Linear Algebra Review

- Vector
- Vector Operations
- Linear Combination
- Span, Bases, Linear Independence
- Length of Vectors
- Dot Product
- Angle between Vectors
- Projection
- Linear Functions
- Linear Transformation
- Scaling, Mirror, Shear, Rotation, Projection
- Composition of Linear Transformations
- Determinent and Inverse
- Change of Bases
- Transformation in Different Bases
- Eigenvalue and Eigenvectors
- Eigenbases and Diagnoalization
- Power of Matrices
- Random Walk and Markov Chain

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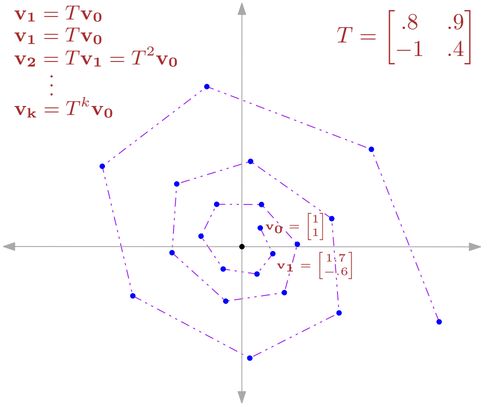
Powers of Matrices

Powers of Matrices

Suppose T represents the change in location of a particle per second

$$
\begin{aligned}
\mathbf{v_1} &= T\mathbf{v_0} \\
\mathbf{v_1} &= T\mathbf{v_0} \\
\mathbf{v_2} &= T\mathbf{v_1} = T^2\mathbf{v_0} \\
&\vdots \\
\mathbf{v_k} &= T^k\mathbf{v_0}
\end{aligned}
\qquad
T = \begin{bmatrix} .8 & .9 \\ -1 & .4 \end{bmatrix}
$$



Find location of the particle after two weeks

Powers of Matrices

Fibonacci numbers F_n, $\quad 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-2} + F_{n-1} & \text{if } n \geq 2 \end{cases}$$

Let $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] \quad \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right] \quad \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 5 \\ 3 \end{smallmatrix}\right] \quad \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 5 \\ 3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 8 \\ 5 \end{smallmatrix}\right] \quad \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 8 \\ 5 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 13 \\ 8 \end{smallmatrix}\right]$

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \qquad \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = T^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} = \frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}}$$

Powers of Matrices

First order linear recurrence relation

$$x_{t+1} = ax_t$$
$$x_0 = 3$$

Coupled system of recurrence relations

$$x_{t+1} = 3x_t + 5y_t$$
$$y_{t+1} = 4x_t - 2y_t$$
$$x_0 = 2,\ y_0 = 3$$

Model many practical scenarios in population dynamics, economics, epidemiology, computing, signal processing

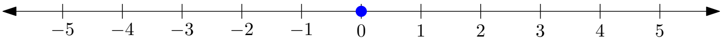
Let $\mathbf{u_t} = \begin{bmatrix} x_t \\ y_t \end{bmatrix}$

$\mathbf{u_0} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$T = \begin{bmatrix} 3 & 5 \\ 4 & -2 \end{bmatrix}$

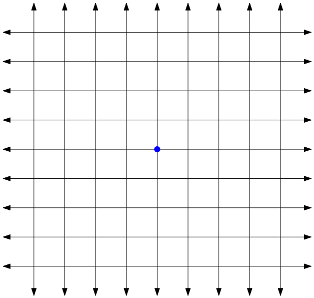
- $\mathbf{u_1} = T\mathbf{u_0}$
- $\mathbf{u_2} = T\mathbf{u_1} = TT\mathbf{u_0} = T^2\mathbf{u_0}$
- $\mathbf{u_3} = T\mathbf{u_2} = TT^2\mathbf{v_0} = T^3\mathbf{u_0}$
- $\vdots$
- $\mathbf{u_k} = T^k\mathbf{u_0}$

