INTRODUCTION TO CATEGORICAL LOGIC

IVAN DI LIBERTI

rules

- Hand your exercises by the **22nd of March** via email. In order to make my life easier, make sure to include the word **CL23 in the subject**.
- Pick at least one exercise from each of the yellow groups.
- You must charge at least **2** full batteries!
  Example. The vector of exercises [3,7,8,13,16] would pass this sheet.

Exercises

categories

Exercise 1 (🔋). How many idempotent monads $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ can you find on the category of sets? Describe them all.

Exercise 2 (🔋). Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a cocontinous monad. Prove or provide a counterexample for the following statement: there exists a monoid M such that $\mathsf{T} \cong \mathsf{M} \times (-)$ as monads.

Exercise 3 (🔋). A *graph* (E, V, s, t) is the data of two sets E, V and two functions $s, t : E \rightrightarrows V$. Morphisms of graphs are defined as expected, and so is the category Gra of graphs. Can you find a full subcategory C containing two objects such that every cocontinuous functor $\mathsf{Gra} \to \mathbf{Set}$ is uniquely determined by its value on C?

Exercise 4 (🔋). Let $\mathcal{A}$ be a category and $a \in \mathcal{A}$ be a dense object, i.e. the family consisting of the single object a forms a dense generating set. Show that $\mathcal{A}$ admits a faithful right adjoint $\mathcal{A} \to \mathbf{Set}$ and exhibit a category $\mathcal{A}$ for which it is not an equivalence of categories.

Exercise 5 (🔋, ☕). Let $\mathcal{A}$ be a cocomplete category with a dense generating set. Show that $\mathcal{A}$ is complete.

Exercise 6 (🔋). In the diagram below all the categories are λ-accessible and so are the functors f, g. Assume also that C is cocomplete. Justify that $\mathsf{lan}_f g$ exists and is accessible too.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{C} \\ \mathcal{B} \arrow[ur, dashed, "\mathsf{lan}_f g"'] & \end{tikzcd}$$

Date: January 19, 2023.

universal algebra

Exercise 7 (🔋, ☕). Consider the categories Grp, Ab of groups and abelian group respectively.

- Describe the Lawvere theories axiomatizing them.
- Show that the inclusion $i : \mathsf{Ab} \hookrightarrow \mathsf{Grp}$ is a morphism of varieties.
- Describe the morphism of Lawvere theories that induce i.

Exercise 8 (🔋, ☕). Consider the categories Grp, Ab of groups and abelian group respectively.

- Describe the **Set** monads axiomatixing them.
- Show that the inclusion $i : \mathsf{Ab} \hookrightarrow \mathsf{Grp}$ is a morphism of varieties.
- Describe the morphism of monads that induce i.

Exercise 9 (🔋, ☕). Recall that the category SLat of suplattices is monadic over $\mathbf{Set}$. Following the standard construction that given a monad produces a (possibly large) algebraic theory, can you describe an equational presentation of the category of suplattices?

Exercise 10 (🔋). Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a finitary monad with some model with two distinct elements, show that its unit is injective.

Exercise 11 (🔋). For every finitary monad $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ construct a finitary polynomial monad $\mathsf{P}_\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ and a morphism of monads $\mathsf{P}_\mathsf{T} \to \mathsf{T}$ which is pointwise surjective.

sketches

Exercise 12 (🔋). Using the technology of the course show that every abelian group embeds in a divisible one.

Exercise 13 (🔋, ☕). Provide a sketch axiomatizing the category of fields. Could it be a limit sketch?

Exercise 14 (🔋, ☕). Given limit sketches $\mathcal{S}_1, \mathcal{S}_2$ define a symmetric tensor product $\mathcal{S}_1 \otimes \mathcal{S}_2$ in such a way that,

$$\mathsf{Mod}(\mathcal{S}_1 \otimes \mathcal{S}_2, \mathbf{Set}) \simeq \mathsf{Mod}(\mathcal{S}_1, \mathsf{Mod}(\mathcal{S}_2, \mathbf{Set})).$$

Exercise 15 (🔋). Show that the category of Banach spaces and non expansive maps is locally presentable. What about the category of Hilbert spaces?

Exercise 16 (🔋). Show that the category of topological spaces and the category of suplattices are not locally presentable.

Exercise 17 (🔋). Show that if $\mathcal{A}$ is locally finitely presentable, so are $\mathcal{A}^{\to}$ and $\mathcal{A}_{/a}$.

The riddle (Givant, ⚠). A finitary monad $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ is *stable* if every algebra is free. (Assuming choice) show that there exactly four families of finitary stable monads $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$. *Comment.* If you solve it with category theoretic methods, you can publish it.