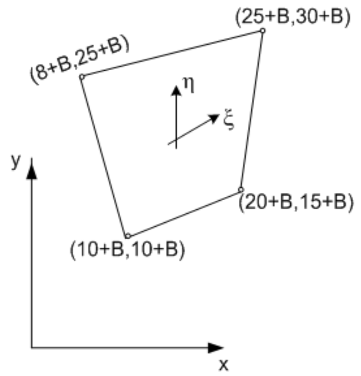
Задание

1. Построить функции формы с помощью аппроксимации Лагранжа и Сирендипова семейства для квадратичного четырехугольного элемента

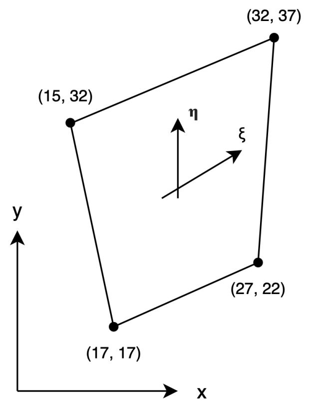
2. Вычислить производные от функций форм $\dfrac{\partial N_i}{\partial x}, \dfrac{\partial N_i}{\partial y}$

3. Вычислить интеграл $\displaystyle\iint\limits_S \left(\frac{\partial N_i}{\partial x} \cdot \frac{\partial N_i}{\partial y}\right) dS$

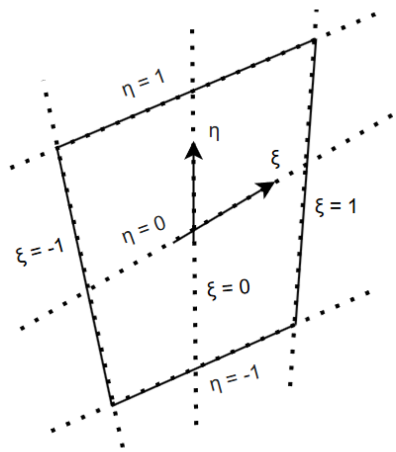


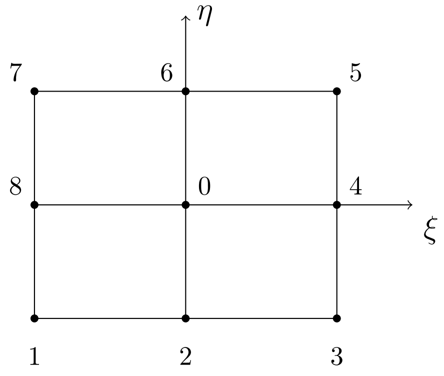
Решение

B = 7



В естественной системе координат:





1. Функции формы

   (a) Сирендипово семейство

$$\varphi_2 = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 y + \alpha_6 xy^2 + \alpha_7 x^2 + \alpha_8 y^2$$

Определим функции формы $N_\beta, \beta = \overline{1,8}$ по формуле:

$$N_\beta = \left[\prod_{j=1}^{4} F_j\right](a_{1\beta} + a_{2\beta}\xi + a_{3\beta}\eta)\,, \quad F_k = \begin{cases} f_k, & \text{если узел } \beta \notin \text{стороне } k, \\ 1, & \text{если узел } \beta \in \text{стороне } k, \end{cases}$$

где $f_1 = (1+\eta), f_2 = (1-\xi), f_3 = (1-\eta), f_4 = (1+\xi)$.
Вычислим функцию формы для элемента 1:

$$N_1 = (1-\eta)(1-\xi)(a_1 + a_2\xi + a_3\eta)$$

$$\begin{cases} N_1(\xi = 0, \eta = -1) = 0 \\ N_1(\xi = -1, \eta = 0) = 0 \\ N_1(\xi = -1, \eta = -1) = 1 \end{cases} \Rightarrow \begin{cases} N_1 = 2(a_1 - a_3) = 0 \\ N_1 = 2(a_1 - a_2) = 0 \\ N_1 = 4(a_1 - a_2 - a_3) = 1 \end{cases}$$

$$\Rightarrow a_1 = a_2 = a_3 = -\frac{1}{4}$$

$$\Rightarrow N_1 = -\frac{1}{4}(1-\eta)(1-\xi)(1+\xi+\eta)$$

Вычислим функцию формы для элемента 2:

$$N_2 = (\alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta)(a_1 + a_2\xi + a_3\eta) = (1+\xi)(1-\xi)(1-\eta)\cdot a \Rightarrow$$

$$\Rightarrow N_2(\xi = 0, \eta = -1) = 2a = 1 \Rightarrow a = \frac{1}{2} \Rightarrow N_2 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