Random Walk

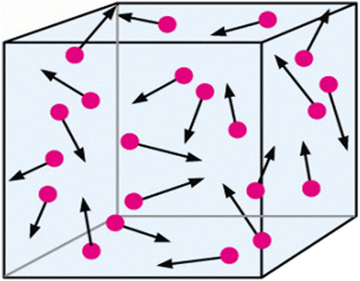


- Suppose the blue dot (•) starts at 0
- At every step if it is at number i, then with probability $1/2$ it goes $i+1$ and and with probability $1/2$ it goes to $i-1$
- How many steps would it take to reach 6 or -8?
- What is root mean squared distance the • covers in n steps?
- Many possible extensions
- Lazy walks: with prob. $1/2$ stay at i, move to $i \pm 1$ each prob $1/4$
- Biased walks: with prob. $3/4$ move to $i+1$ and $1/4$ move to $i-1$
- Biased walks: with prob. $1/2$ move to $i+b$ and $1/2$ move to $i-1$
- Models many things: stock prices fluctuations, gambling outcomes, team results in a game's season, molecules movements

Random Walk Generalizations



At every step • goes $\{Up, Down, Left, Right\}$ with probability $1/4$



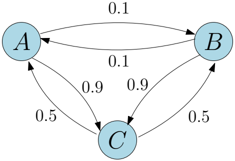
- Random walk on grid
- Random walk in space, often called Brownian motion
  - Model movements of particles in liquid or gas. The particle undertake random walk caused by momentum imparted to it by molecules in random directions

Random Walk on Graphs

- Let $G = (V, E)$ be a graph or digraph
- Let $d(u)$ be the degree of $u \in v$
- A random walker starts at some vertex $v_0 \in V$
- At every step if the walker is at vertex u, it picks randomly moves to a random (out) neighbor of u
- The probability that current vertex is u and next vertex is $v \in N(u)$ is $^1/_{d(u)}$ or $^1/_{d^+(u)}$ (for digraphs)

Markov Chain

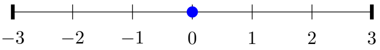
- A Markov chain is a stochastic process defined on finite number of states
- The changes of state of system are called transition
- Transitions probabilities b/w states are given in transition matrix T
- Let X_n be the state of the system at time n
- $T(i,j) := Pr[X_{n+1} = i | X_n = j]$: prob. that system goes from state j to i
- $0 \leq T(i,j) \leq 1$ and columns sum to 1 ▷ column-stochastic
- Memoryless process: $T(i,j)$ does not depend on the history of transitions ▷ Markovian property
- Given present state, the past and future states are independent



$$\begin{bmatrix} 0.0 & 0.1 & 0.5 \\ 0.1 & 0.0 & 0.5 \\ 0.9 & 0.9 & 0.0 \end{bmatrix}$$

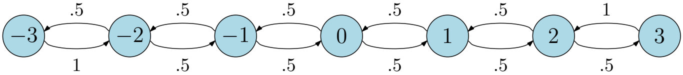
Markov Chain

- Bounded Random Walk on integers $\{-3, \ldots, 3\}$



$$\begin{array}{cc} & \begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & .5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & .5 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 & 0 & 0 \\ 0 & 0 & .5 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & .5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & .5 & 0 \end{bmatrix} \end{array}$$

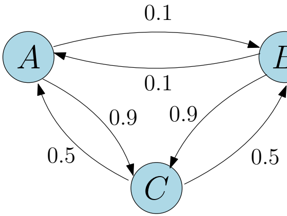
- The • begins at 0
- If • is at ± 3, then with prob. 1 it goes to ± 2
- If • is at $i \neq \pm 3$, then with prob. .5 it goes to $i \pm 1$



Language Recognition System

- Smartphones next words suggestions use language generation
- The first i words are typed, what will be the $(i+1)$st word?
- Model language generation as a Markov chain ▷ not realistic
- States correspond to last used words (say vocabulary has 1000 words)
- Transition probabilities $p_{w_i w_j} := Pr[w_j|w_i] := \dfrac{freq(w_i w_j)}{freq(w_i)}$
- Estimate the 1000×1000 probabilities from a large text corpus
- Probability of generating a text $w_1 w_2 w_3 w_4 w_5$ is $p_{w_1} p_{w_1 w_2} p_{w_2 w_3} p_{w_3 w_4} p_{w_4 w_5}$
- p_{w_i} is (empirical prob) frequency of w_i as first word in the corpus
- Can extend it by estimating $p_{w_i w_j w_k} := Pr[w_k|w_i w_j]$

