

Last time

Limits

- Product
- Equalizers
- Pull back
- General def.

Colimits

- Coprod
- Coequalizer
- Pushout
- General def.

Monos

Epimorphisms

Exa

- Product of sets
- \wedge in posets
- Kernels
- Quotients
- Mons in sets are injective maps
- Epi in sets are surjective maps
- $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epi in Ring.

thm all limits $\Leftrightarrow \prod$ + equalizers
all colimits $\Leftrightarrow \coprod$ + coequalizers

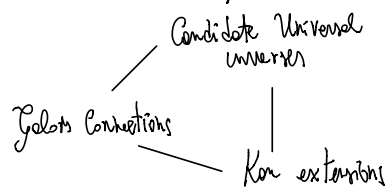
Lecture 3: Adjunctions

Two approaches to adjunctions

Empirical / Historical

"free constructions in universal algebra"

Conceptual / In retrospect



First story (free groups) $\text{Grp}(F(X), G) \cong \text{Set}(X, UG)$
 Second story (f as a generalized inverse).

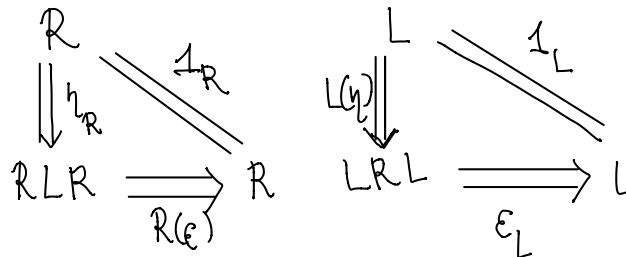
Def Let $L: A \rightleftarrows B: R$ be functors.

We say that L is left adjoint to R $(L \dashv R)$ if there exist natural transformations

$$(\text{unit}) \quad \eta: 1_A \Rightarrow RL$$

$$(\text{counit}) \quad \epsilon: LR \Rightarrow 1_B$$

triangle equations



"Mathematically precise definition"

this is very hard to grasp. sanity check.

$$F : \text{Set} \rightleftarrows \text{Grp} : U$$

$$\begin{array}{c|c} FU(G) \xrightarrow{\epsilon_G} G & X \xrightarrow{\eta_X} UF(X) \\ (w) \mapsto w & x \mapsto g_x \end{array}$$

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & UF(x) \\ \downarrow & \searrow \text{id} & \downarrow \epsilon_x \\ (x) & \xrightarrow{\epsilon_x} & x \end{array}$$

Thm $L : A \rightleftarrows B : R$, then

$$\varphi : A(\underline{x}, R\underline{y}) \xrightarrow{\sim} B(L\underline{x}, \underline{y}) : \varphi^{-1}$$

Proof

$$\begin{array}{ccc} X & & L(X) \\ \downarrow f & & \downarrow g \\ RY & & Y \\ & \nearrow & \\ & L(f) & \\ & \downarrow & \\ & LRY & \\ & \downarrow \epsilon_Y & \\ & Y & \end{array}$$

$$\varphi^{-1} \circ \varphi(f) = \varphi^{-1}(\epsilon_Y \circ L(f)) = \epsilon_Y \circ L(f) \circ \eta_X$$

The bijection above is "natural" in A and B^P .

Other examples

- Vect_k Set
- Grp Set
- Ab Set
- Ab Grp
- $\text{Top} \hookleftarrow \text{Set}$
- cartesian closed category -
- closure operators.
- pointed sets -
- monoidal structure on Ab -

Non example

Fld Set

Rem A functions compose