Test 2 - diff.eq.
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$with(plots):$

Task 1

$Eq_1 := y''-2\,y'-3\,y = x\,e^{4x}\sin(x)$

$$\frac{d^2}{dx^2}\,y(x) - 2\left(\frac{d}{dx}\,y(x)\right) - 3\,y(x) = x\,e^{4x}\sin(x) \quad (1)$$

$dsolve(Eq_1)$

$$y(x) = e^{-x}\,\_C2 + e^{3x}\,\_C1 + \frac{1}{338}\,\left(\left(-39\,x+41\right)\cos(x) + \sin(x)\,\left(26\,x+3\right)\right)e^{4x} \quad (2)$$

Task 2

$Eq_2 := y''' + y'' = 1 - 6\,x^2\,e^{-x}$

$$\frac{d^3}{dx^3}\,y(x) + \frac{d^2}{dx^2}\,y(x) = 1 - 6\,x^2\,e^{-x} \quad (3)$$

$dsolve(Eq_2)$

$$y(x) = -2\,e^{-x}x^3 - 12\,x^2\,e^{-x} - 36\,x\,e^{-x} - 48\,e^{-x} + \frac{1}{2}\,x^2 + \_C1\,e^{-x} + \_C2\,x + \_C3 \quad (4)$$

$dsolve(Eq_2, output = basis)$

$$\left[\left[1, x, e^{-x}\right], \left(-2\,x^3 - 12\,x^2 + e^x - 36\,x + \frac{1}{2}\,x^2\,e^x - x\,e^x - 48\right)e^{-x}\right] \quad (5)$$

Task 3

$Eq_3 := y''' - y' = \tan(x)$

$$\frac{d^3}{dx^3}\,y(x) - \left(\frac{d}{dx}\,y(x)\right) = \tan(x)$$

$dsolve(\{Eq_3, y'(0) = -1, y''(0) = 1\})$

$$y(x) = \int_0^x \left(-e^{-\_z1} + \frac{1}{2}\left(\int_0^{\_z1} e^{-\_z1}\tan(\_z1)\,d\_z1\right)e^{\_z1} - \frac{1}{2}\left(\int_0^{\_z1} e^{\_z1}\tan(\_z1)\,d\_z1\right)e^{-\_z1}\right)d\_z1 + \_C3 \quad (7)$$

Task 4

$Sys_4 := diff(x(t), t\$2) + 5\,diff(x(t), t) + 2\,diff(y(t), t) + y(t) = 0, 3\,diff(x(t), t\$2) + 5\,x(t) + diff(y(t), t) + 3\,y(t) = 0$

$$\frac{d^2}{dt^2}\,x(t) + 5\left(\frac{d}{dt}\,x(t)\right) + 2\left(\frac{d}{dt}\,y(t)\right) + y(t) = 0, 3\left(\frac{d^2}{dt^2}\,x(t)\right) + 5\,x(t) + \frac{d}{dt}\,y(t) + 3\,y(t) = 0 \quad (8)$$

$dsolve(\{Sys_4, x(0) = 1, \ D(x)(0) = 0, y(0) = 1\})$

$$\left\{x(t) = -\frac{1}{2}\,e^{-t} + \frac{3}{2}\,e^t - 2\,e^t\,t, y(t) = 2\,e^{-t} - e^t + 4\,e^t\,t\right\} \quad (9)$$

Task 5

$Eq_5 := y'' + y = y^2$

$$\frac{d^2}{dx^2} y(x) + y(x) = y(x)^2 \tag{10}$$

$dsolve(\{Eq_5, y(0) = 2\,a, D(y)(0) = a\}, y(x), series)$

$$y(x) = 2\,a + a\,x + (2\,a^2 - a)\,x^2 + \left(\frac{2}{3}\,a^2 - \frac{1}{6}\,a\right) x^3 + \left(\frac{2}{3}\,a^3 - \frac{5}{12}\,a^2 + \frac{1}{12}\,a\right) x^4 + \left(-\frac{1}{6}\,a^2 + \frac{1}{120}\,a + \frac{1}{3}\,a^3\right) x^5 + O(x^6) \tag{11}$$

