Linear Algebra Review

- Vector
- Vector Operations
- Linear Combination
- Span, Bases, Linear Independence
- Length of Vectors
- Dot Product
- Angle between Vectors
- Projection
- Linear Functions
- Linear Transformation
- Scaling, Mirror, Shear, Rotation, Projection
- Composition of Linear Transformations
- Determinant and Inverse
- Change of Bases
- Transformation in Different Bases
- Eigenvalue and Eigenvectors
- Eigenbases and Diagnoalization
- Power of Matrices
- Random Walk and Markov Chain

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Change of Bases

Change of Bases

Changing the basis of a vector space involves moving from one set of basis vectors to another

- Basis change is crucial for simplifying matrix operations and understanding different vector space representations

- **Data Compression:** Basis change techniques are used in PCA for data compression and feature extraction
- **Quantum Computing:** In quantum mechanics, changing the basis of quantum states allows for different properties to be observed and measured
- **Computer Graphics:** Change of basis is fundamental in transformations applied in 3D rendering processes

Reversing a basis change gives the original coordinates, e.g., in signal processing for undoing changes made during filtering or compression, and in computer graphics for converting object coordinates back to world coordinates after transformations

Change of Bases

If $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ is a **basis** for $\mathbb{R}^n$, then any vector $\mathbf{x} \in \mathbb{R}^n$

- can be expressed uniquely as $\mathbf{x} = \beta_1\mathbf{b_1} + \beta_2\mathbf{b_2} + \ldots + \beta_n\mathbf{b_n}$
- the scalars $\beta_1, \beta_2, \ldots, \beta_n$ are the coordinates of $\mathbf{x}$ w.r.t the basis B
- $\mathbf{x}$ is denoted by $\mathbf{x}_B = \left[\beta_1, \beta_2, \ldots, \beta_n\right]_B^T$

Let A be the standard basis, $A = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$

Let $\mathbf{x}_A := \begin{bmatrix}\alpha_1 & \alpha_2 & \ldots & \alpha_n\end{bmatrix}_A^T$

To find coordinates of $\mathbf{x}$ w.r.t B, i.e. $\mathbf{x}_B = \begin{bmatrix}\beta_1 & \beta_2 & \ldots & \beta_n\end{bmatrix}_B^T$

Solve the linear system of equations $\mathbf{x} = \beta_1\mathbf{b_1} + \beta_2\mathbf{b_2} + \ldots + \beta_n\mathbf{b_n}$

Change of Bases

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Solve the linear system of equations $\mathbf{x} = \beta_1\mathbf{b_1} + \beta_2\mathbf{b_2} + \ldots + \beta_n\mathbf{b_n}$

B: the matrix with basis vectors as columns $\implies$ B is invertible

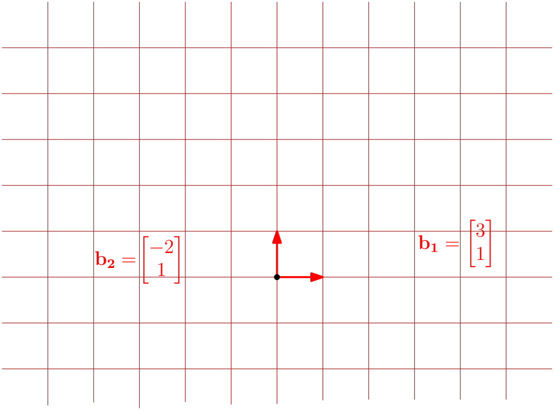
$$\begin{bmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b_2} & \dots & \mathbf{b_n} \\ | & | & & | \end{bmatrix} \begin{bmatrix}\beta_1 \\ \vdots \\ \beta_n\end{bmatrix}_B = \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{bmatrix}_A$$

$\begin{bmatrix}2 & 3\end{bmatrix}_B$ means go 2 and 3 steps in directions $\mathbf{b_1}$ and $\mathbf{b_2}$. We need to know $\mathbf{b_1}$ and $\mathbf{b_2}$ in coordinate system of A. Because in B's coordinates they are $\begin{bmatrix}1 & 0\end{bmatrix}_B^T$ and $\begin{bmatrix}0 & 1\end{bmatrix}_B^T$

$$\begin{bmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b_2} & \dots & \mathbf{b_n} \\ | & | & & | \end{bmatrix}^{-1} \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{bmatrix}_A = \begin{bmatrix}\beta_1 \\ \vdots \\ \beta_n\end{bmatrix}_B$$

Change of Bases

$$\mathbf{b_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Change of Bases

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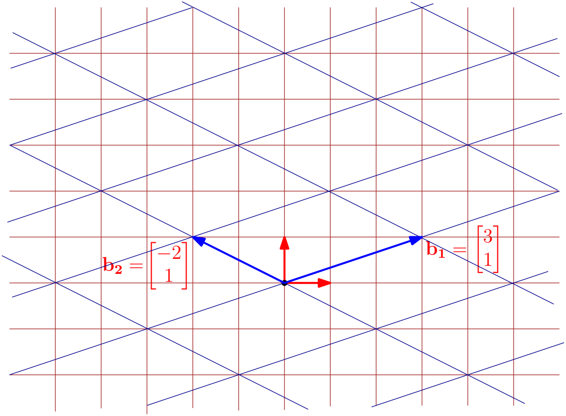


