Linear Algebra Review

- Vector
- Vector Operations
- Linear Combination
- Span, Bases, Linear Independence
- Length of Vectors
- Dot Product
- Angle between Vectors
- Projection
- Linear Functions
- Linear Transformation
- Scaling, Mirror, Shear, Rotation, Projection
- Composition of Linear Transformations
- Determinant and Inverse
- Change of Bases
- Transformation in Different Bases
- Eigenvalue and Eigenvectors
- Eigenbases and Diagnoalization
- Power of Matrices
- Random Walk and Markov Chain

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Linear Transformation

Linear Functions

A function $f : \mathbb{R} \mapsto \mathbb{R}$ is called linear if

1. $f(cx) \;=\; c\,f(x)$
2. $f(x+y) \;=\; f(x)+f(y)$

$$\begin{tikzcd}
\mathbf{x} \arrow[r, "\mathbf{f}"] \arrow[d, "\mathbf{c}"'] & \mathbf{f(u)} \arrow[d, "\mathbf{c}"] \\
\mathbf{cx} \arrow[r, "\mathbf{f}"'] & \mathbf{f(cx) = cf(x)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{x, y} \arrow[r, "\mathbf{f}"] \arrow[d, "+"'] & \mathbf{f(x), f(y)} \arrow[d, "+"] \\
\mathbf{x + y} \arrow[r, "\mathbf{f}"'] & \mathbf{f(x+y) = f(x) + f(y)}
\end{tikzcd}$$

Linear Functions

A function $f : \mathbb{R} \mapsto \mathbb{R}$ is called linear if

1. $f(cx) = cf(x)$
2. $f(x+y) = f(x) + f(y)$

A shorter version: A function $f : \mathbb{R} \mapsto \mathbb{R}$ is linear if

1. $f(ax+by) = af(x) + bf(x)$

- These imply that $f(0) = 0$
- Generally, functions of the form $g(x) = ax + b$ are called linear, which doesn't necessarily imply $g(0) = 0$
- Functions like $g(\cdot)$ are technically and correctly called affine functions, which are linear functions followed by a translation

Dot Product as Linear Functions

For a fixed vector $\mathbf{a} \in \mathbb{R}^n$, define $f_{\mathbf{a}} : \mathbb{R}^n \mapsto \mathbb{R}^1$ as follows

$$f_{\mathbf{a}}(\mathbf{x}) \; := \; \langle \mathbf{a}, \mathbf{x} \rangle \; = \; \mathbf{a} \cdot \mathbf{x}$$

$f_{\mathbf{a}}$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^1$

In fact, it can be shown that these are the only functions that are linear

$$\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$f_{\mathbf{a}}(4\mathbf{x} + 5\mathbf{y}) \; = \; \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 4 * 2 \\ 4 * 3 \end{bmatrix} + \begin{bmatrix} 5 * 1 \\ 5 * 2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ 22 \end{bmatrix} = 39 + 88 = 127$$

$$4f_{\mathbf{a}}(\mathbf{x}) + 5f_{\mathbf{a}}(\mathbf{y}) \; = \; 4 * \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{18} + 5 * \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{11} = 4 * 18 + 5 * 11 = 127$$

Linear Functions on Euclidean Space

- For linear functions of the form $\mathbb{R}^n \mapsto \mathbb{R}^m$, for $m > 1$

  $\triangleright$ vector functions - functions that output vectors in $\mathbb{R}^m$

- Extend the notion of dot product as linear function as follows:

For m fixed vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots \mathbf{a_m} \in \mathbb{R}^n$, define $f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}} : \mathbb{R}^n \mapsto \mathbb{R}^m$ as:

$$f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}(\mathbf{x}) \; := \; \begin{bmatrix} \mathbf{a_1} \cdot \mathbf{x} & \mathbf{a_2} \cdot \mathbf{x} & \ldots & \mathbf{a_m} \cdot \mathbf{x} \end{bmatrix}^T$$

$f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$

Again, it can be shown that these are the only functions that are linear

- $f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}$ is represented by $m \times n$ matrix, $\quad T_f = \begin{bmatrix} \text{---} & \mathbf{a_1} & \text{---} \\ \text{---} & \mathbf{a_2} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a_m} & \text{---} \end{bmatrix}$

- Evaluated by matrix-vector product $\quad f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}(\mathbf{x}) = T_f\,\mathbf{x}$

