Chapter:11

TEST OF HYPOTHESES (OR) TEST OF SIGNIFICANCE

INTRODUCTION

On the basis of sample study, the inductive inference is for deciding about the characteristics of the population. Such a decision involves an element of taking a wrong decision. It is a risky job. But the modern theory of probability plays a vital role in decision making and the method of statistics which helps in arriving at the criterion for such decision is known as "TESTING OF HYPOTHESIS". It was initiated by J. Neyman and E.S. Pearson who employed these techniques to arrive at under decision known uncertainty on the basis of a sample whose size is known in advance.

The Test of Hypothesis is a process of testing of significance regarding parameter of the population on the basis of sample. So that the test of hypothesis is also known as the "TEST OF SIGNIFICANCE".

STATISTICAL HYPOTHESIS

A statistical hypothesis is a statement or assertion about the population or equivalently about the probability distribution characterizing population, which we want to verify on the basis of information available from a sample.

Simple Statistical Hypothesis

If a statistical hypothesis specifies the population completely, it is called a simple statistical hypothesis.

Composite Statistical Hypothesis

If a statistical hypothesis does not specify the population completely, it is called a composite statistical hypothesis.

Null Hypothesis

A definite statement regarding the population parameter without any bias, that is, a neutral statement, is called the null hypothesis. This is denoted by H_0.

Alternative Hypothesis

An alternative statement for every hypothesis which actually we have to test is called the alternative hypothesis and is denoted by H_1.

CRITICAL REGION

Let $x_1, x_2, \ldots, x_n$ be a random sample drawn from a population, with these sample observations we constitute a sample space S. Let the sample space S be expressed in a two-dimensional space. This space is divided in two disjoint parts ω and $\bar{\omega}$. Let X be the statistic from the sample observations, if the value of X falls in the region ω. If H_0 is rejected then the region ω is called critical region or rejection region. If the value of X falls in $\bar{\omega}$, H_1 is rejected or H_0 is accepted then $\bar{\omega}$ is called the acceptance region.

Type-I error: The error of rejecting the null hypothesis when it is true is called Type-I error.

Type-II error: The error of accepting the null hypothesis when it is not true is called Type-II error.

Level of Significance

The probability of Type-I error is called level of significance. It is also called the size of critical region. It is denoted by α.

i.e., $$P[\text{rejecting } H_0 \text{ when } H_0 \text{ is true}] = \alpha$$

ONE-TAILED AND TWO-TAILED TESTS

A test of any statistical hypothesis where the alternatiave hypothesis is one tailed (right tailed or left tailed) is called a one-tailed test.

For example, in a test for testing the mean of a population in a one-tailed test, we assume that the null hypothesis

H_0: $\mu = \mu_0$ against the alternative hypothesis.

H_1: $\mu > \mu_0$ (right tailed)

(OR)

H_1: $\mu < \mu_0$ (left tailed) is called one-tailed test.

In a test of statistical hypothesis where the alternative hypothesis is two tailed, we assume that the null hypothesis.

H_1: $\mu = \mu_0$ against the alternative hypothesis

H_1: $\mu \neq \mu_0$ $[\mu > \mu_0$ or $\mu < \mu_0]$ is called two-tailed test.

Application of one-tailed or two-tailed test for a particular problem depends entirely on the nature of the alternative hypothesis. If the alternative test is two tailed, we apply two-tailed test and if the alternative hypothesis is one tailed we apply one-tailed test.

Example: Consider two population brands of bulbs—one manufactured by routine process (mean μ_1) and the other manufactured by new technique (mean μ_2).

If we want to test whether the bulbs differ significantly then the hypothesis is H_0: $\mu_1 = \mu_2$ and the alternative hypothesis will be H_0: $\mu_1 \neq \mu_2$. This gives us a two-tailed test. Suppose we want to test if the bulbs produced by new process (μ_2) have higher average life than those produced by standard process (μ_1), then we have H_0: $\mu_1 = \mu_2$ and H_1: $\mu_1 < \mu_2$.

In this case we have to adopt a left-tail test.

If we want to test whether the product of new process (μ_2) is inferior to that of standard process (μ_1), then we have H_0: $\mu_1 = \mu_2$ and H_1: $\mu_1 > \mu_2$ which gives a right-tail test.

Hence, the decision about applying a two-tailed test or a single-tailed (left or right) test mainly depends on the problem.

Large sample test: If the sample size $n \geq 30$ it is called a large sample and the test based on it is called a large sample test.

