Assignement # 2
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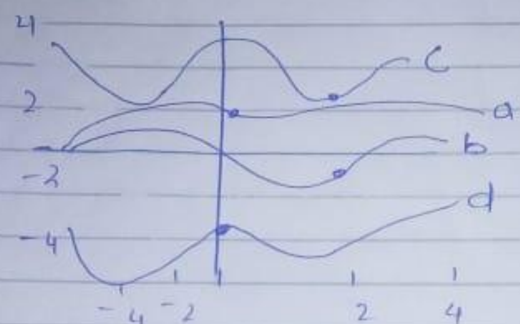
Exercise 2.1

Q#4.

$$\frac{dy}{dx} = (\sin x)(\cos y)$$

a) $y(0)=1$ (b) $y(1)=0$

(c) $y(3)=3$ (d) $y(0)=-5/2$



Q11. $y' = Y = \cos\frac{\pi}{2}x$

left $(x,y) = y - \cos\frac{\pi}{2}x$. The first order partial

darivative is $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y - \cos\frac{\pi}{2}\right)x$

$\rightarrow f(x,y)$, $\frac{\partial f}{\partial y}$ are continuous so curve passes through (x_0, y_0)

a) The curve is passing through $(2,2)$

b) The curve is passing through $(-1,0)$

- **Question 15 (solution)**

From the graph observe that graph of f intersect x-axis at $x = 2, -1, 1, 2$ so, critical points of D.E $\frac{dy}{dx}(fx)$ are $x = -1, -2, 1, 2$ intervals are

$-\infty < x < -2$, $-2 < x < -1$ and

$-1 < x < 1$ and $2 < x < \infty$

In interval

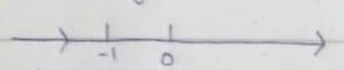
$-\infty < x < -2$, $-1 < x < 1$, $2 < x < \infty$, we have $y' > 0$ hence solution increases and in interval $-2 < x < -1$, $1 < x < 2$, y' decreases

- **Question 27:**

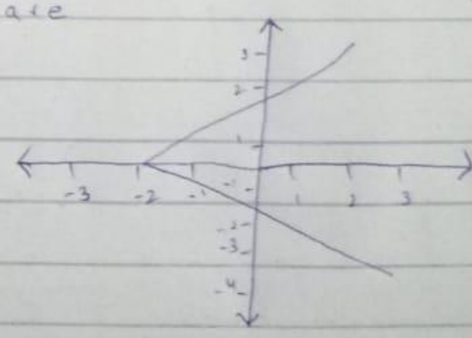
$$\frac{dy}{dx} = y \ln(y+2)$$

$y \ln(y+2) = 0$, $y = 0$, $\ln(y+2) = 0$

$y = 0$, $y + 2 = 1$



point 0 is unstable as arrow are moving away, arrow are moving towards as unstable



Exercise 2.2

Q8

$$e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$$

$$e^y y\, dy = \left(\frac{1+e^{-2x}}{e^x}\right) dx$$

Than take integration

$$\int e^y y = y e^y - e^y + c$$

$$\int \frac{1+e^{-2x}}{e^x} = \int (1+e^{-2x}) e^{-x}$$

$$= -e^{-x} - \frac{1}{3} e^{-3x} + C_2$$

Than the Sol.^{3} is

$$e^y (y-1) = -e^{-x} - \frac{1}{3} e^{-3x} + c$$

Q13

$$(e^y+1)^2 e^{-y} dx + (e^x+1)^3 e^{-x} dy = 0$$

$$(e^x+1)^3 e^{-x} dy = -(e^y+1)^2 e^{-y} dx$$

$$\frac{e^y}{(e^y+1)^2} dy = \frac{-e^x}{(e^x+1)^3} dx$$

By taking integration

$$\int \frac{e^y}{(e^y+1)^2} = \frac{-1}{(e^y+1)} + C_1$$

$$\int \frac{-e^x}{(e^x+1)^3} = \frac{-1}{2(e^x+1)^2} + C_2$$

than the sol. is

$$\frac{-1}{e^y+1} = \frac{-1}{2(e^x+1)^2} + C$$

Q12

$$\sin 3x\, dx + 2y \cos^3 3x\, dy = 0$$

$$-\frac{\sin 3x}{\cos^3 3x} dx = \int 2y\, dy$$

$$= -\tan 3x \sec^2 3x\, dx = 2y\, dy$$

By taking integration

$$= \int -\tan 3x \sec^2 3x\, dx$$

$$= -\frac{1}{3} \frac{(\tan^2(3x))}{2} + C_2$$

$$\int 2y\, dy = 2y^2 + C_1$$

$$= y^2 + C_1$$

Than the Sol. is

$$y^2 = -\frac{1}{6} \tan^2 3x + c$$

Q 22.

