11/10

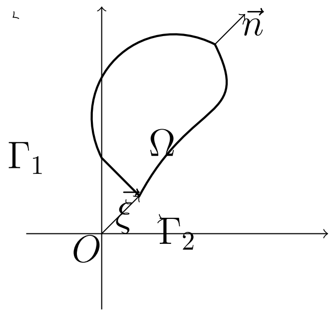
Двумерные краевые задачи

$$-\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right)+bu=f \qquad (1)$$

$K(x,y)$, $b(x,y)$, $f(x,y)$ – заданные (гладкие) функции.

$u(x,y)$ – неизвестная.

Ω – область, где задано уравнение (1), Γ – граница Ω .



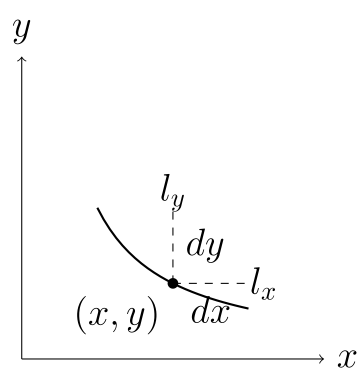
Замкнутый контур Γ – гладкий, за исключением конечного числа угловых точек, в которых внутренний угол $\alpha \in [0;\pi]$.

$$\Gamma = \underbrace{\Gamma_1}_{\text{I рода}} \cup \underbrace{\Gamma_2}_{\text{II/III рода}}$$

$$u(\xi)=\hat{u}(\xi) \text{ – на } \Gamma_1 \text{ (заданное значение).}$$

$$K(\xi)\frac{\partial u}{\partial n}=\hat{\sigma}(\xi) - \text{ на } \Gamma_2$$

$$\frac{\partial u}{\partial n}=\frac{\partial u}{\partial x}l_x+\frac{\partial u}{\partial y}l_y, \quad \begin{cases} l_x=\cos(\alpha x)=\cos(\vec{x},\vec{n}), \\ l_y=\cos(\alpha y)=\cos(\vec{y},\vec{n}), \\ ||\vec{n}||=1 \end{cases}$$



Составим невязку:

$$r(x,y) = -\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) + bu - f = 0$$

$$\iint\limits_{\Omega} r(x,y)\cdot v\,dx\,dy = 0,$$

где $v(x,y)$ – пробная (гладкая) функция; на Γ_1: $v = 0$

$$\iint\limits_{\Omega} \left[-\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) + bu - f\right]\cdot v\,dx\,dy = 0 \tag{2}$$

Рассмотрим первый интеграл:

$$-\iint\limits_{\Omega} \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)\cdot v\,dx\,dy$$

Формула Гаусса-Остроградского:

$$\iint\limits_{\Omega} \frac{\partial}{\partial x}F dxdy = \int\limits_{\Gamma} F\cdot l_x d\xi$$

Представим F в виде $F = uv$:

$$\frac{\partial}{\partial x}F = \frac{\partial}{\partial x}(uv) = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x} \;\Rightarrow\; \frac{\partial u}{\partial x}v = \frac{\partial}{\partial x}(uv) - u\frac{\partial v}{\partial x}$$

Отсюда:

$$\iint\limits_{\Omega} \frac{\partial u}{\partial x}v\,dx\,dy = \iint\limits_{\Omega}\left(\frac{\partial}{\partial x}(uv) - u\frac{\partial v}{\partial x}\right)dx\,dy$$

И по формуле Гаусса-Остроградского:

$$\iint\limits_{\Omega} \frac{\partial u}{\partial x}v\,dx\,dy = \int\limits_{\Gamma} uv\cdot l_x d\xi - \iint\limits_{\Omega}\frac{\partial v}{\partial x}u\,dx\,dy$$

