

# Topoi with enough points

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This talk is based on a joint work with *Morgan Rogers*.

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Motto:propositional first order theories  $\equiv$  Boolean algebras



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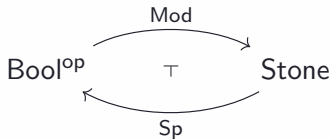
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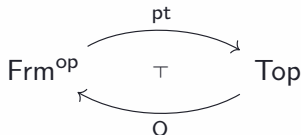


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## Frames



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The canonical example of frame is a topology. When  $(X, \tau)$  is a topological space,  $\tau$  is a frame.

$$O : \mathbf{Top} \rightarrow \mathbf{Frm}^{\text{op}}$$



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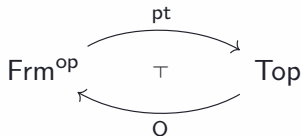
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When a frame has enough points, then  $\text{Opt}(\mathcal{L}) \cong \mathcal{L}$ .



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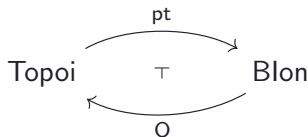
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**Question:** Can we push to the extreme the technology of Deligne and unify every completeness theorem?





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