The geometry of coherent topoi & ultrastructures

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This talk is based on a preprint that you can find on the ArXiv.

- **The geometry of coherent topoi & ultrastructures**, ArXiv:2211.03104.

Plan

1. Motivation: understanding ultraproducts
2. Translate the question into topos theory
3. Back to ultrastructures

First order logic is special in many ways.

- The category of models of an (essentially) algebraic theory is (co)complete.
- This allows for several constructions (free models).
- The category of models of a first order theory may not complete nor cocomplete.
- **Fld** does not have products.
- First order theories are harder to study than (essentially) algebraic ones.

Ultraproducts

Yet, given an X-indexed family of models of a first order theory $\mathbb{T}$

$$M_1, M_2, \dots, M_i \dots$$

there is a way to construct a new model. For U an ultrafilter on X, we can quotient the cartesian product *along* the ultrafilter

$$\Pi M_i/U.$$

This construction is functorial.

$$\int (-) dU : \mathsf{Mod}(\mathbb{T})^X \to \mathsf{Mod}(\mathbb{T}).$$

Ultraproducts are important

Once one acknowledges the existence of ultraproducts one can quickly show:

- Compactness of first order logic
- Completeness of first order logic

So the construction of ultraproducts can be accepted as a *defining* property of first order logic.

Original motivations (and a bit of drama)

Understand ultrastructures.

- Ultrastructures were introduced in *Stone duality for first-order logic* by Makkai in 1987.
- He wanted to capture the construction of ultraproducts.
- They are the main technology to prove the celebrated conceptual completeness.

Conceptual completeness

Let $f : \mathcal{F} \to \mathcal{G}$ be a morphisms of pretopoi. If the induced functor between categories of models is an equivalence of categories, then f is an equivalence too,

$$f^* : \mathsf{Mod}(\mathcal{G}) \to \mathsf{Mod}(\mathcal{F}).$$

Original motivations (and a bit of drama)

Understand ultrastructures.

- Ultrastructures a la Makkai are extremely complicated and technical.
- No news until 1995, Marmolejo's PhD thesis.
- No news until 2019, Lurie's ultracategories.
- Lurie's notion differs from Makkai's one!
- Who's right?!

Idea: let ultrastructures emerge as a necessary structure so that we can isolate the correct definition.

Recap

- Essentially algebraic $\rightsquigarrow$ any (co)limit of models.
- First order $\rightsquigarrow$ ultraproducts and directed colimits of models.
- Geometric $\rightsquigarrow$ directed colimits of models.

Can the existence of some colimits/construction be a *property* of the fragment of logic? How do we even ask this question? We would need an environment in which all these theories sit together in order to compare them...

Classifying topoi

These fragments of logic are classified by a different kinds of topos!

- Essentially algebraic $\rightsquigarrow$ Set^C with C lex.
- First order $\rightsquigarrow$ coherent topoi.
- Geometric $\rightsquigarrow$ topoi.

Remember that the category of points $pt(\mathcal{E})$ of the topos $\mathcal{E}$ are the same of the models of the theory $\mathcal{E}$ classifies

$$pt(\mathcal{E}_{\mathbb{T}}) \simeq Mod(\mathbb{T})$$

.

So, for example, for an essentially algebraic theory $\mathbb{T}$, $pt(\mathcal{E}_{\mathbb{T}})$ is complete and cocomplete.

So coherent topoi should be *special* among topoi?

Spoiler

Yes. But we start from something easier.

Colimits are Kan extensions

When we want to show that a category C had limits of shape D,we can try and prove that the right Kan extension below exists

$$\begin{tikzcd}
D \arrow[r, "\forall f"] \arrow[d] & C \\
1 \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Indeed this is the same of askind that the the diagonal functor $\Delta : C^1 \to C^D$ has a right adjoint.

(Weak Kan Injectivity)

In the recent paper **KZ monads and Kan Injectivity** by Sousa, Lobbia and DL this behaviour is called Weak Kan Injectivity.

Prop. Terminal geometric morphisms can test completeness

If a topos $\mathcal{E}$ is weakly Kan injective with respect to the terminal geometric morphism $\Gamma : Set^D \to Set$, then its category of points has limits of shape D.

$$
\begin{tikzcd}
Set^D \arrow[r, "\forall f"] \arrow[d, "\Gamma"'] & \mathcal{E} \\
Set \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}
$$

Indeed this is the same of askind that the the diagonal functor $pt(\mathcal{E}) = Topoi(Set, \mathcal{E}) \to Topoi(Set^D, \mathcal{E}) = pt(\mathcal{E})^D$ has a right adjoint.

Prop. Essentially algebraic theories are injective

The classifying topos Set^C of an essentially agebraic theory is weakly Kan injective with respect to any geometric morphism and Kan injective with respect to geometric embeddings.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\forall f"] \arrow[d, "x"'] & Set^C \\ \mathcal{G} \arrow[ur, dashed, "\exists h"] & \end{tikzcd}$$

Define $h^* = \mathrm{lan}_y(x_* f^* y)$. One can show that in this case $h_* = \mathrm{lan}_{x_*}(f_*)$.

