Example 11

$$y^{IV} + y''' = 1 - x^2 e^{-x}$$

$$y_c = C_1 + C_2 x + C_3 x^2 + C_4 e^{-x}$$

$$y_p = \underbrace{A}_{y_p} + \underbrace{Bx^2 e^{-x} + Cxe^{-x} + Ee^{-x}}_{y_{p_1}}$$

$$y_p = Ax^3 + Bx^3 e^{-x} + Cx^2 e^{-x} + Exe^{-x}$$

EXERCISE # 4.5

Example 1

(a)- $1 - 5x^2 + 8x^3$

$D^4(1 - 5x^2 + 8x^3) = 0$ As $D^4 x^3 = 0$

(b)- e^{-3x} ; $\alpha = -3$; $n = 1$

$(D + 3)e^{-3x} = 0$

(c)- $4e^{2x} - 10xe^{2x}$

$\alpha = 2$, $n = 2$

$(D-2)^2(4e^{2x} - 10xe^{2x}) = 0$

Example 2

$$5e^{-x}\cos 2x - 9e^{-x}\sin 2x$$

$e^{-x}\cos 2x$ and $e^{-x}\sin 2x$

$\alpha = 1$ & $B = 2$

$D^2 + 2D + 5$ will inihilate the function

$5e^{-x}\cos 2x - 9e^{-x}\sin 2x$

Example 3

$$y'' + 3y' + 2y = 4x^2$$

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$D^3(D^2 + 3D + 2)y = 4D^3 x^2$$

$$D^3(D^2 + 3D + 2)y = 0$$

$$m^3(m^2 + 3m + 2) = 0 \quad \text{or} \quad m^3(m+1)(m+2) = 0$$

$$m_{1,2,3} = 0, \quad m_4 = -1, \quad m_5 = -2$$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + c_5 e^{-2x}$$

$$y_p = A + Bx + Cx^2$$

$$y_p' = B + 2Cx, \quad y_p'' = 2C$$

$$2C + 3B + 6Cx + 12A + 2Bx + 2Cx^2 = 4x^2$$

$$2C = 4, \quad 2B + 6C = 0, \quad 2A + 3B + 2C = 0$$

$$C = 2, \quad B = -6, \quad A = 7$$

$$y_p = 7 - 6x + 2x^2$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2$$

Example 4

$$y'' - 3y' = 8e^{3x} + 4\sin x$$

$$m^2 - 3m = 0$$

$$m = 0, \quad m = 3$$

$$y_c = c_1 + c_2 e^{3x}$$

$$(D-3)e^{3x} = 0 \quad \text{and} \quad (D^2+1)\sin x = 0$$

$(D-3)(D^2+1)(D^2-3D)y=0$

$(m-3)(m^2+1)(m^2-3m)=0$

$y = c_1 + c_2 e^{3x} + c_3 x e^{3x} + c_4 \cos x + c_5 \sin x$

$y_p = Axe^{3x} + B\cos x + C\sin x$

$A = 8/3$, $B = 6/5$, $C = -2/5$

$y_p = 8/3\, xe^{3x} + 6/5 \cos x - 2/5 \sin x$

$y = c_1 + c_2 e^{3x} + 8/3\, xe^{3x} + 6/5 \cos x - 2/5 \sin x$

Example 5

$$y'' + y = x\cos x - \cos x$$

$y_c = c_1 \cos x + c_2 \sin x$

$(D^2+1)(D^2+1)y=0$

$(D^2+1)^3 y = 0$

$y = c_1 \cos x + c_2 \sin x + c_3 x\cos x + c_5 x^2 \cos x$

$+ c_4 x \sin x + c_6 x^2 \sin x$

$y_p = Ax\cos x + Bx\sin x + Cx^2\cos x + Ex^2 \sin x$

$y_p'' + y_p = x\cos x - \cos x$

$y = c_1 \cos x + c_2 \sin x + 1/4 \cos x + 1/4\, x^2 \sin x - 1/2\, x \sin x$

Example 6

$$y'' - 2y' + y = 10e^{-2x}\cos x$$

$y_c = c_1 e^x + c_2 x e^x$

$\alpha = -2$, $\beta = 1$, $n = 1$

$(D^2 + 4D + 5)_2 e^{-2x} \cos x$

$(D^2 + 4D + 5)(D^2 - 2D + 1)y = 0$

roots from A.E $-2 \pm i$, 1 & 1

$y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{-2x}\cos x + c_4 e^{-2x}\sin x$

$y_p = Ae^{-2x}\cos x + Be^{-2x}\sin x$.

Example 7

$$y''' - 4y'' + 4y' = 5x^2 - 6x + 4x^2 e^{2x} + 3e^{5x}$$

$D^3(5x^2 - 6x) = 0$; $(D-2)^3 x^2 e^{2x} = 0$ & $(D-5)e^{5x} = 0$

$D^3(D-2)^3(D-5)(D^3 - 4D^2 + 4D) = 0$

$D^4(D-5)^5(D-5)y = 0$

roots are: 0000, 2, 2, 2, 2, 2 and 5

$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{2x} + c_6 x^{2x} \ldots c_{10} e^{5x}$

$y_p = Ax + Bx^2 + Cx^3 + Ex^2 e^{2x} + Ex^3 e^{2x} + Gx^4 e^{2x} + He^{5x}$.

