Name: Abdul Rehman
Roll No:- 19L-1135
Section:- D
Subject:- Calculus

ASSIGNMENT # 2

EXERCISE : 4.1

Q2) $y = c_1e^{4x} + c_2e^{-x}$, $(-\infty,\infty)$ $y(0)=1$; $y'(0)=2$

$1 = c_1e^0 + c_2e^0$

$\boxed{1 = c_1 + c_2}$ — ①

$y' = 4c_1e^{4x} + (-1)c_2e^{-x}$

$2 = 4c_1 - c_2$

$2 = 4c_1 - (1-c_1)$ from ①

$2 = 4c_1 - 1 + c_1$

$\boxed{3/5 = c_1}$

$c_2 = 1 - 3/5$

$\boxed{c_2 = 2/5}$

$$\boxed{y = \frac{3}{5}e^{4x} + \frac{2}{5}e^{-x}}$$

Q3) $y = c_1x + c_2x\ln x$ $(0,\infty)$, $y(1)=3$, $y'(1)=-1$

$\cancel{1 = 3c_1 + 3}$

$3 = c_1 + c_2\ln(1)$

$\boxed{c_1 = 3}$ — ①

$y' = c_1 + c_2(1+\ln x)$

$-1 = C_1 + C_2(1+\ln(1))$

$-1 = C_1 + C_2$

$-1 = 3 + C_2$

$\boxed{C_2 = -4}$

$$\boxed{y = 3x - 4x\ln x}$$

Q 10) $y'' + (\tan x)y = e^x$, $y(0) = 1$, $y'(0) = 0$

As y'' has no co-efficient in front of it and $\tan x$ is continous on interval $(-\pi/2, \pi/2)$

Q 14) $y = c_1x^2 + c_2x^4 + 3$

(a) $y(-1) = 0$; $y(1) = 4$

$0 = C_1 + C_2 + 3$; $4 = C_1 + C_2 + 3$

$-3 = C_1 + C_2$; $1 = C_1 + C_2$

which is impossible

(b) $y(0) = 1$; $y(1) = 2$

$1 \neq 3$; $2 = C_1 + C_2 + 3$

which is also not possible

(c) $y(0) = 3$; $y(1) = 0$

$\boxed{3 = 3}$; $0 = C_1 + C_2 + 3$

$\boxed{C_1 + C_2 = -3}$

The system have many solutions

(d) $y(1)=3$; $y(2)=15$

$3 = c_1 + c_2 + 3$, $15 = 4c_1 + 16c_2 + 3$

$c_1 + c_2 = 0$, $3 = c_1 + 4c_2$

The system has unique solution

Q21) $f_1(x) = 1+x$, $f_2(x) = x$, $f_3(x) = x^2$

Applying ~~waon~~ wronsfram law:-

$$W(f_1, f_2, f_3) = \begin{vmatrix} 1+x & x & x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$W = 0 + 0 + 2(1+x-x)$

$W = 2$

As $W \neq 0$, So it is linear independent

Q30) $y^4 + y^4 = 0$; $1, x, \cos x, \sin x$

(1) $f(x) = 1$

$\boxed{0+0=0}$ It's a solution

(2) $f(x) = x$

$\boxed{0+0=0}$ It's a solution

(3) $f(x) = \cos x$

$\boxed{\cos x - \cos x = 0}$ It's a solution

(4) $f(x) = \sin x$

$\boxed{\sin x - \sin x = 0}$ It's a solution.

$$W = \begin{vmatrix} 1 & x & \sin x & \cos x \\ 0 & 1 & \cos x & -\sin x \\ 0 & 0 & -\sin x & -\cos x \\ 0 & 0 & -\cos x & \sin x \end{vmatrix}$$

$= 1(\cos^2 x + \sin^2 x)$

$= 1$

$W = 1 \neq 0$, so it is linear independent

The general solution is :-

$$\boxed{y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x}$$

Q31) $y'' - 7y' + 10y = 24e^x$;

$y = c_1 e^{2x} + c_2 e^{5x} + 6e^x \qquad (-\infty, \infty)$

(1) $y_1 = e^{2x}$

$y_1' = 2e^{2x}$

$y_1'' = 4e^{2x}$

$4e^{2x} - 7(2e^{2x}) + 10e^{2x} - \cancel{24e^x} = 0$

$4e^{2x} - 14e^{2x} + 10e^{2x} = 0$

$\boxed{0 = 0}$ Hence, e^{2x} is a solution

(2) $y_2 = e^{5x}$

$y_2' = 5e^{5x}$

$y_2'' = 25e^{5x}$

$25e^{5x} - 7(5e^{5x}) + 10e^{5x} = 0$

$25e^{5x} - 35e^{5x} + 10e^{5x} = 0$

$\boxed{0 = 0}$ Hence e^{5x} is a solution.

(3) $y_3 = 6e^x$

$y_3' = 6e^x$

$y_3'' = 6e^x$

$6e^x - 42e^x + 60e^x = 24e^x$

$\boxed{24e^x = 24e^x}$

So, $y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$ is valid for the interval $(-\infty, \infty)$

EXERCISE # 4.2

Q2) $y'' + 2y' + y = 0 \;;\; y_1 = xe^{-x}$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