Task 6

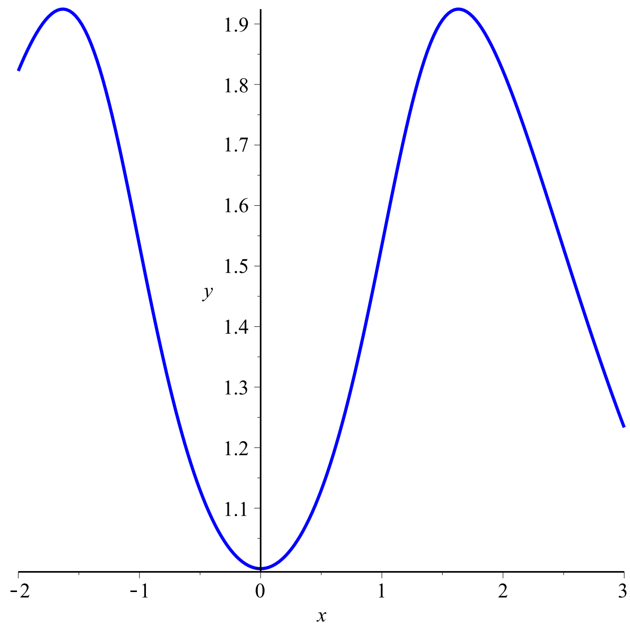
$Eq_6 := y' = \sin(x\,y)$

$$\frac{d}{dx} y(x) = \sin(x\,y(x)) \tag{12}$$

$sol\_num\_6 := dsolve(\{Eq_6, y(0) = 1\}, y(x), numeric)$

proc(*x_rkf45*) ... **end proc** (13)

$p6 := odeplot(sol\_num\_6, [x, y(x)], -2..3, thickness = 2, color = blue):$
$display(p6)$



Task 7

$Eq_7 := y''=x\,y'-y^2$

$$\frac{d^2}{dx^2}\,y(x)=x\left(\frac{d}{dx}\,y(x)\right)-y(x)^2 \tag{14}$$

$cond\_7 := y(0)=1, \mathrm{D}(y)(0)=2$

$$y(0)=1, \mathrm{D}(y)(0)=2 \tag{15}$$

$sol\_num\_7 := dsolve\left(\{Eq_7, cond\_7\}, y(x), numeric\right)$

$$\mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{16}$$

$dsolve\left(\{Eq_7, cond\_7\}, y(x), series\right)$

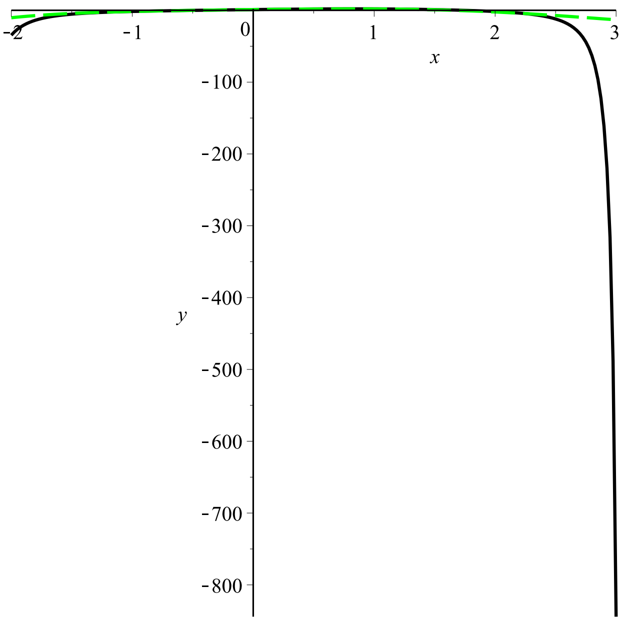
$$y(x)=1+2\,x-\frac{1}{2}\,x^2-\frac{1}{3}\,x^3-\frac{1}{3}\,x^4+\frac{1}{12}\,x^5+\mathrm{O}\left(x^6\right) \tag{17}$$

$sol\_series\_7 := rhs(convert(\mathbf{(17)}, polynom))$

$$1+2\,x-\frac{1}{2}\,x^2-\frac{1}{3}\,x^3-\frac{1}{3}\,x^4+\frac{1}{12}\,x^5 \tag{18}$$

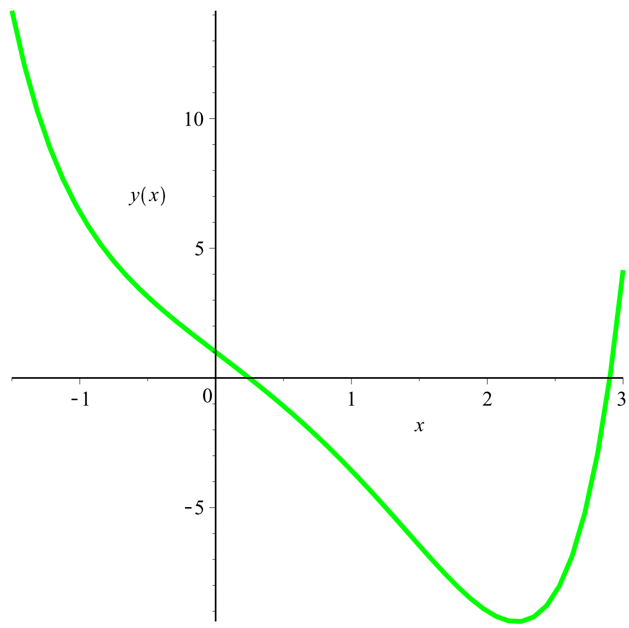
$p71 := odeplot(sol\_num\_7, [x, y(x)], -2..3, thickness=2, color=black) :$

