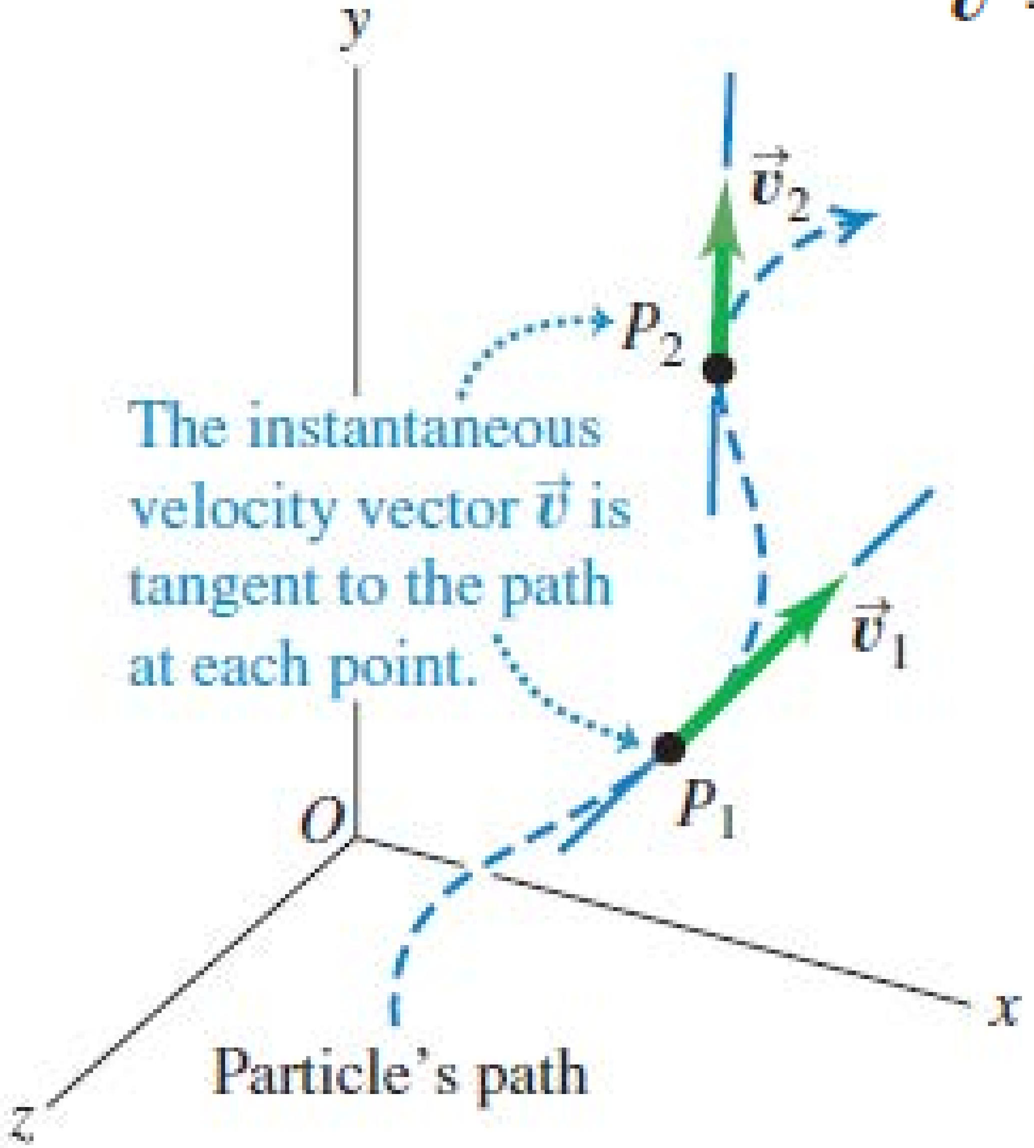
Can be also obtained by

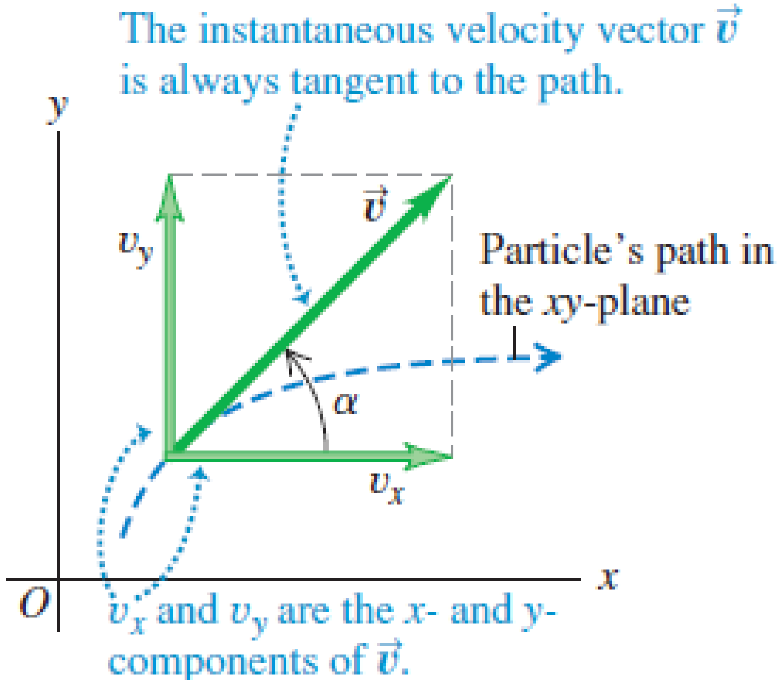
$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt}$$

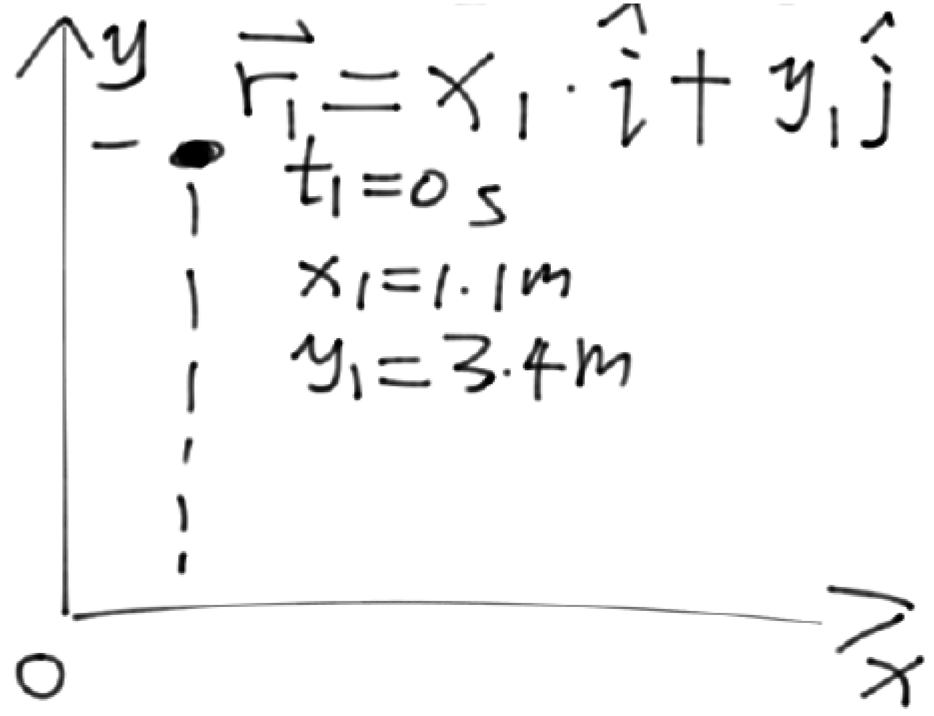
$$|\vec{v}| = v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\tan\alpha = \frac{v_y}{v_x}$$





3.1 • A squirrel has x- and y-coordinates (1.1 m, 3.4 m) at time $t_1 = 0$ and coordinates (5.3 m, −0.5 m) at time $t_2 = 3.0$ s. For this time interval, find (a) the components of the average velocity, and (b) the magnitude and direction of the average velocity.



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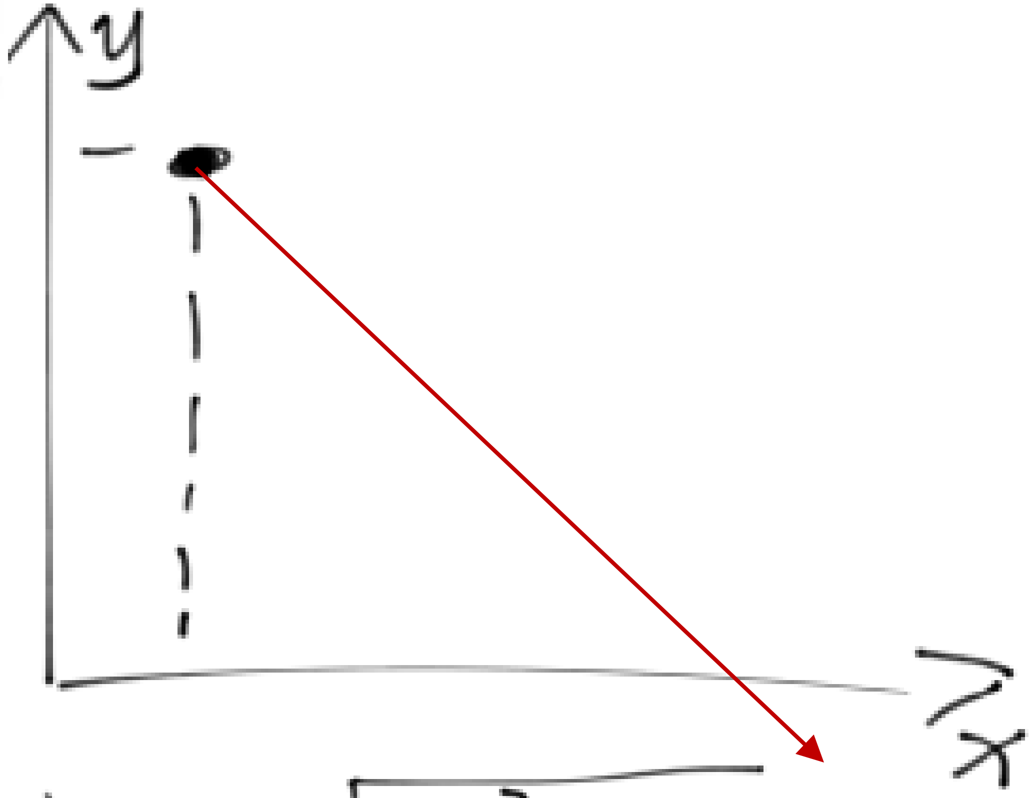


$$\overline{v_x} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$= \frac{5.3\,m - 1.1\,m}{3.0\,s - 0\,s}$$

$$= 1.4\ m/s$$

$$\overline{v_y} = \frac{\Delta y}{\Delta t} = \frac{y_2 - y_1}{t_2 - t_1}$$

$$= \frac{-0.5\,m - 3.4\,m}{3.0\,s} = -1.3\,m/s$$

$$V_{ave} \equiv |\vec{V}_{ave}| = \sqrt{1.4^2 + 1.3^2}\ m/s$$

$$\simeq 1.9\,m/s$$

$$\tan(-\theta) = \frac{13}{14} \Rightarrow \theta = -\tan\left(\frac{13}{14}\right) \simeq -42.9^\circ$$

The Acceleration Vector

Recall (1D) $a_{\text{av-}x} = \dfrac{v_{2x} - v_{1x}}{t_2 - t_1} = \dfrac{\Delta v_x}{\Delta t}$ (average acceleration in ***x***)

In 2D/3D space: $\vec{a}_{\text{av}} = \dfrac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \dfrac{\Delta \vec{v}}{\Delta t}$ (average acceleration vector)

Instantaneous acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \qquad \text{(instantaneous acceleration vector)}$$

CAUTION Any particle following a curved path is accelerating When a particle is moving in a curved path, it always has nonzero acceleration, even when it moves with ***constant speed***. We will have an example soon

The Acceleration Vector

Recall (1D) $$a_{\text{av-}x} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t}$$ (average acceleration in ***x***)

In 2D/3D space: $$\vec{a}_{\text{av}} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$ (average acceleration vector)

Instantaneous acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$ (instantaneous acceleration vector)

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt}$$ (components of instantaneous acceleration)

The Acceleration Vector from Unit Vectors

In terms of unit vectors,

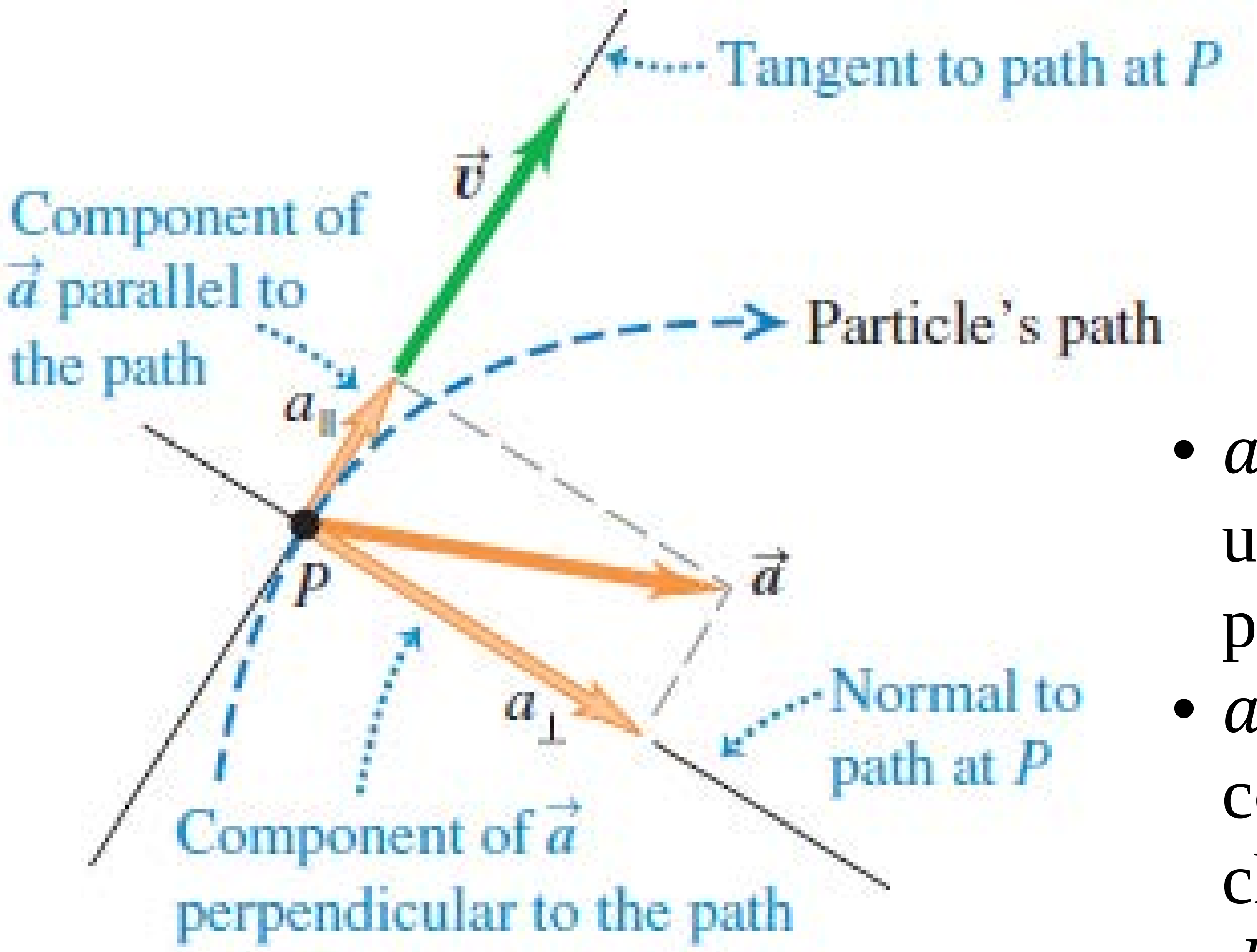
$$\vec{a} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} + \frac{dv_z}{dt}\hat{k} \tag{3.11}$$

The x-component of Eqs. (3.10) and (3.11), $a_x = dv_x/dt$, is the expression from Section 2.3 for instantaneous acceleration in one dimension, Eq. (2.5). Figure 3.8 shows an example of an acceleration vector that has both x- and y-components.

$$a_x = \frac{d^2x}{dt^2} \qquad a_y = \frac{d^2y}{dt^2} \qquad a_z = \frac{d^2z}{dt^2}$$

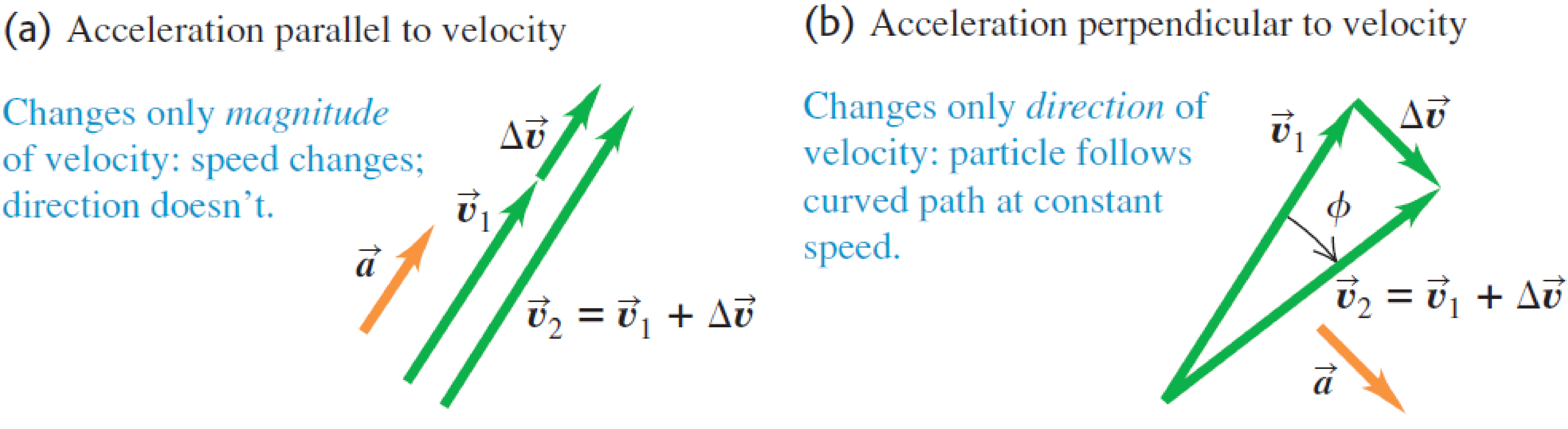
$$\vec{a} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k}$$

Parallel & Perpendicular Components of Acceleration



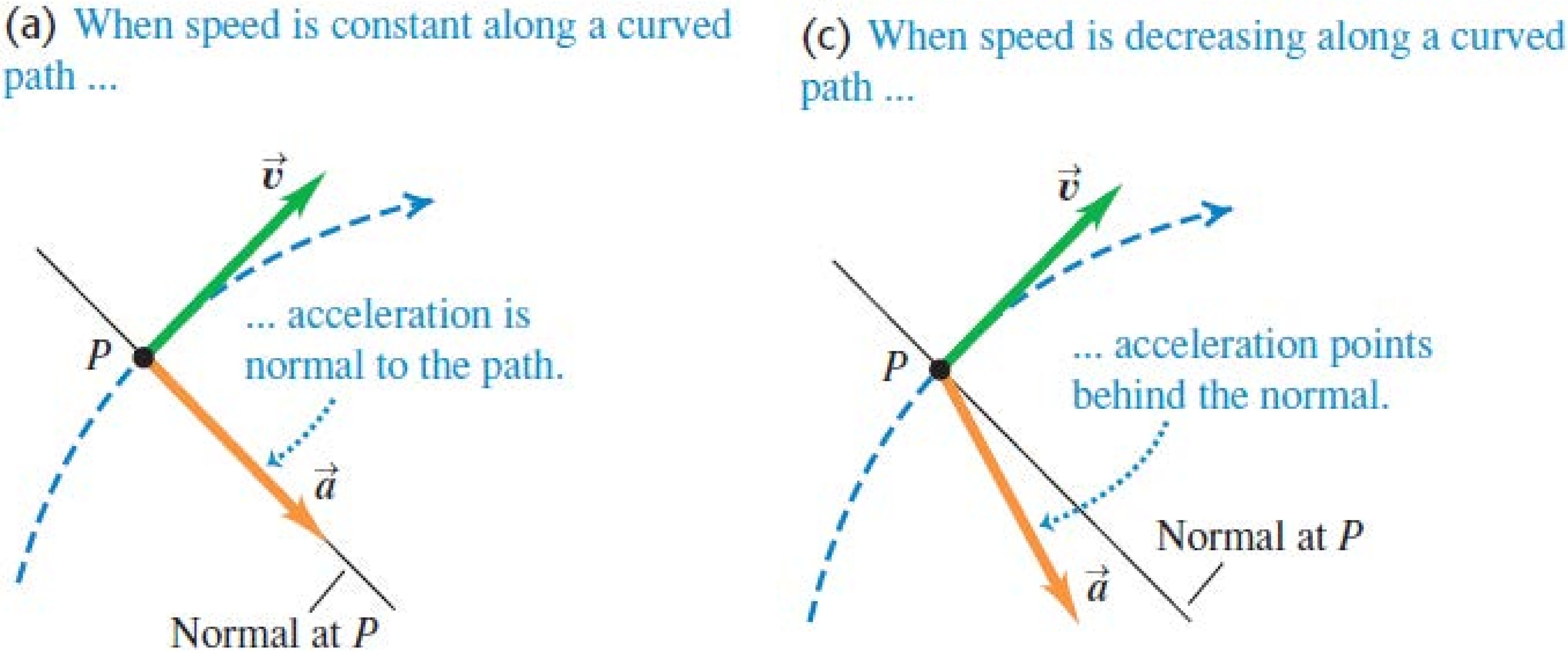
- $a_{\|}$: parallel component tells us about changes in the particle's *speed*
- $a_{\perp}$: perpendicular component tells us about changes in the particle's *direction of motion*.