EXERCISE # 4.6

Example 1

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

$m^2 - 4m + 4 = (m-2)^2 = 0$

$y_c = c_1 e^{2x} + c_2 e^{2x} x$

$y_1 = e^{2x}$, $y_2 = xe^{2x}$

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

$$V_1' = \frac{-(x+1)xe^{4x}}{e^{4x}} = -x^2 - x \quad ; \quad V_2' = \frac{(x+1)}{e^{4x}} = x+1$$

$$V_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2 \quad \text{and} \quad V_2 = \frac{1}{2}x^2 + x$$

$$y_p = \left(-\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)e^{2x} + \left(\tfrac{1}{2}x^2 + x\right)xe^{2x}$$

$$= \tfrac{1}{6}x^3e^{2x} + \tfrac{1}{2}x^2e^{2x}$$

$$y = c_1e^{2x} + c_2xe^{2x} + \tfrac{1}{6}x^3e^{2x} + \tfrac{1}{2}e^{2x}x^2$$

Example 2

$$4y'' + 36y = \operatorname{cosec} 3x$$

$$y'' + 9y = \tfrac{1}{4}\operatorname{cosec} 3x$$

$$m^2 + 9 = 0$$

$$m_1 = 3i \quad , \quad m_2 = -3i$$

$$y_c = c_1\cos 3x + c_2\sin 3x$$

$$f(x) = \tfrac{1}{4}\operatorname{cosec} 3x$$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \tfrac{1}{4}\operatorname{cosec} 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \tfrac{1}{4}\operatorname{cosec} 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

$$V_1' = \frac{-1}{12} \quad , \quad V_2' = \frac{1}{12}\tan 3x$$

$$v_1 = -\frac{1}{12}x \quad , v_2 = \frac{1}{36}\ln|\sin 3x|$$

$$y_p = -1/12\, x\cos 3x + 1/36\,(\sin 3x)\ln|\sin 3x|$$

$$y = c_1\cos 3x + c_2\sin 3x - 1/12\, x\cos 3x + 1/36\,(\sin 3x)\ln(\sin x - 1)$$

Example 3

$$y'' - y = 1/x$$

$$m^2 - 1 = 0$$

$$m_1 = -1 \ , m_2 = 1$$

$$y_c = c_1 e^{x} + c_2 e^{-x}$$

$$W = -2$$

$$v_1' = 1/2 \int_{x_0}^{x} \frac{e^{-t}}{t}\,dt \ , \quad v_2 = -1/2 \int_{x_0}^{x} \frac{e^{t}}{t}\,dt$$

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} e^{x} \int_{x_0}^{x} \frac{e^{t}}{t}\,dt - \frac{1}{2} e^{-x} \int_{x_0}^{x} \frac{e^{t}}{t}\,dt$$

EXERCISE # 4.7

Example 1

$$\frac{x^2 d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

$$\frac{dy}{dx} = mx^{m-1} \ , \quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-1} - 2x(mx^{m-1}) = 0$$

$$m(m-1) - 2m - 4 = 0$$

$$m^2 - m - 2m - 4 = 0$$

$m^2 - 3m - 4 = 0$

$m_1 = -1 \quad , m_2 = 4$

$y = c_1 x^{-1} + c_2 x^4$

Example 2

$$4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} + y = 0$$

$y = x^m$

$4x^2 (m(m-1)x^{m-2}) + 8x (mx^{m-1}) + 1 = 0$

$4m(m-1) + 8m + 1 = 0$

$4m^2 + 4m + 1 = 0$

$(2m+1)^2 = 0$

$m_{1,2} = -1/2$

$y_c = c_1 x^{-1/2} + c_2 x^{-x} \ln(x)$

Example 3

$4x^2 y'' + 17y = 0 \quad ; \; y(1) = -1 \; ; \; y'(1) = -1/2$

$4x^2 (m(m-1)x^{m-2}) + 17x = 0$

$4m(m-1) + 17 = 0$

$4m^2 - 4m + 17 = 0$

$m_{1,2} = 1/2 \pm 2i$

$y = x^{1/2} (c_1 \cos(2\ln(x)) + c_2 \sin(2\ln x))$.

$y = -x^{1/2} \cos(2\ln x)$

Example 4

$$x^3\frac{d^3y}{dx^3} + 5x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = 0$$

$$= x^3(m(m-1)(m-2)x^{m-3}) + 5x^2(m(m-1)x^{m-2})$$

$$+ 7x(m x^{m-1}) + 8x^m = 0$$

$$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$$

$$m_1 = -2, \quad m_2 = 2i \quad \& \quad m_3 = -2i$$

$$y = c_1x^{-2} + c_2\cos(2\ln x) + c_3\sin(2\ln x)$$

Example 5

$$x^2y'' - 3xy' + 3y = 2x^4e^x$$

$$y'' - 3/x\, y' + 3/x^2\, y = 2x^2e^x$$

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x$$

$$y = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$$

Example 6

$$x^2y'' - xy' + y = \ln x$$

$$x = e^t \quad \text{or} \quad t = \ln x$$

$$dy/dx = dy/dt \times dt/dx = 1/x\, dy/dt$$

$$d^2y/dx^2 = 1/x\, d/dx(dy/dt) + dy/dt(-1/x^2)$$

$$-1/x\left(d^2y/dt^2 \cdot 1/x\right) + dy/dt(-1/x^2) = 1/x^2\left(d^2y/dt^2 - \frac{dy}{dt}\right)$$

$$d^2y/dt^2 - 2dy/dt + y = t$$

$$y = c_1e^t + c_2te^t + 2 + t$$

$$y = c_1x + c_2\ln x + 2 + \ln x$$

CHAPTER # 12

EXERCISE # 12.1

Q13) $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

$X''Y + XY'' = 0$

$X''/X + Y''/Y = 0$

$X''/X = -Y''/Y = \lambda^2$

Case I :-

$X'' = X\lambda^2$

$-X\lambda^2 + X'' = 0$

$m^2 + \lambda^2 = 0$

$m = \pm \lambda i$

$X = c_1 \cos \lambda x + c_2 \sin \lambda x$

$Y'' - Y\lambda^2 = 0$

$m^2 - \lambda^2 = 0$

$m = \pm \lambda$

$Y = c_3 \cosh \lambda y + c_4 \sinh y$

$u = XY$

$u = (c_1 \cosh \lambda x + c_3 \sinh \lambda x)(c_3 \cosh \lambda y + c_4 \sin \lambda y)$

