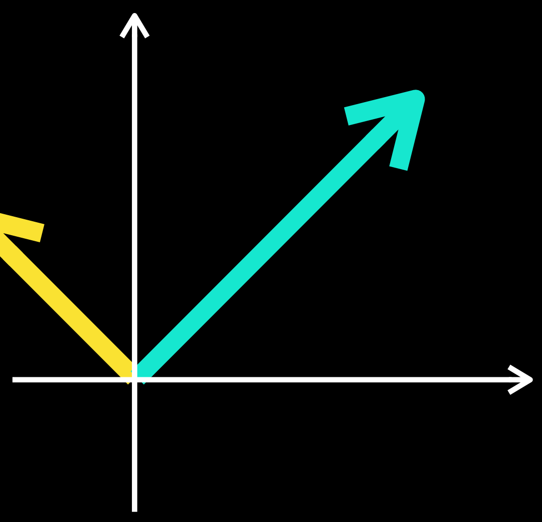
Orthogonal transformations

Linear Algebra Essentials



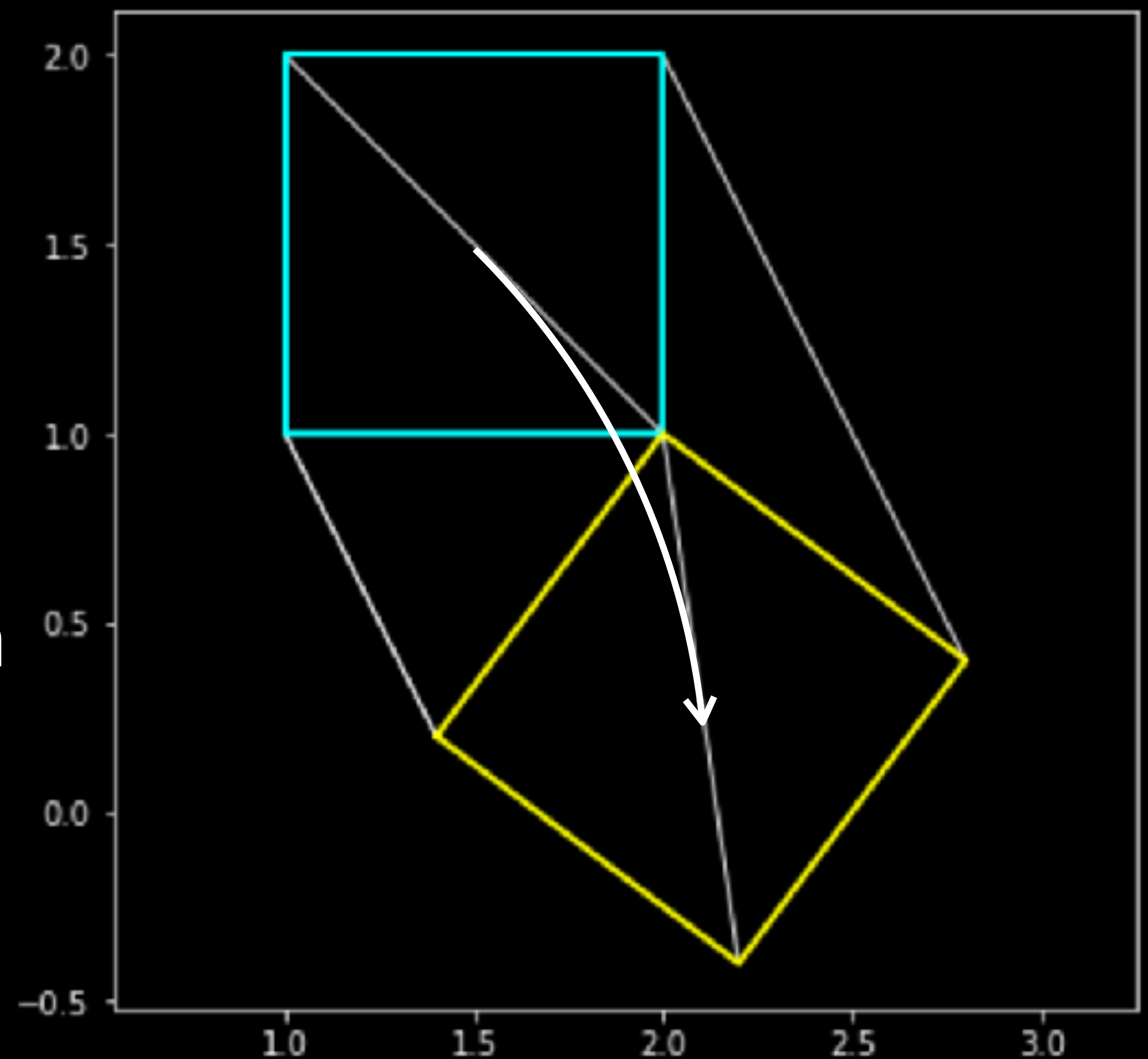
$$A = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$$

