Towards a 2-dimensional spectral construction

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Duality in mathematics

Duality is a common pattern in mathematics
Especially Algebra/Spaces dualities:

$$\text{Algebraic-like objects} \quad \leftrightsquigarrow \quad \text{Space-like objects}$$

Provide spatial intuition for abstract situations
As well as algebraic tools for geometry
But also Syntax/Semantics dualities for propositional and first order logic

Examples of spectral dualities

Grothendieck duality: commutative rings and locally ringed spaces

$$\begin{tikzcd} \mathcal{CR}ing^{op} \arrow[r, bend right, "Spec"'] \arrow[r, phantom, "\perp"] & \mathcal{LRS}paces \arrow[l, bend right, "\Gamma"'] \end{tikzcd}$$

Stone duality: distributive lattices and Stone spaces

$$\begin{tikzcd} \mathcal{DL}at^{op} \arrow[r, bend right, "Spec"'] \arrow[r, phantom, "\simeq_{eq}"] & \mathcal{S}tone \arrow[l, bend right, "\mathcal{K}\Omega(-)"'] \end{tikzcd}$$

Other examples:

- In algebraic geometry: Pierce spectrum, real spectrum
- Stone-like dualities for boolean algebras, Heyting algebras
- Dubuc & Poveda duality for MV-algebras, dualities for residuated lattices, duality for rigs...

General template

Contravariant adjunction between algebras and spaces:

$$\begin{tikzcd} \mathcal{B}^{op} \arrow[r, bend right, "Spec"'] \arrow[r, phantom, "\perp"] & \mathcal{S}tr\mathcal{S}paces \arrow[l, bend right, "\Gamma"'] \end{tikzcd}$$

- a category of **algebraic** objects $\mathcal{B} \simeq \mathbb{T}_{\mathcal{B}}[\mathcal{S}et]$
- $\mathcal{S}$et-valued models of an (essentially) algebraic theory
- with a distinguished subcategory of **"local objects"**
- and a **factorization system** (*etale*, *local*)

- a category of (locally) **structured spaces**
- space-like objects equiped with a sheaf of $\mathcal{B}$-object
- values on opens are in $\mathcal{B}$
- stalks are local objects
- morphisms: underlying continuous maps + comorphisms of sheaves with **"local arrows"** at stalks

- *Spec* associates a structured space to each algebra
- Γ reconstructs algebras as global sections of structural sheaves

General template

Geometry is not intrinsic to the category of algebras
Defined relatively to a choice of **local data**:

- **local objects**, behaving as points
- **local arrows**, behaving as a right class
- **etale arrows**, behaving as a left class

For Grothendieck duality

- $\mathcal{B} = \mathcal{CR}ing$; "structured spaces" $=$ locally ringed spaces
- Local objects $=$ local rings (with unique maximal ideal)
- Local arrows: conservative rings homomorphisms
- Etale arrows: localization of rings

Local data

- Structural sheaves behave locally as local objects
  $\rightarrow$ their stalks are local objects
- Comorphisms behave locally as local maps
  $\rightarrow$ induce local arrows between stalks

Condition of admissibility:

Relates local maps to local objects = entangles factorial and syntactical/topological data
Express a situation of **"multireflection"**
$\rightarrow$ defect of uniqueness of solution for an universal problem
Geometry: produces a topos where uniqueness is restored.
Spectrum as a solution of an universal problem.

The 1-dimensional construction

Different approaches :

- **Cole**: abstract presentation of admissibility
  Spectrum constructed by 2-limits as a classifying object
- **Coste**: syntactical interpretation of Cole's admissibility
  Explicit construction of the spectral site.
- **Anel**: topological behaviours in the opposite category
- **Diers**: more divergent, purely categorical approach
  Abstraction of admissibility into multiadjunction
  Spectrum as a space constructed from its points

$\rightarrow$ In a first part we present the general construction of 1-categorical spectra from Diers and Anel/Coste point of view

Extension of the construction for syntax-semantics dualities

- 1-categorical spectrum associate spaces of points to algebra
- As like as semantics associates "spaces of models" to theories
- Categories of models of first order theories exhibit spatial-like features
- Moreover there exists a correspondence between propositional Stone-like duality, that enjoy spectral approach, and first order syntax-semantics dualities: this invites to a geometrical account of F.O. dualities
- This is already part of the notion of *classifying topos*
- We will try to adapt the construction of spectra to subsume the notion of classifying topos by a more general notion of 2-spectrum

Layout

1. The 1-dimensional spectral construction : notion of amdissibility/geometry and construction of the spectrum
2. Stone-like examples with comparison with Syntax-Semantics dualities
3. Some topological intuitions about first order doctrines
4. Intermezzo on local toposes
5. 2-geometry (ongoing work !)
6. 2-geometry for Lex, Reg, Coh (even more ongoing !)

Syntactic category for $\mathbb{T}$

In the following we fix a finite limit (aka essentially algebraic) theory $\mathbb{T}$

Syntactic category $\mathcal{C}_{\mathbb{T}}$ for $\mathbb{T}$

- Obj: formulas in context $\{\overline{x}, \phi(\overline{x})\}$ in the language of $\mathbb{T}$
- Mor: equivalence classes of functional formulas

$$[\theta(\overline{x}, \overline{y})] : \{\phi, \overline{x}\} \to \{\psi, \overline{y}\} \text{ s.t. } \begin{cases} \theta(\overline{x}, \overline{y}) \vdash_{\mathbb{T}} \psi(\overline{y}) \\ \theta(\overline{x}, \overline{y}) \wedge \theta(\overline{x}, \overline{y'}) \vdash_{\mathbb{T}} \overline{y} = \overline{y'} \\ \phi(\overline{x}) \vdash_{\mathbb{T}} \exists \overline{y} \theta(\overline{x}, \overline{y}) \end{cases}$$

In the following we just write ϕ for $\{x \mid \phi\}$ for concision

Cartesian theory and Lex categories

- If $\mathbb{T}$ is cartesian then $\mathcal{C}_{\mathbb{T}}$ is lex $=$ has finite limite
- Any lex $\mathcal{C}$ is the $\mathcal{C}_{\mathbb{T}}$ of some cartesian theory $\mathbb{T}$

Models in $\mathcal{S}et$

Models as functors

A model of $\mathbb{T}$ is a lex functor $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{S}et$

$$F(\phi) = \underbrace{\{\overline{a} \in F \mid F \models \phi(\overline{a})\}}_{\text{interpretation of the sort } \phi(\overline{x}) \text{ in } F}$$

$$\underbrace{\begin{array}{rccl} F([\theta(\overline{x},\overline{y})]) : & F(\phi) & \to & F(\psi) \\ & \overline{a} & \mapsto & \text{the unique } \overline{b} \text{ such that } \theta[\overline{a},\overline{b}] \end{array}}_{\text{interpretation of the function symbol coded by } [\theta(\overline{x},\overline{y})] \text{ in } F}$$

Then $\int F$ is the model seen as a set with structure.

Locally finitely presentable categories

LFP categories

A locally finitely presentable category $\mathcal{B}$ is a category with:

- small colimits
- a **small** generator $\mathcal{B}_{fp}$ of **finitely presented objects** such that any object B is the filtered colimit of $\mathcal{B}_{fp} \downarrow B$

$$\mathcal{B} \simeq Ind(\mathcal{B}_{fp})$$

Gabriel-Ulmer duality

LFP categories = categories of models of Cartesian theories

$$\begin{array}{rcl} \mathcal{L}ex^{op} & \simeq & \mathcal{LFP} \\ \mathcal{C} = \mathcal{C}_{\mathbb{T}} & \mapsto & \mathcal{L}ex[\mathcal{C}, \mathcal{S}et] = \mathbb{T}[\mathcal{S}et] \\ \mathcal{B}_{fp}^{op} & \leftarrow\!\shortmid & \mathcal{B} \end{array}$$

Syntactic category and finitely presented objects

F.P. generator as dual of the syntactic site

F.P. objects in $\mathcal{B} = Lex[\mathcal{C}_{\mathbb{T}}, \mathcal{S}et]$ are the representable functors
$\rightarrow$ uniquely determined by presentation formula

$$\begin{array}{rcl} \mathcal{B}_{fp}^{op} & \simeq & \mathcal{C}_{\mathbb{T}} \\ K = \langle \overline{x} \rangle / \phi_K(\overline{x}) & \leftrightsquigarrow & \{\phi_K, \overline{x}\} \\ \substack{f:\langle \overline{x} \rangle / \phi(\overline{x}) \to \langle \overline{y} \rangle / \psi(\overline{y}) \\ \text{s.t.}(f(x_i) = \tau_i[\overline{y}])_{i=1,\ldots,n}} & \leftrightsquigarrow & \theta_f(\overline{y}, \overline{x}) \Leftrightarrow \bigwedge_i x_i = \tau_i[\overline{y}] \end{array}$$

In the following we will denote $K_\phi = \langle \overline{x} \rangle / \phi$ the finitely presented model presented by the formula ϕ

F.P. objects as corepresenting objects for sorts

$$\underbrace{\mathcal{B}[K_\phi, B] \simeq B(\phi\}) = \{\overline{a} \in B \mid \ B \models \phi(\overline{a})}_{\text{Each } f:K_\phi \to B \text{ is the name of some } a \in B \text{ s.t. } B \models \phi(\overline{a})}$$

Classifying topos for $\mathbb{T}$

Diaconescu theorem for Lex sites

$$\begin{tikzcd}
\mathcal{C}_{\mathbb{T}} = \mathcal{B}_{fp}^{op} \arrow[r, "F \ lex"] \arrow[d, hook, "よ"'] & \mathcal{S}et \arrow[dl, bend left, "F_*"] \\
\widehat{\mathcal{B}_{fp}^{op}} \arrow[ur, "F^* \ lex"] &
\end{tikzcd}$$

$$\begin{aligned}
\mathcal{B} &\simeq Ind(\mathcal{B}_{fp}) \\
&\simeq Lex[\mathcal{B}_{fp}^{op}, \mathcal{S}et] \\
&\simeq Geom[\mathcal{S}et, \widehat{\mathcal{B}_{fp}^{op}}]
\end{aligned}$$

