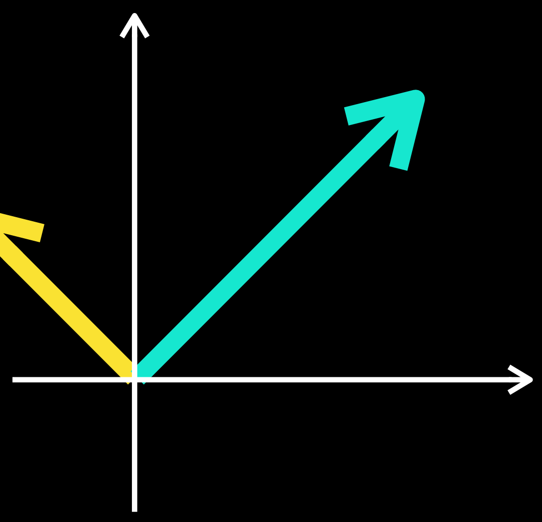
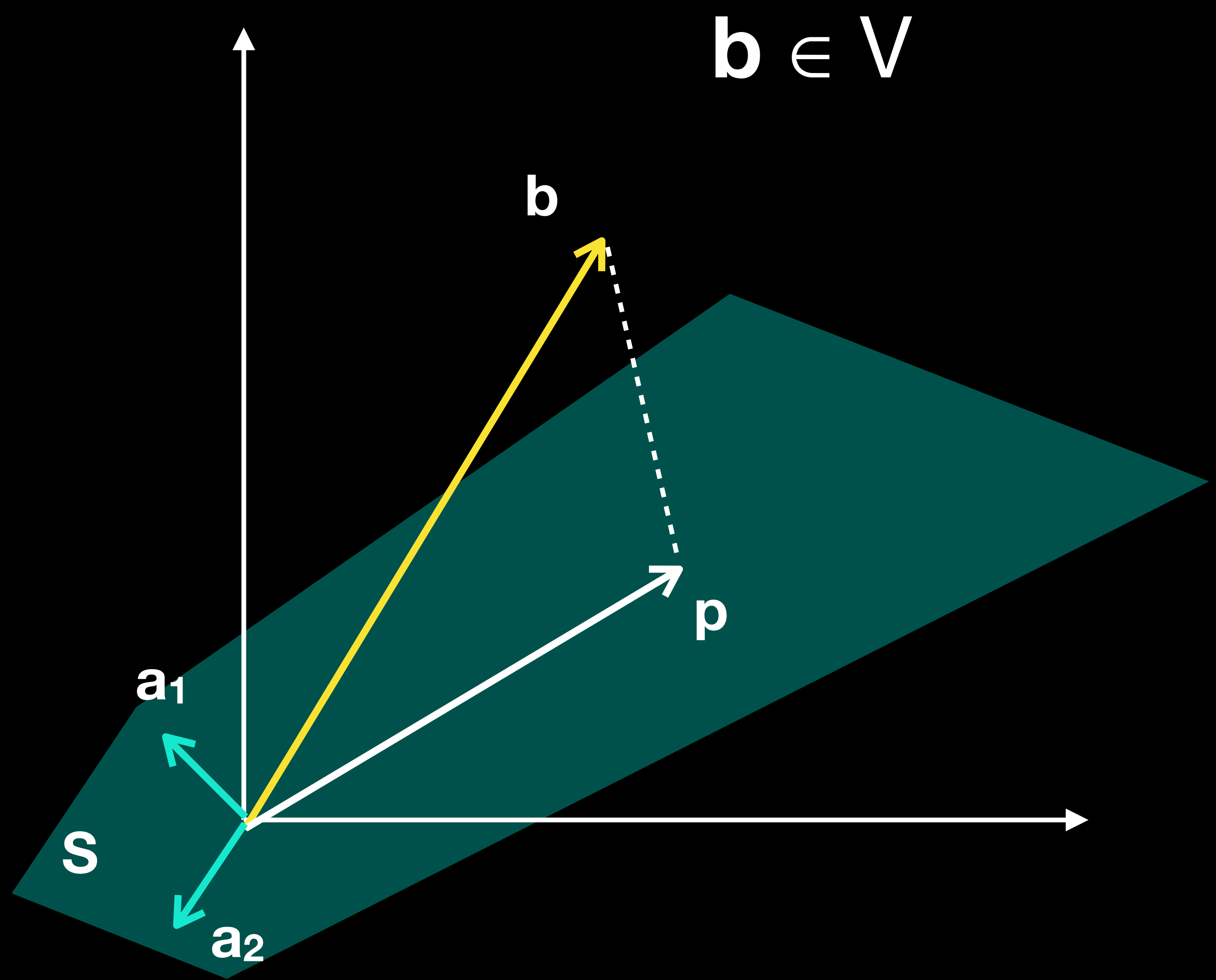
Projection matrix

Linear Algebra Essentials





$\{a_1, a_2, \ldots\}$ - basis of S

Find the projection of vector **b** onto **S**

$\mathbf{p} \in S$

$p = x_1 a_1 + x_2 a_2$

$$p = [a_1 \;\; a_2] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underset{(n\times k)}{A} \, \underset{(k\times 1)}{\vec{x}}$$