$$y_2 = xe^{-x} \int \frac{e^{-2x}}{x^2 e^{-2x}}\,dx$$

$$y_2 = xe^{-x}\left(\frac{-1}{x}\right)$$

$$\boxed{y_2 = -e^{-x}}$$

Q7) $9y'' - 12y' + 4y = 0 \;;\; y_1 = e^{2x/3}$

$$y'' - 4/3\,y' + 4/9\,y = 0$$

$$y_2 = e^{2/3x} \int \frac{e^{-\int -4/3\,dx}}{e^{4/3x}}\,dx$$

$$y_2 = e^{2/3x} \int \frac{e^{4/3x}}{e^{4/3x}}\,dx$$

$$\boxed{y_2 = xe^{2/3}}$$

Q14) $x^2y'' - 3xy' + 5y = 0 \;;\; y_1 = x^2\cos(\ln x)$

$$y'' - 3/x\,y' + 5/x^2\,y = 0$$

$$y_2 = x^2\cos(\ln x) \int \frac{e^{3\ln x}}{x^4\cos^2(\ln x)}\,dx$$

$$y_2 = x^2\cos(\ln x) \int \frac{x^3}{x^4\cos^2(\ln x)}\,dx$$

$$y_2 = x^2\cos(\ln x)\int \frac{1}{x\cos^2(\ln x)}\,dx$$

Let $u = \ln x$

$du = 1/x\,dx$

$$y_2 = x^2\cos(u)\int \frac{1}{\cos^2(u)}\,du$$

$$y_2 = x^2\cos(u)\tan(u)$$

$$y_2 = x^2\cancel{\cos(u)}\cdot\frac{\sin(u)}{\cancel{\cos(u)}}$$

$$\boxed{y_2 = x^2\sin(\ln x)}$$

Q15) $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0;$

$y_1 = x+1$

$$y'' + 2\frac{(1+x)}{-(x^2+2x-1)} - \frac{2}{-(x^2+2x-1)}y = 0$$

$$y_2 = 1+x\int \frac{e^{\int \frac{2+2x}{x^2+2x-1}dx}}{(1+x)^2}\,dx$$

$$y_2 = (1+x)\int \frac{e^{\ln(x^2+2x-1)}}{(1+x)^2}\,dx$$

$$y_2 = (1+x)\int \frac{x^2+2x-1}{(1+x)^2}\,dx$$

$$y_2 = (1+x)\int \frac{x^2+2x+1}{(1+x)^2} - \frac{-2}{(1+x)}\,dx$$

$$y_2 = (1+x)\int \frac{(1+x)^2}{(1+x)^2}\,dx - 2\int \frac{1}{(1+x)^2}\,dx$$

$$y_2 = x(1+x) + \frac{2}{(1+x)}(1+x)$$

$$y_2 = x + x^2 + 2$$

$$\boxed{y_2 = x^2 + x + 2}$$

EXERCISE # 4.3

Ex 4.2

Q 17) $y'' - 4y = 2 \;;\; y_1 = e^{-2x}$

$$y_2 = e^{-2x}\int \frac{e^{4x}}{e^{4x}}\, dx$$

$$\boxed{y_2 = x e^{-2x}} \quad \& \quad \boxed{y_p = -\tfrac{1}{2}}$$

Q7) $12y'' - 5y' - 2y = 0$

$$12m^2 - 5m - 2 = 0$$

$$(3m - 2)(4m + 1) = 0$$

$$m = 2/3 \,,\, -1/4$$

General Solution :- $\boxed{y_2 = C_1 e^{2/3 x} + C_2 e^{-1/4 x}}$

Q 23) $y'''' + y''' + y'' = 0$

$$m^4 + m^3 + m^2 = 0$$

$$m^2(m^2 + m + 1) = 0$$

$$m_1 = 0,\; m_2 = 0$$

$$m^2 + m + 1 = 0$$

$$m = -\tfrac{1}{2} \pm \frac{2\sqrt{3}}{2}$$

$$y_c = c_1 e^{2x} + c_2 x e^{2x} + e^{-\frac{1}{2}x}\left(c_3 \cos \frac{\sqrt{3}}{2}x + c_4 \sin \frac{\sqrt{3}x}{2}\right)$$

$$\boxed{y_c = e^{2x}(c_1 + c_2 x) + e^{-\frac{1}{2}x}\left(c_3 \cos \frac{\sqrt{3}x}{2} + c_4 \sin \frac{\sqrt{3}x}{2}\right)}$$

Q27) $\frac{d^5u}{dr^5} + \frac{5d^4u}{dr^4} - \frac{2d^3u}{dr^3} - 10\frac{d^2u}{dr^2} + \frac{du}{dr} + 5u = 0$

$$m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 = 0$$
$$(m+5)(m^4 - 2m^2 + 1) = 0$$
$$(m+5)(m^2-1)^2 = 0$$
$$m_1 = -5 \; ; \; m_{2,3} = 1 \; , \; m_{4,5} = -1$$
$$y_c = c_1 e^{-5x} + (c_2 + c_3 r)e^{r} + (c_4 + c_5 r)e^{-x}$$

Q30) $\frac{d^2y}{d\theta^2} + y = 0 \; ; \; y(\pi/3) = 0 \; ; \; y'(\frac{\pi}{3}) = 2$

$$m^2 + 1 = 0$$
$$\boxed{m = \pm i}$$
$$\boxed{y_c = c_1 \cos\theta + c_2 \sin\theta}$$
$$0 = c_1 \cos \pi/3 + c_2 \sin \pi/3$$
$$0 = c_2\sqrt{3}/2 + c_1/2$$
$$0 = c_1 + \sqrt{3}\, c_2 \quad \text{— (i)}$$
$$y_1' = -c_1 \sin\theta + c_2 \cos\theta$$
$$2 = -c_1 \sqrt{3}/2 + c_2/2$$
$$\boxed{4 = c_1\sqrt{3} + c_2}$$
$$c_1 = -\sqrt{3} \; , \; c_2 = 1$$

So,

$$\boxed{y_c = -\sqrt{3}\cos\theta + \sin\theta}$$

Q36) $y''' + 2y'' - 5y' - 6y = 0$; $y(0) = y'(0) = 0$,
$y'''(0) = 1$

$m^3 + 2m^2 - 5m - 6 = 0$

$(m+1)(m^2 + m - 6) = 0$

$(m+1)(m-2)(m+3) = 0$

$m_1 = -1$, $m_2 = 2$, $m_3 = -3$

$y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-3x}$

$0 = c_1 + c_2 + c_3$

$y' = c_1 e^{-x} + 2c_2 e^{2x} - 3c_3 e^{-3x}$

$0 = -c_1 + 2c_2 - 3c_3$

$y_c'' = c_1 e^{-x} + 4c_2 e^{2x} + 9c_3 e^{-3x}$

$1 = c_1 + 4c_2 + 9c_3$ from $y''(0) = 1$

$c_1 = -1/6$, $c_2 = 1/15$, $c_3 = 1/10$

So;

