RESEARCH STATEMENT

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I am a **category theorist** motivated by *logic* and *foundations* of mathematics. My scientific interests can be organized in two main categories: **Category theory & Categorical logic** and **(foundations of) geometry**, and my main focus is on exploiting and pushing the connections between the two as far as possible.

1. Research

1.1. **Dualities.** The *fil rouge* of my whole research up to this point has been the topic of dualities, on which I maintain mainly the perspective of a logician, being open to wear the comfortable shoes of the geometer from time to time. Dualities of syntax-semantics type are a pattern of theorems in Categorical Logic whose generic statement takes the form

$$\mathsf{Th}^\circ \cong \mathsf{Sem}.$$

The aim of these results is to provide reconstruction theorems for nice categories that look like categories of models of some theory. This field was initiated in the early days of Category Theory by Lawvere in his PhD thesis [Law63]. He introduced the notion of algebraic theory (nowadays known as Lawvere theory) providing a syntax-semantics duality that axiomatized algebraic varieties[1] (in the sense of universal algebra).

$$\mathsf{Law}^\circ \cong \mathsf{Var}.$$

In the very same fashion one can look at Gabriel-Ulmer duality [CV02],

$$\mathsf{Mod} : \mathsf{Lex}^\circ \leftrightarrows \mathsf{Pres} : \mathsf{Th}.$$

On the left side we have theories that can be axiomatized by *finite limits* and on the left we have locally finitely presentable categories. The topic might sound very much related to logic, but has indeed very deep geometric aspects. In general topology dualities happen to take a different shape and appear most of the time in the form,

$$\mathsf{Alg}^\circ \cong \mathsf{Geom}.$$

An example which might be familiar to the *working mathematician* is the that of affine varieties over $\mathbb{K}$ (in the sense of algebraic geometry) and reduced $\mathbb{K}$-algebras of finite dimension, where the left adjoint maps a variety to its coordinate ring. Another example that is closer to my research and is due to Isbell is the duality between frames and topological spaces [LM94][Chap IX].

$$\mathsf{pt} : \mathsf{Frm}^\circ \leftrightarrows \mathsf{Top} : \mathbb{O}$$

In the past years one of the motivating mottos of my research has been that *dualities of syntax-semantics type are the higher dimension* [2] *analogs of algebra-geometry dualities*. This perspective is not new at all to the my research area, several works of Makkai [Mak88, MP87] build on this intuition, and the same can be said for some works of Awodey and Forssell [AF13]. The unifying principle that makes this story of interaction between geometry and logic so fascinating to me is that in most of the

[1] The proof on the level of morphisms came in fact much later in [ALR03].

[2] In the sense of higher category theory.

relevant examples these dualities are induced by **dualizing objects**. For example, in the case of the Gabriel-Ulmer duality, **Set** induces the duality, while in the case of the duality between frames and spaces the dualizing object is the Sierpinski space $\mathbb{S}$. From the perspective of the category theorist this is even more fascinating, in fact $\mathbb{T} = \{0 < 1\}$ plays among posets the same role that **Set** plays among categories, serving as an *enrichment base*.

1.1.1. *PhD thesis* [Lib19b]. My PhD thesis[3] [Lib19b] is the intense study of a new duality that has both a very geometric and a very logical interpretation.

$$\mathsf{pt} : \mathsf{Topoi} \leftrightarrows {}^4\mathsf{Acc}_\omega : \mathsf{S}$$

On the left we have the 2-category of topoi and geometric morphisms, on the right we have accessible categories with directed colimits and functors preserving them. There are several evidences [MP89, Ros97, BR12] that accessible categories with directed colimits are the right context to study **formal model theory** . The adjunction is still induced by a dualizing objects, which is (not surprisingly) the category of sets. On the geometric side, it offers a categorification of the adjunction between spaces and locales, while on the logical side it offers a candidate axiomatization of an accessible category with directed colimits via geometric logic. Even if the thesis is not finished yet, its content has already been applied twice. Once in [Hen19] by Simon Henry, where the adjunction made its first public appearence, where the author showed that there are AEC that are not axiomatizable in infinitary logic. The second time in [Esp19] by Christian Espindola, where the author claims to have a short proof of Shelah categoricity conjecture through topos theoretic methods.

