Linear Algebra Review

- Vector
- Vector Operations
- Linear Combination
- Span, Bases, Linear Independence
- Length of Vectors
- Dot Product
- Angle between Vectors
- Projection
- Linear Functions
- Linear Transformation
- Scaling, Mirror, Shear, Rotation, Projection
- Composition of Linear Transformations
- Determinent and Inverse
- Change of Bases
- Transformation in Different Bases
- Eigenvalue and Eigenvectors
- Eigenbases and Diagnoalization
- Power of Matrices
- Random Walk and Markov Chain

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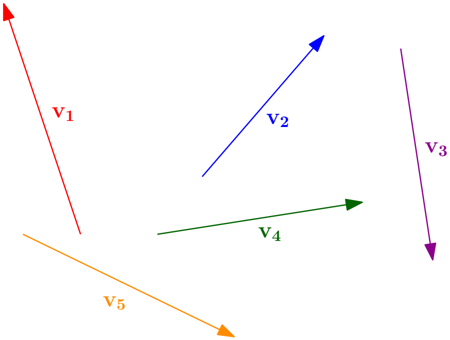
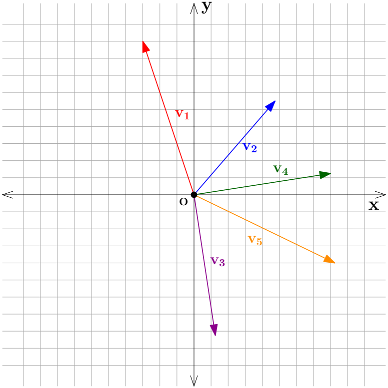
Vectors and Vector Operations

Vector

- Objects with lengths and directions
- Arrows in n-dimensional space, $\mathbb{R}^n$
- Technically, they are called *free vectors*

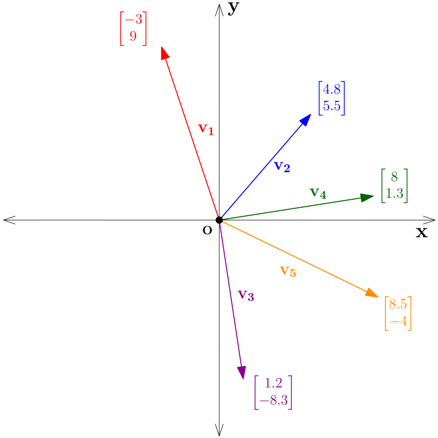


- A coordinate system ▷ Origin and unit length defined
- Look at vectors with tails fixed at the origin
- Displacement in coordinates from the origin



Vector

- A sequence of n numbers, array (ordered list) of numbers
- Bijection: n-length real sequences $\longleftrightarrow$ fixed-tail arrows



Vector

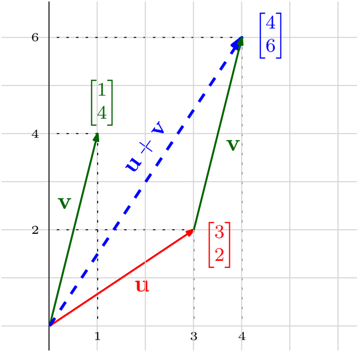
- n-dimensional objects, with the following operations well-defined
- Pairwise addition
- Scalar multiplication (multiplication with a real number)

$$\overset{\mathbf{A}}{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}} + \overset{\mathbf{B}}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}} = \overset{\mathbf{A}+\mathbf{B}}{\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}} \qquad x \overset{\mathbf{A}}{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}} = \overset{x\mathbf{A}}{\begin{bmatrix} xa_1 \\ xa_2 \\ \vdots \\ xa_n \end{bmatrix}}$$

Vector Operations: Addition

- Vectors addition defined numerically as $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$