Critical values of *Z*

Critical value (Z_a)	*Level of significance*		
	1%	5%	10%
Two-tailed test	2.58	1.96	1.645
Right-tailed test	2.33	1.645	1.28
Left-tailed test	–2.33	–1.645	–1.28

PROCEDURE FOR TESTING OF HYPOTHESIS

Null hypothesis: Set up the null hypothesis i.e., H_0.

Alternative hypothesis: Set up the alternative hypothesis i.e., H_1.
This enables us to decide whether we have to use a one tailed (right or left) test or two tailed test.

Level of significance: Choose the appropriate level of significance, α.

Test statistic: Calculate the value of Z (the test statistic) under the null hypothesis.

$$Z = \frac{t - E(t)}{S \cdot E(t)}, \text{ under } H_0$$

Conclusion

Compare the calculated $|Z|$ value with the tabulated value, at the given level of significance, α.

If $|Z| \leq Z_\alpha$ (the table values)

That means that the calculated value $\leq$ table value, then we accept H_0

If $|Z| > Z_\alpha$ (the table values)

That means that the calculated value $>$ table value, then we reject H_0

LARGE AND SMALL SAMPLES

In general, the samples are of two types, which are drawn a population. They are (1) Large sample (2) Small sample.

1. Large Sample: If the sample size $(n) > 30$, then the sample is called a large sample.
2. Small Sample: If the sample size $(n) \leq 30$, then the sample is called a small sample.

TEST FOR SINGLE PROPORTION–LARGE SAMPLES

Let x be the number of persons in a sample of size n persons. Then the proportion of such persons in the sample is given by $p = \dfrac{x}{n}$.

Let P be the proportion in the population. Now the hypothesis that the population proportion is equal to the specified value

$$H_0: P = P_0$$

can be tested as follows against the alternative hypothesis

$$H_1: P \neq P_0$$

The test statistic is $|Z| = \left|\dfrac{x - E(x)}{\sqrt{V(x)}}\right| \sim N(0, 1)$

$$|Z| = \left|\frac{x - nP}{\sqrt{nPQ}}\right| = \left|\frac{\frac{x}{n} - P}{\sqrt{\frac{PQ}{n}}}\right|$$

$$|Z| = \left|\frac{p - E(p)}{\sqrt{V(p)}}\right| \sim N(0, 1)$$

$$|Z| = \left|\frac{p - P}{\sqrt{\frac{PQ}{n}}}\right| \sim N(0, 1)$$

The value of this test statistic is calculated and compared with table value at $\alpha\%$ level of significance.

If calculated $|Z|$ value $\leq$ table value then we accept H_0.
If calculated $|Z|$ value $>$ table value then we reject H_0.

CONFIDENCE LIMITS

95% confidence limits for P are given by $p \pm 1.96 \sqrt{\frac{pq}{n}}$

99% confidence limits for P are given by $p \pm 2.58 \sqrt{\frac{pq}{n}}$

EXAMPLE 11.1: In a sample of 1000 people in Karnataka 540 are rice eaters and the rest are wheat eaters. Can we assume that both rice and wheat are equally popular in this state at 1% level of significance?

Solution: $H_0: P = P_0$

That is, H_0: Both rice and wheat are equally popular in the state.

Given that size of the sample $(n) = 1000$, so that it is a large sample.

Number of rice eaters $(x) = 540$

Sample of proportion of rice eaters$(p) = \frac{x}{n} = \frac{540}{1000} = 0.540$

Sample of proportion of rice eaters$(p) = \frac{x}{n} = \frac{540}{1000} = 0.540$

Population proportion of rice eaters $= P = \frac{1}{2} = 0.5$

The test statistic is

$$P = \frac{1}{2} = 0.5 \text{ \{total possible outcomes means rice + wheat = 2\}}$$

$$Q = 1 - P = 1 - 0.5 = 0.5$$

The test statistic is $|Z| = \left| \frac{p - P}{\sqrt{\frac{PQ}{n}}} \right| \sim N(0, 1)$

$$|Z| = \left| \frac{0.54 - 0.5}{\sqrt{\frac{(0.5)(0.5)}{1000}}} \right| = 2.5298$$

Calculated value of $|Z| = 2.5298$

At 1% level of significance the table value is 2.58

$$2.5298 < 2.58$$

Calculated value < table value then we accept H_0

We conclude that both rice and wheat are equally popular in the state.

EXAMPLE 11.2: A die was thrown 9000 times and of these 3220 yielded a 3 or 4. Is this consistent with the hypothesis that the die was unbiased.

Solution: $H_0: = P = P_0$

That is H_0: The die is unbiased.

Given that size of the sample $(n) = 9000$, so that it is a large sample.

