CATEGORY THEORY 2024

IVAN DI LIBERTI

Exercises

savoir faire: Yoneda, adjunctions and limits

Leinster 1 (). 6.2.20

Leinster 2 (). 6.2.21

Leinster 3 (). 6.3.21(a)

Leinster 4 (). 6.3.22

Leinster 5 (). 6.3.26

Leinster 6 (). 6.3.27

monads

Exercise 7 (). Describe the monads (unit and counit) on Set whose algebras are: monoids, groups, semigroups.

Exercise 8 (). Consider the free-forgetful adjunction $D : \mathbf{Set} \leftrightarrows \mathrm{Top} : \mathrm{U}$, where D equips a set with the discrete topology over it. Compute the algebras for the induced monad over $\mathbf{Set}$.

Exercise 9 (). Show that the category Suplat whose objects are suplattices and morphisms are suplattices morphisms is monadic over $\mathbf{Set}$ via forgetful functor $\mathbb{U} : \mathsf{Suplat} \to \mathsf{Set}$. *Hint*: Guess the monad and prove that an algebra is precisely a suplattice.

Exercise 10 (). A monad T on a category C is idempotent if its multiplication if its multiplication is an isomorphism. Show that the fortgetful functor $U_T : \mathrm{Alg}(T) \to C$ of an idempotent monad is fully faithful.

Exercise 11 (). Let C be a category with coproducts and a terminal object. Can you always put a monad structure on the *maybe endofunctor* $c \mapsto c \coprod 1$?

Exercise 12 (). Show that the category of fields is not monadic over Set.

Exercise 13 (). Show that there is a monad on directed graphs whose algebras are small categories.

Exercise 14 (). Show that there is a monad on the category of small categories (and functors) whose algebras are posets.

Date: March 2, 2024.

Kan extensions

Riehl 15 (Kan extensions have a universal property)**.** Read section 6.1, where a Kan extensions are introduced in a more abstract way and study Thm 6.2.1 which proves that our concrete formula is explicitely computing the Kan extension, when possible.

Riehl 16 (Concepts are Kan extensions)**.** Read section 6.5, where it is shown that many categorical concepts can be phrased in terms of existence of Kan extensions.

Exercise 17 (🔋)**.** Prove[a], when all the functors in the equations are well-defined, that

$$\mathrm{lan}_{fg}(h) \cong \mathrm{lan}_f(\mathrm{lan}_g h).$$

Exercise 18 (🔋)**.** Try to show that if f has a right adjoint g, then

$$\mathrm{lan}_f(1) \cong g.$$

Exercise 19 (🔋)**.** Prove, using our definition, that when g is fully faithful, then $(\mathrm{lan}_g f)\circ g \cong f$.

[a] *Hint*: use that Kan extensions provide left adjoints to precomposition.

Jiří's *treat* (⚠, ☕)**.** Let Suplat^∇ be the category of whose objects are suplattices with a unary operation ∇ satisfying $(\forall x)(x \leq \nabla x)$. Morphisms are suplattices morphisms preserving the unary operation. Show that the forgetful functor

$$\mathbb{U} : \mathsf{Suplat}^\nabla \to \mathsf{Set}$$

preserves limits but does not have a left adjoint. *Hint*: Show that a free algebra over 1 does not exist.

rules

- Hand your exercises before your **oral interview** via email. In order to make my life easier, make sure to include the word **CT24 in the subject**.
- Pick at least one exercise from each of the yellow groups.
- You must charge at least **2** batteries and a half!
  Example. The vector of exercises [2,7,12,19] would pass this sheet.
- The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
- The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.