$(p, b - p) = 0$

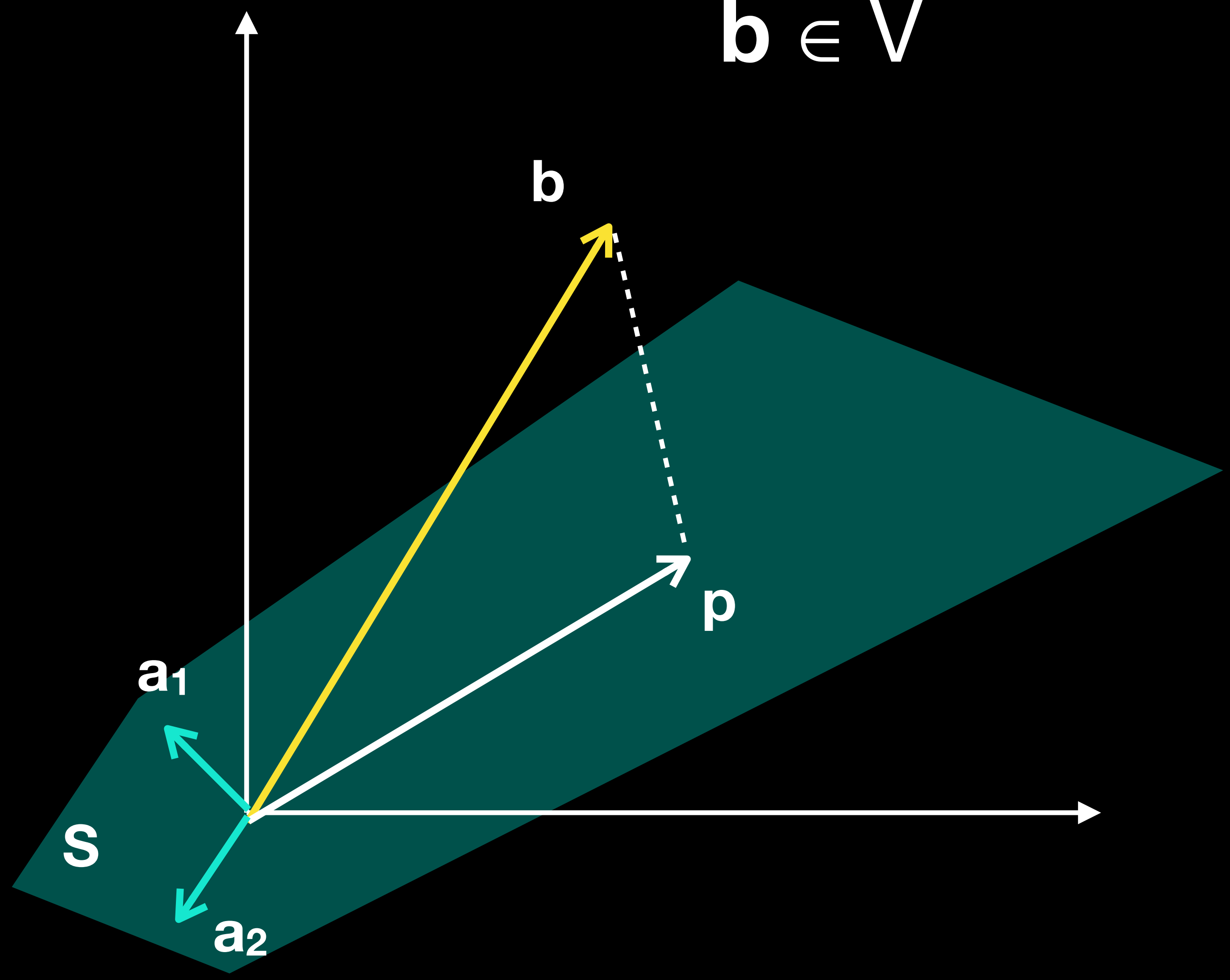
$$\underset{(1\times n)}{p^T} \, \underset{(n\times 1)}{(b - p)} = (Ax)^T (b - Ax) = 0$$

$x^T A^T b - x^T A^T A x = 0$

$x^T \underbrace{(A^T b)} = x^T \underbrace{(A^T A x)}$

$A^T b = A^T A x$

$\boxed{x = (A^T A)^{-1} A^T b}$



$\mathbf{b} \in \mathrm{V}$

$$p = A\vec{x}$$

$$x = (A^T A)^{-1} A^T b$$

$$p = \underbrace{\underset{(n\times k)}{A}\,(\underset{(k\times n)}{A^T}\,\underset{(n\times k)}{A})^{-1}\,\underset{(k\times n)}{A^T}}_{(n\times n)}\,b$$

$$p = P\,b \qquad rank(P) = k$$

$$P = A\,(A^T A)^{-1} A^T$$

P - projection matrix onto a column-space of A

$\{a_1, a_2, \ldots\}$ - basis of S

$$\underset{(n\times k)}{A} = [\,a_1, a_2, \ldots\,]$$

Properties of a projection matrix

1. $P = P^T$

$$P^T = [\, A\, (A^T A)^{-1} A^T \,]^T$$

$$= A^{TT}\, [(A^T A)^{-1}]^T A^T$$

$$= A\, [(A^T A)^T]^{-1} A^T$$

$$= A\, [A^T A^{TT}]^{-1} A^T \quad = A\, [A^T A]^{-1} A^T$$

2. $P^2 = P$

$$P^2 = A\, [A^T A]^{-1} \underbrace{A^T\ A\, [A^T A]^{-1}}_{= I} A^T$$

$$= A\, [A^T A]^{-1} A^T \quad = P$$