Multiphase Flows – WS 2022/23
Problem Session 8 – **Solution**: Single Droplet Heat & Mass Transfer

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Problem Session 8: Single Droplet Heat & Mass Transfer

Problem 1: D^2 Law for Droplet Evaporation

The mass balance for a single evaporating droplet can be written as $\frac{\mathrm{d}m}{\mathrm{d}t} = -k_m A H_m$*, where* k_m *is the mass transport coefficient, A is the surface area of the droplet, and* H_m *is the driving potential for mass transfer.*

a) *From this general equation, derive the ODE for the change of the droplet diameter D over time. Assume the droplet to be spherical. The Sherwood number* $Sh = \frac{k_m D}{\rho_c D_{cv}}$ *be known. Insert the difference of the vapor mass fractions between the droplet surface and the freestream as driving potential.*
Given: $Sh, \rho_c, \rho_d, D_{cv}, Y_{v,\infty}, M_v, M_M, p, T, p_{sat}(T)$

b) *Assuming all relevant fluid properties and the temperature to be constant, derive the D^2 law for droplet evaporation by integrating the ODE.*

Problem 1a : Solution

The mass balance for a single evaporating droplet can be written as $\frac{\mathrm{d}m}{\mathrm{d}t} = -k_m A H_m$, *where* k_m *is the mass transport coefficient, A is the surface area of the droplet, and* H_m *is the driving potential for mass transfer.*

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Given: $Sh, \rho_{\mathrm{c}}, \rho_{\mathrm{d}}, D_{\mathrm{cv}}, Y_{\mathrm{v},\infty}, M_{\mathrm{v}}, M_{\mathrm{M}}, p,\ T, p_{\mathrm{sat}}(T)$

Note: $Sh = \frac{Mass\ transfer\ by\ convection}{Mass\ transfer\ by\ diffusion}$, $Nu = \frac{Heat\ transfer\ by\ convection}{Heat\ transfer\ by\ conduction}$, $\rho_c \to medium, \rho_d \to droplet$

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Problem 1a : Solution

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -k_m A H_m$$

A | B | C | D

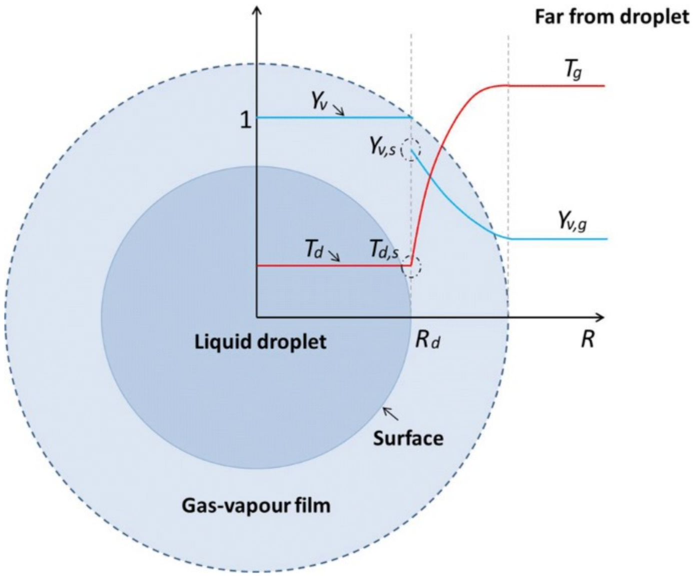
$A: m = \rho_\mathrm{d} V = \rho_\mathrm{d} \frac{\pi D^3}{6}$

$B: Sh = \frac{k_m D}{\rho_\mathrm{c} D_\mathrm{cv}} \Rightarrow \quad k_m = \frac{Sh \cdot \rho_\mathrm{c} D_\mathrm{cv}}{D}$

$C: A = \pi D^2$

E

$D: H_m = Y_\mathrm{v,s} - Y_\mathrm{v,\infty}$



$E: Y_\mathrm{v,s} = \frac{m_\mathrm{v,s}}{m_\mathrm{M,s}} = \frac{M_\mathrm{v}}{M_\mathrm{M}} \frac{n_\mathrm{v,s}}{n_\mathrm{M,s}} = \frac{M_\mathrm{v}}{M_\mathrm{M}} \frac{p_\mathrm{v,s}}{p} = \frac{M_\mathrm{v}}{M_\mathrm{M}} \frac{p_\mathrm{sat}(T)}{p}$

