The Scott adjunction

Towards formal model theory

Ivan Di Liberti

Twas brillig, and the slithy toves
Did gyre and gimble in the wabe;
All mimsy were the borogoves,
And the mome raths outgrabe.

Jabberwocky, Lewis Carroll.

Introduction

The scientific placing of this thesis is a point of contact between logic, geometry and category theory. For this reason, we have decided to provide three introductions. The readers are free to choose which one to read, depending on the their interests and the background. Every introduction covers the whole content of the thesis, but it is shaped on the expected sensibility of the reader. Also, they offer a *sketch of the elephant*.

To the geometer

Grothendieck topoi have been introduced as a generalized notion of locales of open sets of a topological space. The geometric intuition has played a central role since the very early days of this theory and has reached its highest picks in the 90's. In a similar fashion Garner has introduced the notion of ionad. Whereas a topos is a categorified locale, a ionad is a categorified topological space. This thesis studies the relationship between these to different approach to topology and uses this geometric intuition to study accessible categories with directed colimits. **Posets with directed suprema, topological spaces and locales** are tightly connected a provide three different descriptions of a geometric object.

$$\begin{tikzcd}
& \mathrm{Loc} \arrow[dl, bend right, "\mathbb{pt}"'] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathrm{Top} \arrow[ur, dashed, bend left, "\mathbb{O}"] & & \mathrm{Pos}_\omega \arrow[ul, dashed, "\mathsf{S}"'] \arrow[ll, dashed, "\mathsf{ST}"]
\end{tikzcd}$$

We build on this analogy (and the existing literature on the topic) and offer a generalization of the diagram above to **accessible categories with directed colimits, ionads and topoi**. The functors that regulated the interaction between these different approaches to geometry are the main characters of the thesis. These are the **Scott adjuction**, which relates accessible categories with directed colimits to topoi and the **categorified Isbell adjunction**, which relates bounded ionads to topoi. These adjunctions are studied. This topological picture fits the pattern of Stone-like dualities. As the latter is related to completeness results for proposition logic, its categorification is related to syntax-semantics dualities between categories of models and theories. This approach is studied in deep detail in the thesis, providing logical interpretation of ionads (while a logical intuition on topoi and accessible categories is available since years in the literature, since forever in the case of accessible categories).

To the logician

This thesis is mainly devoted to provide a topos theoretic approach to reconstruction results for categorical semantics. Given an accessible category $\mathcal{A}$, that we like to image as the category of models of some theory, we associate to it a geometric theory that is in a precise sense a best approximation of $\mathcal{A}$ among geometric theories. This approach is used to derive structural properties on $\mathcal{A}$ and provide an abstract framework in which to accommodate formal model theory. As Stone-duality offer a valid approach to the study of semantics for propositional logic, we chose to arm ourselves with some topological weapons, of course designed for this framework. In the thesis, topoi are used as alter-egos of geometric theories, while ionads and accessible categories represents spaces.

Main results and structure of the thesis

Promenade. The first chapter presents the Scott adjunction () and prepares the reader to the other chapters, showing to some feature of the Scott adjunction.

Theorem (Thm. 2.1,The Scott adjunction). There is a 2-adjunction,

$$\mathsf{S} : \mathrm{Acc}_{\omega} \leftrightarrows \mathrm{Topoi} : \mathsf{pt}.$$

The first connections with logic and geometry are hinted. It is intended to be a first encounter with the main character of the thesis.

Geometry. This chapter discusses the Scott adjunction from a geometric perspective. We start with a section in general topology dealing with posets, topological spaces and locales. Then we proceed to categorify the topological construction. We introduce the categorified Isbell duality and show that it is idempotent. We introduce the Isbell adjunction.

Theorem (Thm. 3.31, Categorified Isbell adjunction). There is a 2-adjunction,

$$\mathbb{O} : \mathrm{BIon} \leftrightarrows \mathrm{Topoi} : \mathbb{pt}.$$

The left adjoint of this adjunction was found by Garner, in order to find a right adjoint, we must allow for large ionad. Building on the idempotency of the categorified Isbell adjunction (Thm. 3.34), we describe those topoi for which the counit of the Scott adjunction is an equivalence of categories (Thm. 3.44).

Logic. We discuss the connection between classifying topoi, Scott topoi and Isbell topoi (Thm. 4.15 and Thm 4.26). We specialize the Scott adjunction to abstract elementary classes and locally decidable topoi.

Theorem (Thm. 4.34). The Scott adjunction restricts to locally decidable topoi and AECs.

$$\mathsf{S} : \mathrm{AECs} \leftrightarrows \mathrm{LDTopoi} : \mathsf{pt}$$

We introduce *categories of saturated objects* and related them to atomic topoi and categoricity.

Theorem (Thm. 4.50).

(1) Let $\mathscr{A}$ be a category of saturated objects, then $\mathsf{S}(\mathscr{A})$ is an atomic topos.
(2) If in addition $\mathscr{A}$ has the joint embedding property, then $\mathsf{S}(\mathscr{A})$ is boolean and two valued.
(3) If in addition j is isofull and faithful and surjective on objects, then $\mathscr{A}$ is categorical in some presentability rank.

Category Theory. We show that the 2-category of accessible categories with directed colimits is monoidal closed and that the 2-category of topoi is enriched over it. We show that it admits tensors and connect this fact with the Scott adjunction.

Toolbox. This chapter provides technical results that are used in the previous ones.

Intended readers

The process of writing this thesis has raised a quite big issue. On one hand we wanted to provide a mostly original text, which went directly to the point and did not recall too many well established results. This style would have been perfect for the category theorist that is aware of the scientific literature in categorical logic, which is of course a natural candidate reader for this thesis. On the other hand one of the main purposes for this project was to provide a categorical framework to accommodate the classical theory of abstract elementary classes, and thus to use a language which could have been accessible to a more classical logician (model theorist). Unfortunately, due to the scientific inclination of the author the thesis has taken a more and more categorical shift and too many notions should have been introduced to make the thesis completely self contained. It was eventually evident that there is no satisfying solution to this problem, and we hope that eventually a *book* on categorical logic that gathers all the relevant topics will come. Our practical solution to the issue of providing a gentle introduction to the main contents of the thesis was to start with a chapter of background that provides the most relevant definitions and guides the reader through their meaning and common uses. That chapter is thus mostly written for a reader that is a bit far from category theory and introduces many topics redirecting to the most natural references. The intention of the chapter is to introduce notions, concepts and ideas, not to explore them. The rest of the thesis is then written mostly having in mind someone that has a good understanding of the ideas exposed in the background chapter. We hope to have found a good equilibrium between this two souls of the thesis.

The role of the Toolbox

One of the most important chapters of this thesis is without any doubt the Toolbox. It is designed with two main targets in mind. The first one is somehow to be the carpet under which to cover the dust of the thesis, and thus contains all the technical results that do amount conceptual idea. The second one is to leave room in all the other chapters for concepts and relevant proofs. The main intention behind this targets is that of providing a clean and easy to read (and browse text). This chapter is cited everywhere in the chapters that precede it as a blackbox.

Acknowledgments

To be written.

Scientific Acknowledgments

To be written.

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To be written.

Contents

CHAPTER 1

Background

This chapter is devoted to introduce the main definitions and provide the proper references for the most relevant gadgets that will be used in the rest of the thesis. Hence, our intentions are the following:

- hand the reader with an easy-to-check list of definitions of the objects that we use;
- provide the reader that has never met these definitions with enough material and references to familiarize with the content of the thesis;
- meet the different necessities of the *intented readers*;
- fix the notation.

For the reasons above, every section contained in the chapter starts with a crude list of definitions. Then, a list of remarks and subsections will follow and is aimed to contextualize the given definitions and provide references. We stress that this chapter cannot give a complete presentation of the notions that introduces and cannot be considered exhaustive even for the purpose of reading this thesis. Instead it should be seen as an *auxilium* to navigate the existing literature. The end of the chapter fixes the notation.

ACHTUNG! 1.1. If an important definition depends on some others, we procede top-down instead of bottom-up. This means that the important definition is given first, then we procede to define the atoms that partecipate to it.

1. Category theory

This is a thesis in category theory. We decide not to introduce the most basic definitions of category, functors, adjoint functors and so on. Nor do we introduce those auxiliary notions that would distract the reader from the main content of the thesis. We assume complete fluency in the basic notions of category theory and some confidence with more advanced notions. Let us list a collection of references that might be useful along the reading. Several books contain a good exposition of the material that we need, our list does not have the ambition of picking the best reference for each topic.

(1) Basic category theory, [Lei14][Chap. 1-4];
(2) Monads, [Bor94a][Chap. 4].
(3) Kan extensions, [Bor94b][Chap. 3.7] and [Lib19][App. A];
(4) (Simmetric) Monoidal (closed) categories, [Bor94a][Chap. 6].
(5) 2-categories and bicategories, [Bor94b][Chap. 7].

2. Accessible and locally presentable categories

ACHTUNG! 1.2. In this section λ is a regular cardinal.

Definition 1.3 (λ-accessible category). A λ-accessible category $\mathcal{A}$ is a category with λ-directed colimits with a set of λ-presentable objects that generate by λ-directed colimits. An accessible category is a category that is λ-accessible for some λ.

Definition 1.4 (Locally λ-presentable category). A locally λ-presentable category is a cocomplete λ-accessible category. A locally presentable category is a category that is locally λ-presentable for some λ.

Definition 1.5 (λ-presentable object). An object $a \in \mathcal{A}$ is λ-presentable if its covariant hom-functor $\mathcal{A}(a, -) : \mathcal{A} \to \mathbf{Set}$ preserves λ-directed colimits.

Definition 1.6 (λ-directed posets and λ-directed colimits). A poset P is λ-directed if it is non empty and for every λ-small[1] family of elements $\{p_i\} \subset P$, there exists an upper bound. A λ-directed colimit is the colimit of a diagram over a λ-directed poset (seen as a category).

2.1. Literature. There are two main references for the theory of accessible and locally presentable categories, namely [AR94] and [MP89]. The first one is intended for a broader audience and has appeared few years after the second one. The second one is mainly concerned with the logical aspectes of this theory. We mainly reccommend [AR94] because it appears a bit more fresh in style and definetly less demanding in general knowledge of category theory. A more experienced reader (in category theory) that does not fear an outdated style and is mainly interested in logic could choose [MP89]. Even though [AR94] treats some 2-categorical aspects of this topic, [MP89]'s exposition is much more complete in this direction. Another good general exposition is [Bor94a][Chap. 5].

2.2. A short comment on these definitions.

Remark 1.7. The theory of accessible and locally presentable categories has gained quite some popularity along the years because of its natural ubiquity. Most of the categories of the *working mathematician* are accessible, with a few (but still extremely important) exceptions. For example, the category Top of topological spaces is not accessible. In general, categories of algebraic structures are locally $\aleph_0$-presentable and many relevant categories of geometric nature are $\aleph_1$-accessible. A sound rule of thumb is that locally finitely presentable categories correspond to categories of models essentially algebraic theories, in fact this is even a theorem in a proper sense [AR94][Chap. 3]. A similar intuition is available for accessible categories too, but some technical price must be paid [AR94][Chap. 5]. Accessible and locally presentable categories (especially the latter) are *tame* enough to make many categorical wishes come true, that's the case for example of the adjoint funtor theorem, that has a very easy to check version for locally presentable categories.

Remark 1.8. All in all, an accessible category should be seen as a category equipped with a small set of *small* objects such that every object can be obtained as a kind of directed union of them. In the category of topological

[1] This means that its cardinality is strictly less then λ. For example $\aleph_0$-small means finite.

spaces, these small objects are not enogh to recover any other object from them.

EXAMPLE 1.9. To clarify the previous remark, we give list of locally $\aleph_0$-presentable categories. On the right column we indicate the full subcategory of finitely presentable objects.

$\mathscr{K}$	$\mathscr{K}_\omega$
Set	finite sets
Grp	finitely generated groups
Mod(R)	finitely generated modules

It is not surprising at all that a set X is the directed union of its *finite* subsets.

REMARK 1.10. Accessible and locally presentable categories have a *canonical representation*, in terms of *free completions under λ-directed of colimits*. This theory is studied in [AR94][Chap. 2.C]. The free completion of a category C under λ-directed colimits is always indicated by $\mathsf{Ind}_\lambda(C)$ in this thesis.

THEOREM 1.11. A λ-accessible category $\mathscr{A}$ is equivalent to the free completion of $\mathscr{A}_\lambda$ under λ-directed colimits,

$$\mathscr{A} \cong \mathsf{Ind}_\lambda(\mathscr{A}_\lambda).$$

REMARK 1.12. Explicit descriptions of the free completion of a category under λ-directed colimits are indeed available. To be more precise, given a category C one can describe $\mathsf{Ind}(C)$ as the category of flat functors $\mathrm{Flat}(C^\circ, \mathbf{Set})$. In the special case of a category with finite colimits, we have a simpler description of flat functors. Let us state the theorem in this simpler case for the sake of simplicity.

THEOREM 1.13. Let C be a small category with finite colimits, then its free completion under directed colimits is given by the functors preserving finite limits from C° into sets,

$$\mathsf{Ind}(C) \cong \mathsf{Lex}(C^\circ, \mathbf{Set}).$$

2.3. Locally presentable categories and essentially algebraic theories. The connection between locally presentable categories and essentially algebraic theory is made precise in [AR94][Chap. 3]. While algebraic theories axiomatize operational theories, essentially algebraic theories axiomatize operational theories whose operations are only partially defined. Category theorist have an equivalent approach to essentially algebraic theories via categories with finite limits, this approach was initially due to Freyd [Fre02], a seminal work of Coste [Cos76] should be mentioned too.

2.4. Accessible categories and (infinitary) logic. Accessible categories have been connected to (infinitary) logic in several (partially independet) ways. This story is accounted in Chapter 5 of [AR94]. Let us recall two of the most symbolic results of that chapter.

(1) As locally presentable categories, accessible are categories of models of theories, namely *basic* theories [AR94][Def. 5.31, Thm. 5.35].
(2) Given a theory T in L_λ the category $\mathsf{Elem}_\lambda(T)$ of models and λ-elementary embeddings is accessible [AR94][Thm. 5.42].

Unfortunately, it is not true in general that the whole category of models and homomorphisms of a theory in L_λ is accessible. It was later shown by Lieberman [Lie09] and independently by Rosický and Beke [BR12] that abstract elementary classes are accessible too. We will say more about this recent develpments later in the thesis. The reader that is interested in this connection might find interesting [Vas19a], whose language is probably the closest to that of a model theorist.

3. Sketches

Definition 1.14 (Sketch). A sketch is a quadruple $\mathcal{S} = (S, L, C, \sigma)$ where

S is a small category;
L is a class of diagrams in S, called *limit* diagrams;
C is a class of diagrams in S, called *colimit* diagrams;
σ is a function assigning to each diagram in L a cone and to each diagram in C a cocone.

Definition 1.15. A sketch is

- *limit* if C is empty;
- *colimit* if L is empty;
- *mixed* (used only in emphatic sense) if it's not limit, nor colimit.
- *geometric* if each cone is finite.
- *coherent* if it is geometric and and every cocone is either finite or discrete, or it is a regular-epi specification.

Definition 1.16 (Morphism of Sketches). Given two sketches $\mathcal{S}$ and $\mathcal{T}$, a morphism of sketches $f : \mathcal{S} \to \mathcal{T}$ is a functor $f : S \to T$ mapping (co)limit diagrams into (co)limits diagrams and proper (co)cones into (co)cones.

Definition 1.17 (2-category of Sketches). The 2-category of sketches has sketches as objects, morphism of sketches as 1-cells and natural transformations as 2-cells.

Definition 1.18 (Category of models of a sketch). For a sketch $\mathcal{S}$ and a (bicomplete) category $\mathcal{C}$, the category $\mathsf{Mod}_{\mathcal{C}}(\mathcal{S})$ of $\mathcal{C}$-models of the sketch is the full subcategory of $\mathcal{C}^{\mathcal{S}}$ of those functors that are models. If it's not specified, by $\mathsf{Mod}(\mathcal{S})$ we mean $\mathsf{Mod}(\mathcal{S})$.

Definition 1.19 (Model of a sketch). A model of a sketch $\mathcal{S}$ is a category $\mathcal{C}$ is a functor $f : \mathcal{S} \to \mathcal{C}$ mapping each specified (co)cone in a (co)limit. If it's not specified a model is a **Set**-model.

3.1. Literature. There exists a pletora of different and yet completely equivalent approaches to the theory of sketches. We stick to the one that suits best our setting, following mainly [Bor94a][Chap. 5.6] or [AR94][Chap. 2.F]. Other authors, such as [MP89] and [Joh02b] use a different (and more classical) definition involving graphs. Sketches are normally used as generalized notion of theory, from this perspective this approaches are completely equivalent, because the underlying categories of models are the same.

[MP89][page 40] stresses that the graph-definition is a bit more flexible in *daily practise*. Sketches were introduced by C. Ehresmann. Guitart, Lair and Burroni should definitely be mentioned among the relevant contributors. This list of references does not make justice to the french school, that has been extremely prolific on this topic, yet, for the purpose of this thesis the literature above will be more then sufficient.

3.2. Sketches: logic and sketchable categories. Sketches became quite common among category theorists because of their expressivity. In fact, they can be used as a categorical analog of those theories that can be axiomatized by (co)limit properties. For example, in the previous section, essentially algebraic theories are precisely those axiomatizable by finite limits.

3.2.1. *From theories to sketches.* We have mentioned that a sketch can be seen as a kind of theory. This is much more than a motto, or a motivational presentation of sketches. In fact, given a (infinitary) first order theory $\mathbb{T}$, one can always construct in a more or less canonical way a sketch $\mathcal{S}_{\mathbb{T}}$ whose models are precisely the models of $\mathbb{T}$. This is very well explained in [Joh02b][D2.2], for the sake of exemplification, let us state the theorem which is most relevant to our context.

Theorem 1.20. If $\mathbb{T}$ is a (geometric*) (coherent**) theory, there there exists a (geometric*) (coherent**) sketch having the same category of models of $\mathbb{T}$.

Some readers might be unfamiliar with geometric and coherent theories, those are just very specific fragments of first order (infinitary) logic. For a very detailed and clean treatment we suggest [Joh02b][D1.1]. Sketches are a quite handy notion of theory because we can use morphism of sketchs as a notion of translation between theories.

Proposition 1.21 ([Bor94a][Ex. 5.7.14]). If $f : \mathcal{S} \to \mathcal{T}$ is a morphism of sketches, then the composition with f yealds an (accessible) functor $\mathsf{Mod}(\mathcal{S}) \to \mathsf{Mod}(\mathcal{T})$.

3.2.2. *Sketchability.* It should not be surprising that sketches can be used to *axiomatize* accessible and locally presentable too. The two following results appear, for example, in [AR94][2.F].

Theorem 1.22. A category is locally presentable if and only if it's (equivalent to) the category of models of a limits sketch.

Theorem 1.23. A category is accessible if and only if it's (equivalent to) the category of models of a mixed sketch.

4. Topoi

ACHTUNG! 1.24. In this section by topos we mean Grothendieck topos.

DEFINITION 1.25 (Topos). A topos $\mathscr{E}$ is lex-reflective[2] subcategory[3] of a category of presheaves over a small category,

$$i^* : \mathbf{Set}^{C^\circ} \leftrightarrows \mathscr{E} : i_*.$$

DEFINITION 1.26 (Geometric morphism). A geometric morphism of topoi $f : \mathscr{E} \to \mathscr{F}$ is an adjunction $f^* : \mathscr{F} \leftrightarrows \mathscr{E} : f_*$[4] whose left adjoint preserves finite limits (is left exact). We will make extensive use of the following terminology:

f^* is the inverse image functor;
f_* is the direct image functor.

DEFINITION 1.27 (2-category of Topoi). The 2-category of topoi has topoi as objects, geometric morphisms as 1-cells and natural transformations between left adjoints as 2-cells.

4.1. Literature. There are several standard references for the theory of topoi. To the absolute beginner and even the experienced category theorist that does not have much confidence with the topic, we recommend [Lei10]. Most of the technical content of the thesis can be understood via [LM94], a reference that we strongly suggest to start and learn topos theory. Unfortunately, the approach of [LM94] is a bit far for ours, and even though its content is sufficient for this thesis, the intuition that is provided is not 2-categorical enough for our purposes. The reader might have to integrate with the encyclopedic [Joh02a, Joh02b]. A couple of constructions that are quite relevant to us are contained only in [Bor94c], that is very much equivalent to [LM94].

4.2. A comment on these definitions. Topoi were defined by Grothendieck as a *natural* generalization of the category of sheaves $\mathsf{Sh}(X)$ over a topological space X. Their geometric nature was thus the first to be explored and exploited. Yet, with the time, many other properties and facets of them have emerged, making them the main concept in category theory between the 80's and 90's. Johnstone, in the preface of [Joh02a] gives 9 different interpretations of what a topos *can be*. In fact, this multi-faced nature of the concept of topos motivates the title of his book. In this thesis we will concentrate on three main aspects of topos theory.

- A topos is a (categorification of the concept of) locale;
- A topos is a (family of Morita-equivalent) geometric theory;
- A topos is an object in the 2-category of topoi.

The first and the second aspects will be conceptual, and will allow us to infer qualitative results in geometry and logic, the last one will be our *methodological point of view* on topoi, and ultimately the main reason for which [LM94] might not be a sufficient reference for this thesis.

[2]This means that it is a reflective subcategory and that the left adjoint preserves finite limits. Lex stands for *left exact*, and was originally motivated by homological algebra.

[3]Up to equivalence of categories.

[4]Notice that f_* is the right adjoint.

4.3. Site descriptions of topoi. The first definition of topos that has been given was quite different from the one that we have introduced. As we have mentioned, topoi were introduced as category of sheaves over a space, thus the first definition was based on a generalization of this presentation. This is the theory of sites, and the reader of [LM94] will recognize this approach in [LM94][Chap. 3]. In a nutshell, a site (C, J) is the data of a category C together a notion of covering families. For example, in the case of a topological space, C is the locale of open sets of X, and J is given by the open covers. Thus, a topos can be defined to be a category of sheaves over a site,

$$\mathscr{E} \cong \mathsf{Sh}(C, J).$$

$\mathsf{Sh}(C, J)$, is defined as a full subcategory of $\mathbf{Set}^{C^\circ}$, which turn out to be lex-reflective. That's the technical bridge between the site-theoretic description of a topos and the one at the beginning of the section. Site theory is extremely useful in order to study topoi as *categories*, while our approach is much more useful in order to study them as *objects*. We will never use explicitly site theory in the thesis, with the exception of a couple of proofs and a couple of examples.

4.4. Topoi and Geometry. It's a bit hard to convey the relationship between topos theory and geometry in a short subsection. We mainly address the reader to [Lei10]. Let us just mention that to every topological space X, one can associate its category of sheaves $\mathsf{Sh}(X)$ (and this category is a topos), moreover, this assignment uniquely identifies the locale of open sets of X, and thus is a very strong topological invariant. For this reason, the study of $\mathsf{Sh}(X)$ is equivalent to the study of X from the perspective of the topologist, and is very convenient in algebraic geometry and algebraic topology. For example, the category of sets is the topos of sheaves over the one-point-space,

$$\mathbf{Set} \cong \mathsf{Sh}(\bullet)$$

for this reason, algebraic geometers sometime call **Set** *the point*. This intuition is consistent with the fact that **Set** is the terminal object in the category of topoi. Moreover, as a point $p \in X$ of a topological space X is a continuous function $p : \bullet \to X$, a point of a topos $\mathscr{G}$ is a geometric morphism $p : \mathbf{Set} \to \mathscr{G}$. Parallelisms of this kind have motivated most of the definitions of topos theory and most have led to results very similar to those that were achieved in formal topology (namely the theory of locales). The class of points of a topos $\mathscr{E}$ has a structure of category $\mathsf{pt}(\mathscr{E})$ in a natural way, the arrows being natural transformations between the inverse images of the geometric morphisms.

4.5. Topoi and Logic. Geometric logic and topos theory are tightly bounded. Indeed, for a geometric theory $\mathbb{T}$ it is possible to build a topos $\mathbf{Set}[\mathbb{T}]$ (the classifying topos of $\mathbb{T}$) whose category of points is precisely the category of models of $\mathbb{T}$,

$$\mathsf{Mod}(\mathbb{T}) \cong \mathsf{pt}(\mathbf{Set}[\mathbb{T}]).$$

This amounts to the theory of classifying topoi [LM94][Chap. X] and each topos classifies a geometric theory. This gives us a logical interpretation of a topos. Each topos is *geometric theory*, which in fact can be recovered by any of its site of definition. Obviously, for each site that describe the same topos we obtain a different theory. Yet, this theories have the same category of models (in any topos). In this thesis we will exploit the construction of [Bor94c] to show that to each *geometric* sketch (a kind of theory), one can associate a topos whose points are precisely the models of the sketch. This is another way to say that the category of topoi can internalize a geometric logic.

4.5.1. *A couple of words on elementary topoi.* Grothendieck topoi are often treated in parallel with their cousins elementary topoi. The origins of the theory of elementary topoi dates back to Lawvere's *elementary theory of the category of sets* [Law64]. The general idea behind his program was find those axioms that make a category a good place to found mathematics. Eventually, elementary topoi bloomed from the collaboration of Lawvere and Tirney. In a nutshell, an elementary topos is a cartesian closed category with finite limits, and a subobject classifier. It turns out that every Grothendieck topos is an elementary topos, while every cocomplete elementary topos with a generator is a Grothiendieck topos. This two results spot a tight connection between the two concepts. The intuition that we have on them is yet quite different. An elementary topos is *a universe of sets*, while a Grothendieck topos is *a geometric theory* or a *categorified locale*. The theory of elementary topoi is extremely rich, but in this thesis the elementary-topos perspective will never play any role, we are mentioning elementary topoi precisely to stress that this notion will not be used in the thesis.