If $\mathcal{E}$ Grothendieck topos, $\mathbb{T}[\mathcal{E}] = Lex[\mathcal{B}_{fp}^{op}, \mathcal{E}] \simeq Geom[\mathcal{E}, \widehat{\mathcal{B}_{fp}^{op}}]$

$\mathbb{B} = \widehat{\mathcal{B}_{fp}^{op}}$ classifies $\mathbb{T}$-models in arbitrary toposes
$\rightarrow$ representing object for $\mathbb{T}[-]$

Geometric extensions

Geometric theory: constructed with $\wedge, \exists$, **arbitrary** $\bigvee$
$\rightarrow$ has a finite-limit part

Geometric extension of $\mathbb{T}$

- A geometric theory $\mathbb{T}'$ whose finite-limit part is $\mathbb{T}$
- Corresponds to **a Grothendieck topology** J on $\mathcal{C}_{\mathbb{T}} = \mathcal{B}_{fp}^{op}$
- $(\mathcal{B}_{fp}^{op}, J)$ as the syntactic site of $\mathbb{T}'$
- Covering families in J code for disjunction $\bigvee$ in $\mathbb{T}'$

Models of a geometric extension

Models as "J-local" objects

- $\mathbb{T}_J[\mathcal{S}et] = J$-continuous Lex functors $F : (\mathcal{B}_{fp}^{op}, J) \to \mathcal{S}et$

$$\coprod_{i \in I} F(K_i) \overset{\langle F(k_i) \rangle_{i \in I}}{\twoheadrightarrow} F(K)$$

- In particular $\mathbb{T}_J$-models are $\mathbb{T}$-models: $\mathbb{T}_J[\mathcal{S}et] \overset{\textbf{full}}{\hookrightarrow} \mathcal{B}$
- By Yoneda lemma : $\Leftrightarrow$ extension through J-covers

$$\begin{tikzcd}
& K \arrow[rr] \arrow[dl, "k_i"'] \arrow[dr, "k_j"] & & F \\
K_i & \dots & K_j \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Topological intuition

In the opposite category, J-local objects behave as points
A point lies in $\bigcup U_i$ iff it lies in some of the U_i.

Syntactic aspects: local objects

Syntactic interpretation

$$\mathcal{L}oc_J \simeq \mathbb{T}_J[\mathcal{S}et] \simeq pt(\mathcal{S}h(\mathcal{B}_{fp}^{op}, J))$$

Covers = disjunctions of cases for witnesses of domain formulas

$$\mathbb{T}_J = \mathbb{T}_{\mathcal{B}} \cup \left\{ \phi(\overline{x}) \vdash \bigvee_{i \in I} \exists \overline{y}_i (\psi_i(\overline{y}_i) \wedge \theta_{f_i}(\overline{y}_i, \overline{x})) \right\}_{(f_i)_{i \in I} \in J(\langle \overline{x} \rangle / \phi(\overline{x}))}$$

$$\begin{tikzcd}
\langle \overline{x} \rangle_\Sigma / \phi(\overline{x}) \arrow[r, "g = \ulcorner \overline{b} \urcorner"] \arrow[d, "f_i"'] & B \\
\langle \overline{y}_i \rangle_\Sigma / \psi_i(\overline{y}_i) \arrow[ur, dashed, "\exists \ulcorner \overline{b}_i \urcorner"'] &
\end{tikzcd}$$

If $\overline{b} \in B$ such that $B \models \phi(\overline{b})$
then $\exists i \in I$ and $\overline{b}_i \in B$ s.t.
$B \models \psi_i(\overline{b}_i) \wedge \theta_{f_i}(\overline{b}_i, \overline{b}))$

Example of local rings

$$\mathbb{T}_{LocRing} = \mathbb{T}_{CRing} \cup \{x \neq 0 \vdash \exists y(xy = 1) \vee \exists y'((1 - x)y' = 1)\}$$

Classifying topos for a geometric extension

Diaconescu theorem for arbitrary sites

$$\begin{tikzcd}
(\mathcal{B}_{fp}^{op}, J) \arrow[r, "\substack{Flex \\ J-cont}"] \arrow[d, hook, "a_J \text{よ}"'] & \mathcal{S}et \arrow[dl, shift left, "F_*"] \\
\mathcal{S}h(\mathcal{B}_{fp}^{op}, J) \arrow[ur, shift left, "F^*\ lex"] &
\end{tikzcd}$$

$$\begin{aligned}
\mathbb{T}_J[\mathcal{E}] &\simeq Lex_{J-cont}[(\mathcal{B}_{fp}^{op}, J), \mathcal{E}] \\
&\simeq Geom[\mathcal{E}, \mathcal{S}h(\mathcal{B}_{fp}^{op}, J)]
\end{aligned}$$

Exhibits $\mathcal{S}h(\mathcal{B}_{fp}^{op}, J)$ as a representing object for $\mathbb{T}_J[-]$

$$\mathcal{S}h(\mathcal{B}_{fp}^{op}, J) \hookrightarrow \widehat{\mathcal{B}_{fp}^{op}}$$

Problem of the free object

Geometric extensions do not have a good notion of free object.
$\rightarrow$ Several locally free $\mathbb{T}'$-models under a given object

The problem of spectrum

For any B in $\mathcal{B}$ construct:

- a topos $Spec(B)$
- endowed with a free $\mathbb{T}'$-model $\widetilde{B}$ for B

The free model will be

- a sheaf of $\mathbb{B}$-objects in this topos,
- with local objects $=$ $\mathbb{T}'$-models under B as stalks

Cannot process directly: need to precise **factorization data**
Admissibility relates factorial and geometric data

Orthogonality and factorization systems

Factorization systems

A pair $(\mathcal{E}, \mathcal{M})$ s.t. any arrows has a unique factorization

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "n_f \in \mathcal{E}"'] & & C \\ & B_f \arrow[ur, "u_f \in \mathcal{M}"'] & \end{tikzcd}$$

Ortogonality structure

A pair $(\mathcal{E}, \mathcal{M})$ s.t. with diagonalization property:

$$\begin{tikzcd} B \arrow[r] \arrow[d, "n \in \mathcal{E}"'] & A \arrow[d, "u \in \mathcal{M}"] \\ B' \arrow[r] \arrow[ur, dashed, "\exists !"] & B \end{tikzcd}$$

General properties of a factorization system $(\mathcal{E}, \mathcal{M})$

- $\mathcal{E}$ contains iso
- is **left**-cancellable, hence closed by retracts,
- closed by colimits

- $\mathcal{M}$ contains iso
- is **right**-cancellable, hence closed by sections,
- closed by limits

Saturated class

Saturated classes

A saturated class is a $\mathcal{V} \subseteq \overrightarrow{\mathcal{B}_{fp}}$ closed by:

- composition
- pushouts along f.p. arrows
- left cancellation

$$\begin{tikzcd} K \arrow[r, "f\in\mathcal{V}"] \arrow[d, "g\in\mathcal{V}"'] & K' \\ K'' \arrow[ur, "\text{has to be in } \mathcal{V}"'] & \end{tikzcd}$$

In a L.F.P. category

- Orthogonality and factorization systems coincide
- Left generated = determined by $\mathcal{E} \cap \mathcal{B}_{fp}$
- Any saturated class left generates a factorization system

$$\mathcal{V} \mapsto (Ind(\mathcal{V}), \mathcal{V}^{\perp})$$

- Left generated factorization system $\simeq$ saturated classes

Syntactic aspects: etale and local arrows (Coste)

Syntactic interpretation

- Arrows in $\mathcal{V}$ "create witnesses of codomain formulas from witnesses of domain formula"
- Local arrows "reflect witnesses of codomain formulas"

$$\left\{\begin{array}{l} \forall f : K_\phi \to K_\psi \in \mathcal{V} \\ \forall \overline{a} \in A \text{ s.t. } A \models \phi_f(\overline{a}) \\ \forall \overline{b} \in B \text{ s.t. } B \models \psi_f(\overline{b}) \wedge \theta_f(\overline{g(a)}, \overline{b}) \end{array}\right. \Rightarrow \exists! \overline{c} \in A \left\{\begin{array}{l} A \models \psi_f(\overline{c}) \\ A \models \theta_f(\overline{a}, \overline{c}) \\ \overline{g(c)} = \overline{b} \end{array}\right.$$

$$\begin{tikzcd} K_\phi \arrow[r, "\ulcorner a \urcorner"] \arrow[d, "f"'] & A \arrow[d, "g"] \\ K_\psi \arrow[r, "\ulcorner b \urcorner"'] \arrow[ur, dashed, "\exists! \ulcorner c \urcorner"] & B \end{tikzcd}$$

For Grothendieck duality

Etale arrows = localizations: create invertible from nonzero
Local arrows = conservative morphisms: reflect invertibility

Etale and local arrows, admissibility

Geometry (also admissibility structure or Nisnevich context)

A geometry for $\mathcal{B}$ will be determined by a pair $(\mathcal{V}, J)$ with:

- a $\mathcal{V}$ saturated class determining $(\mathcal{E}t_{\mathcal{V}}, \mathcal{L}oc_{\mathcal{V}})$
- a coverage J on $\mathcal{B}^{op}_{fp}$ **with basic covers in $\mathcal{V}$** (encodding the theory of local objects)

Etale arrows: dual of **open inclusions of the geometry**
$\rightarrow$ will constitute the topological part of spectrum
Local forms in $(\mathcal{V}, J)$: etale arrows toward J-local objects
$\rightarrow$ **points of the geometry**

Topological intuition

Etale arrows approximate local forms by filtered colimits
As like as open neighborhood approximate points
Local arrows: residual, non-topological information
Factorization: separate topological from residual data

