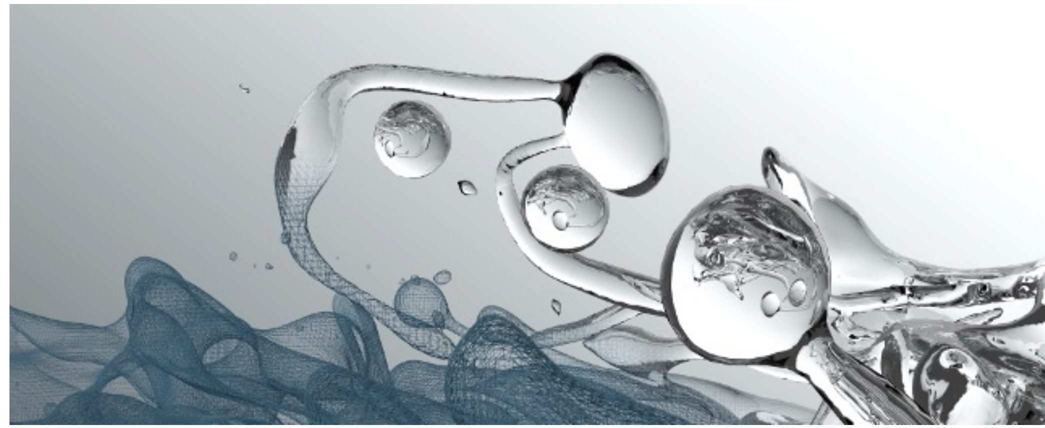
Multiphase Flows – WS 2022/23
Problem Session 11 – **Solution**: Statistical Modeling



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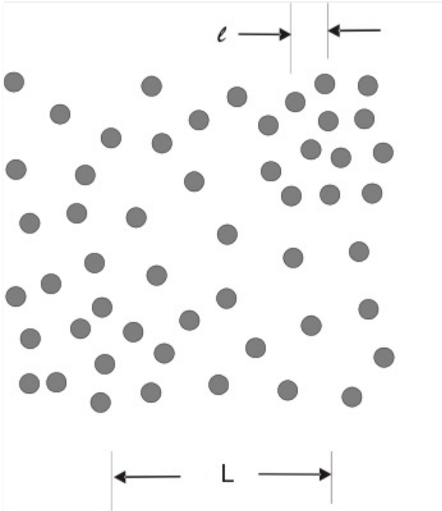
Problem Session 11: Statistical Modeling

Agenda

- Problem Session 10: Applications
- Problem Session 11 (today): Statistical modeling

Volume-Averaged Equations

- Averages of quantity B typically expressed in terms of volume average $\bar{B}$ or phase average $\langle B \rangle$



  - Volume Averaging $$\bar{u} = \frac{1}{V} \int_V u \, dV$$
  - Phase Averaging $$\langle u \rangle(\boldsymbol{x}, t) = \frac{1}{V_c} \int_V u(\boldsymbol{x}, t) dV$$

$$\langle u \rangle(\boldsymbol{x}, t) = \frac{V}{V_c} \frac{1}{V} \int_V u(\boldsymbol{x}, t) \, dV = \frac{1}{\alpha_c} \bar{u}(\boldsymbol{x}, t)$$

[1] Crowe et al., Multiphase Flows with Droplets and Particles, *CRC Press*, 2012.

Problem Session 11: Statistical Modeling

Volume-Averaged Equations

- Averages of quantity B typically expressed in terms of volume average $\bar{B}$ or phase average $\langle B \rangle$

➢ For continuous phase density: $\bar{\rho}_c = \alpha_c \langle \rho_c \rangle$

➢ Volume-averaged gradient operations for quantity B (for detailed derivation: see [1]):

$$\overline{\frac{\partial B}{\partial t}} = \frac{\partial \bar{B}}{\partial t} + \frac{1}{V} \int_{S_d} B(v_i n_i + \dot{r}) dS$$

$$\overline{\frac{\partial B}{\partial x_i}} = \frac{\partial \bar{B}}{\partial x_i} - \frac{1}{V} \int_{S_d} B n_i dS$$

with V: control volume, S_d: surface of dispersed phase in control volume, n_i: normal outward vector of particle surface, r: particle radius

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[1] Crowe et al., Multiphase Flows with Droplets and Particles, *CRC Press*, 2012.