Аналогично для оставшихся элементов:

$$N_3 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_4 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

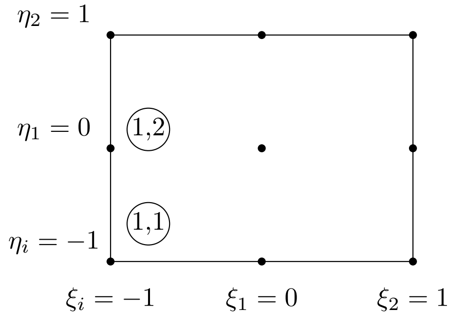
$$N_5 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_6 = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_7 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

(b) Лагранжево семейство



Функция формы:

$$N_{ij} = L_i^n(\xi) L_j^m(\eta)$$

$L_i^n(\xi) L_j^m(\eta)$ - многочлены Лагранжа, n, m - количество разбиений по ξ, η

$$L_i^n(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2) \dots (\xi - \xi_n)}{(\xi_i - \xi_1)(\xi_i - \xi_2) \dots (\xi_i - \xi_n)}$$

$$N_{ij} = L_i^2(\xi) L_j^2(\eta),\ i \neq 1, 2$$

$$L_i^2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_i - \xi_1)(\xi_i - \xi_2)}$$

$$N_1 = N_{11} = \frac{\xi \cdot (\xi - 1)}{-1 \cdot (-2)} \cdot \frac{\eta \cdot (\eta - 1)}{-1 \cdot (-2)} = \frac{1}{4} \cdot \xi \cdot \eta (\xi - 1)(\eta - 1)$$

$$N_8 = N_{12} = \frac{\xi \cdot (\xi - 1)}{2} \cdot \frac{(\eta + 1)(\eta - 1)}{1 \cdot (-1)} = -\frac{1}{2} \xi (\xi - 1)(\eta^2 - 1)$$

$$N_2 = \frac{(1 - \xi^2)\eta(\eta - 1)}{2}$$

$$N_3 = \frac{\xi(\xi + 1)\eta(\eta - 1)}{4}$$

$$N_4 = \frac{\xi(\xi + 1)(1 - \eta^2)}{2}$$

$$N_5 = \frac{\xi(\xi + 1)\eta(\eta + 1)}{4}$$

$$N_6 = \frac{(1 - \xi^2)\eta(\eta + 1)}{2}$$

$$N_7 = \frac{\xi(\xi - 1)\eta(\eta + 1)}{4}$$

$$N_0 = (\xi^2 - 1)(\eta^2 - 1)$$

2. Вычислим производные от функций формы

$$\begin{bmatrix}\frac{\partial N_\beta}{\partial \xi}\\ \frac{\partial N_\beta}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{bmatrix}\begin{bmatrix}\frac{\partial N_\beta}{\partial x}\\ \frac{\partial N_\beta}{\partial y}\end{bmatrix} = J\cdot\begin{bmatrix}\frac{\partial N_\beta}{\partial x}\\ \frac{\partial N_\beta}{\partial y}\end{bmatrix} \Rightarrow \begin{bmatrix}\frac{\partial N_\beta}{\partial x}\\ \frac{\partial N_\beta}{\partial y}\end{bmatrix} = J^{-1}\begin{bmatrix}\frac{\partial N_\beta}{\partial \xi}\\ \frac{\partial N_\beta}{\partial \eta}\end{bmatrix}$$

$$\begin{cases} x = R_1X_1 + R_2X_2 + R_3X_3 + R_4X_4\\ y = R_1Y_1 + R_2Y_2 + R_3Y_3 + R_4Y_4,\end{cases}$$

где X_i, Y_i – координаты вершин четырехугольника,
R_i – линейные интерполяции:

$$\begin{cases} R_1 = \frac{1}{4}(1-\xi)(1-\eta)\\ R_2 = \frac{1}{4}(1+\xi)(1-\eta)\\ R_3 = \frac{1}{4}(1+\xi)(1+\eta)\\ R_4 = \frac{1}{4}(1-\xi)(1+\eta)\end{cases}$$

