Linear Algebra Review

- Vector
- Vector Operations
- Linear Combination
- Span, Bases, Linear Independence
- Length of Vectors
- Dot Product
- Angle between Vectors
- Projection
- Linear Functions
- Linear Transformation
- Scaling, Mirror, Shear, Rotation, Projection
- Composition of Linear Transformations
- Determinent and Inverse
- Change of Bases
- Transformation in Different Bases
- Eigenvalue and Eigenvectors
- Eigenbases and Diagnoalization
- Power of Matrices
- Random Walk and Markov Chain

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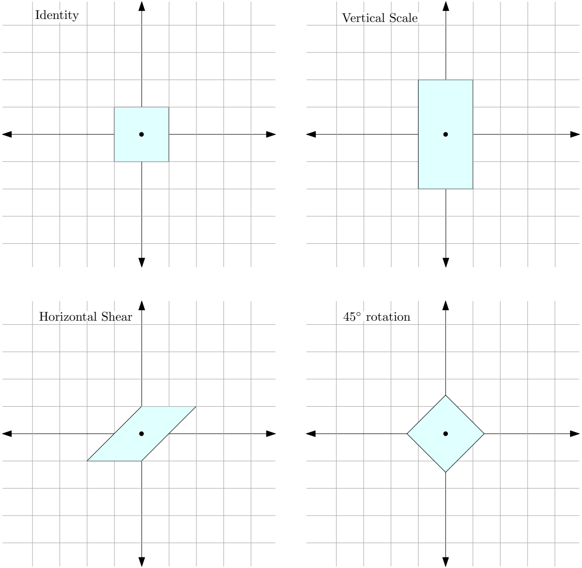
Eigenvalue and Eigenvectors

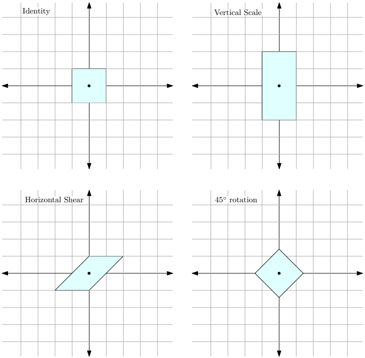
Eigenvalue and Eigenvectors

- Eigenvalue/eigenvectors are extremely important concepts related to linear transformation
- Has fundamental applications in large graph analysis
  - Google's pagerank algorithm and Ask's HITS algorithm
  - Spectral clustering
  - Matrix decomposition
  - Recommender systems
  - Diffusion Processes and Immunization
  - Dynamic systems and many more

Eigenvalue and Eigenvectors: Definition

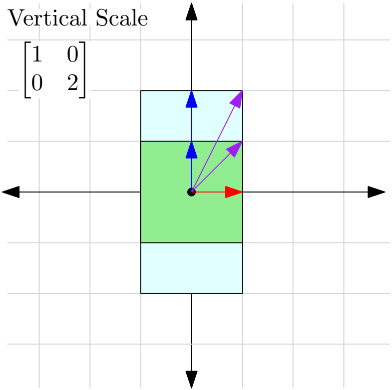


Eigenvalue and Eigenvectors: Definition



- Recall matrices as linear transformation and our view of how the whole space is transformed
- We visualize transformation of the space by observing transformation of the "unit square" (2×2 square centered at the origin)
- Notice some vectors do not change their directions with transformation

Eigenvalue and Eigenvectors: Definition



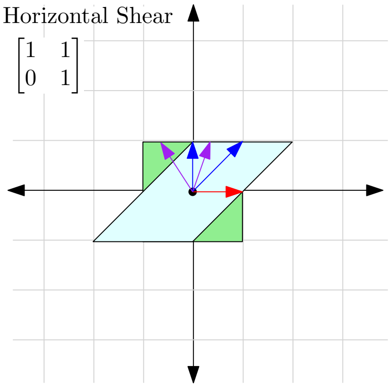
- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not change direction or size
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does not change direction, size is doubled
- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ changes direction and size

The horizontal and vertical vectors are special, they are called eigenvectors

Horizontal vector size does not change so the corresponding eigenvalue is 1

Vertical vector's size is doubled so the corresponding eigenvalue is 2

Eigenvalue and Eigenvectors: Definition

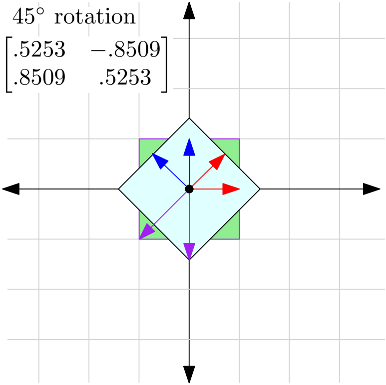


- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not change direction or size
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ changes direction and size
- $\begin{bmatrix} -.6 \\ 1 \end{bmatrix}$ changes direction and size