Parallel & Perpendicular Components of Acceleration



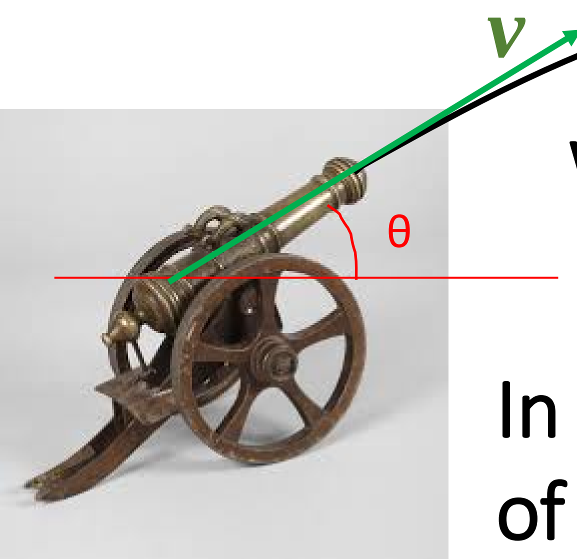
In the most general case, the acceleration $\vec{a}$ has components *both* parallel and perpendicular to the velocity $\vec{v}$ as in Fig. 3.10. Then the particle's speed will change (described by the parallel component) *and* its direction of motion will change (described by the perpendicular component $a_{\perp}$) so that it follows a curved path.

Parallel & Perpendicular Components of Acceleration



Projectile Motion

How to hit the target?



What parameters can we tune?

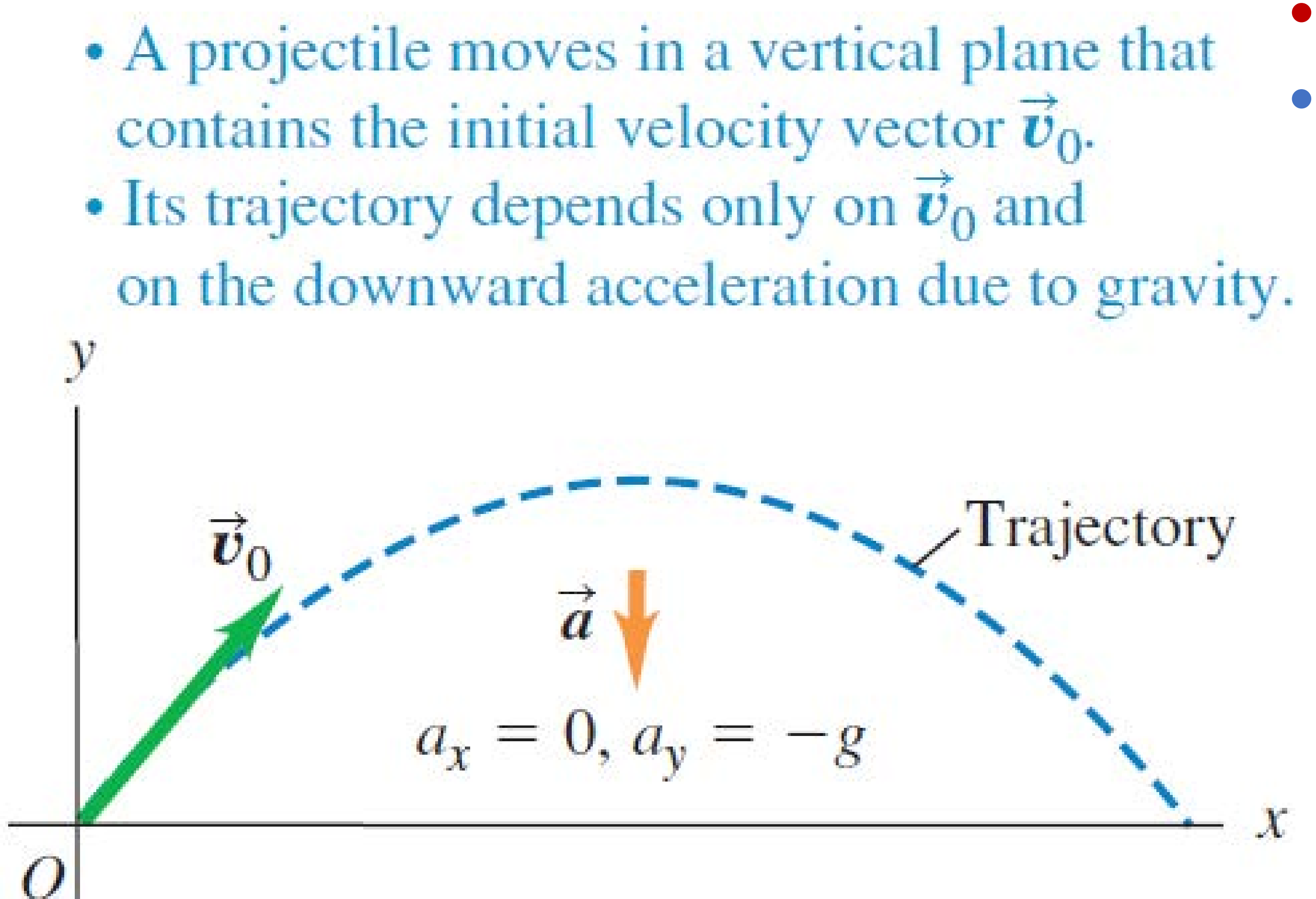
In addition, position of the cannon (x, y, z)

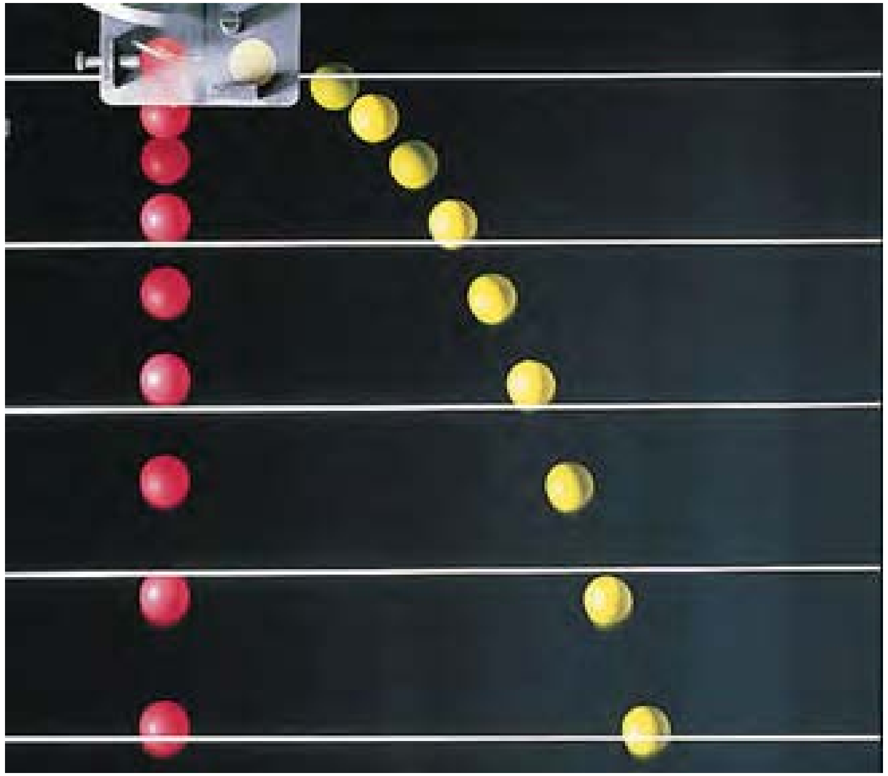
A **projectile** is any body that is given an initial velocity and then follows a path determined entirely by the effects of gravitational acceleration and air resistance. A batted baseball, a thrown football, a package dropped from an airplane, and a bullet shot from a rifle are all projectiles. The path followed by a projectile is called its **trajectory.**

Projectile Motion

Thus projectile motion is *two-dimensional*, which can be described by *xy*-coordinate plane. The ***key*** to analyzing projectile motion is that we can treat the *x*- and *y*-coordinates ***separately***, as a combination of

- horizontal motion with constant ***v***
- vertical motion with constant ***a***