$$(e^x + e^{-x}) \frac{dy}{dx} = y^2$$

$$1/y^2\, dy = \frac{1}{e^x + e^{-x}} dx$$

By taking integration

$$\int \frac{1}{y^2} dy = -\frac{1}{y} + C_1$$

$$\int \frac{1}{e^x + e^{-x}} dx = \int \frac{1}{e^x + \frac{1}{e^x}} dx$$

$$\int \frac{e^x}{e^{2x}+1} dx = \int \frac{e^x}{e^{2x}+1} dx$$

$$\tan^{-1}(e^x) + C_2$$

$$-\frac{1}{y} + C_1 = \tan^{-1}(e^x) + C_2$$

$$y = \frac{-1}{\tan^{-1}(e^x) + c}$$

Q28

$(1+x^4)dy + x(1+4y^2)dx = 0$

$x(1+4y^2)dx = -(1+x^4)dy$

By taking integration

$\int \frac{-x}{1+x^4}dx = \int \frac{-x}{1+(x^2)^2}dx$

$-\frac{1}{2}\int \frac{2x}{1+(x^2)^2} = -\frac{1}{2}\tan^{-1}(x^2)+C_2$

$\int \frac{1}{1+4y^2}dy = \int \frac{1}{1+(2y)^2}dy$

$\frac{1}{2}\tan^{-1}(2y)+C_1$

$\tan^{-1}(2y) = -\tan^{-1}(x^2)+C$ —— (1)

By initial value $y(1)=0$

$\tan^{-1}(2(0)) = -\tan^{-1}(1)^2+C$

$C = \frac{\pi}{4}$

By putting value in (1)

$\tan^{-1}(2y) = -\tan^{-1}(x^2)+\pi/4$

$2y = \tan\left(\frac{\pi}{4} - \tan^{-1}(x^2)\right)$

$= \frac{\tan\frac{\pi}{4} \tan(-\tan^{-1}(x^2))}{1-(\tan\frac{\pi}{4})(\tan(\tan^{-1}(x^2)))}$

$= \frac{-x^2}{1+x^2} \Rightarrow y = \frac{-x^2}{2(1+x^2)}$

Q29

$\frac{dy}{dx} = ye^{-x^2}, \ y(4)=1$

$\frac{dy}{y} = e^{-x^2}dx$

$\int_4^x \frac{dy}{dt} = \int_4^x e^{-t^2}dt$

$\ln(y(t))\Big|_4^x = \int_4^x e^{-t^2}dt$

$\ln(y(x)) - \ln(y(4)) = \int_4^x e^{-t^2}dt$

$\ln y(x) - \ln 1 = \int_4^x e^{-t^2}dt$

$\ln(y(x)) = \int_4^x e^{-t^2}dt$

$y = \exp\left(\int_4^x e^{-t^2}\right)dt$

Q37

$\frac{dy}{dx}(y-1)^2+0.01, \ y(0)=1$

The differential eq can be written as

$\frac{dy}{\sqrt{1-y^2}} = x\,dx$

$\sqrt{1-y^2} = 0$

$1-y^2 = 0$

$y = \pm 1$

So, $y=1, \ y=-1$ are singular solution.

Exercise 2.3

Q3 $\frac{dy}{dx} + y = e^{3x}$

$e^{\int dx} = e^{x}$

$e^{x}\left(\frac{dy}{dx} + y\right) = e^{x}(e^{3x})$

$\frac{dy}{dx}[(e^{x}y)] = e^{4x}$

$\int d(e^{x}y) = \int e^{4x}dx$

$e^{x}y = \frac{1}{4}e^{4x} + C$

$y = \frac{1}{4}e^{3x} + ce^{-x}$

$I: (-\infty, \infty)$

$y = \frac{1}{4}e^{3x} + ce^{-cx}, \ I: (-\infty, \infty)$

ce^{-x} is transient.

Q14

$xy' + (1+x)y = e^{-x}\sin 2x$

$\frac{dy}{dx} + (1 + 1/x)y = \frac{e^{-x}\sin 2x}{x}$

$e^{\int(1+\frac{1}{x})dx} = e^{x+\ln x} = e^{x}e^{\ln x} = xe^{x}$

$xe^{x}\left[\frac{dy}{dx} + \left(1+\frac{1}{x}\right)y\right] = xe^{x}\left[\frac{e^{-x}\sin 2x}{x}\right]$