  $\triangleright$ $T_f\,\mathbf{x}$ is $n \times 1$ vector

Linear Functions on Euclidean Spaces

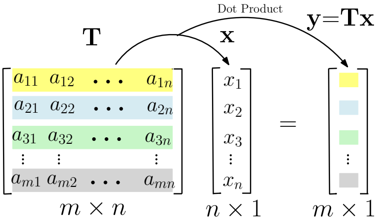
For m fixed vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots \mathbf{a_m} \in \mathbb{R}^n$, define $f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}} : \mathbb{R}^n \mapsto \mathbb{R}^m$ as:

$$f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}(\mathbf{x}) := \begin{bmatrix} \mathbf{a_1} \cdot \mathbf{x} & \mathbf{a_2} \cdot \mathbf{x} & \ldots & \mathbf{a_m} \cdot \mathbf{x} \end{bmatrix}^T$$

$f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$

- $f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}$ is represented by $m \times n$ matrix, $\quad T_f = \begin{bmatrix} \text{---} & \mathbf{a_1} & \text{---} \\ \text{---} & \mathbf{a_2} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a_m} & \text{---} \end{bmatrix}$
- Evaluated by matrix-vector product $\quad f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}(\mathbf{x}) = T_f\,\mathbf{x}$

▷ $T_f\,\mathbf{x}$ is $n \times 1$ vector



For m fixed vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots \mathbf{a_m} \in \mathbb{R}^n$, define $f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}} : \mathbb{R}^n \mapsto \mathbb{R}^m$ as:

$$f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}(\mathbf{x}) := \begin{bmatrix} \mathbf{a_1} \cdot \mathbf{x} & \mathbf{a_2} \cdot \mathbf{x} & \ldots & \mathbf{a_m} \cdot \mathbf{x} \end{bmatrix}^T$$

- $T_f = \begin{bmatrix} \text{---} & \mathbf{a_1} & \text{---} \\ \text{---} & \mathbf{a_2} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a_m} & \text{---} \end{bmatrix}$ $\qquad f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}(\mathbf{x}) = T_f\,\mathbf{x}$

$$\mathbf{a_1} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

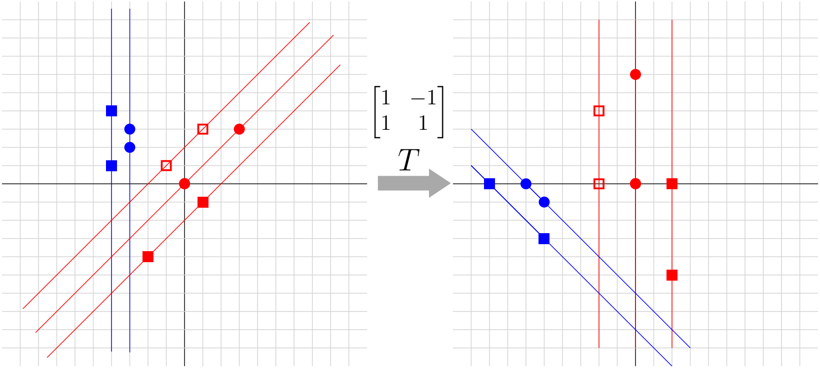
$$T(4\mathbf{x} + 5\mathbf{y}) = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \end{bmatrix} \right) = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 22 \end{bmatrix} = \begin{bmatrix} 127 \\ 48 \end{bmatrix}$$

$$4T\mathbf{x} + 5T\mathbf{y} = 4 \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 72 \\ 28 \end{bmatrix} + \begin{bmatrix} 55 \\ 20 \end{bmatrix} = \begin{bmatrix} 127 \\ 48 \end{bmatrix}$$

Linear Functions on Euclidean Spaces

Geometrically, linear functions (matrix-vector multiplications)

- maps the $\mathbf{0}$ vector (origin) to $\mathbf{0}$
- maps any straight line to a straight lines
- maps any set of parallel lines to a set of parallel lines



Matrices as Linear Transform

- Linear functions, multiplications of $m \times n$ matrices with $n \times 1$ vectors output $m \times 1$ vectors
- For any $m \times n$ matrix T, $\mathbf{y} = T\mathbf{x}$ is a linear function $\mathbb{R}^n \mapsto \mathbb{R}^m$
- Generally called **linear transformation**, because we are interested in **how it transforms the whole space ($\mathbb{R}^n$)**
    - and not in evaluating outputs on specific inputs
    - or its properties as a function (injective, surjective, bijective etc.)
- Just a few quick terminology (while we still call it functions)
- Linear functions on Euclidean space are also called linear maps
- When $m = n$ (same $\mathbb{R}^n \mapsto \mathbb{R}^n$), they are called linear operators
- When the function is bijective (the corresponding matrix is invertible), they are called linear isomorphisms

Matrices as Linear Transform

Meaning of rows of a matrix A as a linear transform

Recall standard bases of $\mathbb{R}^n$ (unit vectors along the axes)

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e_n} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- They help write awkward and wordy things concisely and precisely