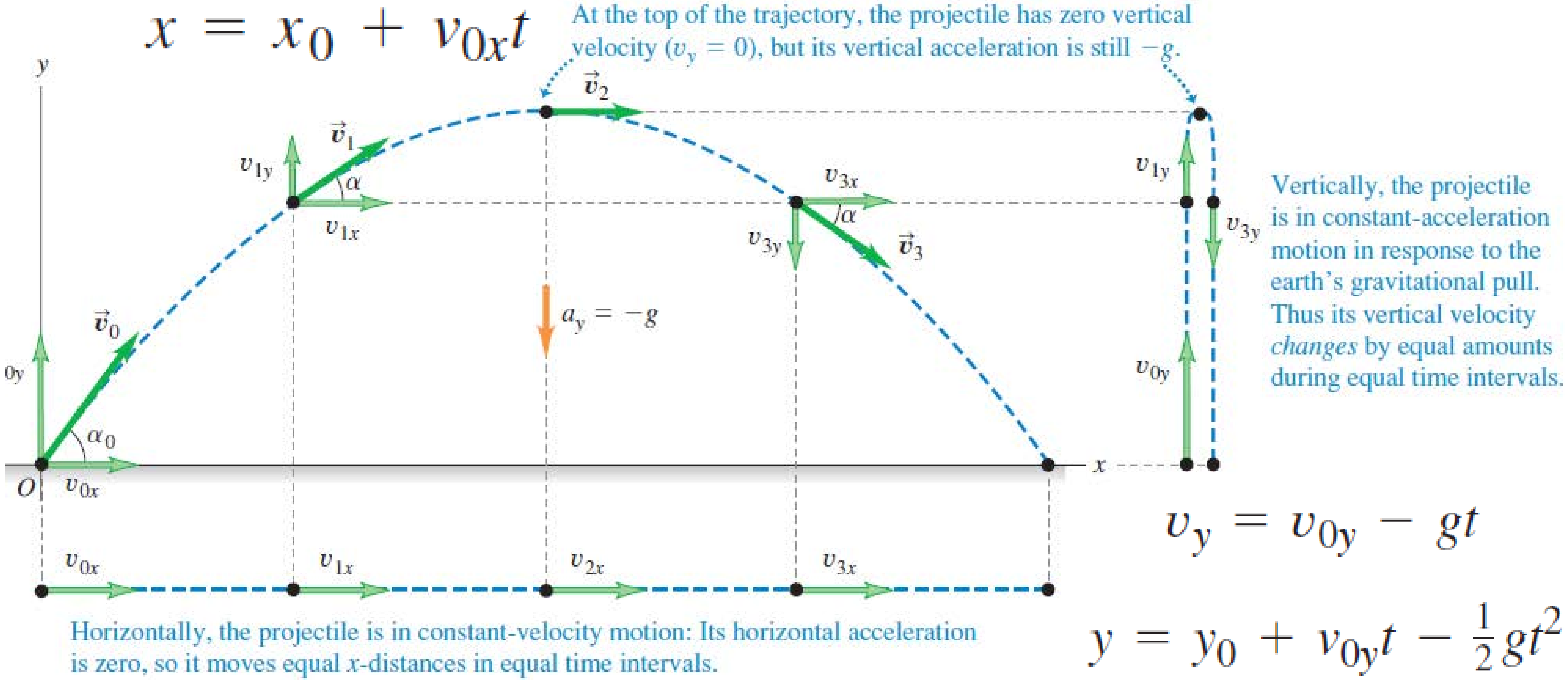




Projectile motion: constant *v* in *x*, constant *a* in *y*

$$v_x = v_{0x}$$

$$x = x_0 + v_{0x}t$$



$$v_y = v_{0y} - gt$$

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

Free Falling with initial velocity in x

Acceleration

$$a_x = 0 \qquad a_y = -g$$

Motion in x

$$v_x = v_{0x}$$

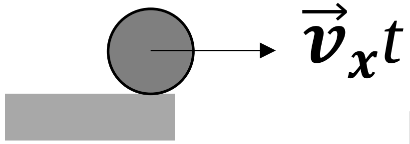
$$x = x_0 + v_{0x}t$$

Motion in y

$$v_y = \underbrace{v_{0y}}_{0} - gt$$

$$y = y_0 + \underbrace{v_{0y}}_{0}t - \frac{1}{2}gt^2$$

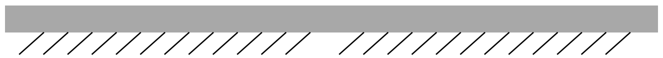
Sphere 1



$v_y = 0$

Gravity

$$\vec{a} = -g\,\vec{j}$$



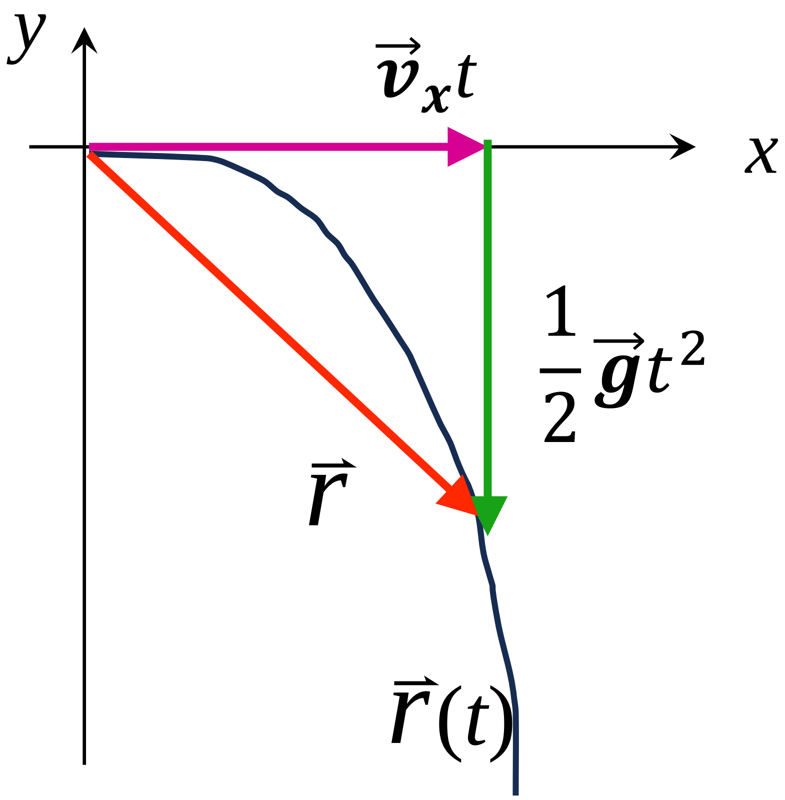
In vector form

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Combine and assume

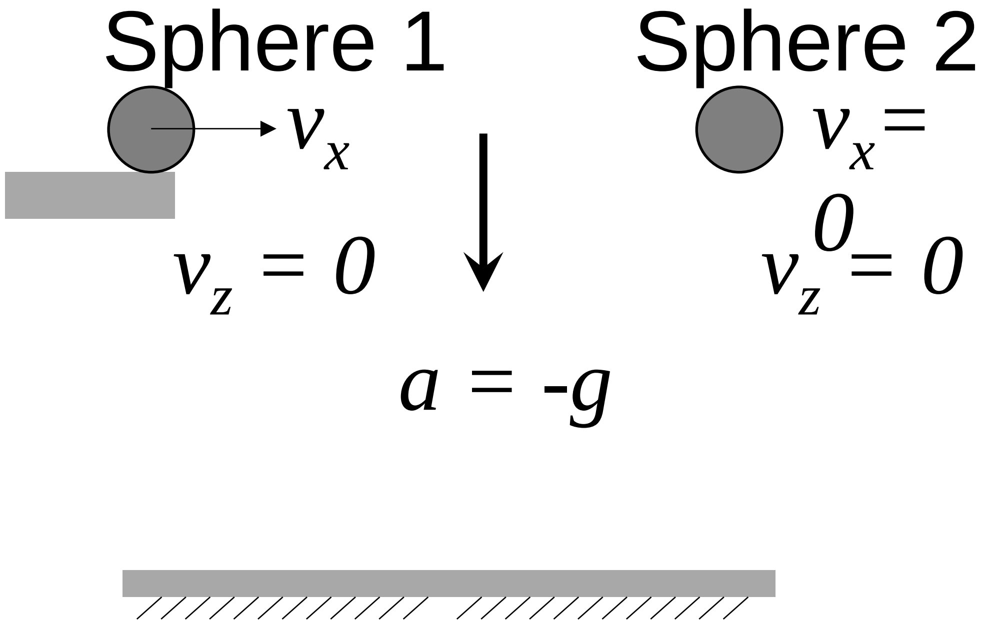
$x_0 = y_0 = 0$

$$\vec{r} = v_x t\,\vec{i} - \frac{1}{2}\vec{g}t^2\,\vec{j}$$



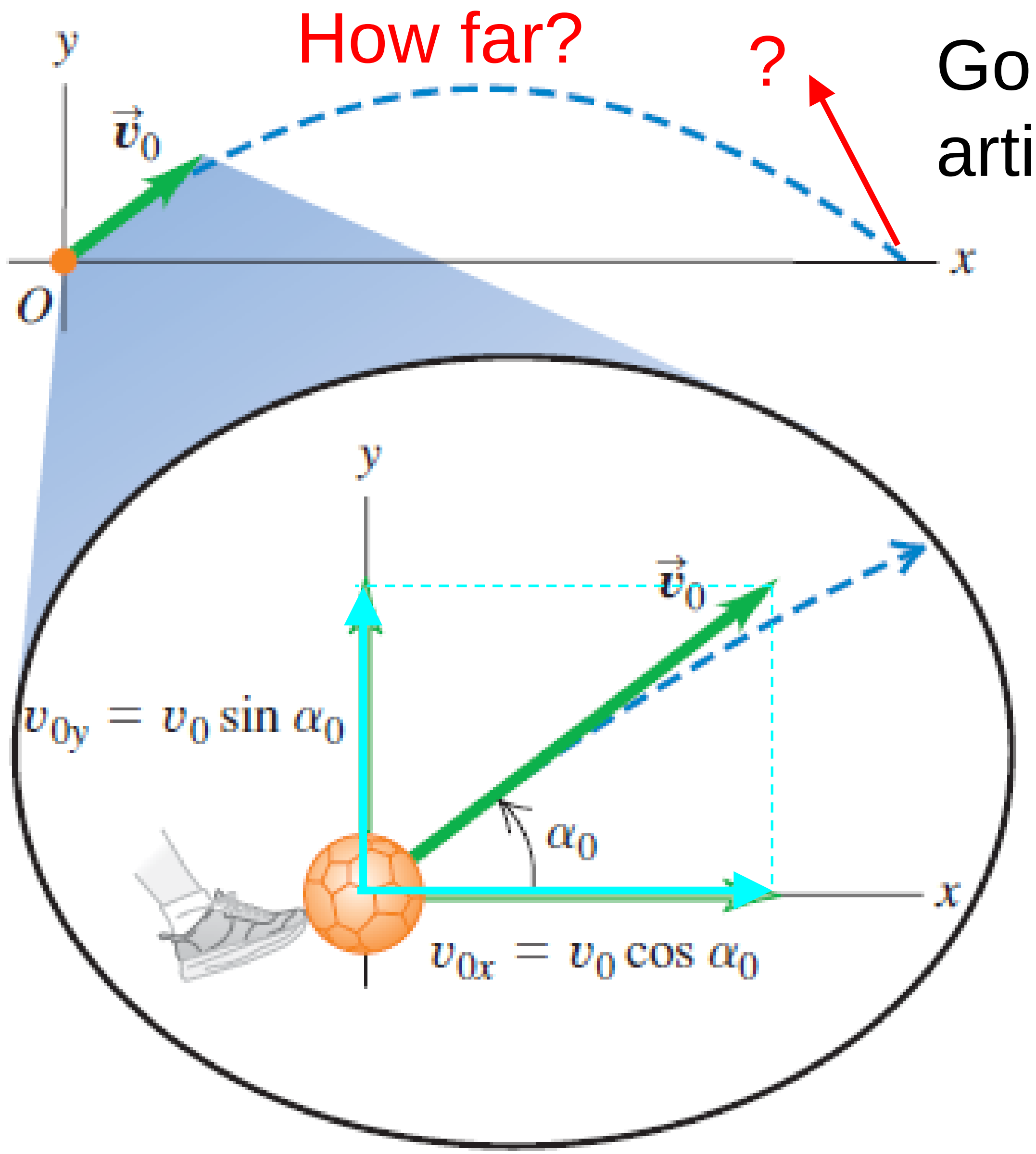
Trajectory

A quick check

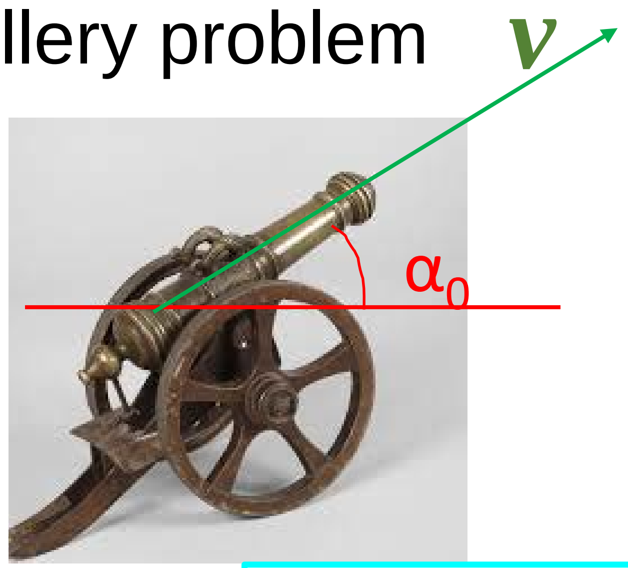


Which of the two spheres will reach the ground first?

Projectile Motion: A Practical Scenario



Going back to this artillery problem



We previously derived:

$$v_x = v_{0x}$$

$$x = x_0 + v_{0x}t$$

$$v_y = v_{0y} - gt$$

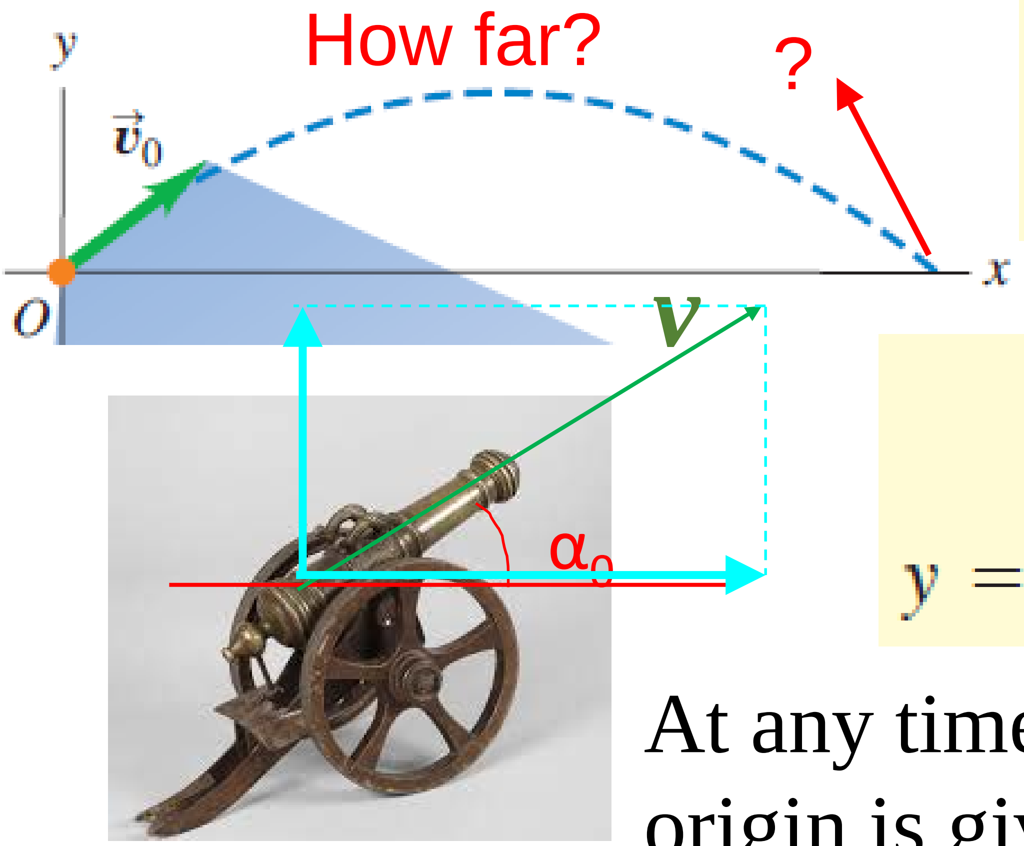
$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

Now: $\boxed{v_{0x} = v_0 \cos\alpha_0 \qquad v_{0y} = v_0 \sin\alpha_0}$

$$v_x = v_0 \cos\alpha_0 \qquad \text{(projectile motion)}$$

$$v_y = v_0 \sin\alpha_0 - gt \qquad \text{(projectile motion)}$$

Projectile Motion: A Practical Scenario



$$v_x = v_0 \cos\alpha_0 \qquad \text{(projectile motion)}$$

$$v_y = v_0 \sin\alpha_0 - gt \qquad \text{(projectile motion)}$$

Integrate and we get displacement

$$x = (v_0 \cos\alpha_0)t \qquad \text{(projectile motion)}$$

$$y = (v_0 \sin\alpha_0)t - \tfrac{1}{2}gt^2 \qquad \text{(projectile motion)}$$

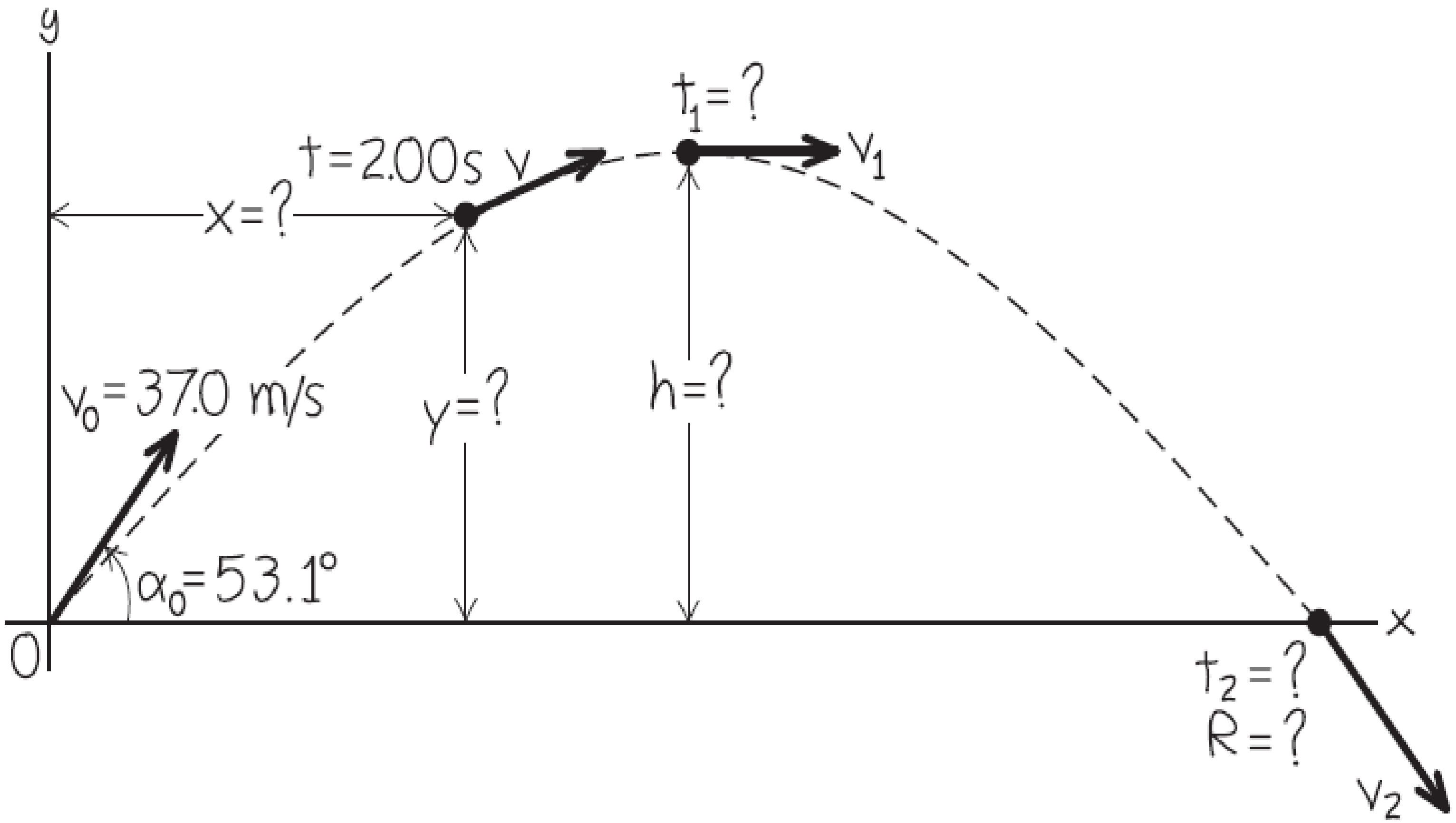
At any time the distance r of the projectile from the origin is given by $r = \sqrt{x^2 + y^2}$

The projectile's speed $v = \sqrt{v_x^2 + v_y^2}$

Direction of the velocity $\tan\alpha = \dfrac{v_y}{v_x}$

Example 3.7 Height and range of a projectile

A batter hits a baseball so that it leaves the bat at speed v_0 = 37.0 m/s at an angle α_0 = 53.1°. (a) Find the position of the ball and its velocity (magnitude and direction) at t = 2.0 s (b) Find the time when the ball reaches the highest point of its flight, and its height *h* at this time. (c) Find the *horizontal range R*—that is, the horizontal distance from the starting point to where the ball hits the ground.



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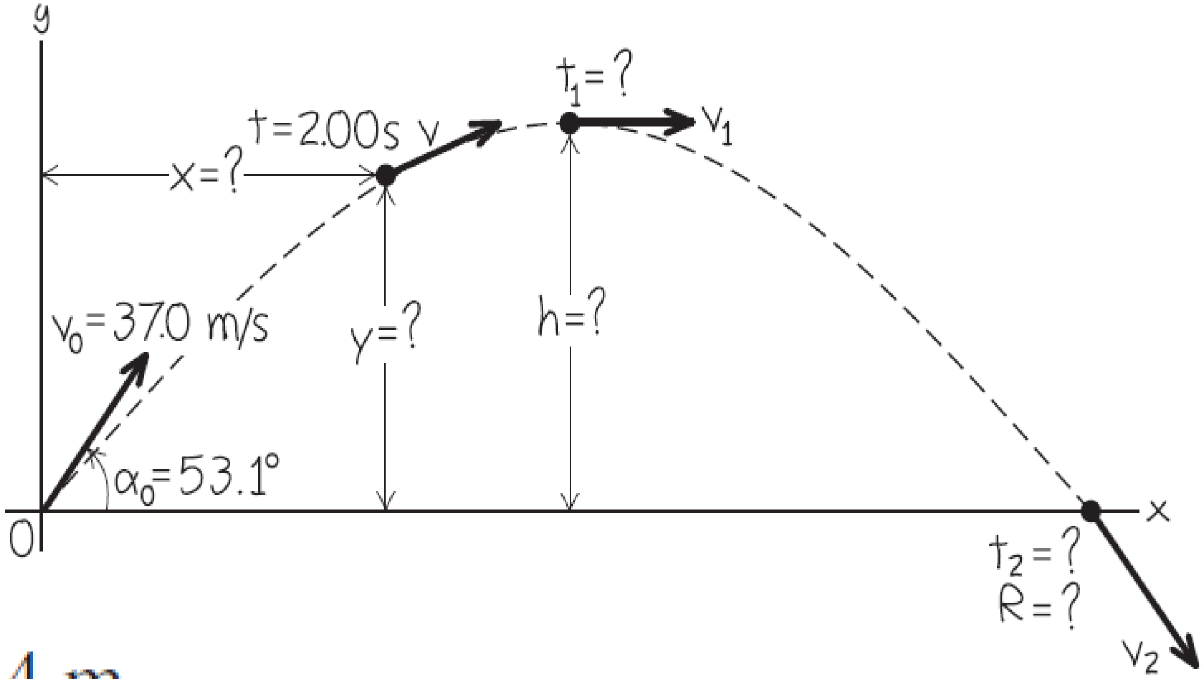

(a) The initial velocity of the ball has components

$$v_{0x} = v_0 \cos \alpha_0 = 22.2 \text{ m/s}$$

$$v_{0y} = v_0 \sin \alpha_0 = 29.6 \text{ m/s}$$

$$x = v_{0x}t = (22.2 \text{ m/s})(2.00 \text{ s}) = 44.4 \text{ m}$$

$$\begin{aligned} y &= v_{0y}t - \tfrac{1}{2}gt^2 \\ &= (29.6 \text{ m/s})(2.00 \text{ s}) - \tfrac{1}{2}(9.80 \text{ m/s}^2)(2.00 \text{ s})^2 \\ &= 39.6 \text{ m} \end{aligned}$$

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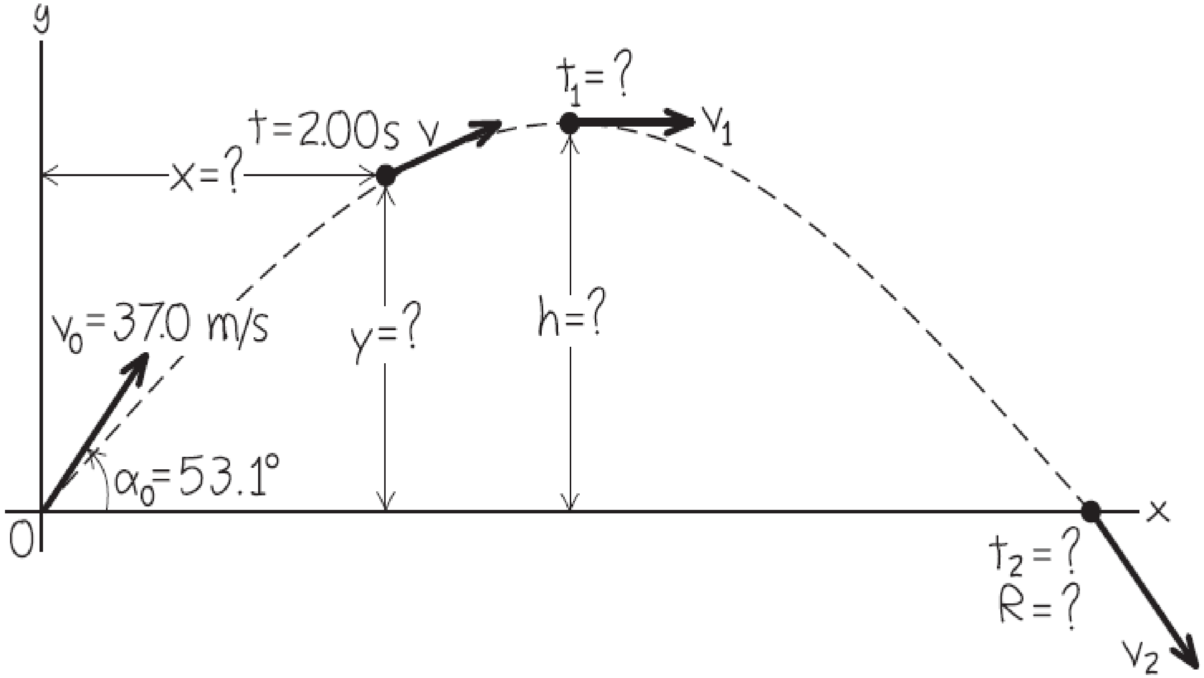