$\frac{d}{dx}(xe^{x}y) = \sin 2x$

$\int d(xe^{x}y) = \int \sin 2x\, dx$

$xe^{x}y = -\frac{1}{2}\cos 2x + C$

$xe^{x}y = -\frac{1}{2}\cos 2x + C$

$y = -\frac{1}{2}x^{-1}e^{-x}\cos 2x + Cx^{-1}e^{-1}$

$I = (0, \infty)$

$y = -\frac{1}{2}x^{-1}e^{-x}\cos 2x + Cx^{-1}e^{-1}, \ I: (0, \infty)$

the sol is transient

Q12. $(1+x)\frac{dy}{dx} - xy = x + x^2$

$\frac{dy}{dx} - \frac{x}{1+x}y = x$

$e^{\int \frac{-x}{1+x}dx} = e^{-\int(1-\frac{1}{1+x})dx}$

$e^{-(x-\ln(1+x))} = e^{-x}e^{\ln(1+x)} =$

$(1+x)e^{-x}$

$(1+x)e^{-x}\left(\frac{dy}{dx} - \frac{x}{1+x}y\right) =$

$(1+x)e^{-x}(x)$

$d/dx\,(1+x)e^{-x}y = (x+x^2)e^{-x}$

$\int d[(1+x)e^{-x}y] = \int(x+x^2)e^{-x}dx$

$y = \frac{-(x^2+3x+3) + ce^{x}}{1+x}$

$I: (-1, \infty)$

$y = \frac{-[x^2+3x+3] + ce^{x}}{1+x}$

Q27. Solution

The eq. becomes

$\frac{dy}{dx} + \frac{y}{x} = \frac{e^{x}}{x}$

$e^{\int \frac{1}{x}dx} = e^{\ln x}$

$x\left(\frac{dy}{dx} + \frac{y}{x}\right) = x\left(\frac{e^{x}}{x}\right)$

$\frac{d}{dx}(xy) = e^{x}$

$\int d(xy) = \int e^{x} = xy = e^{x}$

$y = \frac{e^{x}}{x} + \frac{C}{x}$

$I = (0, \infty)$

$2 = e + C \qquad y = \frac{e^{x} + \ldots}{x}$

$C = 2 - e$

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Q34 sol:-

The standard form becomes

$$\frac{dy}{dx} + \frac{1}{x+1}y = \frac{1}{x(x+1)}$$

Integrating factor

$$e^{\int \frac{1}{x+1}dx} = e^{\ln(x+1)} = x+1$$

$$(x+1)\left(\frac{dy}{dx} + \frac{1}{x+1}y\right) = (x+1)\frac{1}{x(x+1)}$$

$$\frac{d}{dx}(x+1)y = \frac{1}{x}$$

$$\int d(x+1)y = \int \frac{1}{x}dx$$

$$(x+1)y = \ln x + c$$

$$I : (0, \infty)$$

$$e+1 = 1+c$$

$$c = e \Rightarrow (x+1)y = \ln x + e$$

$$y = \frac{\ln x + e}{x+1}$$

Q47 sol:-

$$y = 3x-5, x \to \infty$$

$$\frac{dy}{dx} \Rightarrow y' \Rightarrow 3-ce^{-x}$$

$$-3(-y-3x+5)$$

$$-y+5-3x$$

$$y'+y = 3x-2.$$

Q18.

$2y\sin x\cos x - y + 2y^2 e^{xy^2})dx$

$$M(x,y) = 2y\sin x\cos x y + 2e^{xy^2}y$$

$$\frac{\partial M}{\partial y} = 2\sin x\cos x - 1 + 4e^{xy^2}y - 4xy^3 e^{xy^2}$$

$$\frac{\partial N}{\partial x} = -1 + 2\sin x\cos x + 4ye^{xy^2} + 4xy^3 e^{xy^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$f(x,y) = -xy + y\sin^2 x + 2e^{xy^2}$$

$$g'(x) = 0, \quad g(x) = c$$

$$-xy + y\sin^2 x + 2e^{xy^2} = c$$

Exercise 2.4.

Q#6. $(2y - 1/x + \cos 3x) + (y/x^2 - 4x^3 + 3y\sin 3x)dx = 0$

$$M(x,y) = \frac{y}{x^2} - 4x^3 + 3y\sin 3x$$

$$N(x,y) = 2y - 1/x + \cos 3x$$

$$\frac{\partial M}{\partial y} = \left(\frac{1}{x^2} + 3\sin 3x\right)$$

$$\frac{\partial N}{\partial x} = \left(1/x^2 - 3\sin 3x\right)$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ Givendifferenial is notexist

Q25.

$(y^2\cos x - 3x^2y - 2x)\,dx +$
$(2y\sin x - x^3 + \ln y)\,dy = 0$

$M(x,y) = y^2\cos x - 3x^2y - 2x$

$\frac{\partial M}{\partial y} = 2y\cos x - 3x^2$

$N(x,y) = 2y\sin x - x^3 + \ln y$

$\frac{\partial N}{\partial x} = 2y\cos x - 3x^2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$y^2\sin x - x^3y - x^2 + y\ln y - y = c$

$0 - 0 - 0 + e - e = c$

$c = 0$

$y^2 \sin x - x^3y - x^2 + y\ln y - y = 0.$

. Question 32

$y(x+y+1)dx+(x+2y)dy=0$

$M=xy+y^2+y$, $N=x+2y$, $My=x+2y+1$,

$Mx=1$,

$\frac{My-Nx}{N}=\frac{x+2y+1-1}{x+2y}=\frac{x+2y}{x+2y}=1$

$M(x)=e^{\int dx}=e^x$.

$M(x,y)=e^x(x+2y)$, $\frac{\partial M}{\partial N}=e^x(x+2y+1)$

$\frac{\partial N}{\partial y}=\frac{\partial N}{\partial x}$, $f(x,y)=e^x(xy+y^2)+g(x)$

$\frac{\partial f}{\partial x}=e^x(xy+y^2+y)+g(x)$

$g'(x)=0$, $g(x)=C \Rightarrow e^x(xy+y^2)=C$.