Matrices as Linear Transform: Rows

Meaning of rows of a matrix A as a linear transform

- Standard bases help write wordy things concisely and precisely
- $\mathbf{e_i}^T A$ is the i^{th} row of A $\quad \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$
- $\mathbf{e_i}^T A$ is $\mathbf{a_i}$ in the definition of the function $f_{\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_m}}$ corresponding to A
- $\mathbf{e_i}^T A$ describes how to compute the i^{th} coordinate of result, $\mathbf{y} = A\mathbf{x}$

  $\triangleright$ $\mathbf{y}(i) = \mathbf{e_i}^T A \cdot \mathbf{x}$

Matrices as Linear Transform

Meaning of columns of a matrix A as a linear transform

- $A\mathbf{e_i}$ is the i^{th} column of A $\quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$
- $A\mathbf{e_i}$ is the vector in R^n where $\mathbf{e_i}$ maps to
- So the columns of A are the locations in the range space ($\mathbb{R}^m$), where the standard bases map to by the transform A
- **This is the most important concept to understand!**

Meaning of columns of a matrix A as a linear transform

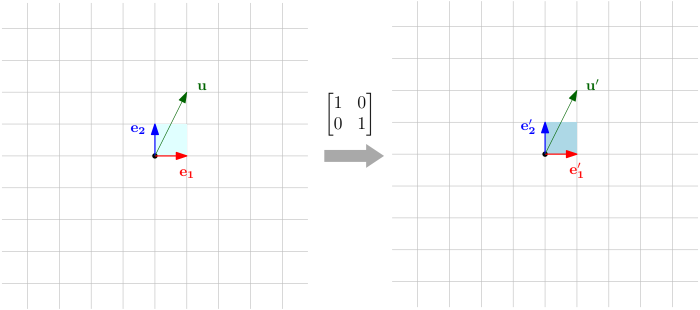
- The columns of A are the locations in the range space ($\mathbb{R}^m$), where the standard bases map to by the transform A
- A linear transform is completely described by knowing where it maps the basis vectors
- Follows from linearity, as $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is actually $\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2}$
- $A\mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} au_1+bu_2 \\ cu_1+du_2 \end{bmatrix}$, By linearity
- $A\mathbf{u} = A(u_1\mathbf{e_1} + u_2\mathbf{e_2}) = u_1A\mathbf{e_1} + u_2A\mathbf{e_2} = u_1\begin{bmatrix} a \\ c \end{bmatrix} + u_2\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} au_1+bu_2 \\ cu_1+du_2 \end{bmatrix}$
- Under A, the image of $\mathbf{u} = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}^T$ is a linear combination of images of basis vectors $(A\mathbf{e_1}, \dots, A\mathbf{e_n})$ with coefficients $u_1, \dots, u_n$

Common Linear Transformation

- We discuss some common transformation to master the concepts
- They are fundamental to computer graphics, image processing, computer vision and other CS disciplines
- In these fields, they mostly need affine transformation, which, as mentioned earlier, is linear transformation followed by translation
- We mainly focus on linear operators ($\mathbb{R}^n \mapsto \mathbb{R}^n$) with $n = 2$, but will mention some others to highlight certain concepts
- We discussed that a linear transformation (matrix) is completely described by its columns - images of standard bases vectors
- We will mainly just show the transformed bases vectors and the image of the 1×1 square in the first quadrant

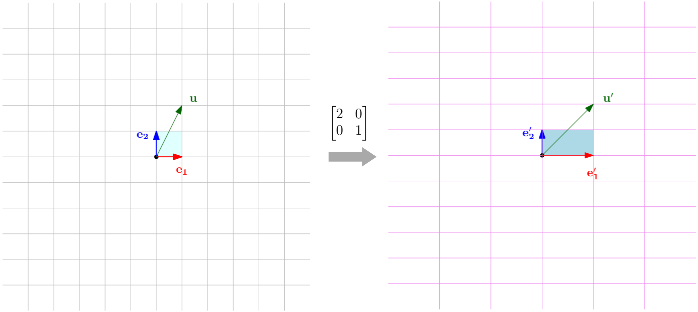
Linear Transformation: Identity

- $A = \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not change any vectors
- $\mathbf{e_1'} = A\mathbf{e_1} = \mathbf{e_1}$ and $\mathbf{e_2'} = A\mathbf{e_2} = \mathbf{e_2}$
- For $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e_1} + y\mathbf{e_2}$, $\quad A\mathbf{u} = x\mathbf{e_1'} + y\mathbf{e_2'} = \mathbf{u}$
- The space does not change, the unit square remains the same



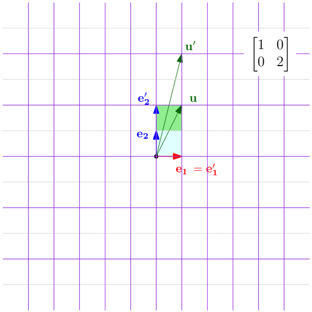
Linear Transformation: Horizontal Scaling