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$$v_{0x} = v_0 \cos\alpha_0 = 22.2 \text{ m/s}$$

$$v_{0y} = v_0 \sin\alpha_0 = 29.6 \text{ m/s}$$

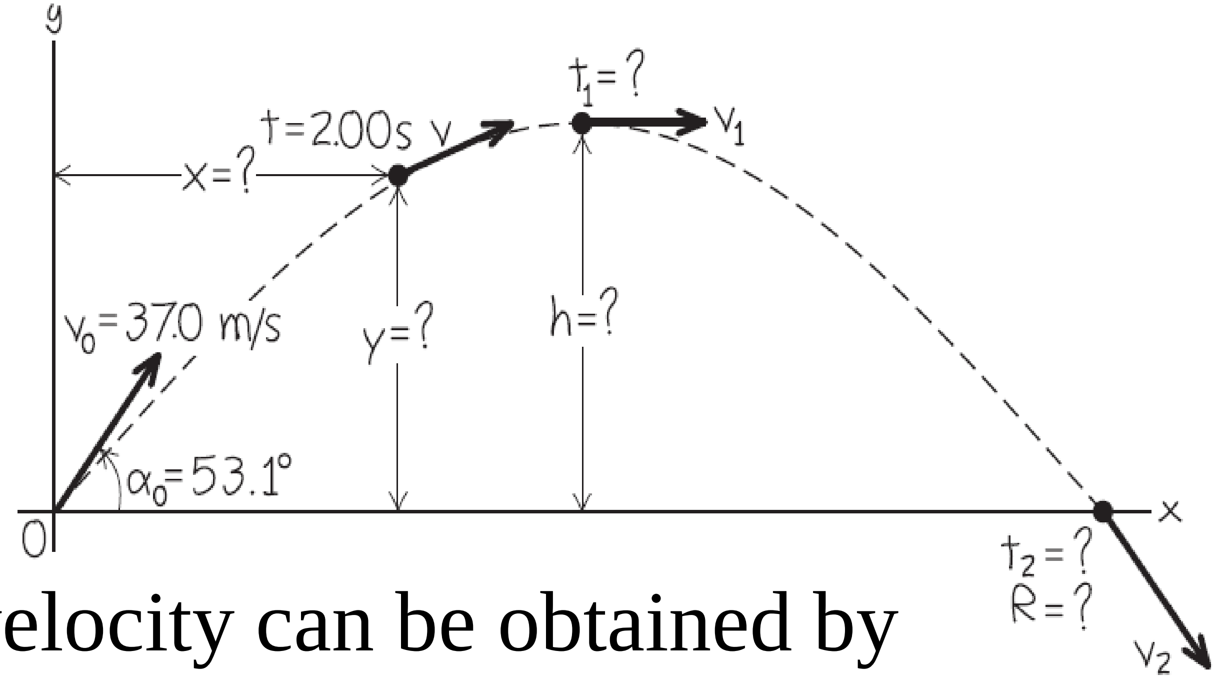
$$v_x = v_{0x} = 22.2 \text{ m/s}$$

$$v_y = v_{0y} - gt = 29.6 \text{ m/s} - (9.80 \text{ m/s}^2)(2.00 \text{ s})$$

$$= 10.0 \text{ m/s}$$

Example 3.7 Height and range of a projectile

A batter hits a baseball so that it leaves the bat at speed $v_0 = 37.0$ m/s at an angle $\alpha_0 = 53.1°$. (a) Find the position of the ball and its velocity (magnitude and direction) at t = 2.0 s



(a) The initial velocity of the ball has components

$$v_{0x} = v_0 \cos\alpha_0 = 22.2 \text{ m/s}$$

$$v_{0y} = v_0 \sin\alpha_0 = 29.6 \text{ m/s}$$

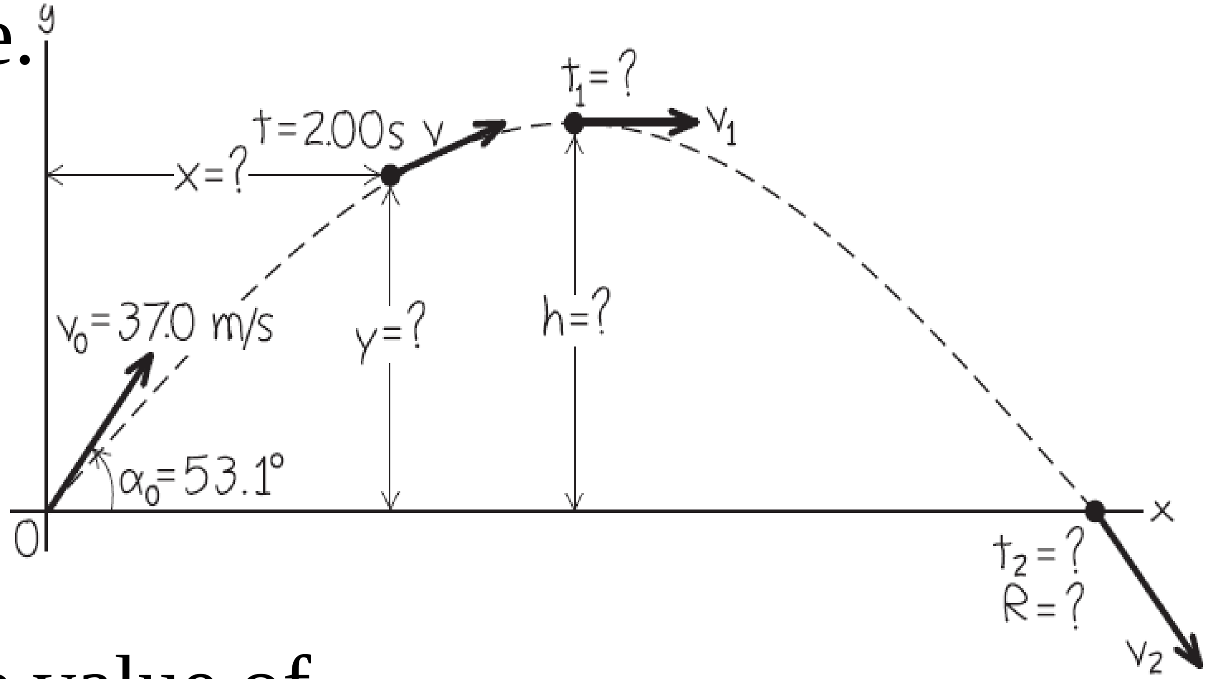
The magnitude and direction of the velocity can be obtained by

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(22.2 \text{ m/s})^2 + (10.0 \text{ m/s})^2} \boxed{= 24.4 \text{ m/s}}$$

$$\tan\alpha = \frac{v_y}{v_x} \Rightarrow \boxed{\alpha = \arctan\left(\frac{10.0 \text{ m/s}}{22.2 \text{ m/s}}\right) = \arctan 0.450 = 24.2°}$$

Example 3.7 Height and range of a projectile

A batter hits a baseball so that it leaves the bat at speed $v_0 = 37.0$ m/s at an angle $\alpha_0 = 53.1°$. (b) Find the time when the ball reaches the highest point of its flight, and its height *h* at this time.



At the highest point, the vertical velocity $v_y = 0$. $v_y = v_{0y} - gt_1 = 0$

$$\boxed{t_1 = \frac{v_{0y}}{g} = \frac{29.6\ \mathrm{m/s}}{9.80\ \mathrm{m/s^2}} = 3.02\ \mathrm{s}}$$

The height *h* at the highest point is the value of *y* at time t_1: plug in the time!

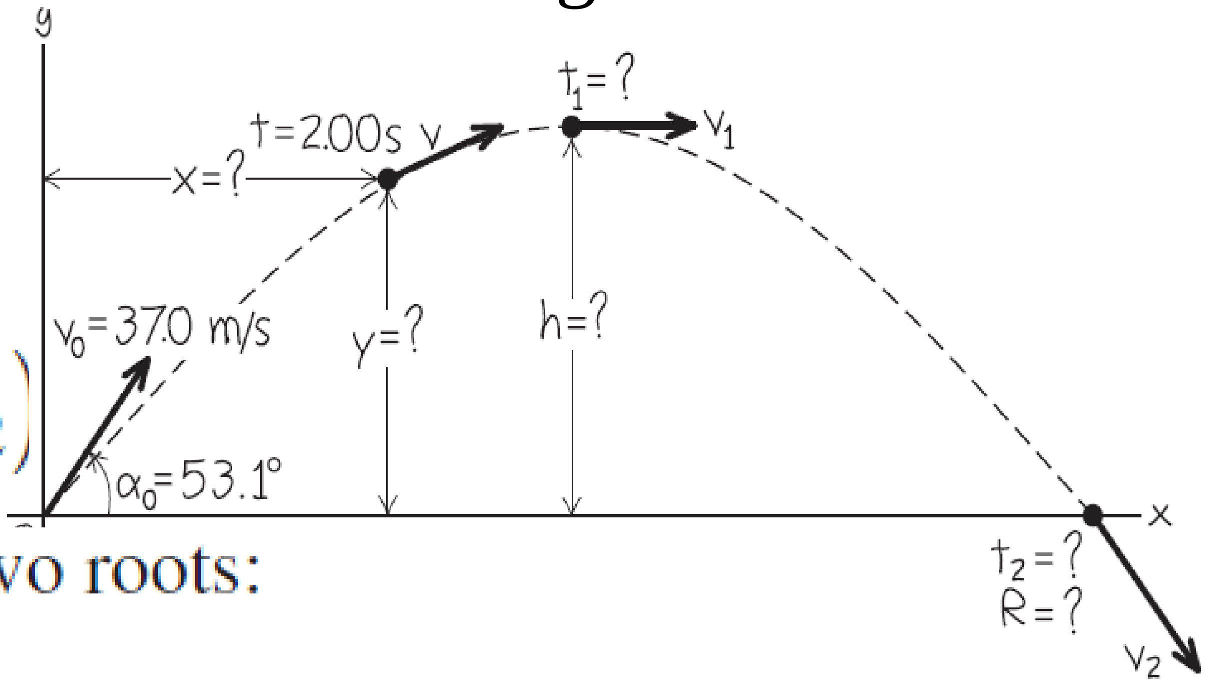
$$h = v_{0y}t_1 - \tfrac{1}{2}gt_1^2$$
$$= (29.6\ \mathrm{m/s})(3.02\ \mathrm{s}) - \tfrac{1}{2}(9.80\ \mathrm{m/s^2})(3.02\ \mathrm{s})^2 = 44.7\ \mathrm{m}$$

Example 3.7 Height and range of a projectile

A batter hits a baseball so that it leaves the bat at speed $v_0 = 37.0$ m/s at an angle $\alpha_0 = 53.1°$. (c) Find the horizontal range *R*—that is, the horizontal distance from the starting point to where the ball hits the ground.

(c) We'll find the horizontal range *R* by finding the time t_2 when $y = 0$ – the ball is at ground level.

$$y = 0 = v_{0y}t_2 - \frac{1}{2}gt_2^2 = t_2\left(v_{0y} - \frac{1}{2}gt_2\right)$$



This is a quadratic equation for t_2. It has two roots:

Leaves ground $\qquad$ *Hits*

$$t_2 = 0 \qquad \text{and} \qquad t_2 = \frac{2v_{0y}}{g} = \frac{2(29.6\ \text{m/s})}{9.80\ \text{m/s}^2} = 6.04\ \text{s}$$

So $\boxed{R = v_{0x}t_2 = (22.2\ \text{m/s})(6.04\ \text{s}) = 134\ \text{m}}$

Example 3.8 Maximum Possible Range

Find the maximum height h and horizontal range R of a projectile launched at v_0 at an initial angle α_0 between 0° and 90°. For a given v_0 **what value of $\boldsymbol{\alpha_0}$** gives maximum height? What value gives maximum horizontal range ***R***?

Recall $v_{0y} = v_0 \sin\alpha_0$ and $v_y = 0$ at the maximum height:

$$t_1 = \frac{v_{0y}}{g} = \frac{v_0 \sin\alpha_0}{g}$$

Plug it in the *h* vs *t* function $h = v_{0y}t_1 - \frac{1}{2}gt_1^2$

$$h = (v_0 \sin\alpha_0)\left(\frac{v_0 \sin\alpha_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\alpha_0}{g}\right)^2 = \frac{v_0^2 \sin^2\alpha_0}{2g}$$

Remember that $\boldsymbol{\alpha_0}$ is an variable we want to tune to maximize *h*!

v_0 and g are constant here, and sin α_0 is 1 at most with **$\boldsymbol{\alpha_0}$ = 90°**

$$\boxed{h = \frac{v_0^2}{2g}}$$

Example 3.8 Maximum Possible Range

Find the maximum height h and horizontal range R of a projectile launched at v_0 at an initial angle α_0 between 0° and 90°. For a given v_0 **what value of $\boldsymbol{\alpha_0}$** gives maximum height? What value gives maximum horizontal range ***R***?

The time t_2 when the projectile hits the ground is, when v_{0y} becomes $-v_{0y}$

$$t_2 = \frac{2v_{0y}}{g} = \frac{2v_0 \sin\alpha_0}{g}$$

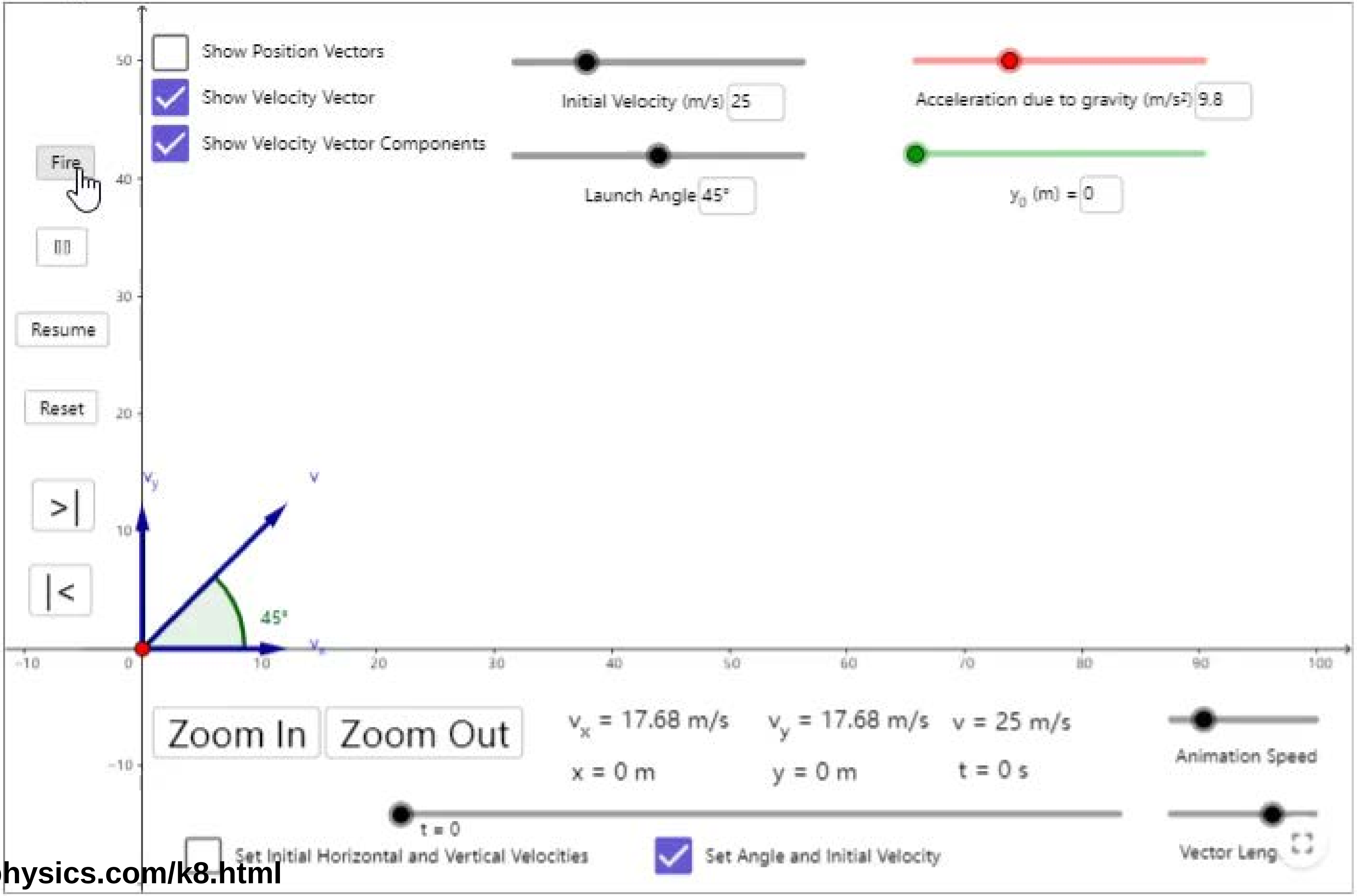
The horizontal range R is the value of x at this time

$$R = (v_0 \cos\alpha_0)t_2 = (v_0 \cos\alpha_0)\frac{2v_0 \sin\alpha_0}{g}$$

$$= \frac{v_0^2 \sin 2\alpha_0}{g}$$

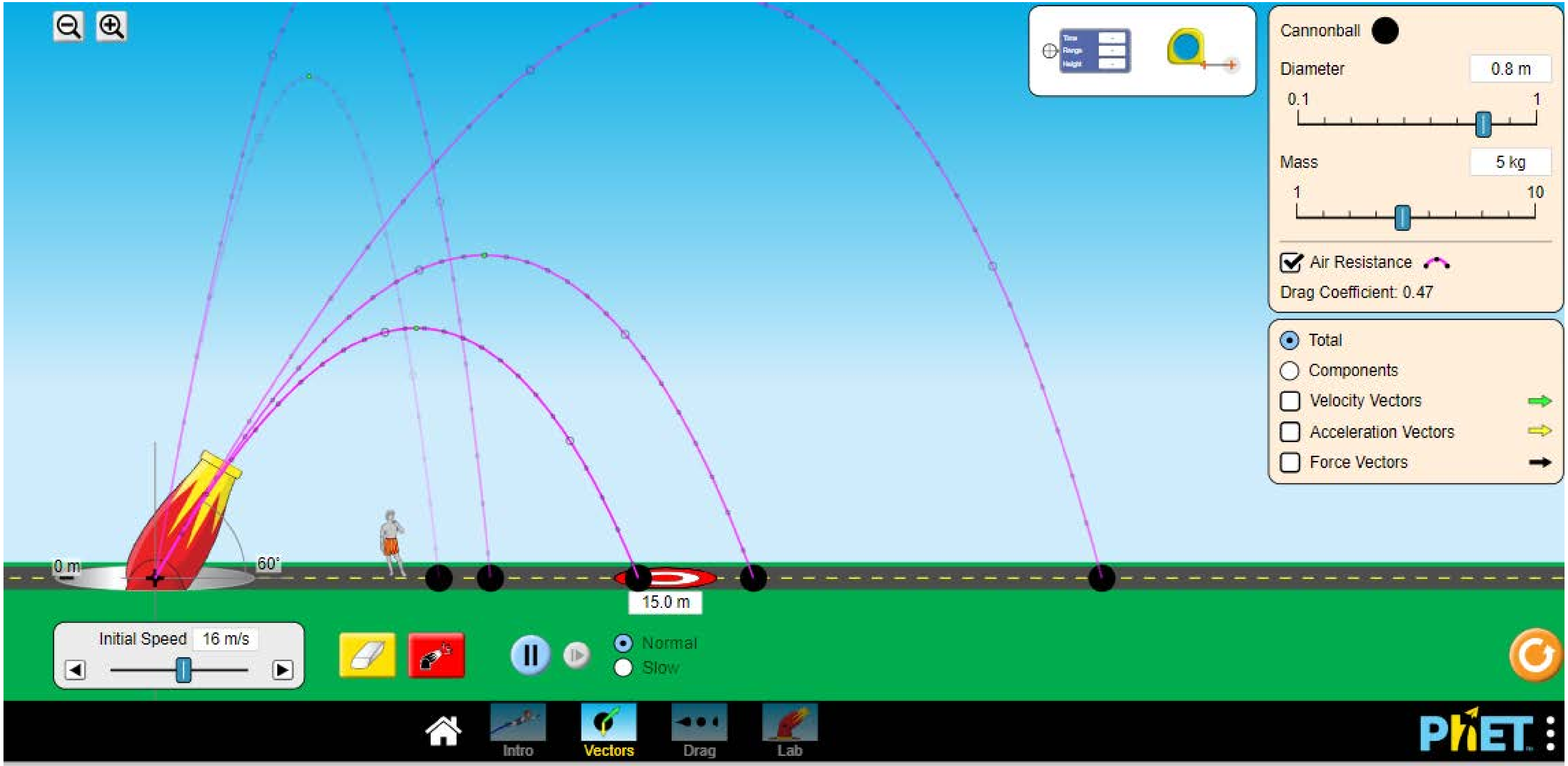
maximizes when $\boldsymbol{\alpha_0}$ **= 45°**

