Multiphase Flows – WS 2022/23
Problem Session 9 - **Solution**:
Linear Momentum Coupling

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Agenda

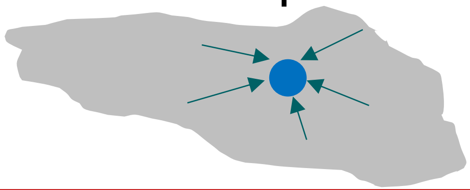
- Problem Session 8: Single droplet heat & mass transfer
- Problem Session 9 (today): Linear momentum coupling (single droplet in air)
- Problem Session 10: Statistical methods

Problem Session 9: Linear Momentum Coupling

Phase Coupling Approaches for Dispersed Flows

- One-way coupling: Only impact of continuous flow field on particle

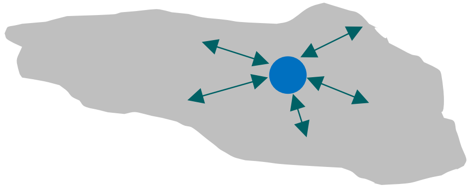
➢ Multiple particles do not "see" each other



Today

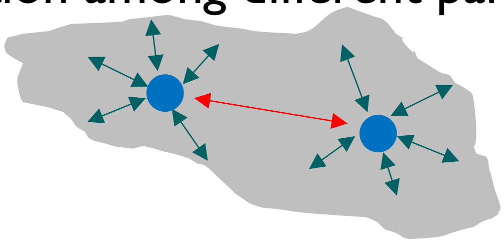
- Two-way coupling: Mutual impact of flow field and particle on each other

➢ In addition to one way coupling: Particle also influences surrounding flow field



- Four-way coupling: Mutual interaction of flow field and different particles

➢ In addition to two-way coupling: also interaction among different particles



Problem Session 9: Linear Momentum Coupling

Recap: BBO Equation

$$m\frac{dv}{dt} = \boxed{3\pi f_1 \mu_c D(u - v)}$$ Drag Force

$$+\boxed{V_d(-\nabla p + \nabla \bullet \tau)}$$ Surface Forces

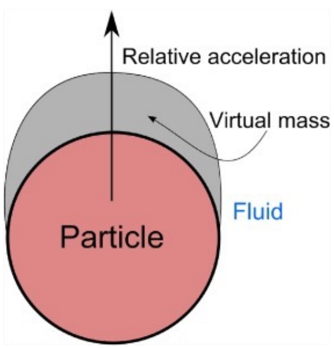
$$+\boxed{\frac{\rho_c V_d}{2}(\dot{u} - \dot{v})}$$ Virtual Mass Force

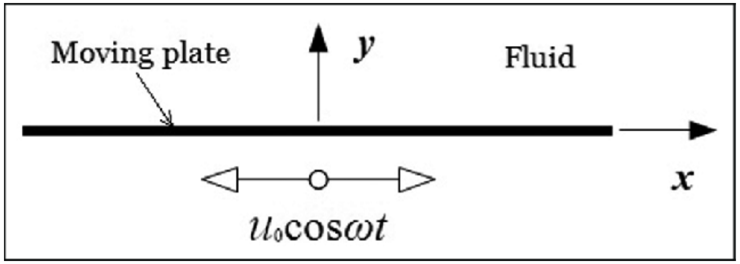
$$+\boxed{\frac{3}{2}D^2\sqrt{\pi\rho_c\mu_c}\left[\int_0^t \frac{\dot{u}-\dot{v}}{\sqrt{t-t'}}dt' + \frac{(u-v)_0}{\sqrt{t}}\right]}$$ History Force

