Machine Learning Exercise Sheet 07

Constrained Optimization

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• Constrained Optimization / Projected GD

Problem (1)

In order to find maximizer, since the plotted domain X is convex we can check the vertices. which we find at the following points:

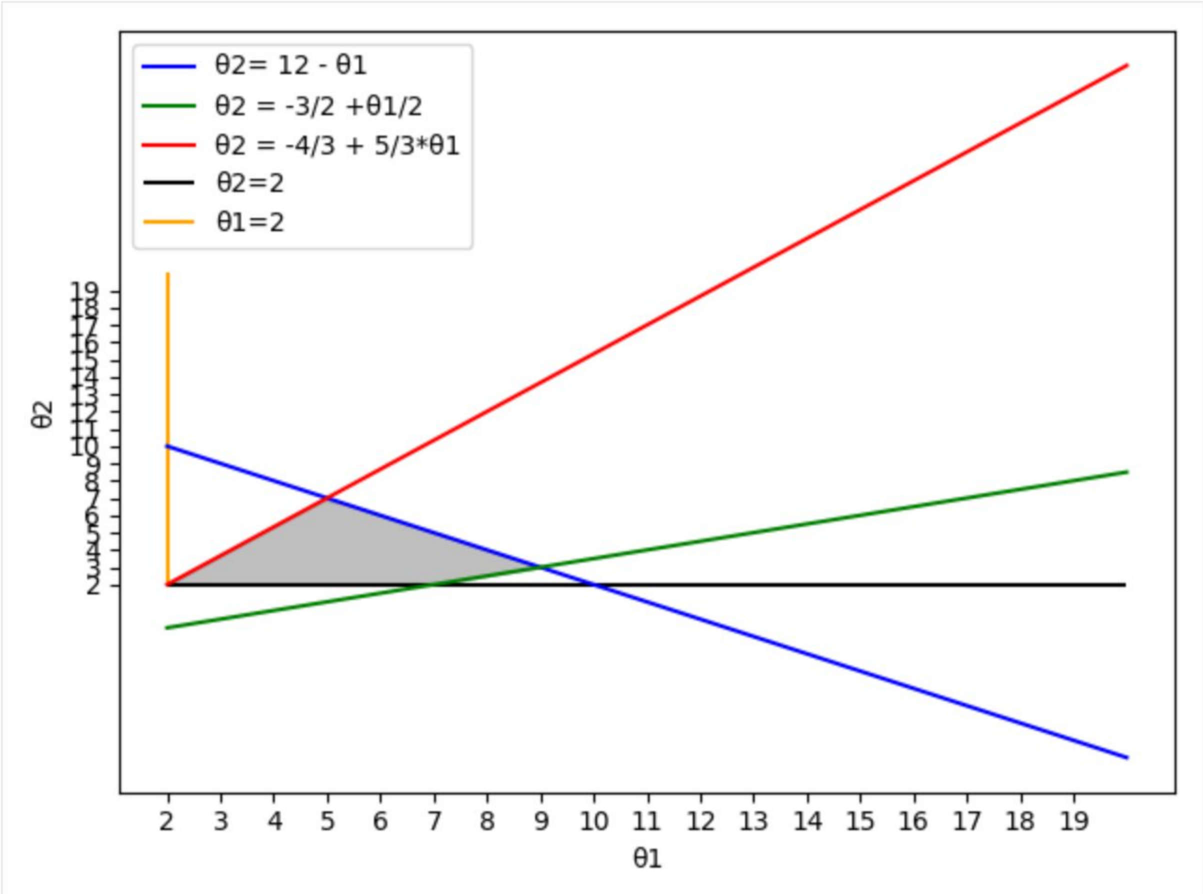
$(7,2) \rightarrow f(7,2) = 2\cdot 7 - 3\cdot 2 = 14 - 6 = 8$