Case II :-

$\lambda = 0$

$X''/X = 0$

$X'' = 0$

$m^2 = 0$

$m_{1,2} = 0, 0$

$X = C_1 + C_2 x$

$-Y''/Y = 0$

$Y'' = 0$

$m_{3,4} = 0$

$Y = C_3 + C_4 Y$

$U = XY$

$U = (C_1 + C_2 X)(C_3 + C_4 Y)$

Case III

$\lambda^2 = -\lambda^2$

$X''/X = -\lambda^2$

$X'' + X\lambda^2 = 0$

$m^2 + \lambda^2 = 0$

$m = \pm i\lambda$

$X = C_1 \cos\lambda x + C_4 \sin\lambda x$

$-Y''/Y = -\lambda^2$

$Y'' - X^2 Y = 0$

$m^2 - \lambda^2 = 0$

$m_{3,2,1} = \pm\lambda$

$Y = C_1 \cosh\lambda y + C_2 \sinh\lambda y$

$U = \pm XY$

$U = (C_1 \cos\lambda x + C_2 \sin\lambda x)(C_1 \cosh y\lambda + C_2 \sinh\lambda y)$

Q17) $\frac{d^2u}{dx^2} + \frac{d^2u}{dxdy} + \frac{d^2u}{dy^2} = 0$

$A = 1$, $B = 1$, $C = 1$

$= B^2 - 4ac$

$= 1 - 4$

$= -3 < 0$ elliptic

Q23) $\frac{du}{dx^2} + \frac{2du}{dxdy} + \frac{d^2u}{dy^2} + \frac{du}{dx} - 6\frac{du}{dy} = 0$

$A = 1$, $B = 2$, $C = 1$

$= B^2 - 4ac$

$= 4 - 4 = 0$ parabolic

Q25) $a^2\frac{d^2u}{dx^2} = \frac{d^2u}{dt^2}$

$A = a^2$, $C = -1$, $B = 0$

$0 + 4a^2$

$4a^2 > 0$ Hyperbolic

EXERCISE #12.2

Equations & Boundary value problems

(i) $K\frac{d^2u}{dx^2} = \frac{du}{dt}$, $K > 0$ parabolic, one dimension

(ii)- $a^2 \frac{d^2u}{dx^2} = \frac{d^2u}{dt^2}$ ⇒ one dimension, hyperbolic

(iii)- $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$ two dimension, elliptic

These equations are one dimension heat equation, one dimension wave equation and Laplace's equation in two dimension respectively.

$$\nabla^2 u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}$$

is called two dimensional laplace of function u.

three dimension : $\nabla^2 u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}$

Heat Equation :-

Conditions for Rod :-

(1)- The flow of heat within the rod takes place only in the x direction

(2)- The lateral or sorted, surface of the rod is insulated that is, no heat escapes from the surface.

(3)- The rod is homogeneous

(4)- The specific heat and thermal conductivity of the material of the rod are constants.

(5)- No heat is being generated within the rod

Wave Equation :-

Conditions for strings :-

(1) The string is perfectly elastic/flexible
(2) The string is homogeneous
(3) The slope of the curve is small at all points
(4) The tension T acts tangent to the string.
(5) The displacements are small as compared to the length of the string.
(6) The tension is large compared with force of gravity.
(7) No other external forces act on string.

Laplace's Equation :-

Laplace equation in two and three dimension occurs in time independent ploblems involving potentials such as electrostatics, gravitational and velocity in field mechanics.

EXERCISE # 12.3

Heat Equation :-

Consider a rod of length L with an initial temperature $f(x)$. Temperature zero for all time $t > 0$

$$K\frac{d^2u}{dx^2} = \frac{du}{dt}, \quad 0 < x < L, \quad t > 0$$

$$K\frac{d^2u}{dx^2} = \frac{du}{dt}, \quad 0 < x < l, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < L$$

Solution :-

$u = XT$ and $-\lambda^2$ as a separation constant

$$X''/X = T'/KT = -\lambda^2$$

$$X'' + \lambda^2 X = 0$$

$$T' + K\lambda^2 T = 0$$

$$X = C_1 \cos\lambda x + C_2 \sin\lambda x$$

$$T = C_3 e^{-K\lambda^2 t}$$

$$u(0,t) = X(0)T(t) = 0 \quad \therefore X(0) = 0$$

$$u(L,t) = X(L)T(t) = 0 \quad \therefore X(L) = 0$$

$$C_1 = 0$$

$$X = C_2 \sin\lambda x$$

Second boundary equation :-

$$X(L) = C_2 \sin\lambda L = 0$$

If $C_2 = 0$ then $X = 0$, $0 = 0$

$$\sin\lambda L = 0$$

$$\lambda L = n\pi \quad \text{~~on~~ or} \quad \lambda = n\pi/L$$

$$\lambda = \frac{n\pi}{L}, \quad n = 1, 2, 3, 4 \ldots$$

$$X = C_2 \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

EXERCISE # 12.4

Wave Equation :-

$$a^2 \frac{d^2u}{dx^2} = \frac{d^2u}{dt^2} \quad , \quad t>0$$

$$u(0,t) = 0 \quad , \quad u(t,t) = 0 \quad , \quad t>0$$

Solution :-

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda^2$$

$$X'' = -\lambda^2 X$$

$$X'' + \lambda^2 X = 0$$

$$T'' + \lambda^2 a^2 T = 0$$

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$T = C_3 \cos \lambda a t + C_4 \sin \lambda a t$$