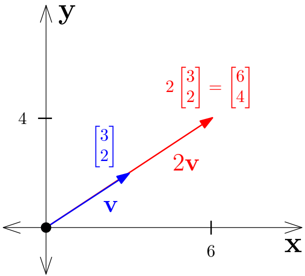
- Geometrically it is the cumulative displacement from origin in each dimension by following the vectors with tip-to-tail joining

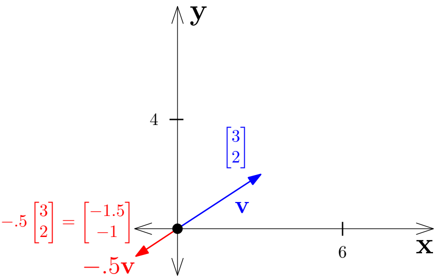


Vector Operations: Scaling

- Vector scaling defined numerically as $\quad x \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} xu_1 \\ \vdots \\ xu_n \end{bmatrix}$

- Geometrically it is the arrow scaled by a factor of x





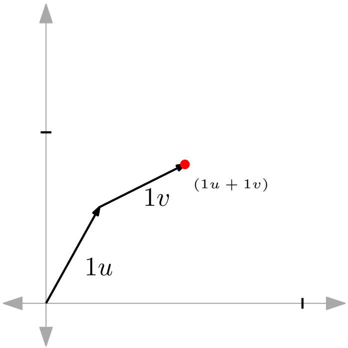
- Vectors subtraction is just combining scaling and addition

Vector Operations: Linear Combination

- Algebraically and geometrically a combination of scaling & addition

$$x \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + y \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} xu_1 + yv_1 \\ \vdots \\ xu_n + yv_n \end{bmatrix}$$

- linear combination $\because$ for fixed x and changing y, $x\mathbf{u} + y\mathbf{v}$ gives a line



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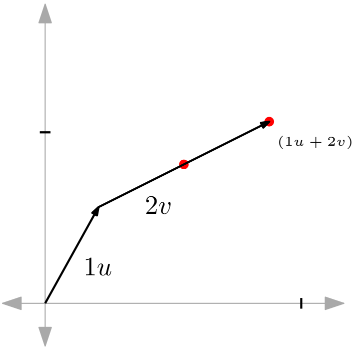

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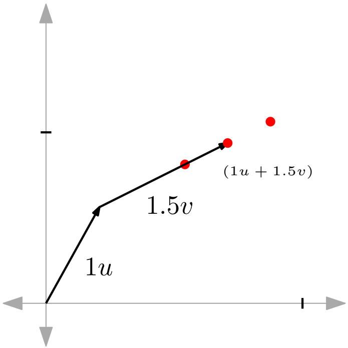

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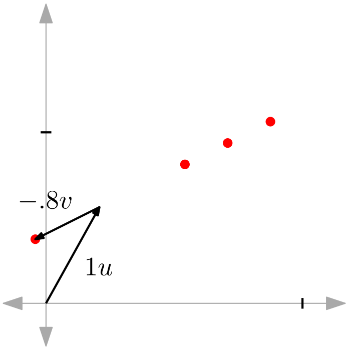

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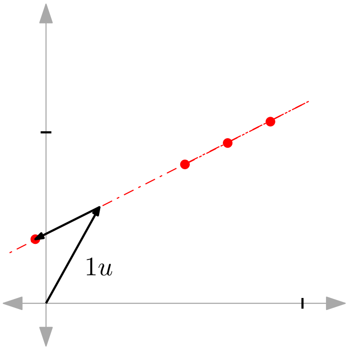

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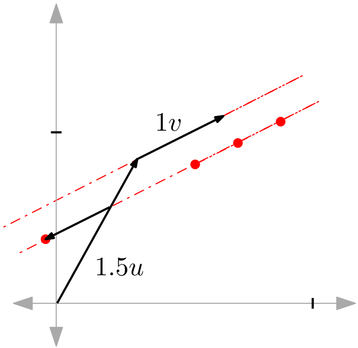