Problem 1 a: Solution

$$\frac{dm}{dt} = \frac{\pi}{6}\rho_\mathrm{d}\frac{dD^3}{dt} = \frac{\pi}{2}\rho_\mathrm{d}D^2\frac{dD}{dt}$$

$$-k_m\,A\,H_m = -\frac{Sh\,\rho_\mathrm{c}\,D_\mathrm{cv}}{D}\pi D^2\left(Y_\mathrm{v,s} - Y_\mathrm{v,\infty}\right)$$

$$=> \frac{dD}{dt} = -\frac{2\cdot Sh\cdot\rho_\mathrm{c}D_\mathrm{cv}}{\rho_d D}\left(Y_\mathrm{v,s} - Y_\mathrm{v,\infty}\right) = -\frac{1}{D}\frac{\lambda}{2}$$

Problem 1 b: Solution

b) $D dD = -\frac{\lambda}{2} dt \quad => \quad \int_{D_0}^{D} D dD = -\frac{\lambda}{2} \int_0^t dt \quad => \boxed{D^2 = D_0^2 - \lambda t}$

D^2 law

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Problem 2: Wet-Bulb Temperature

The energy equation for an evaporating water droplet can be expressed as

$$mc_p \frac{\mathrm{d}T_\mathrm{d}}{\mathrm{d}t} = Nu \cdot \pi D \lambda_\mathrm{c}(T_\infty - T_d) - Sh \cdot \pi \rho_\mathrm{c} D_\mathrm{cv} D\left(Y_{\mathrm{v,s}} - Y_{\mathrm{v},\infty}\right)L,$$

where L is the latent heat of water. The cooling effect of the evaporation dilutes the heating of a droplet in warmer air. Once the droplet reaches the so-called wet-bulb temperature at saturation conditions, it does not heat further until it completely evaporates.

Compute the wet-bulb temperature of a water droplet in air at $1\ \mathrm{bar}$ and $293\ \mathrm{K}$, and at a relative humidity of $40\ \%$. Assume $Nu = Sh$, $Pr = 0.76$, and $Sc = 0.65$.

Further: $c_{p,\mathrm{c}} = 1.005\ \frac{\mathrm{kJ}}{\mathrm{kgK}}$, $L = 2454\ \mathrm{kJ/kg}$, $M_{H_2O} = 18\ \mathrm{g/mol}$, $M_M = 29\ \mathrm{g/mol}$.

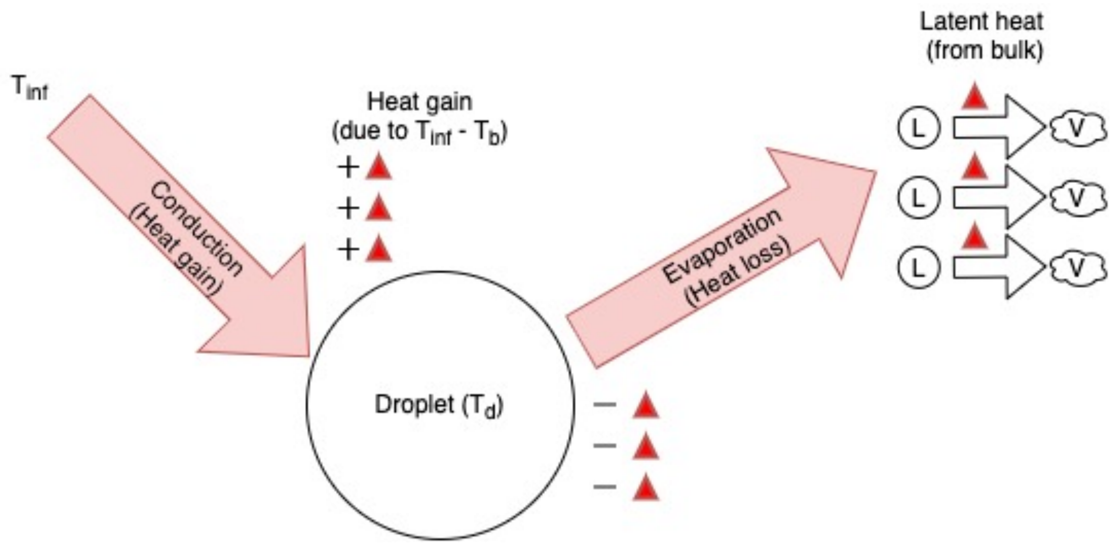
T [K]	280	285	290	293
p_sat [bar]	0.009912	0.01388	0.01919	0.02239

$$\mathrm{Pr} = \frac{\mu c_{p,c}}{\lambda_c}, \quad \mathrm{Sc} = \frac{\mu}{\rho_c D_{cv}}$$

