INTRODUCTION TO CATEGORICAL LOGIC

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rules

- Hand your exercises by the **7th of May**. (CL23 in the subject!)
- Pick at least one exercise from each of the yellow groups.
- You must charge at least **3** batteries!
- At each stage of the exercise sheet, you can (and should) give for granted the statements of all the exercises that come before the one you are solving.

Exercises

doctrines and types

Exercise 1 (Naming, 🔋)**.** Consider the category of sets and the usual powerset doctrine defined over it, $\mathcal{P} : \mathsf{Set}^\circ \to \mathsf{InfLat}$. Using the usual *epi-mono* factorization, we can define a functor $[-]_X : \mathsf{Set}_{/X} \to \mathcal{P}(X)$. Inspired by this construction, for $\mathcal{P} : \mathsf{C}^\circ \to \mathsf{InfLat}$ a doctrine with a sufficient amount of structure[a] construct a (pseudo)natural transformation

$$[-]_{(=)} : \mathsf{C}_{/(=)} \Rightarrow \mathcal{P}(=).$$

Exercise 2 (...and necessity, 🔋)**.** We say that a doctrine has *comprehension schema* if the *naming* functor $[-]_A : \mathsf{C}_{/A} \Rightarrow \mathcal{P}(A)$ of the exercise above has a right adjoint $\{A : -\}$ for all A. Prove that if $\mathcal{P} : \mathsf{C}^\circ \to \mathsf{InfLat}$ is a doctrine with a sufficient amount of structure, there exists a doctrine doctrine $\mathcal{P}^\flat$ defined on its category of elements having comprehension schema,

$$\mathcal{P}^\flat : \mathsf{Elts}(\mathcal{P})^\circ \to \mathsf{InfLat}.$$

Exercise 3 (🔋, ☕)**.** Provide a type theoretic interpretation of a doctrine with comprehension schema.

Exercise 4 (🔋)**.** Provide a translation between the notion of comprehension category and that of category with display maps.

[a] It is enough that C has finite limits and $\mathcal{P}f$ has a left adjoint for all f.

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topoi as spaces

Exercise 5. Show that the category of sheaves over the Sierpinski space is a presheaf topos. Which one?

Exercise 6. Show that $\mathbf{Set}^{\to}$ has a closed subtopos and an open subtopos. Please, provide a full proof that the geometric morphisms you present have the property we require, don't just state it.

Exercise 7. Let X be a compact Hausdorff space. Show that the direct image of the terminal geometric morphism $\Gamma_* : \mathsf{Sh}(X) \to \mathsf{Set}$ preserve directed colimits of monomorphisms.

topoi as sets

Exercise 8. Provide a complete description of the subobject classifier in $\mathbf{Set}^{\mathbb{N}}$, where the category structure of $\mathbb{N}$ is the expected posetal one.

Exercise 9. Show that every topos has a partial map classifier for every object. *Hint:* What are the partial map classifiers in Set?

Exercise 10. Prove that an object of a topos $\mathcal{E}$ is injective (with respect to monos) if and only if it is a retract of Ω^x for some x. Deduce that if e is injective then the functor $[-, e] : \mathcal{E}^\circ \to \mathcal{E}$ preserves reflexive coequalizers.

topoi as theories

Exercise 11. Consider the category of non empty finite sets $\mathsf{Fin}_{>0}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{>0}}$ classify?

Exercise 12. Consider the category of finite sets and monomorphisms $\mathsf{Fin}_{\hookrightarrow}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{\hookrightarrow}}$ classify?

Exercise 13. Consider the category of finite sets and epimorphisms $\mathsf{Fin}_{\twoheadrightarrow}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{\twoheadrightarrow}}$ classify?

Exercise 14. Consider the category of pointed finite sets $\mathsf{Fin}_{\bullet}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{\bullet}}$ classify?

Exercise 15. Consider the comma topos below, and assume comma topoi exist in the bicategory of topoi. Can you describe how does a **Set**-model of the comma topos look like (in terms of models of $\mathbb{T}_1$ and $\mathbb{T}_2$)?

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \mathbf{Set}[\mathbb{T}_1] \arrow[d] \\
\mathbf{Set}[\mathbb{T}_2] \arrow[r] \arrow[ur, Rightarrow, dashed, "\lambda"] & \mathbf{Set}
\end{tikzcd}$$

Feel free to assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are single sorted if you wish.

topoi as objects

Exercise 16 (🔋, ☕)**.** Show that the bicategory of topoi has (pseudo)pushouts. Does the same argument apply to all (pseudo)colimits? *Hint.* This exercise is not as hard as it may seem.

Exercise 17 (🔋, ☕)**.** Show that the bicategory of topoi has (pseudo)pullbacks. *Hint.* Yes, this exercise is too hard.

Exercise 18 (🔋)**.** Show that if a topos $\mathcal{E}$ is localic, then $\mathsf{Topoi}(\mathcal{F}, \mathcal{E})$ is a poset for every $\mathcal{F}$.

Exercise 19 (🔋)**.** Show that open geometric morphisms are pullback stable.

Exercise 20 (🔋)**.** Show that closed geometric morphisms are pullback stable.

learning by gluing

Exercise 21 (🔋, ☕)**.** Show that there is an equivalence of categories between

$$\mathsf{Sub}_{\mathcal{E}}(1) \simeq \mathsf{Topoi}(\mathcal{E}, \mathbf{Set}^{\rightarrow}).$$

Exercise 22 (🔋)**.** Show that the bicategory of topoi has a classifier of closed embeddings, i.e., there exists a closed embedding $p : \mathcal{F}_1 \to \mathcal{F}_2$ such that every closed subtopos can be obtained by pulling back a geometric morphism along p.

$$\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, dashed] & \mathcal{F}_1 \arrow[d, "p"] \\ \mathcal{E} \arrow[r] & \mathcal{F}_2 \end{tikzcd}$$

Prove an anologous statement also for open embeddings. *Hint.* To get the proper intuition, first solve it for spaces, then for locales, and then for topoi. Also, you may want to start with open embeddings.

Exercise 23 (🔋)**.** Show that every open subtopos is *complemented*, i.e. there exists a closed subtopos that is its complement in the lattice of subtopoi.

Riddle. Show that a presheaf topos $\mathsf{Set}^{\mathsf{C}}$ is boolean if and only if C is a groupoid.

Riddle (Freyd)**.** Show that a topos verifies external choice if and only if it is the topos of sheaves over a complete boolean algebra.