Solving Recurrance Relations :-

Iteration Method :-

Example: $\boxed{T(n) = 2T\left(\frac{n}{2}\right) + n}$ —— (A)

Lets expand recurrance:

$T(n) = 2(\qquad) + n$

put $n = \frac{n}{2}$ in (A)

$\therefore T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \frac{n}{2}$

$= 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$

$\boxed{= 4T\left(\frac{n}{4}\right) + n + n}$

$= 4(\qquad) + n + n$

put $n = \frac{n}{4}$ in (A)

$\therefore T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \frac{n}{4}$

$= 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + n + n$

$\boxed{= 8T\left(\frac{n}{8}\right) + n + n + n}$

$= 8(\qquad) + n + n + n$

$T\left(\frac{n}{8}\right) = 2T\left(\frac{n}{16}\right) + \frac{n}{8}$

put $n = \frac{n}{8}$ in (A)

$= 8\left(2T\left(\frac{n}{16}\right) + \frac{n}{8}\right) + n + n + n$

$\boxed{= 16T\left(\frac{n}{16}\right) + n + n + n + n}$

$\therefore$ If n is in power of 2, then we can say,

$n = 2^k$ (i.e. n is in some power of 2)

$n = 2^k$

Take log on both sides,

$\Rightarrow \log_2 n = \log_2 2^k$

$\Rightarrow \log_2 n = k$

$\Rightarrow \boxed{k = \log_2 n}$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + \underbrace{n + n + \cdots + n}_{kn}$$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + kn \quad , \text{(generalized eq.)}$$

put the value of k

$$= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + n\log_2 n$$

$$= n\,T\left(\frac{\cancel{n}}{\cancel{n}}\right) + n\log_2 n$$

$$= n\,T(1) + n\log_2 n$$

$$= n(1) + n\log_2 n$$

$$\boxed{T(n) = n\log_2 n + n}$$

$\therefore\ \theta(n\log_2 n)$

$\Rightarrow$ T(n) is solved to a closed form i.e. $n\log_2 n$

$\boxed{2^{\log_2 n} = n}$

How?

if $n=1$, $2^{\log_2 1} = 1$

if $n=2$, $2^{\log_2 2} = 2$

if $n=4$, $2^{\log_2 4} = 4$

$n=8$, $2^{\log_2 8} = 8$

⋮

Example: Lets have a recurrance relation,

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 3T\left(\frac{n}{4}\right) + n, & \text{otherwise i.e. } n > 1 \end{cases}$$

This is some random recurrance relation, lets find its closed form:

Here we will assume, n is in some power of 4.

i.e.

$$\boxed{n = 4^k \quad \text{and} \quad k = \log_4 n}$$

How? take $\log_4$ on both sides.

$$\boxed{T(n) = 3T\left(\frac{n}{4}\right) + n} \quad \text{—— (A)}$$

$$T(n) = 3(\qquad\qquad) + n$$

put $n = \frac{n}{4}$ in (A)

$$T\left(\frac{n}{4}\right) = 3T\left(\frac{n}{16}\right) + \frac{n}{4}$$

$$T(n) = 3\left(3T\left(\frac{n}{16}\right) + \frac{n}{4}\right) + n$$

$$T(n) = 9T\left(\frac{n}{16}\right) + \left(\frac{3}{4}\right)n + n$$

$$= 9(\qquad\qquad) + \left(\frac{3}{4}\right)n + n$$

put $n = \frac{n}{16}$ in (A)

$$T\left(\frac{n}{16}\right) = 3T\left(\frac{n}{4^3}\right) + \frac{n}{4^2}$$

$$= 9\left(3T\left(\frac{n}{4^3}\right) + \frac{n}{4^2}\right) + \left(\frac{3}{4}\right)n + n$$

$$\boxed{T(n) = 3^3 T\left(\frac{n}{4^3}\right) + \left(\frac{3}{4}\right)^2 n + \left(\frac{3}{4}\right)n + n}$$

$\vdots$

$$T(n) = 3^k T\left(\frac{n}{4^k}\right) + \left(\frac{3}{4}\right)^{k-1} n + \left(\frac{3}{4}\right)^{k-2} n + \dots\dots + \left(\frac{3}{4}\right)^{1} n + \left(\frac{3}{4}\right)^{0} n$$

$$T(n) = 3^k T\left(\frac{n}{4^k}\right) + \sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i n$$

$$T(n) = 3^k T\left(\frac{n}{4^k}\right) + n \sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i$$

$$= 3^k T\left(\frac{\cancel{n}}{\cancel{n}}\right) + n \sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i$$

$\because n = 4^k$ and $k = \log_4 n$

$$= 3^k T(1) + n \sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i$$

Notice: Recurrance solved to Base Case.

i.e. $T(1) = 1$

$$= 3^k + n \sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i$$

$$\boxed{T(n) = 3^{\log_4 n} + n \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{4}\right)^i}$$

putting value of k,

Log property:

$$a^{\log_b n} = n^{\log_b a}$$

$\cancel{\neq n \log}$

$$= n^{\log_4 3} + n \underbrace{\sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{4}\right)^i}$$

closed form.

$\therefore$ Hint: This can be further solved with Geometric Series & gives constant.

$\theta(n)$

Example: $T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(n-2)+1 & \text{otherwise} \end{cases}$

$$\boxed{T(n) = T(n-2)+1} \text{ — (A)}$$

1st iteration:

$T(n) = (\quad\quad) + 1$

put $n = n-2$ in (A)

$T(n-2) = T(n-2-2)+1$
$= T(n-4)+1$

$= (T(n-4)+1)+1$

$$\boxed{T(n) = T(n-4)+2}$$

2nd iteration

$= (\quad\quad) + 2$

put $n = n-4$ in (A)

$T(n-4) = T(n-6)+1$

$= (T(n-6)+1)+2$

$$\boxed{T(n) = T(n-6)+3}$$

3rd iteration

$T(n) = (\quad\quad) + 3$

put $n = n-6$ in (A)

$T(n-6) = T(n-8)+1$

$= (T(n-8)+1)+3$

$$\boxed{T(n) = T(n-8)+4}$$

General form :

$$T(n) = \underline{\underline{T(n-k)}} + \frac{k}{2}$$

we want this to be resolved to base case.

plug in $k = n-1$

$$T(n) = T(n-(n-1)) + k/2$$

$$T(n) = T(\cancel{n}-\cancel{n}+1) + \frac{k}{2}$$

$$= T(1) + \frac{k}{2}$$ putting the value of k,

$$= T(1) + \frac{n-1}{2}$$

$$T(n) = 1 + \frac{n-1}{2}$$

$$\in n.$$