$\boxed{L5}$ The structure - semantics duality

This is the last lecture of the "Algebra" module of this course.

Until this moment we have focus on "finitary" universal algebra, sticking to operations with finite arity, as the intuition suggests

$$A^n \longrightarrow A.$$

In this lecture we break the chains of "finitary algebra". Before doing so, let us recap what we did.

Syntax

category with finite products	abstract theory
Law theory $\mathrm{Fin}_S^{op} \to \mathcal{C}$	Presentation of a theory
finitary monad $T: \mathrm{Set}^S \to \mathrm{Set}^S$	

Semantics

Variety	semantics
variety + forgetful functor to Set^S	structured sets
$\mathrm{Alg}(T)$	

$$\begin{tikzcd} \mathrm{Prod}^{op} \arrow[r, bend left] \arrow[r, phantom, "T"] & \mathrm{Var} \arrow[l, bend left] \end{tikzcd}$$

via slicing we get correspondence between

Law / Varieties equipped with a monadic functor to Set

Reflections

- Starting from a finitary monad T its "theory" is $\mathrm{Kl}_\omega(T)^{op}$? and its algebras are $\mathrm{Alg}(T)$.

In this lecture we "reboot" the course (up to this point) and we try to discard any finitary assumption to see how abstract nonsense makes this theory "easier".

So, lets go back where we started from: the data of a category $\mathcal{K} \xrightarrow{\mathcal{U}} \mathrm{Set}$ equipped with a forgetful functor to Set.

- <u>Unbounded implicit operations</u>, similarly to Lawvere, we define a category whose objects are sets and ~~operations~~ morphisms are given by operations of arity unrestricted

$$\Pi_{\mathcal{U}} = \begin{cases} \text{obj} & \text{sets } I, J \\ \text{morphisms} & \mathrm{Nat}(\mathcal{U}^I, \mathcal{U}^J). \end{cases}$$

Of course, if $\mathcal{U}$ has a left adjoint L, we easily get

$$\mathrm{Nat}(\mathcal{U}^I, \mathcal{U}^J) = \mathcal{K}(L(J), L(I)),$$

similarly to the Grp case that we saw in the first lecture

- So $\Pi_{\mathcal{U}}$ naturally comes equipped with a functor

$$\begin{aligned} \mathrm{Set}^{op} &\longrightarrow \Pi_{\mathcal{U}} \\ I &\longmapsto I \\ \mathrm{Set}(I, J)^{op} &\xrightarrow{\text{"}L\text{"}} \mathcal{K}(L(J), L(I)) \end{aligned}$$

So, given a right adjoint $K \underset{U}{\overset{L}{\rightleftarrows}} \mathrm{Set}$, its "full" algebraic theory is nothing but

$$\mathrm{Set}^{op} \xrightarrow{L^{op}} Kl(T)^{op}$$

(T is the monad associated to $L \dashv U$)

$$\begin{tikzcd}
\text{bo right adjoints } \mathrm{Set}^{op} \xrightarrow{L^{op}} \mathcal{L} \arrow[dr, bend right, "\text{compose adjoints}"'] & & \text{Monadic categories over Set} \arrow[ll, bend right, "Kl"'] \arrow[dl, dashed, "\text{compose adjoints}"] \\
& \text{Monads on Set} \arrow[ul, dashed, "Kl(-)"] \arrow[ur, bend right, "Alg(-)"'] &
\end{tikzcd}$$

• $\underline{\mathrm{Set}}$ ... one should say that, going back to our original example $K \xrightarrow[U]{} \mathrm{Set}$, our fundamental construction

$$\mathbb{T}_U \begin{cases} \to \text{sets} \\ \to \mathrm{Nat}(U^I, U^J) \end{cases}$$

is still available, even in its "finitary" version.

And even more, we will get a comparison functor

$$\begin{tikzcd}
K \arrow[r, "\Theta"] \arrow[dr] & Mod(\mathbb{T}_U) \arrow[d, "\text{monadic functor}"] \\
& \mathrm{Set}
\end{tikzcd}$$

Here models are really functors preserving any product!!

$$\boxed{\Theta : K \longrightarrow U(k)^{(-)}}$$

Here we could both choose the full algebraic $\mathbb{T}$ or its finitary version this observation is due to Linton and in the old times was called "the monadic completion" of a functor $\mathbb{K} \to Set$ in the sense that it provides a correction making $\mathcal{U}$ into the best monadic functor that approximates it.

The structure – semantics adjunction

We can synthetize the the question we have been ~~answer~~ asking by saying that

$$Mnd(Set)^{op} \xrightarrow{Alg(-)} Cat/_{Set}$$

we are trying to find an adjoint for the functor $Alg(-)$, and in a sense we have provided an indirect answer given by the monad associated to $\pi_{\mathcal{U}}$. But there is a more direct way to answer this question.

- Codensity monad.

$$\begin{tikzcd} K \arrow[r, "\mathcal{U}"] \arrow[d, "\mathcal{U}"'] & Set \\ Set \arrow[ur, "\mathrm{Ran}_{\mathcal{U}}\mathcal{U}"'] & \end{tikzcd}$$

- Assume we have no problems in constructing Rans. Let us show that

$\mathrm{Ran}_{\mathcal{U}}\mathcal{U}$ is always a monad (called codensity monad)

- $\mathrm{ran}_{\mathcal{U}} \mathcal{U}$ is a functor. ok.