Тогда:

$$\begin{cases} x = \frac{17(1-\xi)(1-\eta)}{4} + \frac{15(1-\xi)(\eta+1)}{4} + \frac{27(1-\eta)(\xi+1)}{4} + 8\left(\xi+1\right)\left(\eta+1\right),\\ y = \frac{17(1-\xi)(1-\eta)}{4} + 8\left(1-\xi\right)\left(\eta+1\right) + \frac{11(1-\eta)(\xi+1)}{2} + \frac{37(\xi+1)(\eta+1)}{4}.\end{cases}$$

$$\begin{cases} x = \frac{7\xi\eta}{4} + \frac{27\xi}{4} + \frac{3\eta}{4} + \frac{91}{4},\\ y = \frac{5\xi}{2} + \frac{15\eta}{2} + 27.\end{cases}$$

$$\frac{\partial x}{\partial \xi} = \frac{7\eta}{4} + \frac{27}{4},\quad \frac{\partial y}{\partial \xi} = \frac{5}{2},\quad \frac{\partial x}{\partial \eta} = \frac{7\xi}{4} + \frac{3}{4},\quad \frac{\partial y}{\partial \eta} = \frac{15}{2}$$

Матрица Якоби:

$$J = \begin{bmatrix}\frac{7\eta}{4} + \frac{27}{4} & \frac{5}{2}\\ \frac{7\xi}{4} + \frac{3}{4} & \frac{15}{2}\end{bmatrix},\quad |J| = -\frac{35\xi}{8} + \frac{105\eta}{8} + \frac{195}{4}$$

$$J^{-1} = \begin{bmatrix}-\frac{12}{7\xi-21\eta-78} & \frac{4}{7\xi-21\eta-78}\\ \frac{14\xi+6}{35\xi-105\eta-390} & \frac{-14\eta-54}{35\xi-105\eta-390}\end{bmatrix}$$

(a) Сирендипово семейство

$$\begin{bmatrix}\frac{\partial N_1}{\partial \xi}\\ \frac{\partial N_1}{\partial \eta}\end{bmatrix} = -\frac{1}{4}\begin{bmatrix}(\eta-1)(\eta+2\xi)\\ (2\eta+\xi)(\xi-1)\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_2}{\partial \xi}\\ \frac{\partial N_2}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\xi(\eta-1)\\ \frac{1}{2}-\frac{1}{2}\xi^2\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_3}{\partial \xi}\\ \frac{\partial N_3}{\partial \eta}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}(\eta-1)(\eta-2\xi)\\ (2\eta-\xi)(\xi+1)\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_4}{\partial \xi}\\ \frac{\partial N_4}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}-\frac{1}{2}\eta^2\\ -\eta(\xi+1)\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_5}{\partial \xi}\\ \frac{\partial N_5}{\partial \eta}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}(\eta+1)(\eta+2\xi)\\ (2\eta+\xi)(\xi+1)\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_6}{\partial \xi}\\ \frac{\partial N_6}{\partial \eta}\end{bmatrix} = \begin{bmatrix}-\xi(\eta+1)\\ \frac{1}{2}-\frac{1}{2}\xi^2\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_7}{\partial \xi}\\ \frac{\partial N_7}{\partial \eta}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}(\eta+1)(2\xi-\eta)\\ (\xi-2\eta)(\xi-1)\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_8}{\partial \xi}\\ \frac{\partial N_8}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\eta^2-\frac{1}{2}\\ \eta(\xi-1)\end{bmatrix}$$

Итоговые производные:

$$\begin{bmatrix}\frac{\partial N_1}{\partial x}\\ \frac{\partial N_1}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{(\xi-1)(\xi+2\eta)-3(2\xi+\eta)(\eta-1)}{-7\xi+21\eta+78}\\ \frac{-(\xi-1)(\xi+2\eta)(7\eta+27)+(2\xi+\eta)(7\xi+3)(\eta-1)}{10(-7\xi+21\eta+78)}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_2}{\partial x}\\ \frac{\partial N_2}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{-2\xi^2+12\xi(\eta-1)+2.0}{-7\xi+21\eta+78}\\ \frac{-2\xi(7\xi+3)(\eta-1)+\left(\xi^2-1.0\right)(7\eta+27)}{5(-7\xi+21\eta+78)}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_3}{\partial x}\\ \frac{\partial N_3}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{(\xi+1)(\xi-2\eta)-3(2\xi-\eta)(\eta-1)}{-7\xi+21\eta+78}\\ \frac{-(\xi+1)(\xi-2\eta)(7\eta+27)+(2\xi-\eta)(7\xi+3)(\eta-1)}{10(-7\xi+21\eta+78)}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_4}{\partial x}\\ \frac{\partial N_4}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{-6\eta^2+4\eta(\xi+1)+6.0}{-7\xi+21\eta+78}\\ \frac{-2\eta(\xi+1)(7\eta+27)+(7\xi+3)\left(\eta^2-1.0\right)}{5(-7\xi+21\eta+78)}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_5}{\partial x}\\ \frac{\partial N_5}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{-(\xi+1)(\xi+2\eta)+3(2\xi+\eta)(\eta+1)}{-7\xi+21\eta+78}\\ \frac{(\xi+1)(\xi+2\eta)(7\eta+27)-(2\xi+\eta)(7\xi+3)(\eta+1)}{10(-7\xi+21\eta+78)}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_6}{\partial x}\\ \frac{\partial N_6}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{2\xi^2-12\xi(\eta+1)-2.0}{-7\xi+21\eta+78}\\ \frac{2\xi(7\xi+3)(\eta+1)-\left(\xi^2-1.0\right)(7\eta+27)}{5(-7\xi+21\eta+78)}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_7}{\partial x}\\ \frac{\partial N_7}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{-(\xi-1)(\xi-2\eta)+3(2\xi-\eta)(\eta+1)}{-7\xi+21\eta+78}\\ \frac{(\xi-1)(\xi-2\eta)(7\eta+27)-(2\xi-\eta)(7\xi+3)(\eta+1)}{10(-7\xi+21\eta+78)}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_8}{\partial x}\\ \frac{\partial N_8}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{6\eta^2-4\eta(\xi-1)-6.0}{-7\xi+21\eta+78}\\ \frac{2\eta(\xi-1)(7\eta+27)-(7\xi+3)\left(\eta^2-1.0\right)}{5(-7\xi+21\eta+78)}\end{bmatrix}$$

(b) Лагранжево семейство

$$\begin{bmatrix}\frac{\partial N_1}{\partial \xi}\\ \frac{\partial N_1}{\partial \eta}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}\eta(2\xi-1)(\eta-1)\\ (\xi-1)(2\eta-1)\xi\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_2}{\partial \xi}\\ \frac{\partial N_2}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\xi\eta(1-\eta)\\ \left(\eta-\frac{1}{2}\right)(1-\xi^2)\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial N_3}{\partial \xi}\\ \frac{\partial N_3}{\partial \eta}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}\eta(2\xi+1)(\eta-1)\\ (\xi+1)(2\eta-1)\xi\end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_4}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \left(\xi + \frac{1}{2}\right)(1-\eta^2) \\ -\eta(\xi+1)\xi \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_5}{\partial \xi} \\ \frac{\partial N_5}{\partial \eta} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} (\eta+1)(2\xi+1)\eta \\ (2\eta+1)(\xi+1)\xi \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_6}{\partial \xi} \\ \frac{\partial N_6}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -\xi\eta(\eta+1) \\ \left(\eta + \frac{1}{2}\right)(1-\xi^2) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_7}{\partial \xi} \\ \frac{\partial N_7}{\partial \eta} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} (\eta+1)\eta(2\xi-1) \\ (2\eta+1)(\xi-1)\xi \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_8}{\partial \xi} \\ \frac{\partial N_8}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \left(\xi - \frac{1}{2}\right)(1-\eta^2) \\ \xi\eta(1-\xi) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_0}{\partial \xi} \\ \frac{\partial N_0}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 2\xi(\eta^2-1) \\ 2\eta(\xi^2-1) \end{bmatrix}$$

Итоговые производные:

