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Симплексный треугольный КЭ

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$\begin{cases} u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ u_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ u_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k \end{cases} \quad \Rightarrow \quad \Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = 2S_\Delta$$

$$\Delta_1 = \begin{vmatrix} u_i & x_i & y_i \\ u_j & x_j & y_j \\ u_k & x_k & y_k \end{vmatrix} = u_i(x_j y_k - x_k y_j) + u_j(x_k y_i - x_i y_k) + u_k(x_i y_j - x_j y_i)$$

$$\Delta_2 = \begin{vmatrix} 1 & u_i & y_i \\ 1 & u_j & y_j \\ 1 & u_k & y_k \end{vmatrix} = u_i(y_j - y_k) + u_j(y_i - y_k) + u_k(y_i - y_j)$$

$$\Delta_2 = \begin{vmatrix} 1 & x_i & u_i \\ 1 & x_j & u_j \\ 1 & x_k & u_k \end{vmatrix} = u_i(x_k - x_j) + u_j(x_i - x_k) + u_k(x_j - x_i)$$

$$\alpha_1 = \frac{\Delta_1}{\Delta}, \quad \alpha_2 = \frac{\Delta_2}{\Delta}, \quad \alpha_3 = \frac{\Delta_3}{\Delta}$$

$$u = \frac{1}{\Delta}(a_i u_i + a_j u_j + a_k u_k + b_i u_i x + b_j u_j x + b_k u_k x + c_i u_i y + c_j u_j y + c_k u_k y) =$$

$$= \frac{1}{\Delta}\left[(a_i + b_i x + c_i y)u_i + (a_j + b_j x + c_j y)u_j + (a_k + b_k x + c_k y)u_i)\right] =$$

$$= N_i u_i + N_j u_j + N_k u_k = [N]\{\Phi\}$$

$$\begin{cases} N_i = \frac{1}{\Delta}(a_i + b_i x + c_i y) \\ N_j = \frac{1}{\Delta}(a_j + b_j x + c_j y) \\ N_k = \frac{1}{\Delta}(a_k + b_k x + c_k y) \end{cases} \quad - \text{ функции формы, } u = [N]\{\Phi\}$$

Возвращаемся к уравнению:

$$\iint\limits_{\Omega} (\frac{\partial v}{\partial x} \cdot K \cdot \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \cdot K \cdot \frac{\partial u}{\partial y} + buv - fv)dxdy - \int\limits_{\Gamma} (K \cdot \frac{\partial u}{\partial x} \cdot l_x + K \cdot \frac{\partial u}{\partial y} \cdot l_y)vd\xi = 0$$

$$v = N\delta\Phi, \quad \delta\Phi = [v_i, v_j, v_k]^T$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial [N]}{\partial x} \\ \\ \dfrac{\partial [N]}{\partial y} \end{bmatrix} \{\Phi\} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_k}{\partial x} \\ \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_k}{\partial y} \end{bmatrix} \{\Phi\} = \frac{1}{\Delta} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \{\Phi\} = [B] \{\Phi\}$$

$$\begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = [B] \{\delta\Phi\} \qquad D = \begin{bmatrix} K_x & 0 \\ 0 & K_y \end{bmatrix}$$

$$\int\limits_{\Omega} \left(\frac{\partial v}{\partial x} \cdot K_x \cdot \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \cdot K_y \cdot \frac{\partial u}{\partial y} \right) \, dxdy = \int\limits_{\Omega} (B\delta\Phi)^T DB\Phi \, dxdy =$$

$$= \{\delta\Phi\}^T \int\limits_{\Omega} B^T DB \, dxdy \; \{\Phi\}$$

$$\int\limits_{\Omega} buv \, dxdy = \int\limits_{\Omega} (N\delta\Phi)^T bN\Phi \, dxdy = \{\delta\Phi\}^T \int\limits_{\Omega} bN^T DN \, dxdy \; \{\Phi\}$$

$$\int\limits_{\Omega} fv \, dxdy =$$

Считаем, что $K_x, K_y - const$:

$$= \frac{S_1}{(2S_1)^2} \left[K_x \begin{pmatrix} b_i b_i & b_i b_j & b_i b_k \\ b_i b_j & b_j b_j & b_j b_k \\ b_i b_k & b_j b_k & b_k b_k \end{pmatrix} + K_y \begin{pmatrix} c_i c_i & c_i c_j & c_i c_k \\ c_i c_j & c_j c_j & c_j c_k \\ c_i c_k & c_j c_k & c_k c_k \end{pmatrix} \right]$$