

BONUS PROBLEMS

Seminar 2

8) N people; $P = \frac{N_f}{N_t}$

the N hats can be redistributed in $N!$ ways.

$$\Rightarrow N_t = N!$$

N_f is the number of ways to redistribute the hats so that no person gets its own hat back.

to compute N_f , we ^{first} count the permutations where at least one person gets its hat back (formally, the permutations σ where $\exists i \in \overline{1, \dots, N}$ such that $\sigma(i) = i$), let it be denoted N_b .

$$\Rightarrow \cancel{N_f} \neq \cancel{N_b} \quad N_f = N! - N_b$$

Denote a_i all the permutations where the condition $\sigma(i) = i$ is satisfied.

$$\begin{aligned}
 N_b &= |a_1 \cup a_2 \cup \dots \cup a_N| = \\
 &= |a_1| + |a_2| + \dots + |a_N| - (|a_1 \cap a_2| + |a_1 \cap a_3| + \dots + \\
 &\quad + |a_{N-1} \cap a_N|) + \dots = \\
 &= \sum_{k=1}^N (-1)^{k+1} \left(\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq N} |a_{i_1} \cap a_{i_2} \cap \dots \cap a_{i_k}| \right)
 \end{aligned}$$

Due to symmetry

$$|a_1| = |a_2| = \dots = |a_N|$$

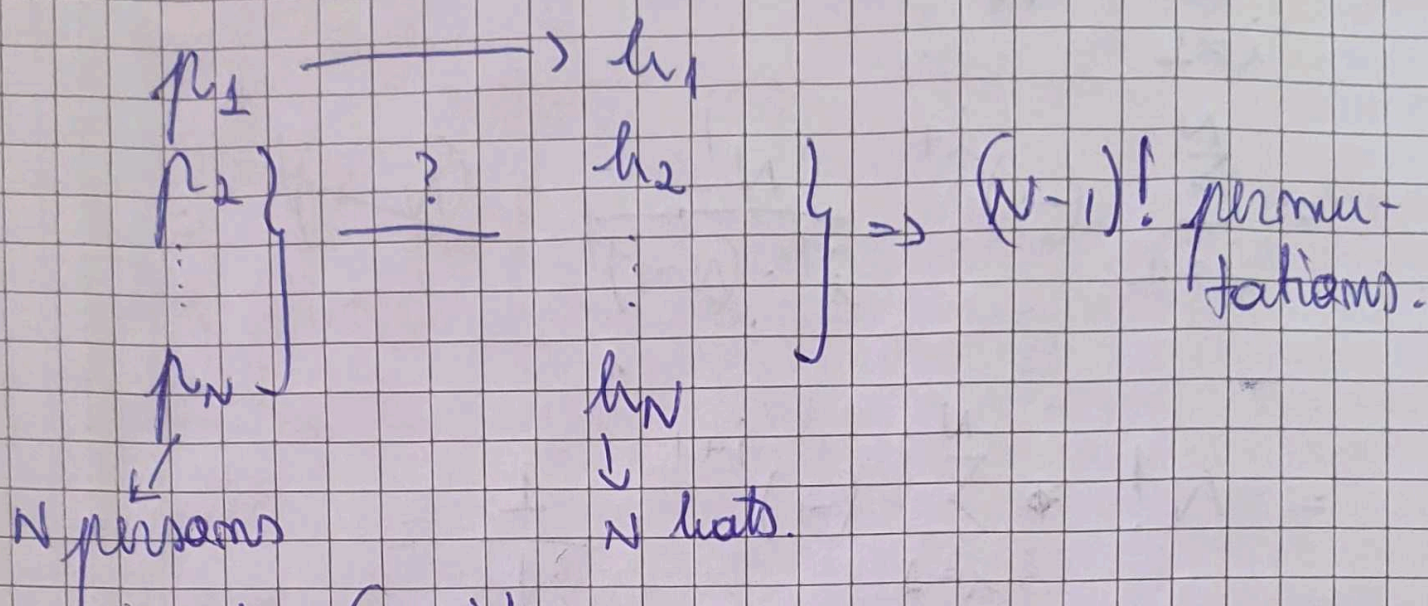
$$|a_1 \cap a_2| = |a_1 \cap a_3| = \dots = |a_{N-1} \cap a_N|$$

