INTRODUCTION TO CATEGORICAL LOGIC

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rules

- Hand your exercises by the midnight of the **7th of May** via email. Make my life easier: include the word **CL23 in the subject**.
- Pick at least one exercise from each of the yellow groups.
- At each stage of the exercise sheet, you can give for granted the statements of all the exercises that come before the one you are solving.
- You must charge at least **3** batteries!
  Example. The vector of exercises [1,5,8,12,15,18] would pass this sheet.

Exercises

doctrines and types

Exercise 1 (Naming, **).** Consider the category of sets and the usual powerset doctrine defined over it, $\mathcal{P} : \mathsf{Set}^\circ \to \mathsf{InfLat}$. Using the usual *epi-mono* factorization, we can define a functor $[-]_X : \mathsf{Set}_{/X} \to \mathcal{P}(X)$. Inspired by this construction, for $\mathcal{P} : \mathsf{C}^\circ \to \mathsf{InfLat}$ a doctrine with a sufficient amount of structure[a] construct a (pseudo)natural transformation

$$[-]_{(=)} : \mathsf{C}_{/(=)} \Rightarrow \mathcal{P}(=).$$

Exercise 2 (...and necessity, **).** We say that a doctrine has *comprehension schema* if the *naming* functor $[-]_{(=)} : \mathsf{C}_{/A} \Rightarrow \mathcal{P}(A)$ of the exercise above has a right adjoint $\{A : -\}$ for all A. Prove that if $\mathcal{P} : \mathsf{C}^\circ \to \mathsf{InfLat}$ is a doctrine with a sufficient amount of structure, the canonical subobject doctrine defined on its category of elements has comprehension schema,

$$\mathsf{Sub} : \mathsf{Elts}(\mathcal{P})^\circ \to \mathsf{InfLat}.$$

Exercise 3 (, **).** Provide a type theoretic interpretation of a doctrine with comprehension schema.

Exercise 4 (**).** Provide a translation between the notion of comprehension category and that of category with display maps.

[a] It is enough that C has finite limits and $\mathcal{P}f$ has a left adjoint.

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topoi as spaces

Exercise 5 (🔋). Show that the category of sheaves over the Sierpinski space is a presheaf topos. Which one?

Exercise 6 (🔋). Show that $\mathbf{Set}^{\rightarrow}$ has a closed subtopos and an open subtopos. Please, provide a full proof that the geometric morphisms you present have the property we require, don't just state it.

Exercise 7 (🔋). Let X be a compact Hausdorff space. Show that the direct image of the terminal geometric morphism $\Gamma_* : \mathsf{Sh}(X) \to \mathsf{Set}$ preserve directed colimits of monomorphisms.

topoi as objects

Exercise 8 (🔋, ☕). Show that the bicategory of topoi has (pseudo)colimits. *Hint.* Yes, this exercise is too hard.

Exercise 9 (🔋, ☕). Show that the bicategory of topoi has (pseudo)pullbacks. *Hint.* Yes, this exercise is too hard.

Exercise 10 (🔋). Show that open geometric morphisms are pullback stable.

Exercise 11 (🔋). Show that closed geometric morphisms are pullback stable.

topoi as objects

Exercise 12 (🔋, ☕). Show that there is a bijective correspondence between

$$\mathsf{Sub}_{\mathcal{E}}(1) \simeq \mathsf{Topoi}(\mathcal{E}, \mathbf{Set}^{\rightarrow}).$$

Exercise 13 (🔋). Show that the bicategory of topoi has a classifier of closed embeddings, i.e., there exists a closed embedding $p : \mathcal{F}_1 \to \mathcal{F}_2$ such that every closed subtopos can be obtained by pulling back a geometric morphism along p.

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \mathcal{F}_1 \arrow[d, "p"] \\
\mathcal{E} \arrow[r] & \mathcal{F}_2
\end{tikzcd}$$

Prove an anologous statement also for open embeddings. *Hint.* To get the proper intuition, first solve it for spaces, then for locales, and then for topoi.

Exercise 14 (🔋). Show that every open subtopos is *complemented*, i.e. there exists a closed subtopos that is its complement in the lattice of subtopoi.

topoi as sets

Exercise 15 (). Provide a complete description of the subobject classifier in $\mathbf{Set}^{\mathbb{N}}$, where the category structure of $\mathbb{N}$ is the expected posetal one.

Exercise 16 (). Show that every topos has a partial map classifier for every object. *Hint:* What are the partial map classifiers in Set?

Exercise 17 (). Prove that an object of a topos $\mathcal{E}$ is injective (with respect to monos) if and only if it is a retract of Ω^x for some x. Deduce that if e is injective then the functor $[-, e] : \mathcal{E}^\circ \to \mathcal{E}$ preserves reflexive coequalizers.

topoi as theories

Exercise 18 (). Consider the category of non empty finite sets $\mathsf{Fin}_{>0}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{>0}}$ classify?

Exercise 19 (). Consider the category of finite sets and monomorphisms $\mathsf{Fin}_{\hookrightarrow}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{\hookrightarrow}}$ classify?

Exercise 20 (). Consider the category of finite sets and epimorphisms $\mathsf{Fin}_{\twoheadrightarrow}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{\twoheadrightarrow}}$ classify?

Exercise 21 (). Consider the category of pointed finite sets $\mathsf{Fin}_{\bullet}$. What theory does $\mathbf{Set}^{\mathsf{Fin}_{\bullet}}$ classify?

Exercise 22 (). Consider the comma topos below, and assume comma topoi exist in the bicategory of topoi. Can you describe how does a $\mathbf{Set}$-model of the comma topos look like (in terms of models of $\mathbb{T}_1$ and $\mathbb{T}_2$)?

$$
\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \mathbf{Set}[\mathbb{T}_1] \arrow[d] \\
\mathbf{Set}[\mathbb{T}_2] \arrow[r] \arrow[ur, Rightarrow, dashed, "\lambda"] & \mathbf{Set}
\end{tikzcd}
$$

Feel free to assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are single sorted if you wish.

The riddle (⚠). Show that a presheaf topos $\mathsf{Set}^{\mathsf{C}}$ is boolean if and only if C is a groupoid.