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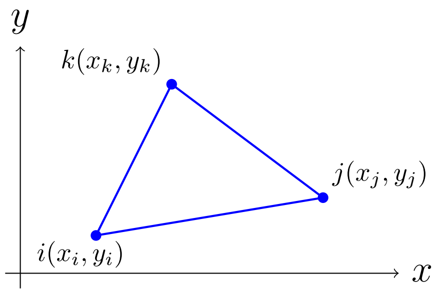
Симплексный треугольный конечный элемент

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$



$$\begin{cases} u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ u_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ u_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k \end{cases} \Rightarrow \quad \Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = 2S_\Delta$$

$$\Delta_1 = \begin{vmatrix} u_i & x_i & y_i \\ u_j & x_j & y_j \\ u_k & x_k & y_k \end{vmatrix} = u_i \underbrace{(x_j y_k - x_k y_j)}_{a_i} + u_j \underbrace{(x_k y_i - x_i y_k)}_{a_j} + u_k \underbrace{(x_i y_j - x_j y_i)}_{a_k}$$

$$\Delta_2 = \begin{vmatrix} 1 & u_i & y_i \\ 1 & u_j & y_j \\ 1 & u_k & y_k \end{vmatrix} = u_i \underbrace{(y_j - y_k)}_{b_i} + u_j \underbrace{(y_k - y_i)}_{b_j} + u_k \underbrace{(y_i - y_j)}_{b_k}$$

$$\Delta_2 = \begin{vmatrix} 1 & x_i & u_i \\ 1 & x_j & u_j \\ 1 & x_k & u_k \end{vmatrix} = u_i \underbrace{(x_k - x_j)}_{c_i} + u_j \underbrace{(x_i - x_k)}_{c_j} + u_k \underbrace{(x_j - x_i)}_{c_k}$$

$$\alpha_1 = \frac{\Delta_1}{\Delta}, \quad \alpha_2 = \frac{\Delta_2}{\Delta}, \quad \alpha_3 = \frac{\Delta_3}{\Delta}$$

$$u = \frac{1}{\Delta}(a_i u_i + a_j u_j + a_k u_k + b_i u_i x + b_j u_j x + b_k u_k x + c_i u_i y + c_j u_j y + c_k u_k y) =$$

$$= \frac{1}{\Delta}\left[(a_i + b_i x + c_i y)u_i + (a_j + b_j x + c_j y)u_j + (a_k + b_k x + c_k y)u_i)\right] =$$

$$= N_i u_i + N_j u_j + N_k u_k = [N]\{\Phi\}$$

$$\{\Phi\} = [u_i,\ u_j,\ u_k]^T, \quad [N] = [N_i,\ N_j,\ N_k]$$

$$\begin{cases} N_i = \frac{1}{\Delta}(a_i + b_i x + c_i y) \\ N_j = \frac{1}{\Delta}(a_j + b_j x + c_j y) \\ N_k = \frac{1}{\Delta}(a_k + b_k x + c_k y) \end{cases} \quad - \text{ функции формы, } u = [N]\{\Phi\}$$

Свойства функций формы:

$$\begin{cases} N_i(x_i, y_i) = 1, \quad N_i(x_j, y_j) = 0, \quad N_i(x_k, y_k) = 0; \\ N_j(x_i, y_i) = 0, \quad N_j(x_j, y_j) = 1, \quad N_j(x_k, y_k) = 0; \\ N_k(x_i, y_i) = 0, \quad N_k(x_j, y_j) = 0, \quad N_k(x_k, y_k) = 1; \\ N_i + N_j + N_k = 1 \end{cases}$$

Возвращаемся к уравнению:

$$\iint_{\Omega} \left(\frac{\partial v}{\partial x} \cdot K_x \cdot \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \cdot K_y \cdot \frac{\partial u}{\partial y} + buv - fv \right) dxdy -$$

$$- \int_{\Gamma} \left(K_x \cdot \frac{\partial u}{\partial x} \cdot l_x + K_y \cdot \frac{\partial u}{\partial y} \cdot l_y \right) v \; d\xi = 0$$

$$v = N\delta\Phi, \quad \delta\Phi = [v_i, v_j, v_k]^T$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial [N]}{\partial x} \\ \\ \dfrac{\partial [N]}{\partial y} \end{bmatrix} \{\Phi\} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_k}{\partial x} \\ \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_k}{\partial y} \end{bmatrix} \{\Phi\} = \frac{1}{\Delta} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \{\Phi\} = [B]\{\Phi\}$$

$$\begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = [B]\{\delta\Phi\} \qquad D = \begin{bmatrix} K_x & 0 \\ 0 & K_y \end{bmatrix}$$

$$\int_{\Omega} \left(\frac{\partial v}{\partial x} \cdot K_x \cdot \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \cdot K_y \cdot \frac{\partial u}{\partial y} \right) \; dxdy = \int_{\Omega} (B\delta\Phi)^T DB\Phi \; dxdy =$$