- $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ stretches each vector by a factor of 2 horizontally
- $\mathbf{e}_1' = A\mathbf{e}_1 = 2\mathbf{e}_1$ and $\mathbf{e}_2' = A\mathbf{e}_2 = \mathbf{e}_2$
- For $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2, \qquad A\mathbf{u} = x\mathbf{e}_1' + y\mathbf{e}_2' = \begin{bmatrix} 2x \\ y \end{bmatrix}$
- grid changes, unit square becomes 2×1 rectangle



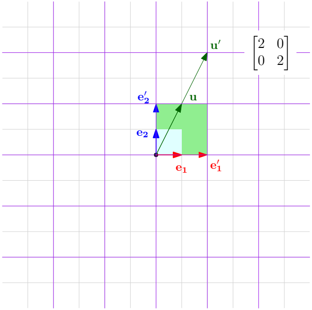
Linear Transformation: Vertical Scaling

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ stretches each vector by a factor of 2 vertically
- $\mathbf{e_1'} = A\mathbf{e_1} = \mathbf{e_1}$ and $\mathbf{e_2'} = A\mathbf{e_2} = 2\mathbf{e_2}$
- For $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e_1} + y\mathbf{e_2}, \quad A\mathbf{u} = x\mathbf{e_1'} + y\mathbf{e_2'} = \begin{bmatrix} x \\ 2y \end{bmatrix}$
- grid changes, unit square becomes 1×2 rectangle



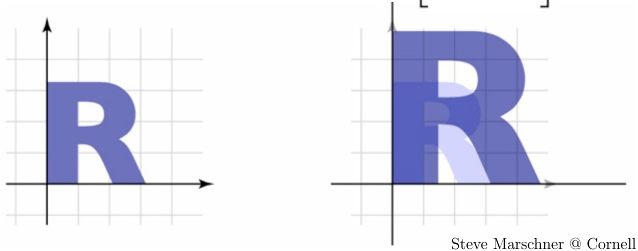
Linear Transformation: Uniform Scaling

- $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ stretches each vector by a factor of 2 in both directions
- $\mathbf{e_1'} = A\mathbf{e_1} = 2\mathbf{e_1}$ and $\mathbf{e_2'} = A\mathbf{e_2} = 2\mathbf{e_2}$
- For $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e_1} + y\mathbf{e_2}$, $\quad A\mathbf{u} = x\mathbf{e_1'} + y\mathbf{e_2'} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$
- grid changes, unit square is uniformly stretched by a factor of 2



Linear Transformation: Uniform Scaling

Uniform Scaling Application



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

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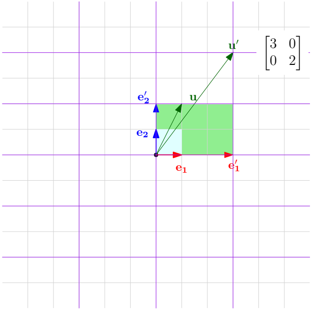
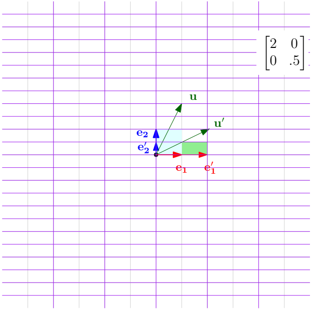
Linear Transformation: Non-Uniform Scaling

- $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ stretches vectors by factors 3 and 2
- $\mathbf{e_1'} = A\mathbf{e_1} = 3\mathbf{e_1}$ and $\mathbf{e_2'} = A\mathbf{e_2} = 2\mathbf{e_2}$
- For $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e_1} + y\mathbf{e_2}$, $\quad A\mathbf{u} = x\mathbf{e_1'} + y\mathbf{e_2'} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$
- grid changes, unit square becomes a 3×2 rectangle



Linear Transformation: Non-Uniform Scaling

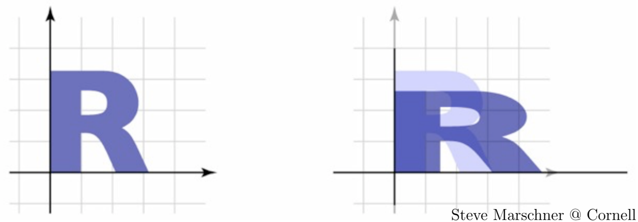
- $A = \begin{bmatrix} 2 & 0 \\ 0 & .5 \end{bmatrix}$ stretches vectors by factor of 3 and $1/2$
- $\mathbf{e_1'} = A\mathbf{e_1} = 2\mathbf{e_1}$ and $\mathbf{e_2'} = A\mathbf{e_2} = 1/2\mathbf{e_2}$
- For $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e_1} + y\mathbf{e_2}, \quad A\mathbf{u} = x\mathbf{e_1'} + y\mathbf{e_2'} = \begin{bmatrix} 2x \\ y/2 \end{bmatrix}$
- grid changes, unit square becomes a $2 \times 1/2$ rectangle