$$R = \frac{v_0^2}{2g}$$



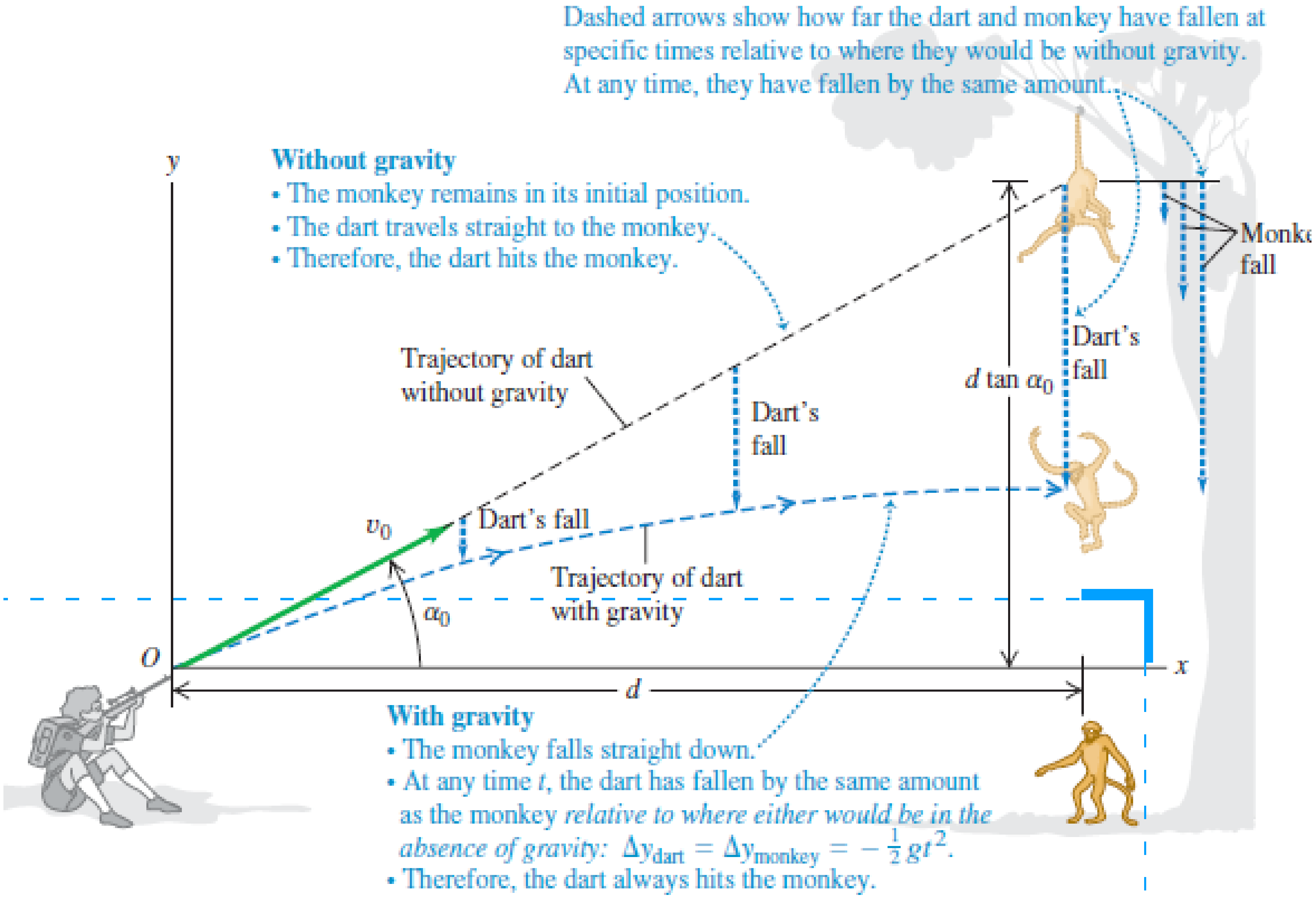
https://ophysics.com/k8.html

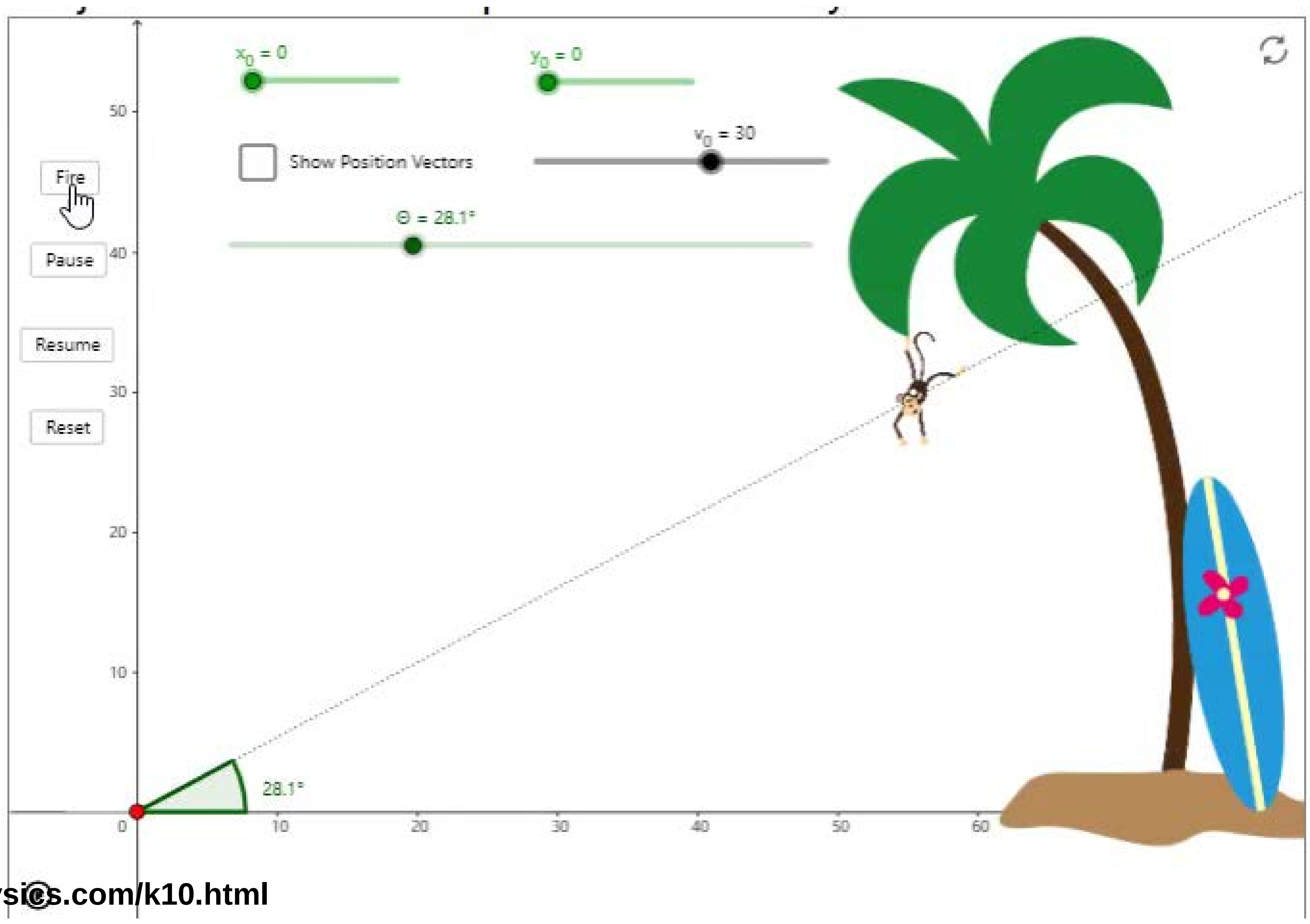
Try it in a Game: Can you get it with 1 attempt?



https://phet.colorado.edu/sims/html/projectile-motion/latest/projectile-motion_all.html

Example 3.10 The zookeeper and the monkey





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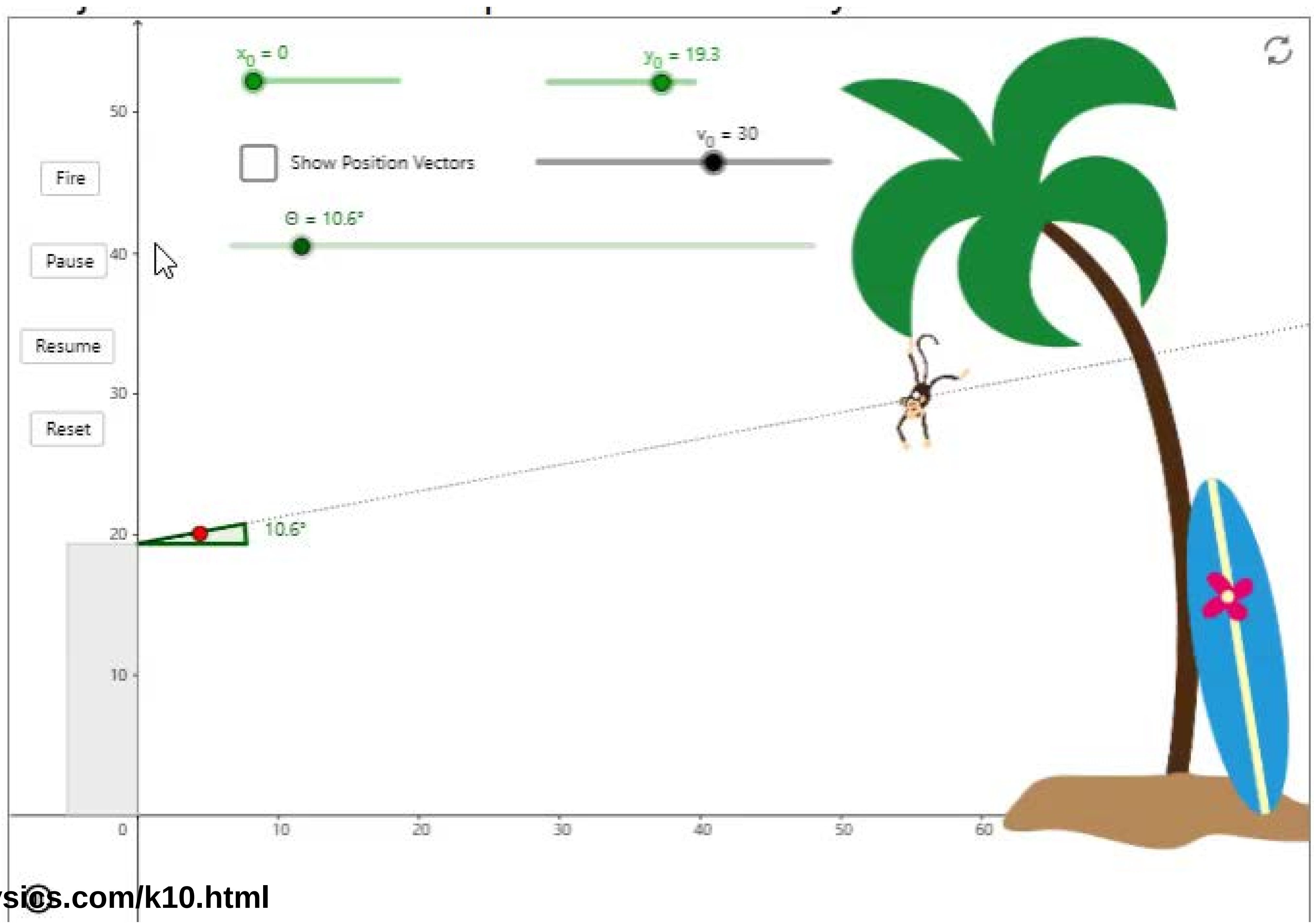
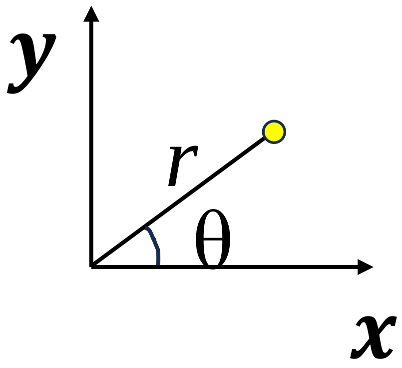


https://ophysics.com/k10.html

Circular Motion: 0. Polar Coordinates

Polar (r, θ)



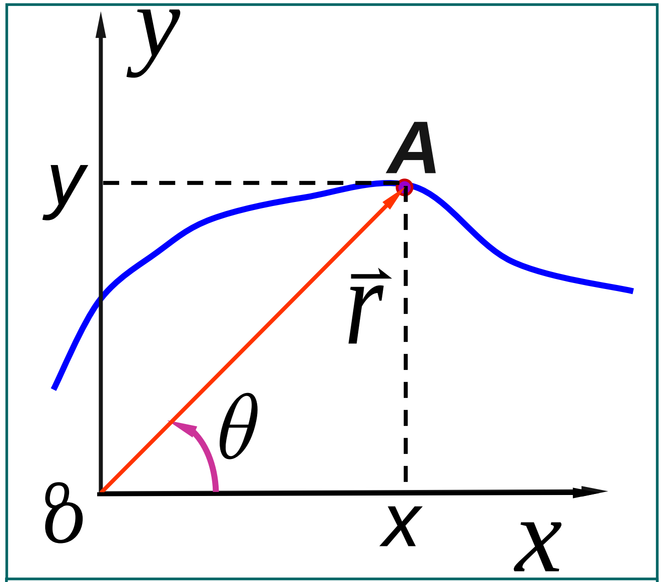
Denote any position A by:

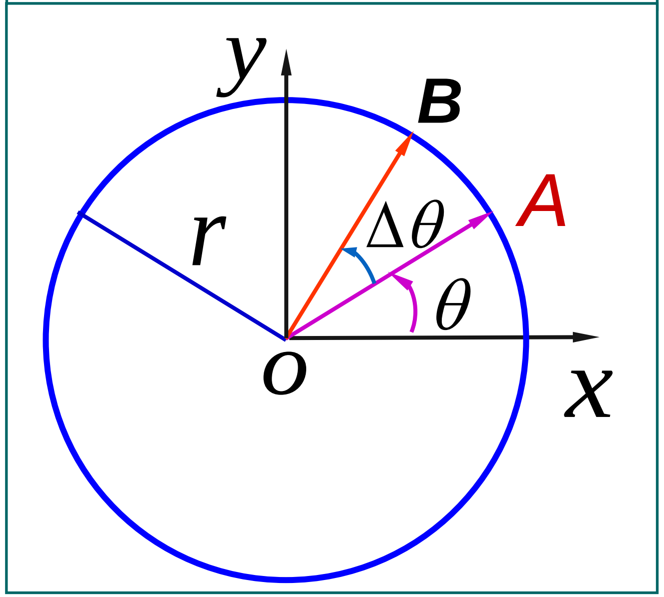
- r: distance to origin
- θ: angle of r from the reference direction (x axis, typically)

Conversion to Cartesian:

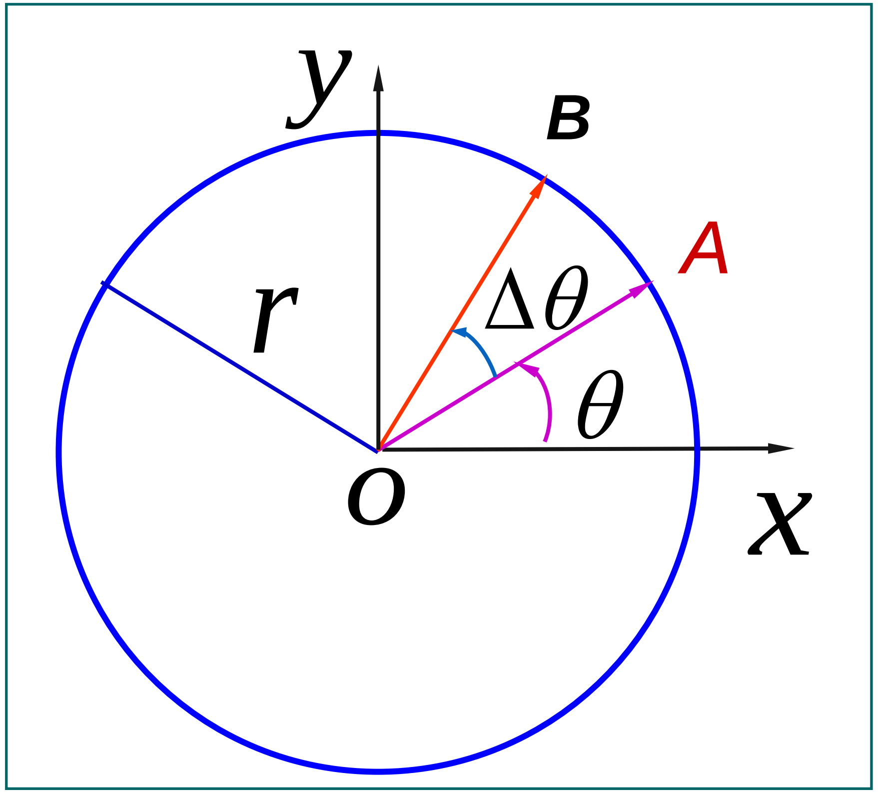
$$\boxed{x = r\cos\theta \quad y = r\sin\theta}$$

Suitable for problems with rotational symmetry





Circular Motion: 1. Define the Concepts



$\boldsymbol{r}$ Called **radial coordinate** or polar angle

$\theta(t)$ Called **angular coordinate** or polar angle

Units: rad (radians)

$\Delta\theta$ Called **angular displacement**

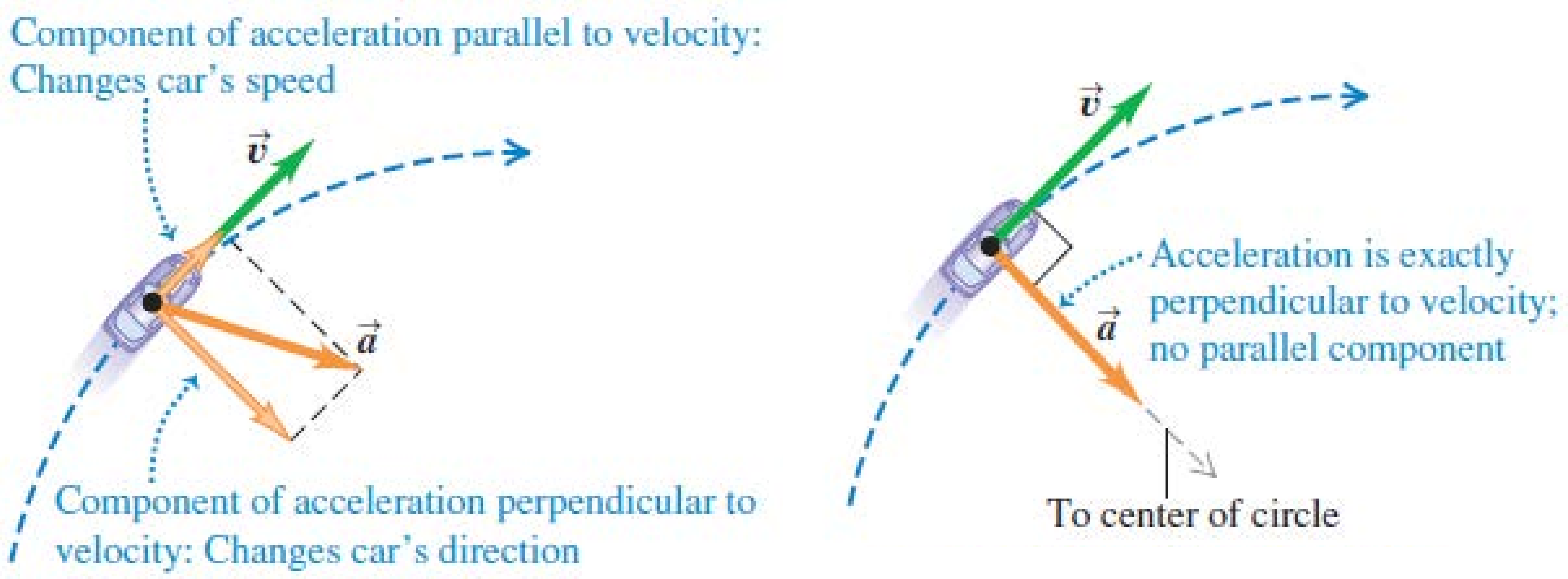
$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$ **Angular velocity**