. Question 33:

$6xy\,dx+(4y+9x^2)dy=0$

$M=6x$, $N=4y+9x^2$, $My=6x$,

$Nx=18x$

$\frac{My-Nx}{N}=\frac{6x-18x}{6xy}=\frac{-12x}{4y+9x^2}$, $\frac{Nx-My}{y}=\frac{18x-6x}{6xy}$

$=\frac{12x}{6xy}=2/y$

$M(y)=e^{\int 2/y\,dy}=e^{2\ln y^2}=y^2$

$6xy^3dx+4y^3+9x^2y^2\,dy=0$,

$M(x,y)=6xy^3$, $\frac{\partial M}{\partial y}=18xy^2$, $N(x,y)=4y^2+9x^2y^2$

$\frac{\partial N}{\partial x}=18xy^2$, $\frac{\partial N}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow f(x,y)=y^4+3x^2y^3+g(x)$

$\frac{\partial f}{\partial x}=6xy^3+g'(x)$, $g'(x)=0$, $g(x)=C$, $y^4+3x^2y^3=C$.

- **Question 37:**

$x\,dx + (x^2y + 4y)\,dy = 0 \; ; \; y(4) = 0$

$M = x, \quad N = x^2y, \quad M_y = 0, \quad N_x = 2xy$

$\frac{M_y - N_x}{M} = \frac{0 - 2xy}{x} = \frac{2xy}{x} = 2y$

$M(y) = e^{\int \frac{N_x - M_y}{M}} = e^{\int 2y\,dy} = e^{y^2}$

$M(y)M + M(y)N = 0, \quad xe^{y^2}dx + x^2y\,e^{y^2}$

$+ 4ye^{2}dy = 0 \implies M = xe^{y^2}, \; M_y = 2xye^{y^2}$

$N = x^2ye^{y^2} + 4ye^{y^2}, \; N_x = 2xye^{y^2}$

$f(x,y) = \int xe^{y^2}dx = \frac{1}{2}x^2e^{y^2} + g(y)$

$\frac{\partial f}{\partial y} = x^2ye^{y^2} + g'(y), \; g'(y) = 4ye^{y^2}$

$g(y) = \int 4ye^{y^2}dy$

Let $u = y^2$, $du = 2y\,dy = 2du = 4y\,dy$

$g(y) = 2\int e^{u}du = 2e^{u} + 2e^{y^2}$

$= \frac{1}{2}e^{x}e^{y^2} + 2e^{y^2} = C \implies \frac{1}{2}(4)^2e^{0} + 2e^{(0)^2} = C$

$\frac{1}{2}(16)(1) + 2(1) = C = 8 + 2 = C$

$\boxed{10 = C}$

$\frac{1}{2}x^2e^{y^2} + 2e^{y^2} = 10$

$x^2e^{y^2} + 4e^{y^2} = 20$

$e^{y^2}(x^2 + 4) = 20.$

- Exercise 2.5:

- Question 9:

$-y\,dx + (x + \sqrt{xy})\,dy = 0$

$x = vy$

$\frac{dx}{dy} = v + y\frac{dv}{dy} \Rightarrow dx = v\,dy + y\,dv$

$-y\,dx + (x + \sqrt{xy})\,dy = 0$

$-y(v\,dy + y\,dv) + vy + \sqrt{(vy)}\,v\,dy = 0$

$-vy\,dy - v^2 dy + vy\,dy + y\sqrt{v}\,dy = 0$

$-y^2 dv + y\sqrt{v}\,dy = 0 \Rightarrow -y\,dv + \sqrt{y}\,dy = 0$

$\frac{1}{\sqrt{v}}dv = \frac{1}{y}dy \Rightarrow \int \frac{1}{\sqrt{v}}dv = \int \frac{1}{\sqrt{y}}dy$

$4x/y = (\ln y + c)^2 \Rightarrow 4x(y)\,(\ln y + c)^2.$

- Question 13:

$x + ye^{y/x} - xe^{y/x}\,dy = 0, \quad y(1) = 0$

$y = Ux, \; \frac{dy}{dx} = v + x\frac{dv}{dx} = dy = v\,dx + x\,dv$

$(x + e^{x}y)\,dx - xe^{y/x}dx = 0$

$x + ((ux)e^{ux/x})\,dx - xe^{vx/x}(v\,dx + x\,dv) = 0$

$(x + uxe^{v})\,dx - e^{v}x(u\,dx + x\,dv) = 0$

$x\,dx + uxe^{v}dx - uxe^{v}dx - x^2e^{v}dv = 0$

$x\,dx - e^{v}x^{v}dv = 0, \quad dx - xe^{v}dv = 0,$

$\int e^{v}dv, \int \frac{1}{x}dx = e^{v} = \ln x + c$

$e^{y/x} = \ln x + c, \quad 1 = 0 + c, \quad c = 1$

$e^{y/x} = \ln x + 1.$

. Question 17:

$$\frac{dy}{dx} = y(xy^3 - 1)$$

$$\frac{dy}{dx} = y(xy^3-1) \Rightarrow \frac{dy}{dx} = xy^4 - 1$$

$$\frac{dy}{dx} + y = xy^4, \quad \frac{1}{y^4}\frac{dy}{dx} + \frac{1}{y^3} = x$$

$$u = \frac{1}{y^3}, \quad \frac{du}{dy} = -\frac{3}{y^4} \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \Rightarrow \frac{dy}{dx} = -\frac{y^3}{3}\frac{du}{dx}$$

$$\frac{1}{y^4}\left[-\frac{y^4}{3}\frac{du}{dx}\right] + u = x$$

$$-\frac{1}{3}\frac{du}{dx} + u = x \Rightarrow \frac{du}{dx} - 3u = -3x$$

$$e^{\int -3x\,dx} = e^{-3x}, \quad e^{-3x}\left(\frac{du}{dx} - 3u\right) = e^{-3x}(-3x)$$

$$\frac{d}{dx}(e^{-3x}u) = \int -3x e^{-3x}dx \Rightarrow e^{-3x}u = xe^{-3x} + \frac{1}{3}e^{-3x} + c$$

$$u = x + \frac{1}{3} + ce^{3x}$$

$$\frac{1}{y^3} = x + \frac{1}{3} + ce^{3x} \Rightarrow \frac{1}{y^3} = x + \frac{1}{3} + ce^{3x}$$

$$y^{-3} = x + \frac{1}{3} + ce^{3x}$$

. Question 21:

$$x^2\frac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$$

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{3}{x^2}y^4 = \frac{1}{y^4}\frac{dy}{dx} - \frac{2}{x}\frac{1}{y^3} = \frac{3}{x^2}$$

$$u = \frac{1}{y^3} = \frac{du}{dy} = -\frac{3}{y^4} \Rightarrow \frac{dy}{du} = -\frac{y^4}{3}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{dy}{dx} = -\frac{y^4}{3}\frac{du}{dx}$$

$$= \frac{1}{y^4}\left(-\frac{y^4}{3}\frac{du}{dx}\right) - \frac{2}{x}u = \frac{3}{x^2}$$

$$-\frac{1}{3}\frac{du}{dx} - \frac{2}{x}u = \frac{3}{x^2}$$

$\frac{du}{dx} + \frac{6}{x}u = -\frac{9}{x^2} \Rightarrow \frac{d}{dx}(x^6 u) = -9x^4$

$\int d(x^6 u) = \int -9x^4 dx \Rightarrow x^6 u = -\frac{9}{5}x^5 + c$

$u = -\frac{9}{5x} + cx^{-6} \Rightarrow y^{-3} = -\frac{9}{5x} + cx^{-6}$

$(-\frac{1}{2})^{-3} = -\frac{9}{5} + c$, $c = 45/9$, $y^{-3} = -\frac{9}{5x} + \frac{49}{5}x^{-6}$

• Question 24:

$\frac{dy}{dx} = (1-x-y) / x+y$

$\frac{dy}{dx} = \frac{1-(x+y)}{x+y}$, $\frac{du}{dx} - 1 = \frac{1-u}{u}$, $\frac{du}{dx} = \frac{1-u}{u} + 1$

$\frac{du}{dx} - \frac{1}{u} = u\,du = dx \Rightarrow \int u\,du = \int dx$

$\frac{1}{2}u^2 = x + C \Rightarrow \frac{1}{2}(x+y)^2 = x + c.$

• Question 30 (Solution)

Let $u = 3x+2y+2$, $\frac{du}{dx} = 3 + 2\frac{dy}{dx} = \frac{dy}{dx}$

$= \frac{1}{2}\frac{du}{dx} - \frac{3}{2} = (\frac{1}{2}\frac{du}{dx} - \frac{3}{2}) = \frac{u-2}{2}$

$\frac{1}{2}\frac{du}{dx} = \frac{u-2}{u} + \frac{3}{2}$, $\frac{du}{dx} = \frac{5u-4}{u}$

$\frac{u}{5u-4} \cdot dx = \frac{1}{25}(5u-4) + 4\ln(5u-4)$

$= x + c \Rightarrow C = 6/25 + 4\ln 19$

$-15x + 10y + 4\ln 115 + 10y = 6 =$

$100 \ln 19.$

Chapter 3:- (Exercise 3.1)

* Question 6. (solution)

We have the amount of radioactive substance as x- milligrams and this amount of this substance x- decays at a time t at rate is propotional to x

$$\frac{dx}{dt} = -kx$$

with the conditions

$$t(x = 100 \text{ milligrams}) = 0$$
$$t(x = 97 \text{ milligrams}) = 6 \text{ hrs.}$$

and we have to obtain the amount of substance after 24 hours than

$$\frac{dx}{dx} = -k\,dt \rightarrow \int \frac{1}{x}\,dx = -k\int dt = \ln x = -kt + c_1$$

$$e^{\ln x} = e^{(-kt+c_1)} \rightarrow x = e^{-kt}e^{c_1} = ce^{-kt} \rightarrow a.$$