Linear Transformation: Non-Uniform Scaling

Non-Uniform Scaling Application

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.8 \end{bmatrix}$$



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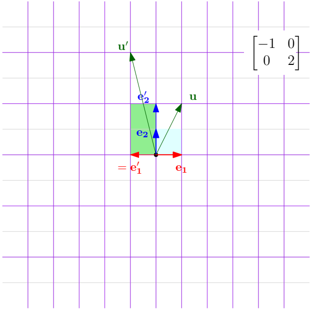
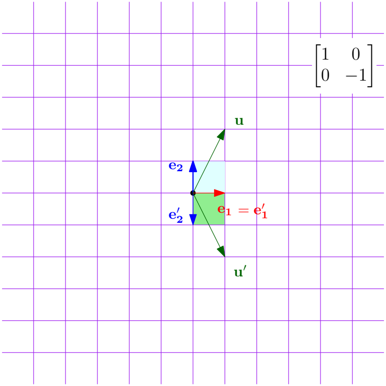
Linear Transformation: Negative Scaling

- $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ stretches each vector by a factor of -1 horizontally and by a factor of 2 vertically
- $\mathbf{e_1'} = A\mathbf{e_1} = -1\mathbf{e_1}$ and $\mathbf{e_2'} = A\mathbf{e_2} = 2\mathbf{e_2}$
- For $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e_1} + y\mathbf{e_2}$, $\quad A\mathbf{u} = x\mathbf{e_1'} + y\mathbf{e_2'} = \begin{bmatrix} -x \\ 2y \end{bmatrix}$
- grid changes, unit square becomes a 1×2 rectangle but flipped across



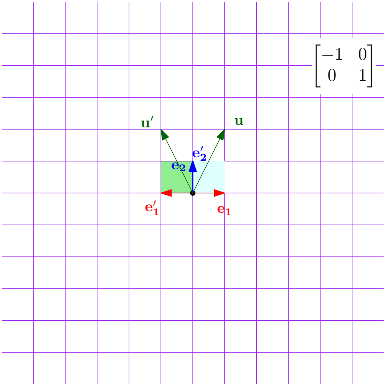
Linear Transformation: Horizontal Mirror

- $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects each vector across vertical axis
- grid stays the same with different orientation, unit square is mirrored through horizontal axis



Linear Transformation: Vertical Mirror

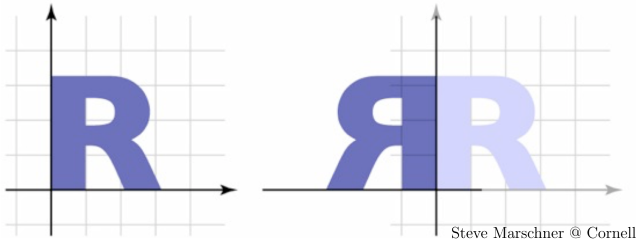
- $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects each vector across vertical axis
- grid stays the same with different orientation, unit square is mirrored through horizontal axis



Linear Transformation: Vertical Mirror

Reflection/Mirror Application

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



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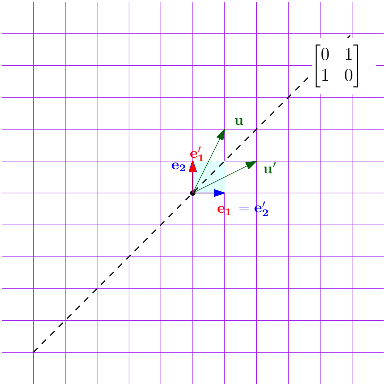
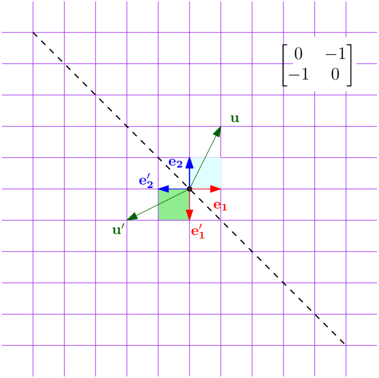
Linear Transformation: Diagonal Mirror

- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflects each vector across 45° mirror
- grid stays the same with different orientation, unit square is mirrored through 45° mirror



Linear Transformation: Other Diagonal Mirror

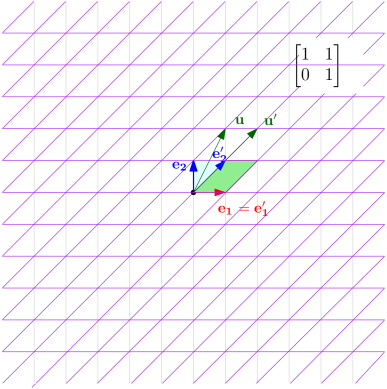
- $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ reflects each vector across 45° mirror
- grid changes, unit square is mirrored through the other diagonal mirror