$$+\boxed{mg}$$ Body Force

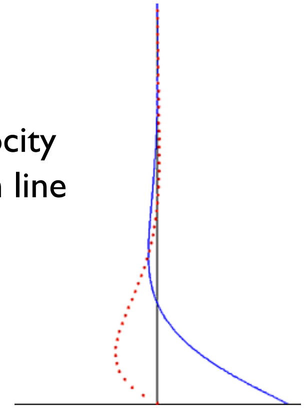
$$F_D = \frac{1}{2}\rho v^2 C_D A$$

$$C_D = f_1 \frac{24}{Re}.$$





-- - velocity
... - path line



Problem Session 9: Linear Momentum Coupling

Re-arranging the terms

- Recap from the lecture: BBO equation for isolated particle

$$\left(1+\frac{1}{2}\frac{\rho_\mathrm{c}}{\rho_\mathrm{d}}\right)\frac{\mathrm{d}\mathbf{u}_\mathrm{d}}{\mathrm{d}t}=\frac{f_1}{\tau_v}(\mathbf{u}_\mathrm{c}-\mathbf{u}_\mathrm{d})+\frac{3}{2}\frac{\rho_\mathrm{c}}{\rho_\mathrm{d}}\dot{\mathbf{u}}_\mathrm{c}+\mathbf{g}\left(1-\frac{\rho_\mathrm{c}}{\rho_\mathrm{d}}\right)$$
$$+\sqrt{\frac{9}{2\pi}\frac{\rho_\mathrm{c}}{\rho_\mathrm{d}}\frac{1}{\tau_v}}\left[\int_0^t\frac{\dot{\mathbf{u}}_\mathrm{c}-\dot{\mathbf{u}}_d}{\sqrt{t-t'}}\mathrm{d}t'+\frac{(\mathbf{u}_\mathrm{c}-\mathbf{u}_\mathrm{d})_0}{\sqrt{t}}\right]$$

- For $\rho_c \ll \rho_d$: unsteady terms in BBO equation negligible
- Only steady-state drag and body forces (gravity…) remaining

$$\frac{\mathrm{d}\mathbf{u}_\mathrm{d}}{\mathrm{d}t}=\frac{f_1}{\tau_v}(\mathbf{u}_\mathrm{c}-\mathbf{u}_\mathrm{d})+\mathbf{g}$$

- In the following problems: also neglection of body forces

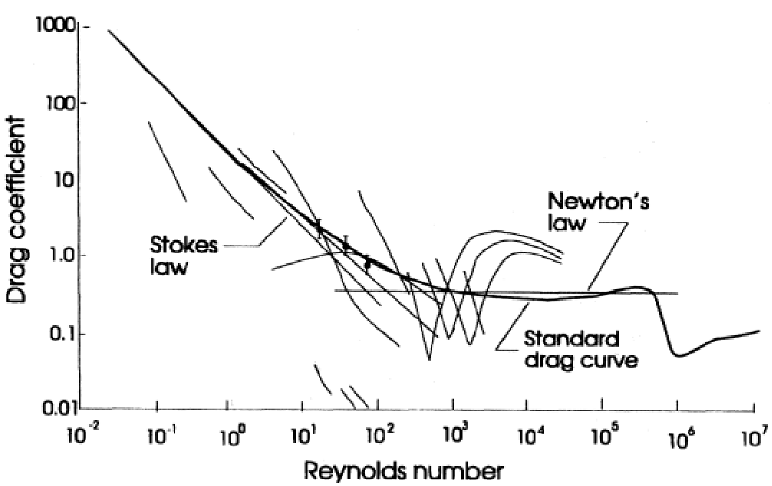
Problem Session 9: Linear Momentum Coupling

Problem 1: Particle Response Time

The drag term in the simplified equation $\frac{\mathrm{d}\mathbf{u}_\mathrm{d}}{\mathrm{d}t} = \frac{f_1}{\tau_v}(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d})$ *contains the particle response time* τ_v*. It can be regarded as the time scale of momentum transfer between the continuous phase and the drop.*

Derive an expression for τ_v*, assuming the drag coefficient to be given as* $C_\mathrm{D} = f_1 \frac{24}{\mathrm{Re}}$.

The drag force is $F_\mathrm{D} = \frac{1}{2}\rho_\mathrm{c} C_\mathrm{D} A |\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d}|(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d})$.



Problem Session 9: Linear Momentum Coupling

Problem 1: Solution

Drag force: $\mathbf{F}_D = \frac{1}{2}\rho_\mathrm{c} C_\mathrm{D} A|\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d}|(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d}) = \frac{1}{2}\rho_\mathrm{c} f_1 \frac{24}{Re} A|\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d}|(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d})$

$$= \frac{1}{2}\rho_\mathrm{c} f_1 \frac{24\nu_c}{|\mathbf{u}_c - \mathbf{u}_d|D} A|\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d}|(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d}) = 3\pi f_1 \mu_\mathrm{c} D(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d})$$