Problem 2: Solution

$$mc_p \frac{\mathrm{d}T_\mathrm{d}}{\mathrm{d}t} = Nu \cdot \pi D \lambda_\mathrm{c} (T_\infty - T_d) - Sh \cdot \pi \rho_\mathrm{c} D_\mathrm{cv} D \left(Y_\mathrm{v,s} - Y_\mathrm{v,\infty}\right) L$$

The cooling effect of the evaporation dilutes the heating of a droplet in warmer air. Once the droplet reaches the so-called wet-bulb temperature at saturation conditions, it does not heat further until it completely evaporates.



Problem 2: Solution

Wet-bulb temperature: T_d stays constant $\frac{dT_\mathrm{d}}{dt} = 0$

$$=> \quad Nu \cdot \lambda_\mathrm{c}(T_\infty - T_\mathrm{d}) = Sh \cdot \rho_\mathrm{c} D_\mathrm{cv}\left(Y_\mathrm{v,s} - Y_{\mathrm{v},\infty}\right)L$$

$$=> \quad Nu \cdot (T_\infty - T_\mathrm{d}) = Sh \cdot \frac{\mu c_{p,c}}{\lambda_c} \frac{\rho_c D_{cv}}{\mu} \frac{L}{c_{p,c}} \left(Y_\mathrm{v,s} - Y_{\mathrm{v},\infty}\right)$$

$$=> \quad Nu \cdot (T_\infty - T_\mathrm{d}) = Sh \cdot \frac{Pr}{Sc} \frac{L}{c_{p,\mathrm{c}}} \left(Y_\mathrm{v,s} - Y_{\mathrm{v},\infty}\right)$$

$$=> \quad T_\infty = T_\mathrm{d} + \frac{Sh}{Nu} \frac{Pr}{Sc} \frac{L}{c_{p,\mathrm{c}}} \left(Y_\mathrm{v,s} - Y_{\mathrm{v},\infty}\right) \qquad (1)$$

$p_\mathrm{sat}(293\ \mathrm{K}) = 0.02239\ \mathrm{bar}, x_{rel} = \frac{p_{H_2O}}{p^*_{H_2O}}$ (actual to equilibrium partial pressure ratio)

$$=> Y_{\mathrm{v},\infty} = x_\mathrm{rel} \frac{p_\mathrm{sat}(T_\infty)}{p} \frac{M_{\mathrm{H_2O}}}{M_\mathrm{M}} = 0.4 \cdot \frac{0.02239\ \mathrm{bar}}{1\ \mathrm{bar}} \cdot \frac{18\,\frac{\mathrm{g}}{\mathrm{mol}}}{29\,\frac{\mathrm{g}}{\mathrm{mol}}} = 0.00556$$

Problem 2: Solution

$$Y_{\mathrm{v,s}}(T_\mathrm{d}) = \frac{p_{\mathrm{sat}}(T_\mathrm{d})}{p}\frac{M_{\mathrm{H_2O}}}{M_\mathrm{M}} = \frac{p_{\mathrm{sat}}(T_\mathrm{d})}{1\ \mathrm{bar}} \cdot \frac{18}{29}$$

T_d	280	285	290
$Y_{v,s}$	0.00615	0.00862	0.01191

From (1): $T_\mathrm{d} - T_\infty + \frac{Sh}{Nu}\frac{Pr}{Sc}\frac{L}{c_{p,c}}\left(Y_{\mathrm{v,s}} - Y_{\mathrm{v},\infty}\right) = 0$

$$=> T_\mathrm{d} - 293 + \frac{0.76}{0.65} \cdot \frac{2454}{1.005}\left(Y_{\mathrm{v,s}} - 0.00556\right) = \delta$$