⋮

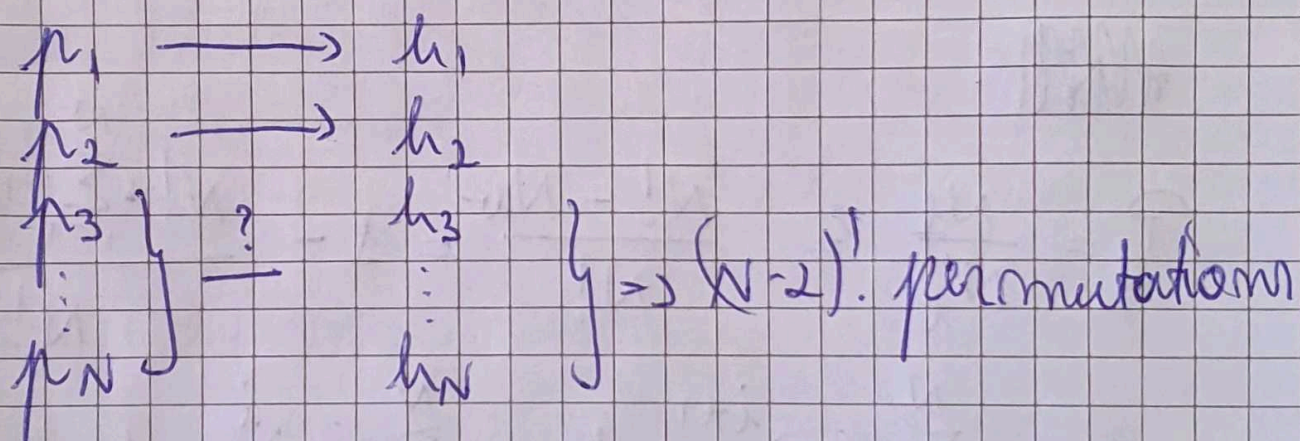
$$N_b = C_N^1 |a_1| - C_N^2 |a_1 \cap a_2| + \dots +$$

$$+ (-1)^{N+1} C_N^N |a_1 \cap a_2 \cap \dots \cap a_N|.$$

$|a_1| = (N-1)!$ (the first person gets its own hat back, therefore there are $(N-1)!$ possible permutations of the other hats)



$$|a_1 \cap a_2| = (N-2)!$$



$$\Rightarrow |a_1 \cap a_2 \cap \dots \cap a_i| = (N-i)! \quad \text{where } i = \overline{1, N}$$

$$N_b = C_N^1 \cdot (N-1)! - C_N^2 (N-2)! + \dots + (-1)^{N+1} C_N^N (0!) =$$

$$= \sum_{i=1}^N (-1)^{i+1} C_N^i (N-i)!, =$$

$$= \sum_{i=1}^N (-1)^{i+1} \frac{N!}{i! \cdot \cancel{(N-i)!}} \cdot \cancel{(N-i)!}^1 =$$

$$= N! \cdot \sum_{i=1}^N (-1)^{i+1} \cdot \frac{1}{i!} =$$

~~When~~

$$P_N = \frac{N_f}{N_t} = \frac{N! - N_b}{N!} = 1 - \frac{N! \cdot \sum_{i=1}^N (-1)^{i+1} \cdot \frac{1}{i!}}{N!} =$$

$$= 1 - \sum_{i=1}^N (-1)^{i+1} \cdot \frac{1}{i!} = \sum_{i=0}^N (-1)^i \cdot \frac{1}{i!}$$

When $N \rightarrow \infty$, P_N tends to $\frac{1}{e}$.

g) We note with

A - the event where the first block has an error

B - the event where the second block has an error.

$$P(A) = 0.2$$

$$P(B) = 0.3.$$

the probability that the program returns an error ~~is~~ is $P(A \cup B)$ (either there is an error in the first block, ~~either~~ ^{or} in the second one).

the probability that there is an error in both blocks is $P(A \cap B)$ (there is an error in the first block and in the second one).

therefore we need to compute

$$P = \frac{P(A \cap B)}{P(A \cup B)}.$$

as A and B are mutually exclusive $P(A \cap B) = P(A) \cdot P(B)$

$$P(A \cap B) = 0,2 \cdot 0,3 = 0,06$$

from the principle of inclusion and exclusion we have

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) = \\ &= 0,2 + 0,3 - 0,06 = \\ &= 0,44 \end{aligned}$$

$$\Rightarrow P = \frac{P(A \cap B)}{P(A \cup B)} = \frac{0,06}{0,44} \approx 0,1364$$