Final Test Task
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$with(LinearAlgebra):$
$with(linalg):$
$with(student):$
$with(DEtools):$
$with(VectorCalculus):$

Linear Algebra

Task 1

$M_1 := Matrix(3, 3, [[x, a+bi, c+di], [a-bi, y, e+fi], [c-di, e-fi, z]])$

$$\begin{bmatrix} x & a+\mathrm{I}\,b & c+\mathrm{I}\,d \\ a-\mathrm{I}\,b & y & e+\mathrm{I}\,f \\ c-\mathrm{I}\,d & e-\mathrm{I}\,f & z \end{bmatrix} \quad \textbf{(1)}$$

$det(M_1)$

$$-a^2 z + 2\,a\,c\,e + 2\,a\,d\,f - b^2 z - 2\,b\,c\,f + 2\,b\,d\,e - c^2 y - d^2 y - e^2 x - f^2 x + x\,y\,z \quad \textbf{(2)}$$

Task 2

$a_1 := \langle 1, 0, 0, 2, 5 \rangle$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 5 \end{bmatrix} \quad \textbf{(3)}$$

$a_2 := \langle 0, 1, 0, 3, 4 \rangle$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 4 \end{bmatrix} \quad \textbf{(4)}$$

$a_3 := \langle 0, 0, 1, 4, 7 \rangle$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 7 \end{bmatrix} \quad \textbf{(5)}$$

$a_4 := \langle 2, -3, 4, 11, 12 \rangle$

$$\begin{bmatrix} 2 \\ -3 \\ 4 \\ 11 \\ 12 \end{bmatrix} \tag{6}$$

$A := Matrix(5, 4, [a_1, a_2, a_3, a_4])$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 4 & 11 \\ 5 & 4 & 7 & 12 \end{bmatrix} \tag{7}$$

Rank of the matrix shows the maximal number of linearly independent colums(vectors) :
$Rank(A)$

$$4 \tag{8}$$

We have 4 vectors, rank $= 4 \Rightarrow$ these vectors are linearly independent.

Task 3

$a_1 := \langle 2, -1, 3, 5 \rangle$

$$\begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \end{bmatrix} \tag{9}$$

$a_2 := \langle 4, -3, 1, 3 \rangle$

$$\begin{bmatrix} 4 \\ -3 \\ 1 \\ 3 \end{bmatrix} \tag{10}$$

$a_3 := \langle 3, -2, 3, 4 \rangle$

$$\begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix} \tag{11}$$

$a_4 := \langle 4, -1, 15, 17 \rangle$

$$\begin{bmatrix} 4 \\ -1 \\ 15 \\ 17 \end{bmatrix} \quad \textbf{(12)}$$

$a_5 := \langle 7, -6, 7, 0 \rangle$

$$\begin{bmatrix} 7 \\ -6 \\ 7 \\ 0 \end{bmatrix} \quad \textbf{(13)}$$

$b := Basis([a_1, a_2, a_3, a_4, a_5])$

$$\left[\begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix} \right] \quad \textbf{(14)}$$

As you can see, the system of vectors{_1,_2,_3,_4,a_5} have a basis **b** = {_1,_2,_3}. The coordinates of the vector **a_i** in the basis **b** are the solution **x** of the matrix equation **Bx = a_i**, where B is matrix of basis b. Let's solve this equation using the LinearSolve function.

$B := Matrix(4, 3, [a_1, a_2, a_3])$

$$\begin{bmatrix} 2 & 4 & 3 \\ -1 & -3 & -2 \\ 3 & 1 & 3 \\ 5 & 3 & 4 \end{bmatrix} \quad \textbf{(15)}$$

$LinearSolve(B, a_1)$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(16)}$$

$LinearSolve(B, a_2)$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \textbf{(17)}$$

$LinearSolve(B, a_3)$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (18)$$

$LinearSolve(B, a_4)$

$$\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \quad (19)$$

$LinearSolve(B, a_5)$

$$\begin{bmatrix} -6 \\ -2 \\ 9 \end{bmatrix} \quad (20)$$

Task 4

Let's solve by the Gauss-Jordan method. To find the fundamental system of solutions of this system of equations, it is necessary to write out the matrix of coefficients and transform the matrix into a triangular one.

