Chain-Matrix Multiplication

1 => Time Complexity in Simple Multiplication of Matrices

2 => What is the Problem

3 => Using Divide and Conquer

4 => Recursion to Solve Problem

5 => Dynamic Programing

$$A = \begin{array}{c} k \\ \left[\begin{matrix} 2 & 2 & 1 \\ 3 & 1 & 4 \end{matrix}\right] \end{array} \qquad B = \left[\begin{matrix} 3 & 6 & 9 & 1 \\ 4 & 7 & 2 & 2 \\ 1 & 8 & 1 & 1 \end{matrix}\right]$$

2×3 3×4

$p \quad q$ $q \quad r$

$$C = \left[\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right] \quad \begin{matrix} 2 \times 4 \\ p \times r \end{matrix}$$

$$C[i,j] = \sum_{k=1}^{q} A[i,k] * B[k,j]$$

$\Theta(q)$ - for one Multiplicati

$\Theta(q \times p \times r)$ for total Multipl.

Problem

lets Consider

$A_1 \quad A_2 \quad A_3$

$2 \times 4 \quad 4 \times 3 \quad 3 \times 5$

Associative Property.

~~Not Comutative~~

$A_1 A_2 \neq A_2 A_1$

$(A_1 A_2) A_3$ $A_1 (A_2 A_3)$

$2 \times 4 \times 3 + 2 \times 3 \times 5$ $2 \times 4 \times 5 + 4 \times 3 \times 5$

$24 + 30$ $40 + 60$

54 100

Motive of this Problem is find way that multiply in minimum cost

Can we apply divide and conquer on this Problem?

$A_1 A_2 A_3 A_4 \ldots A_n$

We need to find

$A_1 \_\_ A_k \quad A_{k+1} \_\_ A_n$

whereas k is not known. we need to find k.

We dont exactly know where we can divide therefore we cannot Apply divide and Conquer further subproblems are not independant.

Direct Solution.

$A_1 (A_2 A_3 A_4 \ldots A_n)$ $\quad k=1$

$A_1 A_2 (A_3 A_4 \_\_ A_n)$ $\quad k=2$

$\vdots$

$A_1 A_2 A_3 \ldots A_{n-1} (A_n)$ $\quad k=n-1$

further different possiblities.

$1 \le k \le n-1$

Same for this

A_{3-n}

$A_2 (A_3 A_4 \ldots A_n)$ $i=2$ $k=2$

$A_2 A_3 (A_4 \ldots A_n)$ $i=2$ $k=3$

$A_2 A_3 A_4 (A_5 \ldots A_n)$ $i=2$ $k=4$

Dynamic Programing

Record the subproblems for future use in table

$A_{i \ldots j}$: (Resultant Matrix)

A_1	A_2	A_3	A_4	$A_{1 \ldots 4}$
5×4	4×2	2×3	3×5	~~m[1,4]~~
P_0 P_1	P_1 P_2	P_2 P_3	P_3 P_4	

$A_i = P_{i-1} \times P_i$ $A_i = P_{i-1} \times P_i$

$$m[1,4] = m[1,k] + m[k+1,4]$$
$$+ \underset{P_{i-1}}{5} \times k \times \underset{P_j}{5}$$

$$m[i,j] = \min_{i \le k < j} \begin{cases} m[i,k] + m[k+1,j] \\ + P_{i-1} \times P_k \times P_j \end{cases}$$

Important

Example

$A_1 \; A_2 \; A_3 \; A_4 \; A_5$

$5\times4 \;\; 4\times6 \;\; 6\times2 \;\; 2\times7 \;\; 7\times3$

$P_0 \; P_1 \; P_1 \; P_2 \; P_2 \; P_3 \; P_3 \; P_4 \; P_4 \; P_5$

$m[i,i] = 0$

i	1	2	3	4	5
1	m[1,1] 0				m[1,5]
2		m[2,2] 0			m[2,5]
3			m[3,3] 0		m[3,5]
4				m[4,4] 0	m[4,5]
5					m[5,5] 0

$$m[1,5] \underset{k=1}{=} \begin{cases} m[1,\underset{k}{1}] + m[2,5] + 5\times4\times3 \\ 0 + 150 + 60 \end{cases} \quad \underset{k=1}{210}$$

$$m[2,5] = m[2,2] + m[3,5] + 4\times6\times3$$
$$0 + 78 + 72$$

$$m[3,5] = m[3,3] + m[4,5] + 6\times2\times3$$
$$0 + 42 + 36$$

$$m[4,5] = m[4,4] + m[5,5] + 2\times7\times3$$
$$0 + 0 + 42$$

$A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5$

$5\times4 \quad 4\times6 \quad 6\times2 \quad 2\times7 \quad 7\times3$

$P_0\times P_1 \quad P_1\times P_2 \quad P_2\times P_3 \quad P_3\times P_4 \quad P_4\times P_5$

1 2 4
4 6 7

1 3 4
4 2 7

i \ j	1	2	3	4	5
1	1,1 0	1,2 120	1,3 88		
2		2,2 0	2,3 48	2,4 104	
3			3,3 0	3,4 84	3,5 78
4				4,4 0	4,5 42
5					5,5 0

$m[1,2] = m[1,1] + m[2,2] + \overset{0}{5}\times\overset{1}{4}\times\overset{2}{6}$

$k = 1 - n-1 \qquad 0 + 0 + 120$

$m[2,3] = m[2,2] + m[3,3] + \overset{1}{4}\times\overset{2}{6}\times\overset{3}{2}$

$0 + 0 + 48$

$m[3,4] = 0 + 0 + \overset{2}{6}\times\overset{3}{2}\times\overset{4}{7}$

$m[4,5] = m[4,4] + m[5,5] + \overset{3}{2}\times\overset{4}{7}\times\overset{5}{3}$

$0 + 0 + 42$

Just to give idea of recursive calls

$$m[1,3] = \begin{cases} m[1,1] + m[2,3] + 5\times4\times2 & k=1 \\ 0 + 48 + 40 & \\ m[1,2] + m[3,3] + 5\times6\times2 & k=2 \\ 120 + 0 + 60 & \end{cases}$$

$$m[2,4] = \min \begin{cases} m[2,2] + m[3,4] + 4\times6\times7 \\ 0 + 84 + \\ m[2,3] + m[4,4] + 4\times2\times7 \\ 48 \quad\quad 0 \end{cases}$$

	1	2	3	4	5
1	0	1	1		
2		0	2	3	
3			0	3	3
4				0	4
5					0

For Backtracking

$$42 + \overset{2\;3\;5}{6\times2\times3} -$$

$$84 + \overset{2\;4\;5}{6\times7\times3}$$