The horizontal vector is special, it is called eigenvector

Horizontal vector size does not change so the corresponding eigenvalue is 1

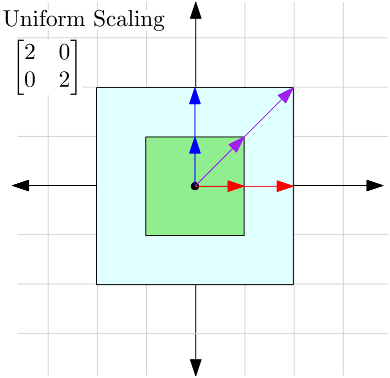
Eigenvalue and Eigenvectors: Definition



- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates by 45°
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates by 45°
- $\begin{bmatrix} -.6 \\ 1 \end{bmatrix}$ rotates by 45°

All vectors change their directions

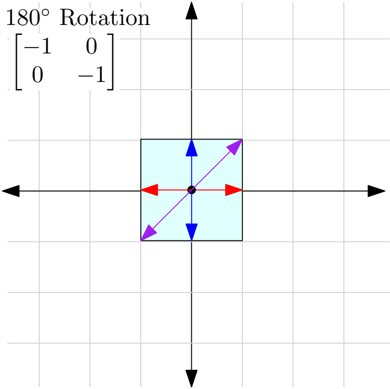
Eigenvalue and Eigenvectors: Definition



- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not change span and size is doubled
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does not change span and size is doubled
- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not change span and size is doubled

All vectors stay on their directions and sizes are doubled

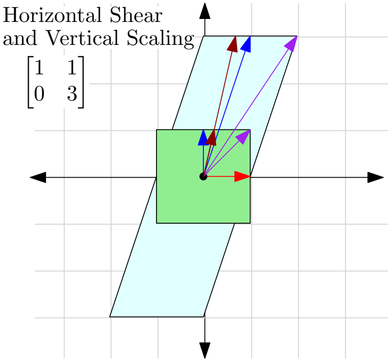
Eigenvalue and Eigenvectors: Definition



- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not change span and size is scaled by -1
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does not change span and size is scaled by -1
- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not change span and size is scaled by -1

All vectors stay on their spans and sizes are scaled by -1

Eigenvalue and Eigenvectors: Definition



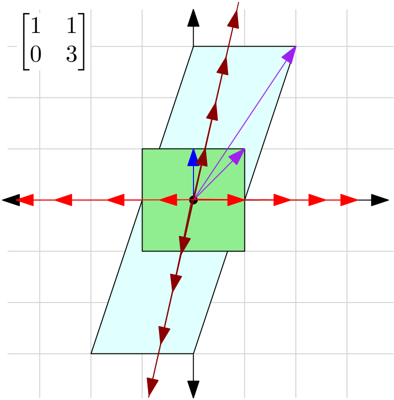
- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not change span and size
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ changes its span and size
- $\begin{bmatrix} .4472 \\ .8944 \end{bmatrix}$ does not change span and size is increased

All other vectors change their span

Eigenvalue and Eigenvectors: Computation

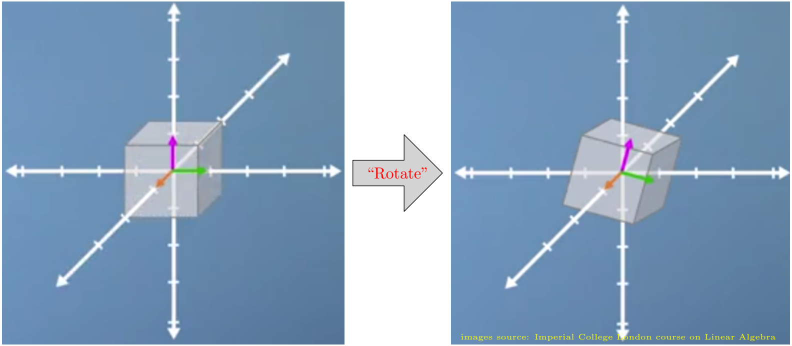
- eigen (German) means “self”’ or “characteristic”
- eigenvectors := “self vectors” or “characteristic vectors”
- Transform the space
- Find vectors that remain on the same span (these are eigenvectors)
- Measure how their lengths have changed (corresponding eigenvalues)
- Clearly, cannot do it geometrically, think of higher dimensions
- For a square matrix A, solve $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$
  - $\mathbf{x}$ is a vector that stays on its span, just scales by a factor of λ
  - There is no change of direction (span) of $\mathbf{x}$
  - **Solutions x’s are called eigenvectors of A**
  - λ **is called the eigenvalue corresponding to x**