Linear Transformation: Horizontal Shear

- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ leaves horizontal dimension intact and skew each vector in vertical dimension (horizontal shear)

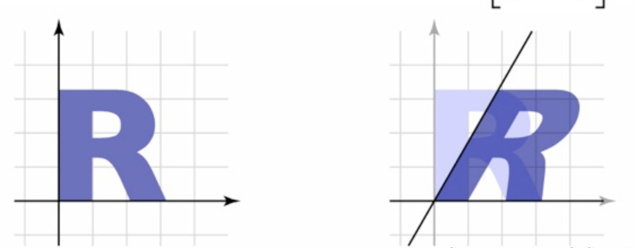
- unit square becomes a parallelogram



Linear Transformation: Horizontal Shear

Horizontal Shear Application

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$



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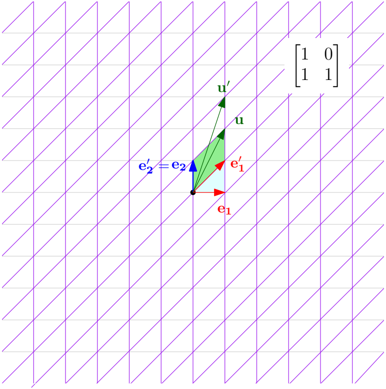
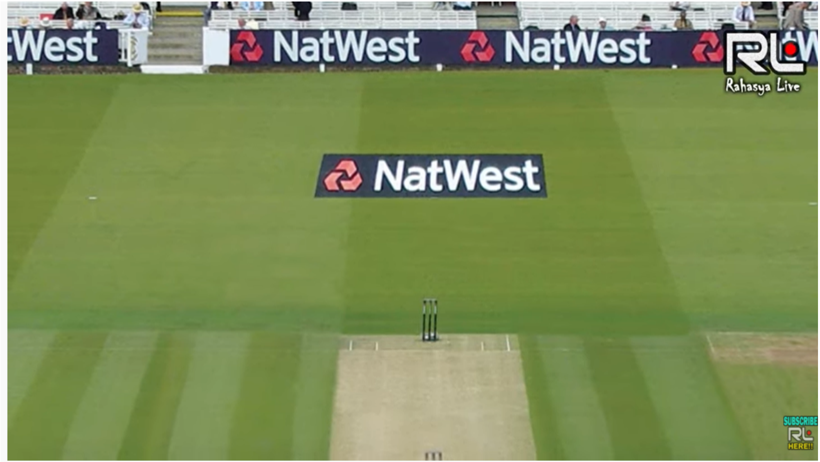
Linear Transformation: Vertical Shear

- $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ leaves vertical dimension intact and skew each vector in horizontal dimension (vertical shear)
- unit square becomes a parallelogram



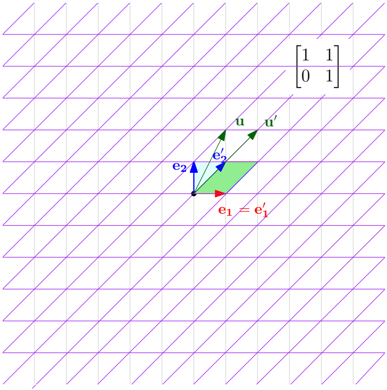
Linear Transformation: Vertical Shear

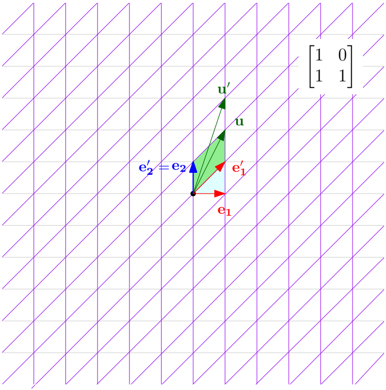
Vertical Shear Application



Linear Transformation: Shear

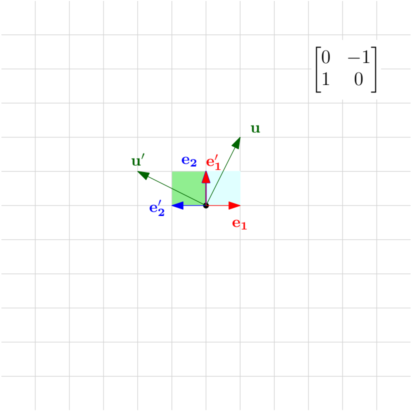
- $A = \begin{bmatrix} 1 & 1 \\ 0 & s \end{bmatrix}$ vertical shear and $A = \begin{bmatrix} s & 0 \\ 1 & 1 \end{bmatrix}$ horizontal shear
- unit square becomes a parallelogram