Number of success $(x) = 3220$

$$\text{Sample proportion of success } (p) = \frac{x}{n} = \frac{3220}{9000} = 0.3578$$

Population proportion of success $= P$

$$P(\text{getting 3 or 4}) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} = 0.3333$$

$$Q = 1 - P = 1 - 0.3333 = 0.6667$$

$$\text{The test statistic is } |Z| = \left| \frac{p - P}{\sqrt{\frac{PQ}{n}}} \right| \sim N(0, 1)$$

$$|Z| = \left| \frac{0.3578 - 0.3333}{\sqrt{\frac{(0.3333)(0.6667)}{9000}}} \right| = 4.9306$$

Calculated value of $|Z| = 4.9306$

At 1% level of significance the table value is 2.58

$$4.9306 > 2.58$$

Calculated value > table value then we accept H_0
Hence, the die is biased.

EXAMPLE 11.3: Among 900 people in a state 90 are found to be chapathi eaters. Constru
99% confidence interval for the true proportion.

Solution: Given that size of the sample (n) = 900, so that it is a large sample.
Number of chapathi eaters (x) = 90

$$\text{Sample proportion } (p) = \frac{x}{n} = \frac{90}{900} = 0.1$$

$$q_1 - p = 1 - 0.1 = 0.9$$

At 99% table value is 2.58

Confidence intervals are $\left[p - Z_\alpha\sqrt{\frac{pq}{n}}, p + Z_\alpha\sqrt{\frac{pq}{n}}\right]$

$$\left[0.1 - 2.58\sqrt{\frac{(0.1)(0.9)}{900}}, 0.1 + 2.58 + \sqrt{\frac{(0.1)(0.9)}{900}}\right]$$

[0.07, 0.13]

TEST OF SIGNIFICANCE FOR DIFFERENCE OF PROPORTIONS- LARGE SAMPLE

Let x_1, x_2 be the number of individuals possessing a given characteristics in two independent random samples of sizes n_1 and n_2 drawn from two populations. Then the proportions of such individuals in the samples are given by $p_1 = {}^{x_1}/_{n_1}$ and $p_2 = {}^{x_2}/_{n_2}$.

Let P_1 and P_2 be denoted respectively by the corresponding proportions in the two populations. Then the hypothesis that the two population proportions are equal i.e.,

$$H_0 : P_1 = P_2 = P$$

Against the alternative hypothesis

$$H_1 : P_1 \neq P_2$$

The test statistics is given by

$$|Z| = \left[\frac{p_1 - p_2}{\sqrt{PQ\left[\frac{1}{n_1} + \frac{1}{n_2}\right]}} \right] \sim N(0,1)$$

Note: If the value of P is not known , it is estimated as follows.

$$\widehat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}$$

And

$$\widehat{Q} = 1 - \widehat{P}$$

The value of this test statistic is calculated and compared with the table value at $\alpha\%$
level of significance .

If calculated value $\leq$ table value then we accept H_0

If calculated value $>$ table value then we reject H_0

Example 11.4: Random samples of 400 men and 600 women were asked whether they would like to have a flyover near their residence. 200 men and 325 women were in favour of the proposal. Test the hypothesis that proportions of men and women in favour of the proposal are same , at 5% level.

Sol: H_0: $P_1 = P_2 = P$ is unknown.
That is H_0 : there is no significant difference between the option of men and women as far as proposal of the flyover is concerned.

Size of the first sample $(n_1) = 400$.
Size of the second sample $(n_2) = 600$.

$n_1, n_2 > 30$, so that the given samples are large samples.

Number of men in favour $(x_1) = 200$; Number of women n favour $(x_2) = 325$;

First sample proportion (men) $p_1 = {}^{x_1}/_{n_1}$ = 200/400 = 0.5
Second sample proportion (women) $p_2 = {}^{x_2}/_{n_2}$ = 325/600 = 0.541

$$|Z| = \left[\frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left[\frac{1}{n_1} + \frac{1}{n_2}\right]}}\right] \sim N(0,1)$$

Where $\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{400*0.5+600*0.541}{400+600} = 0.5246$;

$\hat{Q} = 1 - \hat{P}$ = **1-0.5246= 0.4754.**

|Z| = $\left|\frac{0.5 - 0.541}{\sqrt{0.5246*0.4754\left[\frac{1}{400}+\frac{1}{600}\right]}}\right|$ **= 1.28**

At 5% level of significance the table value is 1.96.

Since 1.28 < 1.96 i.e., Calculated value < table value then we accept H_0.

We conclude that there is no significant difference between the option of men and women as far as proposal of the flyover is concerned.

