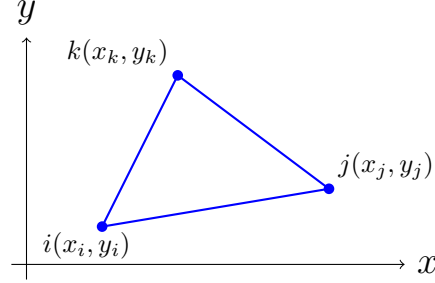


Симплексный треугольный конечный элемент

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$



$$\begin{cases} u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ u_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ u_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k \end{cases} \Rightarrow \Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = 2S_\Delta$$

$$\Delta_1 = \begin{vmatrix} u_i & x_i & y_i \\ u_j & x_j & y_j \\ u_k & x_k & y_k \end{vmatrix} = u_i \underbrace{(x_j y_k - x_k y_j)}_{a_i} + u_j \underbrace{(x_k y_i - x_i y_k)}_{a_j} + u_k \underbrace{(x_i y_j - x_j y_i)}_{a_k}$$

$$\Delta_2 = \begin{vmatrix} 1 & u_i & y_i \\ 1 & u_j & y_j \\ 1 & u_k & y_k \end{vmatrix} = u_i \underbrace{(y_j - y_k)}_{b_i} + u_j \underbrace{(y_k - y_i)}_{b_j} + u_k \underbrace{(y_i - y_j)}_{b_k}$$

$$\Delta_3 = \begin{vmatrix} 1 & x_i & u_i \\ 1 & x_j & u_j \\ 1 & x_k & u_k \end{vmatrix} = u_i \underbrace{(x_k - x_j)}_{c_i} + u_j \underbrace{(x_i - x_k)}_{c_j} + u_k \underbrace{(x_j - x_i)}_{c_k}$$

$$\alpha_1 = \frac{\Delta_1}{\Delta}, \quad \alpha_2 = \frac{\Delta_2}{\Delta}, \quad \alpha_3 = \frac{\Delta_3}{\Delta}$$

$$\begin{aligned} u &= \frac{1}{\Delta} (a_i u_i + a_j u_j + a_k u_k + b_i u_i x + b_j u_j x + b_k u_k x + c_i u_i y + c_j u_j y + c_k u_k y) = \\ &= \frac{1}{\Delta} [(a_i + b_i x + c_i y) u_i + (a_j + b_j x + c_j y) u_j + (a_k + b_k x + c_k y) u_k] = \\ &= N_i u_i + N_j u_j + N_k u_k = [N] \{\Phi\} \end{aligned}$$

$$\{\Phi\} = [u_i, u_j, u_k]^T, \quad [N] = [N_i, N_j, N_k]$$

$$\begin{cases} N_i = \frac{1}{\Delta}(a_i + b_i x + c_i y) \\ N_j = \frac{1}{\Delta}(a_j + b_j x + c_j y) \\ N_k = \frac{1}{\Delta}(a_k + b_k x + c_k y) \end{cases} \quad - \text{ функции формы, } u = [N] \{\Phi\}$$

Свойства функций формы:

$$\begin{cases} N_i(x_i, y_i) = 1, & N_i(x_j, y_j) = 0, & N_i(x_k, y_k) = 0; \\ N_j(x_i, y_i) = 0, & N_j(x_j, y_j) = 1, & N_j(x_k, y_k) = 0; \\ N_k(x_i, y_i) = 0, & N_k(x_j, y_j) = 0, & N_k(x_k, y_k) = 1; \\ N_i + N_j + N_k = 1 \end{cases}$$

Возвращаемся к уравнению:

$$\iint_{\Omega} \left(\frac{\partial v}{\partial x} \cdot K_x \cdot \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \cdot K_y \cdot \frac{\partial u}{\partial y} + buv - fv \right) dx dy -$$

$$- \int_{\Gamma} \left(K_x \cdot \frac{\partial u}{\partial x} \cdot l_x + K_y \cdot \frac{\partial u}{\partial y} \cdot l_y \right) v \, d\xi = 0$$

$$v = N \delta \Phi, \quad \delta \Phi = [v_i, v_j, v_k]^T$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial [N]}{\partial x} \\ \frac{\partial [N]}{\partial y} \end{bmatrix} \{\Phi\} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} \end{bmatrix} \{\Phi\} = \frac{1}{\Delta} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \{\Phi\} = [B] \{\Phi\}$$

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = [B] \{\delta \Phi\} \quad D = \begin{bmatrix} K_x & 0 \\ 0 & K_y \end{bmatrix}$$

$$\int_{\Omega} \left(\frac{\partial v}{\partial x} \cdot K_x \cdot \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \cdot K_y \cdot \frac{\partial u}{\partial y} \right) dx dy = \int_{\Omega} (B \delta \Phi)^T D B \Phi \, dx dy =$$

$$= \{\delta \Phi\}^T \int_{\Omega} B^T D B \, dx dy \, \{\Phi\}$$

$$\int_{\Omega} buv \, dxdy = \int_{\Omega} (N\delta\Phi)^T bN\Phi \, dxdy = \{\delta\Phi\}^T \int_{\Omega} bN^T N \, dxdy \, \{\Phi\}$$

$$\int_{\Omega} fv \, dxdy = \{\delta\Phi\}^T \int_{\Omega} f \cdot N^T \, dxdy$$

$$\int_{\Gamma} \hat{\sigma} \cdot v \, d\xi = \{\delta\Phi\}^T \int_{\Gamma} \hat{\sigma} N^T \, d\xi$$

$$K = \int_{\Omega} (B^T DB + bN^T N) \, dxdy; \quad P = \int_{\Omega} fN^T \, dxdy + \int_{\Gamma} \hat{\sigma} N^T \, d\xi$$

$$\{\delta\Phi\}^T K\Phi = \{\delta\Phi\}^T P \Rightarrow K\Phi = P$$

$$\int_{\Omega} B^T DB \, dxdy = \int_{\Omega} \frac{1}{\Delta} \begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix} \cdot \begin{bmatrix} K_x & 0 \\ 0 & K_y \end{bmatrix} \cdot \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \, dxdy = \odot$$

Считаем, что $K_x, K_y - \text{const}$:

$$\odot = \frac{S_1}{(2S_1)^2} \left[K_x \begin{pmatrix} b_i b_i & b_i b_j & b_i b_k \\ b_i b_j & b_j b_j & b_j b_k \\ b_i b_k & b_j b_k & b_k b_k \end{pmatrix} + K_y \begin{pmatrix} c_i c_i & c_i c_j & c_i c_k \\ c_i c_j & c_j c_j & c_j c_k \\ c_i c_k & c_j c_k & c_k c_k \end{pmatrix} \right]$$

$$\int_{\Omega} bN^T N \, dxdy = \int_{\Omega} b \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix} \cdot \begin{bmatrix} N_i & N_j & N_k \end{bmatrix} \, dxdy =$$

$$= \int_{\Omega} b \cdot \begin{bmatrix} N_i N_i & N_i N_j & N_i N_k \\ N_i N_j & N_j N_j & N_j N_k \\ N_i N_k & N_j N_k & N_k N_k \end{bmatrix} \, dxdy$$

$$P_1 = \int_{\Omega} fN^T \, dxdy = \int_{\Omega} f \cdot \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix} \, dxdy$$

Естественная система координат