Now we find c, k constants

$$100 = ce^{0} \Rightarrow c = 100 \text{ by putting in a}$$

$x = 100e^{-kt}$ than apply point $(x, t) = (97, 6)$

$$97 = 100e^{-6k} \rightarrow 0.97 = e^{-6k} \rightarrow k = \frac{\ln 0.97}{}$$

$$= \frac{0.0305}{-6} = 0.0051,\ x = 100e^{-0.0051t - 6}$$

Substance after 24 hours

$$x = 100e^{-0.0051 \times 24} = 100e^{-0.1224}$$

$$= 100 \times 0.8853 = 88.53 \text{ milligrams}$$

Question 7:-

We have the formula that obtains the amount of radioactive substance as

$x = 100e^{-0.051t}$

and we have to obtain the half life

as $x = 0.5x_0$

$0.5x_0 = 100e^{-0.0051t}$

$0.5 \times 100 = 100e^{-0.0051t} = \ln 0.5 = 0.051t.$

$t = \frac{\ln 0.5}{-0.0051} = \frac{-0.693}{-0.0051}$

$= 135.9$ hrs (half life)

Question 19:-

$T_m = 70°F$ is temperature in which dead body is placed. And this temperature increases at a rate propotional to the difference b/w body's temperature T and room temperature T_m as $\frac{dT}{dt} = k(T - T_m)$ — (1)

with the conditions.

$T(t = 0 \text{ hours}) = 98.6°F$

$T(t = t_1 \text{ hours}) = 85°F$

$T(t = t_1 + 1 \text{ hour}) = 80°F$

From eq. (1)

$$\frac{dT}{T-T_m} = k\,dt \rightarrow \int \frac{1}{T-70^\circ}\,dt = k\int dt \rightarrow \ln(T-70^\circ) = kt + c_1$$

$$e^{\ln(T-70^\circ)} = e^{kt+c_1} \rightarrow T-70^\circ = e^{kt}\, e^{c_1}$$

$$T = e^{kt} \cdot e^{c_1} + 70^\circ \rightarrow ce^{kt} + 70^\circ \rightarrow ②$$

To apply points $(T, t) = (98.6°F, 0 \text{ hours})$ in ② eq.

$$98.6 = ce^{0} + 70^\circ \rightarrow ce^{0} = 98.6 - 70^\circ$$

$$c = 28.6^\circ$$

$$T = 28.6e^{kt} + 70^\circ \rightarrow ③$$

Apply point $(T, t) = (85°F, t_1 \text{ hours})$ into ③ eq.

$$85^\circ = 28.6e^{kt_1} + 70^\circ \rightarrow 85 - 70 = 28.6e^{kt_1}$$

$$e^{kt_1} = \frac{15}{28.6} \rightarrow kt_1 = \ln\left(\frac{15}{28.6}\right) \rightarrow ④$$

Apply point $(T, t) = 85°F, t_1\,1 \text{ hours})$ in ③

$$80^\circ = 28.6e^{k(t_1+1)} + 70^\circ \rightarrow 80 - 70 = 28.6\, e^{k(t_1+1)}$$

$$1 = 28.6e^{kt_1} = e^{k}(t+1) = \frac{10}{28.6}$$

$$k(t_1+1) = \ln\left(\frac{15}{28.6}\right) \rightarrow ⑤$$

By subtracting 4 and 5

$$k(t_1+1) - kt_1 = \ln\left(\frac{10}{28.6}\right) - \ln\left(\frac{15}{28.6}\right)$$

$$kt_1 + k - kt_1 = \ln\left(\frac{10}{28.6}\right) - \ln\left(\frac{15}{28.6}\right)$$

$$k = \ln\left(\frac{10}{28.6}\right) - \ln\left(\frac{15}{28.6}\right) = -1.0508 - (-0.64535)$$

$$= -1.0508 + 0.64535 = -0.405$$

Put in 4 as.

$$-0.405t_1 = \ln\left(\frac{15}{28.6}\right) \rightarrow t_1 = \frac{\ln\left(\frac{15}{28.6}\right)}{-4.05}$$

$$= 1.59 \text{ hours.}$$

Question 38:-

$$W = (125 + 35) = 160 \text{ lb.}$$

Velocity v and h = 15000 ft After 15s parachute open so differential eq is.

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} = g - \frac{k}{m}v \rightarrow eq\ ①$$

With initial conditions $V_0 = 0$ ft/s than

$$\frac{dv}{g-\frac{k}{m}v} = dt \rightarrow \int\frac{1}{g-k/mv} = \int dt \rightarrow -\frac{m}{k}\int\frac{-k/m}{g-k/mv}dv = \int dt$$

$$-\frac{m}{k}\ln\left(g-\frac{k}{m}v\right) = t + c_1 \rightarrow \ln\left(g-\frac{k}{m}v\right) = -\frac{k}{m}t + c_2$$

$$e^{\ln\left(g-\frac{k}{m}v\right)} = e^{(-k/mt + c_2)} \rightarrow g - \frac{k}{m}v = e^{c_2}e^{-k/mt}$$

$$g - \frac{k}{m}v = ce^{-k/mt} \rightarrow \frac{k}{m}(v) = g - ce^{-k/mt}$$

Then we have

$$v(t) = \frac{mg}{k} - se^{-k/mt} \rightarrow eq\ ② \text{ (velocity formula)}$$