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,t) = X(x)\, T(t)$$

$$u(0,t) = X(0)\, T(t) = 0$$

$$u(L,t) = X(L)\, T(t) = L$$

$$C_1 = 0 \quad , \quad C_2 \sin \lambda L = 0$$

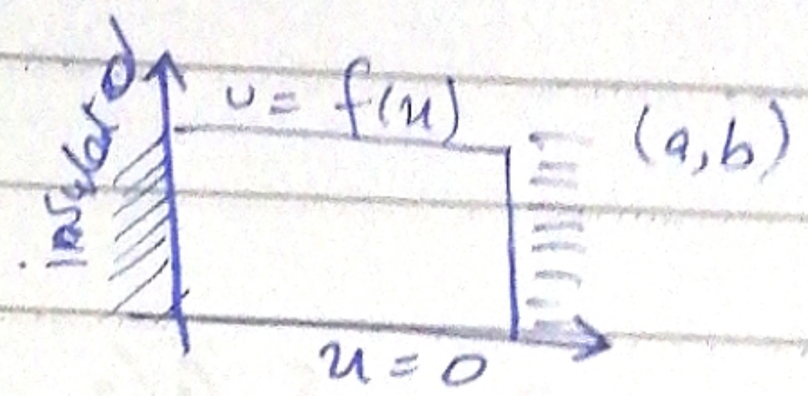
$$X = C_2 \sin \frac{n\pi}{L} a \quad , \quad n = 1, 2, 3, \ldots$$

$$\boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{\pi n x}{L}\, dx}$$

$$\boxed{B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \left(\frac{n\pi x}{L}\right) dx}$$

EXERCISE # 12.5

Laplace's Equation :-



$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0 \quad , 0 < x < a$$

$$X''/X = -Y''/Y = -\lambda^2$$

$$X'' + \lambda^2 X = 0$$

$$Y'' - \lambda^2 Y = 0$$

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x$$

Since $0 < y < b$ is finite interval

$$Y = c_3 \cosh \lambda y + c_4 \sinh \lambda y$$

CHAPTER # 11

EXERCISE # 11.1

Q3) $f_1(x) = e^x \; ; f_2(x) = xe^{-x} - e^{-x} [0,2]$

$$= \int_0^2 e^x (xe^{-x} - e^{-x}) dx$$

$$= \int_0^2 (x - 1) dx$$

$$= x^2/2 - x \Big|_0^2$$

$$= ((4/2) - 2) - 0$$

$$= (2 - 2) - 0$$

$$= 0$$

olthogonal

Q5) $f_1(x) = x$, $f_2(x) = \cos 2x$, $[-\pi/2, \pi/2]$

$$\int_{-\pi/2}^{\pi/2} x \cos 2x \, dx$$

$$\frac{x \cos 2x}{2} \Big|_{-\pi/2}^{\pi/2} - \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sin 2x \, dx$$

$$\frac{x \sin 2x}{2} \Big|_{-\pi/2}^{\pi/2} + \frac{1}{2} \frac{\cos 2x}{2} \Big|_{-\pi/2}^{\pi/2}$$

$$0 + \frac{1}{4} \cos\pi - \cos\pi$$

$$0 + 1/4(-1+1) = 0$$

olthogonol

Q7) $\{\sin x, \sin 3x, \sin 5x, \ldots\}$, $[0, \pi/2]$

$$\int_0^{\pi/2} \sin x \sin 3x \, dx$$

$$-1/2 \int_0^{\pi/2} \cos(3x + x) - \cos|3x - x| \, dx$$

$$-1/2 \left(\cos 4x/4\right)\Big|_0^{\pi/2} - \sin 2x/2 \Big|_0^{\pi/2}$$

$$-1/2 \, (0 - 0)$$

$$= 0$$

olthogonol

Norm:

$Q_0 \quad [x] = \sin x$

$= \int_0^{\pi/2} \sin^2 x \, dx$

$= \frac{1}{2} \int_0^{\pi/2} 1 - \cos 2x \, dx$

$= \frac{1}{2} \left(x \Big|_0^{\pi/2} - \frac{\sin 2x}{2} \Big|_0^{\pi/2} \right)$

$= \frac{1}{2} \left(\pi/2 - \frac{\sin \pi}{2} - 0 \right)$

$= \frac{1}{2} (\pi/2)$

$\|Q_0(x)\|^2 = \pi/4$

norm $= \sqrt{\pi}/2$

Q12) $\left\{ 1, \frac{\cos n\pi x}{p}, \sin m \frac{\pi}{p} x \right\}, [-p, p], n = 1, 2, \ldots$ $m = 1, 2, \ldots$

$\int_{-p}^{p} \frac{\cos n\pi x}{p} dx$

$\frac{\sin n\pi x}{p} \Big|_{-p}^{p}$

$\sin(n\pi) + \sin(\pi n) = 0$ orthogonal

Norm:-

$|Q_0| = 1$

$= \int_{-p}^{p} 1 \, dx$

$= x \Big|_{-p}^{p}$

$= p + p = 2p$

norm $= \sqrt{2p}$

$Q(x) = \frac{\cos(n\pi)}{p} x$

$= \int_{-p}^{p} \frac{\cos(n\pi)x}{p} dx$

$$= \frac{1}{2}\int_{-p}^{p} 1 + \cos\left(\frac{2n\pi x}{p}\right) dx$$

$$= \frac{1}{2}\left(x + \frac{\sin 2n\pi}{p} \times \frac{p}{2n\pi}\right)\Big|_{-p}^{p}$$

$$= \frac{1}{2}(p + 0 + p - 0) \qquad = p$$

$$\boxed{\text{norm} = \sqrt{p}}$$

Q 13) $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$

$W(x) = e^{-x^2}$, $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} 2x\left(e^{-x^2}\right) dx$$

