# TOPOLOGY

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ABSTRACT. This note summarizes the content of the third lesson of tutoring on the course Topology 2020. Also, attached at the end, there is an exercise sheet.

## 1. SEPARATION AXIOMS

Welcome back to our course in topology. I hope you learnt something in these four lessons of introduction to topology, because today it's your day. We will focus separation axioms, examples, counterexamples and simple exercises. The use of the word *separation axioms* for separation properties of topological spaces is not accidental. Depending on what kind of topological spaces a mathematician typically deals with, the exists corresponding notion of *decency* for topological spaces. Every separation property gives different *proof techniques.* The lesson of today will be mostly interactive.

### 1.1. Kolmogorov, seu $\mathbf{T_0}$.

In lesson 2, while studying Alexandroff spaces we were very close to the definition of topological indistinguishability.

**Definition 1.** Two points $x, y$ in a topological space $\mathcal{X}$ are topological indistinguishable $x \equiv y$ if $x$ is near $y$ and viceversa[1].

We observed that topological indistinguishability is an equivalence relation and we used it in order to construct natural examples of Alexandroff spaces. In fact, in that case we were using a variation of indistinguishability, that is called specialization preorder.

**Exercise 1.** A space is $T_0$ if and only if the only point which is topological indistinguishable from $x$ is $x$ itself.

Given a topological space $\mathcal{X}$ we can endow $\mathcal{X}/\equiv$ with the biggest topology $\tau_\equiv$ making the quotient map $\pi : \mathcal{X} \to \mathcal{X}/\equiv$ continuous.

**Exercise 2** (Kolmogorification). $\mathcal{X}/\equiv$ is universal among $T_0$-approximations of $\mathcal{X}$, in the sense that:

(1) $\mathcal{X}/\equiv$ is $T_0$.
(2) Every continuous map $\mathcal{X} \to \mathcal{Y}$, where $\mathcal{Y}$ is a $T_0$-space factors trough $\mathcal{X}/\equiv$.

*Sketch of proof of (2).* When a function is continuous, it preserves nearness. This means that two equivalent points in $\mathcal{X}$ are sent in equivalent points in $\mathcal{Y}$. Since the image $T_0$, $f$ in constant on $\equiv$-equivalence classes. Thus there is a set-theoretical factorization of $f$ along $\pi : \mathcal{X} \to \mathcal{X}/\equiv$, this function is continuous. □

**Example 2.** Most of the topological spaces in nature are much more than $T_0$. For this reason we just make a list of **non** $T_0$-spaces:

(1) the indiscrete topology $\mathsf{Ind}$ on a set with at least two elements is never $T_0$.
(2) consider $\mathbb{R}$ with the euclidean topology $\mathsf{E}$. The topological space $(\mathbb{R}, \mathsf{E}) \times (\mathbb{R}, \mathsf{Ind})$ is not $T_0$, in fact $(a, b)$ and $(a, c)$ are not distinguishable.

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[1] i.e. $x \in \mathrm{cl}(y)$ and viceversa.

1.2. **$\mathrm{T}_1$.** There is no standard name to refer to $\mathrm{T}_1$-spaces. Some people call them **accessible**, or **Tychonoff,** or **Fréchet.** I suggest not to use these names, because they are not very well established, especially the last one might be confused with other terminologies coming from functional analysis.

**Exercise 3.** A space is $\mathrm{T}_1$ if and only if points are closed sets.

**Exercise 4.** $\mathrm{T}_1$-spaces are $\mathrm{T}_0$.

**Exercise 5.** $\mathrm{T}_1$-spaces are stable under products but not under subspaces.

**Exercise 6.** The following is a list of **non** $\mathrm{T}_1$ spaces.

(1) The Sierpiński space $\mathsf{S}$ (the only topology on a set with two points where one is open and the other is closed) is not $\mathrm{T}_1$.
(2) The Zariski topology on a commutative ring is not in general $\mathrm{T}_1$.

1.3. **Hausdorff, seu $\mathrm{T}_2$.**

**Exercise 7.** A space $\mathcal{X}$ is $\mathrm{T}_2$ if and only if the diagonal $\{(x,x)\}_{x\in\mathcal{X}}$ is a closed set in the topological product $\mathcal{X}\times\mathcal{X}$.

**Exercise 8.** $\mathrm{T}_2$-spaces are stable under product and subspaces.

$\mathrm{T}_2$-spaces are separated *enough*, in the sense that points retain some information about the topology. We will see two examples of this behaviour, the first one is the exercise below, the second one is Prop. 1.1, where we show that $\mathbb{Q}$ knows everything about continuous functions defined on $\mathbb{R}$.

**Exercise 9** (Uniqueness of limits in $\mathrm{T}_2$-spaces)**.** Let $\mathcal{X}$ be a $\mathrm{T}_2$-space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}$. Then there is at most one point $x$ that does not belong to and $\{x_n\}_{n\in\mathbb{N}}$ but is near it.