$\mathbf{y} = A\ \mathbf{x}$

Orthogonal transformation preserves *inner product*

$(v_1, v_2) = (A\ v_1, A\ v_2)$

A - is *orthogonal matrix*



Rotations

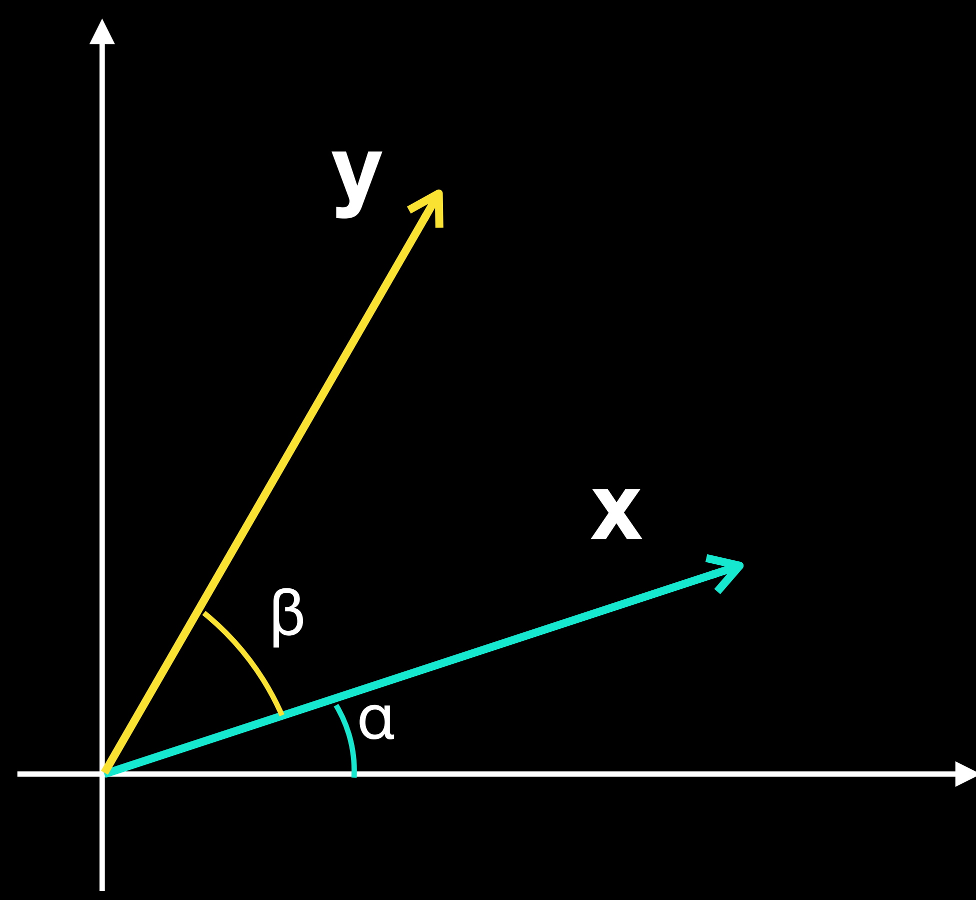
$$Rx = y \qquad R-?$$

$$x = L \cdot \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \end{bmatrix}$$

$$y = L \cdot \begin{bmatrix} cos(\alpha+\beta) \\ sin(\alpha+\beta) \end{bmatrix}$$