For Free fall = $m = \frac{160}{32} = 5$, $K = 0.5$, $g = 32 \, ft/s^2$

$$Vf(t) = \frac{5 \times 32}{0.5} - ce^{-0.5/5 t} = 320 - ce^{-1/10 t} \quad -③$$

To obtain constants we use point $(v, t) = (0, 0)$

$$0 = 320 - 5e^0 \rightarrow s = 320$$

By putting s in eq ②

$$v(t) = 320 - 320e^{-1/10 t}$$

Distance measured from releasing point

$$\frac{ds}{dt} = V \rightarrow \int ds = \int v dt$$

$$s f(t) - \int (320 - 320 e^{-1/10 t}) dt \rightarrow 320 \int dt - 320 \int e^{-1/10 t} dt$$

$$= 320t - 320(-10e^{-1/10 t}) + h$$

$$= 320t + 3200 (e^{-1/10 t}) + h$$

To find h apply point $(s, t) = (0 ft, 0 s)$

$$0 = 320(0) + 3200e^0 + h$$

$$h = -3200$$

$$s(t) = 320t + 3200e^{-1/10 t} - 3200 \rightarrow ⑦$$

For fallong with open parachute.

Put $t = 15$ in ④ and ⑦

$$v(15) = 320 - 320e^{-1/10 \times 15} = 320 - 320e^{-3/2}$$

$$= 248.6 ft/s \text{ and } s(15) = 320 \times 15 + 3200e^{-1/10 \times 15} - 3200$$

$$= 4800 + 3200e^{-3/2} - 3200 = 2314 \text{ ft}$$

$K = 10$ and using (2)

$V_p(t) = \frac{5 \times 32}{10} - S_2 e^{-10/5 t} \rightarrow 16 - Se^{-2t}$ — (8)

To find the value S_2 put $(v, t) = (248.6 ft/s, 0s)$

In (8)

$248.6 = 16 - S_2 e^{0} \rightarrow S_2 e^{-30} = 16 - 248.6$

$S_2 = 232.6 \rightarrow V_p(t) = 16 + 232.6 e^{-2t} \rightarrow$ (9)

velocity after 20 sec.

$V = 16 + 232.6e^{-10}$

$= 16 + 0.0106 = 16.0106 ft/s$

$t = \infty$ in (9)

$V_{ter} = 16 + 232.6e^{-\infty} = 16 + 0$

$= 16 ft/s$

t in (9)

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$$\theta(t) = T + (\theta_0 - T)\, e^{-kt} \rightarrow (b)$$

However the cooling rate is that appears in this expression is as yet unknown we can determine k by making 2nd measurement of the body's temprative at some later time t_1. Suppose $\theta = \theta_1$ and $t = t_1$. By putting these values in (b) we find that,

$$\theta_1 - T = (\theta_0 - T)\, e^{-kt}$$

$$K = -\frac{1}{t_1} \ln \frac{\theta_1 - T}{\theta_0 - T} \rightarrow c$$

where θ_0, θ_1, T are knownas quantiti

Finally to determine td and $\theta = \theta d$ in (b).

Solve td and we obtain

$$td = -\frac{1}{K} \ln \frac{\theta d - T}{\theta_0 - T}$$ where k is given by eq (b).

For example; suppose that the temperatu of a corpse is 85°F when discovered and 74°F two hours later and ambient temperature is 68°F.

Then from (B)

$$K = -\frac{1}{2} \ln \frac{74-68}{85-68} \approx 0.5207 hr^{-1}.$$

and from(c)

$$bd = \frac{1}{0.5207} \ln \frac{98.6-68}{85-68} \approx -1.129$$

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Thus we conclude that the body was discovered approximately 1hr, 8min after death.

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- Exercise 8.3
- Question 4

→ Since w is the first to decay it will only lose atoms from elements w. Radioactive decay is expressed as constant in this case λ_1 is multiplied by its respective element.

$$\frac{dw}{dt} = -\lambda_1 w$$

→ X is gaining atoms from w while at the same time losing atoms due to its own decay. In short $\frac{dw}{dt}$ will xply by -1

$$\frac{dx}{dt} = \lambda_1 w - \lambda_2 x$$

$$\frac{dy}{dt} = (\lambda_1 w - \lambda_1 w - \lambda_2 x) - \lambda_3 y$$

(c) By putting values

$$t = \frac{10^{10}}{4.7526 + 0.5874} \ln\left(\frac{4.7526 + 0.5874}{0.5874} \times 8.5 \times 10^{-7} + 1\right)$$

$$= \frac{10^{10}}{5.34} \ln(9.091 \times 0.574 + 1) \; 5.4 \times 10^{-6} = \ln \frac{2.43 \times 10^{10}}{5.34}$$

$$= 1.64 \times 10^{9} \text{ years}$$

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Question 6 solution.

