Multiphase Flows – WS 2021/22
Problem Session 2 - **Solution**: Multiphase Shock Tube

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Problem Session 2: Multiphase Shock Tube

Problem 1: 4-Equation Model

The 1-D single-phase Euler system is written as

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0$$

$$\text{with } \mathbf{W} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} \text{ and } \mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}$$

In the following, a 4-equation model describing two phases of vapor ("v") and liquid ("l") shall be used to describe a two-phase problem.

a) *Extend the 3-equation single-phase Euler system by an additional conservative equation for the vapor mass density ρY_{v}. Include a source term for inter-phase mass transfer.*

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Problem 1: Mixture Stiffened-Gas Equation of State

In the 4-equation model, only mixture momentum and energy equations are solved. This is possible based on the equilibrium assumptions $u_\mathrm{v} = u_\mathrm{l} = u$, $p_\mathrm{v} = p_\mathrm{l} = p$, *and* $T_\mathrm{v} = T_\mathrm{l} = T$.

b) *Derive an expression* $T = T(p, \rho, Y_\mathrm{v})$.
(Hint: Start from the expression for mixture specific volume $v = Y_\mathrm{v} v_\mathrm{v} + Y_\mathrm{l} v_\mathrm{l}$ *and use the single-phase stiffened-gas EOS for vapor and liquid.)*

c) *Derive an expression* $p = p(\rho, e, Y_\mathrm{v})$.
(Hint: Use the expression for temperature derived in (a) and the expression for mixture internal energy $e = Y_\mathrm{v} e_\mathrm{v} + Y_\mathrm{l} e_\mathrm{l}$*.)*

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Problem 1: Solution

a) Additional conservation equation needed

Conserved quantity: ρY_v

$$=> \quad \frac{\partial \rho Y_\mathrm{v}}{\partial t} + \frac{\partial \rho Y_\mathrm{v} u}{\partial x} = \dot{m}$$

Mass transfer from liquid to vapor phase (assumed to be zero for Problem 2 of Problem Session 2)

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Problem 1: Solution

b) Equilibrium assumptions: $p_\mathrm{v} = p_\mathrm{l} = p,\; T_\mathrm{v} = T_\mathrm{l} = T,\; u_\mathrm{v} = u_\mathrm{l} = u$

Needed: $T = T(p, \rho, Y_\mathrm{v})$

Specific volume (mixture): $v = Y_\mathrm{v} v_\mathrm{v} + Y_\mathrm{l} v_\mathrm{l} = \frac{Y_\mathrm{v}}{\rho_\mathrm{v}} + \frac{Y_\mathrm{l}}{\rho_\mathrm{l}} = \frac{1}{\rho}$

$$\rho_k(p, T) = \frac{p + p_{\infty,k}}{c_k(\gamma_k - 1)T}$$

Solve for T (with $Y_\mathrm{l} = 1 - Y_\mathrm{v}$):

$$T(p, \rho, Y_\mathrm{v}) = \frac{1}{\rho\left(Y_\mathrm{v}\frac{(\gamma_\mathrm{v} - 1)c_\mathrm{v}}{p + p_{\infty,\mathrm{v}}} + (1 - Y_\mathrm{v})\frac{(\gamma_\mathrm{l} - 1)c_\mathrm{l}}{p + p_{\infty,\mathrm{l}}}\right)}$$

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Problem 1: Solution

c) Needed: $p = p(\rho, e, Y_v)$

Mixture internal energy: $e = Y_\mathrm{v} e_\mathrm{v} + Y_\mathrm{l} e_\mathrm{l}$

Insert expression for T from (b)

$$e_k(p,T) = \frac{(p + \gamma_k p_{\infty,k}) c_k T}{(p + p_{\infty,k})} + q_k$$

➢ Quadratic equation in p – to be solved for p:

$$p(\rho, e, Y_\mathrm{v}, Y_\mathrm{l}) = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad (\text{with } Y_\mathrm{l} = 1 - Y_\mathrm{v})$$

$$A = Y_\mathrm{v} c_\mathrm{v} + Y_\mathrm{l} c_\mathrm{l}$$

$$\begin{aligned} B = {} & \rho[Y_\mathrm{v}(\gamma_\mathrm{v} - 1) c_\mathrm{v} + Y_\mathrm{l}(\gamma_\mathrm{l} - 1) c_\mathrm{l}](-e + Y_\mathrm{v} q_\mathrm{v} + Y_\mathrm{l} q_\mathrm{l}) \\ & + Y_\mathrm{v} c_\mathrm{v} \left(p_{\infty,\mathrm{l}} + \gamma_\mathrm{v} p_{\infty,\mathrm{v}}\right) + Y_\mathrm{l} c_\mathrm{l} \left(p_{\infty,\mathrm{v}} + \gamma_\mathrm{l} p_{\infty,\mathrm{l}}\right) \end{aligned}$$

$$\begin{aligned} C = {} & \rho\left[Y_\mathrm{v}(\gamma_\mathrm{v} - 1) c_\mathrm{v} p_{\infty,\mathrm{l}} + Y_\mathrm{l}(\gamma_\mathrm{l} - 1) c_\mathrm{l} p_{\infty,\mathrm{v}}\right](-e + Y_\mathrm{v} q_\mathrm{v} + Y_\mathrm{l} q_\mathrm{l}) \\ & + Y_\mathrm{v} c_\mathrm{v} \gamma_\mathrm{v} p_{\infty,\mathrm{v}} p_{\infty,\mathrm{l}} + Y_\mathrm{l} c_\mathrm{l} \gamma_\mathrm{l} p_{\infty,\mathrm{v}} p_{\infty,\mathrm{l}} \end{aligned}$$

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Problem 2: Multiphase Shock Tube (in MATLAB)

The solution to the Matlab implementation of the 2-Phase shock tube problem of Problem Session 2 is equivalent to the template provided for solution of Problem Session 3.

Thank you for your attention

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