Multiphase Flows – WS 2022/23
Problem Session 1 – **Solution**
Navier-Stokes Equations & Single-Phase Shock Tube

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Problem Session 1: N-S Equations & Single-Phase Shock Tube

Problem 1: Navier-Stokes Equations

The Navier-Stokes Equations (NSE) are given in index notation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_k}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ki}}{\partial x_k} + \rho g_i$$

$$\text{with } \tau_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right)$$

a) *Write down the full NSE for two dimensions.*

b) *Derive the incompressible NSE.*

c) *Rewrite the incompressible NSE in dimensionless form. Neglect body forces.*

d) *What are the advantages in using the dimensionless NSE?*

Problem 1: Solution

a) <u>2-D Navier-Stokes equations:</u>

Mass: $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} = 0$

Momentum (x-direction): $\frac{\partial \rho u_x}{\partial t} + \frac{\partial \rho u_x u_x}{\partial x} + \frac{\partial \rho u_x u_y}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g_x$

with $\frac{\partial \tau_{xx}}{\partial x} = \eta \left(2 \frac{\partial^2 u_x}{\partial x^2} - \frac{2}{3} \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} \right) \right) = \eta \left(\frac{4}{3} \frac{\partial^2 u_x}{\partial x^2} - \frac{2}{3} \frac{\partial^2 u_y}{\partial x \partial y} \right)$

and $\frac{\partial \tau_{xy}}{\partial y} = \eta \left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y} \right)$

In y-direction: Similar expression

Problem 1: Solution

b) <u>Incompressible Navier-Stokes (done here for index notation):</u>

Assumption: $\rho = \text{const.} \quad => \quad \frac{\partial \rho}{\partial t} = 0$

Mass equation: $\frac{\cancel{\partial \rho}}{\cancel{\partial t}} + \rho \frac{\partial u_j}{\partial x_j} = 0. \quad => \quad \frac{\partial u_j}{\partial x_j} = 0$

Momentum: $\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_k}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ki}}{\partial x_k} + \rho g_i$ with $\tau_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \underbrace{\frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k}}_{= 0 \text{ (from mass equation)}} \right)$

$$=> \frac{\partial u_i}{\partial t} + \underbrace{\frac{\partial u_i u_k}{\partial x_k}}_{= u_k \frac{\partial u_i}{\partial x_k} + u_i \cancel{\frac{\partial u_k}{\partial x_k}}} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \underbrace{\frac{\partial}{\partial x_k} \left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right)}_{= \frac{\partial}{\partial x_i} \cancel{\frac{\partial u_k}{\partial x_k}} + \frac{\partial^2 u_i}{\partial x_k^2}} + g_i$$

$$=> \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k^2} + g_i$$

Problem 1: Solution

c) Non-dimensionalization:

Introduce non-dimensional variables:

$$u_i^* = \frac{u_i}{u_0}, x_i^* = \frac{x_i}{L}, p^* = \frac{p}{p_0} = \frac{p}{\rho u_0^2}, t^* = \frac{t u_0}{L}$$

$$=> \frac{\partial u_i^*}{\partial t^*} + u_k^* \frac{\partial u_i^*}{\partial x_k^*} = -\frac{\partial p^*}{\partial x_i^*} + \underbrace{\frac{\nu}{u_0 L}}_{= 1/Re} \frac{\partial^2 u_i^*}{\partial {x_k^*}^2}$$

d) Advantages:

Similarity (scaling of the problem), reduction of number of free parameters, comparability & importance of different terms

Problem 2: Solution

The solution to the Matlab implementation of the 1-Phase shock tube problem of Problem Session 1 is equivalent to the template provided for solution of Problem Session 2.

Thank you for your attention

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