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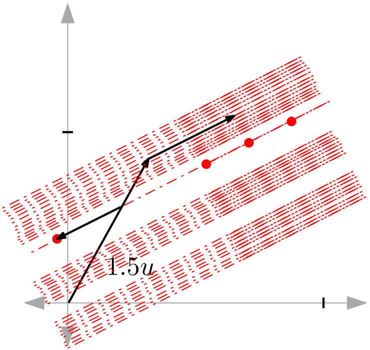
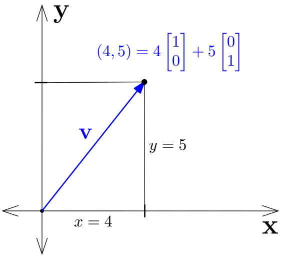


Vector: Standard Bases

- $\mathbf{e_1} = \hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e_2} = \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are standard basis vectors in $\mathbb{R}^2$
- A vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is two scalars expressing how much this vector scales the standard basis vectors $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$
- Each vector in $\mathbb{R}^2$ is a linear combination of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$



Vector: Standard Bases

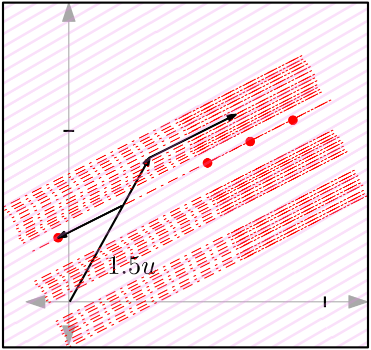
- In $\mathbb{R}^n$, $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots, \mathbf{e_n} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

- The standard bases are unit vectors along the axes

- A vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is $\mathbf{v} = v_1\mathbf{e_1} + v_2\mathbf{e_2} + \ldots + v_n\mathbf{e_n}$

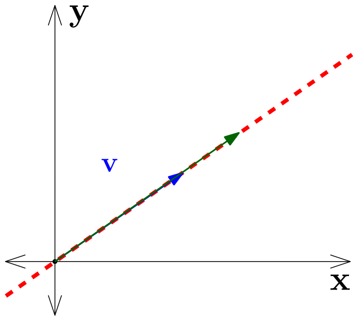
Vector: Different Bases

- Take two different vectors **u** and **v** ($\neq \hat{\mathbf{i}}, \hat{\mathbf{j}}$)
- Consider all linear combinations of **u** and **v**
- Try all combinations of scalars x and y, and check $x\mathbf{u} + y\mathbf{v}$
- Which vectors can you get? In most cases, you get all vectors in $\mathbb{R}^2$



Vector: Span, Bases and linear independence

- Take two different vectors $\mathbf{u}$ and $\mathbf{v}$ ($\neq \hat{\mathbf{i}}, \hat{\mathbf{j}}$)
- Span: space of vectors we get as linear combinations of $\mathbf{u}$ and $\mathbf{v}$
- Generally it is $\mathbb{R}^2$, or $\mathbf{u}$ and $\mathbf{v}$ line up $\implies$ it is a 1-dim subspace of $\mathbb{R}^2$
- $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, otherwise linearly independent
- Or when $\mathbf{u} = \mathbf{v} = \mathbf{0}$, then we get 0-dim subspace