Problem 1: Volume-Averaged Continuity Equation

The general continuity equation for the carrier phase of a two-phase mixture is given as

$$\frac{\partial \rho_c}{\partial t} + \frac{\partial \rho_c u_i}{\partial x_i} = 0.$$

Derive the volume averaged continuity equation of the carrier phase!

(Express in terms of phase averages, and mass averaged velocity)

Problem Session 11: Statistical Modeling

Problem 1: Solution

Local volume average: $\overline{\frac{\partial \rho_c}{\partial t}} + \overline{\frac{\partial \rho_c u_i}{\partial x_i}} = 0$

With gradient operators: $\frac{\partial \overline{\rho_c}}{\partial t} + \frac{1}{V}\int_{S_d} \rho_c (v_i n_i + \dot{r}) dS + \frac{\partial \overline{\rho_c u_i}}{\partial x_i} - \frac{1}{V}\int_{S_d} \rho_c u_i n_i dS = 0$ (1)

At droplet surface – velocity of continuous phase: $u_i = v_i + (\dot{r} + w) n_i$ (2)

Particle velocity

Particle growth

Velocity of medium with respect to surface

Inserting (2) in (1): $\frac{\partial \overline{\rho_c}}{\partial t} + \frac{\partial \overline{\rho_c u_i}}{\partial x_i} = \frac{1}{V}\int_{S_d} \rho_c w dS$

Average mass flux: $\overline{\rho_c u_i} = \frac{1}{V}\int_V \rho_c u_i dV$; mass-aver. velocity: $\langle \rho_c \rangle \tilde{u}_i = \frac{1}{V_c}\int_V \rho_c u_i dV$

$$=> \overline{\rho_c u_i} = \alpha_c \langle \rho_c \rangle \tilde{u}_i$$

Mass transfer rate per particle

Local number density

$$=> \frac{\partial \alpha_c \langle \rho_c \rangle}{\partial t} + \frac{\partial \alpha_c \langle \rho_c \rangle \tilde{u}_i}{\partial x_i} = -n\dot{m}$$

Problem 2: Rosin-Rammler Distribution

The Rosin-Rammler distribution is frequently used to describe droplet size distributions during fuel injection. In terms of the cumulative mass distribution, it can be expressed as

$$F_m(D) = 1 - \exp\left(-\left(\frac{D}{\delta}\right)^n\right)$$

a) *Derive the corresponding PDF.*

b) *For the parameters $\delta = 144\ \mu m$ and $n = 2$, compute the mean diameter and variance by numerical integration in MATLAB. Plot the resulting PDF and CDF.*

Problem 2: Solution

a) PDF: $f_m(D) = \frac{dF_m}{dD} = \exp\left(-\left(\frac{D}{\delta}\right)^n\right)\frac{n}{\delta}\left(\frac{D}{\delta}\right)^{n-1}$

b) Mean diameter: $\mu_m = \int_0^\infty D\, f_m(D)dD = \int_0^\infty D \exp\left(-\left(\frac{D}{\delta}\right)^n\right)\frac{n}{\delta}\left(\frac{D}{\delta}\right)^{n-1} dD$

Variance:

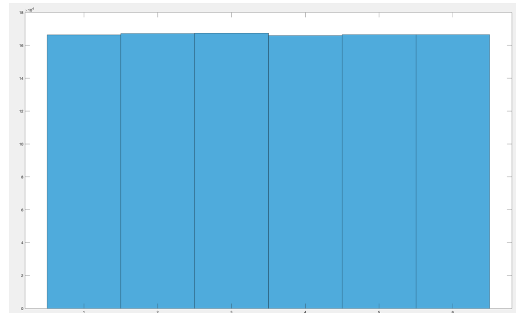
$$\sigma^2 = \int_0^\infty D^2\, f_m(D)dD - \mu_m^2 = \int_0^\infty D^2 \exp\left(-\left(\frac{D}{\delta}\right)^n\right)\frac{n}{\delta}\left(\frac{D}{\delta}\right)^{n-1} dD - \mu_m^2$$

The complete code has been uploaded to Moodle.