4.6. Special classes of topoi. In the thesis we will study some relevant classes of topoi. In this subsection we recall all of them and give a good reference to check further details. These references will be repeted in the relevant chapters.

Topoi	Reference
connected	[Joh02b][C1.5.7]
compact	[Joh02b][C3.2]
atomic	[Joh02b][C3.5]
locally decidable	[Joh02b][C5.4]
coherent	[Joh02b][D3.3]
boolean	[Joh02b][D3.4, D4.5], [Joh02a][A4.5.22]

5. Ionads

5.1. Garner's definitions.

Definition 1.28 (Ionad). A ionad $\mathcal{X} = (X, \mathrm{Int})$ is a set X together with a comonad $\mathrm{Int} : \mathbf{Set}^X \to \mathbf{Set}^X$ preserving finite limits.

Definition 1.29 (Category of opens of a ionad). The category of opens $\mathbb{O}(\mathcal{X})$ of a ionad $\mathcal{X} = (X, \mathrm{Int})$ is the category of coalgebras of Int. We shall denote by $U_{\mathcal{X}}$ the forgetful functor $U_{\mathcal{X}} : \mathbb{O}(\mathcal{X}) \to \mathbf{Set}^X$.

Definition 1.30 (Morphism of Ionads). A morphism of ionads $f : \mathcal{X} \to \mathcal{Y}$ is a couple $(f, f^\sharp)$ where $f : X \to Y$ is a set function and $f^\sharp$ is a lift of f^*,

$$\begin{tikzcd} \mathbb{O}(\mathcal{Y}) \arrow[r, "f^\sharp"] \arrow[d, "U_{\mathcal{Y}}"] & \mathbb{O}(\mathcal{X}) \arrow[d, "U_{\mathcal{X}}"] \\ \mathbf{Set}^Y \arrow[r, "f^*"] & \mathbf{Set}^X \end{tikzcd}$$

Definition 1.31 (Specialization of morphism of ionads). Given two morphism of ionads $f, g : \mathcal{X} \to \mathcal{Y}$, a specialization of morphism of ionads $\alpha : f \Rightarrow g$ is a natural transformation between $f^\sharp$ and $g^\sharp$,

$$\begin{tikzcd} \mathbb{O}(\mathcal{Y}) \arrow[r, bend left, "f^\sharp"] \arrow[r, bend right, "g^\sharp"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{O}(\mathcal{X}) \end{tikzcd}$$

Definition 1.32 (2-category of Ionads). The 2-category of ionads has ionads as objects, morphism of ionads as 1-cells and specializations as 2-cells.

Definition 1.33 (Bounded Ionads). A ionad $\mathcal{X}$ is bounded if $\mathbb{O}(\mathcal{X})$ is a topos.

5.2. Ionads and topological spaces. Ionads were defined by Garner in [Gar12], and to our knowledge that's all the literature available on the topic. His definition is designed to generalize the definition of topological space. Indeed a topological space $\mathcal{X}$ is the data of a set (of points) and an interior operator,

$$\mathrm{Int} : 2^X \to 2^X.$$

Garner builds on the well known analogy between powerset and presheaf categories and extends the notion of interior operator to a presheaf category. The whole theory is extremely consistent with the expectaions: while the poset of (co)algebras for the interior operator is the locale of open sets of a topological space, the category of coalgebras of a ionad is a topos, a natural categorificaion of the concept of locale.

5.3. A generalization and two related propositions. In his paper Garner mentions that in giving the definition of ionad he could have chosen a category intead of a set [Gar12][Rem 2.4], let us quote his own comment on the definition.

> [[Gar12], Rem. 2.4] In the definition of ionad, we have chosen to have a mere *set* of points, rather than a category of them. We do so for a number of reasons. The first is that this choice mirrors most closely the definition of topological space, where we have a set, and not a poset, of points. The second is that we would in fact obtain no extra generality by allowing a category of points. We may see this analogy with the topological case, where to give an interior operator on a poset of points $(X, \leq)$ is equally well to give a topology $\mathcal{O}(X)$ on X such that every open set is upwards-closed with respect to $\leq$. Similarly, to equip a small category C with an interior comonad is equally well to give an interior comonad on $X := \mathrm{ob}C$ together with a factorisation of the forgetful functor $\mathbb{O}(X) \to \mathsf{Set}^X$ through the presheaf category Set^C; this is an easy consequence of Example 2.7 below. However, the most compelling reason for not admitting a category of points is that, if we were to do so, then adjunctions such as that between the category of ionads and the category of topological spaces would no longer exist. Note that, although we do not allow a category of points, the points of any (well-behaved) ionad bear nonetheless a canonical category structure – described in Definition 5.7 and Remark 5.9 below – which may be understood as a generalisation of the specialisation ordering on the points of a space.

We have decided to allow ionads over a category, even a locally small (but possibly large) one. We will need this definition later in the text to establish a connecion between ionads and topoi. While the structure of cateory is somewhat accessory, as Garner observes, the one of proper class will be absolutely needed.

Definition 1.34 (Large Ionads). A (large) ionad $\mathcal{X} = (C, \mathrm{Int})$ is a locally small (but possibly large) category C together with a comonad $\mathrm{Int} : \mathbf{Set}^C \to \mathbf{Set}^C$ preserving finite limits.

Remark 1.35. In analogy with the notion of base for a topology, Garner defines the notion of base of a ionad [Gar12][Def. 3.1, Rem 3.2]. This notion will be a hand technical tool in the thesis and in the forthcoming remarks and proposition we introduce the technology that we plan to use. Our definition is pretty much equivalent to Garner's one (up to the fact that we keep flexibility on the size of the base) and is designed to be easier to handle in our setting.

Definition 1.36 (Base of a ionad). Let $\mathcal{X} = (X, \mathrm{Int})$ be a ionad. We say that a flat functor $e : B \to \mathbf{Set}^X$ generates[5] the ionad if Int is naturally isomorphic to the density comonad of e,

$$\mathrm{Int} \cong \mathsf{lan}_e e.$$

[5] This definition is just a bit different from Garner's original definition [Gar12][Def. 3.1, Rem 3.2], but completely equivalent. We stress that in this definition, we allow for large basis.

REMARK 1.37. We are aware that $\mathbf{Set}^C$ is not a locally small category, but in the thesis this will not give rise to any set-theoretic issue. Moreover, we will always consider large *bounded* ionads, and C will always be an accessible category.

In [Gar12][3.6, 3.7], the author lists three equivalent conditions for a ionad to be bounded. The conceptual one is obviously that the category of opens is a Grothendieck topos, while the other ones are more or less technical. In our treatment the equivalence between the three conditions would be false. But we have the following characterizaition.

REMARK 1.38. Later in the thesis, we will need a practical way to induce morphism of ionads. The following proposition does not appear in [Gar12] and will be our main *morphism generator*. From the perspective of developing technical tool in the theory of ionads, this proposition has an interest per se.

PROPOSITION 1.39. A ionad $\mathcal{X}$ is bounded if any of the following equivalent conditions is verified:

(1) It may be generated (up to isomorphism) from a small base [Gar12][Def. 3.1];
(2) $\mathrm{Int}_{\mathcal{X}}$ is a small functor;
(3) $\mathbb{O}(\mathcal{X})$ is a topos.

PROOF. Similar to [Gar12][3.6, 3.7]. □

PROPOSITION 1.40 (Generator of morphism of ionads). Let $\mathcal{X}$ and $\mathcal{Y}$ be ionads, respectively generated by a base $e_X : B \to \mathbf{Set}^X$ and $e_Y : C \to \mathbf{Set}^Y$. Let $f : X \to Y$ a functor admitting a lift as in the diagram below.

$$\begin{tikzcd} C \arrow[r, dashed, "f^\diamond"] \arrow[d, "e_Y"] & B \arrow[d, "e_X"] \\ \mathbf{Set}^Y \arrow[r, "f^*"] & \mathbf{Set}^X \end{tikzcd}$$

If one of the two following conditions holds, then f induces a morphism of ionads $(f, f^\sharp)$:

(a) e_X is a left adjoint;
(b) e_X is fully faithful;

PROOF.

(a) By the discussion in [Gar12][Exa 4.6, diagram (6)], it is enough to provide a morphism as described in the diagram below.

$$\begin{tikzcd} C \arrow[r, dashed, "f'"] \arrow[d, "e_Y"] & \mathbb{O}(\mathsf{lan}_{e_X} e_X) \arrow[d, "\mathsf{U}_{\mathcal{X}}"] \\ \mathbf{Set}^Y \arrow[r, "f^*"] & \mathbf{Set}^X \end{tikzcd}$$

Now by the universal property of the category of coalgebras for a comonad, since e_X is a left edjoint, there exists a functor making the triangle below commute.

$$\begin{tikzcd}
C \arrow[r, "f^\diamond"] \arrow[d, "e_Y"] \arrow[rr, bend left, dashed, "f'"] & B \arrow[r] \arrow[d, "e_X"] & \mathbb{O}(\mathsf{lan}_{e_X} e_X) \arrow[dl, "\mathsf{U}_{\mathfrak{X}}"] \\
\mathbf{Set}^Y \arrow[r, "f^*"] & \mathbf{Set}^X &
\end{tikzcd}$$

And thus we obtain the required f' by composition.

(b) Using [Gar12][Rem 4.5] and the main idea of [Gar12][Exa 4.6], it is enough to provide a natural transformation

$$f^* \circ \mathsf{lan}_{e_Y} e_Y \Rightarrow \mathsf{lan}_{e_X} e_X \circ f^*.$$

By the universal property of left Kan extensions, such a natural transformation is the same of a natural transformation,

$$f^* \circ e_Y \Rightarrow \mathsf{lan}_{e_X} e_X \circ f^* \circ e_Y.$$

Now, recall that $f^* \circ e_Y = e_X \circ f^\diamond$, and thus it is enough to find a natural transformation,

$$f^* \circ e_Y \Rightarrow (\mathsf{lan}_{e_X} e_X) \circ e_X \circ f^\diamond.$$

Since e_X is fully faithful, the composition $(\mathsf{lan}_{e_X} e_X) \circ e_X$ coincides with e_X itself, and thus we obtain the required arrow from the existence of a natural isomorphism between $f^* \circ e_Y$ and $e_X \circ f^\diamond$.

□

6. Notations and conventions

Most of the notation will be introduced when needed and we will try to make it as natural and intuitive as possible, but we would like to settle some notation.

(1) $\mathcal{A}, \mathcal{B}$ will always be accessible categories, very possibly with directed colimits.
(2) $\mathfrak{X}, \mathcal{Y}$ will always be ionads.
(3) When it appears, κ is a finitely accessible category.
(4) Ind_λ is the free completion under λ-directed colimits.
(5) $\mathcal{A}_\kappa$ is full subcategory of κ-presentable objects of $\mathcal{A}$.
(6) $\mathcal{G}, \mathcal{T}, \mathcal{F}, \mathcal{E}$ will be Grothendieck topoi.
(7) η is the unit of the Scott adjunction.
(8) ϵ is the counit of the Scott adjunction.
(9) A Scott topos is a topos of the form $\mathsf{S}(\mathcal{A})$.
(10) An Isbell topos is a topos of the form $\mathbb{O}(\mathfrak{X})$.

Notation 1.41 (Presentation of a topos). A presentation of a topos $\mathcal{G}$ is the data of a geometric embedding into a presheaf topos $f^* : \mathbf{Set}^C \leftrightarrows \mathcal{G} : f_*$. This means precisely that there is a suitable topology τ_f on C that turns $\mathcal{G}$ into the category of sheaves over τ, in this sense f *presents* the topos as the category of sheaves over the site (C, τ_f).

Notation 1.42 (Canonical presentation). In [Str81], Street showed that a topos is a lextotal category, in our terminology this means that the Yoneda embedding $よ : \mathcal{G} \to \mathbf{Set}^{\mathcal{G}^\circ}$ yields a presentation of $\mathcal{G}$, $よ^* : \mathbf{Set}^{\mathcal{G}^\circ} \leftrightarrows \mathcal{G} : よ_*$,

where the direct image $よ_*$ is the Yoneda embedding. We will refer to this presentation as the *canocical presentation* of $\mathscr{G}$. In this notation there is an abuse of terminology because, strictly speaking, the presheaf category $\mathbf{Set}^{\mathscr{G}^\circ}$ is not a topos[6]. Yet, the terminology is good in the sense that this presentation is terminal among all the presentations of $\mathscr{G}$.

[6]It lacks a generator. This is mainly a set theoretic issue, the category is indeed a cocomplete infinitary pretopos.

CHAPTER 2

Promenade

The Scott adjunction will be the main character of this thesis. This chapter introduces the reader to the essential aspects of the adjunction and hints to those features that will be developed in the following chapters. From a technical point of view we establish an adjunction between accessible categories with directed colimits and Grothedieck topoi. The qualitative content of the adjunction is twofolded. On one hand it has a very clean geometric interpretation, whose roots belong to Stone-like dualities and Scott domains. On the other, it can be seen as a syntax-semantics duality between formal model theory and geometric logic. In this chapter we provide enough information to understand the crude statement of the adjunction and we touch on these contextualizations. With the benefit of hindsight, one could say that this chapter, together with a couple of results that appear in the Toolbox, is a report of our collaboration with Simon Henry [Hen19].

STRUCTURE. The exposition is organized as follows:

- Sec. 1 we introduce the constructions involved in the Scott adjunction;
- Sec. 2 we provide some comments and insights on the first section;
- Sec. 3 we give a quick generalization of the adjunction and discuss its interaction with the standard theorem;
- Sec. 4 we prove the Scott adjunction.

1. The Scott adjunction: definitions and constructions

We begin by giving the crude statement of the adjunction, then we proceed to construct and describe all the objects involved in the theorem. The actual proof of 2.1 will close the chapter.

THEOREM 2.1 ([Hen19][Prop. 2.3] The Scott adjunction). There is a 2-adjunction,

$$\mathsf{S} : \mathrm{Acc}_\omega \leftrightarrows \mathrm{Topoi} : \mathsf{pt}.$$

REMARK 2.2 (Characters on the stage). Acc_ω is the 2-category of accessible categories with directed colimits, a 1-cell is a functor preserving directed colimits, 2-cells are natural transformations. Topoi is the 2-category of Groethendieck topoi. A 1-cell is a geometric morphism and has the direction of the right adjoint. 2-cells are natural transformation between left adjoints.

REMARK 2.3 (2-categorical warnings). Both Acc_ω and Topoi are 2-categories, but most of the time our intuition and our treatment of them will be 1-categorical, we will essentially downgrade the adjunction to a 1-adjunction where everything works *up to equivalence of categories*. We feel free to use any classical result about 1-adjunction, paying the price of decorating any statement with the correct use of the word *pseudo*. For example, right adjoints preserve pseudolimits, and pseudomorphism.

REMARK 2.4 (The functor pt). The functor of points pt is easier than its left adjoint to describe, because it belongs to the literature since much more time, pt is the covariant hom functor $\mathrm{Topoi}(\mathbf{Set}, -)$. It maps a Grothendieck topos $\mathcal{G}$ to its category of points (see Sec. 4),

$$\mathcal{G} \mapsto \mathrm{Cocontlex}(\mathcal{G}, \mathbf{Set}).$$

Of course given a geometric morphism $f : \mathcal{G} \to \mathcal{E}$, we get an induced morphism $\mathsf{pt}(f) : \mathsf{pt}(\mathcal{G}) \to \mathsf{pt}\mathcal{E}$ mapping $p^* \mapsto p^* \circ f^*$. The fact that $\mathrm{Topoi}(\mathbf{Set}, \mathcal{G})$ is an accessible category with directed colimits appears in the classical reference by Borceux as [Bor94c][Cor. 4.3.2], while the fact that $\mathsf{pt}(f)$ preserves directed colimits follows trivially from the definition.

1.1. The Scott construction.

REMARK 2.5 (The Scott construction). We recall the construction of the S from [Hen19]. Let $\mathcal{A}$ be an accessible category with directed colimits. $\mathsf{S}(\mathcal{A})$ is defined as the category the category of functors preserving directed colimits into sets.

$$\mathsf{S}(\mathcal{A}) = \mathrm{Acc}_\omega(\mathcal{A}, \mathbf{Set}).$$

For a functor $f : \mathcal{A} \to \mathcal{B}$ be a 1-cell in Acc_ω, the geometric morphism $\mathsf{S}f$ is defined by precomposition as described below.

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "f"] & \mathsf{S}\mathcal{A} \arrow[d, bend left, "f_*"] \\
\mathcal{B} & \mathsf{S}\mathcal{B} \arrow[u, bend left, "f^*"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

$\mathsf{S}f = (f^* \dashv f_*)$ is defined as follows: f^* is the precomposition functor $f^*(g) = g \circ f$. This is well defined because f preserve directed colimits. f^* is a functor preserving all colimits between locally presentable categories[1] and thus has a right adjoint, that we indicate with f_*. Observe that f^* preserve finite limits because finite limits commute with directed colimits in $\mathbf{Set}$.

REMARK 2.6 ($\mathsf{S}(\mathcal{A})$ is a topos). Together with 2.5 this shows that the Scott construction provides a 2-functor $\mathsf{S} : \mathrm{Acc}_\omega \to \mathrm{Topoi}$. A proof has already appeared in [Hen19][2.2] with a practically identical idea. The proof relies on the fact that, since finite limits commute with directed colimits, the category $\mathsf{S}(\mathcal{A})$ inherits from its inclusion in the category of all functors $\mathcal{A} \to \mathbf{Set}$ all the relevant exactness condition prescribed by Giraud axioms. The rest of the proof is devoted to provide a generator for $\mathsf{S}(\mathcal{A})$. In the proof below we pack in categorical technology the proof-line above.

PROOF. By definition $\mathcal{A}$ must be λ-accessible for some λ. Obviously $\mathrm{Acc}_\omega(\mathcal{A}, \mathbf{Set})$ sits inside λ-$\mathrm{Acc}(\mathcal{A}, \mathbf{Set})$. Recall that λ-$\mathrm{Acc}(\mathcal{A}, \mathbf{Set})$ is equivalent to $\mathbf{Set}^{\mathcal{A}_\lambda}$ by the restricion-Kan extension paradigm and the universal

[1]this is shown in 2.6.

property of Ind_λ-completion. This inclusion $i : \mathrm{Acc}_\omega(\mathscr{A}, \mathbf{Set}) \hookrightarrow \mathbf{Set}^{\mathscr{A}_\lambda}$, preserves all colimits and finite limits, this is easy to show and depends on one hand on how colimits are computed in this category of functors, and on the other hand on the fact that in **Set** directed colimits commute with finite limits. By the adjoint functor theorem, $\mathrm{Acc}_\omega(\mathscr{A}, \mathbf{Set})$ amounts to a coreflective subcategory of a topos whose associated comonad is left exact. By [LM94][V.8 Thm.4], it is a topos. □

REMARK 2.7 (A description of f_*). In order to have a better understainding of the right adjoint f_*, which in the previous remark was shown to exist via a special version of the adjoint functor theorem, we shall fit the adjunction $(f^* \dashv f_*)$ fits in a more general picture. We start by introducing the diagram below.

$$\begin{tikzcd}
\mathsf{S}\mathscr{A} \arrow[d, "\iota_{\mathscr{A}}"] \arrow[rr, bend right, "f_*"'] & & \mathsf{S}\mathscr{B} \arrow[ll, "f^*"'] \arrow[d, "\iota_{\mathscr{B}}"] \\
{[\mathscr{A}, \mathbf{Set}]} \arrow[rr, bend left, "\mathsf{lan}_f"] \arrow[rr, bend right, "\mathsf{ran}_f"'] & & {[\mathscr{B}, \mathbf{Set}]} \arrow[ll, "f^*"']
\end{tikzcd}$$

(1) Observe that the natural inclusion $\iota_{\mathscr{A}}$ of $\mathsf{S}\mathscr{A}$ in $[\mathscr{A}, \mathbf{Set}]$ has a right adjoint[2] $r_{\mathscr{A}}$, namely $\mathsf{S}\mathscr{A}$ is coreflective and it coincides with the algebras for the comonad $\iota_{\mathscr{A}} \circ r_{\mathscr{A}}$. If we ignore the evident size issue for which $[\mathscr{A}, \mathbf{Set}]$ is not a topos, the adjunction $\iota_{\mathscr{A}} \dashv r_{\mathscr{A}}$ amounts to a geometric surjection $r : [\mathscr{A}, \mathbf{Set}] \to \mathsf{S}\mathscr{A}$.

(2) The left adjoint lan_f to f^* does exist because f preserve directed colimits, while in principle ran_f may not exists because it is not possible to cut down the size of the limit in the ran-limit-formula. Yet, for those functors for which it is defined, it provides a right adjoint for f^*. Observe that since the f^* on the top is the restriction of the f^* on the bottom, and $\iota_{\mathscr{A},\mathscr{B}}$ are fully faithful, f_* has to match with $r_{\mathscr{B}} \circ \mathsf{ran}_f \circ \iota_{\mathscr{A}}$, when this composition is well defined,

$$f_* \cong r_{\mathscr{B}} \circ \mathsf{ran}_f \circ \iota_{\mathscr{A}},$$

indeed the left adjoint f^* on the top coincides with $f^* \circ \iota_{\mathscr{B}}$ and by uniqueness of the right adjoint one obtains precisely the equation above. Later in the text this formula will prove to be useful. We can already use it to have some intuition on the behaviour f_*, indeed $f_*(p)$ is the best approximation of $\mathsf{ran}_f(p)$ preserving directed colimits. In particular if it happens for some reason that $\mathsf{ran}_f(p)$ preserves directed colimits, then this is precisely the value of $f_*(p)$.

REMARK 2.8. Let $\mathscr{A}$ be a λ-accessible category, then $\mathsf{S}(\mathscr{A})$ can be described as the full subcategory of $\mathbf{Set}^{\mathscr{A}_\lambda}$ of those functors preserving λ-small ω-filtered colimits. A proof of this observation can be found in [Hen19][2.2], and in fact shows that $\mathsf{S}(\mathscr{A})$ has a generator.

[2]This will be shown in 3.25.

2. The Scott adjunction: comments and suggestions

Remark 2.9 (Cameos in the existing literature). Despite the name, nor the adjunction nor the construction is due to Scott and was presented for the first time in [Hen19], even if it implicitly appeared in special cases both the the very classical literature [Joh02b] and in some recent developent [AL18]. Karazeris introduces the notion of Scott topos of a finitely accessible category $\mathcal{K}$ in [Kar01], this notion coincides with $\mathsf{S}(\mathcal{K})$, as the name suggests. In Chap. 3 we will make the connection with some seminal works of Scott and clarify the reason for which this is the correct categorification of a construction which is due to him. As observed in [Hen19][2.4], the Scott construction is the categorification of the usual Scott topology on a directed complete poset. This will help us to develop a geometric intuition on accessible category with directed colimits, those will look like the underlying set of some topological space. We cannot say to be satisfied with this choice of name for the adjunction, but we did not manage to come up with a better name.

Remark 2.10 (Schizophrenicity). A schizophrenic adjunction is an adjunction that is contravariantly induced by a schizophrenic[3] object. For example, the famous dual adjunction between frames and topological spaces [LM94][Chap IX],

$$\mathbb{O} : \mathsf{Top} \leftrightarrows \mathsf{Frm}^\circ : \mathsf{pt}$$

is induced by the Sierpinski space $\mathbb{T}$. Indeed, since it admits a natural structure of frame, and a natural structure of topological space the adjunction above can be recovered in the form

$$\mathsf{Top}(-, \mathbb{T}) : \mathsf{Top} \leftrightarrows \mathsf{Frm}^\circ : \mathsf{Frm}(-, \mathbb{T}).$$

Most of the known topological dualities are induced in this way. The interested reader might want to consult [PT91]. Makkai has shown ([MP87], [Mak88]) that relevant families of syntax-semantics dualities can be induced in this way using the category of sets as a dualizing object. In this fashion, the content of Rem. 2.5 together with Rem. 2.4 acknowledges that $\mathsf{S} \dashv \mathsf{pt}$ is essentially a schizophrenic 2-adjunction induced by the object **Set** that inhabits both the 2-categories.

Remark 2.11 (Generalized axiomatizations). As was suggested by Joyal, the category Logoi = Topoi$^\circ$ can be seen as the category of geometric theories. Caramello [Car10] pushes on the same idea stressing the fact that a topos is a Morita-equivalence class of geometric theories. In this perspective the Scott adjunction, which in this case is a dual adjunction

$$\mathrm{Acc}_\omega \leftrightarrows \mathrm{Logoi}^\circ,$$

presents $\mathsf{S}(\mathcal{A})$ as a free geometric theory attached to the accessible category $\mathcal{A}$ that is willing to axiomatize $\mathcal{A}$. When $\mathcal{A}$ has a faithful functor preserving directed colimits into the category of sets, $\mathsf{S}(\mathcal{A})$ axiomatizes an envelop of $\mathcal{A}$, as proved in 6.19. A logical understanding of the adjunction will be developed in Chap 4, we connect the Scott adjunction to the theory of classifying topoi and to the seminal works of Lawvere and Linton in

[3] Often called dualizing.

categorical logic. This intuition will be used also to give a topos theoretic approach to abstract elementary classes.