T_d	280	285	290
δ	−11.31	0.74	15.13

=> Interpolate: $T_\mathrm{d}^{\mathrm{wet-bulb}} = 280\ \mathrm{K} + \frac{(285-280)\mathrm{K}\cdot(0+11.31)}{0.74+11.31} = 284.7\ \mathrm{K}$

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Problem 3: Droplet Evaporation (in MATLAB)

a) *For the conditions of Problem 2, plot the droplet diameter and the droplet mass over time for initial diameters of $1000\ \mu\text{m}$, $10\ \mu\text{m}$, and $1\ \mu\text{m}$ based on the D^2 law. Check if D^2 plot is a straight line. The Sherwood number be $Sh = 2$.*

Further fluid properties: $\lambda_c = 0.0257\ \frac{\text{W}}{\text{mK}}, \rho_{\text{H}_2\text{O}} = 1000\ \frac{\text{kg}}{\text{m}^3}$.

Assumptions: $Sh = \text{const.}, Pr = \text{const.}, Sc = \text{const.}, \lambda_\text{c} = \text{const.},$
$\rho_\text{d} = \text{const.}, c_{p,\text{c}} = \text{const.}, x_\text{rel} = \text{const.}$

b) *Plot the droplet evaporation time $t_\text{evap}(p, T_\infty)$ over temperature and pressure of the surrounding air for an initial droplet diameter of $1000\ \mu\text{m}$. The air have 10 % relative humidity. Assume that the droplet always has the wet-bulb temperature at the respective conditions. The fluid properties and dimensionless numbers remain constant. Use the stiffened-gas equation of state for computing the saturation pressures and wet-bulb temperatures.*

Problem 3 a: Solution

a) D^2 law: $D^2 = D_0^2 - \lambda t$

1. Plug in Yv_s and Yv_inf from problem 2 (Yv_inf found out, Yv_s from interpolation)
2. Calculate value of lambda for D^2 equation
3. Find final evaporation time (time =0:t_evap)
4. Plug in expressions for diameter and mass

$$Y_{\mathrm{v,s}} = 0.00615 + \frac{(0.00862 - 0.00615)(284.7 - 280)}{285 - 280} = 0.00847$$

$$\lambda = \frac{4 \cdot Sh \cdot Pr \cdot \lambda_c \left(Y_{\mathrm{v,s}} - Y_{\mathrm{v},\infty}\right)}{Sc \cdot \rho_{\mathrm{d}} \cdot c_{p,\mathrm{c}}} = 6.961 \cdot 10^{-10} \, \frac{\mathrm{m}^2}{\mathrm{s}}$$

$$t_{\mathrm{evap}} = \frac{D_0^2}{\lambda(p, T_\infty)}$$

Mass: $m_{\mathrm{d}} = \rho_d \cdot \frac{\pi D^3}{6} = \rho_d \frac{\pi \left(D_0^2 - \lambda t\right)^{\frac{3}{2}}}{6}$

Problem 3 b: Solution

b) $t_{\mathrm{evap}} = \frac{D_0^2}{\lambda(p,T_\infty)}$

1. Plug in expressions for Yv_s and Yv_inf
2. Plug in expression for lambda for D^2 law
3. Plug in value of delta function for bisection method for calculating wet bulb temp.

$$Y_{\mathrm{v},\infty} = x_{\mathrm{rel}} \frac{p_{\mathrm{sat}}(T_\infty)}{p_\infty} \frac{M_{\mathrm{H_2O}}}{M_{\mathrm{M}}}$$

$$Y_{\mathrm{v,s}} = \frac{p_{\mathrm{sat}}(T_{\mathrm{d}})}{p_\infty} \frac{M_{\mathrm{H_2O}}}{M_{\mathrm{M}}}$$

$$\lambda = \frac{4 \cdot Sh \cdot Pr \cdot \lambda_{\mathrm{c}} \left(Y_{\mathrm{v,s}} - Y_{\mathrm{v},\infty}\right)}{Sc \cdot \rho_{\mathrm{d}} c_{p,\mathrm{c}}}$$

$$T_{\mathrm{d}} - T_{inf} + \frac{Sh}{Nu} \cdot \frac{Pr}{Sc} \frac{L}{c_p} \left(Y_{\mathrm{v,s}} - Y_{v,inf}\right) = \delta$$

Thank you for your attention

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