Q1 Prove that $Z^+ \times Z^+$ are countable.

Assumption: Z^+ is countable set

let Z_1^+ be a set that belongs to Z^+ and it has 'm' elements.

$$Z_1^+(n) = m$$

and $Z_2^+(n) = n$

Then $Z_3^+(n) = m \times n$

$$Z_3^+(n) = m \times n$$
$$Z_1^+ \times Z_2^+ = m \times n$$

where m and $n \in Z^+$
hence, $m \times n \in Z^+$
which is a countable number.

What if $m \to \infty$, $n \to \infty$ then still, it countable as it hold the property of infinitly countable.

Q2 Show that

$$1\times2\times3 + 2\times3\times4 + n(n+1)(n+2)\cdots = \frac{n(n+1)(n+2)(n+3)}{4}$$

$$\sum_1^n K = \frac{n(n+1)}{2}$$

$$\sum_1^n K^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_1^n K^3 = \left[\frac{n+(n+1)}{2}\right]^2$$

$$\sum_{2=1}^{n} 1\times2\times3 + 2\times3\times4 + \cdots n(n+1)(n+2)$$

$$\sum_1^n \cancel{n}(\cancel{n}+1)(\cancel{n}+2)$$

$$(K^2+K)(K+2) = K^3+2K^2+K^2+2K$$

$$\sum_1^n (K^3+3K^2+2K)$$

$$= \sum_1^n K^3 + 3\sum_1^n K^2 + 2\sum_1^n K$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + 3\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{2(n(n+1))}{2}$$

$$= \frac{(n(n+1))^2}{4} + \frac{2[n(n+1)(2n+1)]}{4} + \frac{4(n(n+1))}{4}$$

multiplying numerator and denominator by 2

Taking $\frac{n(n+1)}{4}$ common

$$= \frac{n(n+1)}{4}\left[n(n+1) + 2(2n+1) + 4\right]$$

$$= \frac{n(n+1)}{4}\left[n^2 + n + 4n + 2 + 4\right]$$

$$= \frac{(n(n+1))(n^2 + 5n + 6)}{4}$$

$$\Rightarrow n^2 + 5n + 6 = (n+2)(n+3)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$ Shown.

Q3 By strong induction, prove that $\sqrt{3}$ is irrational.

Ans: let $p(n) : \sqrt{2} \neq \frac{n}{x}$

Basis Step: $p(1) = \frac{1}{x} < \sqrt{3}$

which is true.

let it true for all upto 'k'

Then $p(k+1) = \frac{k+1}{x} = \sqrt{3} \quad x \in \mathbb{N}$

$$\left(\frac{k+1}{x}\right)^2 = 3 \Rightarrow (k+1)^2 = 3x^2$$

This means $k+1$ is factor of 3 for some 'y'

$$3y = k+1$$
$$(3y)^2 = (k+1)^2$$
$$(3y)^2 = 3x^2$$
$$x^2 = \frac{(3y)^2}{3} = \frac{9y^2}{3} = 3y^2$$

Now by the same reasoning 'x' is multiple of 3.

Since 'y' is multiple of 3 and 'x' is multiple of 3 then it contradits that $\frac{k+1}{x}$ $k+1$ and x are not co-prime, they are multiple of each other, Hence, by contradiction,

$$\sqrt{3} \neq \frac{n}{x} \text{ or } \frac{y}{x}$$

where $x, y \in \mathbb{N}^+$