Topology on $\mathcal{B}^{op}$

Induced topology

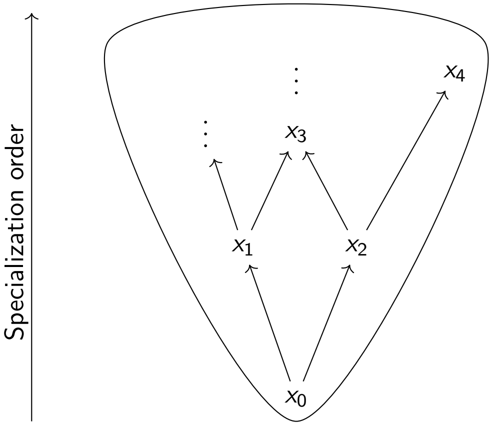
$(\mathcal{V}, J)$ induces a topology on $\mathcal{B}^{op}$
Can transfer J covers under arbitrary objects by pushouts
Define $\widetilde{J}$ whose covers are dual cocones of

$$(B \xrightarrow{f_i} B_i)_{i \in I} \text{ s.t. } \begin{tikzcd} B \arrow[d, "f_i"'] & K \arrow[l] \arrow[d, "k_i \in \mathcal{V}"] \\ B_i \arrow[ur, phantom, "\urcorner", very near start] & K_i \arrow[l] \end{tikzcd} \text{ with } (K \xrightarrow{k_i} K_i) \in J^{op}$$

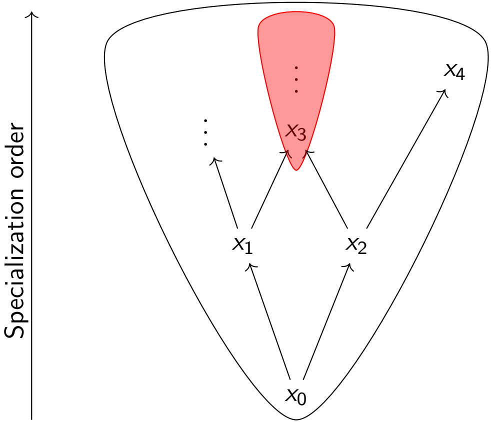
Local objects are $\widetilde{J}$-irreducible $\rightarrow$ lift their own covers

$$\begin{tikzcd} & A \arrow[dl, "f_i"'] \arrow[dr, "f_j"] \arrow[rr, equal] & & A \\ B_i & \dots & B_j \arrow[ur, dashed, "\exists \text{ for some } j"'] & \end{tikzcd}$$

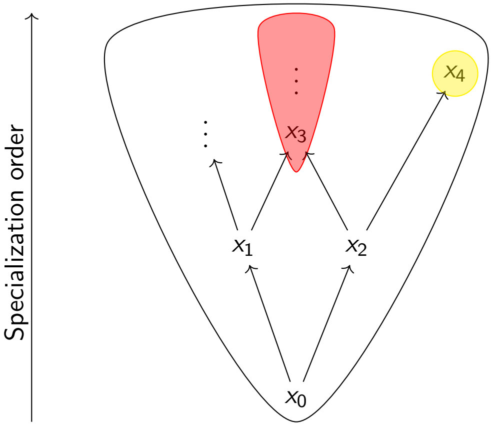
Local objects as focal spaces



Local objects as focal spaces



Local objects as focal spaces



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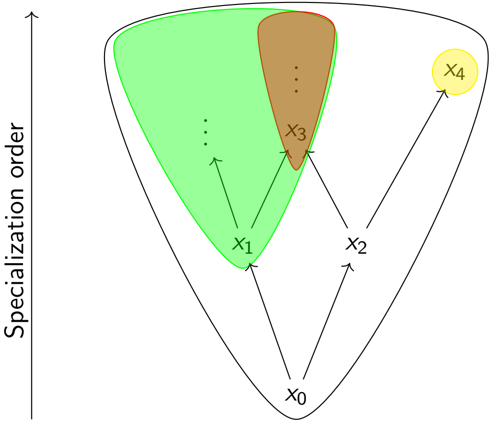

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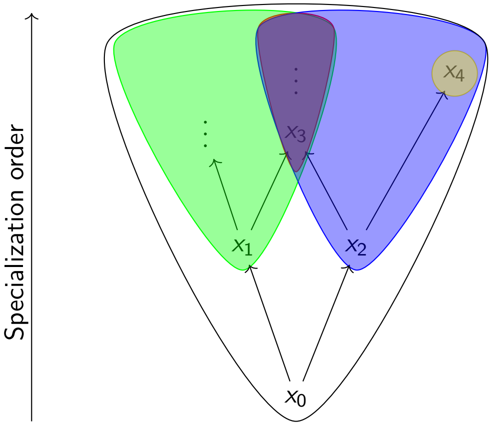

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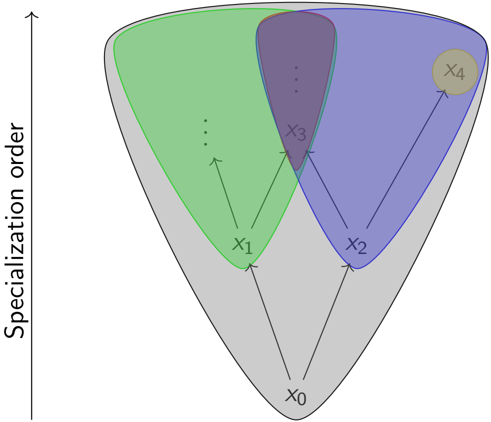


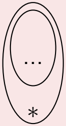
Topological interpretation (Anel)

Through the looking glass

In $\mathcal{B}$ (algebraic side)	$\leftrightarrow$	In $\mathcal{B}^{op}$ (spatial side)
Etale arrow $B \xrightarrow{l} C$	$\leftrightarrow$	Etale open inclusion $C \xrightarrow{l^{op}} B$
Local objects	$\leftrightarrow$	Focal spaces
$\rightarrow$ Lift their own cover:		$\rightarrow$ Have a minimal point:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "l_i"'] & A \\ B_i \arrow[ur, dashed, "\exists r"'] & \end{tikzcd}$$

$\leftrightarrow$



Cone of local units

$$\begin{tikzcd} & B \arrow[dl, "\eta_i"'] \arrow[dr, "\eta_j"] & \\ A_i & \dots & A_j \end{tikzcd}$$

$\leftrightarrow$

Cocone of focal components

$$\begin{tikzcd} & \text{“}Spec(B)\text{”} & \\ \uparrow_{\sqsubseteq} \{x_i\} \arrow[ur, hook] & \dots & \uparrow_{\sqsubseteq} \{x_j\} \arrow[ul, hook'] \end{tikzcd}$$

Local and multi right adjoints

Local right adjoint (cf Diers theory of spectrum)

Let $U : \mathcal{A} \to \mathcal{B}$ a functor:

- U local RAdj if each slice is RAdj: $\begin{tikzcd} \mathcal{A}/_A \arrow[r, bend right, "U/_A"'] \arrow[r, phantom, "\perp"] & \mathcal{B}/_{U(A)} \arrow[l, bend right, "L_A"'] \end{tikzcd}$
- U is multi-RAdj if any B in $\mathcal{B}$ has a small cone of local units

$$(B \xrightarrow{\eta_i} U(A_i))_{i \in I_B}$$

initial in the comma $B \downarrow U$

A Multi-Radj is a stable functor with a solution set

Multireflection

(Non-full) faithful multi RAdj are (non-full) multireflections.

Multireflection induced by admissibility

Admissibility is encoded by the situation of multireflectivity

"Glidding property"

Local objects are downclosed for local maps:
if $u : A \to L$ a local map with L local, then A is local

Multireflection associated to a geometry

Any map form toward a local object factorizes through a local form:

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "n_f"'] & & A \\ & \underbrace{A_f}_{J-local} \arrow[ur, "u_f \in \mathcal{L}oc"'] & \end{tikzcd}$$

$\mathbb{T}_J[\mathcal{S}et]^{\mathcal{L}oc} \hookrightarrow \mathcal{B}$ is multireflective.

Local units correspond to local forms = points

Multireflection and admissibility

Conversely: multiadjunctions produce admissibility.
Defect of uniqueness of the unit
Universal property of reflection **jointly** assumed by the universal cone.

$$\begin{tikzcd} & B \arrow[dl, "\eta_i"'] \arrow[dr, "\eta_j"] \arrow[rr, "f"] & & A \\ A_i & \dots & A_j \arrow[ur, dashed, "\exists! \text{ for some } j"'] & \end{tikzcd}$$

Taking as local maps the right class generated by $\overrightarrow{\mathcal{A}}$:
Multireflectivity says that one of the factorization is admissible
Initial amongst those with a local arrow on the right

Admissibility in arbitrary toposes

Admissibility is inherited in any arbitrary topos $\mathcal{E}$

- In any topos $\mathcal{E}$, $\mathbb{T}[\mathcal{E}]$ inherits a factorization system $(\mathcal{E}t_{\mathcal{E}}, \mathcal{L}oc_{\mathcal{E}})$
- Local objects in $\mathcal{E}$ are "absorbant right to local maps"
- The inclusion $\mathbb{T}_J[\mathcal{E}]^{\mathcal{L}oc} \hookrightarrow \mathbb{T}[\mathcal{E}]$ is multireflective
- In any topos $\mathcal{E}$, a retract of a local object is local

(Locally) modelled topose

Œcumene for $\mathbb{T}$-models

$\mathbb{T}_{\mathcal{B}}\mathcal{T}opos$: $\mathbb{T}_{\mathcal{B}}$-modeled toposes

- Obj: $(\mathcal{E}, E)$ with E in $\mathbb{T}[\mathcal{E}]$
- Arr: $(f, f^\sharp) : (\mathcal{E}, E) \to (\mathcal{F}, F)$ with: $\begin{cases} \mathcal{F} \xrightarrow{f} \mathcal{E} \text{ geom.} \\ \mathrm{f}^* E \xrightarrow{f^\sharp} F \quad \mathbb{T}\text{-morph.} \end{cases}$

$\mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos$: $\mathbb{T}_J$-locally modelled toposes:

- Obj: $(\mathcal{E}, E)$ with E in $\mathbb{T}_J[\mathcal{E}] \Rightarrow$ each E_x local, $x \in pt(\mathcal{E})$
- Arr: $(f, f^\sharp)$ with $f^\sharp$ in $\mathbb{T}_J$ transformation

$$f_x^\sharp : E_{fx} \to F_x \text{ a local arrow in } \mathbb{T}_J[Set]$$

Turning admissibility into reflection

The fundamental adjunction

One wants to construct a left adjoint *Spec* to the inclusion

$$\begin{tikzcd} \mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos \arrow[r, bend right, hook, "w"'] \arrow[r, phantom, "\perp"] & \mathbb{T}_{\mathcal{B}}\mathcal{T}opos \arrow[l, bend right, "Spec_{\mathcal{V},J}"'] \end{tikzcd}$$