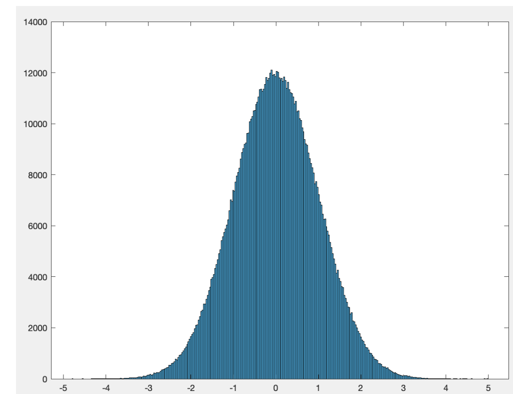
Problem Session 11: Statistical Modeling

Some common PDFs

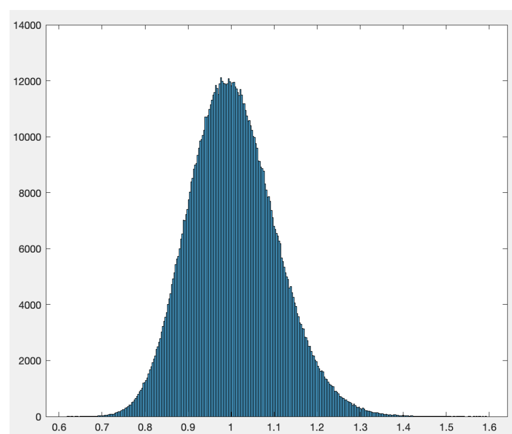
1. Flat PDF (die roll)



2. Normal distribution



3. Log-normal distribution

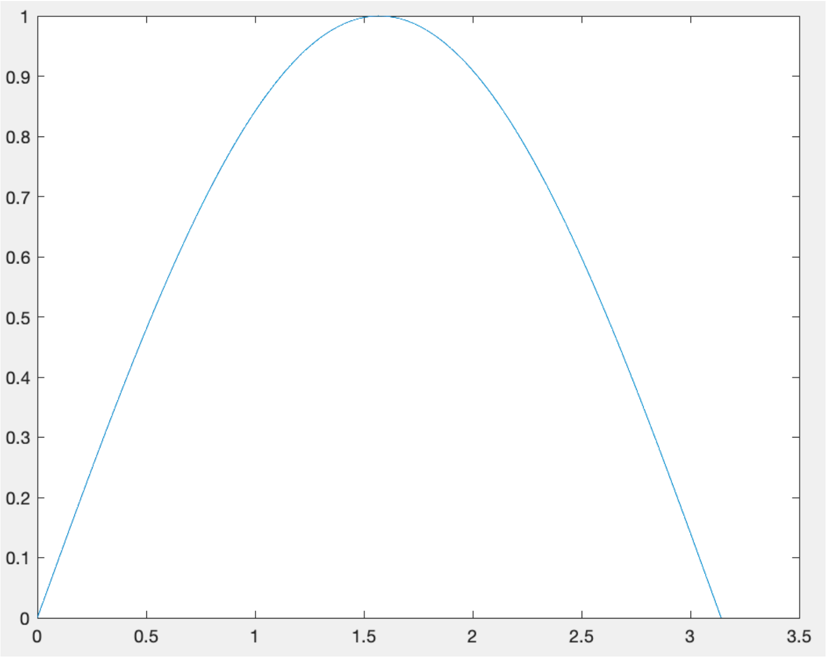


Problem Session 11: Statistical Modeling

Inverse Transform Sampling

- Algorithm to generate any arbitrary distribution

1. Decide on PDF
2. Calculate CDF
3. Seed random numbers in y-axis of CDF
4. Corresponding x-values is your desired PDF



Example pdf = sin(x)

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Inverse Transform Sampling

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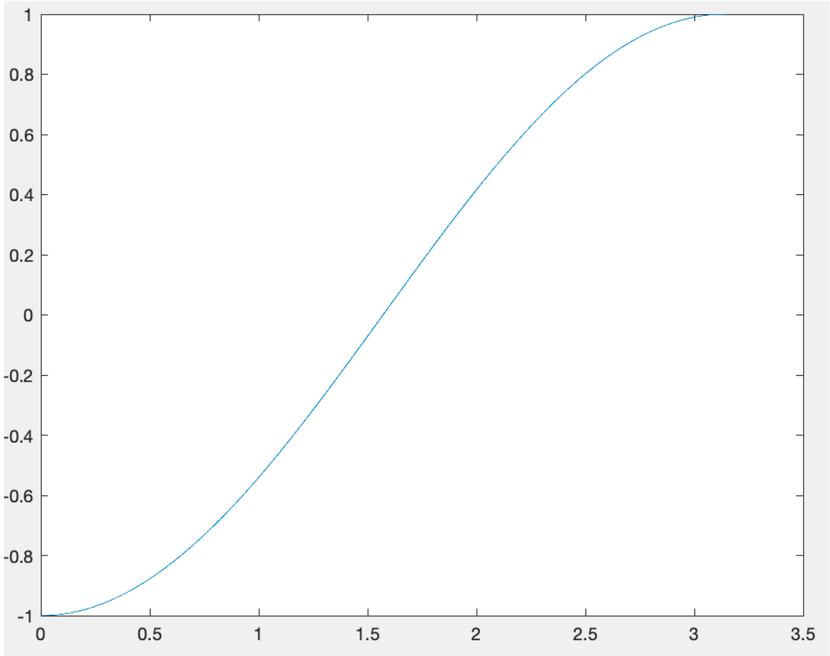