$(9,3) \rightarrow f(9,3) = 2\cdot 9 - 3\cdot 3 = 18 - 9 = 9$

$(5,7) \rightarrow f(5,7) = 2\cdot 5 - 3\cdot 7 = 10 - 21 = -11$

$(2,2) \rightarrow f(2,2) = 2\cdot 2 - 3\cdot 2 = 4 - 6 = -2$

We choose the biggest value out of them which is 9 for ϑ_{max} $(9,3)$



θ2= 12 - θ1
θ2 = -3/2 +θ1/2
θ2 = -4/3 + 5/3*θ1
θ2=2
θ1=2
θ2
θ1
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

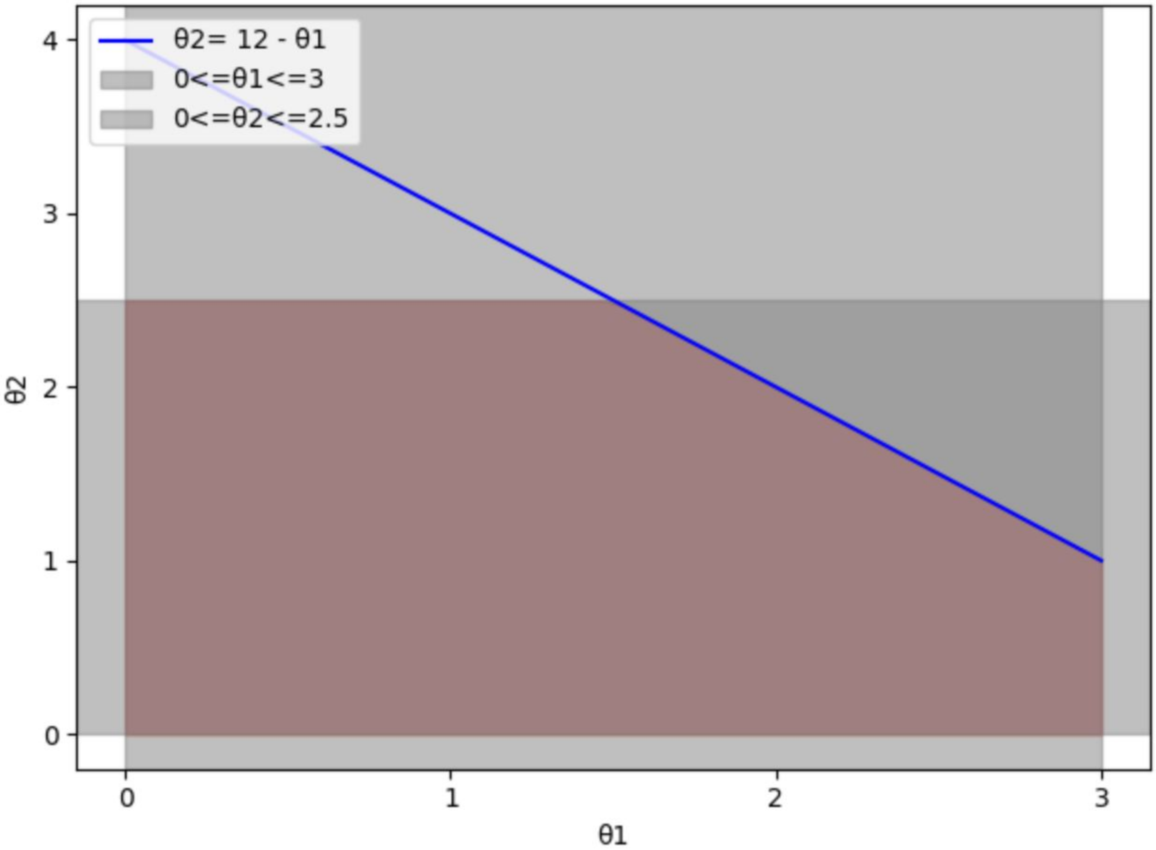
Problem (2):

By looking at the plot it's easy to see that all the points with $\vartheta_2 < 1$ or $\vartheta_1 < 1.5$ we can use box-constraints.