Consider models jointly, regardless of their base topos

Then admissibility turns into proper reflection

One can construct a free local object under a given $\mathbb{T}$-model

If allowed to change of topos

For models in $\mathcal{S}et$

Adjunction for models in $\mathcal{S}et$

In particular if restricting to models over *Set*:

$$\begin{tikzcd} \mathcal{B}^{op} \arrow[r, bend right, "Spec_{\mathcal{V},J}"'] \arrow[r, phantom, "\perp"] & \mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos \arrow[l, bend right, "\Gamma"'] \end{tikzcd}$$

Here Γ applies the direct image part of

$$! : \mathcal{F} \to \mathcal{S}et$$

to the structure sheaf F

Coste's spectrum of a $\mathcal{S}et$-valued model

Spectral site of $B \in \mathcal{B}$

$$\mathcal{V}_B = \left\{ l : B \rightarrow B_l \mid \begin{tikzcd} B \arrow[d, "l"'] & K \arrow[l, "f"'] \arrow[d, "k"] \\ C \arrow[ur, phantom, "\lrcorner" very near start] & K' \arrow[l, "k_* f"] \end{tikzcd} \quad \begin{array}{l} \text{for some } k \in \mathcal{V} \\ \text{and } f : K \rightarrow B \end{array} \right\}$$

$$\underbrace{J_B(l)}_{on\, \mathcal{V}_B^{op}} = \left\{ \left(\begin{tikzcd} B \arrow[r, "l"] \arrow[dr, "n_i"'] & B_l \arrow[d, "m_i"] \\ & B_{n_i} \end{tikzcd} \right)_{i \in I} \mid \begin{tikzcd} B_l \arrow[d, "m_i"'] & K \arrow[l, "u"'] \arrow[d, "k_i"] \\ B_{n_i} \arrow[ur, phantom, "\lrcorner" very near start] & K_i \arrow[l] \end{tikzcd} \right\}$$

Gathers etale arrows under B with relative topology

One can prove that $\mathcal{V}_B$ is closed under finite colimits
$\rightarrow$ hence $(\mathcal{V}_B^{op}, J_B)$ is a lex site.
Objects of $\mathcal{V}_B$ are the finitely presented etale maps under B

Coste's spectrum of a $\mathcal{S}et$-valued model

Spectrum of $B \in \mathcal{B}$

$$Spec_{\mathcal{V},J}(B) = \mathcal{S}h(\mathcal{V}_B^{op}, J_B)$$

$\mathcal{V}_B^{op}$ is a Lex site coding for "basic compact open inclusions"

Etale arrows and spectrum

Etale arrow $l : B \to C$ correspond to etale geometric morphisms

$$Spec(l) : Spec(C) \simeq Spec(B)/a_{J_B}(よ_l) \to Spec(B)$$

Fp etale arrows are stable under pushouts
$\to$ l induces a lex morphism of site $\mathcal{V}_l^{op} : \mathcal{V}_B^{op} \to \mathcal{V}_C^{op}$

$$\begin{tikzcd} B \arrow[r, "l"] \arrow[d, "n"'] & C \arrow[d, "l_* n"] & fp \\ B_n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & l_* B_n & \end{tikzcd}$$

Opens and saturated compacts

Topological interpretation

- Objects of $\mathcal{V}_B$ should be seen as a basis of **compacts opens**
- One has $\mathcal{V}_B^{op} \to \mathcal{S}h(\mathcal{V}_B^{op}, J_B)$
  ($\to$ in particular this is ff when J_D is subcanonical)
- When $\mathcal{V}_B^{op}$ is a poset, object of $\mathcal{S}h(\mathcal{V}_B^{op}, J_B)$ are **opens**
- One has $\mathcal{V}_B \hookrightarrow Ind(\mathcal{V}_B) =$ arbitrary étale maps under B
- When $\mathcal{V}_B$ is a poset, they define **saturated compact** of $Spec(B)$
- In particular, points are saturated compacts.

Points of the spectrum

Points

Points of spectral site of B coincide with local forms under B

$$pt(Spec_{\mathcal{V},J}(B)) \simeq \mathcal{L}ex\mathcal{S}ite[(\mathcal{V}_B^{op}, J_B), \mathcal{S}et]$$

In particular points are Ind-object of $\mathcal{V}_B$
Send J_B-covers on jointly epic families

$$\underset{i \in I}{colim}\; p(B \xrightarrow{n_i} B_{n_i}) \xrightarrow{\langle p(A_l \xrightarrow{m_i} B_{n_i}) \rangle_{i \in I}} p(B \xrightarrow{l} B_l)$$

By Yoneda lemma:

$$\begin{aligned} p(B \xrightarrow{l} B_l) &\simeq Nat[\text{よ}_l, p] \\ &= \{u : B_l \to P \mid \; ul = p\} \end{aligned}$$

$$\begin{tikzcd} & B \arrow[d, "l"] \arrow[ddl, bend right, "n_i"'] \arrow[ddr, bend left, "p"] & \\ & B_l \arrow[dl, "m_i"'] \arrow[dr, "u"] & \\ B_{n_i} \arrow[rr, dashed, "\exists"'] & & A_p \end{tikzcd}$$

$\rightarrow$ Hence A_p is a local object

Points of the spectrum

Points

- Points of spectral site of B are local forms under B
- If $B \xrightarrow{I} C$ etale, any point of $Spec(C)$ is a point of $Spec(B)$
- If A $\mathcal{S}$et-valued local, $Spec(A)$ local topos

At the level of points, etale maps $I : B \rightarrow C$ produces discrete opfibrations

$$pt(Spec(C)) \simeq pt(Spec(B)/\text{よ}_I) \rightarrow pt(Spec(B))$$

In particular when the spectrum is spatial this reduces to an open inclusion (as any I is subterminal in $Spec(B)$)

Structural sheaf of $\mathcal{S}et$-valued model

Structural sheaf of B in $\mathcal{B}$

- $\widetilde{B}$ is a distinguished sheaf of $\mathcal{B}$-objects in $Spec(B)$:

$$\widetilde{B} = a_{J_B}((B \xrightarrow{l} C) \longmapsto C)$$

  Sheafification of the Codomain functor
- At stalks: $\widetilde{B}$ returns local objects under B
  Hence $\widetilde{B}$ is a $\mathbb{T}_J$-model in $Spec(B)$
  $\rightarrow$ This is **the free local object under B**

$\widetilde{B}$ gathers local forms of B as its stalks.

Sheaf representability

One has a sheaf-representation theorem iff J_B is subcanonical
Then $\Gamma\widetilde{B} = \widetilde{B}(1_B) \simeq B$ and the codomain functor $\mathcal{V}_B \rightarrow \mathcal{B}$ already is a sheaf.

Spectrum of a model in an arbitrary topos

A $\mathbb{T}$ model in $\mathcal{E} = \mathcal{S}h(\mathcal{C}, J)$ is a sheaf of $\mathcal{B}$-objects over $(\mathcal{C}, J)$ (because $\mathcal{B}$ is LFP)
One wants to construct $(Spec(E), \widetilde{E})$ so that $\widetilde{E}$ is a $\mathbb{T}_J$ model.
In particular it will return local objects at stalks.

Spectrum as a classifiant of stack

Construct the indexed site

$$\begin{array}{ccc} \mathcal{C}^{op} & \overset{\mathcal{V}^{op}_{E(-)}}{\rightarrow} & \mathcal{L}ex \\ c & \mapsto & \mathcal{V}^{op}_{E(c)} \\ c_1 \overset{u}{\rightarrow} c_2 & \mapsto & \mathcal{V}^{op}_{E(c_2)} \overset{(u_*)^{op}}{\rightarrow} \mathcal{V}^{op}_{E(c_1)} \end{array}$$

This is a *lex stack*, induces a fibration of sites $\int \mathcal{V}^{op}_{E(-)} \rightarrow (\mathcal{C}, J)$.
Then $Spec(E)$ is the classifying topos of this lex stack in the sense of Giraud
$\widetilde{E}$ is the sheafification of the codomain functor $\int \mathcal{V}^{op}_{E(-)} \rightarrow \mathcal{B}$

Zariski Geometry

Stable inclusion for Stone

Define the category $\mathcal{LocDLat}^{1-cons}$ having:

- Obj: local DLat, where $\{1\}$ is prime filter
- Mor: 1-conservative morphisms f s.t. $f^{-1}(\{1\}) = \{1\}$

Then $\mathcal{LocDLat}^{1-cons} \hookrightarrow \mathcal{DLat}$ is a multireflection

Zariski Geometry

Etale maps = *1-minimal quotients* $A \twoheadrightarrow A/\theta$ with θ minimal amongst congruences whose class in 1 is $[1]_\theta$.
One has a factorization system $(MinQuo_1, 1 - Cons)$ on $\mathcal{DLat}$.
Define J_1 on $\mathcal{DLat}_{fp}^{op}$ generated by $(f_i : D \twoheadrightarrow D/\theta_i)$ such that $\bigcap \theta_i = diag_D$.
Now observe that a DLat D is $\mathcal{J}_1$-local if an only if $\{1\}$ is a prime ideal
$\rightarrow$ D has a minimal point $L \rightarrow 2$ sending any $a \neq 1$ on 1.
Local lattices are the points of the topos $\mathcal{S}h(\mathcal{DLat}_{fp}, J_1)$
For a lattice D fp-1-minimal quotient are of the form $D \rightarrow D/\theta_{(a,1)}$

Spectral Stone duality with Zariski

Spectral Stone duality

The associated spectrum for D is

$$(Spec(D) = (\mathcal{F}_D^{Prime}, \tau_D^{Zariski}), \widetilde{D})$$

with $\widetilde{D}$ defined on the basis as $\widetilde{D}(U_a^{coZar}) = D/\theta_{(a,0)}$ for any $a \in D$

$$\begin{tikzcd}
\mathcal{DL}at^{op} \arrow[rr, bend right, "Spec^{Zar}"'] \arrow[rr, phantom, "\perp"] & & \mathcal{DL}at^{*} - \mathcal{S}paces \arrow[ll, bend right, "\Gamma"']
\end{tikzcd}$$

Stone spaces are the underlying spaces of affine $\mathcal{DL}at$-spaces.