$$-\int -2x e^{-x^2} dx$$

$$e^{-x^2}\Big|_{-\infty}^{\infty}$$

$$\frac{1}{e^{x^2}}\Big|_{-\infty}^{\infty}$$

$$\frac{1}{\infty} - \frac{1}{e^{(-\infty)^2}} \Rightarrow \frac{1}{\infty} - \frac{1}{\infty} \Rightarrow 0 - 0 \Rightarrow 0$$

orthogonal

Q 18) $f_1(x) = x$, $f_2(x) = x + c_1 x^2 + c_2 x^3$

$f_3(x) = f_1(x) + c_2 f_2(x) + c_2 x^3$

$f_3(x) = x + c_1 x^2 + c_2 x^3$

$$(f_1, f_2) = \int_{-2}^{2} f_1(x) f_1(x) dx + c_1 \int_{-2}^{2} f_1(x) f_2(x) dx$$
$$+ c_2 \int_{-2}^{2} f(x) x^3 dx$$

By theorem 11.1.3

$$c_1 \int_{-2}^{2} f_1(x) f_2(x) dx = 0$$

$$\int_{-2}^{2} x^2\,dx + C_2 \int_{-2}^{2} x^4\,dx$$

$$0 = 16/3 + 64/5\, C_2$$

$$C_1 = -5/12$$

$$(f_2, f_3) = \int_{-2}^{2} f_2(x) f_1(x) + C_1 \int_{-2}^{2} f_2(x) f_2(x)\,dx + C_2 \int_{-2}^{2} f_2(x) x^3\,dx$$

$$= 0 + C_1 \int_{-2}^{2} x^4\,dx - 5/12 \int_{-2}^{2} x^5\,dx$$

$$= 64/5\, C_1 + 0$$

$$C_1 = 0$$

$$\boxed{f_3(x) = x - \frac{5}{12} x^3}$$

EXERCISE # 11.2

Q7) $f(x) = x + \pi \qquad -\pi < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{(n\pi x)}{p} \right]$$

$$a_0 = \frac{1}{p} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \left(\frac{x^2}{2} + \pi x \Big|_{-\pi}^{\pi} \right)$$

$$= \frac{1}{\pi} \left(\frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2} + \pi^2 \right)$$

$$= \frac{\pi}{2} + \pi - \frac{\pi}{2} + \pi$$

$$= 2\pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) \cos nx\,dx$$

$$= \frac{1}{\pi} \left(\left[\frac{1}{n} (n + \pi) \sin(nx) \right]_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} \sin(nx) \right)$$

$$= \frac{-1}{n\pi} \left| -\frac{\cos(nx)}{n} \right|_{-\pi}^{\pi}$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi + x) \sin(nx)\, dx$$

$$= \frac{1}{\pi} \left(\frac{-1}{n} (\pi + x) \cos x \Big|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos(nx)\, dx \right)$$

$$= \frac{1}{\pi} \left(\frac{-2\pi(-1)^n}{n} + 0 + \frac{1}{n} \left(\frac{\sin(nx)}{n} \right) \Big|_{-\pi}^{\pi} \right)$$

$$\boxed{b_n = \frac{2(-1)^{n+1}}{n}}$$

$$f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^{n+1}}{n} \sin(nx) \right]$$

Average:

$$= \frac{f(\pi -) + f(-\pi +)}{2}$$

$$= \frac{(\pi + \pi) + (\pi - \pi)}{2} = \pi$$

Q9) $f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ \sin x & , 0 \le x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{0} 0\, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin x\, dx$$

$$= \frac{1}{\pi} (-\cos x) \Big|_{0}^{\pi} \Rightarrow \frac{1}{\pi} (1+1) = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \, \frac{\cos(n\pi x)}{\pi} \, dx$$

$$\frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx$$

$$\frac{1}{2\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n} \right]_0^{\pi}$$

$$\frac{1}{2\pi}\left(-\frac{(-1)^{1+n}}{1+n} - \frac{(-1)^{1-n}}{1-n} + \frac{1}{1+n} + \frac{1}{1-n} \right)$$

$$\frac{1}{2\pi}\left(\frac{2(-1)^n + 2}{(1+n)(1-n)} \right)$$

$$\frac{(-1)^n + 1}{\pi(1-n^2)}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx$$

$$a_1 = \frac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx$$

$a_1 = 0$ when $n = 1$

$$a_n = \begin{cases} 0 & n = 1 \\ \frac{(-1)^n + 1}{\pi(1-n^2)} & \text{if } n = 2, 3, \ldots \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin \frac{n\pi x}{\pi} \, dx$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin(nx) \, dx$$

$$b_n = \frac{1}{2\pi}\int_0^{\pi} \cos(1-n)x - \cos(1+n)x) \, dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]\Bigg|_0^{\pi}$$

$$= 0$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$b_1 = \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x)\, dx$$

$$b_1 = \frac{1}{2\pi}\left(x - \frac{\sin 2x}{2}\right)\Bigg|_0^{\pi}$$

$$b_1 = \frac{1}{2}$$

$$b_n = \begin{cases} 1/2 , & n=1 \\ 0 , & n=2,3,4,\ldots \end{cases}$$

$$f(x) = \frac{1}{\pi} + \frac{\sin x}{2} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n+1}{\pi(1-n^2)}\cos(nx)\right]$$