For points between $X_2 = X_1 + 1$ and $X_2 = X_1 - 2$ and above $X_2 = -X_1 + 4$ we can use line projection. In all other cases, we should take $(1.5, 2.5)$ or $(3, 1)$ as they are the nearest to the projection on the line.

$$\Pi_X(p) = \begin{cases} a + \dfrac{(p-a)^T(b-a)}{\|b-a\|_2^2}(b-a) & (\vartheta_1 + \vartheta_2 \leq 4,\ 0 \leq \vartheta_1 \leq 3,\ 0 \leq \vartheta_2 \leq 2.5 \\ (1.5,\ 2.5) & X_1 > 1.5 \text{ and } (X_2 - X_1 \geq 1) \\ (3,\ 1) & X_2 > 1 \text{ and } (X_2 - X_1 \leq -2) \\ \min(\max(l_i, p_i)\, u_i) & \text{else} \end{cases}$$

a, b lay on $X_2 = -X_1 + 4$


θ2= 12 - θ1
0<=θ1<=3
0<=θ2<=2.5
θ2
θ1
0
1
2
3
4

• Lagrangian / Duality

Problem (3) :

$$f_0(\vartheta) = \vartheta_1 - \sqrt{3}\vartheta_2$$
$$f_1(\vartheta) = \vartheta_1^2 + \vartheta_2^2 - 4$$

1. f_0 and f_1 are convex (sum of convex functions)
2. There exists ϑ (eg $[0,0]$) for which $f_1 < 0$

(1), (2) $\Rightarrow$ Slater's constraint qualification is fullfilled. Therefore strong duality holds.!

$$L(\vartheta_1, \vartheta_2, a) = \vartheta_1 - \sqrt{3}\vartheta_2 + a(\vartheta_1^2 + \vartheta_2^2 - 4)$$
$$\nabla_\vartheta L(\vartheta_1, \vartheta_2, a) = [1 + 2a\vartheta_1, -\sqrt{3} + 2a\vartheta_2]$$
$$\nabla_\vartheta L(\vartheta_1, \vartheta_2, a) = 0 \Leftrightarrow \vartheta_1 = \frac{-1}{2a}, \vartheta_2 = \frac{\sqrt{3}}{2a}$$
$$\theta^*(a) = \arg\min_\vartheta L(\vartheta_1, \vartheta_2, a) = \left[\frac{-1}{2a}, \frac{\sqrt{3}}{2a}\right]$$
$$g(a) = L(\vartheta^*(a), a) = -\frac{1}{a} - 4a$$
$$g'(a) = a^{-2} - 4 = 0 \Leftrightarrow a = \frac{1}{2}$$
$$g''(a) = -2a^{-3}$$
$$g''\left(\frac{1}{2}\right) = -16 < 0 \Rightarrow \max(g(a)) = -4 = \min_\vartheta L[\vartheta_1, \vartheta_2]$$

We are going back to gradient of L to find ϑ for which f_0 has minimum.

$$\vartheta_1 = \frac{-1}{2 * \frac{1}{2}} = -1 \qquad \vartheta_2 = \frac{\sqrt{3}}{2 * \frac{1}{2}} =$$

Because strong equality holds we can say that f_0 subject to $f_1 < 0$ has minimum at $[-1, \sqrt{3}]$ with value -4.

Problem (4) :

$$f_0(w, b) = \frac{1}{2} w^T w$$

$$\text{subject to :} \quad f_1(w, b) = y_i\left(w^T x_i + b\right) - 1 \geq 0$$

To show that duality gap is 0 we can use weak Slater's condition which says that strong duality holds when f_0 is convex and f_i are affine.

First is satisfy as quadratic function is convex. Second is satisfy as f_i are affine functions w.r.t w and b.