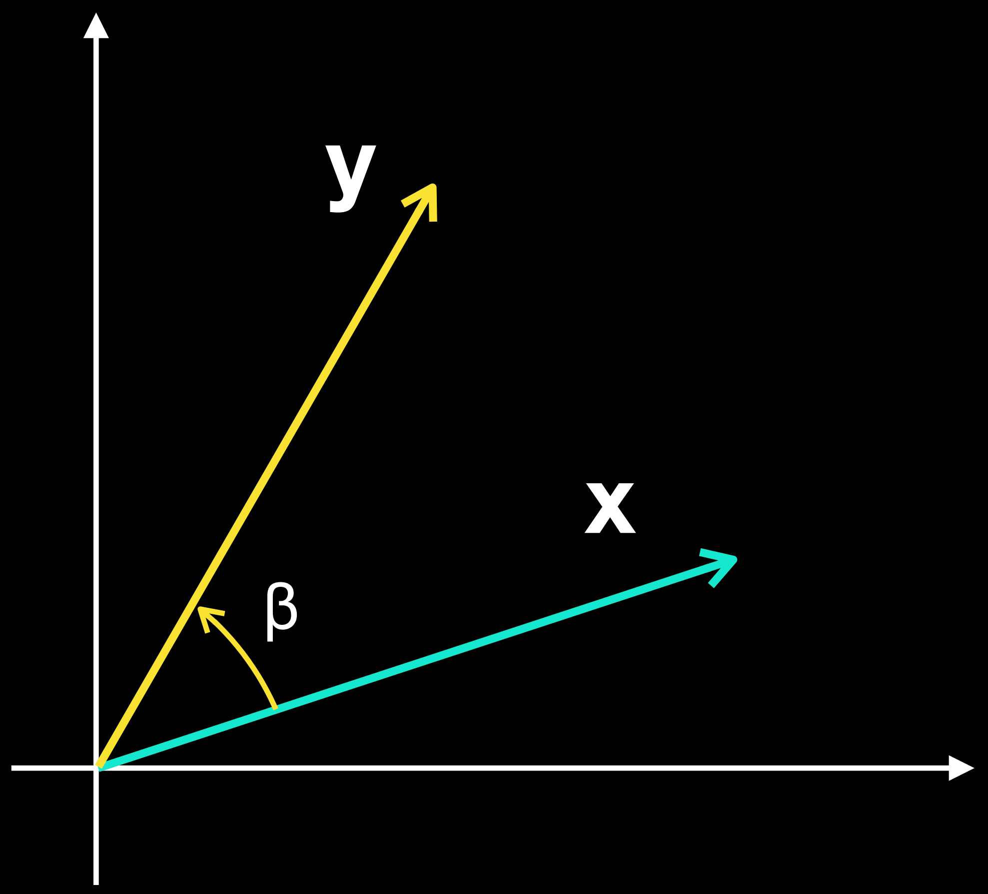
$$y = L \cdot \begin{bmatrix} cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta) \\ sin(\beta)cos(\alpha) + sin(\alpha)cos(\beta) \end{bmatrix}$$

$$= L \cdot \begin{bmatrix} cos(\beta) & -sin(\beta) \\ sin(\beta) & cos(\beta) \end{bmatrix} \cdot \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \end{bmatrix} = \begin{bmatrix} cos(\beta) & -sin(\beta) \\ sin(\beta) & cos(\beta) \end{bmatrix} \cdot x$$



Rotation matrix properties

$$R(\beta) = \begin{bmatrix} cos(\beta) & -sin(\beta) \\ sin(\beta) & cos(\beta) \end{bmatrix}$$



