

LS

# The structure - semantics duality

this is the last lecture of the "Algebre" module of this course.

Until this moment we have focus on "finitary" universal algebra, sticking to operations with finite arity, as the intuition suggests

$$A^n \rightarrow A.$$

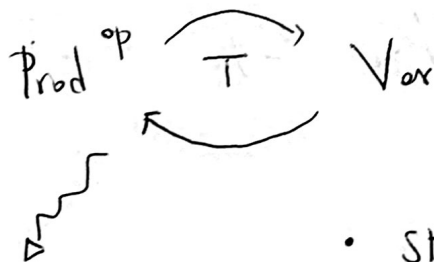
In this lecture we break the chains of "finitary algebra". Before doing so, let us recap what we did -

Syntax

category with products finite	abstract theory
Law $\text{Fin}_S^{\text{op}} \rightarrow \mathcal{C}$ theory	Presentation of a theory
finitary monad $T: \text{Set}^S \rightarrow \text{Set}^S$	

Semantics

Variety	Semantics
variety + forgetful functor to $\text{Set}^S$	Structured sets
$\text{Alg}(T)$	



via slings we get correspondence between

Law / Varieties equipped with a monadic functor to  $\text{Set}$

## Reflections

- Starting from a finitary monad  $T$  its "theory" is  $\text{Kl}(T)^{\text{op}}$  and its algebras are  $\text{Alg}(T)$ .

In this lecture we "reboot" the course (up to this point) and we try to discard any finitary assumption to see how abstract nonsense makes this theory "easier".

So, let's go back where we started from: the data of a category  $K \xrightarrow{u} \mathbf{Set}$  equipped with a forgetful functor to  $\mathbf{Set}$ .

- Unbounded implicit operations, similarly to Lawvere, we define a category whose objects are sets and ~~operations~~ <sup>morphisms</sup> are given by operations of arity unrestricted

$$\Pi_u = \begin{cases} \text{obj sets } I, J \\ \text{morphisms } \text{Nat}(u^I, u^J) \end{cases}$$

of course, if  $u$  has a left adjoint  $L$ , we easily get

$$\text{Nat}(u^I, u^J) = \text{Nat}(L(J)^{\bullet}, L(I)^{\bullet}),$$

analogously to the Grp case that we saw in the first lecture

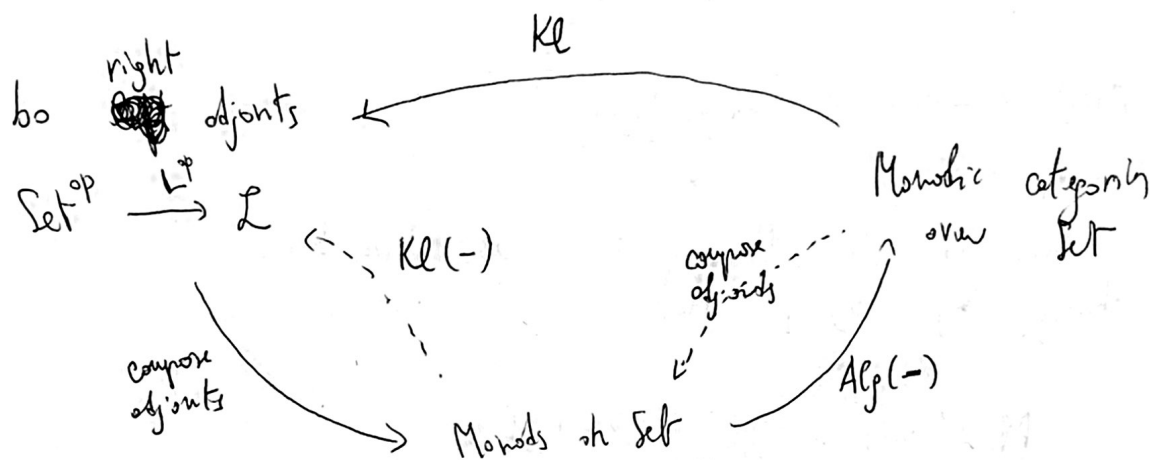
- So  $\Pi_u$  naturally comes equipped with a functor

$$\begin{array}{ccc} \mathbf{Set}^{\mathcal{P}} & \longrightarrow & \Pi_u \\ I & \xrightarrow{\quad \text{"L"} \quad} & I \\ \mathbf{Set}(I, J)^{\mathcal{P}} & \xrightarrow{\quad \text{"L"} \quad} & K(L(J), L(I)) \end{array}$$