- To provide the unit $1 \longrightarrow \mathrm{ran}_{\mathcal{U}} \mathcal{U}$ we use the universal property of ran's

$$
\begin{tikzcd}
\mathrm{Set}^{K} \arrow[rr, bend left, "\mathrm{ran}_{\mathcal{U}}(\,)"] \arrow[rr, phantom, "\top"] & & \mathrm{Set}^{\mathrm{Set}} \arrow[ll, "\mathcal{U}^{*}"]
\end{tikzcd}
$$

with implies that to provide a map

$$1 \xRightarrow{\eta} \mathrm{ran}_{\mathcal{U}} \mathcal{U}$$

is the same of providing a map

$$1 \circ \mathcal{U} \xRightarrow{id_{\mathcal{U}}} \mathcal{U},$$

and we choose the identity.

- to provide the multiplication $\mathrm{ran}_{\mathcal{U}} \mathcal{U} \circ \mathrm{ran}_{\mathcal{U}} \mathcal{U} \Rightarrow \mathrm{ran}_{\mathcal{U}} \mathcal{U}$

we observe that this is the same of providing

$$\mathrm{ran}_{\mathcal{U}} \mathcal{U} \circ \mathrm{ran}_{\mathcal{U}} \mathcal{U} \circ \mathcal{U} \Rightarrow \mathcal{U}$$

and now we use again the property $\mathcal{U}^{*} \dashv \mathrm{ran}_{\mathcal{U}}(-)$ to get

$$\mathrm{ran}_{\mathcal{U}} \mathcal{U} \circ \boxed{\mathrm{ran}_{\mathcal{U}} \mathcal{U} \circ \mathcal{U}} \xRightarrow{\text{Counit}} \mathrm{ran}_{\mathcal{U}} \mathcal{U} \circ 1 \simeq \mathrm{ran}_{\mathcal{U}} \mathcal{U}.$$

thm (1) $\mathrm{Alg}(T_{\mathcal{U}}) \simeq \mathrm{Mod}(\mathbb{T}_{\mathcal{U}})$.

the codensity monad

the "full Lawvere theory" -

thm (2)

$$\begin{tikzcd} \mathrm{Mnd}(\mathrm{Set})^{op} \arrow[rr, bend left, "\mathrm{Alg}(-)"] \arrow[rr, phantom, "\top"] & & \mathrm{Cat}/\mathrm{Set} \arrow[ll, bend left, "\mathrm{ran}_{(-)}(-)"] \end{tikzcd}$$

$\boxed{\text{Structure semantics adjunction}}$

- $\mathrm{ran}_{(-)}(-)$ is fully faithful !

- We cheated a little bit with the existence of Kan extensions $\mathrm{Cat}/\mathrm{Set}$ should be replaced with "functors for which $\mathrm{ran}_{\mathcal{U}}\mathcal{U}$ exists" -

to prove thm (1) there are two strategies -

Strategy 1

Provide a functor

$$\mathrm{Mod}(\mathbb{T}_{\mathcal{U}}) \longrightarrow \mathrm{Alg}(T_{\mathcal{U}})$$

and show that it is an equivalence

Strategy 2

$$\begin{tikzcd} \boxed{\mathrm{Mnd}(\mathrm{Set})^{op}} & & \boxed{\mathrm{Cat}/\mathrm{Set}} \arrow[dl, "\mathbb{T}_{(-)}"] \\ & \boxed{\begin{array}{c}\text{Right adjoints} \\ \mathrm{Set}^{op} \to K\end{array}} \arrow[ul, "\text{composition}"] & \end{tikzcd}$$

Show that the composition above offers a left adjoint for $\mathrm{Alg}(-)$ and use uniqueness of R.A.

• One last word about "arity".

OK but given a functor $\mathbb{K} \xrightarrow{\mathcal{U}} \mathrm{Set}$ we have produced two theories

$\prod^{fin}_{\mathcal{U}}$	$\prod_{\mathcal{U}}$
– only finitary operations.	– also infinitary operations
– Category with products fin	– category with all products.

Which one should we trust? Which one should we invest on?

~~[illegible]~~

~~[illegible]~~

The answer is very simple in the case of Grp we get both that

$$\mathrm{Grp} \simeq \mathrm{Prod}_{fin}\left(\prod^{Fin}_{\mathcal{U}}, \mathrm{Set}\right) \simeq \mathrm{Prod}_{all}\left(\prod_{\mathcal{U}}, \mathrm{Set}\right)$$

and the reason is that the inclusion

$$\prod^{Fin}_{\mathcal{U}} \hookrightarrow \prod_{\mathcal{U}} \quad \text{is dense!!}$$

in full generality this will not happen.

So - $\mathbb{T}^{Fin}_{u}$ will axiomatize all the finite natural generation

- $\mathbb{T}_{u}$ will axiomatize ALL the natural gerations

- None of these might be enough $K = Cat$

- $\mathbb{T}_{u}$ might be enough $K = SLat$

- both might be ok $K = Grp$ -