$$\boxed{y_c = -\frac{1}{6}e^{-x} + \frac{1}{15}e^{2x} + \frac{1}{10}e^{-3x}}$$

Q37) $y'' - 10y' + 25y = 0$; $y(0) = 1$

$m^2 - 10m + 25 = 0$

$m^2 - 5m - 5m + 25 = 0$

$m(m-5) - 5(m-5) = 0$

$(m-5)^2 = 0$

$y_c = (c_1 + c_2 x)e^{5x}$

$0 = (c_1 + c_2)e5$

$\boxed{c_1 + c_2 = 0}$

$\boxed{c_1 = 1}$

$\boxed{c_2 = -1}$ $\Rightarrow$ $\boxed{y_c = (1 + (-1)x)e^{5x}}$

Q41) $y'' - 3y = 0 \; ; y(0) = 1 \; ; y'(0) = 5$

$$m^2 - 3 = 0$$
$$m = \pm\sqrt{3}$$
$$y_c = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}$$
$$1 = c_1 + c_2$$
$$y_c = \sqrt{3}\, c_1 e^{\sqrt{3}x} - \sqrt{3}\, c_2 e^{\sqrt{4}x}$$
$$5 = \sqrt{3}(c_1 \mp c_2)$$
$$5/\sqrt{3} = c_1 - c_2$$
$$c_1 - (1 - c_1) = 5/\sqrt{3}$$
$$c_1 = \frac{5 + \sqrt{3}}{2\sqrt{3}}$$
$$\boxed{c_2 = \frac{-5 - \sqrt{3}}{2\sqrt{3}}}$$
$$\boxed{y_c = \frac{5 + \sqrt{3}}{2\sqrt{3}} e^{\sqrt{3}x} - \left(\frac{5 + \sqrt{3}}{2\sqrt{3}}\right) e^{-\sqrt{3}x}}$$

EXERCISE # 4.4

Q7) $y'' + 3y = -48x^2 e^{3x}$

$$m^2 + 3 = 0$$
$$m = \pm\sqrt{3}\, i$$
$$y_p = (Ax^2 + Bx + C)e^{3x}$$
$$y_p' = Ax^2 e^{3x} + Bxe^{3x} + Ce^{3x}$$
$$y_p'' = 9Ax^2 e^{3x} + (12A + 9B)xe^{3x} + (2A + 6B + 9C)e^{3x}$$
$$-48x^2 e^{3x} = 12Ax^2 e^{3x} + (12A + 12B)xe^{3x} + (2A + 6B + 12C)e^{3x}$$
$$\boxed{A = -4}$$

$12A + 12B = 0$;

$\boxed{B = 4}$

$\boxed{C = -4/3}$

$y_p = (-4x^2 + 4x - 4/3)e^{3x}$

$y = y_c + y_p$

$$\boxed{y = (C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x) + e^{3x}(-4x^2 + 4x - 4/3)}$$

Q21) $y''' - 6y'' = 3 - \cos x$

$m^3 - 6m^2 = 0$

$m^2(m-6) = 0$

$m_1 = 6,\ m_{2,3} = 0$

$y_c = C_3 e^{6x} + (C_2 + C_3 x)e^{0x}$

$y_p = A\cos x + B\sin x + Cx^2$

$y_p'' = -A\cos x - B + \sin x + 2C$

$y_p''' = A\sin x - B\cos x$

$(A\sin x - B\cos x) - 6(-A\cos x - B\sin x + 2C) = 3 - \cos x$

$(6A - B)\cos x + (A + 6B)\sin x - 12C = 3 - \cos x$

$-12C = 3$

$6A - B = -1$

$A + 6B = 0$

$A = -6/37$, $B = 1/37$, $C = -1/4$

$y_p = -\frac{6}{37}\cos x + \frac{1}{37}\sin x + \left(\frac{-1}{4}\right)x^2$

$y = y_c + y_p$

$$\boxed{y = C_1 e^{6x} + C_2 + C_3 x - \frac{6}{37}\cos x + \frac{1}{37}\sin x - \frac{1}{4}x^2}$$

Q32)- $y'' - y = \cosh x$, $y(0) = 2$; $y'(0) = 12$

$m^2 - 1 = 0$

$m = \pm 1$

$y_c = c_1 e^x + c_2 e^{-x}$

$y_p = Axe^x + Bxe^{-x}$

$y_p'' = Axe^x + 2Ae^x + Bxe^{-x} - 2Be^{-x}$

$A = 1/4$, $B = -1/4$

$y_p = 1/4\, xe^x - 1/4\, xe^{-x}$

$y = c_1 e^x + c_2 e^{-x} + 1/4\, e^x x - 1/4\, e^{-x} x$

$y(0) = 2$, $y'(0) = 12$

$c_1 + c_2 = 2$

$c_1 - c_2 = 12$

$c_1 = 7$, $c_2 = -5$

$$y = \frac{1}{2}x\left(\frac{e^x - e^{-x}}{2}\right) + 7e^x - 5e^{-x}$$

$$y = \frac{1}{2}x \sinh x + 7e^x - 5e^{-x}$$

Q37)- $y'' + y = x^2 + 1$; $y(0) = 5$, $y(1) = 0$

$m^2 + 1 = 0$

$m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_p = Ax^2 + Bx + C$

$y_p' = 2xA + B$

$y_p'' = 2A$

$2A + Ax^2 + Bx + C = x^2 + 1$

$\boxed{A = 1}$, $\boxed{B = 0}$, $\boxed{C = -1}$

$y_p = x^2 - 1$

$y = y_c + y_p$

$\boxed{y = c_1 \cos x + c_2 \sin x + x^2 - 1}$

$5 = c_1 - 1$

$c_1 = 6$

$0 = c_1 \cos(1) + c_2 \sin(1) + 1 - 1$

$c_2 = -6 \cot(1)$

$\boxed{y = 6\cos(x) - 6\cot(1)\sin x + x^2 - 1}$

Q 41) $y'' + 4y = g(x)$; $y(0) = 1$; $y'(0) = 2$

$g(x) = \begin{cases} \sin(x), & 0 \le x \le \pi/2 \\ 0, & x > \pi/2 \end{cases}$

$m^2 + 4 = 0$

$m = \pm 2i$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

for $g(x) = \sin x$ where $0 \le x \le \pi/2$

$y_p'' = A\sin(x) - B\sin(x)$

$-A\cos x - B\sin x + 4\cos x + 4B\sin x = \sin x$

$3A\cos x + 3B\sin x = \sin x$

$A = 0$; $B = 1/3$

$y_p = 1/3 \sin x$

$y = c_1\cos 2x + c_2 \sin x + 1/3 \sin x$

$g(x) = 0$ where $x > \pi/2$

$y_p = A$

$4A = 0$

$\boxed{A = 0}$

$\boxed{y = C_1 \cos 2x + C_2 \sin 2x}$

EXERCISE # 4·5

Q7)- $y''' + 2y'' - 13y' + 10y = xe^{-x}$

$(D^3 + 2D^2 - 13D + 10)y = xe^{-x}$

$(D-1)(D^2 + 3D - 10)y = xe^{-x}$

$(D-1)(D+5)(1)-2) = xe^{-x}$

Q13) $(D-2)(D+5)$; $y = e^{2x} + 3e^{-5x}$

$(D-2)(D+5)(e^{2x} + 3e^{-5x})$

$(D^2 + 5D - 2D - 10)(e^{2x} + 3e^{-5x})$

$(D^2 + 3D - 10)(e^{2x} + 3e^{-5x})$

$4e^{2x} + 75e^{-5x} + 6e^{2x} - 45e^{-5x} - 10e^{2x} - 30e^{-5x}$

$(4+6-10)e^{2x} + (75-45-30)e^{-5x}$

$0 + 0 = 0$

The given operator initates y.