Example cdf = -cos(x)

Inverse Transform Sampling

- Algorithm to generate any arbitrary distribution

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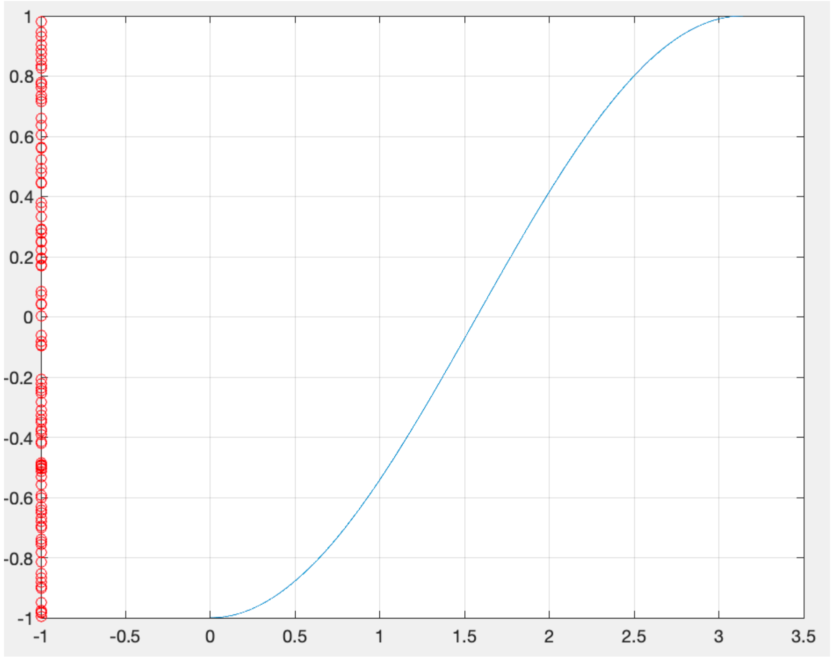

Random seeds (yvals) in y-axis

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Inverse Transform Sampling

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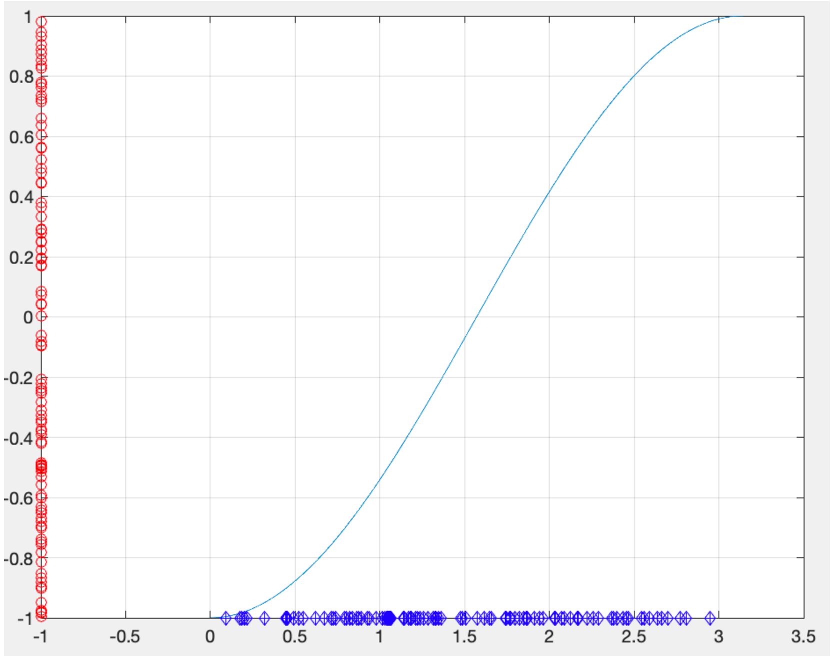

Corresponding xvals found using acos(-yvals)

