11/10

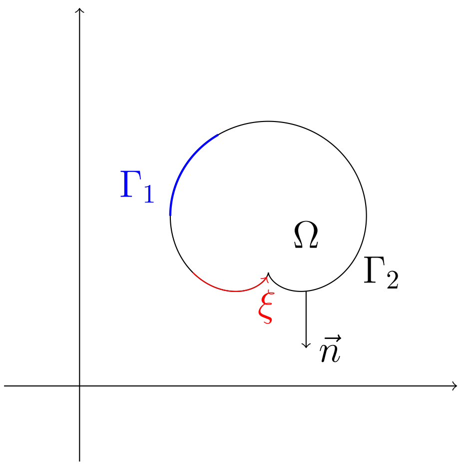
Двумерные краевые задачи

$$-\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right)+bu=f \qquad (1)$$

$K(x,y),\ b(x,y),\ f(x,y)$ – заданные (гладкие) функции.

$u(x,y)$ – неизвестная.

Ω – область, где задано уравнение (1), Γ – граница Ω .



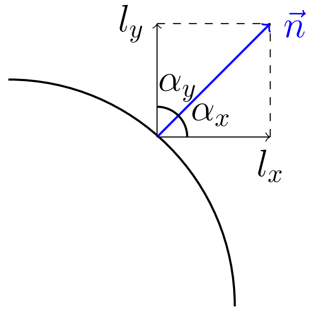
Замкнутый контур Γ – гладкий, за исключением конечного числа угловых точек, в которых внутренний угол $\alpha \in [0;\pi]$.

$$\Gamma = \underbrace{\Gamma_1}_{\text{I рода}} \cup \underbrace{\Gamma_2}_{\text{II/III рода}}$$

$u(\xi)=\hat{u}(\xi)$ – на Γ_1 (заданное значение).

$$K(\xi)\frac{\partial u}{\partial n}=\hat{\sigma}(\xi) - \text{ на } \Gamma_2$$

$$\frac{\partial u}{\partial n}=\frac{\partial u}{\partial x}l_x+\frac{\partial u}{\partial y}l_y,\ \begin{cases} l_x=\cos(\alpha x)=\cos(\vec{x},\vec{n}),\\ l_y=\cos(\alpha y)=\cos(\vec{y},\vec{n}),\\ ||\vec{n}||=1\end{cases}$$



Составим невязку:

$$r(x,y) = -\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) + bu - f = 0$$

$$\iint\limits_{\Omega} r(x,y)\cdot v\,dx\,dy = 0,$$

где $v(x,y)$ – пробная (гладкая) функция; на Γ_1: $v = 0$

$$\iint\limits_{\Omega}\left[-\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) + bu - f\right]\cdot v\,dx\,dy = 0 \tag{2}$$

Рассмотрим первый интеграл:

$$-\iint\limits_{\Omega}\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)\cdot v\,dx\,dy$$

Формула Гаусса-Остроградского:

$$\iint\limits_{\Omega}\frac{\partial}{\partial x}F dxdy = \int\limits_{\Gamma} F\cdot l_x d\xi$$

Представим F в виде $F = uv$:

$$\frac{\partial}{\partial x}F = \frac{\partial}{\partial x}(uv) = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x} \;\Rightarrow\; \frac{\partial u}{\partial x}v = \frac{\partial}{\partial x}(uv) - u\frac{\partial v}{\partial x}$$

Отсюда:

$$\iint\limits_{\Omega}\frac{\partial u}{\partial x}v\,dx\,dy = \iint\limits_{\Omega}\left(\frac{\partial}{\partial x}(uv) - u\frac{\partial v}{\partial x}\right)dx\,dy$$

И по формуле Гаусса-Остроградского:

$$\iint\limits_{\Omega} \frac{\partial u}{\partial x} v \, dx \, dy = \int\limits_{\Gamma} uv \cdot l_x d\xi - \iint\limits_{\Omega} \frac{\partial v}{\partial x} u \, dx \, dy$$

Тогда:

$$-\iint\limits_{\Omega} \frac{\partial}{\partial x}\left(K \frac{\partial u}{\partial x}\right) \cdot v \, dx \, dy = -\int\limits_{\Gamma} K \cdot \frac{\partial u}{\partial x} \cdot v \cdot l_x d\xi + \iint\limits_{\Omega} \frac{\partial v}{\partial x}\left(K \cdot \frac{\partial u}{\partial x}\right) dx \, dy$$

$$-\iint\limits_{\Omega} \frac{\partial}{\partial y}\left(K \frac{\partial u}{\partial y}\right) \cdot v \, dx \, dy = -\int\limits_{\Gamma} K \cdot \frac{\partial u}{\partial y} \cdot v \cdot l_y d\xi + \iint\limits_{\Omega} \frac{\partial v}{\partial y}\left(K \cdot \frac{\partial u}{\partial y}\right) dx \, dy$$