Q23) $e^{-x} + 2xe^{x} - x^2e^{x}$

$(D+1)$ annihlate e^{-4}

$(D-1)^3$ annihilate $(2x - x^2)e^{x}$

So, $(D+1)(D-1)^3 e^{-x} + 2xe^{4} - x^2e^{x} = 0$

$(D+1)(D-1)^3 e^{-x} + 2xe^{x} - x^2e^{x} = 0$

operator is $(D+1)(D-1)^3$

Q29) $(D-6)(2D+3)$

$(D-\alpha)^n$ for $x^{n-1}e^{\alpha x}$

$(D-6)$ we have e^{+6x}

$(2D+3)$ we have $e^{-3/2\,x}$

$(D-6)(2D+3) \Rightarrow e^{6x}, e^{3/2\,x}$

Q63) $y^{iv} - 2y''' + y'' = e^x + 1$

$(D^4 - 2D^3 + y)^2 y = e^x + 1$

$m^4 - 2m^3 + m^2 = 0$

$m^2(m^2 - 2m + 1) = 0$

$m_{1,2} = 0, (m-1)^2 = 0$

$m_{1,2} = 0, m_{3,4} = 1$

$y_c = (C_1 + C_2 x) + (C_3 + C_4 x)e^x$

$y_p = Ax^2e^4 + Bx^2$

$2Ae^x + 2B = e^x + 1$

$2Ae^x + 2B = e^n + 1$

$2A = 1 \;;\; A = 1/2$

$2B = 1 \;;\; B = 1/2$

$y_p = 1/2\, x^2 e^x + 1/2\, x^2$

$y = c_1 + C_2 x + (C_3 + C_4 x)e^x + 1/2\, x^2 e^x + \frac{1}{2}$

Q67) $y'' - 5y' = x - 2 \;;\; y(0) = 0 \;;\; y'(0) = 2$

$y(D^2 - 5D) = x - 2$

$m^2 - 5m = 0$

$m(m-5) = 0$

$m = 0, m = 5$

$y_c = c_1 + c_2 e^{5x}$

$y_p = \cancel{e} A x^2 + Bx$

$y' = 2Ax + B$

$y'' = 2A$

$A = -1/10$

$2A - 5B = -2$

$B = 9/25$

$y = c_1 + c_2 e^{5x} - 1/10\, x^2 + 9/25\, x$

Exercise : 4.6

Q 15) $y'' + 2y' + y = e^{-t} \ln t$

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m_{1,2} = -1$

$y_c = c_1 e^{-t} + c_2 t e^{-t}$

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t}$$

$$W_1 = \begin{vmatrix} 0 & te^{-t} \\ e^{-t}\ln t & e^{-t} - te^{-t} \end{vmatrix} = -e^{-2t}\, t \ln(t)$$

$$W_2 = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & e^{-t}\ln(t) \end{vmatrix} = e^{-2t} \ln(t)$$

$$V_1' = \frac{W_1}{W} = \frac{-\cancel{e^{-2t}}\, t \ln(t)}{\cancel{e^{-2t}}} = -t \ln(t)$$

$$V_2' = \frac{W_2}{W} = \frac{\cancel{e^{-2t}} \ln(t)}{\cancel{e^{-2t}}} = \ln(t)$$

$$V_1 = -\int t\ln(t)\,dt = \frac{-t^2\ln(t)}{2} + \frac{t^2}{4}$$

$$V_2 = \int \ln(t)\,dt = t\ln t - t$$

$$y_p = \left(\frac{-t^2\ln(t)}{2} + \frac{t^2}{4}\right)e^{-t} + (t\ln(t) - t)e^{-t}t$$

$$y = y_c + y_p$$

$$y = c_1 e^{-t} + c_2 t e^{-t} + \tfrac{1}{2}t^2 e^{-t}\ln t - \tfrac{3}{4}t^2 e^{-t}$$

Q18) $4y'' - 4y' + y = e^{x/2}\sqrt{1-x^2}$

$$4m^2 - 4m + 1 = 0$$

$$4m^2 - 2m - 2m + 1 = 0$$

$$2m(m-1) - 1(2m-1) = 0$$

$$(2m-1)(m-1) = 0$$

$$m = 1/2 \quad m = 1$$

$$y_c = c_1 e^{x/2} + c_2 e^{x}$$

$$y'' - y' + y/4 = \frac{e^{x/2}\sqrt{1-x^2}}{4}$$

$$W = \begin{vmatrix} e^{1/2x} & e^{x} \\ \tfrac{1}{2}e^{1/2x} & e^{x} \end{vmatrix} = e^{3/2x} - \frac{1}{2}e^{3/2x} = \frac{1}{2}e^{3/2x}$$

$$W_1 = \begin{vmatrix} 0 & e^{x} \\ \frac{e^{x/2}\sqrt{1-x^2}}{4} & e^{x} \end{vmatrix} = \frac{-e^{3/2x}\sqrt{1-x^2}}{4}$$

$$W_2 = \begin{vmatrix} e^{x/2} & 0 \\ \tfrac{1}{2}e^{x/2} & \frac{e^{x/2}\sqrt{1-x^2}}{4} \end{vmatrix} = \frac{e^{x}\sqrt{1-x^2}}{4}$$

$$V_1' = \frac{W_1}{W} = \frac{-2e^{3/2x}\sqrt{1-x^2}}{4e^{3/2x}} = \frac{-\sqrt{1-x^2}}{2}$$

$$V_2' = \frac{W_2}{W} = \frac{2e^{x}\sqrt{1-x^2}}{e^{3/2x}\times 4} = \frac{\sqrt{1-x^2}}{2e^{3/2x}}$$

$$y = c_1 e^{1/2 x} + c_2 e^{x} + 1/12\, e^{1/2 x} (\sqrt{1-x^2})^3 + \frac{1}{?} e^{1/2 x} (\sin^{-1}(x) + x\sqrt{1-x^2})$$