EXAMPLE 11.5: In a sample of 600 men from a certain city, 450 men are found to be smokers. In a sample of 900 from another city, 450 are found to be smokers. Do the data indicate that the two cities are significantly different with respect to prevalence of smoking habits among men?

Solution: H_0: $P_1 = P_2 = P$ is unknown

That is H_0: There is no significant difference in the smoking habits of two cities.

Size of the first sample $(n_1) = 600$

Size of the second sample $(n_2) = 900$

$n_1, n_2 > 30$, so that the given samples are large samples

Number of smokers of city-I $(x_1) = 450$

Number of smokers of city-II $(x_2) = 450$

First sample proportion (city-I) $p_1 = \dfrac{x_1}{n_1} = \dfrac{450}{600} = 0.75$

Second sample proportion (city-II) $p_2 = \dfrac{x_2}{n_2} = \dfrac{450}{900} = 0.5$

$n_2 = 900$

The test statistic is $|Z| = \left| \dfrac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left[\dfrac{1}{n_1} + \dfrac{1}{n_2}\right]}} \right| \sim N(0, 1)$

where $\hat{P} = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \dfrac{450 + 450}{600 + 900} = 0.6$

$\hat{Q} = 1 - \hat{P} = 1 - 0.6 = 0.4$

$$|Z| = \left| \frac{0.75 - 0.5}{\sqrt{(0.6)(0.4)\left[\dfrac{1}{600} + \dfrac{1}{900}\right]}} \right| = 9.7$$

Calculated value of $|Z| = 9.7$

at 5% level of significance the table value is 1.96

$$9.7 > 1.96$$

Calculated value > table value then we accept H_0.

We conclude that there is significant difference in the smoking habits of the two cities.

TEST FOR THE MEAN OF A POPULATION – LARGE SAMPLE

Let $x_1, x_2, \ldots\ldots, x_n$ be a random sample drawn from a population with mean μ and variance σ^2. Let $\overline{x}$ and s^2 denote the mean and variance of the sample respectively.

Now the hypothesis that the population mean is equal to a specified value (or) there is no significant difference between the sample and the population mean.

i.e.,
$$H_0: \mu = \mu_0$$

Against the alternative hypothesis
$$H_1: \mu \neq \mu_0$$

The test statistic is $|Z| = \left|\frac{\overline{x} - E(\overline{x})}{\sqrt{V(\overline{x})}}\right| \sim N(0,1)$

$$|Z| = \left|\frac{\overline{x} - \mu}{\sigma/\sqrt{n}}\right| \sim N(0,1)$$

If σ is not known the above test statistic can be written as $|Z| = \left|\frac{\overline{x} - \mu}{s/\sqrt{n}}\right| \sim N(0,1)$

where s is the simple standard deviation which can be found from the formula

$$s = \sqrt{\frac{1}{n}\sum(x - \overline{x})^2}$$

The value of this test statistic is calculated and compared with the table value at $\alpha\%$ level of significance .

If calculated value $\leq$ table value then we accept H_0

If calculated value $>$ table value then we reject H_0

Example 11.6: A sample of size 400 was drawn and the sample mean was found to be 99. Test whether this sample could have come from a normal population with mean 100 and standard deviation 8 at 5% level of significance.

Sol: H_0: $\mu = \mu_0$, that is H_0 : the sample has come from a normal population with mean 100 and standard deviation 8.

$$H_0 \text{: } \mu = 100$$

Size of the sample $(n) = 400$; sample mean ($\bar{x}$) = 99; Population mean (μ) = 100; Standard deviation (σ)=8

The test statistic is $|Z| = \left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| \sim N(0,1) = \left|\frac{99-100}{\frac{8}{\sqrt{400}}}\right| = 2.5$

At 5% level of significance the table value is 1.96.

We conclude that the sample has not been drawn from a normal population with mean 100 and standard deviation 8.

Example 11.7: **A sample of 400 items is taken from a population whose standard deviation is 10. The mean of the sample is 40. Test whether the sample has come from a population with mean 38. Also calculate 95% confidence interval.**

Sol. H_0: $\mu = \mu_0$, **that is** H_0: **The sample has come from a normal population with mean 38.**

$$H_0: \mu = 38$$

Size of the sample $(n) = 400$

Sample mean $(\bar{x})$ **= 40**

Population mean $(\mu) = 38$

Sample standard deviation (σ)**=10**

The test statistics is

$$|Z| = \left|\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right| = \left|\frac{40-38}{\frac{10}{\sqrt{40}}}\right| = 4$$

At 5% level of significance the table value is 1.96.

Since $4 > 1.96$

Calculated value > table value , then we reject H_0.

We conclude that the sample has not been drawn from a normal population with mean 38.

