## INTRODUCTION TO CATEGORICAL LOGIC

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rules

- Hand your exercises by the midnight of the **7th of May** via email. Make my life easier: include the word **CL23 in the subject**.
- Pick at least one exercise from each of the yellow groups.
- At each stage of the exercise sheet, you can give for granted the statements of all the exercises that come before the one you are solving.
- You must charge at least 3 batteries! *Example*. The vector of exercises [1,5,8,12,15,18] would pass this sheet.

## **EXERCISES**

doctrines and types

**Exercise 1** (Naming,  $\blacksquare$ ). Consider the category of sets and the usual powerset doctrine defined over it,  $\mathcal{P}: \mathsf{Set}^\circ \to \mathsf{InfLat}$ . Using the usual *epi-mono* factorization, we can define a functor  $[-]_X: \mathsf{Set}_{/X} \to \mathcal{P}(X)$ . Inspired by this construction, for  $\mathcal{P}: \mathsf{C}^\circ \to \mathsf{InfLat}$  a doctrine with a sufficient amount of structure<sup>a</sup> construct a (pseudo)natural transformation

$$[-]_{(=)}$$
:  $C_{/(=)} \Rightarrow \mathcal{P}(=)$ .

**Exercise 2** (...and necessity,  $\blacksquare$ ). We say that a doctrine has *comprehension schema* if the *naming* functor  $[-]_{(=)}: C_{/A} \Rightarrow \mathcal{P}(A)$  of the exercise above has a right adjoint  $\{A: -\}$  for all A. Prove that if  $\mathcal{P}: C^{\circ} \to \mathsf{InfLat}$  is a doctrine with a sufficient amount of structure, the canonical subobject doctrine defined on its category of elements has comprehension schema,

Sub: 
$$\mathsf{Elts}(\mathcal{P})^{\circ} \to \mathsf{InfLat}$$
.

Exercise 3 ( , Provide a type theoretic interpretation of a doctrine with comprehension schema.

Exercise 4 ( ). Provide a translation between the notion of comprehension category and that of category with display maps.

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1

 $<sup>^{</sup>a}$ It is enough that C has finite limits and  $\mathcal{P}f$  has a left adjoint.

topoi as spaces

Exercise 5 ( ). Show that the category of sheaves over the Sierpinski space is a presheaf topos. Which one?

**Exercise 6** ( $\blacksquare$ ). Show that **Set** $^{\rightarrow}$  has a closed subtopos and an open subtopos. Please, provide a full proof that the geometric morphisms you present have the property we require, don't just state it.

**Exercise 7** ( $\blacksquare$ ). Let X be a compact Hausdorff space. Show that the direct image of the terminal geometric morphism  $\Gamma_*: \operatorname{Sh}(X) \to \operatorname{Set}$  preserve directed colimits of monomorphisms.

topoi as objects

Exercise 8 ( , Show that the bicategory of topoi has (pseudo)colimits. *Hint*. Yes, this exercise is too hard.

Exercise 9 (, Show that the bicategory of topoi has (pseudo)pullbacks. *Hint*. Yes, this exercise is too hard.

Exercise 10 (**)**. Show that open geometric morphisms are pullback stable.

Exercise 11 ( ). Show that closed geometric morphisms are pullback stable.

topoi as objects

Exercise 12 ( $\blacksquare$ ). Show that there is a bijective correspondence between  $Sub_{\mathcal{E}}(1) \simeq Topoi(\mathcal{E}, \mathbf{Set}^{\rightarrow}).$ 

**Exercise 13** ( $\blacksquare$ ). Show that the bicategory of topoi has a classifier of closed embeddings, i.e., there exists a closed embedding  $p: \mathcal{F}_1 \to \mathcal{F}_2$  such that every closed subtopos can be obtained by pulling back a geometric morphism along p.

$$\begin{array}{ccc} \bullet & --- & \mathcal{F}_1 \\ \downarrow & & \downarrow^p \\ \downarrow & & \mathcal{E} & \longrightarrow \mathcal{F}_2 \end{array}$$

Prove an anologous statement also for open embeddings. *Hint*. To get the proper intuition, first solve it for spaces, then for locales, and then for topoi.

**Exercise 14** ( ). Show that every open subtopos is *complemented*, i.e. there exists a closed subtopos that is its complement in the lattice of subtopoi.

topoi as sets

**Exercise 15** ( $\blacksquare$ ). Provide a complete description of the subobject classifier in  $\mathbf{Set}^{\mathbb{N}}$ , where the category structure of  $\mathbb{N}$  is the expected posetal one.

Exercise 16 ( ). Show that every topos has a partial map classifier. *Hint:* What is the partial map classifier in Set?

**Exercise 17** ( ). Prove that an object of a topos  $\mathcal{E}$  is injective (with respect to monos) if and only if it is a retract of  $\Omega^x$  for some x. Deduce that if e is injective then the functor  $[-,e]: \mathcal{E}^{\circ} \to \mathcal{E}$  preserves reflexive coequalizers.

topoi as theories

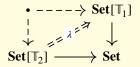
**Exercise 18** ( $\blacksquare$ ). Consider the category of non empty finite sets  $Fin_{>0}$ . What theory does  $\mathbf{Set}^{Fin}_{>0}$  classify?

Exercise 19 ( $\blacksquare$ ). Consider the category of finite sets and monomorphisms Fin $_{\hookrightarrow}$ . What theory does  $\mathbf{Set}^{\mathsf{Fin}_{\hookrightarrow}}$  classify?

Exercise 20 ( $\blacksquare$ ). Consider the category of finite sets and epimorphisms Fin $_{\rightarrow}$ . What theory does Set Fin $_{\rightarrow}$  classify?

Exercise 21 (•). Consider the category of pointed finite sets Fin. What theory does Set Fin. classify?

**Exercise 22** ( $\blacksquare$ ). Consider the comma topos below, and assume comma topoi exist in the bicategory of topoi. Can you describe how does a **Set**-model of the comma topos look like (in terms of models of  $\mathbb{T}_1$  and  $\mathbb{T}_2$ )?



Feel free to assume that  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are single sorted if you wish.

The riddle  $(\triangle)$ . Show that a presheaf topos  $Set^C$  is boolean if and only if C is a groupoid.