We are convinced that Kan injectivity can isolate classes of topoi with special properties.

Next Steps

- Show that coherent topoi are Kan injective with respect to a special class of maps.
- Recover the ultrastructure from such property.

Definition

A geometric morphism $x : \mathcal{F} \to \mathcal{G}$ is flat if x_* preserve finite colimits.

Thm.

Coherent topoi are Kan injective with respect to flat embeddings.

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "f"] \arrow[d, hook, "i"'] & \mathcal{E} \\
\mathcal{L}'
\end{tikzcd}$$

Thm.

Coherent topoi are Kan injective with respect to flat embeddings.

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "f"] \arrow[d, hook, "i"'] & \mathcal{E} \arrow[d, hook, "j"] \\
\mathcal{L}' & Set^C
\end{tikzcd}$$

Embed $\mathcal{E}$ in a presheaf topos with a geometric embedding preserving directed colimits.

Thm.

Coherent topoi are Kan injective with respect to flat embeddings.

$$\begin{tikzcd} \mathcal{L} \arrow[r, "x"] \arrow[d, hook, "i"'] & \mathcal{E} \arrow[d, hook, "j"] \\ \mathcal{L}' \arrow[r, dashed, "h"'] & Set^C \end{tikzcd}$$

If we now show that $h_* \cong j_* j^* h_*$, we are done.

If we now show that $h_* \cong j_* j^* h_*$, we are done.

$$\begin{aligned} j_* j^* h_* &\cong j_* j^* \mathrm{lan}_{i_*}(j_* x_*) \\ &\cong j_* \mathrm{lan}_{i_*}(j^* j_* x_*) \\ &\cong j_* \mathrm{lan}_{i_*}(x_*) \\ (*) &\cong \mathrm{lan}_{i_*}(j_* x_*) \\ &\cong h_*. \end{aligned}$$

Why j_* preserves the Kan extension $\mathrm{lan}_{i_*}(x_*)$?

$$j_* \mathrm{lan}_{i_*}(x_*)(y) \cong j_*(\operatorname*{colim}_{i_*(d) \to y} x_*(d))$$

Because i_* preserve finite colimits, the diagram indexing the colimit is filtered, and thus is preserved by j_*.

So we have shown that coherent topoi are special. Now we should recover the ultrastructure from this property.

Rem.

Let X be a set and let $\beta(X)$ be its space of ultrafilters. Call $i : X \to \beta(X)$ the inclusion mapping each element to the principal ultafilter at that element. Then the induced geometric embedding is flat

$$Sh(X) \to Sh(\beta(X)).$$

Observe that because the topology on X is discrete, $Sh(X)$ is Set^X.

We will need a tautological factorization of the map i in the previous slide.

$$
\begin{tikzcd}
 & (X, disc) \arrow[dl, "j"'] \arrow[d, "i"] \\
(\beta(X), disc) \arrow[r, two heads, "q"'] & (\beta(X), \tau)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Set^X \arrow[dl, "j"'] \arrow[d, "i"] \\
Set^{\beta(X)} \arrow[r, two heads, "q"'] & \mathsf{Sh}(\beta(X))
\end{tikzcd}
$$

Now, consider a coherent topos and recall that we are Kan injective with respect to i.

$$\begin{tikzcd}
 & \mathrm{Set}^X \arrow[r, "f"] \arrow[dl, "j"'] \arrow[d, "i"] & \mathcal{E} \\
\mathrm{Set}^{\beta(X)} \arrow[r, "q"'] & \mathrm{Sh}(\beta(X)) \arrow[ur, dashed, "i_\sharp(f)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{pt}(\mathcal{E})^X \arrow[d, dash, "\simeq"'] & \mathrm{Topoi}(\mathrm{Sh}(\beta(X)), \mathcal{E}) \arrow[r, "q^\sharp"] & \mathrm{Topoi}(\mathrm{Set}^{\beta(X)}, \mathcal{E}) \arrow[d, dash, "\simeq"'] \\
\mathrm{Topoi}(\mathrm{Set}, \mathcal{E})^X \arrow[r, dash, "\simeq"'] & \mathrm{Topoi}(\mathrm{Set}^X, \mathcal{E}) \arrow[u, "i_\sharp"] & \mathrm{pt}(\mathcal{E})^{\beta(X)}
\end{tikzcd}$$

Altogether, and with a bit of abuse of notation that ignores the equivalence of categories, we obtain a functor

$$q_X^\sharp i_\sharp^X : \mathrm{pt}(\mathcal{E})^X \to \mathrm{pt}(\mathcal{E})^{\beta(X)}. \tag{1}$$

If we now transpose this functor, we obtain the pairing below, which we shall denote suggestively by an integral notation,

$$\int_X (-) d(-) : \mathrm{pt}(\mathcal{E})^X \times \beta(X) \to \mathrm{pt}(\mathcal{E}). \tag{2}$$

We have presented the main ideas in the first two sections of the paper. In the rest of the paper we further develop the properties of $\int_X (-) d(-)$ and axiomatise them in our notion of ultrastructure. Which turns out to be Lurie's!. Kinda.