1.1.2. *Accessibility and Presentability in 2-categories* [DL18]. The motivation for [DL18] was to provide a definition of locally presentable object in a 2-category, keeping as a paradigmatic example locally presentable categories in Cat. Locally presentable categories are *tame* objects, and thus provide a good level of generality in which to accommodate a lot of category theory. They happen to have a deep logical meaning, being categories of models of limit theories. The paper contains a very **axiomatic approach to the Gabriel Ulmer duality** in the contect of 2-categories equipped with a suitable replacement of the presheaf construction. This project is a collaboration with Fosco Loregian.

1.1.3. *GU duality for topoi and its relation with site presentation* [LG19]. This [LG19] collaboration with Julia Ramos González is motivated by a very geometric question about exponentiability of Grothendieck categories. In the paper we provide a complete answer to the question *when does the Ind-completion of a category is a Grothendieck topos?* offering a **specialization of the Gabriel-Ulmer duality to the context of topoi**. We also discuss the **relationship** between this presentation of a topos and the usual **site-theoretic description**. The original motivation of this problem has led to further developments that will be discussed later in this note.

1.1.4. *Codensity, Isbell duality, compactness and accessibility* [Lib19a]. This paper [Lib19a] connects the theory of **codensity monads to the Isbell duality** and shows that the Isbell duality is somewhat generic among relevant codensity monads. The Isbell duality can be seen as an archetypal Algebra/Geometry duality as indicated

[3]Whose title has still to be choosen between *The Scott adjunction* and *Topos theoretic approaches to Formal model theory*.

[4]Be careful, the left adjoint is S. In this brief note we decided to put it on the right to be consistent with the previous notation of putting the logical side of the duality of the left.

by [L+86][Sec 7]. The paper contains several other technical results aimed to describe the general behavior of codensity monads induced by full subcategories, including accessibily.

1.2. **Formal category theory.** I was exposed to formal methods in category theory by Fosco Loregian during our collaboration on [DL18]. After a deep skepticism I became a fond supported of this sorcery, even if I never bent my knees to coend calculus.

1.2.1. *On the unicity of formal category theories* [LL19]. From my perspective this collaboration [LL19] with Fosco Loregian sits at the very core of understainding dualities. As we have speculated before, meaningful dualities arise in $\mathcal{V}$-Cat, when we dualize along $\mathcal{V}$. That's the very case of Posets, which are categories enriched over truth values $\mathbb{T}$, or Categories, which are categories enriched over **Set**. The classical Morita theory of modules shows that this pattern is perfectly viable also for Ab-enriched categories. In [LL19] we contribute to the **axiomatic study of the presheaf construction** in $\mathcal{V}$-Cat, that is the key ingredient in any duality-like theorem. In this field there where two very similar but apparently unrelated approach to the presheaf construction: namely Yoneda structures [SW78] and Proarrow equipments. We isolate conditions in which the two approaches are equivalent.

1.3. **Categorical Model Theory.** Since the beginning of my PhD I was involved in the research of Wieslaw Kubiś, who works (among many other things) on categorical approaches to Fraïssé theory. The latter can be seen as the study of universal/saturated/generic objects among finite or finitely presented structures.

1.3.1. *Weak saturation and weak amalgamation property* [Di 19]. The existence of Fraïssé limits in Fraïssé classes is very often related to the amalgamation property. In [Di 19] I show that to a weakening of the amalgamation property corresponds a **weakening of the Fraïssé theory**. This generalizes both some results of Kubiś and Rosický in this direction.

1.4. **Homotopy Theory.**

1.4.1. *Homotopical algebra is not concrete* [DLL18]. This paper [DLL18] was the beginning of my collaboration with Fosco Loregian and offers a very wide generalization of a theorem due to Peter Freyd [Fre69], namely the Homotopy category of topological spaces does not have a **faithful functor** into **Set**. In the paper we generalize this result to a vast class of model categories that do admit a variant of the Eilenberg-Mac Lane spaces. This project may seem very far from the others, but the study of good faithful **Set**-functors (which means good representations) is now a central point of my PhD thesis.

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