$C := Matrix(3, 4, [[2,-4, 5, 3], [3,-6, 4, 2], [4,-8, 17, 11]])$

$$\begin{bmatrix} 2 & -4 & 5 & 3 \\ 3 & -6 & 4 & 2 \\ 4 & -8 & 17 & 11 \end{bmatrix}$$

$gaussjord(C)$

$$\begin{bmatrix} 1 & -2 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & \frac{5}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (22)$$

So, the fundamental system of solutions of this system is these two linear independent vectors X_1=(1, -2,0,-2/7), X_2=(0,0,1,5/7)

Task 5

$A := Matrix(3, 3, [[5, 2,-3], [1, 3,-1], [2, 2,-1]])$

$$\begin{bmatrix} 5 & 2 & -3 \\ 1 & 3 & -1 \\ 2 & 2 & -1 \end{bmatrix} \quad (23)$$

$E := Matrix(3, 3, [[1, 0, 0], [0, 1, 0], [0, 0, 1]])$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (24)$$

$f := A^3 - 7A^2 + 13A - 5E$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{(25)}$$

Task 6

$a_1 := \langle 1, 0, 0, -1 \rangle$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{(26)}$$

$a_2 := \langle 2, 1, 1, 0 \rangle$

$$\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{(27)}$$

$a_3 := \langle 1, 1, 1, 1 \rangle$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{(28)}$$

$a_4 := \langle 1, 2, 3, 4 \rangle$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{(29)}$$

$a_5 := \langle 0, 1, 2, 3 \rangle$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{(30)}$$

$b := \mathit{Basis}\left(\left[a_1, a_2, a_3, a_4, a_5\right]\right)$

$$\left[\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right] \tag{31}$$

The dimension of a linear subspace is defined as the number of elements of its basis, so the dimension equals 3.

Task 7

$a_1 := \langle 1, 2, 1\rangle$

$$\begin{bmatrix}1\\2\\1\end{bmatrix} \tag{32}$$

$a_2 := \langle 1, 1, -1\rangle$

$$\begin{bmatrix}1\\1\\-1\end{bmatrix} \tag{33}$$

$a_3 := \langle 1, 3, 3\rangle$

$$\begin{bmatrix}1\\3\\3\end{bmatrix} \tag{34}$$

$b_1 := \langle 2, 3, -1\rangle$

$$\begin{bmatrix}2\\3\\-1\end{bmatrix} \tag{35}$$

$b_2 := \langle 1, 2, 2\rangle$

$$\begin{bmatrix}1\\2\\2\end{bmatrix} \tag{36}$$

$b_3 := \langle 1, 1, -3\rangle$

$$\begin{bmatrix}1\\1\\-3\end{bmatrix} \tag{37}$$

$SumBasis([[a_1, a_2, a_3], [b_1, b_2, b_3]])$

$$\left[\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\3\\-1\end{bmatrix}\right] \tag{38}$$

$IntersectionBasis([[a_1, a_2, a_3], [b_1, b_2, b_3]])$

$$\left[\begin{bmatrix}3\\5\\1\end{bmatrix}\right] \tag{39}$$

Task 8

$M_{81} := Matrix(3, 3, [[2,-1, 2], [5,-3, 3], [-1, 0,-2]])$

$$\begin{bmatrix}2 & -1 & 2\\5 & -3 & 3\\-1 & 0 & -2\end{bmatrix} \tag{40}$$

$eigenvalues(M_{81})$

$$-1, -1, -1 \tag{41}$$

$eigenvectors(M_{81})$

$$\left[-1, 3, \left\{\begin{bmatrix}-1 & -1 & 1\end{bmatrix}\right\}\right] \tag{42}$$

Calculus

Task 9

$f := \dfrac{1+x^3}{1+x}$

$$\frac{x^3+1}{1+x} \tag{43}$$

$iscont(f, x=-\infty..+\infty)$

$$false \tag{44}$$

$discont(f, x)$

$$\{-1\} \tag{45}$$

Given function is not defined at the point x = -1. Consider one-sided limits at this point:

$limit\left(\dfrac{1+x^3}{1+x}, x=-1, right\right)$

$$3 \tag{46}$$

$limit\left(\dfrac{1+x^3}{1+x}, x=-1, left\right)$

$$3 \tag{47}$$

One-sided limits are finite and equal. Therefore, the function is not continuous and suffers a discontinuous discontinuity at the point x = -1.