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{-\xi(\xi-1)(2\eta-1)+3\eta(2\xi-1)(\eta-1)}{-7\xi+21\eta+78} \\ \frac{x(\xi-1)(2\eta-1)(7\eta+27)-\eta(2\xi-1)(7\xi+3)(\eta-1)}{10(-7\xi+21\eta+78)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{4\left(-3\xi\eta(\eta-1)+\left(\xi^2-1\right)(\eta-0.5)\right)}{-7\xi+21\eta+78} \\ \frac{2\left(\xi\eta(7\xi+3)(\eta-1)-\left(\xi^2-1\right)(\eta-0.5)(7\eta+27)\right)}{5(-7\xi+21\eta+78)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{-\xi(\xi+1)(2\eta-1)+3\eta(2\xi+1)(\eta-1)}{-7\xi+21\eta+78} \\ \frac{x(\xi+1)(2\eta-1)(7\eta+27)-\eta(2\xi+1)(7\xi+3)(\eta-1)}{10(-7\xi+21\eta+78)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_4}{\partial x} \\ \frac{\partial N_4}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{4\left(\xi\eta(\xi+1)-3(\xi+0.5)\left(\eta^2-1\right)\right)}{-7\xi+21\eta+78} \\ \frac{2\left(-\xi\eta(\xi+1)(7\eta+27)+(\xi+0.5)(7\xi+3)\left(\eta^2-1\right)\right)}{5(-7\xi+21\eta+78)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_5}{\partial x} \\ \frac{\partial N_5}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{-\xi(\xi+1)(2\eta+1)+3\eta(2\xi+1)(\eta+1)}{-7\xi+21\eta+78} \\ \frac{x(\xi+1)(2\eta+1)(7\eta+27)-\eta(2\xi+1)(7\xi+3)(\eta+1)}{10(-7\xi+21\eta+78)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_6}{\partial x} \\ \frac{\partial N_6}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{4\left(-3\xi\eta(\eta+1)+\left(\xi^2-1\right)(\eta+0.5)\right)}{-7\xi+21\eta+78} \\ \frac{2\left(\xi\eta(7\xi+3)(\eta+1)-\left(\xi^2-1\right)(\eta+0.5)(7\eta+27)\right)}{5(-7\xi+21\eta+78)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_7}{\partial x} \\ \frac{\partial N_7}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{-\xi(\xi-1)(2\eta+1)+3\eta(2\xi-1)(\eta+1)}{-7\xi+21\eta+78} \\ \frac{x(\xi-1)(2\eta+1)(7\eta+27)-\eta(2\xi-1)(7\xi+3)(\eta+1)}{10(-7\xi+21\eta+78)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_8}{\partial x} \\ \frac{\partial N_8}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{4\left(\xi\eta(\xi-1)-3(\xi-0.5)\left(\eta^2-1\right)\right)}{-7\xi+21\eta+78} \\ \frac{2\left(-\xi\eta(\xi-1)(7\eta+27)+(\xi-0.5)(7\xi+3)\left(\eta^2-1\right)\right)}{5(-7\xi+21\eta+78)} \end{bmatrix}$$

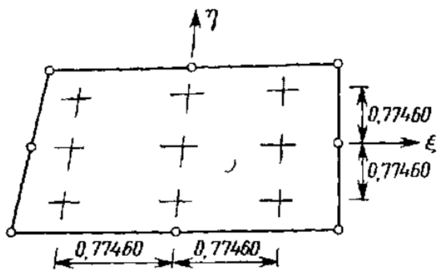
$$\begin{bmatrix} \frac{\partial N_0}{\partial x} \\ \frac{\partial N_0}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{8\left(3\xi\left(\eta^2-1\right)-\eta\left(\xi^2-1\right)\right)}{-7\xi+21\eta+78} \\ \frac{4\left(-\xi(7\xi+3)\left(\eta^2-1\right)+\eta\left(\xi^2-1\right)(7\eta+27)\right)}{5(-7\xi+21\eta+78)} \end{bmatrix}$$

3. Вычислим интеграл $\iint\limits_S \left(\frac{\partial N_i}{\partial x} \cdot \frac{\partial N_i}{\partial y}\right) dS$

$$Z = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\eta d\xi = \sum_{i=1}^{n}\sum_{j=1}^{n} H_i H_j f(\xi_i, \eta_j)$$

Порядок квадратур Гаусса—Лежандра для двумерных элементов

Элемент	$[N]^T\,[N]$	$[B]^T\,[B]$	$[N]^T$
	ξ, η	ξ, η	ξ, η
Линейный	2,2	2,2	1,1
Квадратичный	3,3	2,2	2,2
Кубичный	4,4	3,3	2,2



Координаты узлов и весовые коэффициенты для квадратуры Гаусса—Лежандра до четвертого порядка

n	ξ_1	H_1	n	ξ_1	H_i
2	±0,577350	1,00	4	±0,861136	0,347855
				±0,339981	0,652145
3	0,0	8/9			
	±0,774597	5/9	5	0,0	0,568889
				±0,538469	0,478629
				±0,906180	0,236927

Для Сирендипова семейства получаем $Z = 0.3296$, для Лагранжева 0.5737.