Q 11)- $f(x) = \begin{cases} 0 , & -2 < x < -1 \\ -2 , & -1 \le x < 0 \\ 1 , & 0 \le x < 1 \\ 0 , & 1 \le x < 2 \end{cases}$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$a_0 = \frac{1}{2}\left(\int_{-2}^{2} 0\,dx + \int_{-1}^{0} -2\,dx + \int_0^1 dx + \int_1^2 0\,dx\right)$$

$$a_0 = -1 + \frac{1}{2}$$

$$\boxed{a_0 = \frac{-1}{2}}$$

$$a_n = \frac{1}{2}\left(-2\int_{-1}^{0} \cos\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{1} \cos\left(\frac{n\pi x}{2}\right) dx\right)$$

$$a_n = \frac{1}{2}\left(\frac{-4}{n\pi} \sin\frac{(n\pi)x}{2}\right) + \left(\frac{-2}{n\pi} \frac{\cos(n\pi)x}{2}\right)$$

$$\boxed{a_n = -\frac{\sin(n\pi/2)}{n\pi}}$$

$$b_n = \frac{1}{2}\left[-2\int_{-1}^{0} \sin\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{1} \sin\left(\frac{n\pi x}{2}\right) dx\right]$$

$$= -2\left[\frac{-2\cos\left(\frac{n\pi x}{2}\right)}{n\pi}\right]_{-1}^{0} + \frac{1}{2}\left(\frac{-2\cos\left(\frac{n\pi x}{2}\right)}{n\pi}\right)$$

$$b_n = \frac{-3\cos n\pi/2}{n\pi}$$

$$\boxed{f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty}\left[\frac{-\sin(n\pi/2)}{n\pi}\cos\frac{(\pi n x)}{2} - \frac{3\cos(n\pi/2)}{n\pi} + \frac{\sin \pi x n}{2}\right]}$$

Q19) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

$$f(x) = \pi + \sum_{n=1}^{\infty} \left[\frac{2(-1)^{n+1}}{n} \sin(nx) \right]$$

$$f(\pi/2) = \pi/2 + \pi$$

$$f(\pi/2) = \pi + \sum_{n=1}^{\infty} \left[\frac{2(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right) \right]$$

$$\frac{\pi}{2} + \pi = \pi + 2\left[1 - 0 + \frac{1}{3}(-1) + \frac{1}{5} - \frac{1}{6}(0) + \cdots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Q20) $\frac{\pi}{4} + \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \cdots$

$$f(x) = \frac{1}{\pi} + \frac{\sin x}{2} + \sum_{n=2}^{\infty} \left[\frac{(-1)^n}{\pi(1-n^2)} + 1\cos(nx) \right]$$

$$\frac{\pi}{2} = \sin\frac{\pi}{2} = 1$$

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{1}{\pi}\left[\frac{2}{3} + 0 - \frac{2}{15} + 0 + \frac{2}{35} + \cdots\right]$$

$$\frac{\pi}{2} = \frac{1}{2} + \frac{\pi}{4} + \left[\frac{1}{3} - \frac{1}{15} + \frac{1}{35} + \cdots\right]$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} + \frac{(-1)}{3\cdot 5} + \cdots$$

EXERCISE - 11.3

Q1) $f(x) = \sin 3x$

put $x = -x$

$f(x) = \sin 3x$

$f(-x) = \sin[3(-x)]$

$f(-x) = -f(x)$ odd function

Q13) $f(x) = |x|$, $-\pi < x < \pi$

$f(-x) = |-x|$

$f(-x) = |x|$ even function

Applying cosine series:-

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx$$

$$f(x) = \begin{cases} -x & , \text{if } -\pi < x < 0 \\ x & , \text{if } 0 \le x < \pi \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx$$

$$a_0 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos\left(\frac{n\pi}{\pi}\right)x\,dx$$

$$a_n = \frac{2}{\pi}\left(\frac{1}{n}x\sin(nx)\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)\,dx\right)$$

$$a_n = -\frac{2}{n\pi}\left(-\frac{\cos(nx)}{n}\right)\Big|_0^{\pi}$$

$$a_n = -\frac{2}{n\pi}\left(-\frac{(-1)^n+1}{n}\right)$$

$$a_n = \frac{2(-1)^n - 2}{n^2\pi}$$

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n - 2}{n^2\pi}\cos(n\pi)}$$

Q25) $f(x) = \begin{cases} 1, & 0 < x < 1/2 \\ 0, & 1/2 \le x < 1 \end{cases}$

$$a_0 = 2\int_0^{1/2} dx = 1$$

$$a_n = 2\int_0^{1/2} \cos(n\pi x)\,dx = \frac{2}{n\pi}\sin\frac{(n\pi)}{2}$$

$$b_n = 2\int_0^{1/2} \sin x\,dx = \frac{2}{n\pi}\left(1 - \frac{\cos(n\pi)}{2}\right)$$

Half range cosine expansion

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{(n\pi)}{2}\cdot\cos(n\pi)x$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \frac{\cos(n\pi)}{2}\right)\sin(n\pi x)}$$

Q38) $f(x) = 2 - x \qquad 0 < x < 2$

$$a_0 = \frac{2}{2}\int_0^2 2 - x\,dx$$

$$a_0 = \frac{2}{2}\left(2x - \frac{x^2}{2}\right)\Big|_0^2$$

$$a_0 = \frac{2}{2}\left(4 - \frac{4}{2}\right)$$

$$a_0 = 2$$

$$a_n = \frac{2}{2}\int_0^2 (2-x)\cos\frac{(n\pi x)}{2}\,dx$$

$$a_n = \left(\frac{(2-x)}{n\pi}\sin\frac{(n\pi x)}{2} - \frac{1}{n\pi}\int_0^2 \sin(n\pi x)\,dx\right)$$

$$a_n = 0$$

$$b_n = \int_0^2 (2-x)\sin(n\pi x)\,dx$$

$$b_n = \left[-\frac{1}{n\pi}(2-x)\cos(n\pi x)\right]_0^2 - \frac{1}{n\pi}\int_0^2 \cos(n\pi x)\,dx$$

$$\cancel{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{(n\pi)}{2}\cos(n\pi x)}$$