Zariski site of D

- The spectral site of a DLat D is $(Zar_D^{op}, J_1(D))$ where Zar_D consists of fp-1-minimal quotients of D : $D \twoheadrightarrow D/\theta_{(a,1)}$, and arrows

$$\begin{tikzcd} D \arrow[r, two heads, "q_F"] \arrow[d, two heads, "q_a"'] & D/\theta_F \\ D/\theta_{(a,1)} \arrow[ur, dashed, two heads, "a \in F"'] & \end{tikzcd}$$

  $J_1(D)$ consists of finite families $(D \twoheadrightarrow D/\theta_{(a_i,1)})_{i\in I}$ with $\bigvee a_i = 1$
- Being made of epi, Zar_D is a poset and $Zar_D \simeq D^{op}$
  $J_1(D)$ coincides with the coherent topology on D.
  The spectrum is spatial and is equiped with the Zariski topology which is the frame of filters $\mathcal{F}_D$

Zariski Geometry

- One has $D \simeq Zar_D^{op}$ Opens of Zariski topology form the frame $\tau_{Zar} = Sh(Zar_D^{op}, J_1(D)) = I_D$: Zariski opens corresponds to ideals of D and $D \hookrightarrow I_D$ is a base of compact open of Zariski topology.
- On the other side, $D \hookrightarrow (\mathcal{F}_D)^{op}$, but a filter F of D just define a filtered diagram whose colimit is the 1-minimal quotient at F

$$S \twoheadrightarrow S/\theta_F^{min} = \underset{a \in F}{colimfilt}\ S/\theta(a,1)$$

  Those filters are *saturated compact* of Zariski topology.
- A prime filter x corresponds to the 1-quotient $D \twoheadrightarrow D/\theta_x =$ the saturated compact in x.

CoZariski Geometry

One could have either defined the factorization system $(0 - minQuo, 0 - cons)$ and taken a local object DLat with $\{0\}$ prime. The CoZariski site would have been $(coZar_D^{op}, J_0(D))$ with:

- $coZar_D$ made of the $D \twoheadrightarrow D/\theta_{(a,0)}$
- and $J_0(D)$ defined by $(a_i)_{i \in I}$ such that $\bigwedge a_i = 0$

Then $D \simeq CoZar_D$, so that
$D^{op} \hookrightarrow \tau_{coZar} = Sh(coZar_D^{op}, J_0(D)) \simeq (\mathcal{F}_D)^{op} \simeq I_{D^{op}}$
$\rightarrow$ filters are closed of coZariski topology.
On their side ideals I_D defines filtered colimits of fp-etale maps in $coZar_D$, hence correspond to saturated compacts.

Hochster duality is Isbell for DLat

Actually coZariski is the Hochster dual of Zariski
This is an instance of Lawson duality for stably compact spaces

$$\begin{tikzcd}
 & D \arrow[dl] \arrow[dr] & \\
{\begin{array}{c} I_D = K_{coZar}(D) \\ = \tau_{zar} = Sh(Zar_D^{op}, J_1(D)) \end{array}} \arrow[rr, bend right] \arrow[d] & & {\begin{array}{c} (F_D)^{op} = I_{D^{op}} = \tau_{cozar}^{op} \\ = K_{zar}(D) = Sh(coZar_D^{op}, J_0(D))^{op} \end{array}} \arrow[ll, bend right] \arrow[d] \\
{[D^{op}, 2]} \arrow[rr, bend right] & & {[D, 2]^{op}} \arrow[ll, bend right]
\end{tikzcd}$$

Observe that analogy with Isbell.

HMS spaces

In a space X with specialization order $\sqsubseteq$, a compact open filter is an upset F for $\sqsubseteq$ which is both open and compact.
For X denote $\mathcal{KOF}(X)$ its set of compact open filters.
A point x is basic compact open if $\uparrow$ is a compact open filter.

HMS spaces

Hoffman-Mislove-Stralka spaces are sober spaces X such that $\mathcal{KOF}(X)$ is a basis closed under finite intersection.
Denote $\mathcal{HMS}$ the category of HMS spaces with continuous maps $f : X \to Y$ such that f^{-1} restrict to $\mathcal{KOF}(Y) \to \mathcal{KOF}(X)$.

Any compact open filter of a HMS space has a focal point.
In a HMS any point is a directed join of basic compact open points
The specialization order makes $(X, \sqsubseteq)$ a complete lattice
There are both an initial and terminal points in such a X

Jipsen-Moshier duality

Jipsen-Moshier duality

HMS spaces are dual to $\wedge$-semilattices with unit

$$\wedge - \mathcal{SL}at_1^{op} \simeq \mathcal{HMS}$$

- Defines $Spec(S) = (\mathcal{F}_S, \downarrow S)$ (equivalently, with Scott-topology).
- For X HMS, $\mathcal{KOF}(X)$ is a $\wedge$-slat

Then $S \simeq \mathcal{KOF}(Spec(S))$ and $X = Spec(\mathcal{KOF}(X))$

If S is a $\wedge$-slat, $\mathcal{F}_S \simeq (\mathcal{I}_S^{prime})^{op}$ is a complete lattice.
Any filter of a $\wedge$-semilattice is trivially prime.
This just says that $Spec(S) = \wedge - Slat[S, 2]$

Admissibility structure for J-M

- $(MinQuo,\ 1 - Cons)$ also is a factorization system on $\wedge - \mathcal{SL}at$
- FP-etale maps under a $\wedge$-slat just are principal minimal quotient

$$S \twoheadrightarrow S/\theta(a, 1)$$

  and they always define a basic compact open point
- For a filter F one has a minimal quotient

$$S \twoheadrightarrow S/\theta_F^{min} = \underset{a \in F}{colimfilt}\ S/\theta(a, 1)$$

  (θ_F^{min} is the congruence in F given as $\theta_F^{min} = \bigcap\{\theta \mid [1]_\theta = F\}$)
  This defines a point of $Spec(S)$, and any saturated compact actually has a focal point.

Jipsen-Moshier and Gabriel-Ulmer

$Gabriel - Ulmer$	$Jipsen - Moshier$
$\underline{\mathcal{L}ex^{op}} \quad \simeq_{eq} \quad \underline{\mathcal{LFP}}$ $\mathcal{C} \mapsto \mathcal{L}ex[\mathcal{C}, \mathcal{S}et]$ $\mathcal{C} \xrightarrow{F} \mathcal{D} \mapsto \mathcal{L}ex[\mathcal{F}, \mathcal{S}et]$ $(\mathcal{A}_{fp})^{op} \leftarrow\!\shortmid \mathcal{A}$ $G^* \mid_{fp}^{op} \leftarrow\!\shortmid \mathcal{A} \overset{G^*}{\leftrightarrows} \mathcal{B}$ G_* finitary	$\underline{\bigwedge - \mathcal{SL}at_1{}^{op}} \quad \simeq \quad \mathcal{HMS}$ $L \mapsto \mathcal{F}_L$ $f : L \to M \mapsto f^{-1}$ $\mathcal{KOF}_X \leftarrow\!\shortmid X$ $h^{-1} \mid_{\mathcal{KOF}_Y} \leftarrow\!\shortmid X \xrightarrow{h} Y$
$\mathcal{L}ex[\mathcal{C}, \mathcal{S}et] \simeq \mathcal{C} - Mod_{\mathcal{S}et}$ $F \mapsto \int F$ $F_M \leftarrow\!\shortmid M = (M_c)_{c \in \mathcal{C}}$	$\bigwedge - \mathcal{SL}at_1[L, 2] \simeq \mathcal{F}_L$ $f \mapsto f^{-1}(1) \simeq \int f$ $\chi_F \leftarrow\!\shortmid F$
$\mathcal{LFP}$ categories are complete and cocomplete	$X \in \mathcal{HMS} \Rightarrow (X, \sqsubseteq) \in \mathcal{CL}at$
$K \in \mathcal{A}_{fp} \Leftrightarrow \mathcal{A}[K, -]$is finitary : $\forall f : K \to colim^{\uparrow} X_i,$ $\exists i, g : K \to X_i,\ f : q_i \circ g$	$\uparrow_{\sqsubseteq} x \in \mathcal{KOF}_X \Leftrightarrow \uparrow_{\sqsubseteq} x$ open so $x \sqsubseteq \bigsqcup^{\uparrow} x_i \Leftrightarrow \bigsqcap_{\downarrow} \uparrow_{\sqsubseteq} x_i \subseteq \uparrow_{\sqsubseteq} x$ $\Rightarrow \exists i\ x \sqsubseteq x_i$ because HMS spaces are well filtered
$\mathcal{A}_{fp} \downarrow X$ is filtered	$\uparrow_{\mathcal{KOF}_X} F$ is directed
$\mathcal{A}_{fp} \downarrow X, X \downarrow \mathcal{A}_{fp}$ are LFP	$\uparrow_{\sqsubseteq} x, \downarrow_{\sqsubseteq} x$ are HMS

Semantics as a 2-categorical geometry ?

Example of correspondences

- Jipsen-Moshier
  $\wedge - \mathcal{SL}at_1^{op} \simeq \mathcal{HMS}$
- ???
- Stone
  $\mathcal{DL}at^{op} \simeq \mathcal{S}tone$
- Esakia
  $\mathcal{H}eyt^{op} \simeq \mathcal{E}sa$
- Duality for frames
- Gabriel-Ulmer
  $\mathcal{L}ex^{op} \simeq \mathcal{LFP}$
- Kuber-Rosický
  for Reg/Ex
- Awodey-Forsell, Makkai
  for coherent theories
- Duality for Heyting categories
  ?
- Di Liberti categorified Isbell adjunction

Using a 2-spectrum of models to get categories of models for finite-limit, regular, coherent, geometric theories and recover their classifying topos ?

