Topoi with enough points

Ivan Di Liberti

Logic Colloquium 24
June 2024, Gothenburg.

This talk is based on a joint work with *Morgan Rogers*.

- **Topoi with enough points**, ArXiv:2403.15338.

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Motto:propositional first order theories $\equiv$ Boolean algebras

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The canonical example of frame is a topology. When (X, τ) is a topological space, τ is a frame.

$$\mathrm{O} : \mathrm{Top} \to \mathrm{Frm}^{\mathrm{op}}$$

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When a frame has enough points, then $\mathsf{Opt}(\mathcal{L}) \cong \mathcal{L}$.

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