Q22) $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

$$m^2 - 4m + 4 = 0$$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

$$m_{1,2} = 2$$

$$y_c = (c_1 + c_2 x) e^{2x}$$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (12x^2 - 6x)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = -xe^{2x}(e^{2x}(12x^2 - 6x))$$

$$= -12x^3 e^{4x} + 6x^2 e^{x/4}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (12x^2 - 6x)e^{2x} \end{vmatrix} = e^{4x}(12x^{1/2} - 6x)$$

$$v_1' = \frac{W_1}{W} = -x(12x^3 - 6x)$$

$$v_2' = \frac{W_2}{W} = 12x^2 - 6x$$

$$v_1 = -3x^4 + 2x^3$$

$$v_2 = 4x^3 - 3x^2$$

$$y_p = (-3x^4 + 2x^3)e^{2x} + (4x^3 - 3x^2)e^{2x}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^{2x} + e^{2x}(-3x^4 + 2x^3) + (4x^3 - 3x^2)$$

Q23)- $x^2y'' + xy' + \left(\left(x^2 - \frac{1}{4}\right)y\right) = x^{3/2}$

$y'' + \frac{1}{x}y'\left(1 - \frac{1}{4x^2}\right)y = x^{-1/2}$

$y'' + P(x)y' + g(x)y = f(x)$

$y_1 = x^{-1/2}\cos x$ and $y_2 = x^{-1/2}\sin x$

$y_c = c_1 x^{-1/2}\cos x + c_2 x^{-1/2}\sin x$

$$W = \begin{vmatrix} x^{-1/2}\cos x & x^{-3/2}\sin x \\ -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x & -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x \end{vmatrix}$$

$= -\frac{1}{2}x^{-2}\sin x\cos x + x^{-1}\cos^2 x + \frac{1}{2}x^{-2}\sin x\cos x$

$\cdot \; + x^{-1}\sin^2 x$

$= x^{-1}(\cos^2 x + \sin^2 x)$

$\boxed{W = x^{-1}}$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y & 0 \\ y_1' & f(x) \end{vmatrix} = x^{-1}\cos x$$

$v_1' = -\sin x$

$v_2' = \cos x$

$v_1 = \cos x$

$v_2 = \sin x$

$y_p = \cos x \times x^{-1/2}\cos x + \sin x * x^{-1/2}\sin x$

$y_p = x^{-1/2}(\cos^2 x + \sin^2 x)$

$y_p = x^{-1/2}$

$y = y_c + y_p$

$y = c_1 x^{-1/2}\cos x + c_2 \cdot x^{-1/2}\sin x + x^{-1/2}$

Q26) $y''' + 4y' = \sec 2x$

$m^3 + 4m = 0$

$m(m^2 + 4) = 0$

$m = 0 \quad m = \pm 2i$

$y_c = c_1 + c_2 \cos 2x + c_3 \sin 2x$

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$= (8\sin^2 2x + 8\cos^2 2x)$

$= 8$

$$W_1 = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ f(x) & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$= \sec 2x (2\cos^2 2x + 2\sin^2 2x)$

$= 2\sec 2x$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2\cos 2x \\ 0 & f(x) & -4\sin 2x \end{vmatrix}$$

$= 1(-\sec 2x \cdot 2\cos 2x)$

$W_2 = -2$

$$W_3 = \begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2\sin 2x & 0 \\ 0 & -4\cos 2x & f(x) \end{vmatrix}$$

$= 1(-2\sec 2x \sin 2x)$

$W_3 = -2\tan 2x$

$V_1' = \dfrac{\sec 2x}{4}$

$V_2' = -1/4$

$V_3' = -\frac{\tan 2x}{4}$

$V_1 = 1/2 \ln(\sec 2x + \tan 2x)$

$V_2 = -1/4\, x$

$V_3 = -1/8 \ln(\sec 2x)$

$y_P = 1/8 \ln(\sec 2x + \tan 2x) - 1/4 \ln(\cos 2x) + 1/8 \sin 2x \ln(\cos$

$y = y_P + y_c$

$y = c_1 + c_2 \cos 2x + c_3 \sin 2x + 1/8 \ln\{(\sec 2x) + \tan 2x\}$
$-1/4\, x\cos 2x + 1/8 \sin 2x \ln(\cos 2x)$

Excercise : 4.7

Q16) $x^3y''' + x^2y' - y = 0$

Let $y = x^m$

$y' = mx(m-1)$

$y''' = m(m-1)(m-2)x^{(m-3)}$

$x^3 m(m-1)(m-2)x^{m-3} + xm(x^{m-1}) - x^m = 0$

$m(m-1)(m-2)x^m + mx^m - x^m = 0$

$m(m-1)(m-2) + m - 1 = 0$

$m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)^3 = 0$

$m_{1,2,3} = 1, 1, 1$

$y = c_1x + c_2x(\ln x) + c_3x(\ln x)^2$

Q17) $xy''' + 6y''' = 0$

Let $y = x^m$

$y''' = m(m-1)(m-2)x^{m-3}$

$y'''' = m(m-1)(m-2)(m-3)x^{m-4}$

$x^4 y'''' + 6x^3 y''' = 0$

$0 = x^4(m(m-1)(m-2)(m-3)x^{m-4}) + 6x^3(m(m-1)(m-2)x^{m-3}$

$0 = m(m-1)(m-2)(m-3) + 6m(m-1)(m-2)$

$m(m-1)(m-2)(m-3+6) = 0$

$m(m-1)(m-2)(m+3) = 0$

$m = 0, 1, 2, -3.$

$y = c_1 + c_2 x + c_3 x^2 + c_4 x^{-3}$

Q21) $x^2y'' - xy' + y = 2x$

let $y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$x^2(m(m-1)x^{m-2}) - x(mx^{m-1}) + x^m = 0$

$m(m-1) - m + 1 = 0$

$m^2 - m - m + 1 = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m_{1,2} = 1, 1$

$y_c = c_1 x + c_2 (\ln x)x$

$y'' - {}^1/_x y' + {}^y/_{x^2} = {}^2/_x$

$$W = \begin{vmatrix} x & (\ln x) x \\ 1 & \ln x + 1 \end{vmatrix} = x \ln x + x - \ln x$$

$$W = x$$

$$W_1 = \begin{vmatrix} 0 & (\ln x) x \\ 2/x & \ln x + 1 \end{vmatrix} = -2 \ln x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2/x \end{vmatrix} = 2$$