Topological interpretation

When $\mathbb{T}$ is some geometric theory classified by (subcanonical) site (C, J), then any object ϕ in C defines a discrete opfibration $\mathbb{T}[\mathcal{S}et]$

$$\int ev_\phi \to \mathbb{T}[\mathcal{S}et]$$

This is the discrete opfibration of the points of the etale geometric morphism

$$\mathcal{S}h(C, J)/ よ_\phi \to \mathcal{S}h(C, J)$$

This extend to arbitrary objects of $\mathcal{S}h(C, J)$

Arrows as specialization order

Here the lifts in the opfibration just says that (basic) opens are up-closed for the specialization order :

$$(X, a) \xrightarrow{f} (X', ev_\phi(f)(a))$$

$$X \xrightarrow{\quad f \quad} X'$$

$\to$ a witness a that X is in ϕ is sent to a witness that X' is in ϕ

Topology of doctrines

Classifying sites and points for different doctrines

- A finite limit $\mathbb{T}$ is classified by some $\widehat{L}$ with L lex (no coverage) and models are $pt(\widehat{L}) = Lex[L, \mathcal{S}et]$
- A regular $\mathbb{T}$ is classified by some $\mathcal{S}h(C, J_{reg})$ with J_{reg} generated by single regular epi $C \twoheadrightarrow D$ and models are $pt(\mathcal{S}h(C, J_{reg})) = Reg[C, \mathcal{S}et]$ (lex functor preserving regular epi)
- A coherent $\mathbb{T}$ is classified by some $\mathcal{S}h(C, J_{coh})$ with J_{coh} generated by finite jointly regular epic families, and models are $pt(\mathcal{S}h(C, J_{coh})) = Coh[C, \mathcal{S}et]$ or equivalently lex functors preserving regular epi and finite coproducts.

All those sites are subcanonical:
$\rightarrow$ any ϕ of C can be seen as a open through the representable $よ_\phi$.
Now models can be seen as points (they are point of the classifying topos)
Sheaves are discrete fibrations on the categories of points.

Existence of focal points

Now one could ask when objects of the syntactic site also define *points*
That is, when is the *co*representable $よ^*_\phi = C[c, -]$ a point.
Whenever it is, it defines an fp-model K_ϕ = a *compact* point.

Compact points of theories in different doctrines

- In Lex, any corepresentable $よ^*_\phi = K_\phi$ is Lex
  $\rightarrow$ any compact open has a focal point, which is compact
- In Reg, $よ^*_\phi$ is regular when ϕ is projective in C:
  $\rightarrow$ only compact opens associated to a **projective** ϕ have a focal point.
  In particular there is an initial model when 1 is projective.
- In Coh, $よ^*_\phi$ is coherent if ϕ is *indecomposable* (connected projective):
  $\rightarrow$ only compact opens for a **indecomposable** ϕ have a focal point.
  In particular there is an initial model when 1 is indecomposable.

A theory $\mathbb{T}$ has an initial model when the terminal object 1 of C is J-local.
$\rightarrow$ Then the classifying topos is a **local topos**

Local toposes

Local topos

$\mathcal{E}$ is local if its global section functor Γ admits a ff RAdj

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\Gamma" description] & \mathcal{S}et \arrow[l, bend right=30, "\mathit{disc}\ \text{ff, lex}"'] \arrow[l, bend left=30, "\mathit{codisc}\ \text{ff}"] \end{tikzcd}$$

$=$ $\mathcal{E}$ possesses a minimal point $p_{\mathcal{E}} = (\Gamma \dashv \mathit{Codisc}) : \mathcal{S}et \to \mathcal{E}$ which is initial amongst points of $\mathcal{E}$

Example of local toposes

- $Sh(X)$ with X a focal space
- $Sh(C, J)$ when C lex and 1_C is J-irreductible
- $Spec(A)$ for a geometry where A is a local object

Grothendieck-Verdier localization

Recall that an etale geometric morphism is of the form $\mathcal{E}/X \to E$
For a geometric morphism $\mathcal{F} \to \mathcal{E}$ a factorization through an etale

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "f"] \arrow[dr, dashed, "1 \overset{x}{\to} f(X)"'] & & \mathcal{E} \\
& \mathcal{E}/X \arrow[ur] &
\end{tikzcd}$$

is the name of a global element of $f(X)$. In particular for a point $\mathcal{S}et \to \mathcal{E}$, it can be seen as a witness that X is a neighborhood of p.

Grothendieck-Verdier localization of $\mathcal{E}$ at a point p

$\to$ The local component of $\mathcal{E}$ at p
cofiltered limit over $\int p^*$ ($=$ compact open neighborhoods of F):

$$\mathcal{E}_p = \varprojlim_{(\phi, a) \in \int p^*} \mathcal{E}/\text{よ}_\phi \to \mathcal{E}$$

$\to$ *pro-etale* geometric morphism over $\mathcal{E}$

Grothendieck-Verdier localization

Can also be expressed as the bipullback (see Johnstone&Moerdijk)

$$\begin{tikzcd}
\mathcal{E}_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}^2 \arrow[d, "\partial_0"] \\
Set \arrow[r, "p"] & \mathcal{E}
\end{tikzcd}$$

where ∂_0 is the generic domain functor, which is local
$\rightarrow$ Produces a local topos
The category of points is the coslice:

$$pt(\mathcal{E}_p) \simeq p \uparrow pt(\mathcal{E}) \overset{\text{dopfib}}{\longrightarrow} pt(\mathcal{E})$$

Topological interpretation

$\rightarrow$ one forces p to become initial amongst points of $\mathcal{E}$
$\rightarrow$ select the up-set $\uparrow p$ in the specialization order
$\rightarrow$ constructed by intersection of basic compact open neighborhood

Grothendieck-Verdier localization in Lex

If $\mathbb{T}$ is a finite limit theory with L as syntactic site, then $\mathcal{S}[\mathbb{T}] = \widehat{L}$
Moreover at any object $\mathcal{S}[\mathbb{T}]/よ_{\phi} \simeq \widehat{L}/よ_{\phi} \simeq \widehat{L/\phi}$
Hence GV at finitely presented model K_ϕ trivializes into a basic etale geometric morphism:

$$\mathcal{S}[\mathbb{T}]_{K_\phi} \simeq \mathcal{S}[\mathbb{T}]/よ_{\phi} \to \mathcal{S}[\mathbb{T}]$$

Indeed points of this etale topos form the dopfibration $K_\phi \downarrow \mathbb{T}[\mathcal{S}et]$:

$$\begin{tikzcd}
 & \mathcal{S}et/A(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{L/\phi} \arrow[d] \\
\mathcal{S}et \arrow[ur, "x \in A(\phi)"] \arrow[r, Rightarrow] \arrow[urr, dotted] & \mathcal{S}et \arrow[r, "A"] & \widehat{L}
\end{tikzcd}$$

But a $x \in A(\phi)$ is the same as an arrow $K_\phi \to A$.
This tells us that $(K_\phi \downarrow \mathbb{T}[\mathcal{S}et])^{op}_{fp}$ is L/ϕ

$\to$ Basic compact opens of LFP-spaces have a focal points

Spectral construction for FO dualities

- Geometry on FO doctrines : Lex, Reg, Coh...
- $\rightarrow$ Factorization system on those doctrines ?
- Notion of local objects ? Localizing 2-topology ?
- One must construct a spectral 2-site for categories in those doctrines
- As the spectrum in Stone-like dualities happens to be localic, we expect the 2-spectral of FO theories to be 1-truncated
  $\rightarrow$ should coincide with their classifying topos

Locally presentable 2-categories (Bourke and Street)

2-dimensional analog of locally presentable categories
Surprisingly, one needs only 2-colimits over *1-filtered* diagrams

(finitely)-Accessible 2-category (Bourke)

A 2-category $\mathcal{C}$ with finitely-1-filtered 2-colimits with a small set of (finitely)-presented objects generating $\mathcal{C}$ under (finitely)-filtered 2-colimits.

When $\mathcal{C}$ has power with 2 commuting with filtered 2-colimits then 2-accessibility amounts to accessibility of the underlying 1-category.

Locally finitely presentable 2-category (Bourke)

A 2-category $\mathcal{C}$ wich is finitely 2-accessible, has *flexible* limits and where finite flexible limits commute with finitely-filtered 2-colimits.

(Flexible limits are those generated by PIES-limits
$\rightarrow$ avoid to care about pseudo or strictness of weight...)

Locally presentable 2-categories

"Models of finite PIE-theories"
A convenient context to generalize the spectral construction

Examples

First order doctrines $\mathcal{L}ex$, $\mathcal{R}eg$, $\mathcal{C}oh$... are locally finitely 2-presentable

See also Street theory of *computads*
Finitely presentedness hence just need to be tested relatively to ordinary 1-filtered diagrams
And object are constructed from 1-filtered diagrams of fp-objects.

2-site and Grothendieck 2-topos (Shulman and Street)

2-coverage (Shulman)

A 2-site is a 2-category with a 2-coverage, that is for each object U, a collection of families $(f_i : U_i \to U)_{i \in I}$ such that if $(f_i : U_i \to U)_i$ is a covering family and $g : V \to U$ is a morphism, then there exists a covering family $(h_j : V_j \to V)j$ such that each composite gh_j factors through some f_i, up to isomorphism

Any 2-cover $(f_i : U_i \to U)_{i \in I}$ generate a 2-dimensional nerve

$$\coprod_{i,j,k} f_i \downarrow f_j \downarrow f_k \substack{\to \\ \to \\ \to} \coprod_{i,j} f_i \downarrow f_j \rightrightarrows \coprod U_i \to U$$

Morally, a stack is a 2-functor $C^{op} \to Cat$ sending this diagram to a 2-limit diagram... (cf descent of stacks)
Street also defines a 2-Grothendieck 2-topology as an ordinary 1-topology on the underlying category
The canonical topology J_{can} is the topology of jointly eso families.
A 2-Topos is a category of stack $St(C, J)$ over some 2-site.