Markov Chain

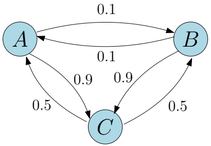


$$\begin{bmatrix} 0.0 & 0.1 & 0.5 \\ 0.1 & 0.0 & 0.5 \\ 0.9 & 0.9 & 0.0 \end{bmatrix}$$

- Instead of thinking that the system is in a given state at time t, consider
- a vector $\mathbf{x}$ specifying probability distribution of system being in all states
- $\mathbf{x}^{(\mathbf{t})}$ is probability distribution at time t, ${\mathbf{x}^{\mathbf{t}}}_i \geq 0$, $\sum_i {\mathbf{x}^{\mathbf{t}}}_i = 1$
- $\mathbf{x}^{(\mathbf{t+1})} = T\mathbf{x}^{(\mathbf{t})}$
- By Markovian property, probability of going from j to i in two steps is $\sum_k T(k,j)T(i,k) = T^2(i,j)$
- probability of going from j to i in s steps is $T^s(i,j)$

Markov Chain

- $\mathbf{x^{(t)}}$: prob. distribution at time t
- $\mathbf{x^{(t+1)}} = T\mathbf{x^{(t)}}$



$$\begin{bmatrix} 0.0 & 0.1 & 0.5 \\ 0.1 & 0.0 & 0.5 \\ 0.9 & 0.9 & 0.0 \end{bmatrix}$$

A distribution π is a stationary distribution for Markov chain T, if

$$T\pi = \pi$$ ▷ eigenvector of T with eigenvalue 1

- The largest eigenvalue of a column stochastic real matrix is real ($\lambda_1 = 1$)

A markov chain is *ergodic* if there is a unique stationary distribution π and for any initial distribution $\mathbf{x}$ we have

$$\lim_{t\to\infty} M^t\mathbf{x} = \pi$$ ▷ always converges to π

Eigenbases and Diagnolization: Applications

Transformation in different Bases

- Translation of vectors and linear transformation between standard bases and another bases B
- Vectors in B are basis vectors (linearly independent) B is invertible

$$B = \begin{bmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b_2} & \dots & \mathbf{b_n} \\ | & | & & | \end{bmatrix}$$

$$\begin{tikzcd} \mathbf{x} \arrow[rr, "T"] \arrow[d, shift right, "B^{-1}"'] & & \mathbf{x}' \arrow[d, shift left, "B^{-1}"] \\ \mathbf{x}_B \arrow[rr, "T_B"'] \arrow[u, shift right, "B"'] & & \mathbf{x}'_B \arrow[u, shift left, "B"] \end{tikzcd}$$

$$T_B = B^{-1} T B \qquad T = B T_B B^{-1}$$

Eigenbases: Diagonalization

- Diagonalization simplifies matrix operations by finding a basis in which the matrix is diagonal
- An eigenbasis makes this possible for linear transformations, leading to efficient computations
- Changing to an eigenbasis simplifies linear transformations to scaling operations along coordinate axes

Eigenbases: Diagonalization

- Let T be a $n \times n$ linear transformation
- Let $B = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ be bases - vectors in B are eigenvectors of T
- For $1 \leq i \leq n$, $\quad T\mathbf{b_i} = \lambda_i \mathbf{b_i}$
- Note there must be n vectors in B

$$B = \begin{bmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b_2} & \dots & \mathbf{b_n} \\ | & | & & | \end{bmatrix}$$

$$\begin{tikzcd} \mathbf{x} \arrow[r, "T"] \arrow[d, shift right, "B^{-1}"'] & \mathbf{x}' \arrow[d, shift left, "B^{-1}"] \\ \mathbf{x}_B \arrow[r, "T_B"'] \arrow[u, shift right, "B"'] & \mathbf{x}'_B \arrow[u, shift left, "B"] \end{tikzcd}$$

$$T_B = B^{-1} T B \qquad\qquad T = B T_B B^{-1}$$

- How does $T\mathbf{x}$ looks like in eigenbasis?