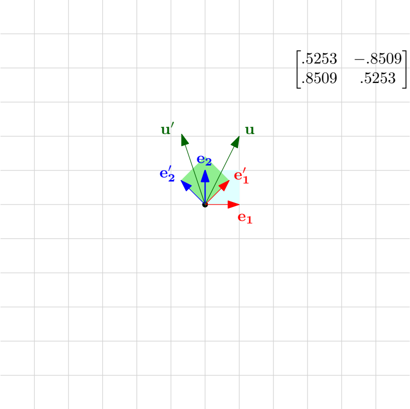
Linear Transformation: Rotation

- $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates every vector by 90° clockwise
- unit square rotates to the adjacent unit square



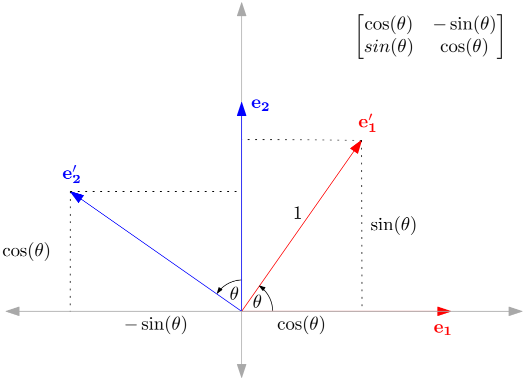
Linear Transformation: Rotation

- $A = \begin{bmatrix} .5253 & -.8509 \\ .8509 & .5253 \end{bmatrix}$ rotates every vector by 45° clockwise
- unit square rotates by 45°



Linear Transformation: Rotation

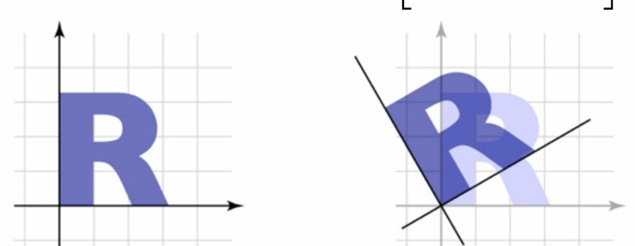
- $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ sin(\theta) & \cos(\theta) \end{bmatrix}$ rotates every vector by θ clockwise
- unit square rotates by θ clockwise



Linear Transformation: Rotation

Rotation Applications

$$\begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$$



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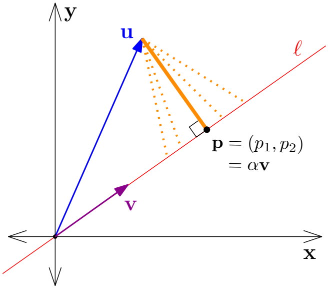
Linear Transformation: Projection

- Let $\mathbf{v}$ be a vector, let ℓ be a line in the direction of $\mathbf{v}$
- Projection of $\mathbf{u}$ on ℓ (or on $\mathbf{v}$) is the point $\mathbf{p}$ on ℓ that is closest to $\mathbf{u}$

- $\mathbf{p}$ is scaled vector $\hat{\mathbf{v}}$, $\quad \mathbf{p} = a\hat{\mathbf{v}}$
  - ▷ a : scalar projection or projection length
- $\mathbf{u} - \mathbf{p} = \mathbf{u} - a\hat{\mathbf{v}}$ is perpendicular on $\mathbf{v}$
  - $\mathbf{v} \cdot (\mathbf{u} - a\mathbf{v}) = 0$
- Hence $\mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot a\mathbf{v} = \mathbf{v} \cdot \mathbf{u} - a\mathbf{v} \cdot \mathbf{v} = 0$
- Which means $a\mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $a = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|}$



The vector projection, $\mathbf{p}$ is given by $\mathbf{p} = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|}\hat{\mathbf{v}} = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|}\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|^2}\mathbf{v}$

Linear Transformation: Projection

The vector projection, $\mathbf{p}$ is given by $\mathbf{p} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|}\hat{\mathbf{v}} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|}\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|^2}\mathbf{v}$

- For unit vector $\hat{\mathbf{v}}$, the vector projection, $\mathbf{p}$ of $\mathbf{u}$ on $\hat{\mathbf{v}}$ is $\mathbf{p} = (\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$

$$\mathbf{p} = (\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} = \left(\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}\right)\begin{bmatrix} a \\ b \end{bmatrix} = (xa + yb)\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} xa^2 + yab \\ xab + yb^2 \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

- $A = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$ projects every vector onto the unit vector $\begin{bmatrix} a \\ b \end{bmatrix}$