$V_1' = -2/x \ln x \; ; \; V_1 = \frac{-2(\ln x)^2}{2} = -(\ln x)^2$

$V_2' = 2/x \; ; \; V_2 = 2 \ln x$

$y_1^3 = (-(\ln x)^2)(x) + (2 \ln x)(x \ln x)$

$y_1^3 = x (\ln x)^2$

$y = c_1 x + c_2 x (\ln x) + x (\ln x)^2$

Q29)- $xy'' + y' = x \quad ; y(1) = 1 \quad ; y'(1) = -1/2$

$x^2 y'' + x y' = x^2$

let $y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$x^2 (m(m-1) x^{m-2}) + x (m x^{m-1}) = 0$

$m(m-1) + m = 0$

$m(m-1+1) = 0$

$m(m) = 0$

$m^2 = 0$

$\boxed{m_{1,2} = 0, 0}$

$y = c_1 + c_2 \ln x$

$y'' + \frac{1}{x} y' = 1$

$$W = \begin{vmatrix} 1 & \ln x \\ 0 & 1/x \end{vmatrix} = \frac{1}{x}$$

$$W_1 = \begin{vmatrix} 0 & \ln x \\ 1 & 1/x \end{vmatrix} = -\ln x$$

$$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$v_1' = -x \ln x \; ; \; v_1 = -x^2/2 \ln x + x^2/4$

$v_2' = x \; ; \; v_2 = x^2/2$

$y_p = v_1 y_1 + v_2 y_2$

$y_p = (1/4 x^2 - 1/2 x^2 \ln x) + x^2/2 \times \ln x$

$y_p = 1/4 x^2$

$y = c_1 + c_2 \ln(x) + 1/4 x^2$

$1 = c_1 + 1/4$

$1 - 1/4 = c_1$

$\boxed{c_1 = 3/4}$

$c_2 + 1/2 = -1/2$

$\boxed{c_2 = -1}$

Ans: $\boxed{y = 3/4 - \ln x + 1/4 x^2}$

Q35) $x^2 y'' - 3xy' + 13y = 4 + 3x$

Let $x = e^t$

$t = \ln x$

$dt/dx = 1/x$

$dy/dx = 1/x \, dy/dt$

$d^2y/dx^2 = d/dx(1/x \, dy/dt)$

$d^2y/dx^2 = 1/x (d/dx \cdot dy/dt) + dy/dt(-1/x^2)$

$d^2y/dx^2 = 1/x\,(dy/dx)(dy/dt) - 1/x^2\, dy/dt$

$d^2y/dx^2 = 1/x\,(dy/dt)(1/x\, dy/dt) - 1/x^2\, dy/dt$

$d^2y/dx^2 = 1/x^2\, d^2y/dt - 1/x^2\, dy/dt$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt} - \frac{dy}{dt}\right)$$

$$x^2 \cdot \left(\frac{1}{x^2}\right)\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 3x\left(\frac{1}{x}\right)\frac{dy}{dt} + 13y = 0$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 3\frac{dy}{dt} + 13y = 0$$

Let $y = e^{mt}$

$$\frac{dy}{dt} = m e^{mt}$$

$$\frac{d^2y}{dt^2} = m^2 e^{mt}$$

$$m^2 e^{mt} - 4me^{mt} + 13e^{mt} = 0$$

$$m^2 - 4m + 13 = 0$$

$$m = 2 \pm 3i$$

$$\boxed{y_c = c_1 e^{2t}\cos 3t + c_2 e^{2t}\sin 3t}$$

$$y_p = Ae^t + B$$

$$\frac{dy_p}{dt^2} = Ae^t$$

$$Ae^t - 4Ae^t + 13(Ae^t + B) = 4 + 3e^t$$

$$10Ae^t + 13B = 4 + 3e^t$$

$10A = 3$; $13B = 4$

$A = \frac{3}{10}$; $B = \frac{4}{13}$

$y_p = \frac{3}{10} e^t + \frac{4}{13}$

$y = e^{2t}(c_1 \cos 3t + c_2 \sin 3t) + 3/10\, e^t + 4/13$

$t = \ln x$

$$\boxed{y = x^2 (c_1 \cos 3(\ln x) + c_2 \sin 3(\ln x)) + 3/10\, x + 4/13}$$

Q37) $4x^2 y'' + y = 0 \quad ; \quad y(-1) = 2 \; ; \; y'(-1) = 4$

Let $y = x^m$

$y'' = m(m-1)x^{m-2}$

$4x^2 (m(m-1)x^{m-2}) + x^m = 0$

$4m(m-1)x^m + x^m = 0$

$4m(m-1) + 1 = 0$

$4m^2 - 4m + 1 = 0$

$4m^2 - 2m - 2m + 1 = 0$

$2m(2m-1) - 1(2m-1) = 0$

$(2m-1)(2m-1) = 0$

$m_{1,2} = 1/2$

$y_c = c_1 x^{1/2} + c x^{1/2}(\ln x)$

$x = -x \quad ; \quad (-\infty, 0)$

$y = 2(-x)^{1/2} - 5(-x)^{1/2} \ln(-x) \quad , \quad x < 0$

Chapter # 4

EXAMPLES

EXERCISE # 4.1

Example : 1

$$3y''' + 5y'' + y' + 7y = 0$$

$$y(1) = 0 \; ; \; y'(1) = 0 \; ; \; y''(1) = 0$$

Theorem 4.1.1 conditions are fulfilled. Hence $y = 0$ is the Solution in interval containing $x = 1$.

Example : 2

$$y''' - 4y = 12x \; ; \; y(0) = 4 \; ; \; y'(0) = 1$$

The co-efficient $g(x) = 12x$ are continous and $a_2(x) \neq 0$ on interval containing $x = 0$. From Theorem 4.1.1 we conclude that given function is unique solution

Example : 3

$$x = c_1 \cos(4t) + c_2 \sin(4t)$$

a)- $x(0) = 0 \Rightarrow x(\pi/2) = 0$

$\boxed{c_1 = 0}$ & $\boxed{c_2 = 0}$

b)- $x(0) = 0$, $x(\pi/8) = 0$

$\boxed{c_1 = 0}$, $c_2 - 1 = c_2 \sin \pi/2 = 0$

c) $x(0)=0$; $x(\pi/2)=1$

$\boxed{c_1=0}$ $\boxed{c_2=0}$

Example 4

$$y_1 = x^2 , y_2 = x^2 ; x^3y''' - 2xy' + 4y = 0$$

The linear combination by superposition condition is : $y = c_1x^2 + c_2x^2\ln(x)$ is also the solution of the equation on the interval.