Remark 2.12 (Trivial behaviours and Diaconescu). S willing to be as simple as possible. If $\mathscr{K}$ is a finitely accessible category $\mathsf{S}(\mathscr{K})$ coincides with the presheaf topos $\mathbf{Set}^{\mathscr{K}_\omega}$, where we indicated with $\mathscr{K}_\omega$ the full subcategory of finitely presentable objects. This follows directly from the following chain of isomorphisms,

$$\mathsf{S}(\mathscr{K}) = \mathrm{Acc}_\omega(\mathscr{K}, \mathbf{Set}) \cong \mathrm{Acc}_\omega(\mathsf{Ind}(\mathscr{K}_\omega), \mathbf{Set}) \cong \mathbf{Set}^{\mathscr{K}_\omega}.$$

As a consequence of Diaconescu theorem [Joh02a][B3.2.7] and the characterization of the Ind-completion via flat functors, when restricted to finitely accessible categories, the Scott adjunction yields an equivalence of categories ω-Acc $\cong$ Presh with the full subcategory of presheaves topoi.

$$\begin{tikzcd}
\mathrm{Acc}_\omega \arrow[rr, bend right, "\mathsf{S}"'] & & \text{Topoi} \arrow[ll, bend right, "\mathsf{pt}"'] \\
\omega\text{-Acc} \arrow[u, hook] \arrow[rr, bend right] & & \text{Presh} \arrow[u, hook] \arrow[ll, bend right]
\end{tikzcd}$$

This observation is not new to literature, the proof of [Joh02b][C4.3.6] spots this special case of the Scott adjunction. It is very surprising that the book does not investigate, or even mention the existence of the Scott adjunction, since it gets very close to define it explicitly.

Theorem 2.13. The Scott adjunction restricts to a biequivalence of 2-categories between the 2-category of finitely accessible categories [4] and the 2-category of presheaf topoi[5].

$$\mathsf{S} : \omega\text{-Acc} \leftrightarrows \text{Presh} : \mathsf{pt}.$$

Proof. The previous remark has show that when $\mathscr{A}$ is finitely accessible, $\mathsf{S}(\mathscr{A})$ is a presheaf topos and that, when $\mathscr{E}$ is a presheaf topos, $\mathsf{pt}(\mathscr{E})$ is finitely accessible. To finish, we show that in this case the unit and the counit of the Scott adjunction are equivalence of categories. This is in fact essentially shown by the previous considerations.

$$(\mathsf{ptS})(\mathsf{Ind}(C)) \cong \mathsf{pt}(\mathbf{Set}^C) \overset{\text{Diac}}{\cong} \mathsf{Ind}(C).$$

$$(\mathsf{Spt})(\mathbf{Set}^C) \overset{\text{Diac}}{\cong} \mathsf{S}(\mathsf{Ind}(C)) \cong \mathbf{Set}^C.$$

$\square$

Remark 2.14. Thus, the Scott adjunction must induce an equivalence of categories between the Cauchy completions of ω-Acc and Presh. The Cauchy completion of ω-Acc is the full subcategory of Acc_ω of *continuous categories*

[4] With finitely accessible functors and natural transformation.

[5] With geometric morphisms and natural transformations between left adjoints.

[JJ82]. Continuous categories are the categorification of the notion of continuous poset and can be characterized as split subobjects of finitely accessible categories in Acc_ω. In [Joh02b][C4.3.6] Johnstone observe that if a continuous category is cocomplete, the corresponding Scott topos is injective (with respect to geometric embeddings) and viceversa.

EXAMPLE 2.15. As a direct consequence of Rem 2.12, we can calculate the Scott topos of **Set**. $\mathsf{S}(\mathbf{Set})$ is $\mathbf{Set}^{\mathrm{FinSet}}$. This topos is very often indicated as $\mathbf{Set}[\mathbb{O}]$, being the classifying topos of the theory of objects, i.e. the functor: $\mathrm{Topoi}(-, \mathbf{Set}[\mathbb{O}]) : \mathrm{Topoi}^\circ \to \mathrm{Cat}$ coincides with the forgetful functor. As a reminder for the reader, we state clearly the equivalence:

$$\mathsf{S}(\mathbf{Set}) \cong \mathbf{Set}[\mathbb{O}].$$

REMARK 2.16 (The Scott adjunction is not a biequivalence: Fields). Whether the Scott adjunction is a biequivalence is a very natural question to ask. Up to this point we noticed that on the subcategory of topoi of presheaf type the counit of the adjunction is in fact an equivalence of categories. Since presheaf topoi are a quite relevant class of topoi one might think that the whole biadjunction amounts to a biequivalence. That's not the case, in this remark provide a topos $\mathcal{F}$ such that the counit

$$\epsilon_{\mathcal{F}} : \mathsf{Spt}\mathcal{F} \to \mathcal{F}$$

is not an equivalence of categories. Let $\mathcal{F}$ be the classifying topos of the theory of geometric fields [Joh02b][D3.1.11(b)]. Its category of points is the category of fields Fld, since this category is finitely accessible the Scott topos $\mathsf{Spt}(\mathcal{F})$ is of presheaf type by 2.12,

$$\mathsf{Spt}(\mathcal{F}) = \mathsf{S}(\mathsf{Fld}) \overset{2.12}{\cong} \mathbf{Set}^{\mathsf{Fld}_\omega}.$$

It was shown in [Bek04][Cor 2.2] that $\mathcal{F}$ cannot be of presheaf type, and thus $\epsilon_{\mathcal{F}}$ cannot be an equivalence of categories.
Chapter 3 builds on a very strong analogy between the Scott adjunction and the locales-topological spaces adjunction. This remark shows that the analogy can be pushed only up to a certain point. $\mathcal{F}$ is a coherent topos (because it classifies a coherent theory) and thus has enough points [Joh77][7.44]. If the analogy was perfect, ϵ should be an equivalence of categories precisely for topoi with enough points. The last part of Chapter 3 dedicates to understand what can be saved of this intuition and studies up to what extent the Scott adjunction is idempotent[6]. We show that $\epsilon_{\mathcal{F}}$ is a geometric surjection and discuss in further detail this example.

3. The κ-Scott adjunction

The most natural generalization of the Scott adjunction is the one in which directed colimits are replaced with κ-filtered colimits and finite limits (ω-small) are replaced with κ-small limits, this unveils the deepest reason for which the Scott adjunction exists: namely κ-directed colimits commute with κ-small limits in the category of sets.

[6]Be careful, this example does not show that the Scott adjunction is not idempotent, at least not in this formulation, in fact we never proved that $\mathcal{F}$ is a Scott topos by any means.

THEOREM 2.17. [Hen19][Prop 3.4] There is an 2-adjunction

$$\mathsf{S}_\kappa : \mathrm{Acc}_\kappa \leftrightarrows \kappa\text{-Topoi} : \mathsf{pt}_\kappa.$$

REMARK 2.18. Acc_κ is the 2-category of accessible categories with κ-directed colimits, a 1-cell is a functor preserving κ-filtered colimits, 2-cells are natural transformations. Topoi_κ is the 2-category of Groethendieck κ-topoi. A 1-cell is a κ-geometric morphism and has the direction of the right adjoint. 2-cells are natural transformation between left adjoints. A κ-topos is a κ-exact localization of a presheaf topos. These creatures are not completely new to the literature but they appear sporadically and a systematic study is still missing. We should reference, though, the works of Espindola [Esp19].

REMARK 2.19. It is quite evident that every remark untill this point finds its direct κ-generalization substituting every occurrence of *directed colimits* with κ*-directed colimits.*

REMARK 2.20. Let $\mathscr{A}$ be a category in Acc_ω. Eventually in κ the Scott adjunction axiomatizes $\mathscr{A}$ (in the sense of Rem 2.11), in fact if $\mathscr{A}$ is κ-accessible $\mathsf{pt}_\kappa \mathsf{S}_\kappa \mathscr{A} \cong \mathscr{A}$, for the κ-version of Diaconescu Theorem, that in this text appears in Rem. 2.12.

REMARK 2.21. It pretty evident that λ-Topoi is a locally fully faithful sub 2-category of κ-topoi when $\lambda \geq \kappa$. The same is true for Acc_ω. This observation leads to a filtration of the categories Topoi and Acc_ω as shown in the following diagram,

$$\begin{tikzcd}
\kappa & \mathrm{Acc}_\kappa \arrow[r, bend right, "\mathsf{S}_\kappa"'] \arrow[d, "\iota_\kappa^\lambda"] & \kappa\text{-Topoi} \arrow[l, bend right, "\mathsf{pt}_\kappa"'] \arrow[d, "i_\kappa^\lambda"] \\
\lambda \arrow[u] & \mathrm{Acc}_\lambda \arrow[r, bend right, "\mathsf{S}_\lambda"'] \arrow[d, "\iota_\lambda^\omega"] & \lambda\text{-Topoi} \arrow[l, bend right, "\mathsf{pt}_\lambda"'] \arrow[d, "i_\lambda^\omega"] \\
\omega \arrow[u] & \mathrm{Acc}_\omega \arrow[r, bend right, "\mathsf{S}"'] & (\omega\text{-})\mathrm{Topoi} \arrow[l, bend right, "\mathsf{pt}"']
\end{tikzcd}$$

REMARK 2.22 (The diagram does **not** commute). We depicted the previous diagram in order to trigger the reader's pattern recogniction and conjecture its commutativity. In this remark we stress that the diagram does **not** commute, meaning that

$$\mathsf{S}_\lambda \circ \iota_\kappa^\lambda \ncong i_\kappa^\lambda \circ \mathsf{S}_\kappa,$$

at least not in general. In fact, once the definitions are spelled out, there is absolutely no reasons for which one should have commutativity in general. The same is true for the right adjoint pt.

REMARK 2.23. In the following diagram we show the interaction between 2.20 and the previous remark. Recall that presheaf categories belong to κ-topoi for every κ.

$$\begin{tikzcd}
\kappa\text{-Acc} \arrow[r] & \mathrm{Acc}_\kappa \arrow[r, bend right, "\mathsf{S}_\kappa"'] \arrow[d, "\iota_\kappa^\lambda"] & \kappa\text{-Topoi} \arrow[l, bend right, "\mathsf{pt}_\kappa"'] \arrow[d, "i_\kappa^\lambda"] & \mathrm{Presh} \arrow[l] \\
\lambda\text{-Acc} \arrow[r] & \mathrm{Acc}_\lambda \arrow[r, bend right, "\mathsf{S}_\lambda"'] \arrow[d, "\iota_\lambda^\omega"] & \lambda\text{-Topoi} \arrow[l, bend right, "\mathsf{pt}_\lambda"'] \arrow[d, "i_\lambda^\omega"] & \mathrm{Presh} \arrow[l] \\
\omega\text{-Acc} \arrow[r] & \mathrm{Acc}_\omega \arrow[r, bend right, "\mathsf{S}"'] & (\omega\text{-})\mathrm{Topoi} \arrow[l, bend right, "\mathsf{pt}"'] & \mathrm{Presh} \arrow[l]
\end{tikzcd}$$

REMARK 2.24. It might be natural to conjecture that Presh happens to be $\bigcap_\kappa \kappa$-Topoi. The author knowns only one counterexample, due to Simon Henry, namely $\mathsf{Sh}([0,1])$.

4. Proof of Thm. 2.1

We end this chapter including a full proof of the Scott adjunction.

FIRST PROOF OF 2.1. We spell out the unit and the counit of the adjunction.

- η For an accessible category with directed colimits $\mathcal{A}$ we must provide a functor $\eta_{\mathcal{A}} : \mathcal{A} \to \mathsf{ptS}(\mathcal{A})$. Define,
$$\eta_{\mathcal{A}}(a)(-) := (-)(a).$$
$\eta_{\mathcal{A}}(a) : \mathsf{S}(\mathcal{A}) \to \mathbf{Set}$ defined in this way is a functor preserving colimits and finite limits and thus defines a point of $\mathsf{S}(\mathcal{A})$.
- ϵ The idea is very similar, for a topos $\mathcal{E}$, we must provide a geometric morphism $\epsilon_{\mathcal{E}} : \mathsf{Spt}(\mathcal{E}) \to \mathcal{E}$. Being a geometric morphism, it's equivalent to provide a cocontinuous and finite limits preserving functor $\epsilon^*_{\mathcal{E}} : \mathcal{E} \to \mathsf{Spt}(\mathcal{E})$. Define,
$$\epsilon^*_{\mathcal{E}}(e)(-) = (-)^*(e).$$

□

SECOND PROOF OF 2.1. We prove that there exists an equivalence of categories,
$$\mathrm{Topoi}(\mathsf{S}(\mathcal{A}), \mathcal{F}) \cong \mathrm{Acc}_\omega(\mathcal{A}, \mathsf{pt}(\mathcal{F})).$$
The proof makes this equivalence evidently natural. This proof strategy is similar to that appearing in [Hen19], even thought it might looks different at first sight.

$$\begin{aligned}
\mathrm{Topoi}(\mathsf{S}(\mathcal{A}), \mathcal{F}) \cong & \mathrm{Cocontlex}(\mathcal{F}, \mathsf{S}(\mathcal{A})) \\
\cong & \mathrm{Cocontlex}(\mathcal{F}, \mathrm{Acc}_\omega(\mathcal{A}, \mathbf{Set})) \\
\cong & \mathrm{Cat}_{\mathrm{cocontlex}, \mathrm{acc}_\omega}(\mathcal{F} \times \mathcal{A}, \mathbf{Set}) \\
\cong & \mathrm{Acc}_\omega(\mathcal{A}, \mathrm{Cocontlex}(\mathcal{F}, \mathbf{Set})) \\
\cong & \mathrm{Acc}_\omega(\mathcal{A}, \mathrm{Topoi}(\mathbf{Set}, \mathcal{F})). \\
\cong & \mathrm{Acc}_\omega(\mathcal{A}, \mathsf{pt}(\mathcal{F})).
\end{aligned}$$

□

Our first encounter with the Scott adjunction is over, the following two chapters will try to give a more precise intuition to the reader, depending on the background.

CHAPTER 3

Geometry

This chapter is dedicated to unveil the geometric flavour of the Scott adjunction. We build on a quite well understood analogy between topoi and locales to bring the geometric intuition on the Scott adjunction. We show that this intuition is well founded and fruitful both in formulating the correct guesses and directing the line of thoughts. This perspective is not new to the existing literature, as anticipated, the analogy between the notion of locale and that of topos was known since the very introduction of the latter. Our contribution is thus a step towards the categorification process of poset theory into actual category theory. The relation with the existing literature will be discussed along the chapter.

STRUCTURE. The chapter is divided in six sections,

- Sec. 1 we account on the concrete topology on which the analogy is built: the Isbell duality, relating topological spaces to locales. We also relate the Isbell duality to Scott's seminal work on the Scott topology, this first part is completely motivational and expository. This section cointains the posetal version of the sections 2, 3 and 4.
- Sec. 2 We introduce the higher dimensional analogs of topological spaces, locales and posets with directed colimits: ionads, topoi and accessible categories with directed colimits. We categorify the Isbell duality building on Garner's work on ionads, and we relate the Scott adjunction to its posetal analog.
- Sec. 3 We study the the categorified version of Isbell duality. This amounts to the notion of sober ionad and topos with enough points. We show that the categorified Isbell adjuction is idempotent.
- Sec. 4 We use the previous section to derive properties of the Scott adjunction.
- Sec. 5 We show that the analogy on which the chapter is build is deeply motivated and we show how to recover the content of the first section from the following ones.
- Sec. 6 In the last section we provide a quite expected generalization of the second one to κ-topoi and κ-ionads.

1. Spaces, locales and posets

Our topological safari will start by the very celebrated adjunction between locales and topological spaces. It was firstly observed by Isbell, whence the name Isbell adjunction/duality. Unluckly this name is already taken by the dual adjunction between presheaves and co-presheaves, this sometimes leads to some terminological confusion. The two adjunction are similar in spirit, but do not generalize, at least not apparently, in any common framework.

This first subsection is mainly expository and we encourage the interested reader to consult [Joh86] and [LM94][Chap. IX] for a proper introduction. The aim of the subsection is not to introduce the reader to these results and objects, it is instead to organize them in a way that is useful to our purposes. More precise references will be given along the section.

1.1. Spaces, Locales and Posets. This subsection tells the story of the diagram below. Let us bring every character to the stage.

$$\begin{tikzcd}
 & \mathrm{Loc} & \\
\mathrm{Top} \arrow[ur, "\mathbb{O}"] & & \mathrm{Pos}_\omega \arrow[ul, "\mathsf{S}"'] \arrow[ll, "\mathsf{ST}"]
\end{tikzcd}$$

REMARK 3.1 (The categories).

- Loc is the category of Locales. It is defined to be the opposite category of frames, where objects are frames and morphisms are morphisms of frames.
- Top is the category of topological spaces and continuous mappings between them.
- Pos_ω is the category of posets with directed suprema and functions preserving directed suprema.

REMARK 3.2 (The functors). The functors in the diagram above are quite well known to the general topologist, we proceed to a short description of them.

- $\mathbb{O}$ associates to a topological space its frame of open sets and to each continuous function its inverse image.
- ST equips a poset with directed suprema with the Scott topology [Joh86][Chap II, 1.9]. This functor is fully faithful, i.e. a function is continuous with respect to the scott topologies if and only if preserves suprema.
- S is the composition of the previous two, in particular the diagram above commutes.

REMARK 3.3 (The Isbell duality and a posetal version of the Scott adjunction). Both the functors $\mathbb{O}$ and S have a right adjoint, we inditicate them by pt and $\mathbb{pt}$, which in both cases stands for *points*.

$$\begin{tikzcd}
 & \mathrm{Loc} \arrow[dl, bend right, "\mathbb{pt}"'] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathrm{Top} \arrow[ur, dashed, bend left, "\mathbb{O}"] & & \mathrm{Pos}_\omega \arrow[ul, dashed, bend right, "\mathsf{S}"'] \arrow[ll, dashed, "\mathsf{ST}"]
\end{tikzcd}$$

In the forthcoming remarks the reason for this clash of names will be motivated, indeed the two functors look alike. The adjunction on the left is the *Isbell duality*, while the one on the right was not named yet to our knowledge and we will refer to it as the *(posetal) Scott adjunction*. Let us mention that there exists a natural transformation,

$$\iota : \mathsf{ST} \circ \mathsf{pt} \Rightarrow \mathbb{pt}$$

which will be completely evident from the description of the functors. We will say more about ι, for the moment we just want to clarify that there is no reason to believe (and indeed it would be a false belief) that ι is an isomorphism.

REMARK 3.4 (The Isbell duality). An account of the adjunction $\mathbb{O} \dashv \mathbb{pt}$ can be found in the very classical reference [LM94][IX.1-3]. While the description of $\mathbb{O}$ is relatively simple, (it associates to a topological space X its frame of opens $\mathbb{O}(X)$), pt is more complicated to define. It associates to a locale L its sets of *formal points* $\mathrm{Loc}(\mathbb{T}, L)$[1] equipped with the topology whose open sets are defined as follows, for every l in L we pose,

$$V(l) := \{p \in \mathrm{Frm}(L, \mathbb{T}) : p(l) = 1\}.$$

Further details in [LM94][IX.2]. In the classical literature, such an adjunction that is induced by a contravariant hom-functor is called schizoprenic. The special object in this case is 2 that inhabits both the categories under the alias of Sierpiński space in the case of Top and the terminal object in the case of Frm. A detailed explanation of the schizoprenicity of this adjunction can be found in [LM94][IX.3, last paragraph]. The schizophrenicity of the adjunction is the reason for which we address it as a duality.

REMARK 3.5 (The (posetal) Scott adjunction). To our knowledge, the adjunction $\mathsf{S} \dashv \mathsf{pt}$ does not appear explicitly in the literature. Let us give a description of both the functors involved.

- pt The underlying set of $\mathsf{pt}(L)$ is the same of the Isbell duality. Its posetal structure is inherited by $\mathbb{T}$, in fact $\mathrm{Frm}(L, \mathbb{T})$ has a natural poset structure with directed unions given by pointwise evaluation [Vic07][1.11].
- S Given a poset P, its Scott locale $\mathsf{S}(P)$ is defined by the frame $\mathrm{Pos}_\omega(P, \mathbb{T})$, it's quite easy to show that this poset is a locale.

Observe that also this adjunction is a dual one, and is induced precisely by the same object of the Isbell duality.

EXAMPLE 3.6 (Scott domains are kind of discrete spaces). If $(P, \leq)$ is a Scott domain, $\mathsf{S}(P)$ coincides with the frame 2^{P_ω}, the power set of compact elements of P. Since $\mathsf{S}(P)$ is a powerset, one might be tempted to think that the Scott topology over P is discrete. This is not the case if one checks the definitions in detail. Instead, it means that the inclusion $(P_\omega, \mathsf{Disc}) \hookrightarrow \mathsf{ST}(P)$ is dense and moreover,

$$\mathrm{Top}(\mathsf{ST}(P), \mathcal{Y}) \cong \mathbf{Set}(P_\omega, \mathcal{Y})$$

for every space $\mathcal{Y}$. In this *skew* sense a Scott domain is a kind of **discrete object** among posets with directed colimits.

[1] $\mathbb{T}$ is the boolean algebra $\{0 < 1\}$.

Remark 3.7 (An unfortunate naming). There are several reasons for which we are not satisfied of the naming choices in this thesis. Already in the topological case, Isbell duality (or adjunction) is a very overloaded name, later in the text we will categorify this adjunction calling it *categorified Isbell duality*, which propagates an unfortunated choice. Also the name of Scott for the Scott adjunction is not completely proper, because he introduced the Scott topology on a set, providing the functor ST. In the previous chapter, we called Scott adjunction the categorification of the posetal Scott adjunction in this section, propagating this incorrect attribution. Yet, we did not manage to find better options, and thus we will stick to these choices.

1.2. Sober spaces and spatial locales. The Isbell adjunction is a very fashinating construction that in principle could even candidate iself to be an equivalence of categories. *Shouln't a spaces be precisely the space of formal points of its locale of open sets?* It turns out that the answer is in general negative, this subsection accounts on how far is this adjunction from being an equivalence of categories.

Remark 3.8 (Unit and counit). Given a space X the unit of the Isbell adjunction $\eta_X : X \to (\mathsf{pt} \circ \mathbb{O})(X)$ might not be injective. This is due to the fact that if two points x, y in X are such that $\mathsf{cl}(x) = \mathsf{cl}(y)$, then η_X will confuse them.

Remark 3.9 (Sober spaces and spatial locales). In the classical literature about this adjunction people have introduced the notion of sober space and locale with enough points, these are precisely those objects on which the (co)unit is an iso. It turns out that even if η_X is not always an iso $\eta_{\mathsf{pt}(L)}$ is always an iso and this characterizes those η that are isomorphisms. An analogue result is true for the counit.

Remark 3.10 (Idempotency). The technical content of the previous remark is summarized in the fact that the Isbell adjunction $\mathbb{O} \dashv \mathsf{pt}$ is idempotent, this is proved in [LM94][IX.3][Prop. 2, Prop. 3 and Cor. 4.]. It might look like this results has no qualitative meaning, istead it means that given a local L, the locale of opens sets of its points $\mathbb{O}\mathsf{pt}(L)$ is the best approximation of L among spatial locales, namely those that are the locale of opens of a space. The same observation is true for a space X and the formal points of its locale of open sets $\mathsf{pt}\mathbb{O}(X)$. In the next two proposition we give a more categorical and more concrete incarnation of this remark.

Theorem 3.11 ([LM94][IX.3.3], Concrete incarnation of 3.10). The following are equivalent:

(1) L has enough points;
(2) the counit $\epsilon_{\mathscr{C}} : (\mathbb{O} \circ \mathsf{pt})(L) \to L$ is an isomorphism of locales;
(3) L is the locale of open sets $\mathbb{O}(X)$ of some topological space X.

Theorem 3.12 (Abstract incarnation of 3.10). The subcategory of locales with enough points is coreclective in the category of locales.

Remark 3.13. An analogue of this result is true also for spaces. This is not surprising, it's far form being true that any adjunction is idempotent, but it is easy to check that given an adjunction whose induced comonad is idempotent, so must be the induced monad (and viceversa).

1.3. From Isbell to Scott: topology.

REMARK 3.14 (Relating the Scott construction to the Isbell duality). Going back to the (non-)commutativity of the diagram in Rem. 3.3, we observe that there exists a natural transformation $\iota : \mathsf{ST} \circ \mathsf{pt} \Rightarrow \mathbb{pt}$.

$$\begin{tikzcd} & \mathrm{Loc} \arrow[dl, "\mathbb{pt}"'] \arrow[dr, "\mathsf{pt}"] & \\ \mathrm{Top} & & \mathrm{Pos}_\omega \arrow[ll, "\mathsf{ST}"] \arrow[ll, phantom, shift left=4, "\Leftarrow \iota"] \end{tikzcd}$$

The natural transformation is pointwise given by the identity (in fact the underlying set is indeed the same), and witnesses by the fact that every Isbell-open (Rem. 3.4) is a Scott-open (Rem. 3.5). This observation is implictely written in [Joh86][II, 1.8].

REMARK 3.15 (Scott is not always sober). In principle ι might be an isomorphism. Unfortunately it was shown by Johnstone in [Joh81] that some Scott-spaces are not sober. Since every space in the image of $\mathbb{pt}$ is sober, ι cannot be an isomorphism at least in those cases. Yet, Johnstone says in [Joh81] that he does know any example of a complete lattice whose Scott topology is not sober. Thus it is natural to conjecture that when $\mathsf{pt}(L)$ is complete, then ι_L is an isomorphism. We will not only show that this is true, but even provide a generalization of this result later.