Composition of Linear Transformation

Any image processing operation (linear) can be described as combination of the above elementary transformation

Composing transformations

- Want to transform an object, then transform it some more

$$\mathbf{u} \mapsto g(\mathbf{u}) \mapsto f(g(\mathbf{u})) \quad := \quad (f \circ g)(\mathbf{u})$$

- Represent $(f \circ g)(\cdot)$ using same representation as for f and g (matrix) $\triangleright$ ("f compose g")
- Let S and T be the corresponding matrices for f and g, resp.
- $f(\mathbf{u}) = S\mathbf{u}$ and $g(\mathbf{u}) = T\mathbf{u}$
- $(f \circ g)(\mathbf{u}) = ST\mathbf{u}$

Composition of Linear Transformation

90° rotation followed by horizontal shear

$$\underbrace{S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{shear} \qquad \underbrace{T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{rotation}$$

$$\mathbf{e}_1' = T\mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2' = T\mathbf{e}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

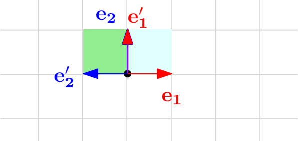
$$S\mathbf{e}_1' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S\mathbf{e}_2' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

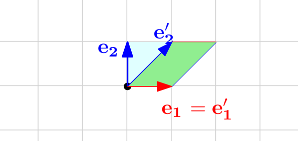
$$ST = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Composition of Linear Transformation

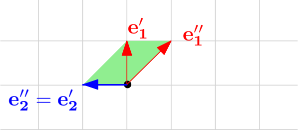
$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

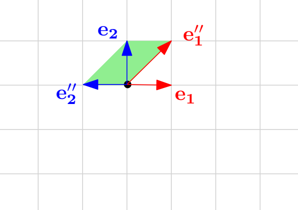


$$T\mathbf{e_1'} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$T\mathbf{e_2'} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$



Composition of Linear Transformation

- Transforming first by T then by S is the same as transforming by ST
- In general, composition is not commutative
- Generally, $ST \neq TS$
- Note that $S \circ T$, is applying T first and S second
- We can compose many transformation $S \circ T \circ R$

Simultaneous Equations: Solving $A\mathbf{x} = \mathbf{b}$

Consider the following scenario

- ISB metro has 3 bridges, 4 stations, 20km length and cost is 20b
- Lahore metro has 2 bridges, 6 stations, 27km length and cost is 27b
- Multan metro has 3 bridges, 5 stations, 22km length and cost is 24b
- You want another metro with 4 bridges, 5 stations and 25km length, what will be the cost?
- If we have cost per bridge, per station, per km then we can solve it

$$\begin{aligned} 3b + 4s + 20\ell &= 20 \\ 2b + 6s + 27\ell &= 27 \\ 3b + 5s + 22\ell &= 24 \end{aligned} \implies \begin{bmatrix} 3 & 4 & 20 \\ 2 & 6 & 27 \\ 3 & 5 & 22 \end{bmatrix} \begin{bmatrix} b \\ s \\ \ell \end{bmatrix} = \begin{bmatrix} 20 \\ 27 \\ 24 \end{bmatrix} := A\mathbf{x} = \mathbf{b}$$

Which vector $\mathbf{x}$, the transformation A maps to $\mathbf{b}$? (the reverse question)

Solving $A\mathbf{x} = \mathbf{b}$

For a matrix A, let A^{-1} be a matrix such that

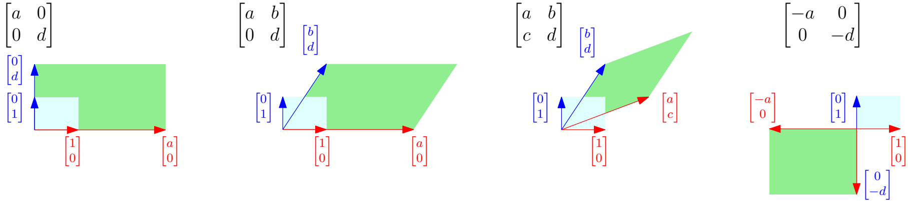
$$A^{-1}A = \mathbb{I}$$

Composing A^{-1} with A gives solution to $A\mathbf{x} = \mathbf{b}$

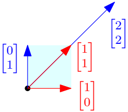
$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \implies \mathbb{I}\mathbf{x} = A^{-1}\mathbf{b}$$

A^{-1} is called the inverse of A, if we can find it then we can solve $A\mathbf{x} = \mathbf{b}$

Linear Transformation: Determinant and Inverse



The area of this new parallelogram (the transformed unit square) $ad - bc$ in $2d$ is called the determinant of the matrix A, $det(A)$



$$det\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = 0$$

- Columns of A are linearly dependent $\implies$ determinant is 0
- This matrix is not invertible