$$\cancel{\boxed{f(x) = \sum_{n=1}^{\infty} \frac{1}{n\pi}\left(1 - \cos\frac{(n\pi)}{2}\right)\sin(n\pi x)}}$$

$$= \left[\frac{0+2}{n\pi}\right]$$

$$b_n = \frac{2}{n\pi}$$

$$\boxed{f(x) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(\pi n x)}$$

Q 39) $f(t) = \begin{cases} 5 , & 0 < t < \pi \\ -5 , & \pi < t < 2\pi \end{cases}$

function is odd

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} [\sin(nt)\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} +(-5)\sin(nt)\,dt$$

$$b_n = \frac{5}{\pi}\left[\frac{-\cos(nt)}{n}\right]_0^{\pi} - \frac{5}{\pi}\left(\frac{-\cos(nt)}{n}\right)\Bigg|_{\pi}^{2\pi}$$

$$= \frac{5}{\pi}\left(\frac{-(-1)^n+1}{n}\right) - \frac{5}{\pi}\left(\frac{-1+(-1)^n}{n}\right)$$

$$= \frac{10(-1)^{n+1}+10}{n\pi}$$

$$f(t) = \sum_{n=1}^{\infty} \frac{10(-1)^{n+1}+10}{n\pi} \cdot \frac{\sin(n\pi)t}{2\pi/2}$$

$$\frac{md^2x}{dt^2} + Kx = \frac{10(-1)^{n+1}+10}{n\pi} \cdot \frac{\sin(n\pi)t}{2\pi/2}$$

$$\frac{d^2x}{dt^2} + 10x = \frac{10(-1)^{n+1}+10}{n\pi} \cdot \frac{\sin(n\pi)t}{2\pi/2}$$

$$x_p = \sum_{n=1}^{\infty} A_x \sin(nt)$$

$$\frac{d^2x}{dt^2} = \sum_{n=1}^{\infty} n^2 A \sin(nt)$$

$$-\sum_{n=1}^{\infty} n^2 A_x \sin xt + 10\sum_{n=1}^{\infty} A_n \sin t_n t$$

$$= \sum_{n=1}^{\infty} \left(\frac{10(-1)^{n+1}+10}{n\pi}\right) \cdot \sin(nt)$$

$$[-n^2+10]A_n = \frac{10(-1)^{n+1}+10}{n\pi}$$

$$A_n = \frac{10(-1)^{n+1} + 10}{n\pi\,[10n^2]}$$

$$\boxed{x_p(t) = \sum_{n=1}^{\infty} \frac{10(-1)^{n+1} + 10}{n\pi\,[10-n^2]} \cdot \sin(nt)}$$

Q51) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$a_0 = \frac{2}{\pi}\left(\frac{x^2}{2}\right)\Big|_0^{\pi}$$

$$a_0 = \frac{2}{\pi} \cdot \frac{\pi^2}{2}$$

$$a_0 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos\left(\frac{n\pi}{\pi}\right) x\,dx$$

$$a_n = \frac{2}{\pi}\left(\frac{x\sin(2nx)}{2n}\Big|_0^{\pi} - \frac{1}{2n}\int_0^{\pi} \sin(2nx)\,dx\right)$$

$$a_n = \frac{1}{n\pi}\left(\sin(2n\pi) - 0 - \frac{1}{2n}\left(\frac{\cos(2nx)}{2n}\right)\Big|_0^{\pi}\right)$$

$$a_n = \frac{1}{n\pi}\left(-\frac{1}{4n}(1-1)\right)$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin 2nx\,dx$$

$$b_n = \frac{2}{n}\left(-\frac{1}{2n}\int_0^{\pi} \frac{\cos 2n}{2}\, dx\right)$$

$$b_n = 0$$

$$f(x) = \frac{\pi}{2}$$

Q53) $$f(x,y) = A_{00} + \sum_{m=1}^{\infty} A_{m0} \cos\left(\frac{m\pi}{b}\right)x + \sum_{n=1}^{\infty} A_{0n} \cos\frac{n\pi}{c} y + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \cos\frac{m\pi}{b} \cos\frac{n\pi}{c} y$$

$$A_{00} = \frac{1}{bc}\int_0^c \int_0^b f(x,y)\, dx\, dy$$

$$A_{m0} = \frac{2}{bc}\int_0^c \int_0^b f(x,y) \cos\left(\frac{m\pi}{b}\right)x\, dx\, dy$$

$$A_{mn} = \frac{4}{bc}\int_0^c \int_0^b f(x,y) \cos\left(\frac{m\pi x}{b}\right)\cos\left(\frac{n\pi}{c}\right) y\, dx\, dy$$

First we have to find A's on their respective functions and then put in $xp(t)$ function to find the u function with respect to t.

CHAPTER # 11

EXAMPLES

EXERCISE # 11-1

Example 1

$$\{1, \cos x, \cos 2x, \dots\} \quad [-\pi, \pi]$$

$$(\varphi_0, \varphi_n) = \int_{-\pi}^{\pi} \cos(nx)\, dx$$

$$= \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} = \frac{1}{n}\left[\sin(n\pi) - \sin(n\pi)\right]$$

$$= 0$$

$$(\varphi_m, \varphi_n) = \int_{-\pi}^{\pi} \varphi_m(x)\, \varphi_n(x)\, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x + \cos(m-n)x\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]\Bigg|_{-\pi}^{\pi} = 0$$