REMARK 3.16 (Scott from Isbell). Let us conclude with a version of [LM94][IX.3.3] for the Scott adjunction. This has guided us in looking understanding the correct idempotency of the Scott adjunction.

THEOREM 3.17 (Consequence of [LM94][IX.3.3]). The following are equivalent:

(1) L has enough points and ι is an isomorphism;
(2) the counit of the Scott adjunction is an isomorphism of locales;

PROOF. It follows directly from [LM94][IX.3.3] and the fact that ST is fully faithful. □

2. Ionads, Topoi and Accessible categories

Now we come to the 2-dimensional conterpart of the previous section. As the whole previous one, this section is dedicated to describing the properties of a diagram.

$$\begin{tikzcd} & \mathrm{Topoi} & \\ \mathrm{BIon} \arrow[ur, "\mathbb{O}"] & & \mathrm{Acc}_\omega \arrow[ul, "\mathsf{S}"'] \arrow[ll, "\mathsf{ST}"] \end{tikzcd}$$

2.1. Motivations. There are no dubts that we drew a triangle which is quite simiar to the one in the previous section, but *in what sense these two triangles are related?* There is a long tradition behind this question and too many papers showld be cited. In this very short subsection we provide an intuitive account on this question.

REMARK 3.18 (Replacing posets with categories). There is a well known analogy[2] between the category of posets Pos and the category of categories Cat. A part of this analogy is very natural: joins and colimits, meets and limits, monotone functions and functors. Another part might appear a bit counter intuitive at first sight. The poset of truth values $\mathbb{T} = \{0 < 1\}$ plays the role of the category of sets. The inclusion of a poset $i : P \to \mathbb{T}^{P^\circ}$ in its poset of ideals plays the same role of the Yoneda embedding.

Pos	Cat
$P \to \mathbb{T}^{P^\circ}$	$C \to \mathbf{Set}^{C^\circ}$
joins	colimits
meets	limits
monotone function	functor

REMARK 3.19 (Pos_ω and Acc_ω). Following the previous remark one would be tempted to say that posets with directed colimits correspond to categories with directed colimits. Thus, *why to put the accessibility condition on categories?* The reason is that in the case of posets the accessibility condiction is even stronger, even if hidden. In fact a poset is a small (!) poclass, poclasses with directed joins would be the correct analog of categories with directed colimits. Being accessible is a way to have control on the category without requesting smallness, which would be a too strong assumption.

REMARK 3.20 (Locales and Topoi). Quite surprisingly the infinitary distributivity rules which characterizes locales has a description in term of the posetal *Yoneda* inclusion. Locales can be described as those posets whose (Yoneda) inclusion has a left adjoint preserving finite joins,

$$L : \mathbb{T}^{P^\circ} \leftrightarrows P : i.$$

In the same fashion, Street [Str81] proved that Topoi can be described as those categories whose Yoneda embedding $\mathcal{G} \to \mathbf{Set}^{\mathcal{G}^\circ}$ has a left adjoint preserving finite limits. In analogy with the case of locales this property is reflected by a kind of interaction been limits and colimits which is called *descent*. In this sense a topos is a kind of $\mathbf{Set}$-locale, while a locale is a $\mathbb{T}$-locales. In the next section we will show that there is an interplay between this two *notions of locale*.

REMARK 3.21 (Spaces and Ionads). While topoi are the categorification of locales, ionads are the categorification of topological spaces. Recall that a topological space, afterall, is just the data of its interior operator

$$\mathsf{int} : 2^X \to 2^X.$$

[2]It is much more than an analogy, but this remark is designed to be short, motivational and inspirational. Being more precise and mentioning enriched categories over truth values would not give a more accessible description the the generic reader.

This is an idempotent operator preserving finite meets, we will see that ionads are defined in a very similar way, following the pattern of the previous remarks.

2.2. Categorification. Now that we have given some motivation for this to be the correct categorification of the Isbell duality, we have to present in more mathematical detail all the ingredients that are involved. A part of the triangle in this section is just the Scott adjunction, that we understood quite well in the previous section. Here we have to introduce ionads and all the functors in which they are involved.

$$\begin{tikzcd} & \text{Topoi} & \\ \text{BIon} \arrow[ur, "\mathbb{O}"] & & \text{Acc}_\omega \arrow[ul, "\mathsf{S}"'] \arrow[ll, "\mathsf{ST}"] \end{tikzcd}$$

REMARK 3.22 (The (2-)categories).

Topoi is 2-category of topoi.

BIon is the 2-category of (possibly large) bounded ionads.

Acc_ω is 2-category of accessible categories with directed colimits and functors preserving them.

2.2.1. *The functors.*

REMARK 3.23 ($\mathbb{O}$). Let us briefly recall the relevant definitions of Chap. 1. A (possibly large) bounded ionad $\mathcal{X} = (C, \text{Int})$ is a category C together with a comonad $\text{Int} : \mathbf{Set}^C \to \mathbf{Set}^C$ preserving finite limits whose category of coalgebras is a Grothendieck topos. $\mathbb{O}$ was described by Garner in [Gar12][Rem. 5.2], it maps a bounded ionad to its category of opens, that is the category of coalgebras for the interior operator.

REMARK 3.24 (ST). The construction is based on the Scott adjuction, we map $\mathcal{A}$ to the bounded Ionad $(\mathcal{A}, r_{\mathcal{A}} i_{\mathcal{A}})$, as described in 2.7. Unfortunately, we still have to show that $\mathsf{S}(\mathcal{A})$ is coreflective in $\mathbf{Set}^{\mathcal{A}}$, this will be done in the remark below. A functor $f : \mathcal{A} \to \mathcal{B}$ is sent to the morphism of ionads $(f, f^\sharp)$ below, where $f^\sharp$ coincides with the inverse image of $\mathsf{S}(f)$.

$$\begin{tikzcd} \mathsf{S}(\mathcal{B}) \arrow[r, dashed, "f^\sharp"] \arrow[d] & \mathsf{S}(\mathcal{B}) \arrow[d] \\ \mathbf{Set}^{\mathcal{B}} \arrow[r, "f^*"] & \mathbf{Set}^{\mathcal{A}} \end{tikzcd}$$

REMARK 3.25 ($\mathsf{S}(\mathcal{A})$ is coreflective in $\mathbf{Set}^{\mathcal{A}}$). We would have liked to have a one-line-motivation of the fact that the inclusion $i_{\mathcal{A}} : \mathsf{S}(\mathcal{A}) \to \mathbf{Set}^{\mathcal{A}}$ has a right adjoint $r_{\mathcal{A}}$, unfortunately this result is true for a rather technical argument. By a general result of Kelly, $i_{\mathcal{A}}$ has a right adjoint if and only if $\mathsf{lan}_{i_{\mathcal{A}}}(1_{\mathsf{S}(\mathcal{A})})$ exists and $i_{\mathcal{A}}$ preserves it. Since $\mathsf{S}(\mathcal{A})$ is small cocomplete, if $\mathsf{lan}_{i_{\mathcal{A}}}(1_{\mathsf{S}(\mathcal{A})})$ exists, it must be pointwise and thus i will preserve it because it is a cocontinuous functor. Thus it is enough to prove that $\mathsf{lan}_{i_{\mathscr{A}}}(1_{\mathsf{S}(\mathscr{A})})$ exists. Anyone would be tempted to apply [Bor94b][3.7.2], unfortunately $\mathsf{S}(\mathscr{A})$ is not a small category. In order to cut down this size issue, we use the fact that $\mathsf{S}(\mathscr{A})$ is a topos and thus have a dense generator $j : G \to \mathsf{S}(\mathscr{A})$. Now, we observe that

$$\mathsf{lan}_{i_{\mathscr{A}}}(1_{\mathsf{S}(\mathscr{A})}) = \mathsf{lan}_{i_{\mathscr{A}}}(\mathsf{lan}_j(j)) = \mathsf{lan}_{i_{\mathscr{A}} \circ j} j.$$

Finally, on the latter left Kan extension we can apply [Bor94b][3.7.2], because G is a small category.

Remark 3.26 (S). This was introduced in the previous chapter in detail. Coherently with the previous section, it is quite easy to notice that $\mathsf{S} \cong \mathbb{O} \circ \mathsf{ST}$. Let us show it,

$$\mathbb{O} \circ \mathsf{ST}(\mathscr{A}) = \mathbb{O}(\mathsf{ST}(\mathscr{A})) \overset{2.7}{=} \mathsf{coalg}(r_{\mathscr{A}} i_{\mathscr{A}}) \cong \mathsf{S}(\mathscr{A}).$$

2.2.2. *Points.*

Remark 3.27 (Categorified Isbell duality and the Scott Adjunction). As in the previous section, both the functors $\mathbb{O}$ and S have a right adjoint. We indicate them by pt and $\mathbb{pt}$, which in both cases stands for points. pt has of course been introduced in the previous chapter and correspond to the right adjoint in the Scott adjunction. The other one will be more delicate to describe.

$$\begin{tikzcd}
& \text{Topoi} \arrow[dl, "\mathbb{pt}"] \arrow[dr, "\mathsf{pt}"] & \\
\text{BIon} \arrow[ur, dashed, bend left, "\mathbb{O}"] & & \text{Acc}_\omega \arrow[ll, dashed, "\mathsf{ST}"] \arrow[ul, dashed, "\mathsf{S}"]
\end{tikzcd}$$

This subsubsection will be mostly dedicated to the construction of $\mathbb{pt}$ and to show that it is a right adjoint for $\mathbb{O}$. Let us mention though that there exists a natural functor

$$\iota : \mathsf{ST} \circ \mathsf{pt} \Rightarrow \mathbb{pt}$$

which is not in general an equivalence of categories.

Remark 3.28 (Topoi $\rightsquigarrow$ Bion: every topos induces a possibly large bounded ionad over its points). For a topos $\mathscr{E}$, there exists a natural evaluation pairing

$$\mathsf{ev} : \mathscr{E} \times \mathsf{pt}(\mathscr{E}) \to \mathbf{Set},$$

mapping the couple (e, p) to its evaluation $p^*(e)$. This construction preserves colimits and finite limits in the first coordinate, because p^* is an inverse image functor. This means that its mate functor $ev^* : \mathscr{E} \to \mathbf{Set}^{\mathsf{pt}(\mathscr{E})}$, preserves colimits and finite limits. Since a topos is a total category, ev^* must have a right adjoint ev_*[3]. Thus we get an adjunction,

$$ev^* : \mathscr{E} \leftrightarrows \mathbf{Set}^{\mathsf{pt}(\mathscr{E})} : ev_*.$$

[3]For a total category the adjoint functor theorem reduces to check that the candidate left adjoint preserves colimits.

Since the left adjoit preserve finite limits, the comonad ev^*ev_* is lex and thus induces a ionad over $\mathsf{pt}(\mathscr{E})$. This ionad is bounded because any site of $\mathscr{E}$ provides a (small) base for it. Clearly $ev^* : \mathscr{E} \to \mathbf{Set}^{\mathsf{pt}(\mathscr{E})}$ is a base for this ionad by construction.

Remark 3.29 (Topoi $\rightsquigarrow$ Bion: every geometric morphism induces a morphism of ionads). Observe that given a geometric morphism $f : \mathscr{E} \to \mathscr{F}$, $\mathsf{pt}(f) : \mathsf{pt}(\mathscr{E}) \to \mathsf{pt}(\mathscr{F})$ induces a morphism of ionads $(\mathsf{pt}(f), \mathsf{pt}(f)^\sharp)$ between $\mathbb{pt}(\mathscr{E})$ and $\mathbb{pt}(\mathscr{F})$. In order to describe $\mathsf{pt}(f)^\sharp$, we invoke Prop. 1.40[(a)]. Thus, it is enough to provide a functor making the diagram below commutative (up to natural isomorphism).

$$\begin{tikzcd}
\mathscr{F} \arrow[r, dashed, "f^\diamond"] \arrow[d, "ev^*_{\mathscr{F}}"] & \mathscr{E} \arrow[d, "ev^*_{\mathscr{E}}"] \\
\mathbf{Set}^{\mathsf{pt}(\mathscr{F})} \arrow[r, "\mathsf{pt}(f)^*"] & \mathbf{Set}^{\mathsf{pt}(\mathscr{E})}
\end{tikzcd}$$

Indeed such a functor exists and coincides with the inverse image f^* of the geometric morphism f.

Remark 3.30 (The 2-functor $\mathbb{pt}$). $\mathbb{pt}(\mathscr{E})$ is defined to be the ionad $(\mathsf{pt}(\mathscr{E}), ev^*ev_*)$, as described in the two previous remarks.

Theorem 3.31 (Categorified Isbell adjunction, $\mathbb{O} \dashv \mathbb{pt}$).

$$\mathbb{O} : \text{BIon} \leftrightarrows \text{Topoi} : \mathbb{pt}$$

Proof. We provide the unit and the counit of this adjunction. This means that we need to provide geometric morphism $\rho : \mathbb{O}\mathbb{pt}(\mathscr{E}) \to \mathscr{E}$ and a morphism of ionads $\lambda : \mathfrak{X} \to \mathbb{pt}\mathbb{O}\mathfrak{X}$. Let's study the two problems separately.

(ρ) As in the case of any geometric morphism, it is enough to provide the inverse image functor $\rho^* : \mathscr{E} \to \mathbb{O}\mathbb{pt}(\mathscr{E})$. Now, recall that the interior operator over $\mathbb{pt}(\mathscr{E})$ is induced by the adjunction $ev^* : \mathscr{E} \leftrightarrows \mathbf{Set}^{\mathsf{pt}(\mathscr{E})} : ev_*$ as described in the remark above. By the universal property of the category of algebras, the adjunction $\mathsf{U} : \mathbb{O}\mathbb{pt}(\mathscr{E}) \leftrightarrows \mathbf{Set}^{\mathsf{pt}}(\mathscr{E}) : \mathsf{F}$ is terminal among those adjunctions that induce the comonad ev^*ev_*. This means that there exists a functor ρ^* lifting e^* along U as in the diagram below.

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "\rho^*"] \arrow[dr, "ev^*"'] & \mathbb{O}\mathbb{pt}(\mathscr{E}) \arrow[d, "\mathsf{U}"] \\
 & \mathbf{Set}^{\mathbb{pt}(\mathscr{E})}
\end{tikzcd}$$

It is easy to show that ρ^* is cocontinuous and preserve finite limits and is thus the inverse image functor of a geometric morphism $\rho : \mathbb{O}\mathbb{pt}(\mathscr{E}) \to \mathscr{E}$ as desired.

(λ) Recall that a morphism of ionads $\lambda : \mathfrak{X} \to \mathbb{pt}\mathbb{O}\mathfrak{X}$ is the data of a functor $\lambda : X \to \mathsf{pt}\mathbb{O}\mathfrak{X}$ together with a lifting $\lambda^\sharp : \mathbb{O}\mathfrak{X} \to \mathbb{O}\mathsf{pt}\mathbb{O}\mathfrak{X}$. We only provide $\lambda : X \to \mathsf{pt}\mathbb{O}\mathfrak{X}$, $\lambda^s harp$ is induced by

1.40. Indeed such a functor is the same of a functor $\lambda : X \to \text{Cocontlex}(\mathbb{O}\mathfrak{X}, \mathbf{Set})$. Define,

$$\lambda(x)(s) = (\mathsf{U}(s))(x).$$

From a more conceptual point of wies, λ is just given by the composition of the functors,

$$X \xrightarrow{\mathsf{eval}} \text{Cocontlex}(\mathbf{Set}^X, \mathbf{Set}) \xrightarrow{-\circ\mathsf{U}} \text{Cocontlex}(\mathbb{O}\mathfrak{X}, \mathbf{Set}).$$

□

3. Sober ionads and topoi with enough points

In this section we show that the categorified Isbell adjunction is idempotent, providing a categorification of 1.2. As the notion of sober space, the notion of sober ionad is a bit unsatisfactory and lacks an intrinsic description. Topoi with enough points have been studied very much in the literature. Let us give (or recall) the two definitions.

Definition 3.32 (Sober ionad). A ionad is sober if λ is an equivalence of ionads.

Definition 3.33 (Topos with enough points). A topos has enough points if the inverse image functors from all of its points are jointly conservative.

Theorem 3.34 (Idempotency of the categorified Isbell duality). The following are equivalent:

(1) $\mathcal{E}$ has enough points;
(2) $\rho : \mathbb{O}\mathbb{pt}(\mathcal{E}) \to \mathcal{E}$ is an equivalence of categories;
(3) $\mathcal{E}$ is of the form $\mathbb{O}(\mathfrak{X})$ for some bounded ionad $\mathfrak{X}$.

Proof.

(1) ⇒ (2) Going back to the definition of ρ in 3.31, it's enough to show that ev^* is comonadic. Since it preserves finite limits, it's enough to show that it is conservative to apply Beck's (co)monadicity theorem. Yet, that is just a reformulation of having *enough points*.

(2) ⇒ (3) Trivial.

(3) ⇒ (1) [Gar12][Rem. 2.5].

□

Theorem 3.35. The following are equivalent:

(1) $\mathfrak{X}$ is sober;
(2) $\mathfrak{X}$ is of the form $\mathbb{pt}(\mathcal{E})$ for some topos $\mathcal{E}$.

Proof. For any adjunction, it is enough to show that either the monad or the comonad is idempotent, to obtain the same result for the other one.

□

4. From Isbell to Scott: categorified

This section is a categorification of its analog 1.3 and shows how to infer results about the tightness of the Scott adjunction from the Isbell adjunction. We mentioned in 3.27 that there exists a natural transformation as described by the diagram below.

$$\begin{tikzcd}
 & \text{Topoi} \arrow[dl, "\mathbb{pt}"'] \arrow[dr, "\mathsf{pt}"] & \\
\text{BIon} & & \text{Acc}_\omega \arrow[ll, "\mathsf{ST}"] \arrow[ll, phantom, shift left=5, "\overset{\iota}{\Leftarrow}"]
\end{tikzcd}$$

Let us describe it. Spelling out the content of the diagram, ι should be a morphism of ionads

$$\iota : \mathsf{STpt}(\mathscr{E}) \to \mathbb{pt}(\mathscr{E}).$$

Recall that the underling category of these two ionads is $\mathsf{pt}(\mathscr{E})$ in both cases. We define ι to be the identity on the underlying categories, $\iota = (1_{\mathbb{pt}(\mathscr{E})}, \iota^\sharp)$ while $\iota^\sharp : \mathscr{E} \to \mathsf{Spt}(\mathscr{E})$ is given by the following functor

$$\iota^\sharp(x)(p) = p^*(x).$$

Remark 3.36 ($\iota^\sharp$ is the counit of the Scott adjunction). The reader might have noticed that $\iota^\sharp$ is precisely the counit of the Scott adjunction.

Theorem 3.37 (From Isbell to Scott, cheap version). *The following are equivalent:*

(1) *$\mathscr{E}$ has enough points and ι is an equivalence of ionads.*
(2) *The counit of the Scott adjunction is an equivalence of categories.*

Proof. This is completely obvious from the previous discussion. □

Yet, this result is quite disappointing in practice and we cannot accept it as it is. On the other hand, having understood that the Scott adjunction is not precisely the Isbell's one was a very important conceptual progress in order to guess the correct statement for the Scott adjunction. In the next sections we provide a more handy version of the previous theorem.

4.1. Covers. In order to provide a satisfying version of Thm 3.37, we need to introduce a tiny bit of technology, namely finitely accessible covers. For an accessible category with directed colimits $\mathscr{A}$ a (finitely accessible cover) *cover*

$$\mathscr{L} : \mathsf{Ind}(C) \to \mathscr{A}$$

will be a cocontinuous (pseudo)epimorphism (in Cat) having many good properties. Those will be helpful for us in the discussion. Covers were exploited for the first time in [LR14][2.5].

Remark 3.38 (Generating covers). Every object $\mathscr{A}$ in Acc_ω has a (proper class of) finitely accessible cover. Let $\mathscr{A}$ be κ-accessible. Let us focus on the following diagram.

$$\begin{tikzcd}
 & \mathscr{A}_\kappa \arrow[dl, "\alpha"'] \arrow[dr, "\iota"] & \\
\text{Ind}\ \mathscr{A}_\kappa \arrow[rr, dotted, bend right, "\mathsf{lan}_\alpha \iota"'] & & \mathscr{A} \arrow[ll, dotted, "\mathscr{A}(\iota -, -)"']
\end{tikzcd}$$

$\mathcal{A}(\iota -, -)$, also known as *the nerve of f*, is fully faithful because $\mathcal{A}_\kappa$ is a (dense) generator in $\mathcal{A}$, this functor does not lay in Acc_ω because it is just κ-accessible in general. $\mathsf{lan}_\alpha(\iota)$ exisists by the universal property of of the Ind completion, indeed $\mathcal{A}$ has directed colimits by definition. For a concrete perspective $\mathsf{lan}_\alpha(\iota)$ is evaluating a formal directed colimit on the actual directed colimit in $\mathcal{A}$. These two maps yield an adjunction

$$\mathsf{lan}_\alpha(\iota) \dashv \mathcal{A}(\iota -, -)$$

that establishes $\mathcal{A}$ as a reflective embedded subcategory of $\mathrm{Ind}\ \mathcal{A}_\kappa$. Since $\mathsf{lan}_\alpha(\iota)$ is a left adjoint, it lies in Acc_ω. At this level of generality $\mathcal{A}(\iota -, -)$ will just be κ-accessible and thus we do not know if it lies in Acc_ω.

Definition 3.39. When $\mathcal{A}$ is a κ-accessible category we will indicate with $\mathcal{L}^\kappa_{\mathcal{A}}$ the map that here we indicated with $\mathsf{lan}_\alpha(\iota)$ in the previous remark and we call it a *cover* of $\mathcal{A}$.

$$\mathcal{L}^\kappa_{\mathcal{A}} : \mathrm{Ind}\ \mathcal{A}_\kappa \to \mathcal{A}.$$

Notation 3.40. When it's evident from the context, or it is not relevant at all we will not specify the cardinal on the top, thus we will just write $\mathcal{L}_{\mathcal{A}}$ instead of $\mathcal{L}^\kappa_{\mathcal{A}}$.

Remark 3.41. When $\lambda \geq \kappa$, $\mathcal{L}^\lambda_{\mathcal{A}} \cong \mathcal{L}^\kappa_{\mathcal{A}} \mathcal{L}^\lambda_\kappa$ for some transition map $\mathcal{L}^\lambda_\kappa$. We did not find any application for it, thus we decided not to go in the details of this construction.

Remark 3.42. This construction appeared for the first time in [LR14][2.5], where it is presented as the analogous of Shelah presentation theorem for AECs [LR14][2.6], the reader that is not familiar with formal category theory might find the original presentation more down to earth. In [LR14][2.5] they also show that under interesting circumstances the cover is faithful.

4.2. On the (non) idempotency of the Scott adjunction. This subsection provides a better version of Thm. 3.37. It is based on a technical notion (topological embeddings) that we define and study in the Toolbox chapter (see 6).

Theorem 3.43. A Scott topos $\mathcal{G} \cong \mathsf{S}(\mathcal{A})$ has enough points.

Proof. It is enough to show that $\mathcal{G}$ admits a geometric surjection from a presheaf topos [Joh02b][2.2.12]. Let κ be a cardinal such that $\mathcal{A}$ is κ-accessible. We claim that the 1-cell $\mathsf{S}(\mathcal{L}^\kappa_{\mathcal{A}})$ described in Rem. 3.39 is the desired geometric surjection for the Scott topos $\mathsf{S}(\mathcal{A})$. By 2.12, the domain of $\mathsf{S}(\mathcal{L}^\kappa_{\mathcal{A}})$ is indeed a presheaf topos. By 6.7, it is enough to prove that $\mathcal{L}^\kappa_{\mathcal{A}}$ is a pseudo epimorphism in Cat. But that is obvious, because it has even a section in Cat, namely $\mathcal{A}(\iota, 1)$. □

Theorem 3.44. The following are equivalent.

(1) The counit $\epsilon : \mathsf{Spt}(\mathcal{E}) \to \mathcal{E}$ is an equivalence of categories.
(2) $\mathcal{E}$ has enough points and for all presentations $i^* : \mathbf{Set}^X \leftrightarrows \mathcal{E} : i_*$, $\mathsf{pt}(i)$ is a topological embedding.
(3) $\mathcal{E}$ has enough points and $\mathsf{pt}(y)$ is a topological embedding, where y is the canonical presentation[4] $y : \mathcal{E} \to \mathbf{Set}^{\mathcal{G}^\circ}$.

[4]Notation 1.41.

(4) $\mathscr{E}$ has enough points and there exists a presentation $i^* : \mathbf{Set}^X \leftrightarrows \mathscr{E} : i_*$ such that $\mathsf{pt}(i)$ is a topological embedding.

Proof. As expectable, we follow the proof strategy hinted by the enumeration.