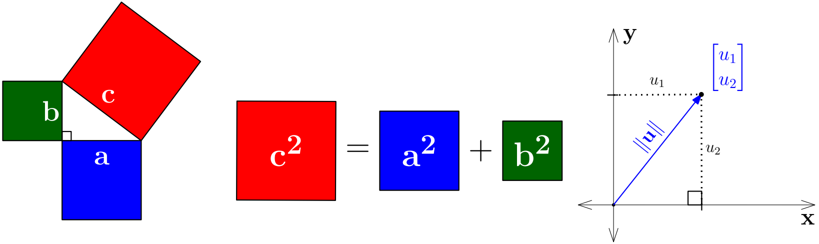


Vector: Span, Bases and linear independence

- Span of a vector $\mathbf{v} \in \mathbb{R}^2$ (actually any space) is a line (unless $\mathbf{v} = \mathbf{0}$)
- Span of 2 vectors in $\mathbb{R}^3$ is a plane (unless they line up)
- Span of 3 vectors in $\mathbb{R}^3$ is the whole $\mathbb{R}^3$ (unless one vector is in the plane spanned by the other two)
- Technically given k vectors if a vector can be removed without reducing the span, then they are linearly dependent
- That is if one vector can be expressed as linear combination of the others, then they are linearly dependent
- Otherwise, they are linearly independent, *every vector really adds another dimension*
- Basis of a vector space (or a space) is a set of linearly independent vectors that spans the whole space

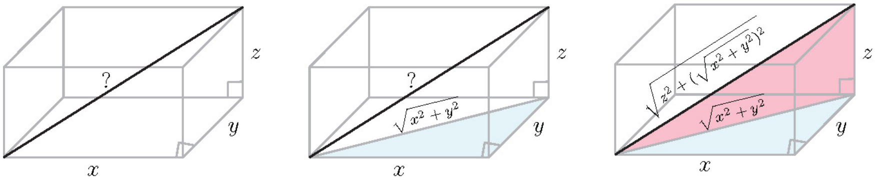
Length of vectors

Length of $\mathbf{u}$, denoted by $\|\mathbf{u}\|$, comes from the Pythagoras theorem



- $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, then $\|\mathbf{u}\|^2 = u_1^2 + u_2^2 \qquad \|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}$
- For $\mathbf{u} \in \mathbb{R}^n$, $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2} = \sqrt{\sum_{i=1}^n u_i^2}$
- By inductively applying the Pythagoras theorem $n-1$ times

Length of vectors



- $\mathbf{u} = \begin{bmatrix} x \ y \ z \end{bmatrix}^T$ is diagonal of the cube
- $\mathbf{u}' = \begin{bmatrix} x \ y \ 0 \end{bmatrix}^T$ is a vector in the $x - y$ plane
- length of base and perpendicular is u_1 and u_2, so $\|\mathbf{u}\| = \sqrt{x^2 + y^2}$
- $\mathbf{u}$ makes a right triangle $\mathbf{u}'$ (base) and $\begin{bmatrix} 0 \ 0 \ z \end{bmatrix}$ (perpendicular)
- So $\|\mathbf{u}\| = \sqrt{\|\mathbf{u}'\|^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$

▷ by a second application of the Pythagoras theorem

Unit Vector

- A vector $\mathbf{u}$ is called a unit vector, if $\|\mathbf{u}\| = 1$

- For any vector $\mathbf{u}$ we can get the unit vector in the direction of $\mathbf{u}$ by scaling it to have length 1

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

- Verify that $\hat{\mathbf{u}}$ has length 1

Dot Product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^t \mathbf{v} = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

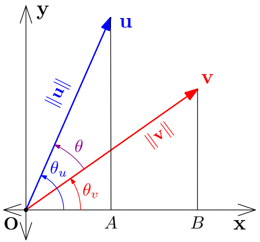
- It takes two vectors and returns a scalar (function)
- Also called inner product, scalar product, projection product
- Many names because it is a really fundamental operation
- Many concepts can be expressed in terms of dot-product
- Note that $\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ (length of vectors from dot-product)

Angle between vectors

Angle θ between vectors $\mathbf{u}$ and $\mathbf{v}$ is related to their dot-product

Let $\mathbf{u}$ and $\mathbf{v}$ make angles θ_u and θ_v resp. with $\mathbf{e_1}$ or x-axis

- From $\triangle OAU$
  - $\sin\theta_u = \dfrac{u_2}{\|\mathbf{u}\|} \qquad \cos\theta_u = \dfrac{u_1}{\|\mathbf{u}\|}$
- From $\triangle OBV$
  - $\sin\theta_v = \dfrac{v_2}{\|\mathbf{v}\|} \qquad \cos\theta_v = \dfrac{v_1}{\|\mathbf{v}\|}$