**Exercise 10.** Let $f,g : \mathcal{X} \rightrightarrows \mathcal{Y}$ be two continuous functions and $\mathcal{Y}$ be a $\mathrm{T}_2$-space. Then the subspace of $\mathsf{E}(f,g)\subset\mathcal{X}$ defined by

$$\mathsf{E}(f,g) \coloneqq \{x\in\mathcal{X} : f(x)=g(x)\},$$

is a closed subset of $\mathcal{X}$.

*Proof.* Consider the continuous function $f\times g : \mathcal{X}\to\mathcal{Y}^2$ mapping $x\mapsto(f(x),g(x))$. Since $\mathcal{Y}$ is $\mathrm{T}_2$, the diagonal $\Delta_{\mathcal{Y}}$ is closed in the product $\mathcal{Y}^2$. To finish, observe that

$$\mathsf{E}(f,g) = (f\times g)^{-1}\Delta_{\mathcal{Y}}$$

and thus must be closed because $f\times g$ is continuous. □

**Definition 3.** A subset $D$ of a topological space is dense if its closure is the whole space.

**Proposition 1.1.** If $D$ is a dense subset of $\mathcal{X}$ and $\mathcal{Y}$ is a Hausdorff space, then every continuous function $f : D\to\mathcal{Y}$ extends in at most one way to a continuous function from $\mathcal{X}$ to $\mathcal{Y}$ .

*Proof.* Consider two extensions $g,h : \mathcal{X}\to\mathcal{Y}$. $\mathsf{E}(g,h)$ is a closed set containing $D$, thus it contains its closure, that is the whole space. This proves that $\mathcal{X}\subset\mathsf{E}(g,h)$, or equivalently that $g$ coincides with $h$ on $\mathcal{X}$. □

**Example 4.** The Zariski topology is not $\mathrm{T}_2$.

1.4. **Regular Hausdorff and higher separation axioms.** I will not go very much into higher notion of separation, you should know that one can go at least as far as $\mathrm{T}_6$, passing trough $\mathrm{T}_{3^1/_2}$. I shall say something on the notion of $\mathrm{T}_3$-spaces, also known as *regular*. Regularity is the *correct* notion to study abstractly metrizability.

**Example 5.** Metric spaces are regular.

**Theorem 1.2** (Uryshon)**.** A space $\mathcal{X}$ is metrizable if and only if it is regular and *second countable*.

**Exercise 11.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous function, where $\mathcal{Y}$ is Hausdorff. Prove that the *graph*

$$\Gamma(f) := \{(x, f(x)) \in \mathcal{X} \times \mathcal{Y}\}$$

is closed in $\mathcal{X} \times \mathcal{Y}$ (with the product topology).

**Exercise 12.** Let $C$ be a closed subset of the $\mathrm{T}_3$ space $\mathcal{X}$. Let $\sim$ be the equivalence relation on $\mathcal{X}$ defined by

$$x \sim y \text{ iff } x = y \text{ or } \{x, y\} \subset A.$$

Let $\mathcal{X}/A$ denote the quotient space $\mathcal{X}/\sim$. Prove that $\mathcal{X}/A$ is Hausdorff.

**The Book (📘).** 15.9x

**Exercise 13.** Reformulate $T_1$-ness in terms of the nearness relation induced by the topology[a].

[a]The answer: *A nearness relation is T1 iff the induced topology is T1* will not be accepted.

**Exercise 14.** Let $G$ be a group. We define a *closure-like* operator on $G$, mapping a subset $A \subset G$ as follows,

$$A \mapsto \bigcap_{H<G\,:\,A\subset G} H.$$

Does this *closure-like* operator define a topology on $G$?

**Exercise 15.** Show that the pointwise convergence topology on $\mathbb{R}^{\mathbb{R}}$ is $T_2$.

**Exercise 16.**

(1) Find a continuous function that does not map closed sets into closed sets.
(2) Find a continuous bijective function which is not a homeomorphism.

**Exercise 17.** Prove that the set of fixed points of a continuous map from a Hausdorff space to itself is a closed set.

**The riddle of the week (☕).** Let $\mathsf{S}$ be the Sierpinski space and let $\mathcal{X} = (x, \chi)$ be any $\mathrm{T}_1$-space. Prove that there is an embedding $e : \mathcal{X} \to \mathsf{S}^{\chi}$ .

- 🔴 the exercises in the red group are mandatory.
- 🟡 pick at least one exercise from each of the yellow groups.
- 🟢 The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- ☕ useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- ⚠ challenging.
- 📘 comes from **Elementary Topology Problem Textbook**, by *Viro, Ivanov, Netsvetaev and Kharlamov*.