1) ⇒ 2) By Thm. 3.43, $\mathscr{E}$ has enough points. We only need to show that for all presentations $i^* : \mathbf{Set}^X \leftrightarrows \mathscr{E} : i_*$, $\mathsf{pt}(i)$ is a topological embedding. In order to do so, consider the following diagram,

$$\begin{tikzcd} \mathsf{Spt}\mathscr{E} \arrow[r, "\mathsf{Spt}(i)"] \arrow[d, "\epsilon_{\mathscr{E}}"] & \mathsf{Spt}\,\mathbf{Set}^X \arrow[d, "\epsilon_{\mathbf{Set}^X}"] \\ \mathscr{E} \arrow[r, "i"] & \mathbf{Set}^X \end{tikzcd}$$

By Rem. 2.12 and the hypotheses of the theorem, one obtains that $\mathsf{Spt}(i)$ is naturally isomorphic to a composition of geometric embedding

$$\mathsf{Spt}(i) \cong \epsilon^{-1}_{\mathbf{Set}^X} \circ i \circ \epsilon_{\mathscr{E}},$$

and thus is a geometric embedding. This shows precisely that $\mathsf{pt}(i)$ is a topological embedding.

2) ⇒ 3) Obvious.

3) ⇒ 4) Obvious.

4) ⇒ 1) It is enough to prove that $\epsilon_{\mathscr{G}}$ is both a surjection and a geometric embedding of topoi. $\epsilon_{\mathscr{G}}$ is a surjection, indeed since $\mathscr{G}$ has enough points, there exist a surjection $q : \mathbf{Set}^X \twoheadrightarrow \mathscr{G}$, now we apply the comonad Spt and we look at the following diagram,

$$\begin{tikzcd} \mathsf{Spt}\,\mathbf{Set}^X \arrow[r, "\epsilon_{\mathbf{Set}^X}"] \arrow[d, "\mathsf{Spt}(q)"] & \mathbf{Set}^X \arrow[d, two heads, "q"] \\ \mathsf{Spt}\mathscr{G} \arrow[r, "\epsilon_{\mathscr{G}}"] & \mathscr{G} \end{tikzcd}$$

Now, the counit arrow on the top is an isomorphism, because $\mathbf{Set}^X$ is a presheaf topos. Thus $\epsilon_{\mathscr{G}} \circ (\mathsf{Spt})(q)$ is (essentially) a factorization of q. Since q is a geometric surjection so must be $\epsilon_{\mathscr{G}}$. In order to show that $\epsilon_{\mathscr{G}}$ is a geometric embedding, we use again the following diagram over the existing presentation i.

$$\begin{tikzcd} \mathsf{Spt}\mathscr{E} \arrow[r, "\mathsf{Spt}(i)"] \arrow[d, "\epsilon_{\mathscr{E}}"] & \mathsf{Spt}\,\mathbf{Set}^X \arrow[d, "\epsilon_{\mathbf{Set}^X}"] \\ \mathscr{E} \arrow[r, "i"] & \mathbf{Set}^X \end{tikzcd}$$

This time we know that $\mathsf{Spt}(i)$ and i are geometric embeddings, and thus $\epsilon_{\mathscr{E}}$ has to be so.

□

Remark 3.45. The version above might look a quite technical but not very useful improvement of Thm. 3.37. Instead, in the following Corollary we prove a quite non-trivial result based on a characterization (partial but useful) of topological embeddings contained in the Toolbox.

Corollary 3.46. Let $\mathscr{E}$ be a topos with enough points, together with a presentation $i : \mathscr{E} \to \mathbf{Set}^C$. If $\mathsf{pt}(\mathscr{E})$ is complete and $\mathsf{pt}(i)$ preserve limits. Then $\epsilon_{\mathscr{E}}$ is an equivalence of categories.

Proof. We verify the condition (4) of the previous theorem. Since $\mathsf{pt}(i)$ preserve all limits, fully faithful, and $\mathsf{pt}(\mathscr{E})$ must be cocomplete, $\mathsf{pt}(i)$ has a left adjoint which establish $\mathsf{pt}\mathscr{E}$ as a reflective subcategory of $\mathsf{pt}\,\mathbf{Set}^C$. By 6.11, $\mathsf{pt}(i)$ must be a topological embedding. $\square$

Remark 3.47 (Scott is not aways sober). The previous corollary is coherent with Johnstone's observation that when a poset is (co)complete its Scott topology is generically sober, and thus the Scott adjunction should reduce to Isbell's duality.

5. Interaction

In this section we shall convince the reader that the posetal version of this Scott-Isbell story *embeds* in the categorical one.

$$\begin{tikzcd}
& \text{Loc} \arrow[rr, dashed, "\mathsf{Sh}"] \arrow[dl] \arrow[dd] & & \text{Topoi} \arrow[dl] \arrow[dd] \\
\text{Top} \arrow[rr, dashed] & & \text{BIon} & \\
& \text{Pos}_\omega \arrow[ul] \arrow[rr, dashed, "\iota"] & & \text{Acc}_\omega \arrow[ul]
\end{tikzcd}$$

We have no applications for this observation, thus we do not provide all the details that would amount to an enormous amount of functors relating all the categories that we have mentioned. Yet, we show the easiest aspect of this phaenomenon. Let us introduce and describe the following diagram,

$$\begin{tikzcd}
\text{Loc} \arrow[r, "\mathsf{Sh}"] \arrow[d, "\mathsf{pt}"] & \text{Topoi} \arrow[d, "\mathsf{pt}"] \\
\text{Pos}_\omega \arrow[r, "\mathsf{i}"] & \text{Acc}_\omega
\end{tikzcd}$$

Remark 3.48 (Sh and i).

- Sh It is well known that the sheashification functor $\mathsf{Sh} : \text{Loc} \to \text{Topoi}$ establishes Loc as a full subcategory of Topoi in a sense made precise in [LM94][IX.5, Prop. 2 and 3].
- i This is very easy to describe. Indeed and posed with directed suprema is an accessible category with directed colimits and a function preserving directed suprema is precisely a functor preserving directed colimits.

Proposition 3.49. The diagram above commutes.

Proof. This is more or less tautological from the point of view of [LM94][IX.5, Prop. 2 and 3]. In fact,

$$\mathsf{pt}(L) = \mathrm{Loc}(\mathbb{T}, L) \cong \mathrm{Topoi}(\mathsf{Sh}(\mathbb{T}), \mathsf{Sh}(L)) \cong \mathsf{pt}(\mathsf{Sh}(L)).$$

□

6. The κ-case

Without any surprise, it is possible to generalize the whole content of this chapter to the κ-case. The notion of κ-ionad is completely strightforward and every construction lifts to infinte cardinalds without any effort. For the sake of completeness, we report the κ-version of the main theorems that we saw in the chapter, but we omit the proof, which would be identical to the finitary case.

Remark 3.50. The following diagram accounts on the κ-version of the relevant adjuctions.

$$\begin{tikzcd}
& \kappa\text{-Topoi} \arrow[dl, bend left, "\mathbb{pt}_\kappa"] \arrow[dr, bend left, "\mathsf{pt}_\kappa"] & \\
\kappa\text{-BIon} \arrow[ur, bend left, "\mathbb{O}_\kappa"] & & \mathrm{Acc}_\kappa \arrow[ul, bend left, "\mathsf{S}_\kappa"] \arrow[ll, "\mathsf{ST}_\kappa"]
\end{tikzcd}$$

Theorem 3.51 (Idempotency of the categorified κ-Isbell duality). Let $\mathcal{E}$ be a κ-topos. The following are equivalent:

(1) $\mathcal{E}$ has enough κ-points;
(2) $\rho : \mathbb{O}_\kappa \mathbb{pt}\kappa(\mathcal{E}) \to \mathcal{E}$ is an equivalence of categories;
(3) $\mathcal{E}$ is of the form $\mathbb{O}_\kappa(\mathfrak{X})$ for some bounded κ-ionad $\mathfrak{X}$.

Theorem 3.52. Let $\mathcal{E}$ be a κ-topos. The following are equivalent.

(1) The counit $\epsilon : \mathsf{S}_\kappa \mathsf{pt}_\kappa(\mathcal{E}) \to \mathcal{E}$ is an equivalence of categories.
(2) $\mathcal{E}$ has enough κ-points and for all presentations $i^* : \mathbf{Set}^X \leftrightarrows \mathcal{E} : i_*$, $\mathsf{pt}_\kappa(i)$ is a topological embedding.
(3) $\mathcal{E}$ has enough κ-points and $\mathsf{pt}_\kappa(y)$ is a topological embedding, where y is the canonical presentation[5] $y : \mathcal{E} \to \mathbf{Set}^{\mathcal{E}^\circ}$.
(4) $\mathcal{E}$ has enough κ-points and there exists a presentation $i^* : \mathbf{Set}^X \leftrightarrows \mathcal{E} : i_*$ such that $\mathsf{pt}(i)$ is a topological embedding.

Corollary 3.53. Let $\mathcal{E}$ be a κ-topos with enough κ-points such that $\mathsf{pt}_\kappa(\mathcal{E})$ is cocomplete. Then $\epsilon_{\mathcal{E}}$ is an equivalence of categories.

[5]Notation 1.41.

CHAPTER 4

Logic

The aim of this chapter is to give a logical account on the Scott adjunction. The reader will notice that once propertly formulated, the statements of this chapter follow directly from the previous chapter. This should not be surprising and echoes the fact that once Stone-like dualities are proven, completeness-like theorems for propositional logic follow almost on the spot.

STRUCTURE. The exposition is organized as follows:

- Sec. 1 The first section will push the claim that the Scott topos $\mathsf{S}(\mathscr{A})$ is a kind of very weak notion of theory naturally attached to the accessible category which is a candidate geometric axiomatization of $\mathscr{A}$. We will see how this traces back to the seminal works of Linton and Lawvere on algebraic theories and algebraic varieties.
- Sec. 2 The second section inspects a very natural guess that might pop up in the mind of the topos theorist: *is there any relation between Scott topoi* and classifying topoi? The question will have a partially affirmative answer in the first subsection. The second one subsumes these partial results. Indeed every theory $\mathcal{S}$ has a category of models $\mathsf{Mod}(\mathcal{S})$, but this category does not retain enough information to recover the theory, even when the theory has enough points. That's why the Scott adjunction is not sharp enough. Nevertheless, every theory has a ionad of models $\mathbb{M}\mathbb{o}\mathbb{d}(\mathcal{S})$, the category of opens of such a ionad $\mathbb{O}\mathbb{M}\mathbb{o}\mathbb{d}(\mathcal{S})$ recovers theories with enough points.
- Sec. 3 This section describes the relation between the Scott adjunction and abstract elementary classes, providing a restriction of the Scott adjunction between accessible categories where every map is a monomorphism and locally decidable topoi.
- Sec. 4 In this section we give the definition of *category of saturated objects* (CSO) and show that the Scott adjunction restricts to an adjunction between CSO and atomic topoi. This section can be understood as an attempt to conceptualize the main result in [Hen19].

1. Generalized axiomatizations

ACHTUNG! 4.1. The content of this section is substantially inspired by some private conversations with Jiří Rosický and he should be credited for it.

REMARK 4.2. Let Grp be the category of groups and $\mathsf{U} : \mathsf{Grp} \to \mathbf{Set}$ be the forgetful functor. The historical starting point of a categorical understanding of universal algebra was precisely that one can recover the category of groups from U. Consider all the natural tranformations of the form

$$\mu : \mathsf{U}^n \Rightarrow \mathsf{U}^m,$$

these can be seen as implictely defined operations of groups. If we gather these operations in an equational theory $\mathbb{T}_{\mathsf{U}}$, we see that the functor U lifts to the category of models $\mathsf{Mod}(\mathbb{T}_{\mathsf{U}})$ as indicated by the diagram below.

$$\begin{tikzcd}
\mathsf{Grp} \arrow[r, dotted] \arrow[dr, "\mathsf{U}"'] & \mathsf{Mod}(\mathbb{T}_{\mathsf{U}}) \arrow[d, "|-|"] \\
& \mathbf{Set}
\end{tikzcd}$$

It is a quite classical result that the comparison functor above is fully faithful and essentially surjective, thus we have axiomatized the category of groups (probably with a non minimal family of operations).

Remark 4.3. The idea above was introduced in Lawvere's PhD thesis [Law63] and later developed in great generality by Linton [Lin66, Lin69]. The interested reader might find interesting [AR94][Chap. 3] and the expository paper [HP07]. Nowadays this is a standard technique in categorical logic and some generalizations of it were presented in [Ros81] by Rosický and later again in [LR14][Rem. 3.5].

Remark 4.4 (Rosický's remark). Rem 4.2 ascertains that the collection of functor $\{\mathsf{U}^n\}_{n\in\mathbb{N}}$, together with all the natural transformations between them, retains all the informations about the category of groups. Observe that in this specific case, the functors U^n all preserve directed colimits, because finite limits commute with directed colimits. This means that this small category $\{\mathsf{U}^n\}_{n\in\mathbb{N}}$ is a full subcategory of the Scott topos of the category of groups. In fact the vucabolary of the theory that we used to axiomatize the category of groups is made up of symbols coming from a full subcategory of the Scott topos.

Remark 4.5 (Lieberman-Rosický construction). In [LR14][Rem. 3.5] ginven a couple $(\mathcal{A}, \mathsf{U})$ where $\mathcal{A}$ is an a accessible category with directed colimits together with a faithful functor $\mathsf{U} : \mathcal{A} \to \mathbf{Set}$ preserving directed colimits, the authors form a category $\mathbb{U}$ whose objects are finitely accessible subfunctors of U^n and arrows are natural transformation between them. Of course there is a naturally attached signature Σ_U and a naturally attached first order theory $\mathbb{T}_{\mathsf{U}}$. In the same fashion of the previous remarks one finds a comparison functor $\mathcal{A} \to \Sigma_{\mathsf{U}}$-Str. In [LR14][Rem. 3.5] the authors stress that is the most natural candidate to axiomatize $\mathcal{A}$. A model of $\mathbb{T}_{\mathsf{U}}$ is the same of a functor $\mathbb{U} \to \mathbf{Set}$ preserving products and subobjects. Of course the functor $\mathcal{A} \to \Sigma_{\mathsf{U}}$-Str factors trough $\mathrm{Mod}(\mathbb{U})$ (seen as a sketch)

$$l : \mathcal{A} \to \mathrm{Mod}(\mathbb{U}),$$

but in [LR14][Rem. 3.5] this was not the main concern of the authors.

Remark 4.6 (Generalized axiomatizations). The generalized axiomatization of Lieberman and Rosický amounts to a sketch $\mathbb{U}$. As we mentioned, there exists an obious inclusion of $\mathbb{U}$ in the Scott topos of $\mathcal{A}$,

$$i : \mathbb{U} \to \mathsf{S}(A)$$

which is a flat functor because finite limits in $\mathsf{S}(\mathcal{A})$ are computed pointwise in $\mathbf{Set}^{\mathcal{A}}$. Thus, every point $p : \mathbf{Set} \to \mathsf{S}(\mathcal{A})$ induces a model of the sketch $\mathbb{U}$ by composition,

$$i^* : \mathsf{pt}(\mathsf{S}\mathcal{A}) \to \mathrm{Mod}(\mathbb{U})$$
$$p \mapsto p^* \circ i.$$

In particular this shows that the unit of the Scott adjunction lifts the comparison functor between $\mathcal{A}$ and $\mathrm{Mod}(\mathbb{U})$ along i^* and thus the Scott topos provides a *sharper* axiomatization of $\mathbb{T}_{\mathsf{U}}$.

$$\begin{tikzcd}
& \mathcal{A} \arrow[dl, bend right, "\eta_{\mathcal{A}}"'] \arrow[dr, bend left, "l"] & \\
\mathsf{ptS}(\mathcal{A}) \arrow[rr, "i^*"] & & \mathrm{Mod}(\mathbb{U})
\end{tikzcd}$$

Remark 4.7 (Faithful functors are likely to generate the Scott topos). Yet, it should be noticed that when $\mathbb{U}$ is a generator in $\mathsf{S}(\mathcal{A})$, the functor i^* is an equivalence of categories. As unlikely as it may sound, in all the examples that we can think of, a generator of the Scott topos is always given by a faithful forgetful functor $\mathsf{U} : \mathcal{A} \to \mathbf{Set}$. This phaenomenon is so pervasive that the author has believed for quite some time that an object in the Scott topos $\mathsf{S}(\mathcal{A})$ is a generator if and only if it is faithful and conservative. We still lack a counterexample, or a theorem proving such a statement.

2. Classifying topoi

This section is devoted to specify the connection between Scott topoi, Isbell topoi (what is a Scott topos? What is a Isbell topos?) and classifying topoi. Recall that for a geometric theory $\mathbb{T}$, a classifying topos $\mathbf{Set}[\mathbb{T}]$ is a topos representing the functor of models in topoi,

$$\mathsf{Mod}_{(-)}(\mathbb{T}) \cong \mathrm{Topoi}(-, \mathbf{Set}[\mathbb{T}]).$$

The theory of classifying topoi allows to internalize geometric logic in the internal logic of the 2-category of topoi.

2.1. Categories of models, Scott topoi and classifying topoi.

The Scott topos $\mathsf{S}(\mathsf{Grp})$ of the category of groups is $\mathbf{Set}^{\mathsf{Grp}_\omega}$, this follow for 2.12 and applies to $\mathsf{Mod}(\mathbb{T})$ for every Lawvere theory. It is well known that $\mathbf{Set}^{\mathsf{Grp}_\omega}$ is also the classifying topos of the theory of groups. This section is devoted to understaing if this is just a coincidense, or if the Scott topos is actually related to the classifying topos.

Remark 4.8. Let $\mathcal{A}$ be an accessible category with directed colimits. In order to properly ask the question *is* $\mathsf{S}(A)$ *the classifying topos?*, we should answer the question *the classifying topos of what?* Indeed $\mathcal{A}$ is just a category, while one can compute classifying topoi of theories. Our strategy is to introduce a quite general notion of theory that fits in the following diagram,

$$\begin{tikzcd}
\mathrm{Acc}_\omega \arrow[rr, bend left=10, "\mathsf{S}"'] & & \mathrm{Topoi} \arrow[ll, bend right=10, "\mathsf{pt}"'] \\
& \mathsf{Theories} \arrow[ul, dotted, "\mathsf{Mod}(-)"] \arrow[ur, dotted, "\daleth(-)"'] &
\end{tikzcd}$$

in such a way that:

(1) $\daleth(\mathbb{T})$ gives the classifying topos of $\mathbb{T}$;
(2) $\mathsf{Mod}(-) \cong \mathsf{pt}\daleth(-)$.

In this new setting we can reformulate our previous question in the following mathematical conjecture:

$$\daleth(-) \overset{?}{\cong} \mathsf{SMod}(-)$$

Remark 4.9 (Geometric Sketches). The notion of theory that we plan to use is that of geometric sketch. The category of (small) sketches was described in [MP89][3.1], while a detailed study of geometric sketches was conducted in [AJMR97, AR96].

$$\begin{tikzcd}
\mathrm{Acc}_\omega \arrow[rr, bend left=10, "\mathsf{S}"'] & & \mathrm{Topoi} \arrow[ll, bend right=10, "\mathsf{pt}"'] \\
& \mathsf{GSketches} \arrow[ul, "\mathsf{Mod}(-)"] \arrow[ur, "\daleth(-)"'] &
\end{tikzcd}$$

Remark 4.10. Following [MP89], there exists a natural way to generate a sketch from any accessible category. This construction, in principle, gives even a left adjoint for the functor $\mathsf{Mod}(-)$, but does land in large sketches. Thus it is indeed true that for each accessible category there exist a sketch (a theory) canonically associated to it. We do not follow this line because the notion of large sketch, form a philosophycal perspective, is a bit unnatural. Syntax should always be very frugal. From an operative perspective, presentations should always be as small as possible. It is possible to cut down the size of the sketch, but this construction cannot be defined functorially on the whole category of accessible categories with directed colimits. Since elegance and naturality is one of the main motivations for this treatment of syntax-semantics dualities, we decided to avoid any kind of non-natural construction.

Remark 4.11. Geometric sketches contain coherent sketches. In the dictionary between logic and geometry that is well motivated in the indicated paper these three classes correspond respectively to geometric and coherent theories. The latter essentially contain all first order theories via the process of Morleyzation. These observations make our choice of geometric sketches a very general notion of theory and makes us confident that it's a good notion to look at.

We now proceed to describe the two functors labelled with the name of Mod and $\beth$.

REMARK 4.12 (Mod). This 2-functor is very easy to describe. To each sketch $\mathcal{S}$ we associate its category of Set-models, while it is quite evident that a morphism of sketches induces by composition a functor preserving directed colimits (see Sec. 3 in the Background chapter).

REMARK 4.13 ($\beth$). The topos completion of a geometric sketch is a highly nontrivial object to describe. Among the possible construction that appear in the literature, we refer to [Bor94c][4.3]. Briefly, the idea behind this construction is the following.

[Bor94c][4.3.3] Every sketch $\mathcal{S}$ can be completed to a sketch $\bar{S}$ whose underlying category is cartesian.

[Bor94c][4.3.6] This construction if functorial and does not change the model of the sketch over any Grothendieck topos.

[Bor94c][4.3.8] The completion of the sketch has a natural topology $\bar{J}$.

Step 4 Thus the correspondence $\mathcal{S} \mapsto \bar{\mathcal{S}} \mapsto (\bar{\mathcal{S}}, \bar{J})$ transformas geometric sketches into sites and morphism of sketches into morphism of sites.

Step 5 We compute sheaves over the site $(\bar{\mathcal{S}}, \bar{J})$.

Step 6 Define $\beth$ to be $\mathcal{S} \mapsto \bar{\mathcal{S}} \mapsto (\bar{\mathcal{S}}, \bar{J}) \mapsto \mathsf{Sh}(\bar{\mathcal{S}}, \bar{J})$.

REMARK 4.14. While [Bor94c][4.3.6] proves that $\mathsf{Mod}(-) \cong \mathsf{pt}\beth(-)$, and [Bor94c][4.3.8] prove that $\beth(\mathcal{S})$ is the classifying topos of $\mathcal{S}$ among Grothendieck topoi, the main question of this section remains completely open, is $\beth(\mathcal{S})$ isomorphic to the Scott topos $\mathsf{SMod}(-)$ of the category of Set models of S? We answer this question with the following theorem.

THEOREM 4.15. If the counit $\epsilon_{\beth(S)}$ of the Scott adjunction is an equivalence of categories on $\beth(S)$, then $\beth(S)$ coincides with $\mathsf{SMod}(S)$.

PROOF. We introduced enough technology to make this proof incredibly slick. Recall the counit

$$\mathsf{Spt}(\beth(S)) \to \beth(S)$$

and assume that it is an equivalence of categories. Now, since $\mathsf{Mod}(-) \cong \mathsf{pt}\beth(-)$, we obtaing that

$$\beth(S) \cong \mathsf{SMod}(S),$$

which indeed it our thesis. □

REMARK 4.16. Thm. 3.44 characterizes those topoi for which the counit is an equivalence of categories, providing a full description of those geometric sketches for which $\beth(S)$ coincides with $\mathsf{SMod}(S)$. Since Thm. 4.15 might not look satisfactory, in the following comment we use Cor. 3.46 to derive a nice looking statement.

COROLLARY 4.17. Assume $\beth(\mathcal{S})$ has enough points and $\mathsf{Mod}(S)$ is complete. Let $i : \beth(\mathcal{S}) \to \mathbf{Set}^C$ be a presentation such that $\mathsf{pt}(i)$ preserve limits. then $\beth(S)$ coincides with $\mathsf{SMod}(\mathcal{S})$.

PROOF. Apply Cor. 3.46 to Thm 4.15. □

Remark 4.18 (Coherent Sketches). Observe that it is a result due to Deligne that if S is a coherent sketch its associated classigying topos $\beth(S)$ has enough points, thus at least in the case of coherent sketches whose category of models is cocomplete, we always have that the Scott topos of the set models of a coherent sketch coincide with the classifying topos of the sketch.

2.2. Ionads of models, Isbell topoi and classifying topoi. Indeed the main result of this section up to this point has been partially unsatisfactory. As it happens sometimes, the answer is not as nice as expected because the question in first place did not take in consideration some relevant factors. The category of models of a sketch does not retain enough information on the sketch. Fortunately, we will show that every sketch has a ionad of models (not just a category) and the category of opens of this ionad is a much better approximation of the classifying topos. In this subsection, we switch diagram of study to the one below.

$$\begin{tikzcd}
\text{BIon} \arrow[rr, bend right, "\mathbb{O}"'] & & \text{Topoi} \arrow[ll, bend right, "\mathbb{pt}"'] \\
& \mathsf{LGSketches} \arrow[ul, "\mathbb{Mod}(-)"] \arrow[ur, "\beth(-)"'] &
\end{tikzcd}$$

Of corse, in order to study it, we need to introduce all its nodes and legs. We should say what we mean by $\mathsf{LGSketches}$ and $\mathbb{Mod}(-)$. Whatever they will be, the main point of the section is to show that this diagram fixes the one of the previous section, in the sense that we will obtain the following result.

Theorem. The following are equivalent:

- $\beth(\mathcal{S})$ has enough points;
- $\beth(\mathcal{S})$ coincides with $\mathbb{O}\mathbb{Mod}(\mathcal{S})$.

We decided to present this theorem separately from the previous one because indeed a ionad of models is a much more complex object to study than a category of models, thus the results of the previous section are indeed very interesting, because easier to handle.