Identity is a rotation matrix

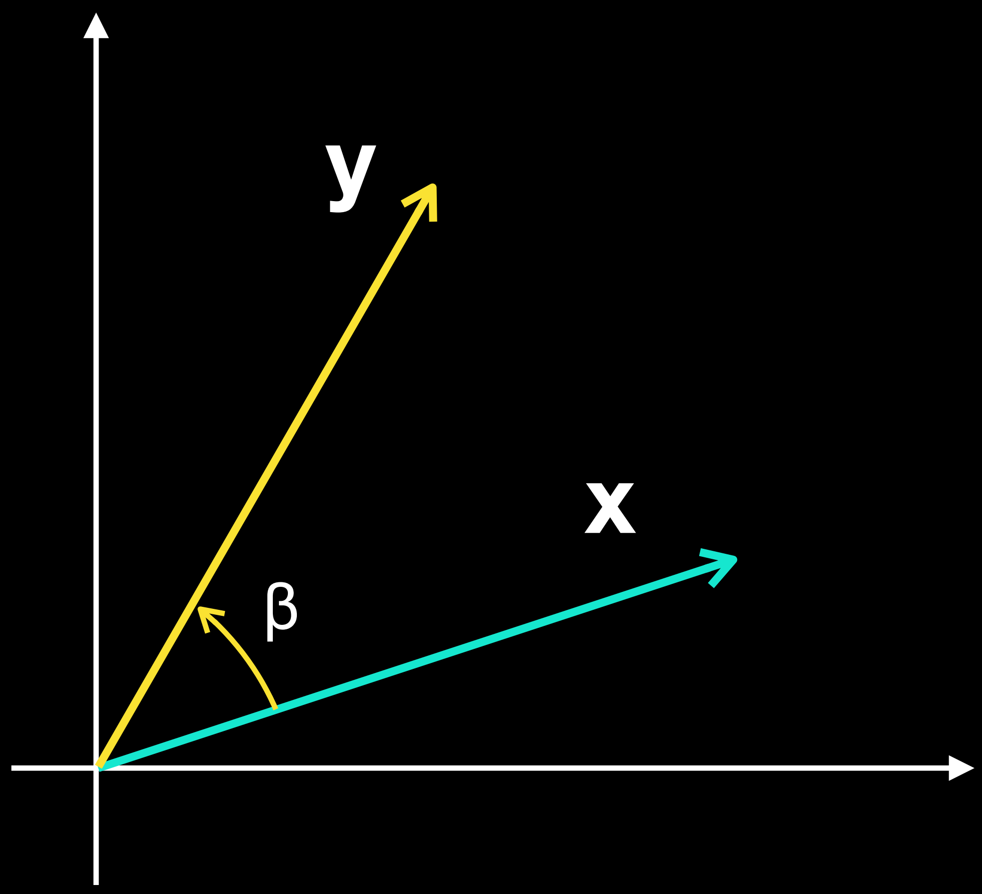
$R_{(0)} = I \qquad R_{(2\pi)} = I$

Commutativity

$R_{(a)}\, R_{(b)} = R_{(b)}\, R_{(a)} = R_{(a+b)} = R_{(b+a)}$

Rotation matrix properties

$$R(\beta) = \begin{bmatrix} cos(\beta) & -sin(\beta) \\ sin(\beta) & cos(\beta) \end{bmatrix}$$