Examples

The 2-category of large categories $\mathcal{C}at$ is the terminal 2-topos. It is equiped with its canonical topology of jointly eso families

1-Truncated 2-topos

- Recall that a 1-topos $\mathcal{E}$ is localic (or (0,1)-truncated) if it is equivalent to a category of sheaves on a locale, $\mathcal{E} \simeq \mathcal{S}h(L)$
- This is equivalent to ask $\mathcal{E}$ to be generated under colimits by subterminal object $U \hookrightarrow 1$ (opens)
- In a 2-category $\mathcal{C}$, an object X is discrete if any 2-cell $\begin{tikzcd} Y \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & X \end{tikzcd}$ actually is an equality

Discrete opfibration are discrete

For instance discrete (op)fibrations over X are discrete in the 2-slices $\mathcal{C}/X$

- A 2-topos is 1-truncated if its is eso-generated by discrete objects, that is if for any X there is an eso morphism $\coprod D_i \twoheadrightarrow X$ with D_i discrete objects
- Equivalently, if it is of the form $St(X)$

Factorization system on 2-categories

2-orthogonality and factorization

In a 2-category, $n : A \to B$ is left orthogonal to $u : C \to D$ if

$$\begin{tikzcd} \mathcal{C}[B,C] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}[A,C] \arrow[d, "{\mathcal{C}[A,u]}"] \\ \mathcal{C}[B,D] \arrow[r, "{\mathcal{C}[n,D]}"'] & \mathcal{C}[A,D] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] \arrow[d, "n"'] & C \arrow[d, "u"] \\ B \arrow[r] \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] \arrow[ur, phantom, "\Downarrow !"] & D \end{tikzcd}$$

A pair (Et, Loc) with $Loc = Et^{\perp}$ is a factorization system if any map f there is a factorization $f = u \circ n$ in (Et, Loc) unique up to unique eq

---

$\to$ A 2-factorization system (Et, Loc) will be left generated if

$$Et = Ind(Et_{fp}) \quad Loc = (Et_{fp})^{\perp}$$

(localness can be tested relatively to finitely presented maps)

(cf : power with 2 and coma of LP 2-categories are LP 2-categories)

2-geometry

2-Geometry

Consider a locally presentable 2-category $\mathcal{C}$, equiped with:

- a factorization system (Et, Loc) which is **left generated**
- a 2-coverage J on $\mathcal{C}_{fp}^{op}$ generated in Et_{fp}

Local object

An object L in $\mathcal{C}$ is J-local if $\text{よ}_L : \mathcal{C}^{op} \to Cat$ sends J-covers $(C_i \to C)$ in $\mathcal{C}^{op}$ to jointly eso-families

$$\coprod \text{よ}_L(C_i) \overset{eso}{\twoheadrightarrow} \text{よ}_L(C)$$

This is equivalent to say that for any $C \to L$ and $C \to C_i$ in $\mathcal{C}$ there is an extension for some i

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr] & L \\ C_i & C_j \arrow[u, dashed] \end{tikzcd}$$

Observations

- Observe that the condition as stated does not implies that a 2-cell

$$\begin{tikzcd} C \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \text{ comes from a 2-cell } \begin{tikzcd} C_i \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \text{ for some } i.$$

  Either f_1 and f_2 extends through some (eventually distinct) members of the cover, but eso condition does not implies the 2-cell itself induces a 2-cell between them.
- Also a problem : does this condition captures all *2-points* of $Spec(B)$? One do not yet have a 2-dimensional Diaconescu theorem describing exactly how 2-points arise, nor a theorem ensuring the existence of a standard 2-site of definition with finite PIE-limits. However seems legit !
- When the 2-spectrum is actually truncated, its points form an honnest categories and are the points of its 1-truncation.

The spectral 2-site

Spectral 2-site

For C in $\mathcal{C}$ define the 2-category $\mathcal{V}_B$ as having

- 0-cells: fp-etale maps $n : C \to C'$
- 1-cells: equality 2-cells

$$\begin{tikzcd} C \arrow[r, "n_2"] \arrow[d, "n_1"'] & C_2 \\ C_1 \arrow[ur, "l"'] & \end{tikzcd}$$

- 2-cells :

$$\begin{tikzcd} C \arrow[r, "n_2"] \arrow[d, "n_1"'] & C_2 \\ C_1 \arrow[ur, bend left, "l_1"] \arrow[ur, bend right, "l_2"'] \arrow[ur, phantom, "\Downarrow\alpha"] & \end{tikzcd}$$

J_B is the topology on $\mathcal{V}_B$ induced from J by pushouts of J-cover.
Then $Spec(B) = St(\mathcal{V}_B^{op}, J_B)$ (category of stacks)

Factorization of 2-cells

How looks the factorization of a 2-cell $\begin{tikzcd} C \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$?

As $\mathcal{C}$ is LP, it has cotensor with 2 in each object $\begin{tikzcd} B^2 \arrow[r, bend left, "\partial_0"] \arrow[r, bend right, "\partial_1"'] \arrow[r, phantom, "\Downarrow\mu"] & B \end{tikzcd}$

such that for any 2-cell α as above one has a unique factorization

$$\begin{tikzcd} C \arrow[r, dashed, "\bar{\alpha}"] \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] \arrow[r, phantom, bend left, "\Downarrow\alpha"] & B^2 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Now there are 4 situations

1. both ∂_0, ∂_1 are local maps for each B
2. ∂_0 is a local map for each B
3. ∂_1 is a local map for each B
4. nor ∂_0, ∂_1 are local maps in general

Each of those situations produce a certain shape of spectral site

Connected geometry

Case 1 : both $\partial_0,\ \partial_1$ are local maps for each B
Then after factorization of $\bar{\alpha}$, $\partial_0 u_{\bar{\alpha}}$ and $\partial_1 u_{\bar{\alpha}}$ both are local, so by unicity of factorization one has $C_{\bar{\alpha}} = C_{f_1} = C_{f_2}$

$$
\begin{tikzcd}
 & C_{f_1} \arrow[d, equal] \arrow[rrd, bend left, "u_{f_1}"] & & \\
C \arrow[ru, "n_{f_1}"] \arrow[r, "n_{\bar{\alpha}}"] \arrow[rd, "n_{f_2}"'] & C_{\bar{\alpha}} \arrow[r, "u_{\bar{\alpha}}"] \arrow[d, equal] & B^2 \arrow[r, shift left, bend left] \arrow[r, shift right, bend right] \arrow[r, phantom, "\Downarrow"] & B \\
 & C_{f_2} \arrow[rru, bend right, "u_{f_2}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "n_\alpha"] & C_\alpha \arrow[r, bend left, "u_{f_1}"] \arrow[r, bend right, "u_{f_2}"'] \arrow[r, phantom, "u_\alpha^* \mu \Downarrow"] & B
\end{tikzcd}
$$

In this case, the local forms of an object C must be seen as connected components as they do not distinguish points that are connected by a morphism.

Focal geometry

Case 2 : ∂_0, is local maps for each B
Then after factorization of $\bar{\alpha}$, $\partial_0 u_{\bar{\alpha}}$ is local, so by unicity of factorization one has $C_{\bar{\alpha}} = C_{f_1}$; moreover, as $(n_{f_2}, \partial_1 u_{f_2})$ is a factorization of f_2 with an etale map on the left, one has an an etale intermediate arrow l_α

$$
\begin{tikzcd}
& C_{f_1} \arrow[d, equal] \arrow[drr, bend left, "u_{f_1}"] & & \\
C \arrow[ur, "n_{f_1}"] \arrow[r, "n_{\bar{\alpha}}"] \arrow[dr, "n_{f_2}"'] & C_{\bar{\alpha}} \arrow[r, "u_{\bar{\alpha}}"] \arrow[d, dashed, "l_\alpha"] & B^2 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \\
& C_{f_2} \arrow[urr, bend right, "u_{f_2}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "n_{f_1}"] \arrow[d, "n_{f_1}"'] & C_{f_2} \\
C_{f_1} \arrow[ur, "l_\alpha"'] &
\end{tikzcd}
$$

In this case, local form must be seen as focal components as an arrow α between points induces a map in the same direction in the site, one must see as dual of inclusion of the focal neighborhood.

Totally connected geometry

Case 3 : ∂_1, is local maps for each B
Then after factorization of $\bar{\alpha}$, $\partial_1 u_{\bar{\alpha}}$ is local, so by unicity of factorization one has $C_{\bar{\alpha}} = C_{f_2}$; moreover, as $(n_{f_1}, \partial_0 u_{f_1})$ is a factorization of f_1 with an etale map on the left, one has an an etale intermediate arrow l_α

$$
\begin{tikzcd}
 & C_{f_1} \arrow[drr, bend left, "u_{f_1}"] & & \\
C \arrow[ur, "n_{f_1}"] \arrow[r, "n_{\bar{\alpha}}"] \arrow[dr, "n_{f_2}"'] & C_{\bar{\alpha}} \arrow[u, dashed, "l_\alpha"] \arrow[r, "u_{\bar{\alpha}}"] \arrow[d, equal] & B^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow"] & B \\
 & C_{f_2} \arrow[urr, bend right, "u_{f_2}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "n_{f_1}"] \arrow[d, "n_{f_1}"'] & C_{f_2} \arrow[dl, "l_\alpha"] \\
C_{f_1} &
\end{tikzcd}
$$

In this case, local form must be seen as totally connected components as an arrow α between points induces a map in the same direction in the site, one must see as dual of inclusion of the closure of the correspoding points.