Example 4.19 (Motivating ionads of models: Ultracategories). We are not completely used to think about ionads of models. Indeed a (bounded) ionad is a quite complex data, and we do not completely have a logical intuition on its interior operator. In which sense does the interior operator equip a category of models with a topology? One very interesting example, that hasn't appeared in the literature to our knowledge is the case of ultracategories. Ultracategories where introduced by Makkai in [AF13] and later simplified by Lurie in [Lur]. These objects are the data of a category $\mathcal{A}$ together with an ultrastructure, that is a family of functors

$$\int_X : \beta(X) \times \mathcal{A}^X \to \mathcal{A}.$$

We redired to [Lur] for the precise definition. In a nutshell, each of these functors $\int_X$ defines a way to compute the ultraproduct of an X-indexed family of objects along some ultrafilter. Of course there is a notion of morphism of ultrastrucategories, that is a functor $\mathscr{A} \to \mathscr{B}$ which is compatible with the ultrastructure [Lur][Def. 1.41]. Since the category of sets has a natural ultrastructure, for every ultracategory $\mathscr{A}$ one can define $\mathrm{Ult}(\mathscr{A}, \mathbf{Set})$ which obviously sits inside $\mathbf{Set}^{\mathscr{A}}$. Lurie observes that the inclusion

$$\iota : \mathrm{Ult}(\mathscr{A}, \mathbf{Set}) \to \mathbf{Set}^{\mathscr{A}}$$

preserves all colimits [Lur][War. 1.4.4], and in fact also finite limits (the proof is the same). In particular, when $\mathscr{A}$ is accessible, an ultrastructure over $\mathscr{A}$ defines a idempotent lex comonad over $\mathbf{Set}^{\mathscr{A}}$ by the adjoint functor theorem. This shows that every accessible ultracategory yelds a ionad, which is also compact in the sense that its category of opens is a compact (coherent) topos. This example is really a step towards a categorified Stone duality involving compact ionads and boolean topoi.

2.2.1. $\mathsf{LGSketches}$ *and* $\mathbb{M}\mathrm{od}(-)$.

Definition 4.20. A geometric sketch $\mathcal{S}$ is lex if its underling category has finite limits and every limiting cone is the the limit class.

Remark 4.21 (Lex sketches are *enough*). [Bor94c][4.3.3] shows that every geometric sketch can be replaced with a lex geometric sketches in such a way that their underling category of models, even their classifiny topos, does not change. In this sense this full subcategory of geometric sketches is as expressive as the whole category of geometric sketches.

Proposition 4.22 ($\mathbb{M}\mathrm{od}(-)$ on objects). Every lex geometric sketch $\mathcal{S}$ induces a ionad $\mathbb{M}\mathrm{od}(\mathcal{S})$ over its category of models $\mathsf{Mod}(\mathcal{S})$.

Proof. The underling category of the ionad $\mathbb{M}\mathrm{od}(\mathcal{S})$ is $\mathsf{Mod}(\mathcal{S})$. We must provide an interior operator (a lex comonad),

$$\mathrm{Int}_{\mathcal{S}} : \mathbf{Set}^{\mathsf{Mod}(\mathcal{S})} \to \mathbf{Set}^{\mathsf{Mod}(\mathcal{S})}.$$

In order to do so, we consider the evaluation pairing $\mathsf{eval} : \mathcal{S} \times \mathsf{Mod}(S) \to \mathbf{Set}$ mapping $(s, p) \mapsto p(s)$. Let $\mathsf{ev} : \mathcal{S} \to \mathbf{Set}^{\mathsf{Mod}(S)}$ be its mate. Because $\mathcal{S}$ is a lex sketch, this functor must preserve finite limits. Indeed,

$$\mathsf{ev}(\lim s_i)(-) \cong (-)(\lim s_i) \cong \lim((-)(s_i)) \cong \lim \mathsf{ev}(s_i)(-).$$

Now, the left Kan extension $\mathsf{lan}_y\mathsf{ev}$ (see diagram below) is left exact because $\mathbf{Set}^{\mathsf{Mod}(\mathcal{S})}$ is an infinitary pretopos and ev preserve finite limits.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\mathsf{ev}"] \arrow[d, "y"'] & \mathbf{Set}^{\mathsf{Mod}(\mathcal{S})} \\
\mathbf{Set}^{\mathcal{S}^\circ} \arrow[ur, dashed, bend right, "\mathsf{lan}_y\mathsf{ev}"'] &
\end{tikzcd}$$

Moreover it is cocontinuous because of the universal property of the presheaf construction. Because $\mathbf{Set}^{\mathcal{S}^\circ}$ is a total category, $\mathsf{lan}_y\mathsf{ev}$ must have a right adjoint (and it must coincide with $\mathsf{lan}_{\mathsf{ev}}y$). The induced comonad must be left exact, because the left adjoint is left exact. Define

$$\mathrm{Int}_{\mathcal{S}} := \mathsf{lan}_y\mathsf{ev} \circ \mathsf{lan}_{\mathsf{ev}}y.$$

Observe that $\mathrm{Int}_{\mathcal{S}}$ coincides with the density comonad of ev by [Lib19][A.7]. □

REMARK 4.23 ($\mathbb{M}\mathrm{od}(-)$ on morphism of sketches). This definition will not be given explicitely, in fact we will use the following remark to show that the ionad above is isomorphic to the one induces by $\daleth(\mathcal{S})$, and thus there exists a natural way to define $\mathbb{M}\mathrm{od}(-)$ on morphisms.

2.2.2. *Ionads of models and theories with enough points.*

REMARK 4.24. In the main result of previous section, a relevant role was plaied by the fact that $\mathsf{pt}\daleth \cong \mathsf{Mod}$. The same must be true in this one. Thus we should show that $\mathbb{pt}\daleth \cong \mathbb{M}\mathrm{od}$. Indeed we only need to show that the interior operator is the same, because its underlying category is the same by the discussion in the previous section.

PROPOSITION 4.25.

$$\mathbb{pt} \circ \daleth \cong \mathbb{M}\mathrm{od}.$$

PROOF. Let $\mathcal{S}$ be a lex geometric sketch. Of course there is a map $j : S \to \daleth\mathcal{S}$, because S is a site of definition of $\daleth\mathcal{S}$. Moreover, j is obviously dense. In particular the evaluation functon that defines the ionad $\mathbb{pt} \circ \daleth$ given by $ev^* : \daleth(\mathcal{S}) \to \mathbf{Set}^{\mathsf{pt}\circ\daleth(\mathcal{S})}$ is uniquely determined by its composition with j. This means that the comonad ev^*ev_* is isomorphic to the density comonad of the composition $ev^* \circ j$. Indeed,

$$ev^*ev_* \cong \mathsf{lan}_{ev^*}ev^* \cong \mathsf{lan}_{ev^*}(\mathsf{lan}_j(ev^*j)) \cong \mathsf{lan}_{ev^*j}(ev^*j).$$

Yet, ev^*j is evidently ev, and thus $ev^*ev_* \cong \mathrm{Int}_{\mathcal{S}}$ as desired. □

THEOREM 4.26. The following are equivalent:

- $\daleth(\mathcal{S})$ has enough points;
- $\daleth(\mathcal{S})$ coincides with $\mathbb{O}\mathbb{M}\mathrm{od}(\mathcal{S})$.

PROOF. By Thm 3.34, $\daleth(S)$ has enough points if and only if the counit of the categorified Isbell duality $\rho : \mathbb{O}\mathbb{pt}(\daleth)(\mathcal{S}) \to \mathcal{S}$ is an equivalence of topoi. Now, since $\mathbb{pt} \circ \daleth \cong \mathbb{M}\mathrm{od}$, we obtain the thesis. □

3. Abstract elementary classes and locally decidable topoi

3.1. A general discussion. This section is dedicated to the interaction between Abstract elementary classes and the Scott adjunction. Abstract elementary classes were introduced in the 70's by Shelah as a framework to encompass infinitary logics within the language of model theorist. In principle, an abstract elementary class $\mathcal{A}$ should look like the category of models of a first order infinitary theory whose morphisms are elementary embeddings. The problem of relating abstract elementary classes and accessible categories has been challenged by Lieberman [Lie11], Beke and Rosický [BR12], and lately have attracted the interest of some model theorists such as Vasey, Boney and Grossberg [BGL+15]. There are many partial, even very convincing results, in this characterization. Let us recall one at least one of them. For us, this characterization will be the very definition of abstract elementary class.

THEOREM 4.27 ([BR12](5.7)). A category $\mathcal{A}$ is equivalent to an abstract elementary class if and only if it is an accessible category with directed colimits whose morphisms are monomorphisms and which admits a full with respect to isomorphisms and nearly full embedding U into a finitely accessible category preserving directed colimits and monomorphisms.

DEFINITION 4.28. A functor $\mathsf{U} : \mathcal{A} \to \mathcal{B}$ is nearly full if, given a commutative diagram like in figure,

$$\begin{tikzcd} \mathsf{U}(a) \arrow[dd, "h"] \arrow[dr, "\mathsf{U}(f)"] & \\ & \mathsf{U}(c) \\ \mathsf{U}(b) \arrow[ur, "\mathsf{U}(g)"'] & \end{tikzcd}$$

there is a map $\bar{h}$ such that $h = \mathsf{U}(\bar{h})$ and the corresponding diagram in $\mathcal{A}$ commutes. Observe that when U is faithful such a filling has to be unique.

REMARK 4.29. In some reference the notion of nearly-full functor was called coherent, referring directly to the *coherence axiom* of AECs that it incarnates. This terminology was abandoned because the word coherent is overloaded in category theory.

EXAMPLE 4.30 ($\mathsf{pt}(\mathcal{E})$ is likely to be an AEC). Let $\mathcal{G}$ be a Grothendieck topos and $f^* : \mathbf{Set}^{\mathcal{C}} \leftrightarrows \mathcal{G} : f_*$ be a presentation of $\mathcal{G}$. Applying the funtor pt we get a fully faithful functor

$$\mathsf{pt}(\mathcal{E}) \overset{6.2}{\to} \mathsf{pt}(\mathcal{E}) \overset{2.12}{\cong} \mathsf{Ind}(\mathcal{C})$$

into a finitely accessibly category. Thus when every map in $\mathsf{pt}(\mathcal{E})$ is a monomorphism we obtain that $\mathsf{pt}(\mathcal{E})$ is an AEC via 4.27. We will see in the next section **??** that this happens when $\mathcal{E}$ is locally decidable, thus the category of points of a locally decidable topos is always an AEC.

EXAMPLE 4.31 ($\eta_{\mathcal{A}}$ behaves nicely on AECs). When $\mathcal{A}$ is an abstract elementary class, the unit of the Scott adjunction $\eta_{\mathcal{A}} : \mathcal{A} \to \mathsf{ptS}(\mathcal{A})$ is faithful and iso-full. This follows directly from Prop. 6.19.

REMARK 4.32. Even if this is the sharpest (available) categorical characterization of AECs it is not hard to see how unsatisfactory it is. Among the most evident problems, one can see that it is hard to provide a categorical *understanding* of nearly full and full with respect to isomorphisms. Of course, an other problem is that the list of requirements is pretty long and very hard to check: *when such a U exists?*

It is very hard to understand when such a pseudo monomorphism exist. That is why it is very useful to have a testing lemma for its existence.

THEOREM 4.33 (Testing lemma). Let $\mathcal{A}$ be an object in Acc_ω where every morphism is a monomorphism. If $\eta_{\mathcal{A}}$ is a nearly-full pseudo monomorphism, then $\mathcal{A}$ is an AEC.

PROOF. The proof is relatively easy, choose a presentation $f^* : \mathbf{Set}^{\mathscr{C}} \leftrightarrows \mathsf{S}(A) : f_*$ of $\mathsf{S}(A)$. Now

$$\mathcal{A} \stackrel{\eta_{\mathcal{A}}}{\to} \mathsf{ptS}(\mathcal{A}) \stackrel{6.2}{\to} \mathsf{pt}(\mathbf{Set}^{\mathscr{C}}) \stackrel{2.12}{\cong} \mathsf{Ind}(\mathscr{C}),$$

the composition if a faithful and nearly full functor preserving directed colimits from an accessible category to a finitely accessible category, and thus $\mathcal{A}$ is an AEC because of 4.27. □

3.2. Locally decidable topoi and AECs. The main result of this subsection relates locally decidable topoi to AECs. The full subcategory of Acc_ω whose objects are AECs will be indicated by AECs. As in the previous chapters, let us give the precise statement and then discuss it in better detail.

THEOREM 4.34. The Scott adjunction restricts to locally decidable topoi and AECs.

$$\mathsf{S} : \text{AECs} \leftrightarrows \text{LDTopoi} : \mathsf{pt}$$

3.2.1. *Locally decidable topoi.* The definition of locally decidable topos will appear obscure at first sight.

DEFINITION 4.35 (Locally decidable topos). An object e in a topos $\mathscr{E}$ is called locally decidable iff there is an epimorphism $e' \twoheadrightarrow e$ such that e' is a decidable object. $\mathscr{E}$ is locally decidable if every object is locally decidable.

In order to make the definiton above clear we should really define decidable objects and discuss its meaning. This is carried out in the literature and it is not our intention to recast the whole theory of locally decidable topoi. Let us cast and discuss the following characterization, that we may take as a definition.

THEOREM 4.36 ([Joh02b][C5.4.4], Characterization of loc. dec. topoi). The following are equivalent:

(1) $\mathscr{E}$ is locally decidable;
(2) there exists a site (C, J) of presentation where every map is epic;
(3) there exists a logicalic geometric morphis into a Boolean topos.

REMARK 4.37. Recall that a localic topos $\mathscr{E}$ is a topos of sheaves over a locale. The theorem above (which is due to Freyd) shows that a locally decidable topos is still a topos of sheaves over a locale, but the locale is not in **Set**. It is instead in some boolean topos. A boolean topos is the closest kind of topos we can think of to the category of sets itself. For more details, we redirect the reader to the Background chapter, where we give references to the literature.

3.2.2. *Proof of 4.34.*

PROOF OF THM. 4.34.

- Let $\mathcal{E}$ be a locally decidable topos. By 4.30, it is enough to show that every map in $\mathsf{pt}(\mathcal{E})$ is a monomorphism. This is more or less a folklore result, let us give the shortest path to it given our techlogy. Recall that one of the possible characterization of localic topos it that it has a localic geometric morphism into a boolean topos $\mathcal{E} \to \mathcal{B}$. If $\mathcal{B}$ is a boolean topos, then every map in $\mathsf{pt}(\mathcal{G})$ is a monomorphism [Joh02b][D1.2.10, last paragraph]. Now, the induce morphism below,

$$\mathsf{pt}(\mathcal{E}) \to \mathsf{pt}(\mathcal{G}),$$

is faithful by 6.4. Thus every map in $\mathsf{pt}\mathcal{E}$ must be a monomorphism.
- Let's show that for an accessible category with directed colimits $\mathcal{A}$, its Scott topos is locally decidable. By [Joh02b][C5.4.4], it's enough to prove that $\mathsf{S}\mathcal{A}$ has a site where every map is an epimorphism. Using Rem 2.8, A_κ° is a site of definition of $\mathsf{S}\mathcal{A}$, and since every map in $\mathcal{A}$ is a monorphism, every map in A_κ° is epic.

□

The previous theorem admits an even sharper version.

THEOREM 4.38. Let $\mathcal{A}$ be an accessible category with concrete directed colimits, then if $\mathsf{S}\mathcal{A}$ is locally decidable, then every map in $\mathcal{A}$ is a monomorphism;

PROOF. And builds on the idea of the previous theorem.

Step 1 If $\mathcal{G}$ is a boolean topos, then every map in $\mathsf{pt}(\mathcal{G})$ is a monomorphism [Joh02b][D1.2.10, last paragraph].

Step 2 Recall that one of the possible characterization of localic topos it that it has a localic geometric morphism into a boolean topos $\mathcal{E} \to \mathcal{G}$.

Step 3 In the following diagram

$$\mathcal{A} \stackrel{\eta_\mathcal{A}}{\to} \mathsf{ptS}(\mathcal{A}) \stackrel{6.4}{\to} \mathsf{pt}(\mathcal{G}),$$

The composition is a faithful functor, thus $\mathcal{A}$ has a faithful functor into a category where every map is a monomorphism. As a result every map in $\mathcal{A}$ is a monomorphism.

□

COROLLARY 4.39 (Continuous categories and AECs). Let $\mathcal{A}$ be a continuous category, the following are equivalent:

(1) $\mathcal{A}$ is an AEC.
(2) every map in $\mathcal{A}$ is a monomorphism.
(3) $\mathsf{S}(\mathcal{A})$ is locally decidable.

PROOF. Since it's a split subobject in Acc_ω of a finitely accessible category, the hypotheses of [BR12][5.7] are met. □

4. Categories of saturated objects, atomicity and categoricity

REMARK 4.40. In this section we define categories of saturated objects and study their connection with atomic topoi and categoricity. The connection between atomic topoi and categoricity was pointed out in [Car12]. This section corresponds to a kind of syntax-free counterpart of [Car12]. In the definition of *category of saturated objects* we axiomatize the relevant properties of the inclusion $\iota : \mathbf{Set}_\kappa \to \mathbf{Set}$ and we prove the following two theorems.

THEOREM.

(1) Let $\mathcal{A}$ be a category of saturated objects, then $\mathsf{S}(\mathcal{A})$ is an atomic topos.
(2) If in addition $\mathcal{A}$ has the joint embedding property, then $\mathsf{S}(\mathcal{A})$ is boolean and two valued.
(3) If in addition j is isofull and faithful and surjective on objects, then $\mathcal{A}$ is categorical in some presentability rank.

THEOREM. If $\mathcal{E}$ is an atomic topos, then $\mathsf{pt}(\mathcal{E})$ is a *candidate* category of saturated objects.

Let us recall (or introduce) the notion of ω-saturated object in a an accessible category.

DEFINITION 4.41. Let $\mathcal{A}$ be an accessible category. We say that $s \in \mathcal{A}$ is ω-saturated if it is injective with respect to maps between finitely presentable objects. Than it, given a morphism between finitely presentable objects $f : p \to p'$ and a map $p \to s$, there exists a lift as in the diagram below.

$$\begin{tikzcd} s & \\ p \arrow[u] \arrow[r] & p' \arrow[ul, dashed] \end{tikzcd}$$

REMARK 4.42. In general, when we look at accessible categories from the perspective of model theory, every map in $\mathcal{A}$ is a monomorphism, and this definition is implicitely adding the hypothesis that every morphism is *injective.*

REMARK 4.43. A very good paper to understand the categorical approach to saturation is [Ros97a].

REMARK 4.44. In [Hen19], Herny proves that there are AECs that cannot appear as the category of points of a topos, that means that cannot be axiomatized in $\mathrm{L}_{\infty,\omega}$. This answers a question initially asked by Rosický at the CT2014 and makes a step towards our understanding of the connection between accessible categories with directed colimits and axiomatizable classes. The main tool that allows him to achieve this result is called in the paper the *Scott construction*; he proves the Scott topos of $\mathbf{Set}_\kappa$[1] is atomic. Even if we developed together the rudiments of the Scott construction the reason for which this result was true appeared to me enigmatic and mysterious. With this motivation in mind the author came to the conclusion that the Scott topos of $\mathbf{Set}_\kappa$ is atomic because of fact that $\mathbf{Set}_\kappa$ appears as a subcategory of saturated objects in $\mathbf{Set}$.

[1]The category of sets of cardinality at least κ and injective functions

Remark 4.45. As a direct corollary of the theorems in this section one gets back the main result of [Hen19], but this is not the main accomplishment of this section. Our main contribution is to present a conceptual understanding of [Hen19] and a neat technical simplification of his proofs. We also improve our poor knowledge of the Scott adjunction, trying to collect and underline its main features. We feel that the Scott adjunction might serve as a tool to have a categorical understanding of the Shelah categoricity conjecture for accessible categories with directed colimits.

Remark 4.46 (What is categoricity and what about the categoricity conjecture?). Recall that a category of models of some theory is categorical in some cardinality κ if it has precisely one model of cardinality κ. Morley has show that if a category of models is categorical in some cardinal κ, then it must be categorical in any cardinal above and in any cardinal below up to ω_1. When Abstract elementary classes were introduced in 70's, Shelah has chosen Morley's theorem as a sanity check result for his definition. Since then, many approximations of these results has appeared in the literature. The most updated to our knowledge is contained in [Vas19b]. We recommend the paper also as an introduction to this topic.

Remark 4.47. The notion of *category of saturated objects* axiomatize the properties of the inclusion $\mathscr{A} \hookrightarrow \mathrm{Sat}_\omega(\mathscr{K}) \hookrightarrow \mathscr{K}$, our motivating example was the inclusion of $\mathbf{Set}_\kappa \hookrightarrow \mathbf{Set}_\omega \hookrightarrow \mathbf{Set}$. The fact that every object in $\mathbf{Set}_\kappa$ is injective with respect to finite sets is essentially the axiom of choice. [Ros97b] spots a direct connection between saturation and amalgamation property, which was also implied in [Car12].

Definition 4.48 ((Candidate) categories of (ω-)saturated objects). Let $\mathscr{A}$ be a category in Acc_ω. We say that $\mathscr{A}$ is a category of (finitely) saturated objects if there is topological embedding $j : \mathscr{A} \to \mathscr{K}$ in Acc_ω such that:

(1) $\mathscr{K}$ is a finitely accessible category.
(2) $j\mathscr{A} \subset \mathrm{Sat}_\omega(\mathscr{K})$.
(3) $\mathscr{K}_\omega$ has the amalgamation property.

We say that $\mathscr{A}$ is a candidate category of (finitely) saturated objects if j is not a topological embedding.

In [Car12], Caramello proves - essentially - that the category of points of an atomic topos is a category of saturated objects and she observes that it is countable categorical. This shows that there is a deep connection between categoricity, saturation and atomic topoi. We recall the last notion before going on with the exposition.

Definition 4.49 (Characterization of atomic topoi). Let $\mathscr{G}$ be a Grothendieck topos, then the following are equivalent:

(1) $\mathscr{G}$ is atomic.
(2) $\mathscr{G}$ is the category of sheaves over an atomic site.
(3) The subobject lattice of every object is a complete atomic boolean algebra.
(4) Every object can be written as a disjoint union of atoms.

Theorem 4.50.

(1) Let $\mathcal{A}$ be a category of saturated objects, then $\mathsf{S}(\mathcal{A})$ is an atomic topos.
(2) If in addition $\mathcal{A}$ has the joint embedding property, then $\mathsf{S}(\mathcal{A})$ is boolean and two valued.
(3) If in addition j is isofull and faithful and surjective on objects, then $\mathcal{A}$ is categorical in some presentability rank.

PROOF.
(1) Let $\mathcal{A}$ be a category of saturated objects $j : \mathcal{A} \to \mathcal{K}$, we must show that $\mathsf{S}(\mathcal{A})$ is atomic. The idea of proof is very simple, we will show that:
   a $\mathsf{S}j$ presents $\mathcal{A}$ as $j^* : \mathbf{Set}^{\mathcal{K}_\omega} \leftrightarrows \mathsf{S}(\mathcal{A}) : j_*$;
   b The induced topology on $\mathcal{K}_\omega$ is atomic.
   (a) follows directed by 6.2 and 2.12. (b) goes identically to [Hen19][Cor. 4.9]: note that for any map $k \to k' \in \mathcal{K}_\omega$, the induced map $j^*yk' \to j^*yk$ is an epimorphisms: indeed any map $k \to ja$ with $a \in \mathcal{A}$ can be extended k' because j makes $\mathcal{A}$ a category of saturated objects. So the induced topology on $\mathcal{K}_\omega$ is the atomic topology (every non-empty sieve is a cover). The fact that $\mathcal{K}_\omega$ has the amalgamation property is needed to make the atomic topology a proper topology.
(2) Because $\mathcal{A}$ has the joint embedding property, its Scott topos is connected by Cor. 7.9. Then, $\mathsf{S}(\mathcal{A})$ is atomic and connected. By [Car18][4.2.17] it is boolean two-valued.
(3) This follows from 6.19 and [Car12]. In fact, Caramello has shown that $\mathsf{ptS}(\mathcal{A})$ must be countably categorical and the countable object is saturated (by construction). Thus, the unit of the Scott adjunction must reflect the (essential) unicity of such object.

□

THEOREM 4.51. If $\mathcal{E}$ is an atomic topos, then $\mathsf{pt}(\mathcal{E})$ is a *candidate* category of saturated objects.

PROOF. Let $\mathcal{E}$ be an atomic topos and $i : \mathcal{E} \to \mathbf{Set}^C$ be a presentation of $\mathcal{E}$ by an atomic site. It follows from [Car12] that $\mathsf{pt}(i)$ presents $\mathsf{pt}(\mathcal{E})$ as is a candidate category of saturated objects. □

4.1. Categories of κ-saturated objects. Obviously the previous definitions can be generalized to the κ-case of the Scott adjunction, obtaining analogous results. Let us boldly state them.

DEFINITION 4.52 ((Candidate) categories of (κ-)saturated objects). Let $\mathcal{A}$ be a category in Acc_κ. We say that $\mathcal{A}$ is a category of κ-saturated objects if there is topological embedding (for the S_κ-adjunction) $j : \mathcal{A} \to \mathcal{K}$ in Acc_κ such that:

(1) $\mathcal{K}$ is a κ-accessible category.
(2) $j\mathcal{A} \subset \text{Sat}_\kappa(\mathcal{K})$.
(3) $\mathcal{K}_\kappa$ has the amalgamation property.

We say that $\mathcal{A}$ is a candidate category of κ-saturated objects if j is not a topological embedding.

THEOREM 4.53.

(1) Let $\mathscr{A}$ be a category of κ-saturated objects, then $\mathsf{S}_\kappa(\mathscr{A})$ is an atomic κ-topos.
(2) If in addition $\mathscr{A}$ has the joint embedding property, then $\mathsf{S}_\kappa(\mathscr{A})$ is boolean and two valued.
(3) If in addition j is isofull and faithful and surjective on objects, then $\mathscr{A}$ is categorical in some presentability rank.

