

INTRODUCTION TO CATEGORICAL LOGIC

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
rules

- Hand your exercises by the midnight of the **7th of May** via email. Make my life easier: include the word **CL23 in the subject**.
- Pick at least one exercise from each of the yellow groups.
- At each stage of the exercise sheet, you can give for granted the statements of all the exercises that come before the one you are solving.
- You must charge at least **3** batteries!


Example. The vector of exercises [1,5,8,12,15,18] would pass this sheet.

EXERCISES



doctrines and types


Exercise 1 (Naming, ) Consider the category of sets and the usual powerset doctrine defined over it, $\mathcal{P} : \mathbf{Set}^\circ \rightarrow \mathbf{InfLat}$. Using the usual *epi-mono* factorization, we can define a functor $[-]_X : \mathbf{Set}_{/X} \rightarrow \mathcal{P}(X)$. Inspired by this construction, for $\mathcal{P} : \mathbf{C}^\circ \rightarrow \mathbf{InfLat}$ a doctrine with a sufficient amount of structure^a construct a (pseudo)natural transformation

$$[-]_{(=)} : \mathbf{C}_{/(=)} \Rightarrow \mathcal{P}(=).$$

Exercise 2 (...and necessity, ) We say that a doctrine has *comprehension schema* if the *naming* functor $[-]_{(=)} : \mathbf{C}_{/(=)} \Rightarrow \mathcal{P}(=)$ of the exercise above has a right adjoint $\{A : -\}$ for all A . Prove that if $\mathcal{P} : \mathbf{C}^\circ \rightarrow \mathbf{InfLat}$ is a doctrine with a sufficient amount of structure, the canonical subobject doctrine defined on its category of elements has comprehension schema,

$$\mathbf{Sub} : \mathbf{Elts}(\mathcal{P})^\circ \rightarrow \mathbf{InfLat}.$$

Exercise 3 (, ) Provide a type theoretic interpretation of a doctrine with comprehension schema.

Exercise 4 () Provide a translation between the notion of comprehension category and that of category with display maps.

^aIt is enough that \mathbf{C} has finite limits and $\mathcal{P}f$ has a left adjoint.

topoi as spaces

Exercise 5 (□). Show that the category of sheaves over the Sierpinski space is a presheaf topos. Which one?

Exercise 6 (□). Show that \mathbf{Set}^\rightarrow has a closed subtopos and an open subtopos. Please, provide a full proof that the geometric morphisms you present have the property we require, don't just state it.

Exercise 7 (□). Let X be a compact Hausdorff space. Show that the direct image of the terminal geometric morphism $\Gamma_* : \mathbf{Sh}(X) \rightarrow \mathbf{Set}$ preserve directed colimits of monomorphisms.

topoi as objects

Exercise 8 (□, ▢). Show that the bicategory of topoi has (pseudo)colimits. *Hint.* This exercise is not as hard as it may seem.

Exercise 9 (□, ▢). Show that the bicategory of topoi has (pseudo)pullbacks. *Hint.* Yes, this exercise is too hard.

Exercise 10 (□). Show that open geometric morphisms are pullback stable.

Exercise 11 (□). Show that closed geometric morphisms are pullback stable.

topoi as objects

Exercise 12 (□, ▢). Show that there is a bijective correspondence between

$$\mathbf{Sub}_{\mathcal{E}}(1) \simeq \mathbf{Topoi}(\mathcal{E}, \mathbf{Set}^\rightarrow).$$

Exercise 13 (□). Show that the bicategory of topoi has a classifier of closed embeddings, i.e., there exists a closed embedding $p : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that every closed subtopos can be obtained by pulling back a geometric morphism along p .

$$\begin{array}{ccc} \bullet & \dashrightarrow & \mathcal{F}_1 \\ \downarrow & & \downarrow p \\ \mathcal{E} & \longrightarrow & \mathcal{F}_2 \end{array}$$

Prove an analogous statement also for open embeddings. *Hint.* To get the proper intuition, first solve it for spaces, then for locales, and then for topoi.

Exercise 14 (□). Show that every open subtopos is *complemented*, i.e. there exists a closed subtopos that is its complement in the lattice of subtopoi.

topoi as sets

Exercise 15 (▣). Provide a complete description of the subobject classifier in $\mathbf{Set}^{\mathbb{N}}$, where the category structure of \mathbb{N} is the expected posetal one.

Exercise 16 (▣). Show that every topos has a partial map classifier for every object. *Hint:* What are the partial map classifiers in \mathbf{Set} ?

Exercise 17 (▣). Prove that an object of a topos \mathcal{E} is injective (with respect to monos) if and only if it is a retract of Ω^x for some x . Deduce that if e is injective then the functor $[-, e] : \mathcal{E}^{\circ} \rightarrow \mathcal{E}$ preserves reflexive coequalizers.

topoi as theories

Exercise 18 (▣). Consider the category of non empty finite sets $\mathbf{Fin}_{>0}$. What theory does $\mathbf{Set}^{\mathbf{Fin}_{>0}}$ classify?

Exercise 19 (▣). Consider the category of finite sets and monomorphisms $\mathbf{Fin}_{\hookrightarrow}$. What theory does $\mathbf{Set}^{\mathbf{Fin}_{\hookrightarrow}}$ classify?

Exercise 20 (▣). Consider the category of finite sets and epimorphisms $\mathbf{Fin}_{\twoheadrightarrow}$. What theory does $\mathbf{Set}^{\mathbf{Fin}_{\twoheadrightarrow}}$ classify?

Exercise 21 (▣). Consider the category of pointed finite sets \mathbf{Fin}_{\bullet} . What theory does $\mathbf{Set}^{\mathbf{Fin}_{\bullet}}$ classify?

Exercise 22 (▣). Consider the comma topos below, and assume comma topos exist in the bicategory of topoi. Can you describe how does a **Set**-model of the comma topos look like (in terms of models of \mathbb{T}_1 and \mathbb{T}_2)?

$$\begin{array}{ccc}
 \bullet & \xrightarrow{\quad\quad\quad} & \mathbf{Set}[\mathbb{T}_1] \\
 \downarrow & \searrow \lambda & \downarrow \\
 \mathbf{Set}[\mathbb{T}_2] & \xrightarrow{\quad\quad\quad} & \mathbf{Set}
 \end{array}$$

Feel free to assume that \mathbb{T}_1 and \mathbb{T}_2 are single sorted if you wish.

The riddle (▲). Show that a presheaf topos $\mathbf{Set}^{\mathbf{C}}$ is boolean if and only if \mathbf{C} is a groupoid.