Momentum equation: $m_\mathrm{d}\frac{d\mathbf{u}_\mathrm{d}}{dt} = \mathbf{F}_\mathrm{D}$

$$=> \frac{d\mathbf{u}_\mathrm{d}}{dt} = \frac{\mathbf{F}_\mathrm{D}}{\rho_\mathrm{d}\frac{\pi D^3}{6}} = \frac{3\pi f_1 \mu_\mathrm{c} D(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d})}{\rho_\mathrm{d}\frac{\pi D^3}{6}} = \frac{18 f_1 \mu_\mathrm{c}(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d})}{\rho_\mathrm{d} D^2}$$

$$=> \frac{d\mathbf{u}_\mathrm{d}}{dt} = \frac{f_1}{\tau_v}(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d}) \quad \text{with} \quad \tau_v = \frac{\rho_\mathrm{d} D^2}{18\mu_\mathrm{c}}$$

Problem 2: Vortex Flow Field (in MATLAB)

We will later simulate the movement of the drop in a given 2D flow field. The continuous air flow field shall consist of steady-state rotating vortices such that the velocity field can be described in an analytic form in Cartesian coordinates.

The velocity field is given here as

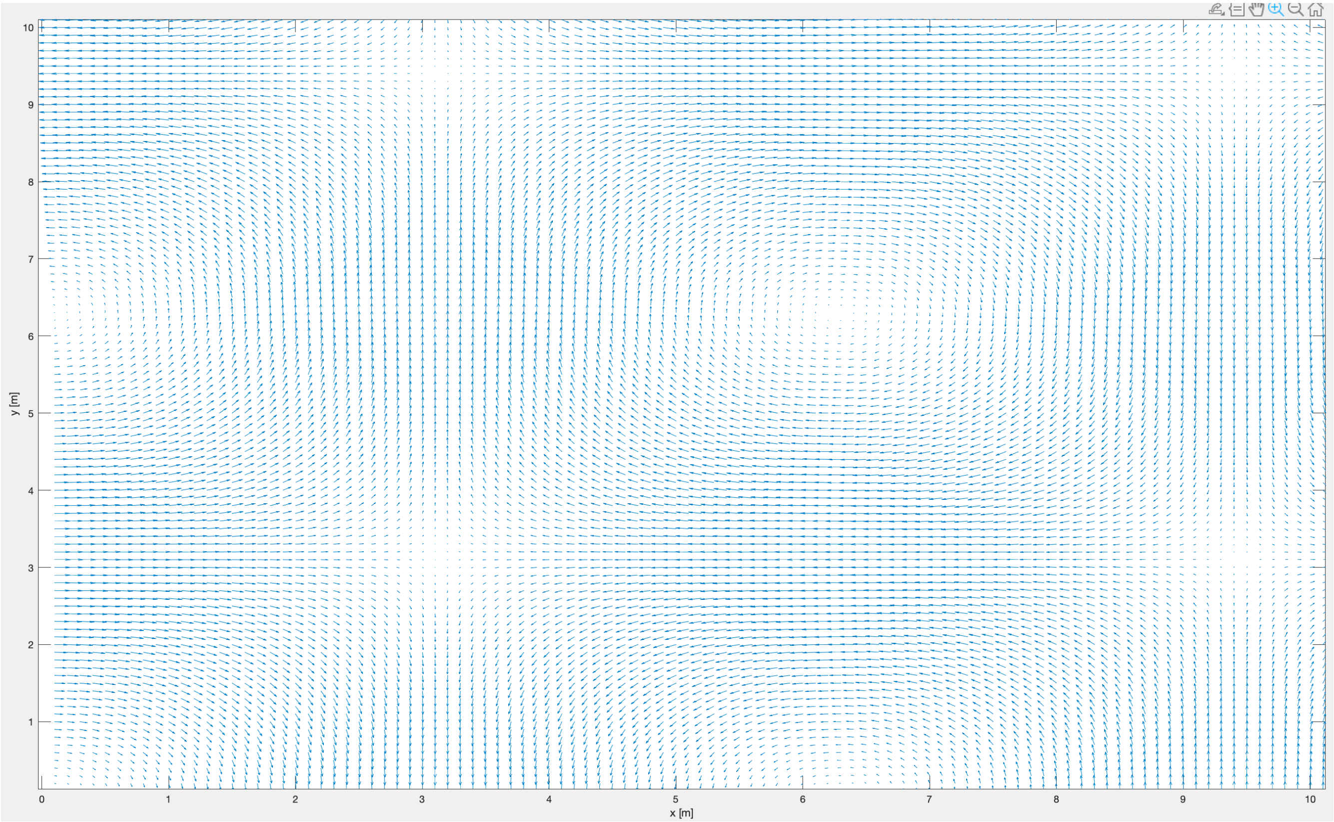
$$u_{\mathrm{c}}(x, y, t) = A\cos(x/L)\sin(y/L)$$
$$v_{\mathrm{c}}(x, y, t) = -A\sin(x/L)\cos(y/L)$$

where A is a constant specifying the velocity magnitude and L is a length scale of the flow field.