Eigenbases: Diagonalization

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$$T_B = B^{-1}TB \qquad \begin{tikzcd} \mathbf{x} \arrow[r, "T"] \arrow[d, shift right, "B^{-1}"'] & \mathbf{x}' \arrow[d, shift left, "B^{-1}"] \\ \mathbf{x}_B \arrow[r, "T_B"'] \arrow[u, shift right, "B"'] & \mathbf{x}'_B \arrow[u, shift left, "B"] \end{tikzcd} \qquad T = BT_BB^{-1}$$

- How does $T\mathbf{x}$ looks like in eigenbasis?

$$T\mathbf{x} = T(\alpha_1\mathbf{e_1} + \ldots + \alpha_n\mathbf{e_n}) \quad = \quad T(\beta_1\mathbf{b_1} + \ldots + \beta_n\mathbf{b_n})$$

$$= \beta_1 T\mathbf{b_1} + \ldots + \beta_n T\mathbf{b_n} \quad = \quad \beta_1\lambda_1\mathbf{b_1} + \ldots + \beta_n\lambda_n\mathbf{b_n}$$

$$= \begin{bmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b_2} & \ldots & \mathbf{b_n} \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \quad = \quad BD\mathbf{x_B} \quad = \quad BDB^{-1}\mathbf{x}$$

- Let T be a $n \times n$ linear transformation
- Let $B = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be bases - vectors in b are eigenvectors of T
- For $1 \leq i \leq n, \quad T\mathbf{b_i} = \lambda_i \mathbf{b_i}$

$$B = \begin{bmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b_2} & \ldots & \mathbf{b_n} \\ | & | & & | \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$T\mathbf{x} \; = \; BDB^{-1}\mathbf{x}$$

Very easy to take T to a higher power (compose it many times)

- $T \; = \; BDB^{-1}$
- $T^2 \; = \; BDB^{-1}BDB^{-1} \; = \; BDDB^{-1} \; = \; BD^2B^{-1}$
- $T^3 \; = \; BD^2B^{-1}BDB^{-1} \; = \; BD^2DB^{-1} \; = \; BD^3B^{-1}$
- $T^4 \; = \; BD^3B^{-1}BDB^{-1} \; = \; BD^3DB^{-1} \; = \; BD^4B^{-1}$
- $T^k \; = \; \ldots \; = \; BD^kB^{-1}$

Eigenbases: Diagonalization

- Let T be a $n \times n$ linear transformation
- Let $B = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be bases - vectors in b are eigenvectors of T
- For $1 \leq i \leq n$, $\quad T\mathbf{b_i} = \lambda_i \mathbf{b_i}$

$$B = \begin{bmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b_2} & \ldots & \mathbf{b_n} \\ | & | & & | \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$T\mathbf{x} \;=\; BDB^{-1}\mathbf{x}$$

- $T^k \;=\; BD^kB^{-1}$

$$D^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

Diagonalizing a Matrix

Find eigenvalues, eigenvectors of, and diagonalize $A = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$

Eigenvalues: Solve $\det(A - \lambda I) = 0$

For A, $\det \begin{bmatrix} 4-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) = 0$

Eigenvalues $\lambda_1 = 4$, $\lambda_2 = 3$

Eigenvectors: For $\lambda_1 = 4$: $(A - 4I)\mathbf{v} = 0 \implies \mathbf{v_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$

- For $\lambda_2 = 3$: $(A - 3I)\mathbf{v} = 0 \implies \mathbf{v_2} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$

Diagonal Matrix: D and Change of Basis Matrix P:

$D = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, columns are v_1 and v_2

Diagonalization: Verify $P^{-1}AP = D$

Powers of Matrices in Eigenbases

In an eigenbasis, computing powers of a matrix is greatly simplified, facilitating the analysis of systems over time

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$, with corresponding eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The diagonal matrix $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ and the change of basis matrix $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ with $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

To compute A^n, we use $A^n = PD^nP^{-1}$

For $n = 2$, $D^2 = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$, thus $A^2 = PD^2P^{-1} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$