Task 10

For an implicit function F(x,y):

$$y' = -\frac{F'_x}{F'_y}$$

Then:

$F_x := diff\left(x\,y - \arctan\left(\frac{x}{y}\right), x\right)$

$$y - \frac{1}{y\left(1 + \frac{x^2}{y^2}\right)} \qquad \textbf{(48)}$$

$F_y := diff\left(x\,y - \arctan\left(\frac{x}{y}\right), y\right)$

$$x + \frac{x}{y^2\left(1 + \frac{x^2}{y^2}\right)} \qquad \textbf{(49)}$$

$y_{diff} := simplify\left(-\frac{F_x}{F_y}\right)$

$$-\frac{(x^2 + y^2 - 1)\,y}{(x^2 + y^2 + 1)\,x} \qquad \textbf{(50)}$$

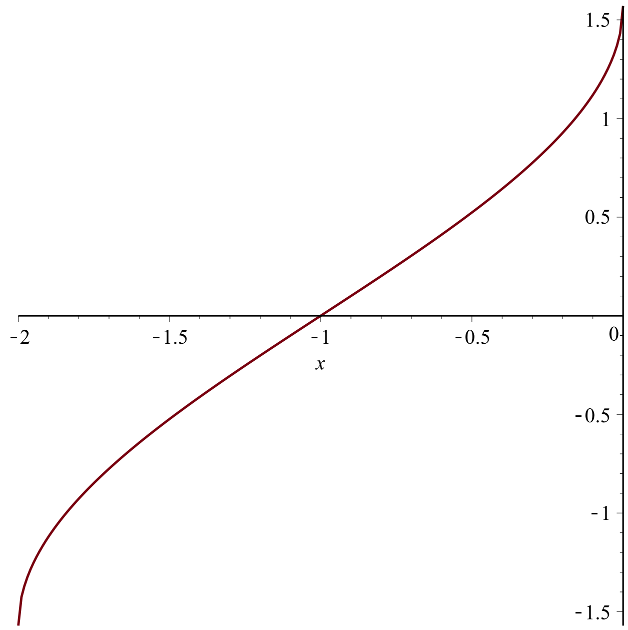
Task 11

$y_{11} := \arcsin(1 + x)$

$$\arcsin(1 + x) \qquad \textbf{(51)}$$

As known, D(arcsin(x))=[-1,1], so D(arcsin(1+x))=[-2,0]

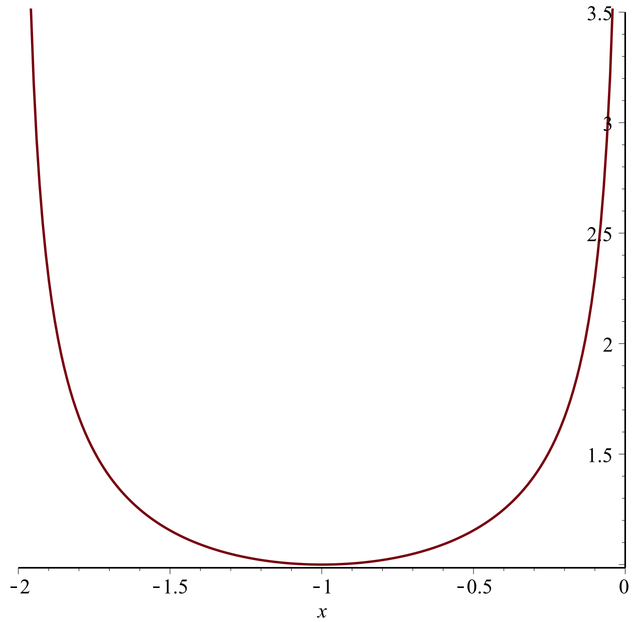
$plot(y_{11}, x = -2\,..0)$



$y_{111} := diff\left(y_{11}, x\right)$

$$\frac{1}{\sqrt{1-(1+x)^2}} \tag{52}$$

$plot\left(y_{111}, x=-2..0\right)$



$solve(y_{111}=0)$

No zeros of derivative.

$subs(x=-1, y_{111})$

$$1 \tag{53}$$

In conclusion, given function increases monotonically over [-2,0].