Inverse matrix

$R_{(b)}{}^{-1} = R_{(-b)}$

$R_{(b)}\ R_{(-b)} = R_{(b-b)} = R_{(0)} = I$

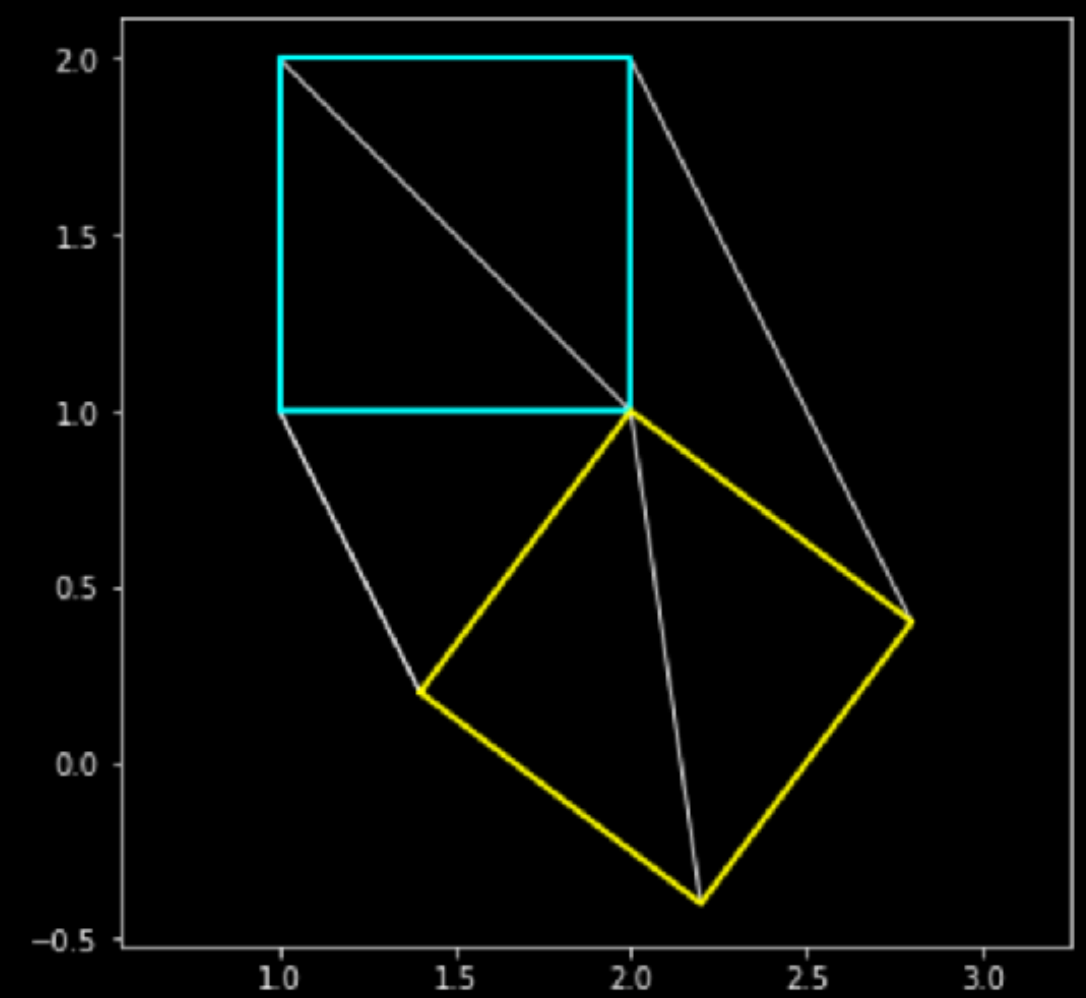
Matrix transpose

$$R(-\beta) = \begin{bmatrix} cos(-\beta) & -sin(-\beta) \\ sin(-\beta) & cos(-\beta) \end{bmatrix} = \begin{bmatrix} cos(\beta) & sin(\beta) \\ -sin(\beta) & cos(\beta) \end{bmatrix}$$

$R_{(b)}{}^{T} = R_{(-b)} = R_{(b)}{}^{-1}$

Rotation matrix properties

$$R(\beta) = \begin{bmatrix} cos(\beta) & -sin(\beta) \\ sin(\beta) & cos(\beta) \end{bmatrix}$$



Determinant = 1

det ($R_{(b)}$) =
= $\cos(b)^2 + \sin(b)^2 = 1$

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} \neq I$$

Conclusions

Orthogonal transformation preserves *inner product, distances, and angles*

$$R(\beta) = \begin{bmatrix} cos(\beta) & -sin(\beta) \\ sin(\beta) & cos(\beta) \end{bmatrix}$$

Commutativity: $R_1 R_2 = R_2 R_1$

Inverse matrix: $R^{-1} = R^T$

Determinant = 1