Eigenvalue and Eigenvectors: Definition



By linearity, vectors on a line map to a line, all vectors on the span of an eigenvectors are also eigenvectors

Eigenvalue and Eigenvectors: Definition



- In 2d rotation all vectors change their spans (except 180° rotation)
- In 3d x-axis and y-axis change their spans but z-axis does not
- These are eigenvectors of this rotation
- Physically, this is the axis of rotation

Eigenvalue and Eigenvectors: Computation

$$[\mathbf{x}, \lambda] \text{ is an eigen pair} \quad \Leftrightarrow \quad A\mathbf{x} = \lambda\mathbf{x}$$

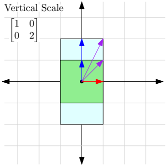
- LHS is matrix-vector product, RHS is scalar-vector product
- Convert RHS to $\lambda\mathbb{I}\mathbf{x}$ ($\lambda\mathbb{I}$ is the uniform scaling matrix)
- This makes the math work but does not change the meaning

$$[\mathbf{x}, \lambda] \text{ is an eigen pair} \quad \Leftrightarrow \quad A\mathbf{x} - \lambda\mathbb{I}\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad (A - \lambda\mathbb{I})\mathbf{x} = \mathbf{0}$$

- $\mathbf{x} = \mathbf{0}$ is a trivial solution (no length or direction)
- We want $\mathbf{x}$ that is mapped to $\mathbf{0}$ by the linear transform $(A - \lambda\mathbb{I})$
- A transformation maps a non-zero vector to $\mathbf{0}$ only if it's determinant is 0
- $\therefore$ we find λ such that $det(A - \lambda\mathbb{I}) = 0$
- Once we get the transformation, solve the system of linear equation to $(A - \lambda\mathbb{I})\mathbf{x} = \mathbf{0}$ to find $\mathbf{x}$

Eigenvalue and Eigenvectors: Computation

- $det\left(\begin{bmatrix}1-\lambda & 0\\ 0 & 2-\lambda\end{bmatrix}\right) = (1-\lambda)(2-\lambda)$
- $(1-\lambda)(2-\lambda) = 0 \implies \lambda = 1 \text{ or } \lambda = 2$



$$\underbrace{\begin{bmatrix}1-1 & 0\\ 0 & 2-1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}}_{@\lambda=1:}$$

$$\implies \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\implies \begin{bmatrix}0\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \implies \mathbf{x} = \begin{bmatrix}t\\ 0\end{bmatrix}$$

$[1, \begin{bmatrix}t\\ 0\end{bmatrix}]$ is an eigenpair

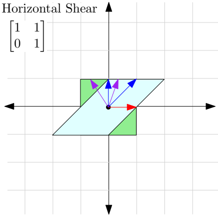
$$\underbrace{\begin{bmatrix}1-2 & 0\\ 0 & 2-2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}}_{@\lambda=2:}$$

$$\implies \begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\implies \begin{bmatrix}-x_1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \implies \mathbf{x} = \begin{bmatrix}0\\ t\end{bmatrix}$$

$[2, \begin{bmatrix}0\\ t\end{bmatrix}]$ is an eigenpair

Eigenvalue and Eigenvectors: Computation



- $det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2$
- $(1-\lambda)^2 = 0 \implies \lambda = 1$

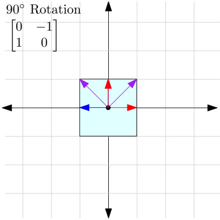
$$\underbrace{\begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{@\lambda=1:}$$

$$\implies \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{x} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$[1, \begin{bmatrix} t \\ 0 \end{bmatrix}]$ is an eigenpair

Eigenvalue and Eigenvectors: Computation

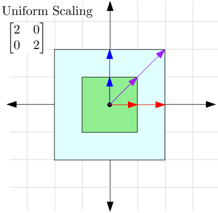
- $det \left(\begin{bmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{bmatrix} \right) = (0-\lambda)^2$ -(1)(-1)
- $(-\lambda)^2 + 1 = 0 \implies \lambda^2 = -1$
- No real λ as solution



Hence no real eigenvectors

Eigenvalue and Eigenvectors: Computation

- $det\left(\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2$
- $(2-\lambda)^2 = 0 \implies \lambda = 2$



$$\underbrace{\begin{bmatrix} 2-2 & 0 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{@\lambda=2:}$$

$$\implies \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{x} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

All vectors are eigenvectors with eigenvalue 2