Task 12

$maximize\left(\frac{x}{1+x^2}\right)$

$$\frac{1}{2} \tag{54}$$

$minimize\left(\frac{x}{1+x^2}\right)$

$$-\frac{1}{2} \tag{55}$$

$maximize(x^3, x=-1..3)$

$$27 \tag{56}$$

$minimize(x^3, x=-1..3)$

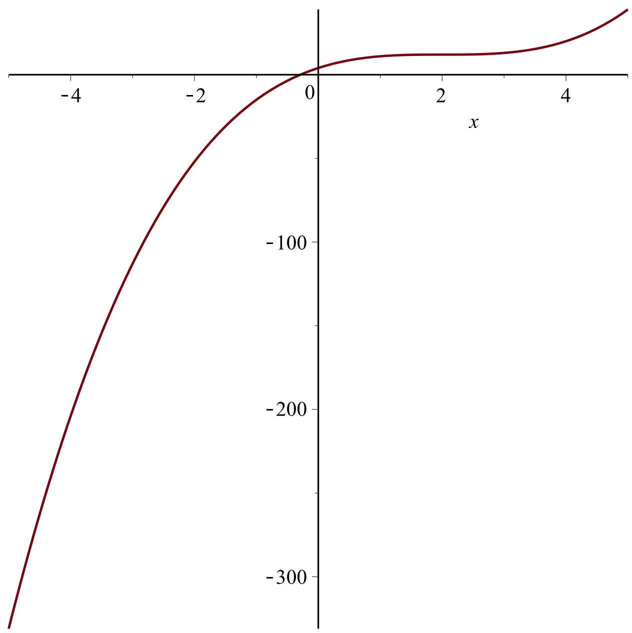
$$-1 \quad (57)$$

Task 13

$y_{13} := x^3 - 6x^2 + 12x + 4$

$$x^3 - 6x^2 + 12x + 4 \quad (58)$$

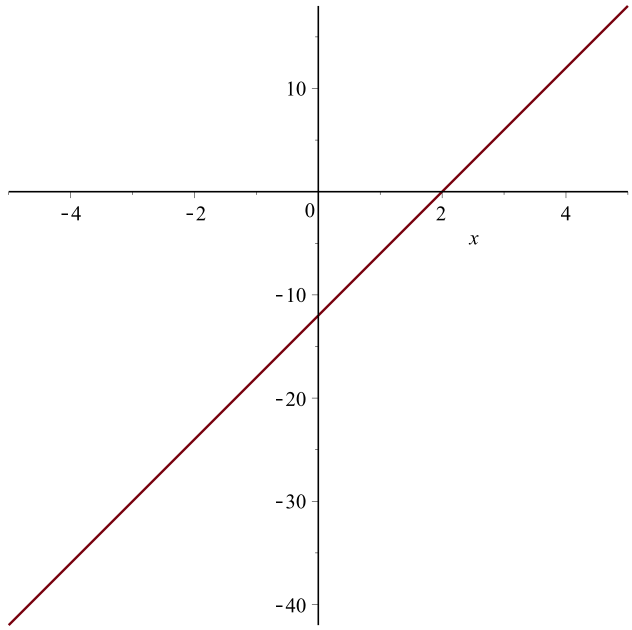
$plot(y_{13}, x = -5..5)$



$y_{131} := diff(y_{13}, x\$2)$

$$6x - 12 \quad (59)$$

$plot(y_{131}, x = -5..5)$



$solve(y_{131}=0)$

$$2 \tag{60}$$

Examine intervals (-∞; 2), (2; +∞) for convexity / concavity:
$subs(x=1, y_{131})$

$$-6 \tag{61}$$

$subs(x=3, y_{131})$

$$6 \tag{62}$$

As a result, the function is concave on the intervals (2; +∞), the function is convex on the intervals (-∞; 2).

Task 14

$Int\left(\frac{1}{(x-1)^2}, x=0..3\right) = int\left(\frac{1}{(x-1)^2}, x=0..3\right)$

$$\int_0^3 \frac{1}{(x-1)^2}\,dx = \infty \tag{63}$$

This improper integral diverges.

$$Int\left(\frac{1}{\sqrt[3]{1-x^4}}, x=0..1\right) = int\left(\frac{1}{\sqrt[3]{1-x^4}}, x=0..1\right)$$

$$\int_0^1 \frac{1}{\left(-x^4+1\right)^{1/3}}\,\mathrm{d}x = \frac{1}{4}\,\mathrm{B}\left(\frac{1}{4}, \frac{2}{3}\right) \tag{64}$$