THEOREM 4.54. If $\mathscr{E}$ is an atomic κ-topos, then $\mathsf{pt}_\kappa(\mathscr{E})$ is a *candidate* category of κ-saturated objects.

CHAPTER 5

Category theory

This chapter is dedicated to a 2-categorical perspective on the Scott adjunction and its main characters. We provide an overview of the categorical properties of Acc_ω and Topoi. Mainly, we show that the 2-category of topoi is enriched over Acc_ω and has copowers. We show that this observation generalizes the Scott adjunction in a precise sense. We discuss the 2-categorial properties of both the 2-categories, but this work is not original. We will provide references along the discussion.

1. 2-categorical properties of $\mathbf{Acc}_\omega$

1.1. (co)Limits in $\mathbf{Acc}_\omega$. The literature contains a variety of results on the 2-dimensional structore of the 2-category Accof accessible categories and accessible functors. Among the others, one should mention [MP89] for lax and pseudo-limits in Acc and [PR13] for colimits. Indeed our main object of study, namely Acc_ω, is a (non-full) subcategory of Acc_ω, and thus its a bit tricky to infer its properties from the existing literature. Most of the work was successfully accomplished in [LR15]. Let us list the main results of these references that are related to Acc_ω.

Proposition 5.1 ([LR15][2.2]). Acc_ω is closed under pie-limits[1] in Acc (and thus in the illigitime 2-category of locally small categories).

Proposition 5.2 (Slight refinement of [PR13][2.1]). Every directed diagram of accessible categories and full embeddings preserving directed colimits has colimit in Cat, and is in fact the colimit in Acc_ω.

1.2. $\mathbf{Acc}_\omega$ is monoidal closed. This subsection discusses a monoidal closed structure on Acc_ω. The reader should keep in mind the monoidal product of modules over a ring, because the construction is similar in spirit, at the end of the subsection we will provide an hopefully convincing argument in order to show that the construction are similar for a quite quantitative reason. The main result of the section should be seen as a slight variation of [KK82][6.5] where the enrichement base is obviously the category of Sets and $\mathcal{F}$-cocontinuity is replaces by preservation of directed colimits. Our result doesn't technically follows from Kelly's one because of size issue, but the proof line and the general idea of the proof belongs to those pages. Moreover, we found cleared to provide an explicit construction of the tensor product in our specific case. The reader is encouraged to check [hdl], where Brandeburg provides a concise presentations of the Kelly's construction.

[1]These are those limits can be reduced to products, inserters and equifiers.

Remark 5.3 (A natural internal hom). Indeed given two accessible categories $\mathcal{A}, \mathcal{B}$ in Acc_ω, the category of functors preserving directed colimits $\mathrm{Acc}_\omega(\mathcal{A}, \mathcal{B})$ has directed colimits and they are computed pointwise, thus we obtain a 2-functor,

$$[-,-] : \mathrm{Acc}_\omega^\circ \times \mathrm{Acc}_\omega \to \mathrm{Acc}_\omega.$$

In our analogy, this corresponds to the fact that the set of morphisms between two modules over a ring $\mathsf{Mod}(M, N)$ has a (pointwise) structure of module.

Remark 5.4 (Looking for a tensor product: the universal property). Assume for a moment that the tensorial structure that we are looking for exists, then we would obtain a family of natural equivalence of categories,

$$\mathrm{Acc}_\omega(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathrm{Acc}_\omega(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \cong \omega\mathrm{Bicocont}(\mathcal{A} \times \mathcal{B}, \mathcal{C}).$$

In the display we wrote -Bicocont$(\mathcal{A} \times \mathcal{B}, \mathcal{C})$ to mean the category of twose functors that preserve directed colimits in each variable. The equation gives us the universal property that should define $\mathcal{A} \otimes \mathcal{B}$ up to equivalence of categories and is consistent with our ongoing analogy of modules over a ring, indeed the tensor product classifies *bilinear maps*.

Remark 5.5 (Looking for a tensor product: the consturction). Let $よ : \mathcal{A} \times \mathcal{B} \to \mathcal{P}(\mathcal{A} \times \mathcal{B})$ be the Yoneda embedding of $\mathcal{A} \times \mathcal{B}$ corestricted to the full subcategory of small presheaves. Let Biflat be the full subcategory of $\mathcal{P}(\mathcal{A} \times \mathcal{B})$ that are flat in both variables. The inclusion $i : \mathrm{Biflat} \hookrightarrow \mathcal{P}(\mathcal{A} \times \mathcal{B})$ defines a small-orthogonality class[2] in $\mathcal{P}(\mathcal{A} \times \mathcal{B})$ and is thus reflective [AR94][1.37, 1.38]. Let L be the left adjoint of the inclusion, as a result we obtain an adjunction,

$$L : \mathcal{P}(\mathcal{A} \times \mathcal{B}) \leftrightarrows \mathrm{Biflat} : i.$$

Now define $\mathcal{A} \otimes \mathcal{B}$ to be the smallest full subcategory of $\mathcal{P}(\mathcal{A} \times \mathcal{B})$ closed under directed colimits and containing the image of $L \circ よ$. Thm. [KK82][6.23] in Kelly ensures that $\mathcal{A} \otimes \mathcal{B}$ has the universal property described in 5.4 and thus is our tensor product. It might be a bit hard to see but this construction still follows our analogy, the tensor product of two modules is indeed built from free module on the product and the *bilinear* relations.

Theorem 5.6. Acc_ω, toghether with the tensor product $\otimes$ defined in 5.5 and the internal hom defined in 5.3 is a monoidal closed category[3] in the sense that there is an equivalnce of categories

$$\mathrm{Acc}_\omega(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathrm{Acc}_\omega(\mathcal{A}, [\mathcal{B}, \mathcal{C}]),$$

whch is natural in $\mathcal{C}$.

Proof. Follows directly from the discussion above. $\square$

[2]Here we are using that $\mathcal{A}$ and $\mathcal{B}$ are accessible in order to cut down the size of the orthogonality.

[3]This is not strictly true, because the definition of monoidal closed category does not allow for equivalence of categories. We did not find a precise terminology in the literature and we felt non-useful to introduct a new concept for such a small discrepancy.

Remark 5.7 (Up to iso/up to equivalence). As in [KK82][6.5], we will not distinguish between the properties of this monoidal structure (where everything is true up to equivalence of categories) and a usual one, where everything is true up to isomorphism. In our study this distinction never plays a role, thus we will use the usual terminology about monoidal structures.

Remark 5.8 (The unit). The unit of the abovementioned monoidal structure is the terminal category in Cat, which is also terminal in Acc_ω.

Remark 5.9 (Looking for a tensor product: an abstract overview). Along this subsection we kept using the case of modules over a ring as a kind of analogy/motivating examples. In this remark we shall convince the reader that the analogy can be pushed much further. Let's start by the observation that R-Mod is the category of algebras for the monad $R[-] : \mathbf{Set} \to \mathbf{Set}$. The monoidal closed structure of Mod can be recovered from the one of the category of sets $(\mathbf{Set}, 1, \times, [-,-])$. via a quite classical theorem proved by Seal in 2012 and motivated by a series of publication of Kock in the seventies. It would not make the tractation more readable to cite all the papers that are involved in this story, thus we mention the PhD thesis of Brandeburg [Bra14][Chap. 6] which provides a very cohese and elegant presentation of the literature.

Theorem 5.10. (Seal, 6.5.1 in [Bra14]) Let T be a coherent (symmetric) monoidal monad on a (symmetric) monoidal category C. Then $\mathsf{Mod}(T)$ becomes a (symmetric) monoidal category.

Now similarly to $\mathsf{Mod}(R)$ the 2-category of categories with directed colimits and functors preserving them is the category of (pseudo)algebras for the KZ monad of the Ind-completion over locally small categories

$$\mathsf{Ind} : \text{Cat} \to \text{Cat}.$$

[Bou17][6.7] provides a version of Seal's theorem for monads over Cat. While it's quite easy to show that the completion under directed colimits meets many of Bourke's hypoteses, we do not believe that it meets all of them, thus we did not manage to apply a Kock-like result to derive Thm. 5.6. Yet, we think we provided enough evidence that the analogy is not just motivational.

2. 2-categorical properties of Topoi

2.1. (co)Limits in Topoi. The 2-categorical properties of the category of topoi has been studied in deep detail in the literature. We mention [Joh02b][B1.4] and [Lur09] as a main reference.

2.2. Enrichment over $\mathbf{Acc}_\omega$, tensor and powers.

2.2.1. *The enrichment.* The main content of this subsubsection will be to show that the category of topoi and geometric morphisms (in the direction of the right adjoint) is enriched over Acc_ω.

Remark 5.11. Recall that to provide such an enrichment means to

(1) show that given two topoi $\mathcal{E}, \mathcal{F}$, the set of geometric morphisms $\text{Topoi}(\mathcal{E}, \mathcal{F})$ admits a structure of accessible category with directed colimits.

(2) provide, for each triple of topoi $\mathcal{E}, \mathcal{F}, \mathcal{G}$, a functor preserving directed colimits

$$\circ : \text{Topoi}(\mathcal{E}, \mathcal{F}) \otimes \text{Topoi}(\mathcal{F}, \mathcal{G}) \to \text{Topoi}(\mathcal{E}, \mathcal{G}),$$

making the relevant diagrams commute.

(1) will be shown in 5.12, while (2) will be shown in 5.13.

Proposition 5.12. Let $\mathcal{E}, \mathcal{F}$ be two topoi. Then the category of geometric morphisms Cocontlex$(\mathcal{E}, \mathcal{F})$, whose objects are cocontinuous left exact functors and morphisms are natural transformations is an accessible category with directed colimits.

Proof. The proof goes as in [Bor94c][Cor.4.3.2], **Set** plays no role in the proof. What matters is that finite limits commute with directed colimits in a topos. □

Proposition 5.13. for each triple of topoi $\mathcal{E}, \mathcal{F}, \mathcal{G}$, a functor preserving directed colimits

$$\circ : \text{Topoi}(\mathcal{E}, \mathcal{F}) \otimes \text{Topoi}(\mathcal{F}, \mathcal{G}) \to \text{Topoi}(\mathcal{E}, \mathcal{G}),$$

making the relevant diagrams commute.

Proof. We will only provide the composition. The relevant diagrams commute trivially from the presentation of the composition. Recall that by 5.4 a map of the form $\circ : \text{Topoi}(\mathcal{E}, \mathcal{F}) \otimes \text{Topoi}(\mathcal{F}, \mathcal{G}) \to \text{Topoi}(\mathcal{E}, \mathcal{G})$, preserving directed colimits is the same of a functor

$$\circ : \text{Topoi}(\mathcal{E}, \mathcal{F}) \times \text{Topoi}(\mathcal{F}, \mathcal{G}) \to \text{Topoi}(\mathcal{E}, \mathcal{G})$$

preserving directed colimits in each variables. Obviously, since left adjoints can be composed in Cat, we already have such a composition. It's enough to show that it preserves directed colimits in each variables. Indeed this is the case, because directed colimits in these categories are computed pointwise. □

Theorem 5.14. The category of topoi is enriched over Acc_ω.

2.3. Tensors. In this subsection we show that the category of topoi has tensors (copowers) with respect to the enrichment of the previous section.

Remark 5.15. Let us recall what means to have tensors for a category K enriched over sets (that is, just a locally small category). To have tensors in this case means that we can define a functor $\boxtimes : \mathbf{Set} \times \mathsf{K} \to \mathsf{K}$ in such a way that,

$$\mathsf{K}(S \boxtimes k, h) \cong \mathbf{Set}(S, \mathsf{K}(k, h)).$$

For example, the category of modules over a ring has tensors given by the formula $S \boxtimes M := \oplus_S M$, indeed it is straightforward to observe that

$$R\text{-}\mathsf{Mod}(\oplus_S M, N) \cong \mathbf{Set}(S, R\text{-}\mathsf{Mod}(M, N)).$$

In this case, this follows from the universal property of the coproduct.

Remark 5.16 (The construction of tensors). We shall define a 2-functor $\boxtimes : \text{Acc}_\omega \times \text{Topoi} \to \text{Topoi}$. Our construction is reminiscent of the Scott adjunction, and we will see that there is an extremely tight connection between the two. Given a topos $\mathcal{E}$ and an accessible category with directed colimits $\mathcal{A}$ we define,

$$\mathcal{A} \boxtimes \mathcal{E} := \mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E}).$$

In order to make this construction meaningful we need to accomplish two tasks:

(1) show that the construction is well defined (on the level of objects), that is, show that $\mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E})$ is a topos.
(2) describe the action of $\boxtimes$ on functors.

We split these two tasks in two different remarks.

Remark 5.17 ($\mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E})$ is a topos). By definition $\mathcal{A}$ must be λ-accessible for some λ. Obviously $\mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E})$ sits inside $\lambda\text{-}\mathrm{Acc}(\mathcal{A}, \mathcal{E})$. Recall that $\lambda\text{-}\mathrm{Acc}(\mathcal{A}, \mathcal{E})$ is equivalent to $\mathcal{E}^{\mathcal{A}_\lambda}$ by the restricion-Kan extension paradigm and the universal property of Ind_λ-completion. This inclusion $i : \mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E}) \hookrightarrow \mathcal{E}^{\mathcal{A}_\lambda}$, preserves all colimits and finite limits, this is easy to show and depends on one hand on how colimits are computed in this category of functors, and on the other hand on the fact that in a topos directed colimits commute with finite limits. Thus $\mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E})$ amounts to a coreflective subcategory of a topos whose associated comonad is left exact. By [LM94][V.8 Thm.4], it is a topos.

Remark 5.18 (Action of $\boxtimes$ on functors). Let $f : \mathcal{A} \to \mathcal{B}$ be a functor in $\mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E})$ and $g : \mathcal{E} \to \mathcal{F}$ be a geometric morphism, we must define a geometric morphism

$$f \boxtimes g : \mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E}) \to \mathrm{Acc}_\omega(\mathcal{B}, \mathcal{F}).$$

We shall describe the left adjoint $(f \boxtimes g)^*$ (which goes in the opposite direction $(f \boxtimes g)^* : \mathrm{Acc}_\omega(\mathcal{B}, \mathcal{F}) \to \mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E})$) by the following equation:

$$(f \boxtimes g)^*(s) = g^* \circ s \circ f.$$

Proposition 5.19. Topoi has tensors over Acc_ω.

Proof. Putting together the content of 5.16, 5.17 and 5.18, we only need to show that $\boxtimes$ has the correct universal property, that is:

$$\mathrm{Topoi}(\mathcal{A} \boxtimes \mathcal{E}, \mathcal{F}) \cong \mathrm{Acc}_\omega(\mathcal{A}, \mathrm{Topoi}(\mathcal{E}, \mathcal{F})).$$

When we spell out the actual meaning of the equation above, we discover tha we did all the relvan job in the previous remarks. Indeed the biggest obstruction was the well-posedness of the definition.

$$\begin{aligned}
\mathrm{Topoi}(\mathcal{A} \boxtimes \mathcal{E}, \mathcal{F}) \cong& \mathrm{Cocontlex}(\mathcal{F}, \mathcal{A} \boxtimes \mathcal{E}) \\
\cong& \mathrm{Cocontlex}(\mathcal{F}, \mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E})) \\
\cong& \mathrm{Cat}_{\mathrm{cocontlex}, \mathrm{acc}_\omega}(\mathcal{F} \times \mathcal{A}, \mathcal{E}) \\
\cong& \mathrm{Acc}_\omega(\mathcal{A}, \mathrm{Cocontlex}(\mathcal{F}, \mathcal{E})) \\
\cong& \mathrm{Acc}_\omega(\mathcal{A}, \mathrm{Topoi}(\mathcal{E}, \mathcal{F})).
\end{aligned}$$

□

3. The Scott adjunction revisited

Remark 5.20 (Yet another proof of the Scott adjunction). Let us start be mentioning that we can reobtain the Scott adjunction directly from the fact that Topoi is tensored over Acc_ω. Indeed if we evaluate the equation in 5.19 when $\mathcal{E}$ is the terminal topos Set,

$$\text{Topoi}(\mathcal{A} \boxtimes \mathbf{Set}, \mathcal{F}) \cong \text{Acc}_\omega(\mathcal{A}, \text{Topoi}(\mathbf{Set}, \mathcal{F}))$$

we obtain precisely the statement of the Scott adjunction,

$$\text{Topoi}(\mathsf{S}(\mathcal{A}), \mathcal{F}) \cong \text{Acc}_\omega(\mathcal{A}, \mathsf{pt}(\mathcal{F})).$$

Being tensored over Acc_ω means in a way to have a relative version of the Scott adjunction.

Remark 5.21. Among natural numbers, we find extremely familiar the following formula,

$$(30 \times 5) \times 6 = 30 \times (5 \times 6).$$

Yet, this formula yelds an important property of the category of sets. Indeed **Set** is tensored over itself and the tensorial structure is given by the product. The formula above tells us that the tensorial structure of set associates over its product. The same property is true for the tensorial structure of topoi over accessible categories with directed colimits.

Remark 5.22 (Associativity of $\boxtimes$ with respect to $\times$). Recall that the category of topoi has products, but they are very far from being computed as in Cat. Pitts has shown [Pit85] that $\mathcal{E} \times \mathcal{F} \cong \text{Cont}(\mathcal{E}^\circ, \mathcal{F})$. This description, later rediscovered by Lurie, is crucial to get a slick proof of the statement below. In the next proposition we show that $\boxtimes$ is associative with respect to the product of topoi.

Proposition 5.23.

$$\mathcal{A} \boxtimes (\mathcal{E} \times \mathcal{F}) \cong (\mathcal{A} \boxtimes \mathcal{E}) \times \mathcal{F}.$$

Proof. We show it by direct computation.

$$\begin{aligned}\mathcal{A} \boxtimes (\mathcal{E} \times \mathcal{F}) \cong & \text{Acc}_\omega(\mathcal{A}, \text{Cont}(\mathcal{E}^\circ, \mathcal{F})) \\ \cong & \text{Cont}(\mathcal{E}^\circ, \text{Acc}_\omega(\mathcal{A}, \mathcal{F})) \\ \cong & (\mathcal{A} \boxtimes \mathcal{F}) \times \mathcal{E}.\end{aligned}$$

□

Remark 5.24. Similarly to Rem. 5.21, the following display will appear completely trivial,

$$(30 \times 1) \times 6 = 30 \times (1 \times 6).$$

Yet, we can get inspiration from it, to unveil an important simplification of the tensor $\mathcal{A} \boxtimes \mathcal{E}$. We will show that it is enough to know the Scott topos $\mathsf{S}(\mathcal{A})$ to compute $\mathcal{A} \boxtimes \mathcal{E}$.

Proposition 5.25 (Interaction between $\boxtimes$ and Scott).

$$\mathcal{A} \boxtimes (-) \cong \mathsf{S}(\mathcal{A}) \times (-).$$

PROOF.

$$\mathscr{A} \boxtimes (-) \cong \mathscr{A} \boxtimes (\mathbf{Set} \times -) \cong (\mathscr{A} \boxtimes \mathbf{Set}) \times (-).$$

$\square$

PROPOSITION 5.26 (Powers and exponentiable Scott topoi). Let $\mathscr{A}$ be an accessible category with directed colimits. The following are equivalent:

(1) Topoi has powers with respect to $\mathscr{A}$,
(2) $\mathsf{S}(\mathscr{A})$ is a continuous category.

Morover, in this positive case, $\mathscr{E}^{\mathscr{A}}$ is given by the expontial topos $\mathscr{E}^{\mathsf{S}(\mathscr{A})}$.

PROOF. The universal property of the power object $\mathscr{E}^{\mathscr{A}}$ is expressed by the following equation,

$$\text{Topoi}(\mathscr{F}, \mathscr{E}^{\mathscr{A}}) \cong \text{Acc}_{\omega}(\mathscr{A}, \text{Topoi}(\mathscr{F}, \mathscr{E})).$$

Now, because we have tensors, this is saying that $\text{Topoi}(\mathscr{F}, \mathscr{E}^{\mathscr{A}}) \cong \text{Topoi}(\mathscr{A} \boxtimes \mathscr{F}, \mathscr{E}))$. Because of the previous proposition, we can gather this observation in the following equation.

$$\text{Topoi}(\mathscr{F}, \mathscr{E}^{\mathscr{A}}) \cong \text{Topoi}(\mathsf{S}(\mathscr{A}) \times \mathscr{F}, \mathscr{E})).$$

This means that $\mathscr{E}^{\mathscr{A}}$ has the same universal property of the topos $\mathscr{E}^{\mathsf{S}(\mathscr{A})}$ and thus exists if and only if the latter exists. By the well known characterization of exponentiable topoi, this happens if and only if $\mathsf{S}(\mathscr{A})$ is exponentiable. $\square$

REMARK 5.27 (Finitely accessible categories are exponentiable). Since finitely accessible categories are continuous, the previous result applies to them. We shall spell out this concrete example, because the Scott topos is easy to compute in this case.

REMARK 5.28 (Classifying topoi for categories of diagrams). The previous discussion can be used to give a proof in our framework of a well known fact in topos theory, namely the category of diagrams over the category of points of a topos can be axiomatized by a geometric theory. This means that there exists $\mathscr{F}$ a topos such that

$$\mathsf{pt}(\mathscr{E})^{C} \cong \mathsf{pt}(\mathscr{F}).$$

With our technology, this is quite easy to show, in fact:

$$\begin{aligned}
\mathsf{pt}(\mathscr{E})^{C} &= \text{Cat}(C, \mathsf{pt}(\mathscr{E})) \\
&\cong \text{Acc}_{\omega}(\mathsf{Ind}(C), \mathsf{pt}(\mathscr{E})) \\
&\cong \text{Topoi}(\mathbf{Set}, \mathscr{E}^{\mathsf{Ind}(C)}) \\
&\cong \mathsf{pt}(\mathscr{E}^{\mathsf{Ind}(C)}).
\end{aligned}$$

It is not hard to show that $\mathscr{E}^{\mathsf{Ind}(C)}$ coincides with the category of functors from C° into $\mathscr{E}$ itself, $\mathscr{E}^{\mathsf{Ind}(C)} \cong \mathscr{E}^{C^{\circ}}$.

CHAPTER 6

Toolbox

This chapter contains technical results on the Scott adjunction that will be extensively employed later for more qualitative results. We study the behaviour pt, S and η, trying to spot all their relevant properties. Before continuing, we briefly list the main results that we will reach in order to facilitate the consultation.

- 6.2 pt transforms geometric embeddings in fully faithful functors.
- 6.4 pt transforms localic morphisms in faithful functors.
- 6.7 S maps pseudo-epis (of Cat) to geometric surjections.
- 6.11 S maps (weak) reflections to geometric embeddings.
- 1.3 Introduces and studies the notion of topological embeddings between accessible categories.
- 6.19 η is faithful (and iso-full) if and only if $\mathcal{A}$ has a faithful (and iso-full) functor into a finitely accessible category.

1. Embeddings & surjections

Remark 6.1. Observe that since S is a left adjoint, it preserve pseudo epimorphisms, analogously pt preserves pseudo-monomorphisms. Props. 6.2 and 6.7 might be consequences of this observation, but we lack of an explicit description of pseudo monomorphisms and pseudo epimorphisms in both categories. Notice that, instead 6.14 and 6.11 represent a surprising behaviour of S.

1.1. On the behaviour of pt.

The functor pt behaves nicely with various notions *injective* or *locally injective* geometric morphism.

1.1.1. *Geometric embeddings.* Geometric embeddings between topoi are a key objct in topos theory. Intuitively, they represent the proper notion of subtopos. The most common example of geometric embedding is the one of presentation of a topos in the sense of 1.41. It is a well kwon fact that subtopoi of a presheaf topos $\mathbf{Set}^C$ correspond to Grothendieck topologies on C bijectively.

Proposition 6.2. pt transforms geometric embeddings in fully faithful functors.

Proof. This is a relatively trivial consequence of the fact that the direct image functor is fully faithful but we shall include the proof in order to show a standard way of thinking. Let $i^* : \mathcal{G} \leftrightarrows \mathcal{E} : i_*$ be a geometric embedding. Recall, this means precisely that the $\mathcal{E}$ is reflective in $\mathcal{G}$ via this adjunction, i.e. the direct image is fully faithful. Let $p, q : \mathbf{Set} \rightrightarrows \mathcal{E}$ be two points, or equivalently let $p^*, q^* : \mathcal{E} \rightrightarrows \mathbf{Set}$ be two cocontinuous functors preserving finite limits. And let $\mu, \nu : p^* \Rightarrow q^*$ be two natural transformation between the points.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, phantom, "\eta\Downarrow \quad \Downarrow\nu"] & \mathscr{E} \arrow[l, bend right, "p^*"'] \arrow[l, bend left, "q^*"] \arrow[r, hook, bend right, "i_*"'] & \mathscr{G} \arrow[l, bend right, "i^*"'] \end{tikzcd}$$

The action of pt on i is the following. It maps p^* to p^*i^* while the $\mathsf{pt}(i)(\mu)$ is defined by wiskering μ with i^* as pictured by the diagram below.

$$\begin{tikzcd} p^*(a) \arrow[d, Rightarrow, bend right, "\mu"'] \arrow[d, Rightarrow, bend left, "\nu"] & p^*i^*(g) \arrow[d, Rightarrow, bend right, "\nu_{i^*g}"] \arrow[d, Rightarrow, bend left, "\mu_{i^*g}"] \\ q^*(a) & q^*i^*(g) \end{tikzcd}$$

Now, observe that $\mu = \mu_{i^*i_*}$ because i^*i_* is the identity, this proves that $\mathsf{pt}(i)$ is faithful, in fact $\mathsf{pt}(i)(\mu) = \mathsf{pt}(i)(\nu)$ means that $\mu_{i^*} = \nu_{i^*}$, this implies that $\mu_{i^*i_*} = \nu_{i^*i_*}$, that is precisely $\mu = \nu$. A similar argument shows that it is full (using that i_* is full). $\square$

1.1.2. *Localic morphisms.* Localic topoi are those topoi that appear as the category of sheaves over a locale. Those topoi have a clear topological meaning and represent a quite concrete notion of generalized space. Localic morphisms are used to generalize the notion of localic topos, intuitively (and in fact this is also a theorem) a localic morphism $f : \mathscr{G} \to \mathscr{E}$ attests that there exist an internal locale L in $\mathscr{E}$ such that $\mathscr{G} \cong \mathsf{Sh}(L, \mathscr{E})$. In accordance with this observation, a topos $\mathscr{G}$ is localic if and only if the terminal geometric morphism $\mathscr{G} \to \mathbf{Set}$ is localic.