Units: rad/s

Intuition: on a wheel rotating at **1 round/s**, points on the wheel (except the center) have an angular frequency of $\boldsymbol{\omega} = \mathbf{2\pi}$ rad/s

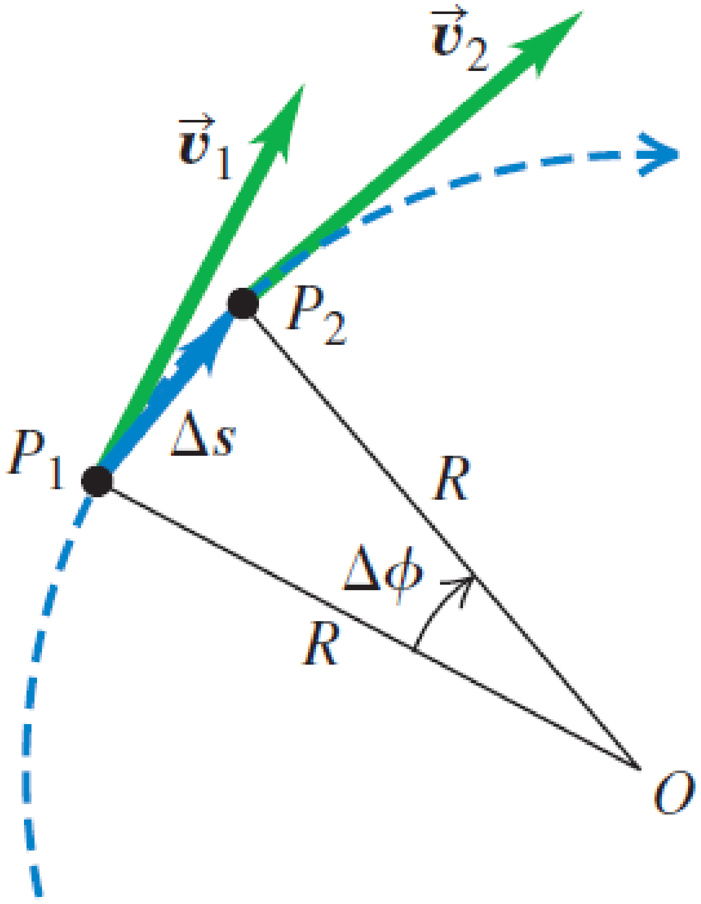
Uniform Circular Motion

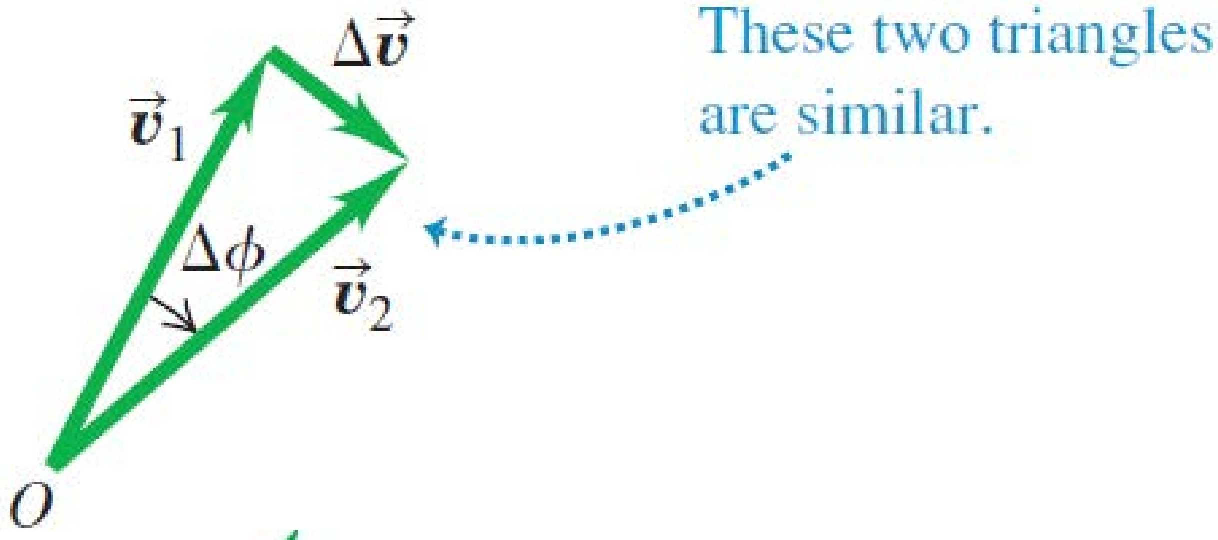
When a particle moves in a circle with *constant speed* (not velocity), the motion is called **uniform circular motion.**

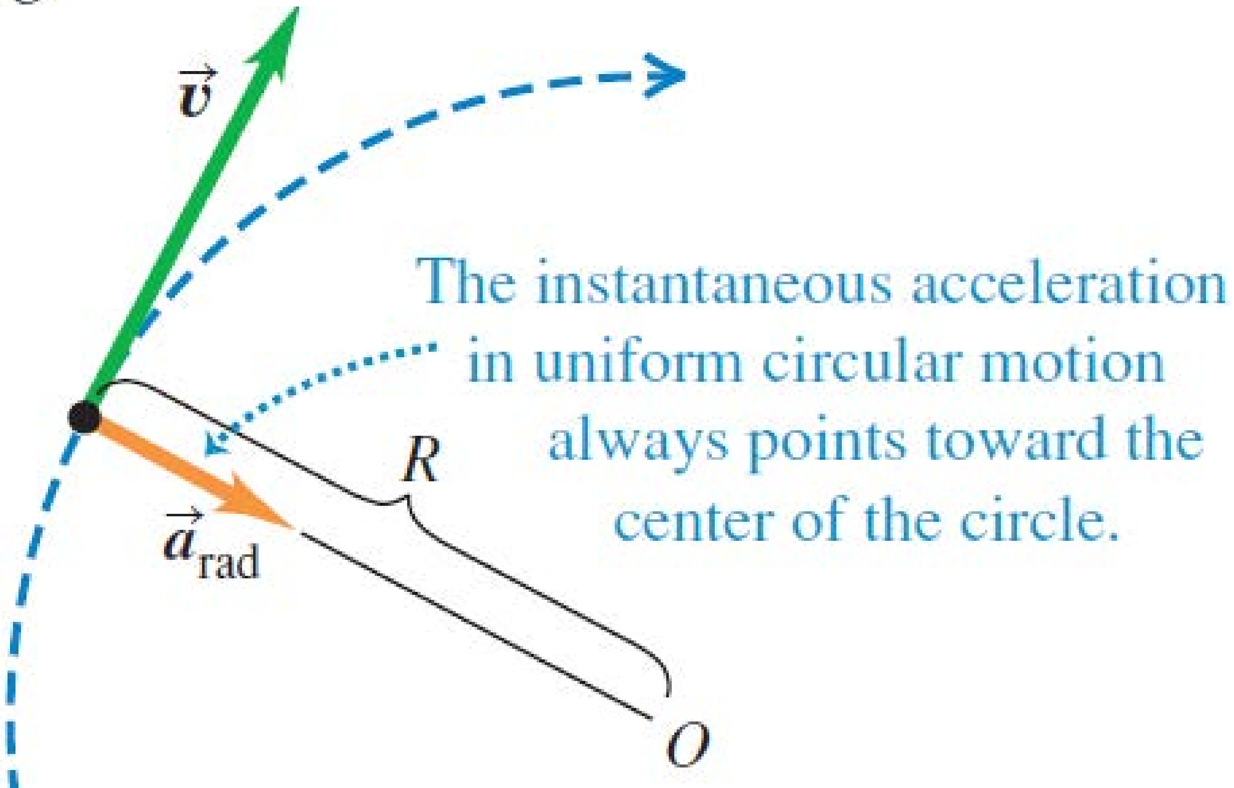


Uniform Circular Motion

A particle moves a distance Δs at constant speed along a circular path.

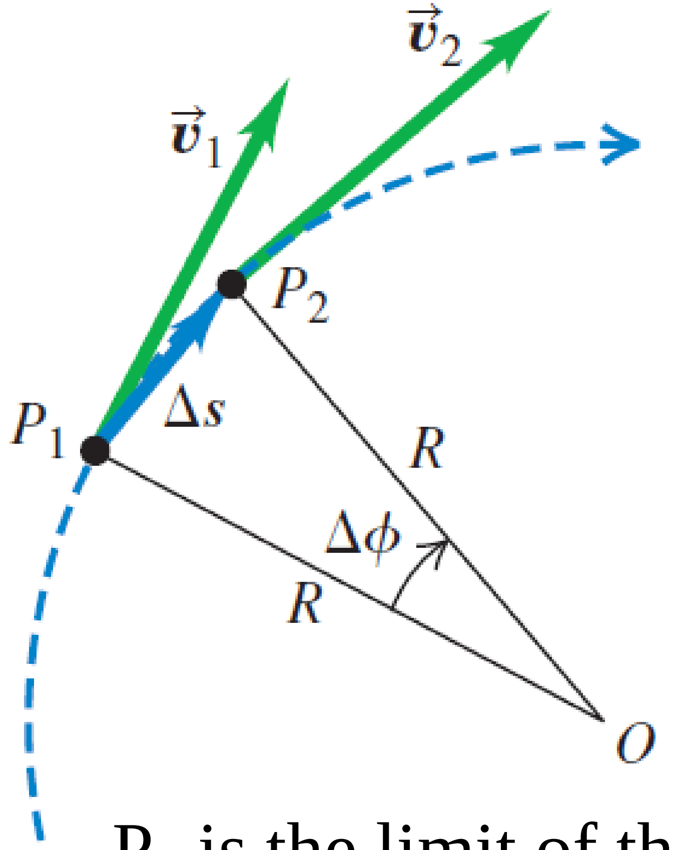






Uniform Circular Motion

The angles labeled $\Delta\phi$ in Figs. 3.28a and 3.28b are the same because $\vec{v}_1$ is perpendicular to the line OP_1 and $\vec{v}_2$ is perpendicular to the line OP_2. Hence the triangles in Figs. 3.28a and 3.28b are *similar*. The ratios of corresponding sides of similar triangles are equal, so



$$\frac{|\Delta\vec{v}|}{v_1} = \frac{\Delta s}{R} \quad \text{or} \quad |\Delta\vec{v}| = \frac{v_1}{R}\Delta s$$

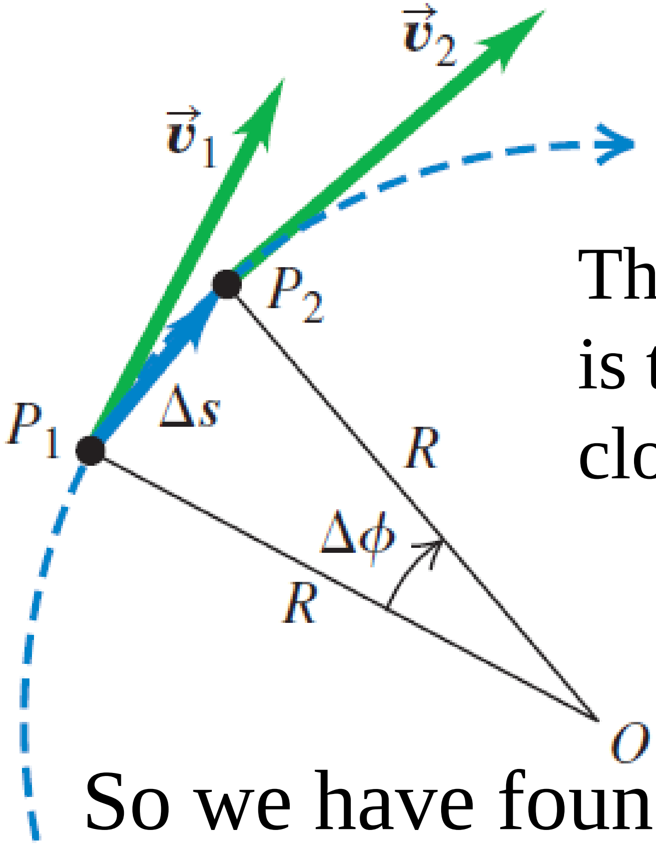
The magnitude a_{av} of the average acceleration during Δt is therefore

$$a_{av} = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v_1}{R}\frac{\Delta s}{\Delta t}$$

The magnitude a of the *instantaneous* acceleration a at point P_1 is the limit of this expression as we take point P_2 closer and closer to point P_1:

Uniform Circular Motion

The magnitude a_{av} of the average acceleration during Δt is therefore

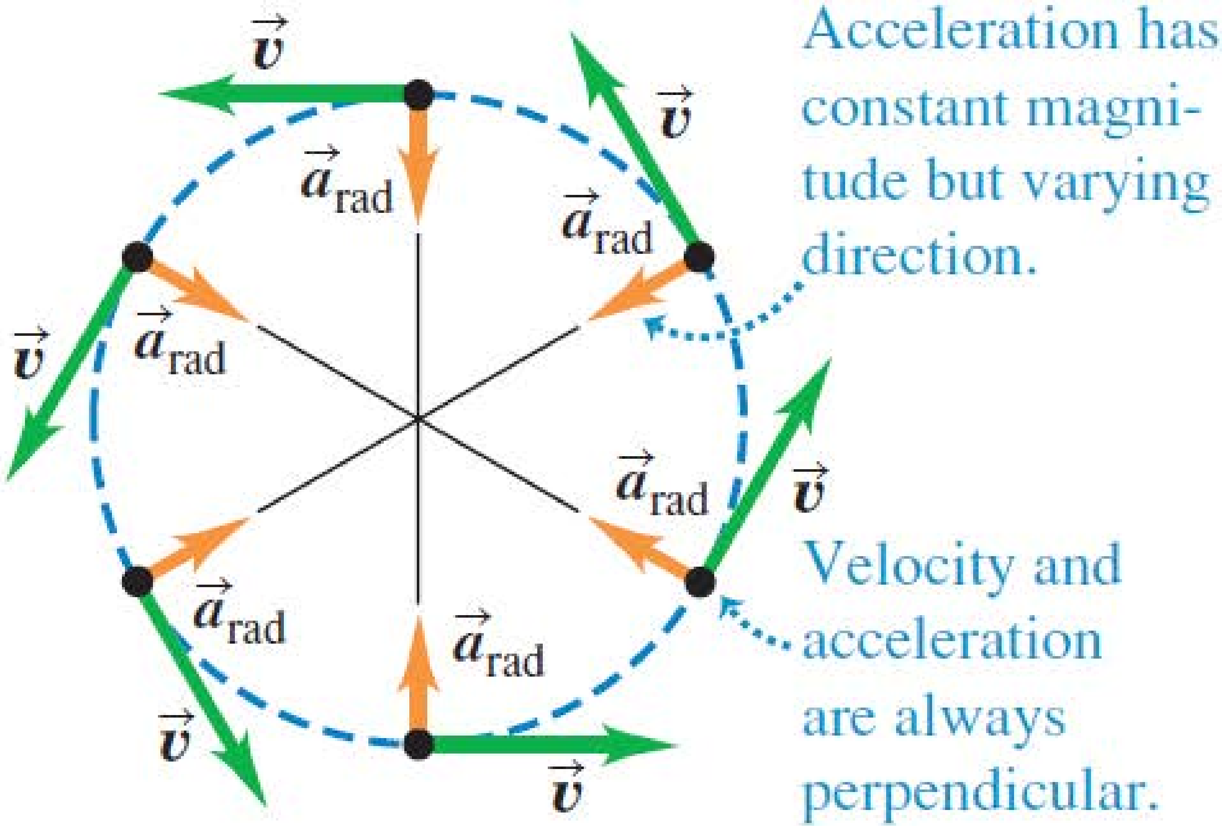


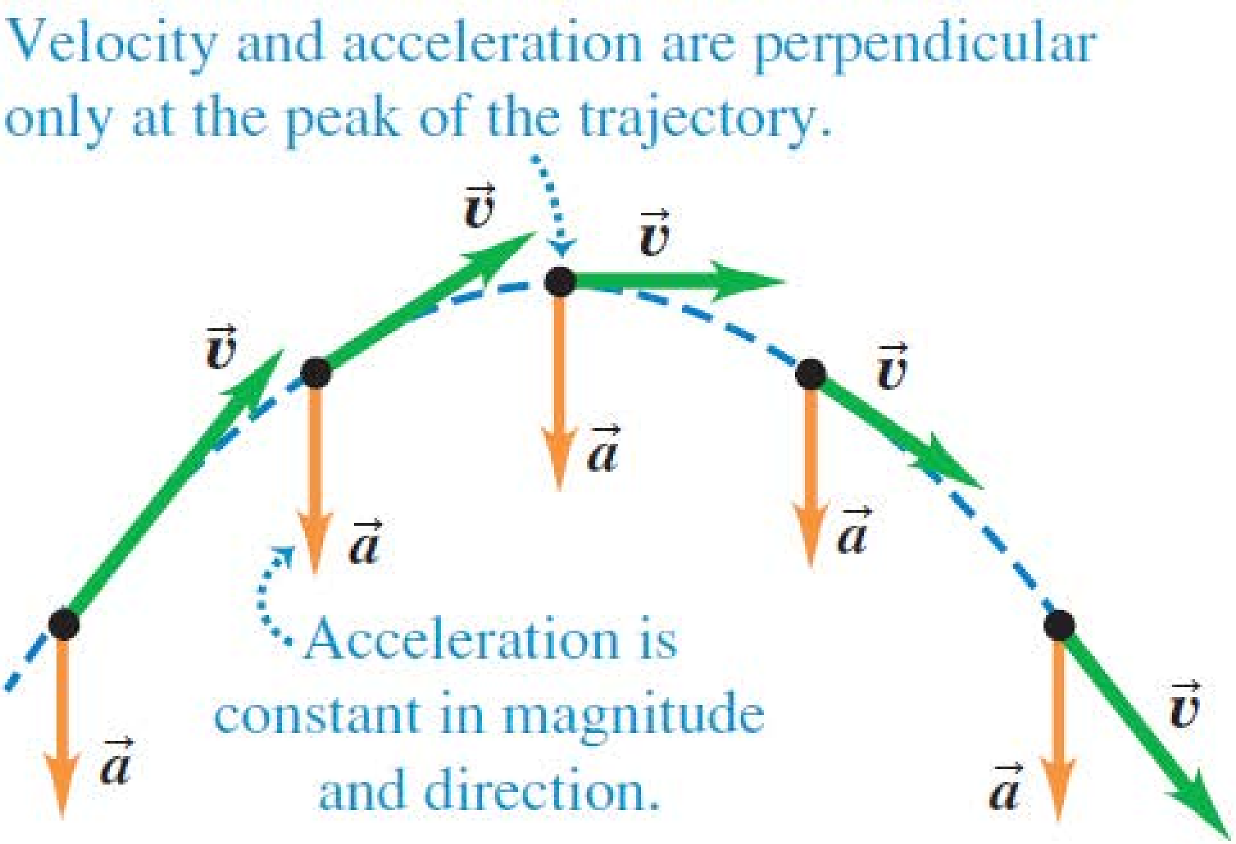
$$a_{av} = \frac{|\Delta \vec{v}|}{\Delta t} = \frac{v_1}{R}\frac{\Delta s}{\Delta t}$$

The magnitude a of the *instantaneous* acceleration a at point P_1 is the limit of this expression as we take point P_2 closer and closer to point P_1:

$$a = \lim_{\Delta t \to 0} \frac{v_1}{R}\frac{\Delta s}{\Delta t} = \frac{v_1}{R}\lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t} \Rightarrow \boxed{a_{rad} = \frac{v^2}{R}}$$

So we have found that *in uniform circular motion, the magnitude of the instantaneous acceleration is equal to the square of the speed divided by the radius R of the circle. Its direction is perpendicular to and inward along the radius – called* **centripetal acceleration**





- **Uniform circular motion vs. projectile motion** Uniform circular motion the ***direction of a*** changes continuously so that it always points toward the center of the circle. (At the top of the circle the acceleration points down; at the bottom of the circle the acceleration points up.)
- Projectile motion: the ***direction*** of *a* remains the same at all times.