Task 15

$$u := x^2 + y^2 + z^2 - x\,y + x - 2\,z$$

$$x^2 - x\,y + y^2 + z^2 + x - 2\,z \tag{65}$$

$$extrema\left(x^2 + y^2 + z^2 - x\,y + x - 2\,z,\ \{x, y, z\}\right)$$

$$\{0\} \tag{66}$$

Task 16

$$Doubleint\left(x\,y, y=0..\sqrt{1-(x-2)^2}, x=1..3\right)$$

$$\int_1^3\int_0^{\sqrt{1-(x-2)^2}} x\,y\,\mathrm{d}y\,\mathrm{d}x \tag{67}$$

$$value(\textbf{(67)})$$

$$\frac{4}{3} \tag{68}$$

Task 17

$$int\left(int\left(Int\left(x, z=0..\sqrt{\frac{4\,x-y^2}{2}}\right), y=0..2\sqrt{x}\right), x=0..2\right)$$

$$\int_0^2\int_0^{2\sqrt{x}}\int_0^{\frac{1}{2}\sqrt{-2\,y^2+8\,x}} x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \tag{69}$$

$$value(\textbf{(69)})$$

$$\frac{4}{3}\,\pi\sqrt{2} \tag{70}$$

Task 18

2470. According to the Leibniz criterion for the convergence of alternating series:

$$limit\left(\left|\frac{(-1)^{n-1}}{2\,n-1}\right|, n=+\infty\right)$$

$$0 \tag{71}$$

The series converges conditionally. Let us check the absolute convergence, i.e. convergence of a series of modules:

$$\sum_{n=1}^{\infty}\frac{1}{2\,n-1}$$

When comparing this series with the harmonic series, it is obvious that the series of modulesdiverges. As a result, the original series does not converge absolutely.

2471. According to the Leibniz criterion for the convergence of alternating series:

$$limit\left(\left|\frac{(-1)^{n-1}}{\sqrt{n}}\right|, n=+\infty\right)$$

$$0 \tag{72}$$

The series converges conditionally. Let us check the absolute convergence, i.e. convergence of a series of modules:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This is a special case of the generalized harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

with α=1/2. The generalized harmonic series converges when α>1 and diverges when α<=1, therefore, the series of modules diverges. As a result, the original series does not converge absolutely.

Task 19

$$a := n \rightarrow \frac{1}{n^x}$$

$$n \rightarrow \frac{1}{n^x} \tag{73}$$

$$R := \operatorname{abs}\left(\frac{a(n+1)}{a(n)}\right)$$

$$\left|\frac{n^x}{(n+1)^x}\right| \tag{74}$$

$$L := limit(R, n=\infty)$$

$$1 \tag{75}$$

$$interval_{conv} := solve(L < 1, x)$$

This series always diverges.

$$a := n \rightarrow \frac{\sqrt{n}}{(x-2)^n}$$

$$n \rightarrow \frac{\sqrt{n}}{(x-2)^n} \tag{76}$$

$$R := \operatorname{abs}\left(\frac{a(n+1)}{a(n)}\right)$$

$$\left|\frac{\sqrt{n+1}\,(x-2)^n}{(x-2)^{n+1}\sqrt{n}}\right| \tag{77}$$

$$L := limit(R, n=\infty)$$

$$\frac{1}{|x-2|} \tag{78}$$

$$interval_{conv} := solve(L < 1, x)$$

$$RealRange(Open(3), \infty), RealRange(-\infty, Open(1)) \tag{79}$$

Task 20

$$a := n \rightarrow \frac{(-1)^{(n-1)} x^n}{n}$$

$$n \rightarrow \frac{(-1)^{n-1} x^n}{n} \tag{80}$$

$$R := \text{abs}\left(\frac{a(n+1)}{a(n)}\right)$$

$$\left|\frac{(-1)^n x^{n+1} n}{(n+1)\,(-1)^{n-1} x^n}\right| \tag{81}$$

$$L := limit(R, n = \infty)$$

$$|x| \tag{82}$$

$$interval_{conv} := solve(L < 1, x)$$

$$RealRange(Open(-1), Open(1)) \tag{83}$$

$$a := n \rightarrow \frac{(n+1)^5 x^{2n}}{2n+1}$$

$$n \rightarrow \frac{(n+1)^5 x^{2n}}{2n+1} \tag{84}$$

$$R := \text{abs}\left(\frac{a(n+1)}{a(n)}\right)$$

$$\left|\frac{(n+2)^5 x^{2n+2} (2n+1)}{(2n+3)\,(n+1)^5 x^{2n}}\right| \tag{85}$$

$$L := limit(R, n = \infty)$$

$$|x|^2 \tag{86}$$

$$interval_{conv} := solve(L < 1, x)$$

$$RealRange(Open(-1), Open(1)) \tag{87}$$

Task 21

$fourierseries :=$ **proc**$(f, x, x1, x2, n)$ **local** $k, l,$
$a, b, s;$

$$l := \frac{(x2 - x1)}{2};$$

$$a[0] := \frac{int(f, x = x1..x2)}{l};$$

$$a[k] := \frac{int\left(f \cdot \cos\left(\frac{k \cdot \pi \cdot x}{l}\right), x = x1..x2\right)}{l};$$