Confidence interval

$$\left(\overline{x} - z_\alpha \frac{\sigma}{\sqrt{n}}, \overline{x} + z_\alpha \frac{\sigma}{\sqrt{n}}\right)$$

$$= \left(40 - 1.96 * {}^{10}\!/_{\sqrt{400}}, 40 + 1.96 * {}^{10}\!/_{\sqrt{400}}\right)$$

$$= (39.02, 40.98)$$

Therefore intervals are 39.02 and 40.98.

TEST FOR THE EQUALITY OF TWO POPULATION MEANS- LARGE SAMPLES

Let $\overline{x_1}$ be the mean of sample size $n_1$drawn from a population with mean μ_1 and variance $(\sigma^2)_1$.
Let $\overline{x_2}$ be the mean of sample size $n_1$drawn from a population with mean μ_2 and variance $(\sigma^2)_2$.
Let ${s_1}^2$ and ${s_2}^2$ be the variances of first and second sample respectively.
Now the hypothesis that the means of the two populations are equal or there is no significant difference between the sample means.
i.e., H_0: $\mu_1 = \mu_2$ against the alternative hypothesis H_1: $\mu_1 \neq \mu_2$ can be tested as follows.

The test statistic is

$$|Z| = \left| \frac{(\overline{x_1} - \overline{x_2})}{\sqrt{\left(\frac{\sigma_1{}^2}{n_1} - \frac{\sigma_2{}^2}{n_2}\right)}} \right| \sim N(0,1)$$

If the population variances are not known they will be estimated as sample variance.

The test statistic is

$$|Z| = \left| \frac{(\overline{x_1} - \overline{x_2})}{\sqrt{\left(\frac{s_1{}^2}{n_1} - \frac{s_2{}^2}{n_2}\right)}} \right| \sim N(0,1)$$

The value of this test statistic is calculated and compared with the table value at $\alpha\%$ level of significance.

If calculated value $\leq$ table value then we accept H_0

If calculated value $>$ table value then we reject H_0

EXAMPLE 11.8: The mean height of two large samples of sizes 1000 and 2000 members are 67.5 inches and 68.0 inches respectively. Can the samples be regarded as drawn form the same population of standard deviation 2.5 inches.

Solution: H_0: $\mu_1 = \mu_2$ (σ is known)

i.e., H_0: The samples have been drawn from the same population of SD 2.5.

Given that size of the first sample (n_1) = 1000

Size of the second sample (n_2) = 2000

Since n_1, $n_2 > 30$, so that the given samples are large samples.

First sample mean ($\overline{x}_1$) = 67.5 and second sample mean ($\overline{x}_2$) = 68.0

Combined standard deviation $\sigma = 2.5$

$$H_0: \mu_1 = \mu_2 \ (\sigma_1 = \sigma_2 = \sigma \text{ is known})$$

The test statistic is $$|Z| = \left| \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} - \dfrac{\sigma_2^2}{n_2}}} \right| \sim N(0, 1)$$

$$|Z| = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\sigma^2\left[\dfrac{1}{n_1} - \dfrac{1}{n_2}\right]}} = \frac{67.5 - 68}{\sqrt{(2.5)^2\left[\dfrac{1}{1000} + \dfrac{1}{2000}\right]}} = 5.16$$

Here $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Calculated value $|Z| = 5.16$

At 5% level of significance the table value is 1.96

$$5.16 > 1.96$$

Calculated value > table value then we reject H_0

We conclude that the samples are not drawn from the same population of the SD 2.5 inches.

EXAMPLE 11.9: Intelligence test of two groups of boys and girls gave the following results.

	Mean	*Standard deviation*	*Sample size*
Girls	75	8	60
Boys	73	10	100

Examine if the difference between the mean scores is significant.

Sol:

Given that size of the first sample (girls) $(n_1) = 60 > 30$
Size of the second sample (boys) $(n_2) = 100 > 30$

So that the given samples are large samples.

First sample mean (girls) $(\bar{x}_1) = 75$ and second sample mean (boys) $(\bar{x}_2) = 73$

Standard deviation of the first sample (girls) $(s_1) = 8$
Standard deviation of the first sample (boys) $(s_2) = 8$

The test statistic is $|Z| = \left| \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} \right| \sim N(0, 1)$

$$|Z| = \left| \frac{75 - 73}{\sqrt{\dfrac{(8)^2}{60} + \dfrac{(10)^2}{100}}} \right| = 1.39$$

Calculated value $|Z| = 1.39$
At 5% level of significance the table value is 1.96

$$1.39 < 1.96$$

Calculated value < table value then we accept H_0.
We conclude that there is no significant difference between the mean scores.

END