Example 5

$$f_1(x) = \cos^2 x ; f_2(x) = \sin^2 x ; f_3(x) = \sec^2 x$$

$$f_4(x) = \tan^2 x$$

$$c_1\cos^2 x + c_2\sin^2 x + c_3\sec^2 x + c_4\tan^2 x$$

Now put $c_1 = c_2 = 1$, $c_3 = -1$, $c_4 = 1$.

$$-1 + 1 = 0$$

$f_1(x), f(x), f_3(x) \ldots f(x)$ is linearly dependent on an interval if atleast one function is expressed as linearly combination.

Example 6

$$f_1(x) = \sqrt{x} + 5 ; f_2(x) = \sqrt{x} + 5x ; f_3(x) = x - 1$$

$$f_4(x) = x^2$$

$f_2(x)$ can be written as linear combination of f_1, f_3, f_4 $\therefore$ $f_2(x) = 1 \cdot f_1(x) + 5f_3(x) + 0 \cdot f_4(x)$

$$\sqrt{x} + 5x = \sqrt{x} + 5 + 5x - 5$$

$$\sqrt{x} + 5x = \sqrt{x} + 5x$$

Example 7:-

$$y_1 = e^{3x}, \quad y_2 = e^{-3x}$$

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}$$

$$W = -3-3 = -6 \neq 0$$

So, it is linear independent

Example 8:-

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

$$c_1 = 2, \quad c_2 = 7$$

$$y = 2e^{3x} - 7e^{-3x}$$

$$y = 2e^{3x} - 2e^{-3x} - 5e^{-3x} = 4\left(\frac{e^{3x} - e^{-3x}}{2}\right)$$

$$y = 4\sinh 3x - 5e^{-3x}$$

Example 9

$$y_1 = e^{x}, \quad y_2 = e^{2x}, \quad y_3 = e^{3x}$$

$$V = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$= 2e^{6x} \neq 0$, it is not linear independent

Example 10

$$y_p = -11/12 - 1/2\,x \quad \& \quad y''' - 6y'' + 11y' + 6y^2 = 0$$

$$y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

$$y = y_c + y_p$$

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - 11/2 - 1/2 x$$

Example 11

$y_{p_1} = -4x^2$ sol of $y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{p_2} = e^{2x}$ sol of $y'' - 3y' + 4y = 2e^{2x}$

$y_{p_3} = xe^{x}$ sol of $y'' - 3y' + 4y = 2xe^{x} - e^{x}$

$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x}$

is a solution of

$$y'' - 3y + 4y = \underbrace{-16x^2 + 24 - 8}_{g_1(x)} \underbrace{+2e^{2x}}_{g_2(x)} \underbrace{+2xe^{x} - e^{x}}_{g_3(x)}$$

EXERCISE # 4.2

Example 1:

$y_1 = e^{x}$; $y = ?$; $y'' - y = 0$

$y_2 = vy_1$

$y = ve^{x}$

$y' = ve^{x} + e^{x}v'$

$y'' = ve^{x} + 2e^{x}v' + e^{x}v''$

$y'' - y = 0$

$e^{x}(v'' + 2v') = 0$

$v'' + 2v' = 0$

$w' = v'$

$$w' + 2w = 0$$
$$I.F = e^{2x}$$
$$w = ce^{-2x}$$
$$u' = c_1 e^{-2x}$$
$$u = -\tfrac{1}{2} c_1 e^{-2x} + c_2$$
$$y = -\tfrac{c_1}{2} e^{-x} + c_2 e^{x}$$
$$c_1 = 0, \; c_2 = -2$$
$$y_1 = e^{x}, \; y_2 = e^{-x}$$

Example 2

$$y_1 = x^2 \; ; \; x^2 y'' - 3y'x + 4y = 0$$
$$y'' - \tfrac{3}{x} y' + \tfrac{4}{x^2} y = 0$$
$$y_2 = x^2 \int \frac{e^{-3\int -1/x \, dx}}{x^4} \, dx$$
$$y_2 = x^2 \int \frac{x^3}{x^4} \, dx$$
$$y_2 = x^2 \int \tfrac{1}{x} \, dx$$
$$y_2 = x^2 \ln x$$

EXERCISE # 4.3

a)- $2y'' - 5y' - 3y = 0$

$$2m^2 - 5m - 3 = (2m+1)(m-3) = 0$$
$$m = -\tfrac{1}{2}, 3$$
$$y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$$

b)- $y'' - 10y' + 25y = 0$

$m^2 - 10m + 25 = 0 = (m-5)^2$

$m = 5 \Rightarrow y = c_1 e^{5x} + c_2 e^{5x} x$.