Тогда:

$$-\iint\limits_{\Omega} \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)\cdot v\,dx\,dy = -\int\limits_{\Gamma} K\cdot\frac{\partial u}{\partial x}\cdot v\cdot l_x d\xi + \iint\limits_{\Omega}\frac{\partial v}{\partial x}\left(K\cdot\frac{\partial u}{\partial x}\right)\,dx\,dy$$

$$-\iint\limits_{\Omega} \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right)\cdot v\,dx\,dy = -\int\limits_{\Gamma} K\cdot\frac{\partial u}{\partial y}\cdot v\cdot l_y d\xi + \iint\limits_{\Omega}\frac{\partial v}{\partial y}\left(K\cdot\frac{\partial u}{\partial y}\right)\,dx\,dy$$

Подставим в (2):

$$\iint\limits_{\Omega}\left[\frac{\partial v}{\partial x}\cdot K\cdot\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\cdot K\cdot\frac{\partial u}{\partial y}+bu\cdot v-f\cdot v\right]\,dx\,dy-$$

$$-\int\limits_{\Gamma_2} K\cdot v\cdot\left(\frac{\partial u}{\partial x}\cdot l_x+\frac{\partial u}{\partial y}\cdot l_y\right)=0$$

<u>Задача упругости</u> $u(x,y)$ - перемещения

$$\varepsilon=[\varepsilon_x \quad \varepsilon_y]^T, \quad \varepsilon_x=-\frac{\partial u}{\partial y}, \quad \varepsilon_y=-\frac{\partial u}{\partial y}$$

$$\varepsilon=Lu, \quad L=\left[-\frac{\partial}{\partial x} \quad -\frac{\partial}{\partial y}\right]^T$$

$$\sigma=k\varepsilon=kLu$$

$$d\Omega=dxdy$$

$$\sigma_x dy+\sigma_y dx+(f-bu)dxdy=\left(\sigma_x+\frac{\partial\sigma_x}{\partial x}dx\right)dy+\left(\sigma_y+\frac{\partial\sigma_y}{\partial y}dy\right)dx$$

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\sigma_y}{\partial y}=f-bu$$

$$L_*=\left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}\right], \quad L_*=-L^T$$

$$L_*\sigma=L_*kLu=f-bu$$

$$\left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}\right]\cdot k\cdot\begin{bmatrix}-\frac{\partial u}{\partial x}\\ -\frac{\partial u}{\partial y}\end{bmatrix}=f-bu$$

$$-\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right)=f\cdot bu \Leftrightarrow (1)$$

$$\Gamma_1 : u(\xi) = \hat{u}(\xi)$$

$$dx = \cos\alpha_y d\xi = l_y d\xi, \quad dy = \cos\alpha_x d\xi = l_x d\xi$$

$$\sigma_y dx + \sigma_x dy + \hat{\sigma} d\xi = 0$$

$$\sigma_x = K\varepsilon_x = -K\frac{\partial u}{\partial x},\ \sigma_y = K\varepsilon_y = -K\frac{\partial u}{\partial y}$$

Тогда:

$$-K\frac{\partial u}{\partial y}dx - K\frac{\partial u}{\partial x}dy + \hat{\sigma} d\xi = 0$$

$$-K\frac{\partial u}{\partial y}l_y d\xi - K\frac{\partial u}{\partial x}l_x d\xi + \hat{\sigma} d\xi = 0$$

$$K\frac{\partial u}{\partial n} = \hat{\sigma}$$

Отсюда:

$$\iint\limits_{\Omega} \left(\frac{\partial v}{\partial x}K\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}K\frac{\partial u}{\partial y} + bu\cdot v - f\cdot v\right) dxdy - \int\limits_{\Gamma} \underbrace{\left(K\frac{\partial u}{\partial x}l_x + K\frac{\partial u}{\partial y}l_y\right)}_{K\cdot\frac{\partial u}{\partial n} = \hat{\sigma}(\xi)} \cdot v d\xi = 0 \tag{3}$$

$$\iint\limits_{\Omega} ((Lu)^T K(Lu) + v^T bu - v^T f) dxdy - \int\limits_{\Gamma_2} v^T \hat{\sigma} d\xi = 0$$