$p72 := plot(sol\_series\_7, x = -2 .. 3, thickness = 2, linestyle = 3, color = green)$:
$display(p71, p72)$



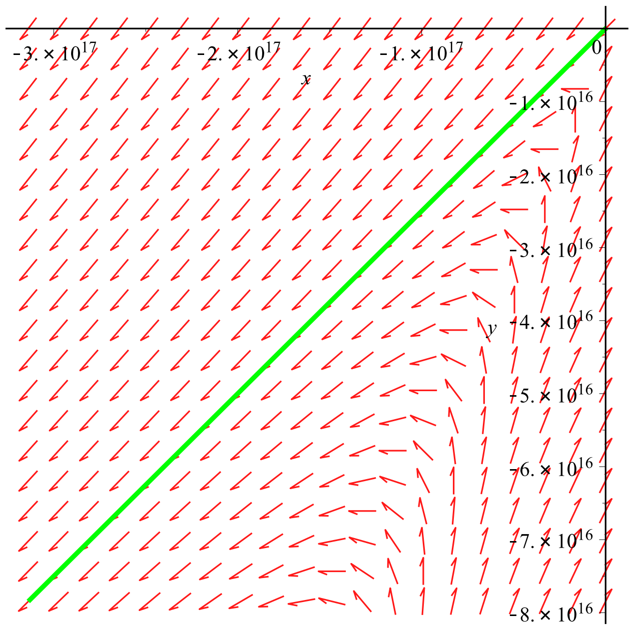
Task 8

$with(DEtools)$:
$DEplot(y'' - x\,y' + x\,y = 0, \{y(x)\}, x = -1.5 .. 3, [[y(0) = 1, \mathrm{D}(y)(0) = -4]], stepsize = 1, linecolor = green, thickness = 3)$



Task 9

$DEplot(\{diff(x(t), t) = 3\,x(t) - 4\,y(t), diff(y(t), t) = x(t) - 2\,y(t)\}, [x(t), y(t)], t = 0\,..20, [[0, 1, 2]], stepsize = 0.1, linecolor = green)$



Task 10

$with(VectorCalculus):$
$with(LinearAlgebra):$
$x_{pr} := e^{x} - e^{-3z}:$
$y_{pr} := 3^{x} - \ln(1 + y^2) - \cos(z)^2:$
$z_{pr} := x\tan(y^2) + \sin(x + y + z^2):$
$F_{pr} := Vector([x_{pr}, y_{pr}, z_{pr}])$

$$(e^{x} - e^{-3z})e_x + (3^{x} - \ln(y^2+1) - \cos(z)^2)e_y + (x\tan(y^2) + \sin(z^2 + x + y))e_z \tag{19}$$

$J_{pr} := Jacobian(F_{pr}, [x, y, z])$

$$\begin{bmatrix} e^{x} & 0 & 3\,e^{-3z} \\ 3^{x}\ln(3) & -\frac{2y}{y^2+1} & 2\cos(z)\sin(z) \\ \tan(y^2) + \cos(z^2+x+y) & 2x\left(1+\tan(y^2)^2\right)y + \cos(z^2+x+y) & 2\cos(z^2+x+y)\,z \end{bmatrix} \tag{20}$$

$J_{pr} := Jacobian(F_{pr}, [x, y, z] = [0, 0, 0])$

$$\begin{bmatrix} 1 & 0 & 3 \\ \ln(3) & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \tag{21}$$

$eigen := evalf(Eigenvalues(\mathbf{(21)}))$

$$\begin{bmatrix} 2.622869892 \\ -0.8114349457 + 0.7734015165\,\mathrm{I} \\ -0.8114349457 - 0.7734015165\,\mathrm{I} \end{bmatrix}$$

$\max\left(\mathrm{Re}\left(eigen_1\right), \mathrm{Re}\left(eigen_2\right), \mathrm{Re}\left(eigen_3\right)\right)$

$$2.622869892 \tag{23}$$

$evalb(\mathbf{(23)} > 0)$

$$true \tag{24}$$

Maximum of real parts of eigenvalues is greater than zero, therefore (by Lyapunov's theorem) "zero" solution *is not stable.*