To determine the required boundary condition for this result to hold, let us consider the functions $f(x) = J_v(\alpha x)$ and $g(x) = J_v(\beta x)$, which, as willbe proved below, respectively satisfy the equations

$$x^2 f + xf + (\alpha^2 x^2 - v^2) f = 0,$$
$$x^2 g + xg + (\beta^2 x^2 - v^2) f = 0,$$

$\boxed{\text{Show that } f(x) = J_y(\alpha x)\text{satisfies (18.85).}}$

If $f(x) = nJ_y(ax)$ and we write $w = \alpha x$, then

$$\frac{df}{dx} = \alpha \frac{dJ_y(w)}{dw}$$

and

$$\frac{d^2 f}{dx^2} = \alpha^2 \frac{d^2 J_v(w)}{dw^2}$$

.

When these expressions are substitutes into 918.85), its LHS becomes
$x^2\alpha^2 \frac{d^2 J_v(w)}{dw^2} + x\alpha \frac{dJ_v(w)}{dw} + (\alpha^2 x^2 = v^2) J_v(w)$

$$= w^2 \frac{d^2 J_v(w)}{dw^2} + w \frac{dJ_v(w)}{dw} + (w^2 - v^2) J_v(w)$$

But, from Bessel's equation itself, this final expression is equal to zero, thus verifying that$f(x)$does satisfy(18.85) $\triangle$

Now multiplying (18,85) by $f(x)$ and (18,85) by $g(x)$ and subtracting them gives

$$\frac{d}{dx} x \Big[(fg' - gf') \Big] = (\alpha^2 - \beta^2) xfg$$

where we have used the fact that

$$\frac{d}{dx} \Big[(x(fg' - gf') \Big] = x(fg'' - gf'') + (fg' - gf').$$

By integrating (18,87)over any given range $x = \alpha$ to $x = \beta$, we obtain

$$\int_a^b xf(x)g(x)dx = \frac{1}{\alpha^2 - \beta^2} \Big[xf(x)g'(x) - xg(x)f'(x) \Big]_a^b$$

,

which, on setting $f(x)J_v(\alpha x)$ and $g(f) = J_v(\beta x)$, becomes

$$\int_a^b xJ_v(\alpha x)J_v(\beta x)dx = \frac{1}{\alpha^2 - \beta^2} \Big[\beta x J_v(\alpha x) J_v'(\beta x) = \alpha x J_v(\beta x) J_v'(\alpha x) \Big]_a^b (18,88)$$

If $x \neq \beta$, and the interval [a,b] is such that the expression on the RHS of (18,88) equals zero, then we obtain the orthogonality condition (18,84). This happends, for example, if $J_v(xx)$ and $J_v()$ vanish at $x = a$ and $x = b$,or if $J_v(xx)$ $J_v()$ vanish at $x = a)$ and $x = b$, or more for many general conditions. It shoud be noted that the boundary term is automatically zero at the point $x = 0$, as one might expect from the fact that the Sturm-Liouville from of Bessel's equation has $p(x) = x$.

If $\alpha = \beta$, the RHS of (18.88) takes the indeterminant from 0/0. This may be