Uniform Circular Motion

We can also express the magnitude of the acceleration in uniform circular motion in terms of the **period** *T* of the motion, the time for one revolution (one complete trip around the circle). In a time *T* the particle travels a distance equal to the circumference $2\pi R$ of the circle, so its speed is

$$v = \frac{2\pi R}{T}$$

When we substitute this into $a_{\text{rad}} = \frac{v^2}{R}$, we obtain the alternative expression:

$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2}$$

Example 3.12 Centripetal acceleration on a carnival ride

Passengers on a ride move at constant speed in a horizontal circle of radius 5.0 m, making a complete circle in 4.0 s. What is their acceleration?

Analysis: The speed is constant, so this is uniform circular motion. We are given the radius R = 5.0 m and the period T = 4.0 s

$$\boxed{a_{\mathrm{rad}} = \frac{4\pi^2 R}{T^2}} \Rightarrow a_{\mathrm{rad}} = \frac{4\pi^2(5.0\ \mathrm{m})}{(4.0\ \mathrm{s})^2} = 12\ \mathrm{m/s^2} = 1.3g$$

Second approach

$$v = \frac{2\pi R}{T} = \frac{2\pi(5.0\ \mathrm{m})}{4.0\ \mathrm{s}} = 7.9\ \mathrm{m/s}$$

$$a_{\mathrm{rad}} = \frac{v^2}{R} = \frac{(7.9\ \mathrm{m/s})^2}{5.0\ \mathrm{m}} = 12\ \mathrm{m/s^2}$$

3.29 • A Ferris wheel with radius 14.0 m is turning about a horizontal axis through its center (Fig. E3.29). The linear speed of a passenger on the rim is constant and equal to 7.00 m/s. What are the magnitude and direction of the passenger's acceleration as she passes through (a) the lowest point in her circular motion? (b) The highest point in her circular motion? (c) How much time does it take the Ferris wheel to make one revolution?

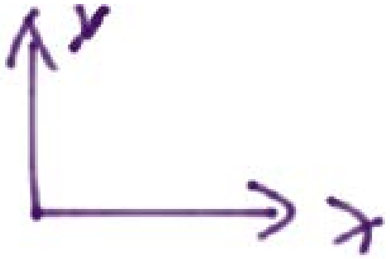
Figure **E3.29**

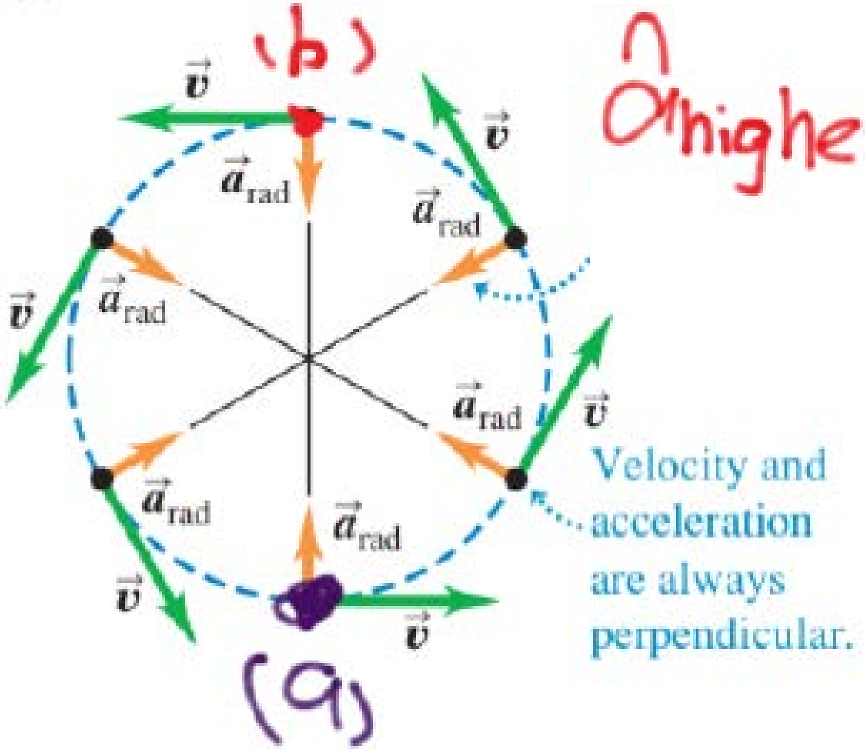
Ferris Wheel

$a_{rad} = \frac{v^2}{R}$ 3.28 for uniform Circular motion

For both (a) and (b).

$$|\vec{a}_{rad}| = \frac{v^2}{R} = \frac{(7.0\,m/s)^2}{14\,m} = 3.5\,m/s^2$$





$\hat{a}_{highest}$ ↓ in $-y$ direction

↑ $\hat{a}_{lowest}$ in $+y$ direction

Ferris Wheel

$a_{rad} = \frac{v^2}{R}$ 3.28 for uniform circular motion

For both (a) and (b).

$$|\vec{a}_{rad}| = \frac{v^2}{R} = \frac{(7.0\,m/s)^2}{14\,m} = 3.5\,m/s^2$$

$t = \frac{S}{v}$ ← distance to travel in one round

$$S = 2\pi R$$

$$t = \frac{2 \cdot 3.14159 \cdot 14\,m}{7\,m/s} \simeq 12.57\,s$$

Nonuniform Circular Motion

If the speed varies, we call the motion **nonuniform circular motion.**

- $a_{rad} = v^2/R$ still gives the *radial* component of acceleration which is always ***perpendicular*** to the instantaneous velocity and directed toward the center of the circle
- a_{tan}: a component of acceleration that is ***parallel*** to the instantaneous velocity

$$a_{rad} = \frac{v^2}{R} \quad \text{and} \quad a_{tan} = \frac{d|\vec{v}|}{dt} \quad \text{(nonuniform circular motion)}$$

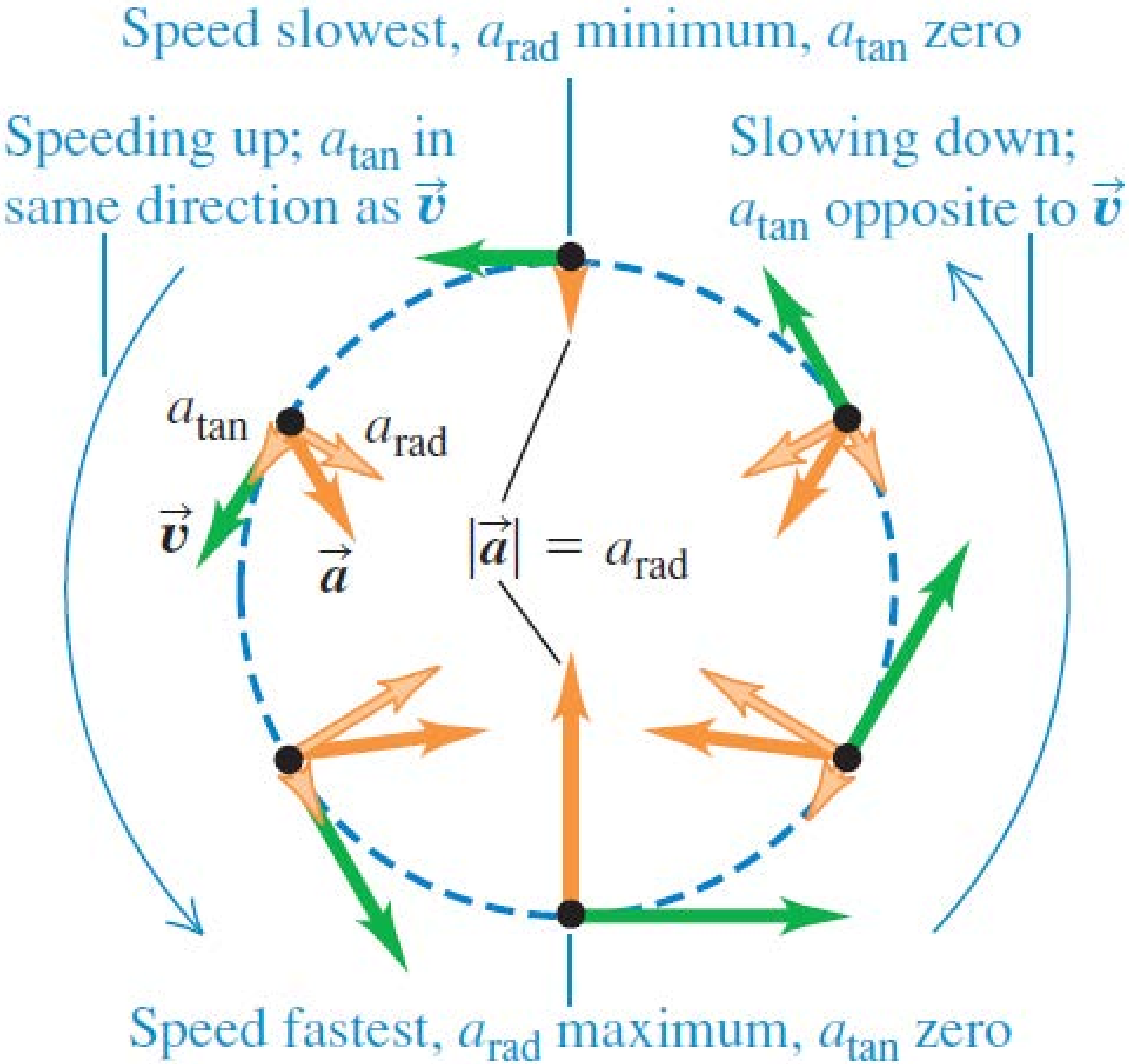
Uniform vs. nonuniform circular motion: These two are NOT the same:

a_{tan} the rate of change of speed; 0 whenever a particle moves with constant ***speed*** (e.g. *uniform* circular motion)

$$\frac{d|\vec{v}|}{dt} \quad \text{vs} \quad \left|\frac{d\vec{v}}{dt}\right|$$

magnitude of the vector acceleration; it is zero only when the particle's acceleration *vector* is zero—that is, when the particle moves in a straight line with constant speed.

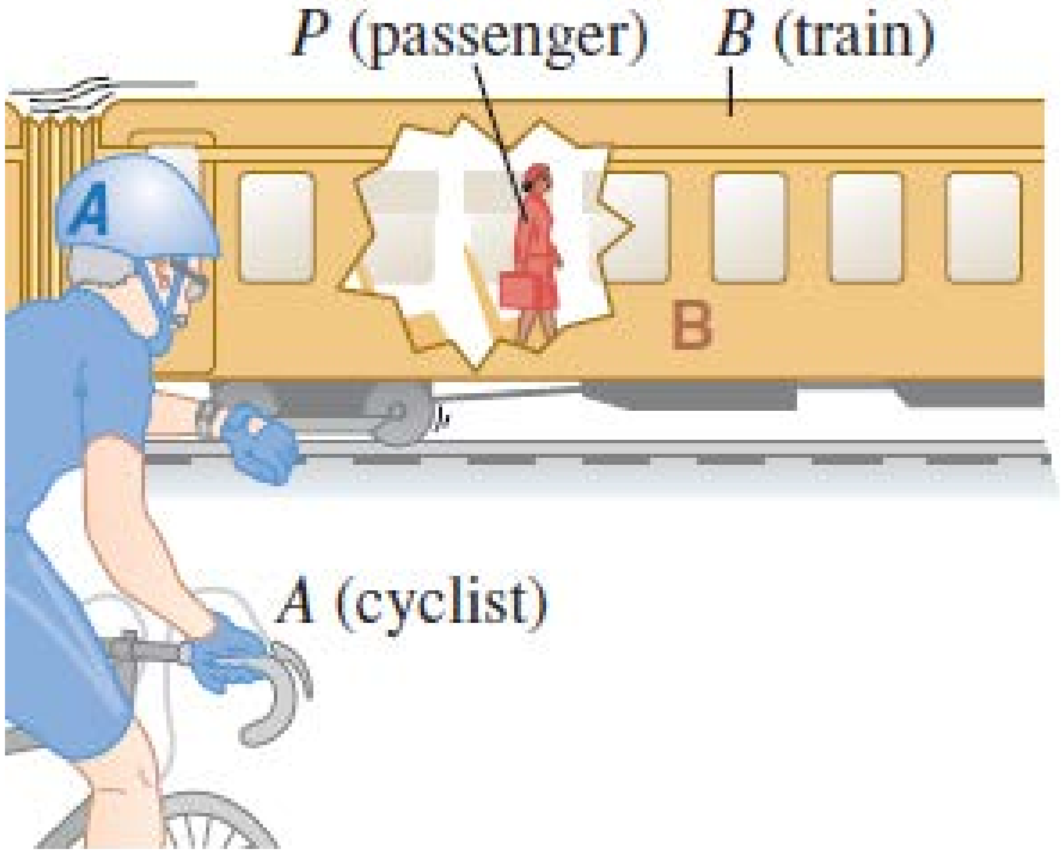
Nonuniform Circular Motion

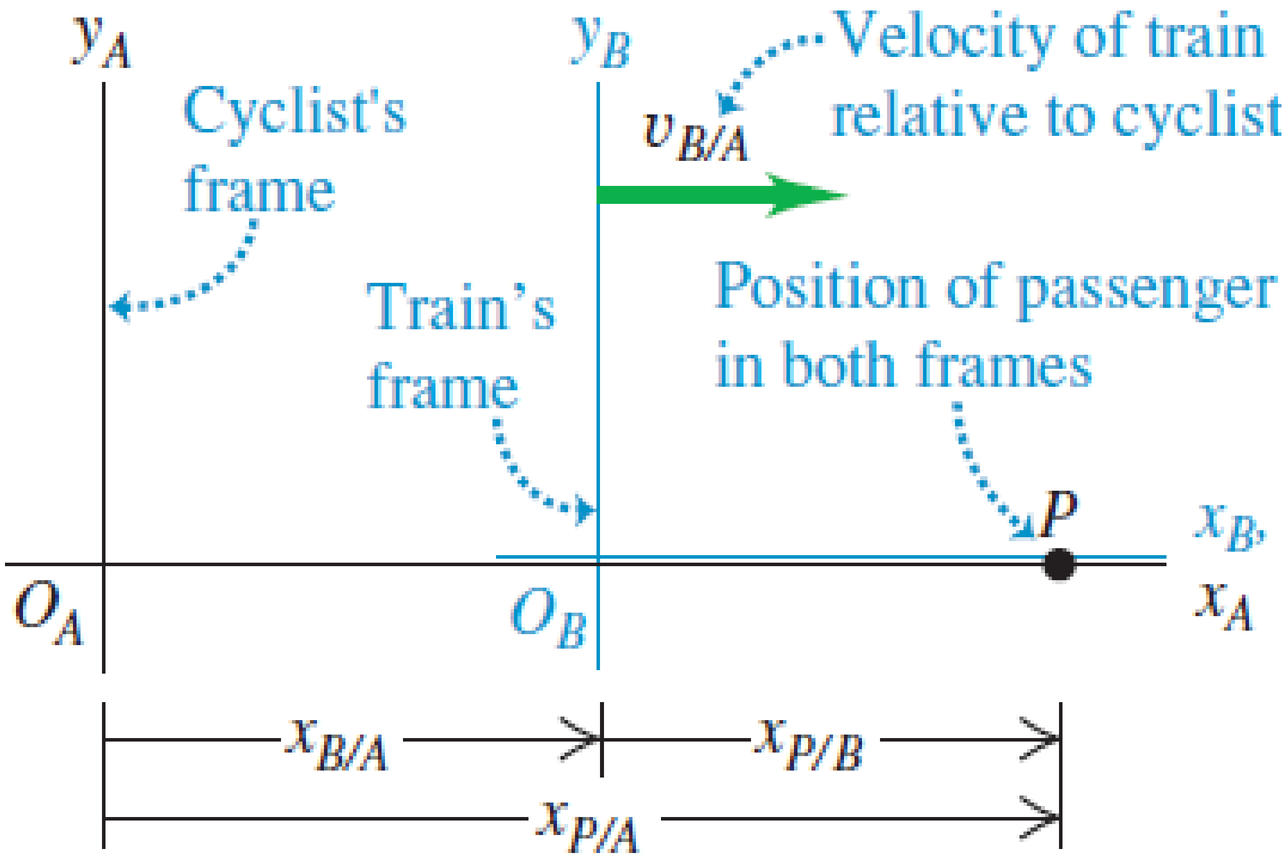


$$\left|d\vec{v}/dt\right| = \sqrt{a_{rad}{}^2 + a_{tan}{}^2}$$

Relative Motion

The velocity seen by a particular observer is called the velocity *relative* to that observer, or simply **relative velocity.**



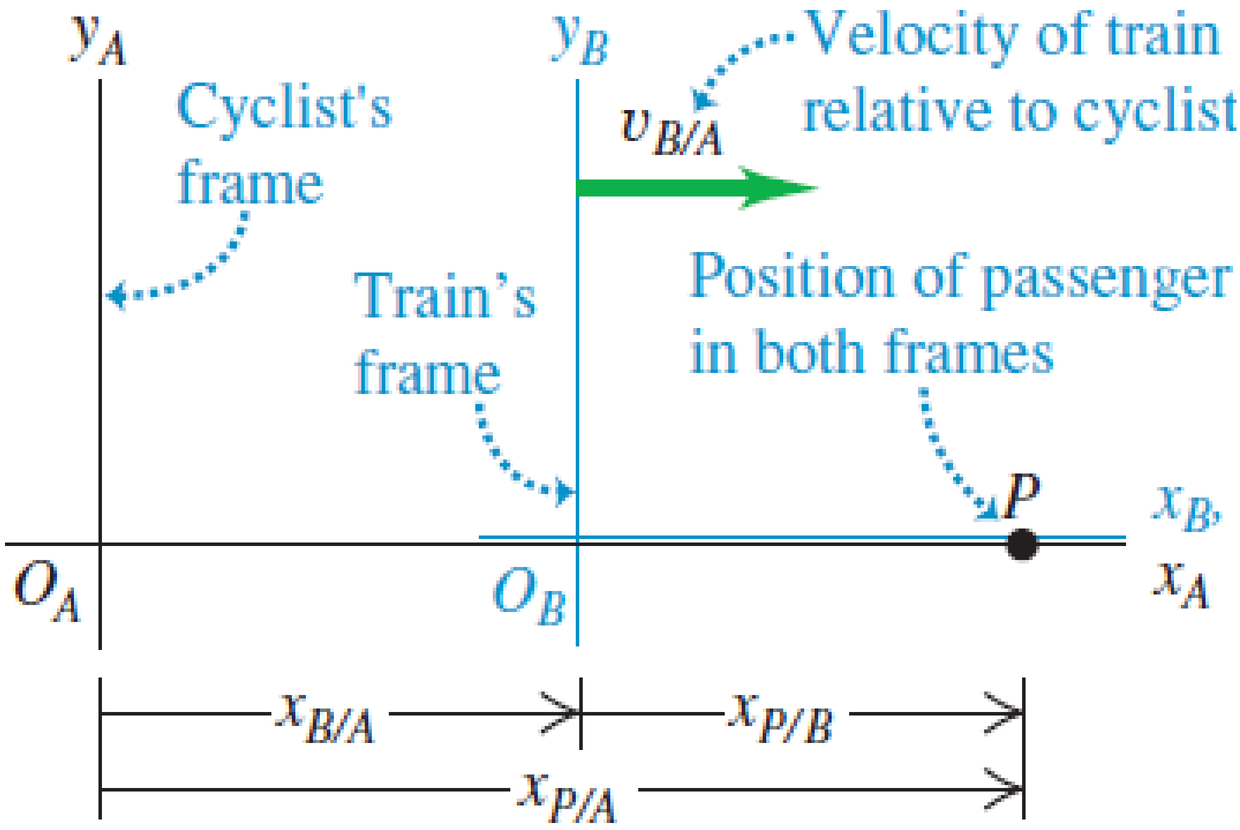


A passenger walks with a velocity of 1.0 m/s along the aisle of a train that is moving with a velocity of 3.0 m/s. What is the passenger's velocity?