Visualize this flow field for $A = 50\frac{m}{s}$ and $L = 2m$!

Problem Session 9: Linear Momentum Coupling

Problem 2: Solution



The complete code has been uploaded to Moodle.

Problem 3: Particle Trajectory (in MATLAB)

A small water drop is inserted in the vortex flow field of air.

Compute and visualize the particle trajectory $\mathbf{x}_\mathrm{p}(t)$ using the equation of motion $\frac{\mathrm{d}\mathbf{u}_\mathrm{d}}{\mathrm{d}t} = \frac{f_1}{\tau_v}(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d})$, where the particle response time was derived in Problem 1. Mass transfer is neglected. Evaluate the impact of the different initial conditions and fluid properties on the particle trajectory by modifying them.

Hint: Set up a system of 1st order ODEs and use one of MATLAB's built-in ODE solvers (e.g., ode45).

Given: $\rho_\mathrm{d} = 1000\,\frac{kg}{m^3}, \mu_\mathrm{c} = 1.846 \cdot 10^{-5}\,\frac{kg}{m \cdot s}, D = 1mm, \rho_\mathrm{c} = 1.2\,\frac{kg}{m^3}$

Initial conditions: $x_\mathrm{d}^0 = 4m,\ y_\mathrm{d}^0 = 6m,\ u_\mathrm{d}^0 = v_\mathrm{d}^0 = 0\,\frac{m}{s}$

$$f_1 = \begin{cases} 1, & \mathrm{Re} < 1 \\ 1 + 0.15\mathrm{Re}^{0.687}, & 1 \leq \mathrm{Re} < 800 \\ 1 + 0.15\mathrm{Re}^{0.687} + \dfrac{0.0175\mathrm{Re}}{1 + 42500\mathrm{Re}^{-1.16}}, & 800 \leq \mathrm{Re} < 2 \cdot 10^5 \end{cases}$$

Problem 3: Solution

$$\frac{d\mathbf{u}_\mathrm{d}}{dt} = \frac{f_1}{\tau_v}(\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d}) \quad \text{with} \quad \tau_v = \frac{\rho_\mathrm{d} D^2}{18\mu_\mathrm{c}} = 3.01\ \mathrm{s}$$

$$Re = \frac{\rho_\mathrm{c}|\mathbf{u}_\mathrm{c} - \mathbf{u}_\mathrm{d}|D}{\mu_\mathrm{c}} = \frac{\rho_\mathrm{c}\sqrt{(u_\mathrm{c} - u_\mathrm{d})^2 + (v_\mathrm{c} - v_\mathrm{d})^2}D}{\mu_\mathrm{c}} \quad => \quad f_1 = f(Re)$$

System of 1st order ODEs:

$$\frac{du_\mathrm{d}}{dt} = \frac{f_1(u_\mathrm{d}, v_\mathrm{d})}{\tau_v}(u_\mathrm{c} - u_\mathrm{d})$$

$$\frac{dv_\mathrm{d}}{dt} = \frac{f_1(u_\mathrm{d}, v_\mathrm{d})}{\tau_v}(v_\mathrm{c} - v_\mathrm{d})$$