Example 2 Norm

$$\varphi_0(x) = 1$$

$$\|\varphi_0(x)\|^2 = \int_{-\pi}^{\pi} 1\, dx = 2\pi$$

$$\varphi_0(x) = \sqrt{2\pi}$$

$$\varphi_0(x) = \cos(nx)$$

$$\|Q_n(x)\|^2 = \int_{-\pi}^{\pi} \cos^2(nx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos 2nx \; dx$$

$$= \frac{1}{2}(2\pi)$$

$$= \pi$$

Norm of $Q_n(x) = \sqrt{\pi}$

EXERCISE # 11.2

Example 1

$$f(x) = \begin{cases} 0 & , \; -\pi < x < 0 \\ \pi - x & , \; 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \pi x \; dx$$

$$= \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)\Big|_0^{\pi}$$

$$= \pi - \frac{\pi^2}{2} = \sqrt{\frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\int_{\pi}^{0} 0\,dx + \int_{0}^{\pi} (\pi - x)\cos(nx)\,dx\right)$$

$$= -\frac{1}{n\pi}\frac{\cos(nx)}{n}\Big|_0^{\pi}$$

$$a_n = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin(nx)\,dx = \frac{1}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos(nx) + \frac{1}{n}\sin x\right\}$$

Example 2

Average at 0

$$\frac{f(0+)+f(0-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$

Example 3

$$\frac{f(0+)+f(0-)}{2} = \frac{\pi}{2}$$

At π

$$\frac{f(\pi-)+f(-\pi+)}{2} = 0$$

EXERCISE # 11.3

Example 1

a) $f(x) = x^2$ is even

$f(-x) = (-x)^2 = x^2$

even

b) $f(x) = x^3$ is odd

$f(-x) = (-x)^3 = -x^3$

odd

Example 2

$\cos x = \cos(-x)$, even

$\sin(-x) = -\sin(x)$, odd

Example 3

$f(x) \Rightarrow -2 < x < 2$

As it is an odd function

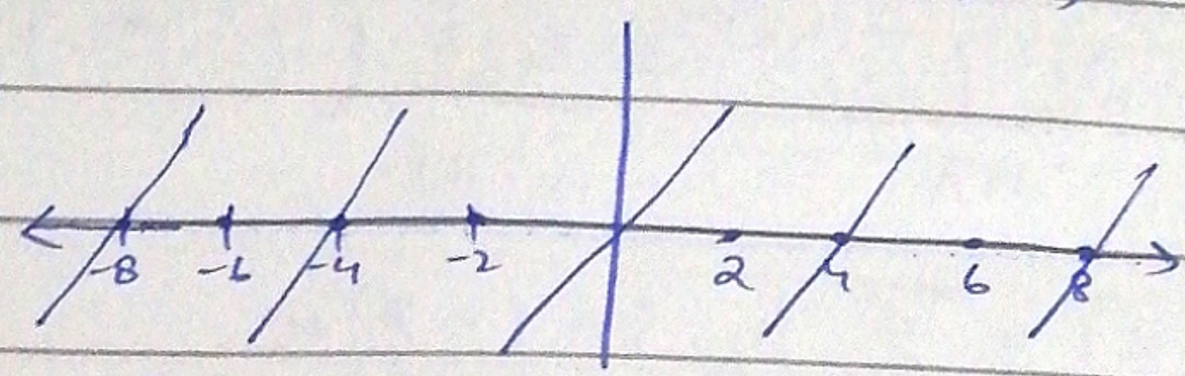
$$b_n = \frac{2}{2}\int_0^2 x \sin\left(\frac{n\pi}{2}\right)x \, dx$$

$$b_n = \frac{4(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{-1}{n}\right)^{n+1} \sin\left(\frac{n\pi}{2}\right)x$$

Example 4

$$f(x) = x \quad , \quad (-2, 2)$$



Example 5

$$f(x) = \begin{cases} -1 & , \ -\pi < x < 0 \\ 1 & , \ 0 \leq x < \pi \end{cases}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} (1) \sin(nx)\, dx -$$

$$= \frac{2}{\pi} \left(\frac{-(-1)^n}{n} \right)$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{-(-1)}{n} \right)^n \sin(nx)$$

Example 6

$$f(x) = x^2 \qquad 0 < x < L$$

(a) - $a_0 = \frac{2}{L} \int_0^L x^2\, dx = \frac{2}{3} L^2$

$$a_n = \frac{2}{L} \int_0^L x^2 \cos\left(\frac{n\pi}{L}\right) x\, dx = \frac{2}{L} \left(x^2 \sin\left(\frac{n\pi x}{L}\right) \right) \Big|_0^L$$

$$= -\frac{2L}{n\pi} \int_0^L x \sin\left(\frac{n\pi}{L}\right) x\, dx$$

$$a_n = \frac{4L^2 (-1)^n}{n^2 \pi^2}$$

$$f(x) = \frac{L^2}{3} + 4L^2 \sum_{n=1}^{\infty} \frac{(-1)}{n^2} \cos \frac{n\pi x}{L}$$

(b) $b_n = \frac{2}{L}\int_0^L x^2 \sin\left(\frac{\pi x}{L}\right) dx$

$$b_n = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^2}\left[(-1)^n - 1\right]$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{(n\pi)x}{L}$$

(c) $a_0 = \frac{2}{L}\int_0^L x^2 dx = \frac{2L^2}{3}$

$$a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{2n\pi x}{L} dx = \frac{L^2}{n^2\pi^2}$$

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{(2n\pi)x}{L} dx = \frac{L^2}{n^2\pi^2}$$

$$\boxed{f(x) = \frac{L^2}{3} + \frac{L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\cos\frac{2\pi nx}{L} - \frac{1}{n}\sin\left(\frac{2\pi nx}{2}\right)\right\}}$$

Example 7

$$b_n = 2\int_0^1 \pi + \sin(\pi t)\, dt = \frac{2(-1)^{n+1}}{n}$$

$$\frac{1}{10}\frac{d^2x}{dt^2} + 4x + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(n\pi t)$$

$$x_p(t) = \sum_{n=1}^{\infty}\frac{32(-1)^{n+1}}{m(64 - n^2\pi^2)}\sin(n\pi t)$$