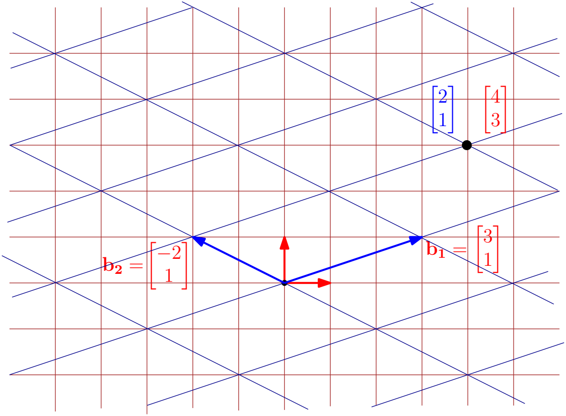
Change of Bases

$$\mathbf{b_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} .2 & .4 \\ -.2 & .6 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Change of Bases in $\mathbb{R}^3$

Let A be the standard basis, $A = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$

Let $B = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\} \subset \mathbb{R}^3$ be $\quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{b_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{b_3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

To find the change of basis matrix $P_{E \to B}$, we express each new basis vector as a linear combination of the standard basis vectors

The change of basis matrix $P_{E \to B}$ is constructed as columns of new basis coordinates in terms of E:

$$P_{E \to B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}$$

Computing $(P_{E \to B})^{-1}$ gives us the matrix to convert coordinates from B back to E:

$$P_{B \to E} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1 & 2 & -1 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

Transformation in different Bases

- A transformation in linear algebra can be represented differently in various bases
- Changing the basis can simplify the representation of a linear transformation, making computations more efficient
- When a linear transformation matrix in a certain basis is diagonal, computing its powers becomes straightforward, significantly reducing computational complexity

Transformation in different Bases

- Apply transformation T to vector $\mathbf{x}_B$
- T is given in coordinate system of A, we cannot do $T\mathbf{x}_B$
- Previously we translated vector from one coordinates system to other
- Now we need to do it for transformation

$$\underbrace{B^{-1}\,\underbrace{T\,\underbrace{B\,\mathbf{x}_B}_{\mathbf{x}_A}}_{\mathbf{x}'_A}}_{\mathbf{x}'_B}$$

- Let T_B be the transformation in B coordinate system then

$$T_B = B^{-1}TB$$

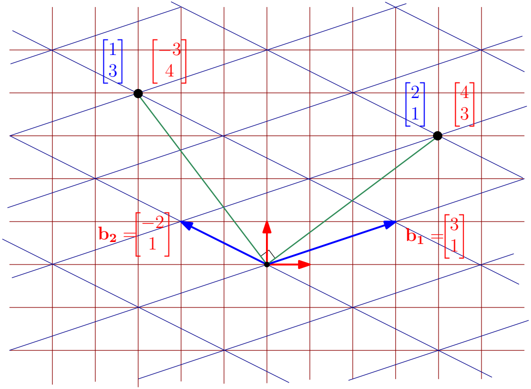
- By the same reasoning

$$T = BT_BB^{-1}$$

Transformation in different Bases

$$\underbrace{B^{-1}\,\underbrace{T\,\underbrace{B\,\mathbf{x}_B}_{\mathbf{x}_A}}_{\mathbf{x}'_A}}_{\mathbf{x}'_B}$$

$$\begin{bmatrix} .2 & .4 \\ -.2 & .6 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{90^\circ \text{ rotation}} \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



Transformation in different Bases

- Translation of vectors and linear transformation between standard bases and another basis B
- Vectors in B are bases vectors (linearly independent) i.e. B is invertible

$$B = \begin{bmatrix} | & | & & | \\ \mathbf{b_1} & \mathbf{b_2} & \dots & \mathbf{b_n} \\ | & | & & | \end{bmatrix}$$

$$\begin{tikzcd}
\mathbf{x} \arrow[rr, "T"] \arrow[d, shift right=1ex, "B^{-1}"'] & & \mathbf{x}' \arrow[d, shift left=1ex, "B^{-1}"] \\
\mathbf{x}_B \arrow[rr, "T_B"'] \arrow[u, shift right=1ex, "B"'] & & \mathbf{x}'_B \arrow[u, shift left=1ex, "B"]
\end{tikzcd}$$

$$T_B = B^{-1} T B \qquad\qquad T = B T_B B^{-1}$$

Transformation in different Bases

Consider a linear transformation T in $\mathbb{R}^2$ represented in standard basis E

In basis $B = \{(2,0),(0,2)\}$, the transformation T might have a simpler matrix representation

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented in the standard basis E as $T_E = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

We want to find its representation in a new basis $B = \{(2,0),(0,2)\}$

The change of basis matrix $P_{E\to B}$ is $\quad P_{E\to B} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

The transformation matrix in basis B, T_B, is given by:

$$T_B = P_{B\to E} T_E P_{E\to B} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$