Last time

Limits
- Product
- Equalizers
- Pull back.
- General def.

Colimits
- Coprod
- Coequalizer
- Push out
- General def

Monos

Epimorphisms

Exa
- Product of sets
- $\wedge$ in posets
- Kernels
- Quotients
- Mons in sets are injective maps
- Epi in sets are surjective maps
- $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epi in Ring.

thm all limits $\Longleftrightarrow$ $\prod$ + equalizers

all colimits $\Longleftrightarrow$ $\coprod$ + coequalizers

Lecture 3: Adjunctions

Two approaches to adjunctions

Empirical / Historical	Conceptual / In retrospect
"free constructions in universal algebras"	Candidate universal universes
	Galois connections
	Kan extensions

<u>First story</u> (free groups) $\quad Grp(F(X), G) \cong Set(X, U(G))$

<u>Second story</u> (f^{-1} as a generalized inverse).

"Mathematically precise definition"

Def Let $L : A \rightleftarrows B : R$ be functors. We say that **L is left adjoint to R** $(L \dashv R)$ if there exist natural transformations

$$(\text{unit}) \qquad \eta : 1_A \Longrightarrow RL$$

$$(\text{counit}) \qquad \varepsilon : LR \Longrightarrow 1_B$$

triangle equalities

$$\begin{tikzcd} R \arrow[d, Rightarrow, "\eta_R"'] \arrow[dr, equal, "1_R"] & \\ RLR \arrow[r, Rightarrow, "R(\varepsilon)"'] & R \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, Rightarrow, "L(\eta)"'] \arrow[dr, equal, "1_L"] & \\ LRL \arrow[r, Rightarrow, "\varepsilon_L"'] & L \end{tikzcd}$$

this is very hard to grasp. Sanity check.

$$F : \mathrm{Set} \rightleftarrows \mathrm{Grp} : \mathcal{U}$$

$$FU(G) \xrightarrow{\varepsilon_G} G \qquad\Big|\qquad X \xrightarrow{\eta_X} \mathcal{U}F(X)$$

$$(w) \longmapsto w \qquad\Big|\qquad x \longmapsto g_x$$

$$\begin{tikzcd}
 & \mathcal{U}(X) \arrow[d, "\eta_{\mathcal{U}}"'] \arrow[dr, dashed, "id."] & \\
 & \mathcal{U}F\mathcal{U}(X) \arrow[r, "\mathcal{U}(\varepsilon)"'] & \mathcal{U}(X)
\end{tikzcd}$$

$$x \longmapsto (x) \longmapsto x$$

$$\boxed{\begin{array}{l} \textbf{Thm} \quad L : A \rightleftarrows B : R, \quad \text{then} \\ \varphi : A(\underline{x}, R\underline{y}) \overset{\sim}{\rightleftarrows} B(L\underline{x}, \underline{y}) : \varphi^{-1} \end{array}}$$

Proof

$$\begin{tikzcd}
X \arrow[d, "f"'] & & & L(X) \arrow[d] \\
RY & L(X) \arrow[d, "L(f)"] & & y \\
 & LRY \arrow[d, "\varepsilon_y"] & & \\
 & y & &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[d] & X \arrow[d, "\eta_X"'] & L(X) \arrow[d] \\
RY & RLX \arrow[d, "\rho"'] & y \\
 & RY &
\end{tikzcd}$$

1

$\shortparallel \quad \triangle$

$$\varphi^{-1}\circ\underbrace{\varphi(f)} = \varphi^{-1}(\varepsilon_y \circ L(f)) = \varepsilon_y \circ L(f) \circ \eta_X$$

The bijection above is "natural in A and B".

<u>Other examples</u>

- Vect_k Set
- Grp Set
- Ab Set
- Ab Grp
- Top $\rightleftarrows$ Set
- cartesian closed category.
- Closure operators.
- Pointed sets.
- Monoidal structure on Ab.

Non example

Fld Set

<u>Rem</u> Adjunctions compose