Подставим в (2):

$$\iint\limits_{\Omega} \left[\frac{\partial v}{\partial x} \cdot K \cdot \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \cdot K \cdot \frac{\partial u}{\partial y} + bu \cdot v - f \cdot v\right] dx \, dy -$$

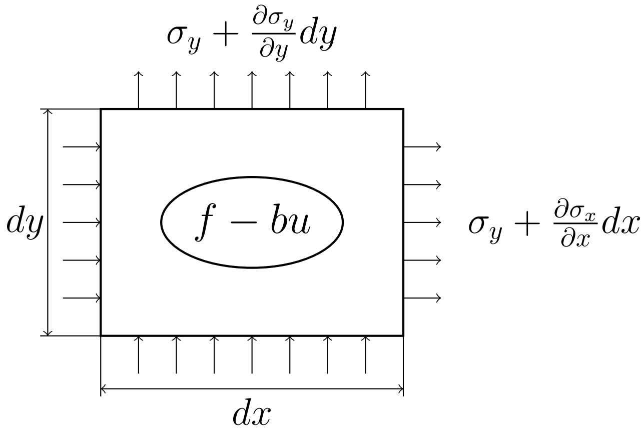
$$-\int\limits_{\Gamma_2} K \cdot v \cdot \left(\frac{\partial u}{\partial x} \cdot l_x + \frac{\partial u}{\partial y} \cdot l_y\right) = 0$$

<u>Задача упругости</u> $u(x, y)$ - перемещения

$$\varepsilon = [\varepsilon_x \quad \varepsilon_y]^T, \quad \varepsilon_x = -\frac{\partial u}{\partial y}, \quad \varepsilon_y = -\frac{\partial u}{\partial y}$$

$$\varepsilon = Lu, \quad L = \left[-\frac{\partial}{\partial x} \quad -\frac{\partial}{\partial y}\right]^T$$

$$\sigma = k\varepsilon = kLu$$



$$d\Omega = dxdy$$

$$\sigma_x dy + \sigma_y dx + (f - bu)dxdy = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\right)dy + \left(\sigma_y + \frac{\partial \sigma_y}{\partial y}dy\right)dx$$

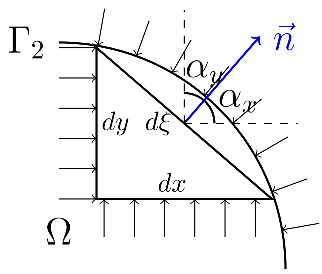
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} = f - bu$$

$$L_* = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}, \quad L_* = -L^T$$

$$L_*\sigma = L_* k L u = f - bu$$

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \cdot k \cdot \begin{bmatrix} -\frac{\partial u}{\partial x} \\ -\frac{\partial u}{\partial y} \end{bmatrix} = f - bu$$

$$-\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) = f \cdot bu \Leftrightarrow (1)$$



$$\Gamma_1 : u(\xi) = \hat{u}(\xi)$$

$$dx = \cos\alpha_y d\xi = l_y d\xi, \quad dy = \cos\alpha_x d\xi = l_x d\xi$$

$$\sigma_y dx + \sigma_x dy + \hat{\sigma} d\xi = 0$$

$$\sigma_x = K\varepsilon_x = -K\frac{\partial u}{\partial x}, \ \sigma_y = K\varepsilon_y = -K\frac{\partial u}{\partial y}$$

Тогда:

$$-K\frac{\partial u}{\partial y}dx - K\frac{\partial u}{\partial x}dy + \hat{\sigma} d\xi = 0$$

$$-K\frac{\partial u}{\partial y}l_y d\xi - K\frac{\partial u}{\partial x}l_x d\xi + \hat{\sigma} d\xi = 0$$

$$K\frac{\partial u}{\partial n} = \hat{\sigma}$$

Отсюда:

$$\iint\limits_{\Omega} \left(\frac{\partial v}{\partial x} K \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} K \frac{\partial u}{\partial y} + bu \cdot v - f \cdot v \right) dxdy - \int\limits_{\Gamma} \underbrace{\left(K \frac{\partial u}{\partial x} l_x + K \frac{\partial u}{\partial y} l_y \right)}_{K \cdot \frac{\partial u}{\partial n} = \hat{\sigma}(\xi)} \cdot v d\xi = 0 \tag{3}$$

$$\iint\limits_{\Omega} ((Lu)^T K (Lu) + v^T bu - v^T f) dxdy - \int\limits_{\Gamma_2} v^T \hat{\sigma} d\xi = 0$$