c)- $m^2 + 4m + 7 = 0$

$m_1 = -2 + \sqrt{3}i \;; m_2 = -2 - \sqrt{3}i$

$y = e^{-2x}(c_1 \cos\sqrt{3}x$

Example 2

$4y'' + 4y' + 17y = 0 \;; y(0) = -1 \;; y'(0) = 2$

$4m^2 + 4m + 17 = 0$

$m_1 = -1/2 \pm 2i$

$y_c = e^{-1/2 x}(c_1 \cos 2x + c_2 \sin 2x)$

$c_1 = -1$ from $y(0) = -1$

$c_2 = 3/4$

$y = e^{-x/2}(-\cos 2x + 3/4 \sin 2x)$

Example 3

$y''' + 3y'' - 4y = 0$

$m^3 + 3m^2 - 4 = 0$

$(m-1)(m^2 + 4m + 4) = 0$

$(m-1)(m+2)^2 = 0$

$m = 1, \; m_{2,3} = \pm 2$

$y = c_1 e^x + c_2 e^{-2x} + c_3 e^{-2x} x$

Example 4

$$\frac{d^4y}{dx^4} + \frac{2d^2y}{dx^2} + y = 0$$

$$m^4 + 2m^2 + 1 = 0$$

$$(m^2+1)^2 = 0$$

$$m_{1,3} = i \quad , m_{2,4} = -i$$

$$y = C_1 e\cos x + C_2 \sin x + C_3 x\cos x + C_4 x\sin x$$

EXERCISE # 4.4

Example 1

$$y''' + 4y' - 2y = 0$$

$$m^2 + 4m - 2 = 0$$

$$m_{1,2} = -2 \pm \sqrt{6}$$

$$y_1 = C_1 e^{(-2+\sqrt{6})x} + C_2 e^{(-2-\sqrt{6})x}$$

$$y_p = Ax^2 + Bx + c$$

$$y_p' = DA + B$$

$$y_p'' = 2A$$

$$2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$$

$$-2A = 2 \quad , 8A - 2B = -3 \quad , 2A + 4B - 2C = 0$$

$$A = -1 \quad , B = -5/2 \quad , C = -9$$

$$y_p = -x^2 - 5/2x - 9$$

$$y = C_1 e^{(2+\sqrt{6})x} + C_2 e^{(-2-\sqrt{6})x} - x^2 - \frac{5x}{2} - 9$$

Example 2

$$y'' - y' + y = 2\sin 3x$$

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x$$
$$= 2\sin 3x$$

$$-8 - 3B = 0 \;,\; 3A - 8B = 2$$

$$A = 6/73 \;,\; B = -16/73$$

$$y_p = 6/73 \cos 3x - 16/73 \sin 3x$$

Example 3

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

$$y_c = c_1 e^{-x} + c_2 e^{3x}$$

$$g(x) = g_1(x) + g_2(x)$$

$$y_p = y_{p_1} + y_{p_2}$$

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

$$y''_p - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x}$$
$$= 4x - 5 + 6xe^{2x}$$

$$-3A = 4 \;,\; -2A - 3B = -5 \;,\; -3C = 0 \;,\; 2C - 3E = 0$$

$$A = -4/3 \;,\; B = 23/9 \;,\; C = -2 \;,\; E = -4/3$$

$$y_p = -4/3\, x + 23/9 - 2xe^{2x} - 4/3\, e^{2x}$$

$$y = c_1 e^{-x} + c_2 e^{3x} - 4/3\, x + 23/9 - (2x + 4/3)e^{2x}$$

Example 4

$$y'' - 5y' + 4y = 8e^{x}$$

$$y_c = c_1 + e^{x} + c_2 e^{4x}$$

$$y_p = Ae^{x}$$

$$y_p^* = Axe^{x}$$

$y_p' = Axe^{4} + Ae^{4}$

$y_p'' = Axe^{4} + 2Ae^{4}$

$-3Ae^{4} = 8e^{4}$

$-3A = 8$

$\boxed{A = -8/3}$

$y_p = -8/3\, xe^{x}$

$y = c_1 + e^{x} + c_2 e^{4x} - 8/3\, xe^{4}$

Example 5

(a)- $y'' - 8y' + 25y = 5x^3e^{-x} - 7e^{-x}$

$g(x) = (5x^3 - 7)e^{-x}$

$y_p = (Ax^3 + Bx^2 + Cx + E)e^{-x}$

$y_c = e^{4x}(c_1\cos 3x + c_2 \sin 3x)$

(b) $g(x) = x\cos x$

$y_p = (Ax + B)\cos x + (Cx + E)\sin x$

$y_c = c_1\cos 2x + c_2 \sin 2x$

Example 6

$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}$

corresponding to $3x^2$ we assume $y_p = Ax^2 + Bx + C$

corresponding to $-5\sin 2x$ we assume $y_p = E\cos 2x + F\sin 2x$

corresponding to $7xe^{6x}$ we assume $y_p = (Gx + H)e^{6x}$

$y_p = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}$

no duplication in term $y_c = c_1e^{2x} + ce^{7x}$

Example 7

$$y'' - 2y' + y = e^x$$

$y_c = c_1 e^x + c_2 x e^x$

$y_p = Ae^x$

$y_p = Axe^x$

$y_p = Ax^2 e^x$ ~~for sales~~

$2Ae^x = e^x$ so, $A = 1/2$

$y_p = 1/2 x^2 e^x$

Example 8

$$y''' + y = 4x + 10\sin x \; ; y(\pi) = 0 \; ; y'(\pi) = 2$$

$y_c = c_1 \cos x + c_2 \sin x$

$g(x) = 4x + 10\sin x$

$y_p = Ax + B + (C\cos x + E\sin x)$

$y_p = Ax + B + Cx\cos x + Ex\sin x$

$y''_p + y_p = Ax + B - 2C\sin x + 2E\cos x = 4x + 10\sin x$

$A = 4, \; B = 0, \; C = -5, \; E = 0$

$y = y_c + y_p$

$\boxed{y = c_1 \cos x + c_2 \sin x + 4x - 5x\cos x}$

$c_1 = 9\pi$ from $y(\pi) = 0$

$c_2 = 7$ from $y'(\pi) = 2$

$y = 9\pi \cos x + 7\sin x + 4x - 5x\cos x$

Example 9

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

$$y_p = Ax^2 + Bx + C + Ee^{x}$$

$$y_p'' - 6y_p' + 9y_p' = 9Ax^2 + (-12A + 9B)^{x} + 2A - 6B + 9C + 2Ee^{3x}$$

$$= 6x^2 + 2 - 12e^{3x}$$

$$A = 2/3, \quad B = 8/9, \quad C = 2/3, \quad E = -6$$

$$y = c_1 e^{3x} + c_2 x e^{3x} + 2/3 x^2 + 8/9 x + 2/3 - 6x^2 e^{3x}$$

Example 10

$$y''' + y'' = e^{x} \cos x$$

$$m^3 + m^2 = 0$$

$$m_1 = m_2 = 0, \quad m_3 = -1$$

$$y_c = c_1 + c_2 x + c_3 e^{-x}$$

$$g(x) = e^{x} \cos x$$

$$y_p = Ae^{x} \cos x + Be^{x} \sin x$$

$$y_p''' + y_p'' = (-2A + 4B) e^{x} \cos x + (-4A - 2B) e^{x}$$

$$= e^{x} \cos x$$

$$-2A + 4B = 1, \quad -4A - 2B = 0$$

$$A = -1/10, \quad B = 1/5$$

$$y_p = -1/10\, e^{x} \cos x + 1/5\, e^{x} \sin x$$

$$y = c_1 + c_2 x + c_3 e^{-x} - 1/10\, e^{x} \cos x + 1/5\, e^{x} \sin x$$