Relative Motion

Each observer, equipped with a meter stick and a stopwatch, forms what we call a **frame of reference:** a coordinate system + a time scale.

- Symbol *A* for the cyclist's frame of reference (at rest with respect to the ground)
- Symbol *B* for the frame of reference of the moving train.
- $x_{P/A}$: straight-line motion the position of a point *P* relative to frame *A* (the position of *P* with respect to *A*)
- $x_{P/B}$: position of *P* relative to frame *B*.
- $x_{B/A}$: position of the origin of *B* with respect to the origin



$$\boxed{x_{P/A} = x_{P/B} + x_{B/A}}$$

Relative Motion

Taking the derivative $x_{P/A} = x_{P/B} + x_{B/A}$

$$\frac{dx_{P/A}}{dt} = \frac{dx_{P/B}}{dt} + \frac{dx_{B/A}}{dt} \Rightarrow \boxed{v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x}}$$

$$v_{P/B\text{-}x} = +1.0\ \text{m/s} \qquad v_{B/A\text{-}x} = +3.0\ \text{m/s}$$

$$v_{P/A\text{-}x} = +1.0\ \text{m/s} + 3.0\ \text{m/s} = +4.0\ \text{m/s}$$

When the passenger looks out the window, the stationary cyclist on the ground appears to her to be moving backward; we can call the cyclist's velocity relative to her $v_{A/P\text{-}x}$. Clearly, this is just the negative of the *passenger's* velocity relative to the *cyclist*, $v_{P/A\text{-}x}$

In general, if *A* and *B* are any two points or frames: $v_{A/B\text{-}x} = -v_{B/A\text{-}x}$

Relative Velocity in Two or Three Dimensions

$$\boxed{x_{P/A} = x_{P/B} + x_{B/A}}$$

$$\boxed{\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A}}$$

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A} \qquad \text{(relative velocity in space)}$$

known as the *Galilean velocity transformation*

It relates the velocity of a body *P* with respect to frame *A* and its velocity with respect to frame *B*

Example 3.14 Flying in a crosswind

An airplane's compass indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at 240 *km*/*h*. If there is a 100-*km*/*h* wind from west to east, what is the velocity of the airplane relative to the earth?

Two dimensions (northward and eastward), a relative velocity problem using vectors. Given the magnitude and direction of the velocity of the plane (*P*) relative to the air (*A*). We are also given the magnitude and direction of the wind velocity, which is the velocity of the air A with respect to the earth (*E*):

$$\vec{v}_{P/A} = 240 \text{ km/h} \qquad \text{due north}$$
$$\vec{v}_{A/E} = 100 \text{ km/h} \qquad \text{due east}$$

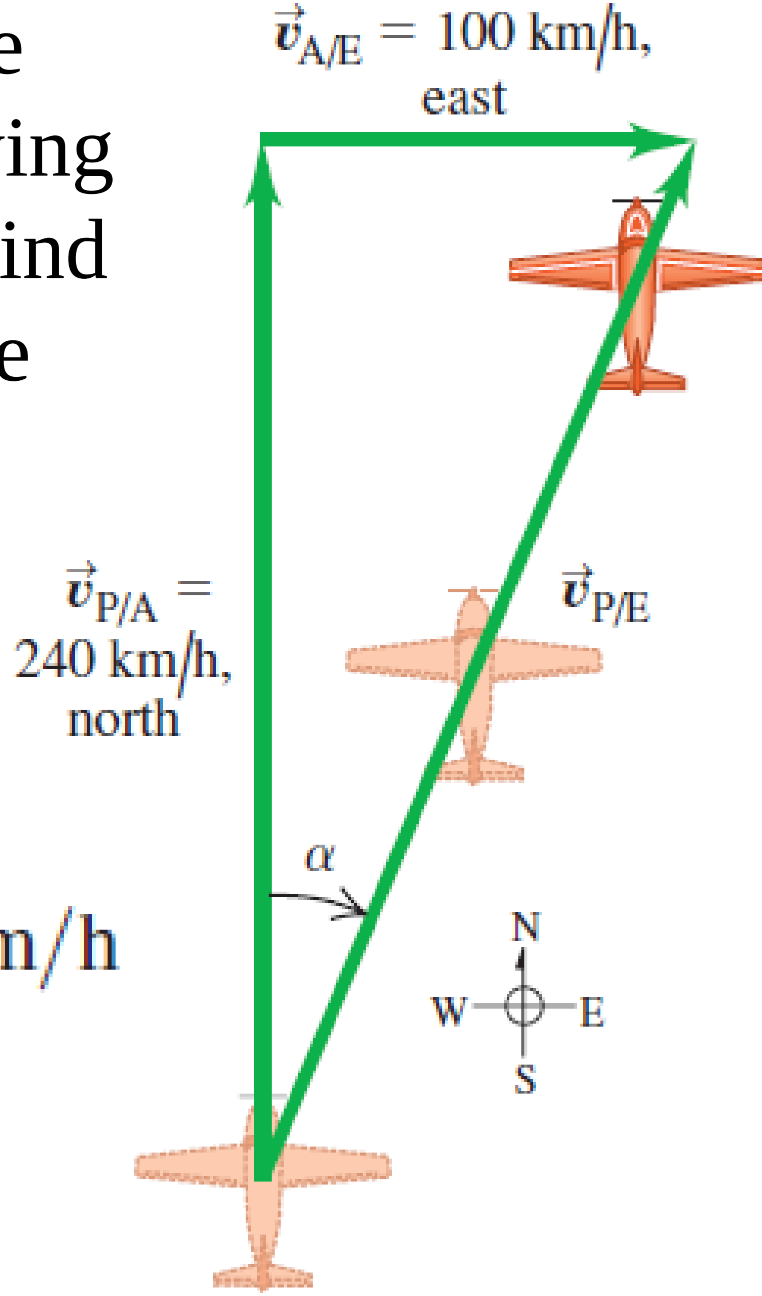
Example 3.14 Flying in a crosswind

An airplane's compass indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at 240 *km*/*h*. If there is a 100-*km*/*h* wind from west to east, what is the velocity of the airplane relative to the earth?

$$\vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E}$$

$$v_{P/E} = \sqrt{(240\ \text{km/h})^2 + (100\ \text{km/h})^2} = 260\ \text{km/h}$$

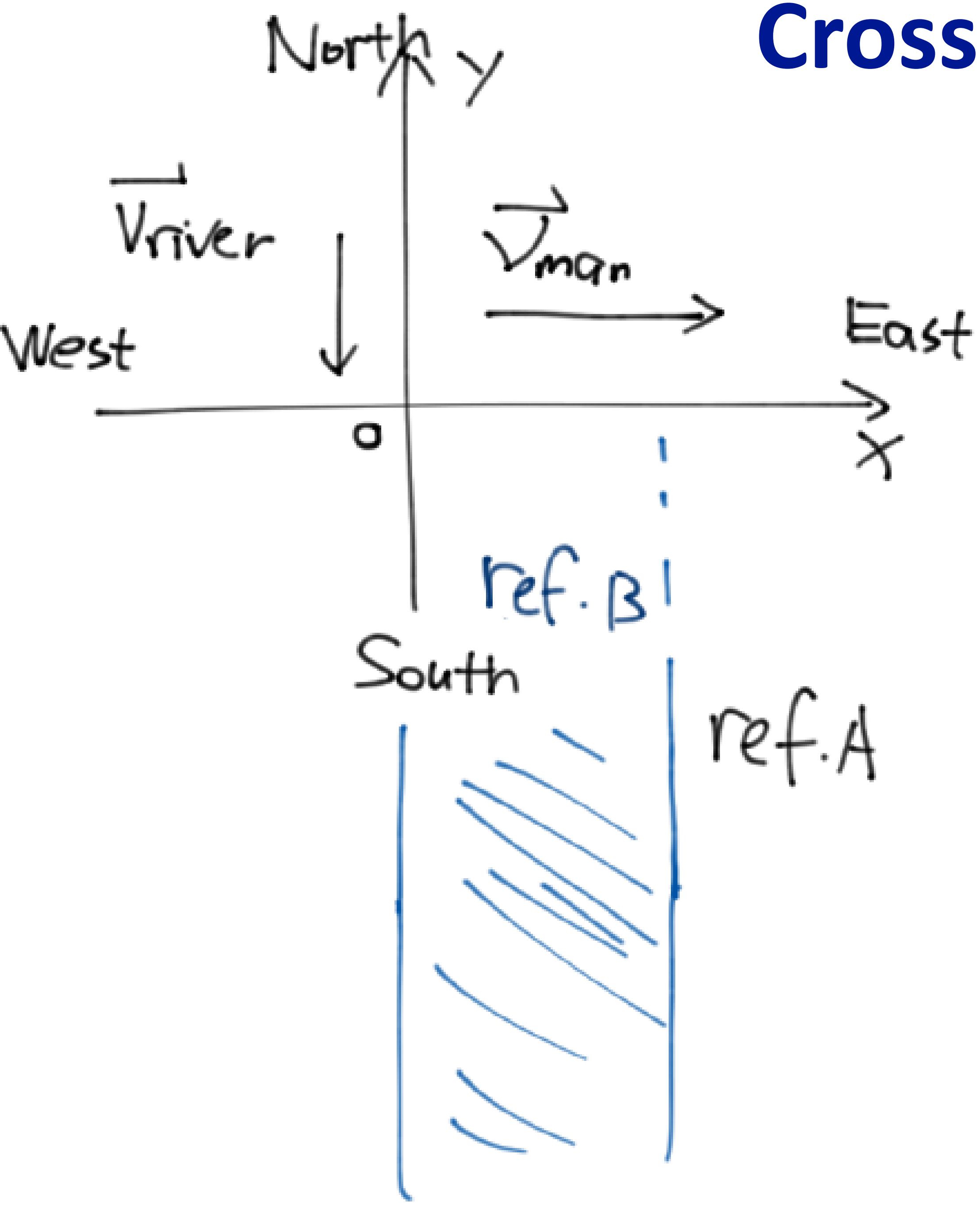
$$\alpha = \arctan\left(\frac{100\ \text{km/h}}{240\ \text{km/h}}\right) = 23^\circ\ \text{E of N}$$



3.35 • Crossing the River I. A river flows due south with a speed of 2.0 m/s. A man steers a motorboat across the river; his velocity relative to the water is 4.2 m/s due east. The river is 800 m wide. (a) What is his velocity (magnitude and direction) relative to the earth? (b) How much time is required to cross the river? (c) How far south of his starting point will he reach the opposite bank?

Again, if you don’t know what to do, draw things out!

Crossing the River



$$\underset{\text{ground}}{\vec{V}_{river}} = \vec{V}_r = -2.0\,m/s \cdot \hat{j}$$

$$= \vec{V}_{B/A}$$

velocity of B (river) relative to A (ground)

$$\underset{\text{river}}{\vec{V}_{man}} = \vec{V}_{P/B} = 4.2\,m/s \cdot \hat{i}$$

$$\vec{V}_{P/A} = \vec{V}_{P/B} + \vec{V}_{B/A}$$

$$= 4.2\,m/s\,\hat{i} - 2.0\,m/s \cdot \hat{j}$$

Crossing the River

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

$$= 4.2\,\text{m/s}\,\hat{i} - 2.0\,\text{m/s}\cdot\hat{j}$$

(a) To get to the other side, $\Delta x = 800\,\text{m}$

or $\vec{r} = 800\,\text{m}\cdot\hat{i} + y\cdot\hat{j}$

don't need in (a)

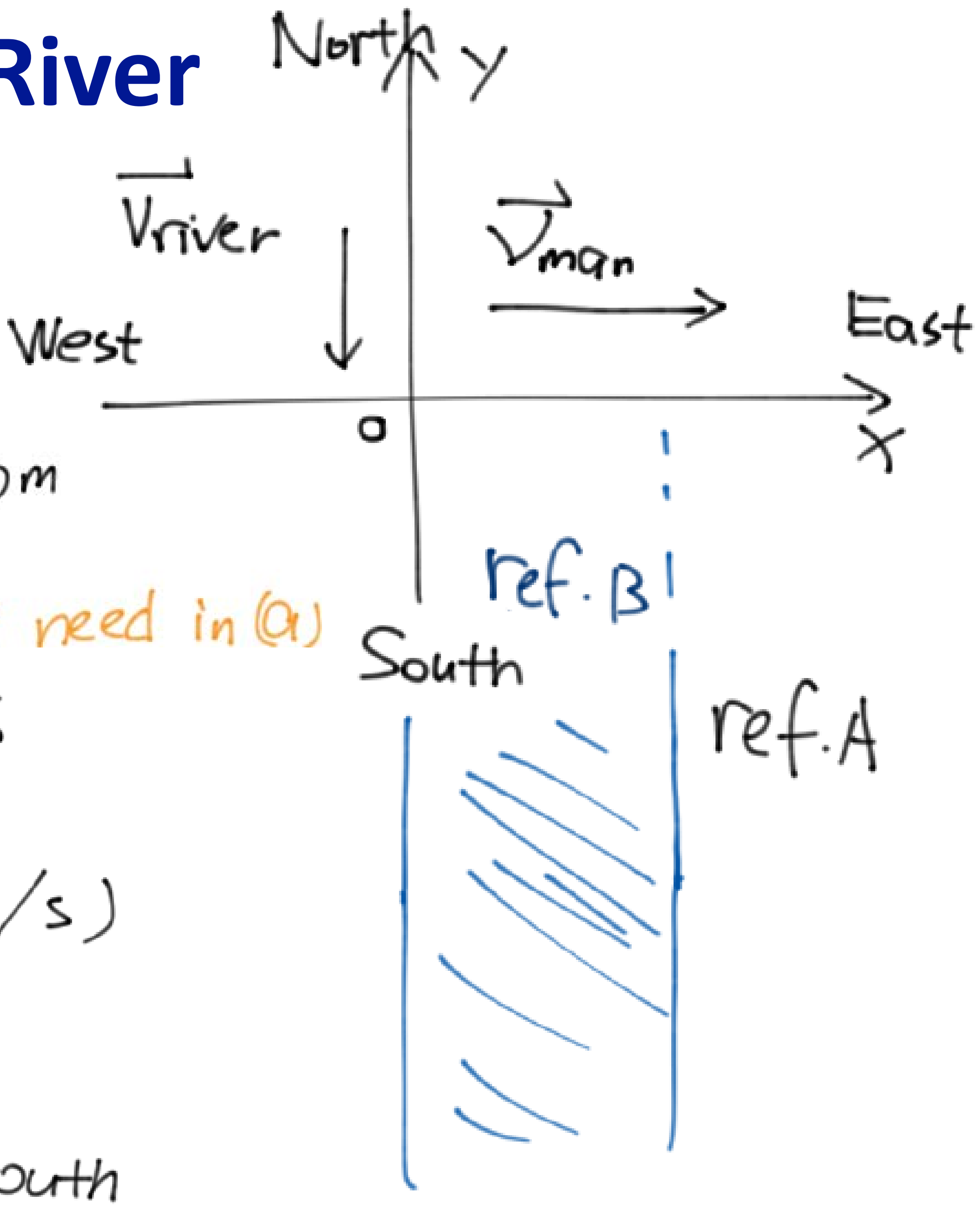
$$\Delta t = \frac{\Delta x}{v_x} = \frac{800\,\text{m}}{4.2\,\text{m/s}} = 190.5\,\text{s}$$

(b) $\Delta y = v_y \cdot \Delta t = 190.5\,\text{s} \times (-2.0\,\text{m/s})$

$$\simeq -381.0\,\text{m}$$

we took north as $+y$

so the guy went 381.0 m south



Summary

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t}$$

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt}$$

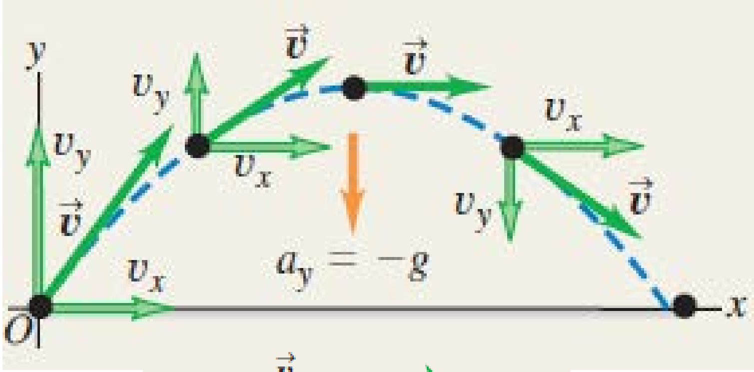
$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$a_x = \frac{dv_x}{dt}$$

$$a_y = \frac{dv_y}{dt}$$

$$a_z = \frac{dv_z}{dt}$$

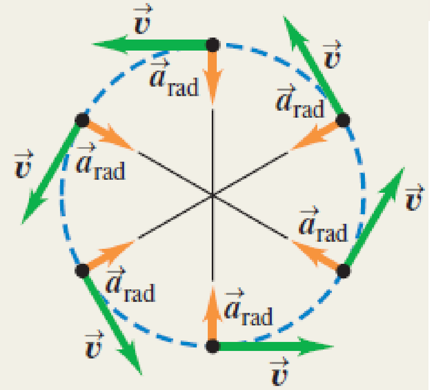


$$x = (v_0 \cos\alpha_0)t$$

$$y = (v_0 \sin\alpha_0)t - \tfrac{1}{2}gt^2$$

$$v_x = v_0 \cos\alpha_0$$

$$v_y = v_0 \sin\alpha_0 - gt$$



$$a_{rad} = \frac{v^2}{R}$$

$$a_{rad} = \frac{4\pi^2 R}{T^2}$$

$$v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x}$$

(relative velocity along a line

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

(relative velocity in space)