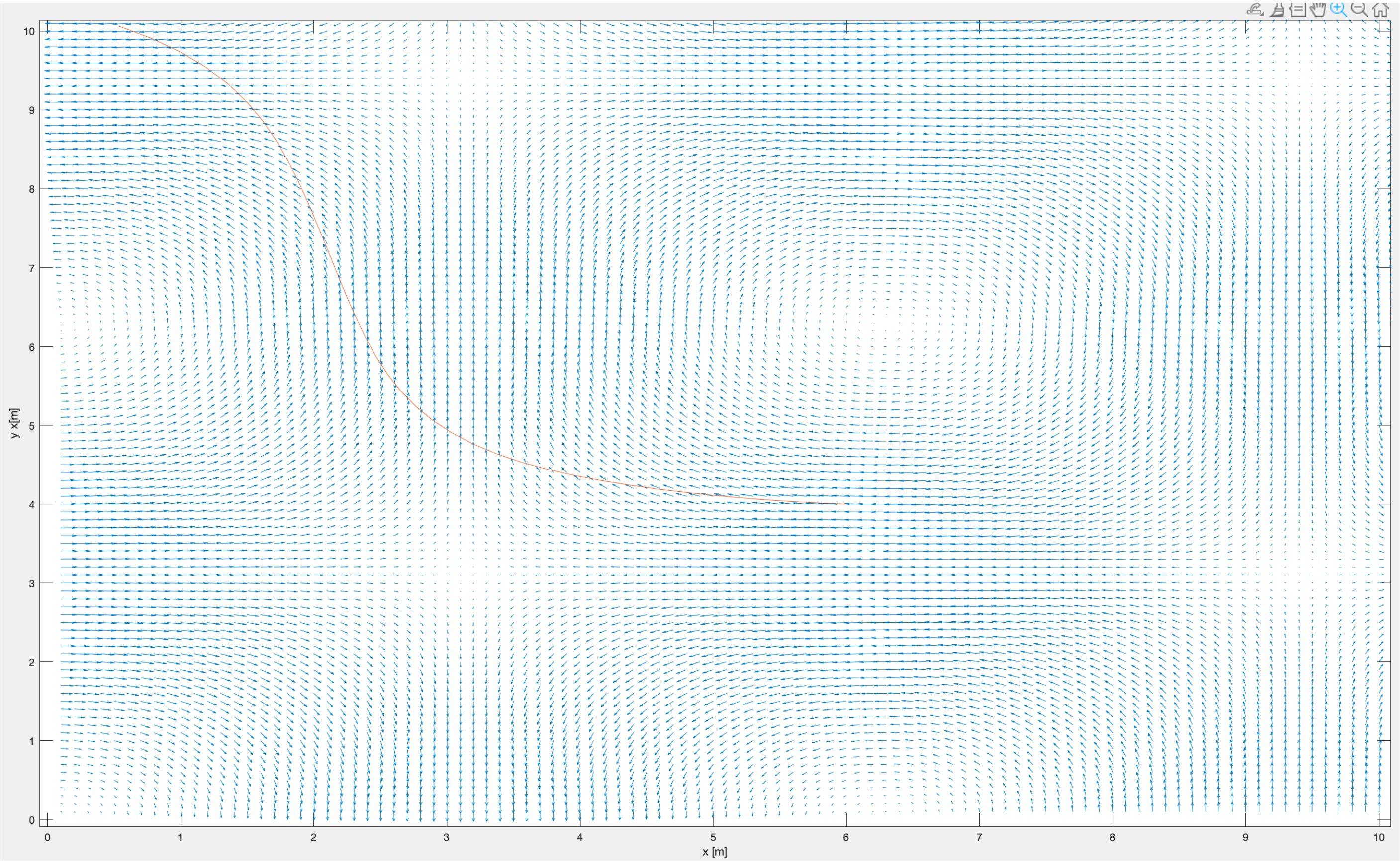
$$\frac{dx_\mathrm{d}}{dt} = u_\mathrm{d}$$

$$\frac{dy_\mathrm{d}}{dt} = v_\mathrm{d}$$

The complete code has been uploaded to Moodle.

Problem Session 9: Linear Momentum Coupling

Problem 3: Solution



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Problem 4 (EXTRA TASK): Inclusion of Drop Evaporation

The water drop of Problem 3 now evaporates in the flow field. This process shall be described here by the very simple D^2 law that you know from the last Problem Session