So, given a right adjoint  $K \overset{\overset{1}{\dots}}{\underset{u}{\longrightarrow}} \text{Set}$ , its "full" algebraic theory is nothing but

$$\text{Set}^{\mathcal{P}} \xrightarrow{L^{\mathcal{P}}} \text{Rel}(T)^{\mathcal{P}}$$

( $T$  is the method associated to  $L + u$ )



- Let ... one should say that, going back to our original example  $K \xrightarrow{u} \text{Set}$ , our fundamental construction

$$\pi_u \begin{cases} \rightarrow \text{sets} \\ \rightarrow \text{Net}(u^I, u^J) \end{cases}$$

is still available, even in its "finitary" version.

And even more, we will get a comparison  
functor. Here made

Hand-drawn diagram illustrating a mapping from a set  $K$  to the model class  $\text{Mod}(\text{Th})$ . The mapping is labeled with a function symbol  $\ominus$ . A crossed-out arrow points from  $K$  to  $\text{Set}$ , indicating that the mapping is not to the set of all sets. A note next to the arrow to  $\text{Mod}(\text{Th})$  says "methodic functor".

Here models are  
really functions preserving  
product !!

$$\oplus: K \rightarrow \mathcal{U}(K)^{(-)}$$

Here we  
could both choose  
the full algebraic  $\Pi$  or  
its finitary version

this observation is due to Linton and in the old times was called "the methodic completion" of a functor  $K \rightarrow \text{Set}$  in the sense that it provides a completion making it into the best free object that approximates it.

### The structure-semantic adjunction

We can rephrase the question we have been asking by saying that

$$\text{Mod}(\text{Set})^{\text{op}} \xrightarrow{\text{Alg}(-)} \text{Cat}/\text{Set}$$

we are trying to find an adjoint for the functor  $\text{Alg}(-)$ , and in a sense we have provided an indirect answer given by the monad associated to  $\Pi$ . But there is a more direct way to answer this question.

#### • Codensity monad.

$$\begin{array}{ccc} K & \xrightarrow{u} & \text{Set} \\ u \downarrow & \nearrow \text{ran } u & \\ \text{Set} & & \end{array}$$

- Assume we have no problems in constructing rens. let us show that

$\text{ran } u$  is always a monad (called codensity monad)

- $\text{ran } \mathcal{U}$  is a functor. ok.

- To provide the unit  $1 \rightarrow \text{ran } \mathcal{U}$  we use the universal property or  $\mathcal{U}$  ran's

$$\begin{array}{ccc} & \text{ran}(\cdot) & \\ \mathcal{K} \swarrow & \xrightarrow{\quad \mathcal{T} \mathcal{U} \quad} & \searrow \text{Set} \\ \text{Set} & & \text{Set} \\ & \mathcal{U} & \end{array}$$

with implies that to provide a map

$$1 \xrightarrow{\quad \eta \quad} \text{ran } \mathcal{U}$$

is the same of providing a map

$$\text{Id}_{\mathcal{U}} \xrightarrow{\quad \text{id}_{\mathcal{U}} \quad} \mathcal{U}, \text{ and we choose the identity.}$$