$$= \{\delta\Phi\}^T \int_{\Omega} B^T DB \; dxdy \;\; \{\Phi\}$$

$$\int\limits_{\Omega} buv\ dxdy = \int\limits_{\Omega} (N\delta\Phi)^T bN\Phi\ dxdy = \{\delta\Phi\}^T \int\limits_{\Omega} bN^T N\ dxdy\ \ \{\Phi\}$$

$$\int\limits_{\Omega} fv\ dxdy = \{\delta\Phi\}^T \int\limits_{\Omega} f \cdot N^T\ dxdy$$

$$\int\limits_{\Gamma} \hat{\sigma} \cdot v\ d\xi = \{\delta\Phi\}^T \int\limits_{\Gamma} \hat{\sigma} N^T\ d\xi$$

$$K = \int\limits_{\Omega} (B^T DB + bN^T N)\ dxdy; \quad P = \int\limits_{\Omega} fN^T\ dxdy + \int\limits_{\Gamma} \hat{\sigma} N^T\ d\xi$$

$$\{\delta\Phi\}^T K\Phi = \{\delta\Phi\}^T P \Rightarrow K\Phi = P$$

$$\int\limits_{\Omega} B^T DB\ dxdy = \int\limits_{\Omega} \frac{1}{\Delta} \begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix} \cdot \begin{bmatrix} K_x & 0 \\ 0 & K_y \end{bmatrix} \cdot \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}\ dxdy = \smiley$$

Считаем, что K_x, K_y – const:

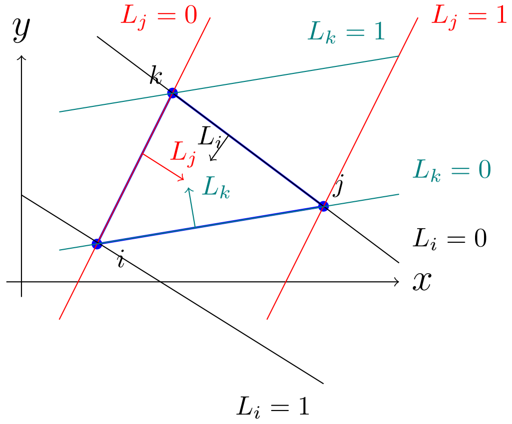
$$\smiley = \frac{S_1}{(2S_1)^2} \left[K_x \begin{pmatrix} b_i b_i & b_i b_j & b_i b_k \\ b_i b_j & b_j b_j & b_j b_k \\ b_i b_k & b_j b_k & b_k b_k \end{pmatrix} + K_y \begin{pmatrix} c_i c_i & c_i c_j & c_i c_k \\ c_i c_j & c_j c_j & c_j c_k \\ c_i c_k & c_j c_k & c_k c_k \end{pmatrix} \right]$$

$$\int\limits_{\Omega} bN^T N\ dxdy = \int\limits_{\Omega} b \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix} \cdot \begin{bmatrix} N_i & N_j & N_k \end{bmatrix}\ dxdy =$$

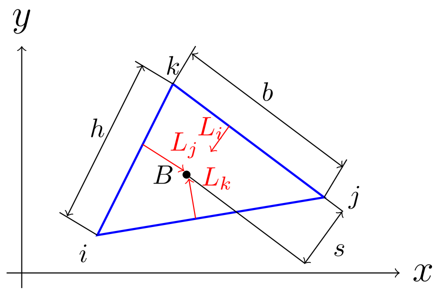
$$= \int\limits_{\Omega} b \cdot \begin{bmatrix} N_i N_i & N_i N_j & N_i N_k \\ N_i N_j & N_j N_j & N_j N_k \\ N_i N_k & N_j N_k & N_k N_k \end{bmatrix}\ dxdy$$

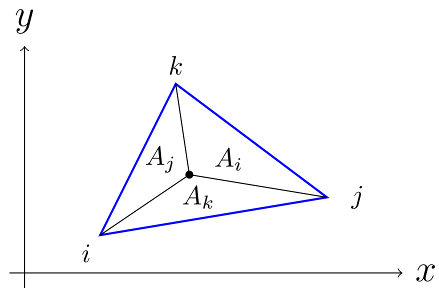
$$P_1 = \int\limits_{\Omega} fN^T\ dxdy = \int\limits_{\Omega} f \cdot \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix}\ dxdy$$

Естественная система координат



$$\begin{cases} 0 \leq L_i \leq 1; \\ 0 \leq L_j \leq 1; \\ 0 \leq L_k \leq 1. \end{cases}$$





$$\begin{cases} S_\triangle = \frac{1}{2}bh, \\ S_{A_i} = \frac{1}{2}bs \end{cases} \Rightarrow L_i = \frac{S_{A_i}}{S_\triangle} = \frac{s}{h}$$

$$L_j = \frac{S_{A_j}}{S_\triangle}, \quad L_k = \frac{S_{A_k}}{S_\triangle}$$

$$L_i + L_j + L_k = \frac{S_{A_i} + S_{A_j} + S_{A_k}}{S_\triangle} = 1$$

$$N_i = L_i,\ N_j = L_j,\ N_k = L_k$$