- $\cos\theta \;=\; \cos(\theta_u - \theta_v) \;=\; \cos\theta_v\cos\theta_u + \sin\theta_v\sin\theta_u$
- $\cos\theta \;=\; \dfrac{u_1}{\|\mathbf{u}\|}\dfrac{v_1}{\|\mathbf{v}\|} + \dfrac{u_2}{\|\mathbf{u}\|}\dfrac{v_2}{\|\mathbf{v}\|} \;=\; \dfrac{u_1v_1 + u_2v_2}{\|\mathbf{u}\|\|\mathbf{v}\|} \;=\; \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$
- What happens if we (negatively) scale one or both vectors?

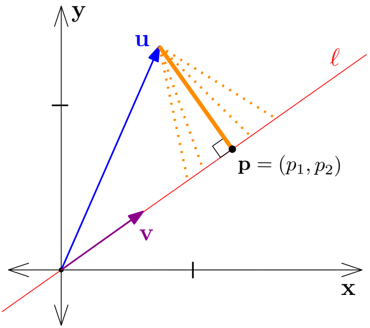
Orthogonal Vectors, Orthonormal Basis

$\mathbf{u}$ and $\mathbf{v}$ are called **orthogonal**, if $\mathbf{u} \cdot \mathbf{v} = 0$

- They are perpendicular to each other, angle θ between them is 90°
- $\mathbf{u} \cdot \mathbf{v} \; = \; \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \; = \; \|\mathbf{u}\| \|\mathbf{v}\| \cos 90^\circ \; = \; 0$
- If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then they are linearly independent
- If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are pairwise orthogonal, they are all linearly independent
- If bases of a space are all pairwise orthogonal, they are called **orthogonal bases**
- If they are unit vectors, they are called **orthonormal basis**
- Verify that the standard bases make orthonormal bases of $\mathbb{R}^n$

Projection

- Let **v** be a unit vector, let ℓ be a line in the direction of **v**
- Find the point **p** on ℓ that is closest to a vector **u**
- The line connecting **u** to **p** is perpendicular to **v**
- Otherwise **p** will not be the closest point (Pythagoras theorem)
- The point (vector) **p** is called the the projection of **u** on **v**



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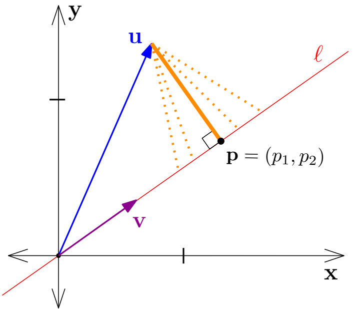
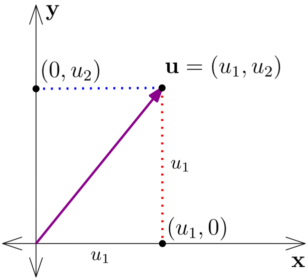
- Finding projection of **v** on the standard basis vectors is easy





Dot product and Projection

- Find the projection $\mathbf{p}$ of $\mathbf{u}$ on $\mathbf{v}$
- For general vectors we derive it from dot product
- $\mathbf{p}$ is just scaled vector $\mathbf{v}$, $\mathbf{p} = a\mathbf{v}$, find that scalar a

- $\mathbf{u} - \mathbf{p} = \mathbf{u} - a\mathbf{v}$ is perpendicular on $\mathbf{v}$
  - $\mathbf{v} \cdot (\mathbf{u} - a\mathbf{v}) = 0$
- Hence $\mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot a\mathbf{v} = \mathbf{v} \cdot \mathbf{u} - a\mathbf{v} \cdot \mathbf{v} = 0$
- Which means $a\mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $a = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|}$