$$D^2 = D_0^2 - \lambda t,$$

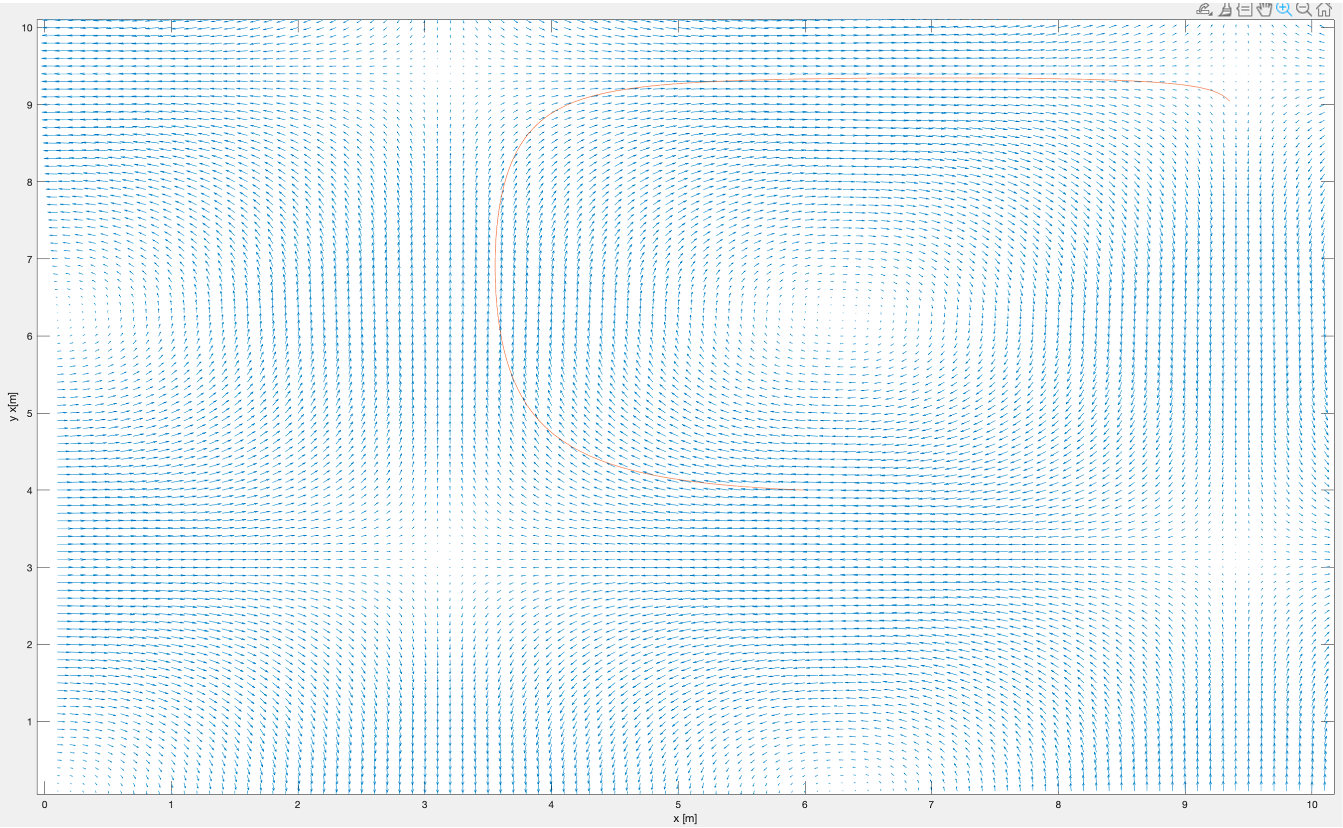
where $\lambda = 7 \cdot 10^{-10}\,\frac{\mathrm{m}^2}{\mathrm{s}}$ is assumed.

The impact of the relative velocity between droplet and surrounding flow on the evaporation rate is neglected. The impact of the Stefan flow, which corresponds to the droplet evaporation, on the particle motion is neglected as well.

Compute and visualize the new particle trajectory. Does the inclusion of evaporation make a significant difference? How does this change with smaller initial droplet diameters? Is the D^2 law adequate for this problem?

Problem Session 9: Linear Momentum Coupling

Problem 4: Solution



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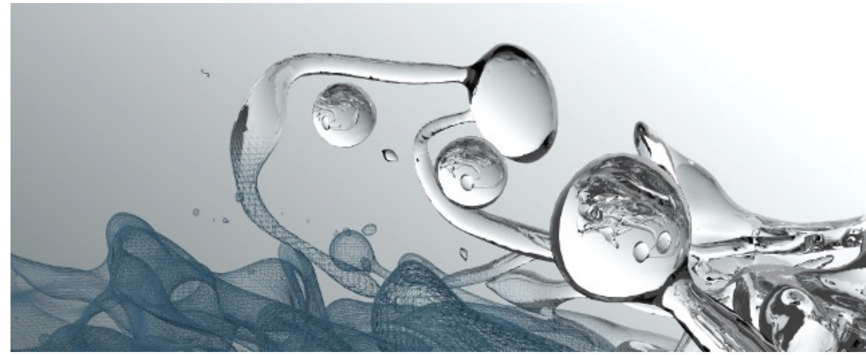

Thank you for your attention

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