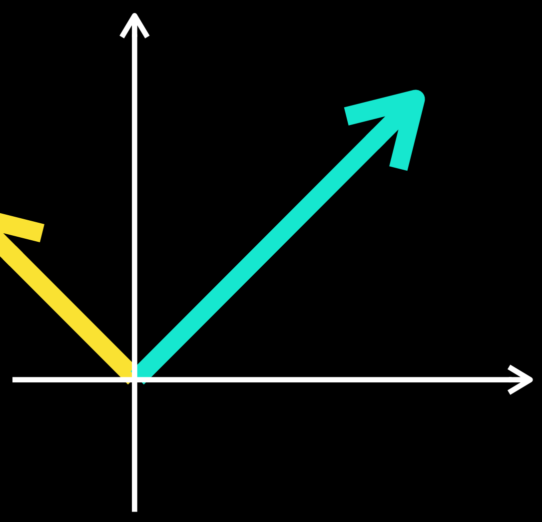
Least squares

Linear Algebra Essentials



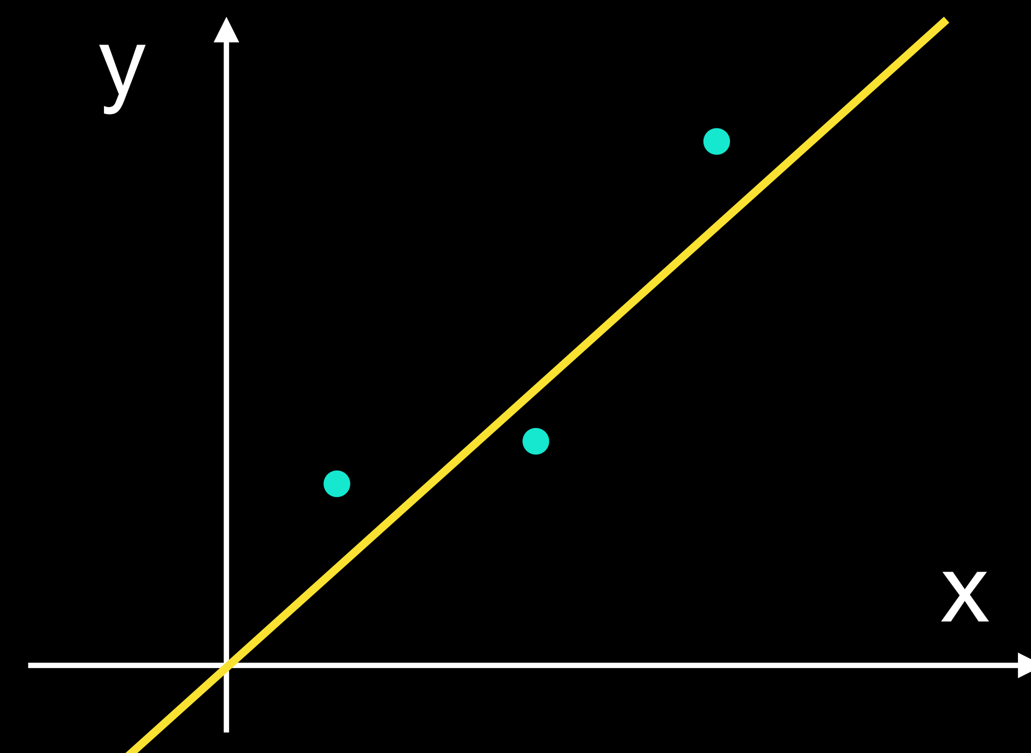
Linear least squares

{ ($\mathbf{x}_1$, y_1), … , ($\mathbf{x}_k$, y_k) }

n < k

$= X^T$

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ & & \dots & \\ x_{k1} & x_{k2} & \dots & x_{kn} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} \simeq \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_k \end{bmatrix}$$



$$\|(X^T a - y)\| \rightarrow min$$

$$L = (X^T a - y)^T (X^T a - y) \rightarrow min$$

$$\frac{\partial L}{\partial a} = \begin{bmatrix} \frac{\partial L}{\partial a_1} \\ \dots \\ \frac{\partial L}{\partial a_n} \end{bmatrix} = \overrightarrow{0}$$

$$L = (X^T a - y)^T (X^T a - y)$$ - Loss function

$$\frac{\partial L}{\partial a} = \frac{\partial}{\partial a}(X^T a - y)^T (X^T a - y)$$

$$= X^{TT}(X^T a - y) + (X^T a - y)^T X^T$$

$$= XX^T a - Xy + a^T XX^T - (Xy)^T$$

$$= 2(XX^T a - Xy) = \vec{0}$$

$$XX^T a = Xy \qquad a = (XX^T)^{-1} Xy$$

XX^T - symmetric

$$(XX^T)^T = (X^T)^T X^T$$
$$= XX^T$$

$$v^T \left[\frac{\partial^2 L}{\partial a_i \partial a_j} \right] v > 0$$

$$v^T XX^T v = (X^T v)^T X^T v$$

$$z = X^T v \qquad z^T z > 0$$

Example

```python
data = [(1, 3), (3, 4), (4, 5)]
```

```python
X, y = zip(*data)
X = np.array(X)
y = np.array(y)

X = np.vstack([np.ones(shape=(len(X)),), X])
X.T
```

```
array([[1., 1.],
       [1., 3.],
       [1., 4.]])
```

$$\boxed{a = (XX^T)^{-1}Xy}$$

```python
a = np.linalg.inv(X.dot(X.T)).dot(X).dot(y)
a
```

```
array([2.28571429, 0.64285714])
```

a_0 - intercept

$\vec{a}$