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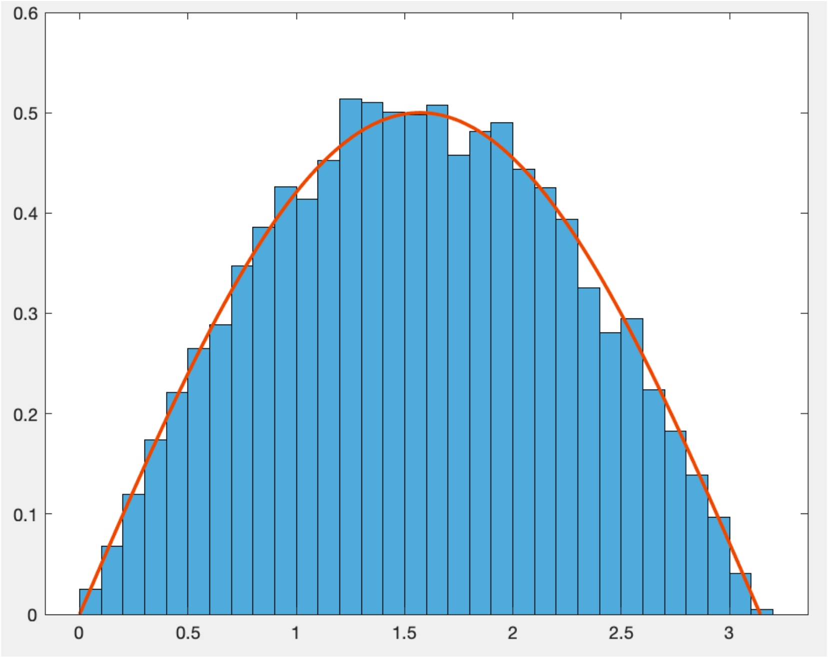


Histogram of xvals

Problem Session 11: Statistical Modeling

Problem 3: Monte-Carlo Sampling (in MATLAB)

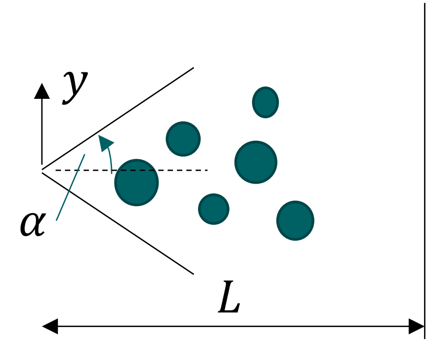
Solid particles are injected into a large chamber. Their size is assumed to be log-normally distributed inside their minimum and maximum bounds of $0.2\ mm$ *and* $10\ mm$*, respectively, with parameters* $D_m = 2mm$ *and variance* $\nu = 0.2\ mm^2$*. The particles are ejected by the nozzle at a constant absolute velocity of* $u_0 = 100\ m/s$*, while their injection angle* α *is uniformly distributed between* $-\pi/8$ *and* $+\pi/8$*. The carrier phase has a constant velocity in y-direction of* $v_c = 10\ m/s$*.*

Simulate the droplet motion by Monte-Carlo sampling with the given distributions of particle diameter D *and injection angle* α*. Visualize the particle distribution in terms of vertical position* y_L*, residence time in the chamber* t_L*, and x-velocity* u_L *at* $x = L = 1m$ *as well as their joint distributions. Compute the corresponding mean values and standard deviations.*

Use MATLAB's built-in functions rand() and lognrnd() for sampling.

Other parameters:

$$\rho_\mathrm{d} = 1000\,\frac{kg}{m^3},\ \mu_\mathrm{c} = 1.846 \cdot 10^{-5}\,\frac{kg}{m\cdot s},\ \rho_\mathrm{c} = 1.2\,\frac{kg}{m^3}$$



Problem 3: Solution

Definition log-normal distribution:

$$f_n(D) = \frac{1}{\sqrt{2\pi}\nu D} \exp\left[-\frac{1}{2}\left(\frac{\ln D - \ln m}{\nu}\right)^2\right]$$

Here: m and ν are mean and variance of the log-normal distribution of D
$\rightarrow$ but mean μ and standard deviation σ of the corresponding normal distribution of $\ln D$ are required for Matlab

with $\mu = \ln\left(\frac{m^2}{\sqrt{\nu+m^2}}\right)$ and $\sigma = \sqrt{\ln\left(\frac{\nu}{m^2}+1\right)}$

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Thank you for your attention

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