$$b[k] := \frac{int\left(f \cdot \sin\left(\frac{k \cdot \pi \cdot x}{l}\right), x = x1..x2\right)}{l};$$

$$s := \frac{a[0]}{2} + sum\left(a[k] \cdot \cos\left(\frac{k \cdot \pi \cdot x}{l}\right) + b[k] \cdot \sin\left(\frac{k \cdot \pi \cdot x}{l}\right), k = 1..n\right);$$

end;

proc$(f, x, x1, x2, n)$ **(88)**

local $k, l, a, b, s;$

$l := 1/2*x2 - 1/2*x1;$
$a[0] := int(f, x = x1 .. x2)/l;$
$a[k] := int(f*\cos(k*\pi*x/l), x = x1 .. x2)/l;$
$b[k] := int(f*\sin(k*\pi*x/l), x = x1 .. x2)/l;$
$s := 1/2*a[0] + sum(a[k]*\cos(k*\pi*x/l) + b[k]*\sin(k*\pi*x/l), k = 1 .. n)$

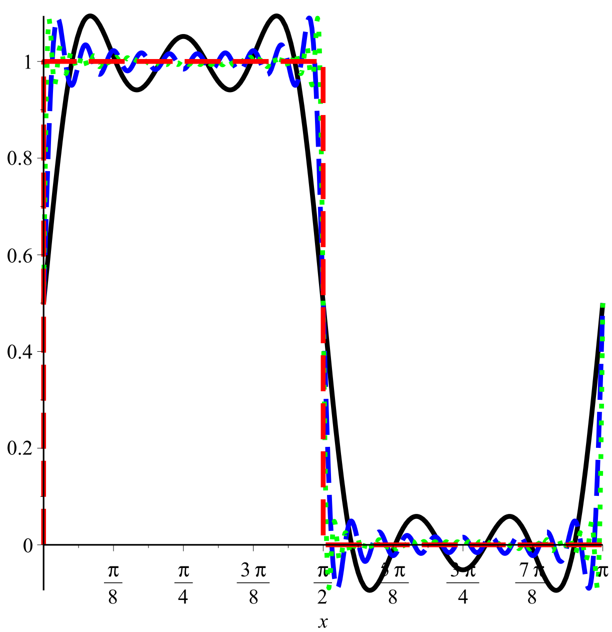
end proc

$f := piecewise\left(x < \frac{\pi}{2} \text{ and } x > 0, 1, \frac{\pi}{2} \le x \text{ and } x \le \pi, 0\right) : x1 := 0 : x2 := \pi :$

$fr_{gen5} := fourierseries(f, x, x1, x2, 5);$

$$\frac{1}{2} + \frac{2\sin(2x)}{\pi} + \frac{2}{3}\frac{\sin(6x)}{\pi} + \frac{2}{5}\frac{\sin(10x)}{\pi} \tag{89}$$

$fr_{gen20} := fourierseries(f, x, x1, x2, 20) :$
$fr_{gen50} := fourierseries(f, x, x1, x2, 50) :$
$plot(\{f, fr_{gen5}, fr_{gen20}, fr_{gen50}\}, x = x1 .. x2, color = [black, blue, green, red], thickness = 3, linestyle = [1, 3, 2, 3])$



Differential equations

Task 22

$Eq_{22} := y' = \frac{y+1}{x}$

$$\frac{d}{dx}\, y(x) = \frac{y(x)+1}{x}$$

$dsolve(\{Eq_{22}, y(1)=0, \})$

$$y(x) = x - 1 \qquad (91)$$

Task 23

$dsolve(\{y'' + 4\,y = 2\sin(2\,x) - 3\cos(2\,x) + 1\})$

$$\left\{y(x) = \sin(2\,x)\,\_C2 + \cos(2\,x)\,\_C1 - \frac{1}{2}\cos(2\,x)\,x - \frac{3}{8}\cos(2\,x) - \frac{3}{4}\sin(2\,x)\,x + \frac{1}{4}\right\} \qquad (92)$$

Task 24

$Eq_{24} := (1+x^2)\, y'' - 2\,x\,y' = 0$

$$(x^2+1)\left(\frac{d^2}{dx^2}\, y(x)\right) - 2\left(\frac{d}{dx}\, y(x)\right) x = 0$$