Factorization at a model in LEX

To determine the etale and local class first look at the factorization of a model in *LEX* (including large lex cat)

Let L be small lex, $\mathcal{A} = Lex[L, Set] = Ind(L^{op})$ the associated LFP
A model $A : L \to Set$ in *LEX* corresponds $\ulcorner A \urcorner : Set \to \mathcal{A}$ in LFP
Then GV localization provides the factorization

$$\begin{tikzcd} Set \arrow[rr, "\ulcorner A \urcorner"] \arrow[dr, "\ulcorner 1_A \urcorner"'] & & \mathcal{A} \\ & A \downarrow \mathcal{A} \arrow[ur, "\partial_1"'] & \end{tikzcd}$$

- On the left = name of the initial object of the coslice : *initial functor*
- On the right = codomain functor : *discrete opfibration*

Determining Etale and Local maps on Lex

Recall that *LFP* as PIES-limits as in cat, and in particular coma objects. For a general map between LFP take the *comprehensive factorization*

(Initial functor, Discrete opFibration)

of Street and Walter

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "F"] \arrow[dr, "1_{F(-)}"'] & & \mathcal{A} \\ & F \downarrow \mathcal{A} \arrow[ur, "\partial_1"'] & \end{tikzcd}$$

Where the intermediate LFP is the coma, $1_{F(-)}$ sends B in $\mathcal{B}$ to $1_F(B)$. Both $1_{F(-)}$ and ∂_1 are finitary and continuous as F is and by computation of limits in coslices and filtered colimit in arrow category.

Determining Etale and Local maps on Lex

Fp objects of $F \downarrow \mathcal{A}$ are $(k : K \to K' \in \mathcal{A}^2_{fp}, B \in \mathcal{B}, u : K \to F(B))$ codding for pushout of fp-map of $\mathcal{A}$ under objects $F(B)$ in the strict image of F

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & K' \arrow[d] \\ F(B) \arrow[r] & u_* K' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Then the left adoints of ∂_1 and $1_{F(-)}$ are

$$\begin{tikzcd} \mathcal{B}_{fp} & & \mathcal{A}_{fp} \arrow[ll, "F^*"'] \arrow[dl, "K \mapsto (0 \to K, 1 = F(1), !_1)"] \\ & (F \downarrow \mathcal{A})_{fp} \arrow[ul, "(k:K \to K', B, u) \mapsto F^*(K')"] & \end{tikzcd}$$

In their opposite categories, this provide the factorization in Lex by posing $L_F = (F \downarrow \mathcal{A})^{op}_{fp}$

Basic etale maps in Lex

Let ϕ in L corresponding to a finitely presented model K_ϕ
In Topos we had the GV localization at the basic point K_ϕ

$$\begin{tikzcd} \mathcal{S}et \arrow[rr, "K_\phi"] \arrow[dr, "1_{K_\phi}"'] & & \widehat{L} \\ & \widehat{L}/\text{よ}_\phi \arrow[ur, "Et_{K_\phi}"'] & \end{tikzcd}$$

But we saw that the site of $K_\phi \downarrow \mathcal{A}$ was just the slice L/ϕ

$$\begin{tikzcd} L \arrow[rr, "K_\phi"] \arrow[dr, "\phi^*"'] & & \mathcal{S}et \\ & L/\phi \arrow[ur, "1_{K_\phi}"'] & \end{tikzcd} \qquad \begin{tikzcd} Set \arrow[rr, "K_\phi"] \arrow[dr, "1_{K_\phi}"'] & & \mathcal{A} \\ & K_\phi \downarrow \mathcal{A} \arrow[ur, "\partial_1"'] & \end{tikzcd}$$

The basic etale map of L at ϕ is given by $\phi^* : L \to L/\phi$ sending $\psi \mapsto \psi \times \phi$ which is Lex.
The right part of the factorization is the name of the unit map 1_{K_ϕ}

Analogy with J-M

- In JM fp-etale maps were 1-minimal quotients forcing an element a to become 1
- Here we force an object ϕ to become terminal in the sense that 1_ϕ is the new terminal object of L/ϕ
- Observe also that here each object ϕ defines both a fp-point and a basic open
- As well as 1-minimal quotients were generated by principal quotient (which were in bijection with element of the semilattice), discrete opfibrations are generated by representable discrete opfibrations

Spectral site of a Lex category

Any LFP category has an initial model: hence any Lex theory is local. $\rightarrow$ The localizing topology is trivial.

Spectral site of L

The spectral site is $\mathcal{V}_L^{op}$ where $\mathcal{V}_L$ is the category whose

- objects are lex functors $L \rightarrow L/\phi$ with ϕ an object of L
- morphisms are triangles as below with $u : \phi_2 \rightarrow \phi_1$ in L

$$\begin{tikzcd} L \arrow[d, "\phi_1^*"'] \arrow[dr, "\phi_2^*"] & \\ L/\phi_1 \arrow[r, "u^*"'] & L/\phi_2 \end{tikzcd}$$

What about 2-cells ?

Spectral site of a Lex category

$$\begin{tikzcd}
L \arrow[rr, bend left, "A_1"] \arrow[rr, "A_2"'] \arrow[rr, phantom, bend left=20, "\Downarrow f"] \arrow[dr, "n_{A_1}"'] & & L' & \\
& L_{A_1} \arrow[ur, "u_{A_1}"] \arrow[ur, bend right, "f"'] \arrow[ur, phantom, "\Rightarrow !"] \arrow[rr, "n_f"'] & & \begin{array}{c}(L_{A_1})_u \\ \simeq L_{A_2}\end{array} \arrow[ul, "u_f"']
\end{tikzcd}$$

$$\begin{tikzcd}
L \arrow[d, "n_{A_1}"'] \arrow[dr, "n_{A_2}"] & \\
L_{A_1} \arrow[r, "L_u"'] & L_{A_2}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{S}et \\
L_{A_1} \arrow[ur, "u_{A_1}"] \arrow[r, "L_u"'] \arrow[r, phantom, shift left=3, "\overset{u_f}{\Rightarrow}"] & L_{A_2} \arrow[u, "u_{A_2}"']
\end{tikzcd}$$

$\rightarrow$ 2-cells are managed in the local part
We are in a local geometry

The spectrum coincides with the classifying topos

- As $Lex^{op} \simeq LFP$ and the ϕ^* are dual to discrete opfibrations over $\mathcal{A}$, which are in particular discrete objects in the slices $LFP/\mathcal{A}$. Hence the ϕ^* are discrete morphisms in Lex, making $\mathcal{V}_L^{op}$ a 1-site
- In fact $\mathcal{V}_L^{op} \simeq L$
- The spectrum is

$$Spec(L) = St[\mathcal{V}_L^{op}] \simeq St[L] = [L^{op}, Cat]$$

  But it is 1-truncated
- Discrete objects form an eso-generator of $[L^{op}, Cat]$, but

$$Disc([L^{op}, Cat]) = [L^{op}, Set] = \widehat{L}$$

  $\rightarrow$ recognize the classifying topos

Local regular category

A regular category is local when 1 is regular projective.
We just have to externalize the regular topology of C:
Define J_{Reg} on Reg_{fp} as generated by singletons

$$\{C \to C/\phi\} \text{ with } \phi \twoheadrightarrow 1 \text{ regular in C}$$

Then L is J_{Reg}-local if for any C fp, any regular $f : C \to L$ extends through

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\phi^*"'] & L \\ C/\phi \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

But $f(\phi) \twoheadrightarrow 1 = f(1)$ in L, and $C/\phi \to L$ is the name of some $1 \to f(\phi)$.
In particular, for $C = Lex[1]$ and f is a name of an object c in L, this says that 1 is projective.

Spectral Site of a regular category

Observe that pushing the topology J_{Reg} under C just externalizes the regular topology of C in $\mathcal{V}_C$

Spectral site

If C is small regular, define $\mathcal{V}_C$ as for lex
Covers of $J_{reg}(C)$ on $\mathcal{V}_C^{op}$ just are

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C/\phi_2 \\ C/\phi_1 \arrow[ur, "u"'] & \end{tikzcd}$$

where u is the name of a regular epi $\phi_2 \twoheadrightarrow \phi_1$

Again $Spec(C) = St(\mathcal{V}_C^{op}, J_{Reg}(C))$ is truncated
With truncation $\mathcal{S}h(C, J_{Reg})$, the classifying topos
An fp-local form is the name of a projective object
and a local form is the same as a regular functor $C \to Set$

Local coherent theory

Again we just process by externalization
J_{Coh} on Coh_{fp} is generated from finite families

$$(C \to C/\phi_i) \text{ with } \coprod_{i=1,\ldots,n} C_i \twoheadrightarrow \phi$$

Local object are L that lift through cover

$$\begin{tikzcd}
 & & C \arrow[dll] \arrow[dr] \arrow[rr, "f"] & & L \\
C/\phi_1 & & \ldots & C/\phi_n \arrow[ur, dashed, "1 \to f(\phi_n)"'] &
\end{tikzcd}$$

This exactly says that 1 is indecomposable in L
Then one can define J_{coh} on $\mathcal{V}_C^{op} \simeq C$ with coincide with the coherent topology on C and again the spectrum is truncated, equivalent to $\mathcal{S}h(C, J_{coh})$.

Externalization of syntactic coverage and self indexation

Structural stack of a lex category

The structural stack of L over $\mathcal{V}_L^{op} \simeq L$ is the self-indexation

$$L/(-) : L^{op} \to Lex$$

Structural stack for reg and coh

Also equivalent to the self-indexation

$$C/(-) : L^{op} \to Reg \quad C/(-) : L^{op} \to Coh$$

Because the self indexation is a stack for the regular and coherent topology

Why all of this ?

Those examples are trivial because of the kind of etale maps we chosen
However this exhibits links with the comprehensive factorization system
Moreover we have 2-multireflections

$$Lex^{opinitial} \hookrightarrow Lex$$

$$LocReg^{opinitial} \hookrightarrow Reg$$

$$LocCoh^{opinitial} \hookrightarrow Coh$$

But Diers could infer characterization of the topology of the spectra by functorial specificity of a given multireflection
$\rightarrow$ could help for topological precision on the dual of Reg and Coh ?

Perspective

- Other 2-cat than F.O. doctrines, or doctrine for exotic fragments of logics
  (e.g. : Dubuc-Poveda duality for MV-algbra $\rightarrow$ spectrum for F.O. Łukaśiewicz logics ?
- Replace local topos with another kinf of topos (totally connected, hyperconnected...) to have other semantical geometries
- When Spec is truncated, it behaves as a classifying topos : generalize classifying topos out of geometric logics ?
- Extend definition of Spec for models of PIE-theories in arbitrary 2-topos
  (e.g. for internal Lex, Reg, Coh in other 2-toposes than CAT or in ordinary 1-categories)

Thanks for your attention !