INTRODUCTION TO CATEGORICAL LOGIC

IVAN DI LIBERTI

rules

- Hand your exercises by the midnight of the **7th of May** via email. Make my life easier: include the word **CL23 in the subject**.
- Pick at least one exercise from each of the yellow groups.
- At each stage of the exercise sheet, you can give for granted the statements of all the exercises that come before the one you are solving.
- You must charge at least **3** batteries!
  Example. The vector of exercises [1,5,8,12,15,18] would pass this sheet.

EXERCISES

doctrines and types

Exercise 1 (Naming,). Consider the category of sets and the usual powerset doctrine defined over it, $\mathcal{P} : \mathsf{Set}^\circ \to \mathsf{InfLat}$. Using the usual *epi-mono* factorization, we can define a functor $[-]_X : \mathsf{Set}_{/X} \to \mathcal{P}(X)$. Inspired by this construction, for $\mathcal{P} : \mathsf{C}^\circ \to \mathsf{InfLat}$ a doctrine with a sufficient amount of structure[a] construct a (pseudo)natural transformation

$$[-]_{(=)} : \mathsf{C}_{/(=)} \Rightarrow \mathcal{P}(=).$$

Exercise 2 (...and necessity,). We say that a doctrine has *comprehension schema* if the *naming* functor $[-]_{(=)} : \mathsf{C}_{/A} \Rightarrow \mathcal{P}(A)$ of the exercise above has a right adjoint $\{A : -\}$ for all A. Prove that if $\mathcal{P} : \mathsf{C}^\circ \to \mathsf{InfLat}$ is a doctrine with a sufficient amount of structure, the canonical subobject doctrine defined on its category of elements has comprehension schema,

$$\mathsf{Sub} : \mathsf{Elts}(\mathcal{P})^\circ \to \mathsf{InfLat}.$$

Exercise 3 (,). Provide a type theoretic interpretation of a doctrine with comprehension schema.

Exercise 4 (). Provide a translation between the notion of comprehension category and that of category with display maps.

[a] It is enough that C has finite limits and $\mathcal{P}f$ has a left adjoint.

Date: March 6, 2023.

topoi as spaces

Exercise 5 (). Show that the category of sheaves over the Sierpinski space is a presheaf topos. Which one?

Exercise 6 (). Show that $\mathbf{Set}^{\to}$ has a closed subtopos and an open subtopos. Please, provide a full proof that the geometric morphisms you present have the property we require, don't just state it.

Exercise 7 (). Let X be a compact Hausdorff space. Show that the direct image of the terminal geometric morphism $\Gamma_* : \mathsf{Sh}(X) \to \mathsf{Set}$ preserve directed colimits of monomorphisms.

topoi as objects

Exercise 8 (,). Show that the bicategory of topoi has (pseudo)colimits. *Hint.* Yes, this exercise is too hard.

Exercise 9 (,). Show that the bicategory of topoi has (pseudo)pullbacks. *Hint.* Yes, this exercise is too hard.

Exercise 10 (). Show that open geometric morphisms are pullback stable.

Exercise 11 (). Show that closed geometric morphisms are pullback stable.

topoi as objects

Exercise 12 (,). Show that there is a bijective correspondence between

$$\mathsf{Sub}_{\mathcal{E}}(1) \simeq \mathsf{Topoi}(\mathcal{E}, \mathbf{Set}^{\to}).$$

Exercise 13 (). Show that the bicategory of topoi has a classifier of closed embeddings, i.e., there exists a closed embedding $p : \mathcal{F}_1 \to \mathcal{F}_2$ such that every closed subtopos can be obtained by pulling back a geometric morphism along p.

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \mathcal{F}_1 \arrow[d, "p"] \\
\mathcal{E} \arrow[r] & \mathcal{F}_2
\end{tikzcd}$$

Prove an anologous statement also for open embeddings. *Hint.* To get the proper intuition, first solve it for spaces, then for locales, and then for topoi.

Exercise 14 (). Show that every open subtopos is *complemented*, i.e. there exists a closed subtopos that is its complement in the lattice of subtopoi.

topoi as sets

Exercise 15. Provide a complete description of the subobject classifier in $\mathbf{Set}^{\mathbb{N}}$, where the category structure of $\mathbb{N}$ is the expected posetal one.

Exercise 16. Show that every topos has a partial map classifier. *Hint:* What is the partial map classifier in Set?

Exercise 17. Prove that an object of a topos $\mathcal{E}$ is injective (with respect to monos) if and only if it is a retract of Ω^x for some x. Deduce that if e is injective then the functor $[-, e] : \mathcal{E}^\circ \to \mathcal{E}$ preserves reflexive coequalizers.

topoi as theories

Exercise 18. Consider the category of non empty finite sets $\mathsf{Fin}_{>0}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{>0}}$ classify?

Exercise 19. Consider the category of finite sets and monomorphisms $\mathsf{Fin}_{\hookrightarrow}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{\hookrightarrow}}$ classify?

Exercise 20. Consider the category of finite sets and epimorphisms $\mathsf{Fin}_{\twoheadrightarrow}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{\twoheadrightarrow}}$ classify?

Exercise 21. Consider the category of pointed finite sets $\mathsf{Fin}_{\bullet}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{\bullet}}$ classify?

Exercise 22. Consider the comma topos below, and assume comma topoi exist in the bicategory of topoi. Can you describe how does a $\mathbf{Set}$-model of the comma topos look like (in terms of models of $\mathbb{T}_1$ and $\mathbb{T}_2$)?

$$\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, dashed] & \mathbf{Set}[\mathbb{T}_1] \arrow[d] \\ \mathbf{Set}[\mathbb{T}_2] \arrow[r] \arrow[ur, Rightarrow, dashed, "\lambda"] & \mathbf{Set} \end{tikzcd}$$

Feel free to assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are single sorted if you wish.

The riddle (⚠). Show that a presheaf topos $\mathsf{Set}^{\mathsf{C}}$ is boolean if and only if C is a groupoid.