$dsolve(\{Eq_{24}, y(0)=0, y'(0)=3\})$

$$y(x) = x^3 + 3\,x \qquad (94)$$

Task 25

$Sys_{25} := diff(y(x), x) = z(x), diff(z(x), x) = -y(x)$

$$\frac{d}{dx}\, y(x) = z(x), \frac{d}{dx}\, z(x) = -y(x) \qquad (95)$$

$dsolve(\{Sys_{25}\})$

$$\{y(x) = \_C1\sin(x) + \_C2\cos(x), z(x) = \_C1\cos(x) - \_C2\sin(x)\} \qquad (96)$$

$Sys_{251} := diff(y(x), x) + 2\,y(x) + z(x) = \sin(x), diff(z(x), x) - 4\,y(x) - 2\,z(x) = \cos(x)$

$$\frac{d}{dx}\, y(x) + 2\,y(x) + z(x) = \sin(x), \frac{d}{dx}\, z(x) - 4\,y(x) - 2\,z(x) = \cos(x) \qquad (97)$$

$dsolve(\{Sys_{251}\})$

$$\{y(x) = 2\sin(x) + x\,\_C1 + \_C2, z(x) = -2\cos(x) - \_C1 - 3\sin(x) - 2\,x\,\_C1 - 2\,\_C2\} \qquad (98)$$

Task 26

$x_{pr} := 2\,e^{-x} - \sqrt{4 + a\,y}$

$$2\,e^{-x} - \sqrt{a\,y + 4} \qquad (99)$$

$y_{pr} := \ln(1 + x + a\,y)$

$$\ln(a\,y + x + 1) \qquad (100)$$

$F_{pr} := Vector([x_{pr}, y_{pr}])$

$$\left(2\,e^{-x} - \sqrt{a\,y+4}\right)e_x + (\ln(a\,y + x + 1))e_y \qquad (101)$$

$J_{pr} := Jacobian(F_{pr}, [x, y])$

$$\begin{bmatrix} -2\,e^{-x} & -\frac{1}{2}\frac{a}{\sqrt{a\,y+4}} \\ \frac{1}{a\,y+x+1} & \frac{a}{a\,y+x+1} \end{bmatrix} \qquad \textbf{(102)}$$

$J_{pr} := Jacobian(F_{pr}, [x, y] = [0, 0])$

$$\begin{bmatrix} -2 & -\frac{1}{8}\sqrt{4}\,a \\ 1 & a \end{bmatrix} \qquad \textbf{(103)}$$

$eigen := eval(eigenvalues(\textbf{(103)}))$

$$\frac{1}{2}\,a-1+\frac{1}{2}\sqrt{a^2+3\,a+4},\ \frac{1}{2}\,a-1-\frac{1}{2}\sqrt{a^2+3\,a+4} \qquad \textbf{(104)}$$

In order for the zero solution *to be stable* by the Lyapunov theorem, the maximum of the real parts of the eigenvalues must be less than 0. Obviously, the real parts of both eigenvalues must be less than 0.

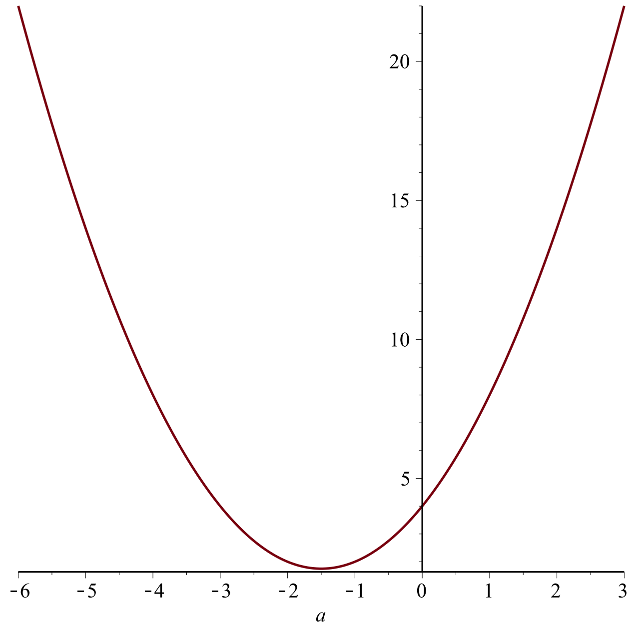
$eigen_1$

$$\frac{1}{2}\,a-1+\frac{1}{2}\sqrt{a^2+3\,a+4} \qquad \textbf{(105)}$$

$eigen_2$

$$\frac{1}{2}\,a-1-\frac{1}{2}\sqrt{a^2+3\,a+4} \qquad \textbf{(106)}$$

$plot(a^2+3\,a+4, a=-6..3)$



Expression under a square root is always positiv, so square root is always a real number. In this case:
$solve(\mathrm{eigen}_1 < 0)$