Definition 6.3. A morphism of topoi $f : \mathscr{G} \to \mathscr{E}$ is localic if every object in $\mathscr{G}$ is a subquotient of an object in the inverse image of f.

Proposition 6.4.
pt transforms localic geometric morphisms into faithful functors.

Proof. Consider a localic geometric morphism $f : \mathscr{G} \to \mathscr{E}$. We shall prove that $\mathsf{pt}f$ is faithful on points. In order to do so, let $p, q : \mathbf{Set} \rightrightarrows \mathscr{E}$ be two points, or equivalently let $p^*, q^* : \mathscr{E} \rightrightarrows \mathbf{Set}$ be two cocontinuous functors preserving finite limits. And let $\mu, \nu : p^* \Rightarrow q^*$ be two natural transformation between the points.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, phantom, "\eta\Downarrow \quad \Downarrow\nu"] & \mathscr{E} \arrow[l, bend right, "p^*"'] \arrow[l, bend left, "q^*"] \arrow[r, hook, bend right, "f_*"'] & \mathscr{G} \arrow[l, bend right, "f^*"'] \end{tikzcd}$$

We need to prove that if $\mu_{i^*} = \nu_{i^*}$, then $\mu = \nu$, this is in fact the concrete rephrasing of the faithfulness of $\mathsf{pt}f$. In order to do so, let e be an object in $\mathscr{E}$. Since f is a localic morphism, there is an object $g \in \mathscr{G}$ and an epimorphism $l : f^*(g) \twoheadrightarrow e$[1].

[1]This is not quite true, we know that e is a subquotient of $f^*(g)$, in the general case the proof gets a bit messier to follow, for this reason we will cover in detail just this case.

$$\begin{tikzcd}
p^*(e) \arrow[d, Rightarrow, bend right, "\mu"'] \arrow[d, Rightarrow, bend left, "\nu"] & p^*i^*(g) \arrow[l, "p^*(l)"] \arrow[d, Rightarrow, bend right, "\nu_{i^*g}"'] \arrow[d, Rightarrow, bend left, "\mu_{i^*g}"] \\
q^*(e) & q^*i^*(g) \arrow[l, "q^*(l)"]
\end{tikzcd}$$

Now, we know that $\mu \circ p^*(l) = q^*(l) \circ \mu_{i^*g}$ and $\nu \circ p^*(l) = q^*(l) \circ \nu_{i^*g}$, because of the naturality of μ and ν. Since $\mu_{i^*} = \nu_{i^*}$, we get

$$\mu \circ p^*(l) = q^*(l) \circ \mu_{i^*g} = q^*(l) \circ \nu_{i^*g} = \nu \circ p^*(l).$$

Finally observe that $p^*(l)$ is an epi, because p^* preserves epis, and thus we can cancel it, obtaining the thesis.

□

1.1.3. *Geometric surjections.*

REMARK 6.5. The behaviour of pt on geometric surjection might be very wild, to convince the reader of this terrible fact, let $\mathcal{T}$ be a topos without points and $\pi_{\mathcal{E}} : \mathcal{E} \times \mathcal{T} \twoheadrightarrow \mathcal{E}$ be natural projection, we claim that $\mathsf{pt}\pi_{\mathcal{E}}$ is highly non surjective on points, and that's evident because $\mathcal{E} \times \mathcal{T}$ does not have points.

PROPOSITION 6.6. Let $f : \mathcal{G} \to \mathcal{E}$ be a geometric morphism, then the following are equivalent.

- For every point $j : \mathbf{Set} \to \mathcal{E}$ the pullback $\mathcal{G} \times_{\mathcal{E}} \mathbf{Set}$ has a point.
- $\mathsf{pt}(f)$ is surjective on objects.

PROOF. Trivial. □

1.2. On the behaviour of S. The functor S behaves nicely with epis, as expected. It does not behave nicely with any notion of monomorphism. In the next section we study those accessible functors f such that $\mathsf{S}(f)$ is a geometric embedding.

PROPOSITION 6.7. S maps pseudo-epis (of Cat) to geometric surjections.

PROOF. Look [ABS01][4.2]. □

1.3. Topological embbeddings.

DEFINITION 6.8. Let $f : \mathcal{A} \to \mathcal{B}$ be a 1-cell in Acc_ω. We say that f is a topological embedding if $\mathsf{S}(f)$ is a geometric embedding.

This subsection is devoted to describe topological embeddings between accessible categories with directed colimits. The reader should expect this description to be highly untrivial and rather technical, because S is a left adjoint and is not expected to have a nice behaviour on any kind of monomorphism.
Fortunately we will manage to provide some useful partial results. Let us list the lemmas that we are going to prove.

1.3.1 a necessary condition for a functor to admit a topological embedding into a finitely accessible category
1.3.2 a full description of topological embeddings into finitely accessible categories
1.3.3 a sufficient and quite easy to check criterion for a functor to be a topological embedding

Remark 6.9. Topological embeddings into finitely accessible categories $i : \mathscr{A} \to \mathsf{Ind}(C)$ are very important because $\mathsf{S}(i)$ will describe, by definition, a subtopos of $\mathbf{Set}^C$. This means that there exist a topology J on C such that $\mathsf{S}(\mathscr{A})$ is equivalent to $\mathsf{Sh}(C, J)$, this leads to concrete presentations of the Scott topos.

1.3.1. *A necessary condition.*

Lemma 6.10 (A necessary condition). If $\mathscr{A}$ has a fully faithful topological embedding $f : \mathscr{A} \to \mathsf{Ind}(C)$ into a finitely accessible category, then η_A is fully faithful.

Proof. Assume that If $\mathscr{A}$ has a topological embedding $f : \mathscr{A} \to \mathsf{Ind}(C)$ into a finitely accessible category. This means that $\mathsf{S}(f)$ is a geometric embedding. Now, we look at the following diagram.

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "f"] \arrow[d, "\eta_{\mathscr{A}}"] & & \mathsf{Ind}(C) \arrow[d, hook, two heads, "\eta_{\mathsf{Ind}(C)}"] \\
\mathsf{ptS}(\mathscr{A}) \arrow[rr, "\mathsf{ptS}f"] & & \mathsf{ptS}(\mathsf{Ind}(C))
\end{tikzcd}$$

2.12 implies that $\eta_{\mathsf{Ind}(C)}$ is an equivalence of categories, while 6.2 implies that $\mathsf{ptS}(f)$ is fully faithful. Since also f is fully faithfull, $\eta_{\mathscr{A}}$ is forced to be fully faithful. □

1.3.2. *A sufficient condition.* The main result of this section is 6.14, even if it is a quite technical statement. In oder to make it sound as natural as possible we guide the reader through an approximation of it.

Theorem 6.11. Let $i : \mathscr{A} \to \mathscr{B}$ be a 1-cell in Acc_ω exhibiting $\mathscr{A}$ as a reflective subcategory of $\mathscr{B}$

$$L : \mathscr{B} \leftrightarrows \mathscr{A} : i.$$

Then i is a topological embedding.

Proof. We want to show that $\mathsf{S}(i)$ is a geometric embedding. This is equivalent to show that the counit $i^* i_*(-) \Rightarrow (-)$ is an isomorphism. Going back to 2.7, we write down the obvious computations,

$$i^* i_*(-) \cong (i^* \circ r_{\mathscr{B}} \circ \mathsf{ran}_i \circ \iota_{\mathscr{A}})(-).$$

Now, observe that since i has a left adjiont L the operator ran_i just coincides with $(-) \circ L$, thus we can elaborate the previous equation as follows.

$$(i^* \circ r_{\mathscr{B}} \circ \mathsf{ran}_i \circ \iota_{\mathscr{A}})(-) \cong (i^* \circ r_{\mathscr{B}})(- \circ L),$$

Now, $(- \circ L)$ will preserve directed colimits because is the composition of a cocontinuous functor with a functor preserving directed colimits. This means that it is a fixed point of $r_{\mathcal{B}}$.

$$(i^* \circ r_{\mathcal{B}})((-) \circ L) \cong i^*((-) \circ L) \cong (-) \circ L \circ i \cong (-).$$

The latter isomorphism is just the definition of reflective subcategory. This concludes the proof. $\square$

DEFINITION 6.12. A full subcategory $i : \mathcal{A} \to \mathcal{B}$ is algebraically weakly reflective if there exists a functor $L : \mathcal{B} \to \mathcal{A}$ and a natural split surjection ϕ,

$$\phi : \mathcal{A}(L-,-) \leftrightarrows \mathcal{B}(-,i-) : \psi.$$

We call L an algebraic weak left adjoint of i. If the induced natural transformation $Li \Rightarrow 1$ is an isomorphism we say that (L, i) is a normal algebraic weak reflection.

REMARK 6.13. It's hard to say where the notion of weak left adjoint has appeared for the first time. Among the relevant papers we mention Kainen [Kai71]. Indeed the topic is related to multireflective subcategories and [AR94][Chap. 4] should be mentioned too. In [LR12] the theory of weak left adjoints is treated in a very satisfying categorical fashion. To our knowledge, the notion that we have introduced above has never appeared in the literature, but possibly fits in the framework of [LR12] when the class $\mathcal{E}$ in the paper is chosen to be split epimorphisms.

THEOREM 6.14. Let $i : \mathcal{A} \to \mathcal{B}$ be a 1-cell in Acc_ω exhibiting $\mathcal{A}$ as a normal algebraically weakly reflective subcategory of $\mathcal{B}$. Assume that the algebraic weak left adjoint L preserve directed colimits, then i is a topological embedding.

PROOF. Following the lines of the previous proof, it is enough to show that the operator $\mathsf{ran}_i(-)$ coincides with $(-) \circ L$. In order to do so, we write the limit-fomula of the right Kan extension,

$$\mathsf{ran}_i(p)(-) \cong \lim_{d \to i-} p(d).$$

Using the algebraic weak reflection we can manipulate the latter equation into the following

$$\mathsf{ran}_i(p)(-) \cong \lim_{Ld \to -} p(Ld) \cong pL(-),$$

which is the thesis. $\square$

1.3.3. *Into finitely accessible categories.*

THEOREM 6.15. $f : \mathcal{A} \to \mathsf{Ind}(C)$ is a topological embedding into a finitely accessible category if and only if, for all $p : \mathcal{A} \to \mathbf{Set}$ the following equation holds,

$$\mathsf{lan}_i(\mathsf{ran}_f(p) \circ i) \circ f \cong p.$$

PROOF. The result follows from the discussion below. $\square$

Remark 6.16 (f_* and finitely accessible categories). Given a 1-cell $f : \mathscr{A} \to \mathscr{B}$ in Acc_ω, we experinced that it can be quite painful to give an explicit formula for the direct image functor f_*. In this remark we improve the formula provided in 2.7 in the special case that the codomain if finitely accessible. In order to do so we study the diagram of 2.7. To settle the notation, call $f : \mathscr{A} \to \mathsf{Ind}(C)$ our object of study and $i : C \to \mathsf{Ind}(C)$ the obvious inclusion.

$$\begin{tikzcd}
\mathsf{S}\mathscr{A} \arrow[d, "\iota_{\mathscr{A}}"] \arrow[rr, bend right, "f_*"] & & \mathsf{SInd}(C) \arrow[ll, "f^*"'] \arrow[d, "\iota_{\mathsf{Ind}(C)}"] \\
{[\mathscr{A}, \mathbf{Set}]} \arrow[rr, bend left, "\mathsf{lan}_f"] \arrow[rr, bend right, "\mathsf{ran}_f"'] & & {[\mathsf{Ind}(C), \mathbf{Set}]} \arrow[ll, "f^*"]
\end{tikzcd}$$

We are use to this diagram from 2.7, where we learnt also the following formula

$$f_* \cong r_{\mathscr{B}} \circ \mathsf{ran}_f \circ \iota_{\mathscr{A}}.$$

We now use the following diagram to give a better description of the previous equation.

$$\begin{tikzcd}
\mathsf{S}\mathscr{A} \arrow[d, "\iota_{\mathscr{A}}"] \arrow[rr, bend right, "f_*"] & & \mathsf{SInd}(C) \arrow[ll, "f^*"'] \arrow[d, "\iota_{\mathsf{Ind}(C)}"] \arrow[rr, bend right, tail, "i^*"] & & \mathbf{Set}^C \arrow[ll, tail, two heads, "\mathsf{lan}_i"'] \arrow[dll, "\mathsf{lan}_i"] \\
{[\mathscr{A}, \mathbf{Set}]} \arrow[rr, bend left, "\mathsf{lan}_f"] \arrow[rr, bend right, "\mathsf{ran}_f"'] & & {[\mathsf{Ind}(C), \mathbf{Set}]} \arrow[ll, "f^*"] \arrow[urr, bend right, "i^*"'] & &
\end{tikzcd}$$

We claim that in the notations of the diagram above, we can describe the direct image f_* by the following formula,

$$f_* \cong \mathsf{lan}_i \circ i^* \circ \mathsf{ran}_f \circ \iota_{\mathscr{A}},$$

this follows from the observation that $r_{\mathsf{Ind}(C)}$ coincides with $\mathsf{lan}_i \circ i^*$ in the diagram about.

2. A study of the unit $\eta_{\mathscr{A}}$

This section is devoted to a focus on the unit of the Scott adjunction. We will show that good properties of $\eta_{\mathscr{A}}$ are related to the existence of *finitely accessible representation* of $\mathscr{A}$. A weaker version of the following proposition appeared in [Hen19][2.6]. Here we give a different proof, that we find more elegant and provide a stronger statement.

Proposition 6.17. The following are equivalent:

(1) The unit of the adjunction $\mathscr{A} \to \mathsf{ptS}\mathscr{A}$ is faithful (and iso-full);
(2) $\mathscr{A}$ admits a faithful (and iso-full) functor $f : \mathscr{A} \to \mathsf{Ind}(C)$ preserving directed colimits;

PROOF. 1) $\Rightarrow$ 2) Assume that $\eta_{\mathcal{A}}$ is faithful. Recall that any topos admis a geometric embedding in a presheaf category, this is true in particular for $\mathsf{S}(\mathcal{A})$. Let us call ι this geoemtric embedding $\iota : \mathsf{S}(\mathcal{A}) \to \mathbf{Set}^X$. Following 6.2 and 2.12, $\mathsf{pt}(\iota)$ is a fully faithful functor into a finitely accessible category $\mathsf{pt}(\iota) : \mathsf{S}(\mathcal{A}) \to \mathsf{Ind}(C)$. Thus the composition $\mathsf{pt}(\iota) \circ \eta_A$ is a faithful functor into a finitely accessible category

$$\mathcal{A} \to \mathsf{ptS}\mathcal{A} \to \mathsf{Ind}(C).$$

Oberseve that if $\eta_{\mathcal{A}}$ is isofull, so is the composition $\mathsf{pt}(\iota) \circ \eta_A$.

2) $\Rightarrow$ 1) Assume that $\mathcal{A}$ admits a faithful functor $f : \mathcal{A} \to \mathsf{Ind}(C)$ preserving directed colimits. Now we apply the monad ptS obtaining the following diagram.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d, "\eta_{\mathcal{A}}"] & \mathsf{Ind}(C) \arrow[d, hook, two heads, "\eta_{\mathsf{Ind}(C)}"] \\
\mathsf{ptS}(\mathcal{A}) \arrow[r, "\mathsf{ptS}f"] & \mathsf{ptS}(\mathsf{Ind}(C))
\end{tikzcd}$$

2.12 implies that $\eta_{\mathsf{Ind}(C)}$ is an equivalence of categories, thus $\mathsf{ptS}(f) \circ \eta_{\mathcal{A}}$ is (essentially) a factorication of f. In particular, if f is faithful, so has to be $\eta_{\mathcal{A}}$. Moverover, if f is isoful and faithful, so must be $\eta_{\mathcal{A}}$, because this characterizes pseudomonomorphisms in Cat (and by direct verification also in Acc_ω). $\square$

REMARK 6.18. If we remove iso-fullness from the statement we can reduce the range of f from any finitely accessible category to the category of sets.

PROPOSITION 6.19. The following are equivalent:

(1) $\mathcal{A}$ admits a faithful functor $f : \mathcal{A} \to \mathbf{Set}$ preserving directed colimits.
(2) $\mathcal{A}$ admits a faithful functor $f : \mathcal{A} \to \mathsf{Ind}(C)$ preserving directed colimits;

PROOF. The proof is very simple. 1) $\Rightarrow$ 2) is completely evident. In order to prove 2) $\Rightarrow$ 1), oberseve that since $\mathsf{Ind}(C)$ is finitely accessible, there is a faithful functor $\mathcal{Y} : \mathsf{Ind}(C) \to \mathbf{Set}$ preserving directed colimits given by

$$\mathcal{Y} := \coprod_{p \in C} \mathsf{Ind}(C)(p, -).$$

The composition $g := \mathcal{Y} \circ f$ is the desired functor into $\mathbf{Set}$. $\square$

CHAPTER 7

Final remarks and Open Problems

Remark 7.1 (The initial project). The original intention of this thesis was to exploit the Scott adjunction to obtain a better understanding of categorical model theory, that is the abstract study of accessible categories with directed colimits from a logical perspective. We believe to have partially contributed to this general plan, yet some observations must be made. Indeed, after some while, we came to the conclusion that a lot of foundational issue about our approach needed to be discussed in order to understand what kind of tool we were developing. For example, in the early days of our work, we naively confused the Scott adjunction with the categorified Isbell duality.

Remark 7.2. As a result, we mainly devoted our project to the creation a very foundational framework, where it would have been possible to organize categorical model theory. In particular, this means that a big part of our original plan is still to be pursued, and of course we tried to do so. Unfortunately, a family of improvement of technical results is needed in order to approach sharp and sophisticated problems like *Shelah categoricity conjecture*.

Remark 7.3 (The role of geometry). Also, moving to a more foundational framework has raised the importance of the geometric intuition on our work. The interplay between the geometric and logic aspects of our quest has proven to be inavoidable. Ionads, for example, offer a perfect ground to study semantics with a syntax-free approach, and yet the intuition that we had studying them was completely geometric.

Remark 7.4. For theses reasons, we think that it is time to disclose our work to the community and offer to everybody the chance of improving our results. This chapter is dedicated to list some of the possible further directions of our work. Obviously, some open problems are just curiosities that haven't find and answers, some others instead are quite relevant and rather technical open problems that would lead to a better understanding of categorical model theory if properly solved.

1. Geometry

1.1. Connected topoi.

Definition 7.5. A topos $\mathcal{E}$ is connected if the inverse image of the terminal geometric morphism $\mathcal{E} \to \mathbf{Set}$ is fully faithful.

Remark 7.6. [Joh02b][C1.5.7] accounts on the general and relevant properties of connected topoi and connected geometric morphisms. For the sake of this subsection, we can think of a connected topos as the locale of opens of a connected topological space. Indeed, the definition can be reduced to this intuition.

QUESTION 7.7.
(1) What kind of Scott topoi are connected?
(2) What kind of Isbell topoi are connected?

In the direction of the first question, we can offer a first approximation of the result.

THEOREM 7.8. If $\mathscr{A}$ is connected then its Scott topos $\mathsf{S}(\mathscr{A})$ is connected.

PROOF. The terminal map $t : \mathsf{S}(\mathscr{A}) \to \mathbf{Set}$ appears at the $\mathsf{S}(\tau)$, where τ is the terminal map $\tau : \mathscr{A} \to \cdot$. When $\mathscr{A}$ is connected τ is a lax-epi, and f^* is fully faithful, [ABS01]. □

COROLLARY 7.9 (The JEP implies connectedness of the Scott topos). Let $\mathscr{A}$ be an accessible category where every map is a monomorphism. If $\mathscr{A}$ has the joint embedding property, then $\mathsf{S}(\mathscr{A})$ is connected.

PROOF. Obviously if $\mathscr{A}$ has the JEP, it is connected. □

2. Logic

2.1. Categoricity spectra and Shelah's conjecture. We have already mentioned in Chap. 4 that we believe it is possible to combine our technology to approach Shelah categoricity conjecture. Let us recall the generic shape of the conjecture.

CONJECTURE 7.10. Let $\mathscr{A}$ be a nice accessible category with directed colimits having just a model of presentabily rank κ, then $\mathscr{A}$ must have at most one model of presentability rank λ for every regular cardinal $\lambda \geq \kappa$.

REMARK 7.11. We should mention that several approximations of this results have appeared in the literature yet. Unfortunately, none of these proofs is deeply categorical, or appears natural to us. Our initial aim was not only to proof Shelah's categoricity conjecture, but also to accommodate it in a framework in which the statement and the proof could look natural.

REMARK 7.12 (A possible strategy). Given an accessible category $\mathscr{A}$, a possible strategy to prove the conjecture could involve the full subcategory $\mathscr{A}_{\geq\lambda}$ of objects of cardinality λ. In fact, if the inclusion $\mathscr{A}_{\geq\lambda} \to \mathscr{A}$ is a topological embedding (for some γ-Scott adjunction such that $\mathscr{A}$ is γ-accessible) in the hypotheses of Thm 4.53, then $\mathscr{A}_{\geq\lambda}$ must be categorical in some presentability rank.

REMARK 7.13. The previous remark shows one of the most interesting open problems of the thesis, that is to provide a refinement of Thm. 6.14 in the case in which the inclusion $i : \mathscr{A} \to \mathscr{B}$ is not weakly algebraically reflective.

2.2. Cosimplicial sets and Indiscernibles. Indiscerbles are a very classical tool in classical model theory and were designed by Morley while working on what now is knows under the name of Morley's categoricity theorem. Makkai rephrased this result in [MP89] in the following way.

THEOREM 7.14. Let $\mathsf{pt}(\mathscr{E})$ be a large category of points of some topos $\mathscr{E}$. Then there is a faithful functor $\mathsf{U} : \text{Lin} \to \mathsf{pt}(\mathscr{E})$, where Lin is the category of linear orders and monotone maps.

A categorical *understanding* of this result was firstly attempted in an unpublished work by Beke and Rosický. We wish we could fit this topic in our framework. The starting point of this process is the following observation.

THEOREM 7.15. $\mathsf{S}(\text{Lin})$ is $\mathbf{Set}^{\Delta}$.

PROOF. There is not much to prove, Lin is finitely accessible and Δ coincides with its full subcategory of finitely presentable objects, thus the theorem is a consequence of Rem. 2.12. □

REMARK 7.16. By the Scott adjunction, any functor of the form $\text{Lin} \to \mathsf{pt}(\mathscr{E})$ corresponds to a geometric morphism $\mathbf{Set}^{\Delta} \to \mathscr{E}$.

QUESTION 7.17.

(1) Is it any easier to show the existence of U by providing a geometric morphism $\mathbf{Set}^{\Delta} \to \mathscr{E}$?
(2) Is it true that every such U is of the form $\mathsf{pt}(f)$ for a localic geometric morphism $f : \mathbf{Set}^{\Delta} \to \mathscr{E}$.

3. Category Theory

3.1. The $\mathscr{E}$-Scott adjunction. It might be possible to develop $\mathscr{E}$-relative version of the Scott adjunction, where $\mathscr{E}$ is a grothendieck topos. The relevant technical setting should be containted in [BQ96] and [BQR98].

CONJECTURE 7.18. There is a 2-adjunction

$$\mathsf{S}_{\mathscr{E}} : \mathscr{E}\text{-Acc}_{\omega} \leftrightarrows \text{Topoi}_{/\mathscr{E}} : \mathsf{pt}_{\mathscr{E}}$$

Moreover, if $\mathscr{E}$ is a κ-topos, the relative version of λ-Scott adjunction holds for every $\omega \leq \lambda \leq \kappa$.

REMARK 7.19. Let us clarify what is understood and what is to be understood of the previous conjecture.

(1) By $\text{Topoi}_{/\mathscr{E}}$ we mean the slice category over $\mathscr{E}$. Recall that Topoi coincides with $\text{Topoi}_{/\mathbf{Set}}$.
(2) By $\mathsf{pt}_{\mathscr{E}}$ we mean the relative points functor $\text{Topoi}_{/\mathscr{E}}(\mathscr{E}, -)$.
(3) It is a bit unclear how $\text{Topoi}_{/\mathscr{E}}(\mathscr{E}, -)$ showld be enriched over $\mathscr{E}$, maybe it is fibered over it?

4. Sparse questions

QUESTION 7.20.

(1) If f is faithful, is$\mathsf{S}(f)$ is localic?
(2) Is always η a topological embedding?

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