- to provide the multiplication  $\text{ran } \mathcal{U} \circ \text{ran } \mathcal{U} \Rightarrow \text{ran } \mathcal{U}$

we observe that this is the same of providing

$$\text{ran } \mathcal{U} \circ \text{ran } \mathcal{U} \circ \mathcal{U} \Rightarrow \mathcal{U}$$

and now we use again the property  $\mathcal{U}^* \vdash \text{ran}(-)$  to get

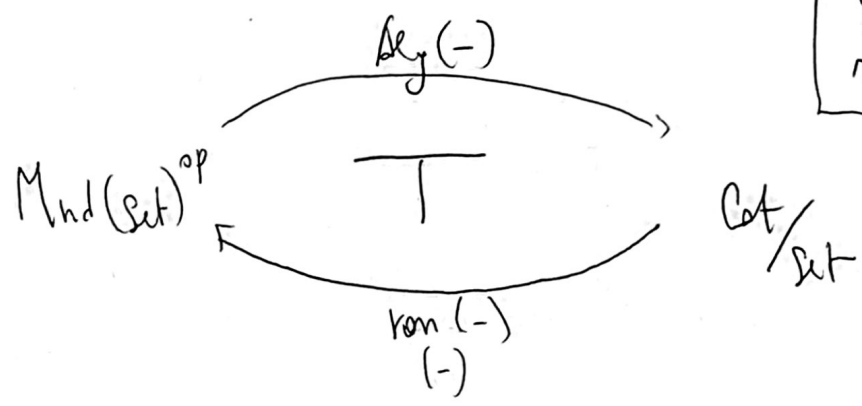
$$\text{ran } \mathcal{U} \circ \boxed{\text{ran } \mathcal{U} \circ \mathcal{U}} \xRightarrow{\text{counit}} \text{ran } \mathcal{U} \circ 1 \simeq \text{ran } \mathcal{U}$$

Thm (1)  $Alg(T_u) \simeq Mod(\Pi_u)$ .

$\nwarrow$   
 the codensity monad

$\nwarrow$   
 the "full low core theory" -

Thm (2)



Structure reverts to adjunction

- $yon(-)$  is fully faithful!
- We checked a little bit with the existence of Kan extensions  $Cat/Set$  should be replaced with "functors for which  $yon_u$  exists"

to prove thm (1) there are two strategies -

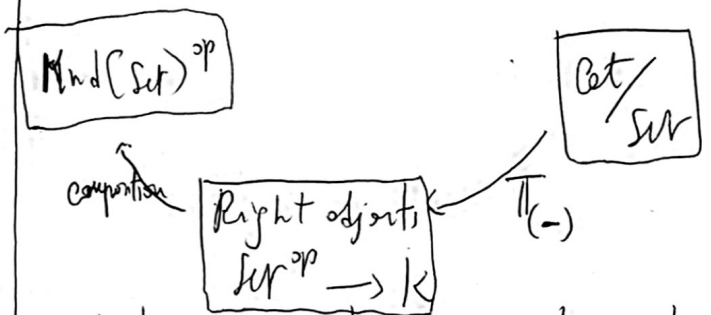
Strategy 1

Provide a functor

$$Mod(\Pi_u) \longrightarrow Alg(T_u)$$

and show that it is an equivalence

Strategy 2



show that the composition above offers a left adjoint for  $Alg(-)$  and use uniqueness of R.A.

• One last word about "arity".

OK but given a functor  $K \xrightarrow{u} \text{Set}$  we have produced two theories

$\Pi_{\text{fin}}^{\text{fin}}$   
 $u$   
 $\uparrow$   
 $\{$   
 - only finitary operations.  
 - category with products fin

$\Pi^{\text{fin}}$   
 $u$   
 $\uparrow$   
 $\{$   
 - also infinitary operations  
 - category with all products.

Which one should we trust? Which one should we invest on?

~~the answer is very simple in the case of Grp we get both that~~

the answer is very simple in the case of Grp we get both that

$$\text{Grp} \simeq \text{Prod}_{\text{fin}}(\Pi_{\text{fin}}^{\text{fin}}, \text{Set}) \simeq \text{Prod}_{\text{all}}(\Pi^{\text{fin}}, \text{Set})$$

and the reason is that the inclusion

$$\Pi_{\text{fin}}^{\text{fin}} \hookrightarrow \Pi^{\text{fin}} \text{ is dense!!}$$

in full generality this will not happen.

So  $\prod_n^{\text{Fin}}$  will axiomatize all the finite model generation

-  $\prod_n$  will axiomatize ALL the natural generation

- None of these might be enough  $K = \text{Cat}$

-  $\prod_n$  might be enough  $K = \text{Set}$

- both might be ok  $K = \text{Grp}$