$$RealRange(-\infty, Open(0)) \tag{107}$$

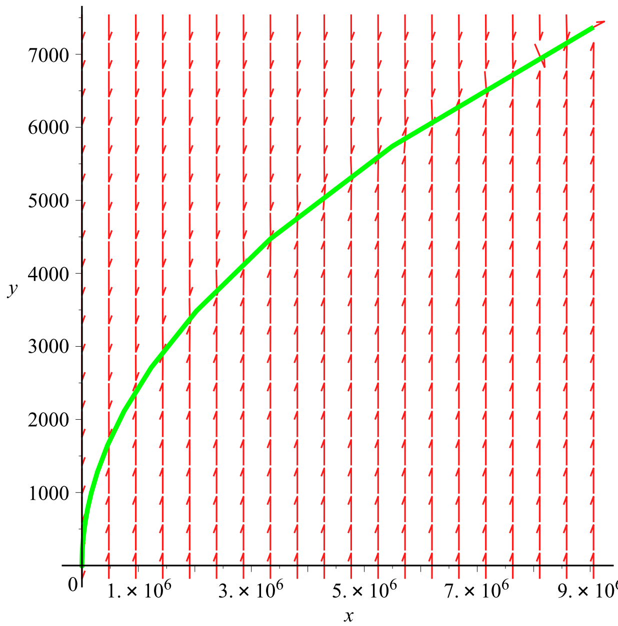
$solve(\mathrm{eigen}_2 < 0)$

$$a \tag{108}$$

what means eigen_2 is always less than 0
So for a from (-∞;0) solution of given system is stable.

Task 27

$DEplot\left(\left\{diff(x(t), t) = 2\,x(t) + (y(t))^2 - 1, diff(y(t), t) = 6\,x(t) - (y(t))^2 + 1\right\}, [x(t), y(t)], t = 0 ..3, [[0, 0, 0]], stepsize = 0.1, linecolor = green\right)$



Task 28

$Order := 4 :$

$Eq_{28} := y' = y + x\,\mathrm{e}^{y}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\, y(x) = y(x) + x\,\mathrm{e}^{y(x)} \tag{109}$$

$dsolve(\{Eq_{28}, y(0) = 0\}, y(x), series)$

$$y(x) = \frac{1}{2}\, x^2 + \frac{1}{6}\, x^3 + \mathrm{O}(x^4) \tag{110}$$

Task 29

$Sys_{29} := diff(y(x), x) = \frac{(y(x))^2}{z(x) - x},\, diff(z(x), x) = y(x) + 1$

$$\frac{\mathrm{d}}{\mathrm{d}x}\, y(x) = \frac{y(x)^2}{z(x) - x},\ \frac{\mathrm{d}}{\mathrm{d}x}\, z(x) = y(x) + 1 \tag{111}$$

$dsolve(\{Sys_{29}\})$

$$\left[\left\{z(x) = \frac{\_C2}{e^{x\_C1}} + x\right\}, \left\{y(x) = \frac{d}{dx} z(x) - 1\right\}\right] \tag{112}$$

Task 30

$Eq_{30} := x^2 y'' + 2\,x\,y' + 4\,y = -x^3$

$$x^2 \left(\frac{d^2}{dx^2} y(x)\right) + 2 \left(\frac{d}{dx} y(x)\right) x + 4\,y(x) = -x^3$$

$dsolve(\{Eq_{30}, y(1) = 7, y(4) = -1\})$

$$y(x) = -\frac{1}{16} \frac{\sin\left(\frac{1}{2}\sqrt{15}\,\ln(x)\right)\left(-96 + 113\cos\left(\sqrt{15}\,\ln(2)\right)\right)}{\sqrt{x}\,\sin\left(\sqrt{15}\,\ln(2)\right)} + \frac{113}{16} \frac{\cos\left(\frac{1}{2}\sqrt{15}\,\ln(x)\right)}{\sqrt{x}} - \frac{1}{16} x^3 \tag{114}$$