BONUS PROBLEMS

Seminar 2

8) N people ; $P = \frac{N_f}{N_t}$

the N hats can be redistributed in N! ways.

$\Rightarrow N_t = N!$

N_f is the number of ways to redistribute the hats so that no person gets its own hat back.

to compute N_f, we first count the permutations where at least one person gets its hat back (formally, the permutations σ where $\exists i \in \overline{1, N}$ such that $\sigma(i) = i$), let it be denoted N_b.

$\Rightarrow N_f = N! - N_b$

Denote a_i all the permutations where the condition $\sigma(i) = i$ is satisfied.

$$N_b = |a_1 \cup a_2 \cup \dots \cup a_N| =$$

$$= |a_1| + |a_2| + \dots + |a_N| - (|a_1 \cap a_2| + |a_1 \cap a_3| + \dots +$$

$$+ |a_{N-1} \cap a_N|) + \dots =$$

$$= \sum_{k=1}^{N} (-1)^{k+1} \left(\sum_{1 \le i_1 < i_2 < \dots < i_k \le m} |a_{i_1} \cap a_{i_2} \cap \dots \cap a_{i_k}| \right)$$

Due to symmetry

$$|a_1| = |a_2| = \dots = |a_N|$$

$$|a_1 \cap a_2| = |a_1 \cap a_3| = \dots = |a_{N-1} \cap a_N|$$

$$\vdots$$

$$N_b = C_N^1 |a_1| - C_N^2 |a_1 \cap a_2| + \dots +$$

$$+ (-1)^{N+1} C_N^N |a_1 \cap a_2 \cap \dots \cap a_N|.$$

$|a_1| = (N-1)!$ (the first person gets its own hat back, therefore there are $(N-1)!$ possible permutations of the other hats)

$$\begin{array}{ccc} p_1 & \longrightarrow & h_1 \\ \left.\begin{array}{c} p_2 \\ \vdots \\ p_N \end{array}\right\} & \xrightarrow{?} & \left.\begin{array}{c} h_2 \\ \vdots \\ h_N \end{array}\right\} \Rightarrow (N-1)! \text{ permutations.} \\ \downarrow & & \downarrow \\ N \text{ persons} & & N \text{ hats.} \end{array}$$

$|a_1 \cap a_2| = (N-2)!$.

$$\begin{array}{ccc} p_1 & \longrightarrow & h_1 \\ p_2 & \longrightarrow & h_2 \\ \left.\begin{array}{c} p_3 \\ \vdots \\ p_N \end{array}\right\} & \xrightarrow{?} & \left.\begin{array}{c} h_3 \\ \vdots \\ h_N \end{array}\right\} \Rightarrow (N-2)! \text{ permutations.} \end{array}$$

$\Rightarrow |a_1 \cap a_2 \cap \dots a_i| = (N-i)!$ where $i = \overline{1,N}$

$$N_b = C_N^1 \cdot (N-1)! - C_N^2 (N-2)! + \dots + (-1)^{N+1} C_N^N (0!) =$$

$$= \sum_{i=1}^{N} (-1)^{i+1} C_N^i (N-i)! =$$

$$= \sum_{i=1}^{N} (-1)^{i+1} \frac{N!}{i! \cdot \cancelto{1}{(N-i)!}} \cdot \cancelto{1}{(N-i)!} =$$

$$= N! \cdot \sum_{i=1}^{N} (-1)^{i+1} \cdot \frac{1}{i!} =$$

$$P_N = \frac{N_f}{N_t} = \frac{N! - N_b}{N!} = 1 - \frac{N! \cdot \sum_{i=1}^{N} (-1)^{i+1} \cdot \frac{1}{i!}}{N!} =$$

$$= 1 - \sum_{i=1}^{N} (-1)^{i+1} \cdot \frac{1}{i!} = \sum_{i=0}^{N} (-1)^{i} \cdot \frac{1}{i!}$$

When $N \to \infty$, P_N tends to $\frac{1}{e}$.

9) We note with

A - the event where the first block has an error

B - the event where the second block has an error.

$P(A) = 0.2$

$P(B) = 0.3$

the probability that the program returns an error ~~(scribble)~~ is $P(A \cup B)$ (either there is an error in the first block, ~~either~~ or in the second one).

the probability that there is an error in both blocks is $P(A \cap B)$ (there is an error in the first block and in the second one).

therefore we need to compute

$$P = \frac{P(A \cap B)}{P(A \cup B)}.$$

as A and B are mutually exclusive $P(A \cap B) = P(A) \cdot P(B)$

$$P(A \cap B) = 0{,}2 \cdot 0{,}3 = 0{,}06$$

from the principle of inclusion and exclusion we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) =$$

$$= 0{,}2 + 0{,}3 - 0{,}06 =$$

$$= 0{,}44$$

$$\Rightarrow P = \frac{P(A \cap B)}{P(A \cup B)} = \frac{0{,}06}{0{,}44} \simeq 0{,}1364$$