Fitting a generalized Pareto distribution to burned area data

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December 16, 2024

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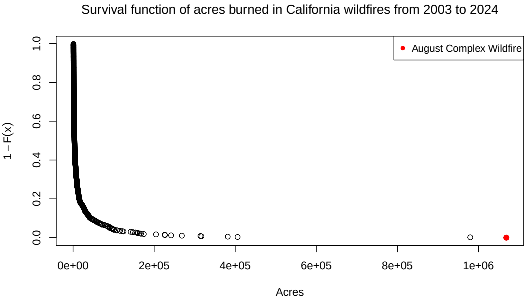

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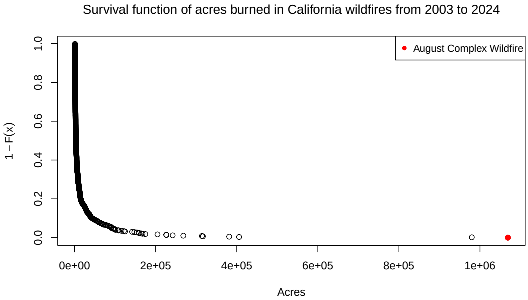

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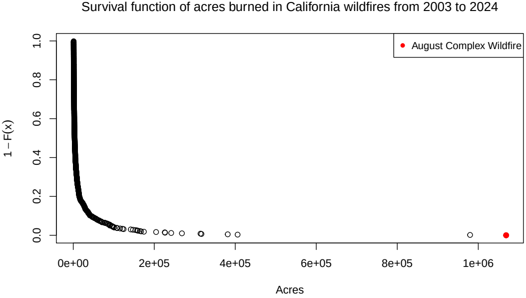

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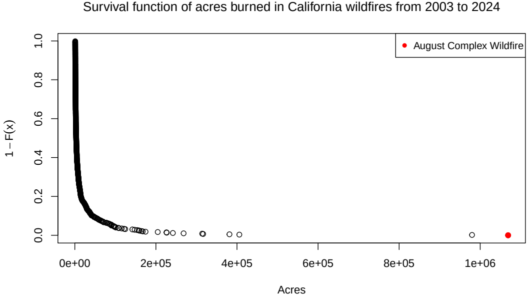

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- however this estimation is based on small sample of observations

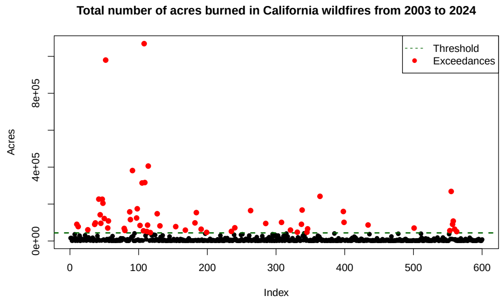
Peaks over thresholds

- Let $X_1, \dots, X_n$ denote the number of acres burned on California wildfires from 2003 to 2024
- We use the following model

$$\lim_{u \to \infty} \mathsf{P}\left(X - u \leqslant x \mid X > u\right) = G(x) \qquad (1)$$

→ Condition $X > u$: "peaks over thresholds"

→ $Y = X - u$ is the exceedance loss above the high threshold u



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Theorem (Gnedenko (1943); Fisher and Tippett (1928))

The limit distribution G in (1) is necessarily a generalized Pareto distribution (gpd):

$$G_{\gamma,\alpha}(x) = \begin{cases} 1 - (1 + \gamma x/\alpha)^{-1/\gamma}, & \gamma \neq 0, \\ 1 - e^{-x/\alpha}, & \gamma \neq 0, \end{cases}$$

where $\alpha > 0$, and $1 + \gamma x/\alpha > 0$ when $\gamma \neq 0$.

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- We obtain the log-likelihood function

$$l((\gamma, \alpha); Y_1, \dots, Y_{N_u}) = \begin{cases} -N_u \ln(\alpha) - (1 + 1/\gamma) \sum_{i=1}^{N_u} \ln\left(1 + \gamma Y_i/\alpha\right), & \gamma \neq 0, \\ \alpha^{-1} \left(N_u(1-u) - \sum_{i=1}^{N_u} Y_i \right), & \gamma \neq 0, \end{cases} \tag{2}$$

which we maximize subject to $\alpha > 0$, and $1 + \gamma Y_i/\alpha$ for all i, when $\gamma \neq 0$

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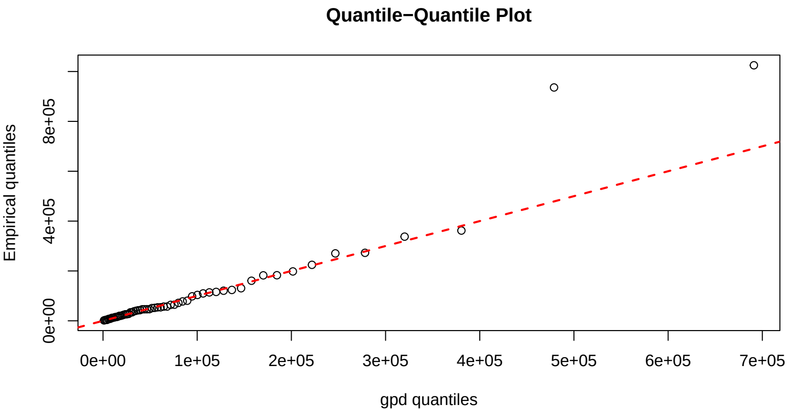
- We consider the QQ-plot

$$\left(Q_{\hat{\gamma},\hat{\alpha}}(\frac{i}{N_u+1}), Y_{i,N_u}\right), \qquad i = \ldots, N_u,$$

  where Y_{i,N_u} denote the i-th ranked observation
- See the plot on the next slide

QQ-plot of the fitting: Part 2



How unusual is the August Complex wildfire?

- Again, we assume that the exceedance losses $Y_1, \ldots, Y_{N_u} \sim G_{\hat{\gamma},\hat{\alpha}}$
- Let $x > u$. In our example, x is the number of acres burned in the August Complex wildfire and recall u, the 0.9 quantile

$$\begin{aligned}
\mathsf{P}(X > x) &= \mathsf{P}(X > u)\,\mathsf{P}(X > x \mid X > u) \\
&= 0.9 \times \mathsf{P}(X - u > x - u \mid X > u) \\
&= 0.9 \times \mathsf{P}(Y > x - u \mid X > u) \\
&= 0.9 \times \left(1 + \hat{\gamma}\frac{x-u}{\hat{\alpha}}\right)^{-1/\hat{\gamma}} \approx 0.006648479
\end{aligned}$$

Bibliography

Bücher, A. and J. Segers (2017). On the maximum likelihood estimator for the generalized extreme-value distribution. *Extremes 20*, 839–872.

Fisher, R. A. and L. H. C. Tippett (1928). Limiting forms of the frequency distribution of the largest or smallest member of a sample. In *Mathematical proceedings of the Cambridge philosophical society*, Volume 24, pp. 180–190. Cambridge University Press.

Gnedenko, B. (1943). Sur la distribution limite du terme maximum d'une serie aleatoire. *Annals of mathematics 44*(3), 423–453.