We get the amount of pottasium k-40 and A-40 from (p.5) at time $t - K(t)$

$= K_0 e^{-(\lambda_1 + \lambda_2)t}$

a) Now ration $\frac{A(t)}{K(t)}$

as:

$$\frac{A(t)}{K(t)} = \frac{\frac{\lambda_2 K_0 (1 - e^{-(\lambda_1 + \lambda_2)t})}{\lambda_1 + \lambda_2}}{K_0 e^{-(\lambda_1 + \lambda_2)t}}$$

$$\frac{A(t)}{K(t)} = \frac{\lambda^2}{\lambda_1 + \lambda_2} K_0 \left(e^{(\lambda_1 + \lambda_2)t} - 1\right)$$

b) Now $t = \frac{1}{\lambda_1 + \lambda_2}$

$\ln\left(1 + \frac{\lambda_1 + \lambda_2}{\lambda_2}\right)$ using above eq

$$\frac{\lambda_1 + \lambda_2}{\lambda_2} \frac{A(t)}{K(t)} = K_0 e^{(\lambda_1 + \lambda_2)t} - K_0$$

$$K_0 e^{(\lambda_1 + \lambda_2)t} = \frac{\lambda_1 + \lambda_2}{\lambda_2} \frac{A(t)}{K(t)} + 1$$

$$\ln e^{(\lambda_1 + \lambda_2)} = \ln \frac{\lambda_1 + \lambda_2}{\lambda_2} \frac{1}{K_0} \frac{A(t)}{K(t)} + 1$$

$$(\lambda_1 + \lambda_2)t = \ln \frac{\lambda_1 + \lambda_2}{\lambda_2} \frac{1}{K_0} \frac{A(t)}{K t} + 1$$

$$t = \frac{1}{\lambda_1 + \lambda_2} \ln \left(\lambda_1 + \lambda_2 \frac{A(t)}{K(t)} + 1\right)$$

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Questing 8 (solution).

$$\frac{dx^1}{dt} = (4\,gal/min)(0\,lb/gal) - (4\,gal/min)\left(\frac{1}{200}x4\ lb/gal\right)$$
$$= \frac{-1}{50}x$$

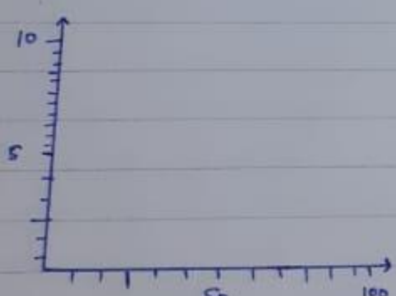
$$\frac{dx_2}{dt} = (4\,gal/min)\left(\frac{1}{200}x\ \frac{lb}{1gal}\right) - (4\,gal/min)(1/150\ x_2\ lb/gal)$$

$$= \frac{1}{50}x_1 - \frac{2}{75}x_2$$

$$\frac{dx_3}{dt} = (4\,gal/min)(1/150\ x_2\ lb/gal) - (4\,gal/min)(1/100\ x_3\ lb/gal)$$

$$= \frac{2}{75}x_2 - \frac{1}{25}x_3.$$

Question: 9 solution.



zooming is on the graph it can be seen that populations are first equal at $t = 5.6$. The approximate period of x and y are both 43.

Question #13 (solution)

$i_1 = i_2 + i_3$, $E(t) = iR_1 + L_1 \frac{di^2}{dt} + i_3 R_3$

$$E(t) = iR_1 + L_2 \frac{di_3}{dt} + i_3 R_3 .$$

The above equation can be obtained by applying Kirch off second law to each Loop, Then

By combining three equations:

$$L_1 \frac{di_2}{} + (R_1 + R_2) i + R_1 i_3 = E$$
$$L_2 \; dt + R_1 i_2 + (R_1 + R_3) i_3 = E .$$

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* **Example 1**

The initial value problem

$3y''' + 5y'' - y' + 7y = 0, \quad y(1) = 0,$

$y'(1) = 0, \quad y''(1) = 0$

It possesses the trivial solution $y = 0$, because the third order equation is linear with constant coefficients, it follows that all conditions of theorem are fulfilled.

Hence $y = 0$ is the only solution on any interval containing $x = 1$.

* **Example 2**

$y'' - 4y = 12y, \quad y(0) = 4$

$y'(0) = 1$

$x^2y'' - 2xy' + 2y = 6$

$y(0) = 3$

$y'(0) = 1$

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$y(a) = y_0, \quad y(b) = y_1$

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* Example 7

The function $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$.

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

---

* Example 8

The function $y = 4\sinh 3x - 5e^{3x}$ is a solution of equation in example 7.

$$y = 2e^{3x} - 2e^{-3x} - 5e^{-3x} = 4\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 5e^{-3x}$$

The last expression is recognized as $y = 4\sinh 3x - 5e^{-3x}$.

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$a_1 y(a) + \beta_1 y'(a) = \gamma_1$

$a_2 y(b) + \beta_2 y'(b) = \gamma_2.$

*** Example 3**

$x'' + 16x = 0$ is differential equation

$x = c_1 \cos 4t + c_2 \sin 4t$

$x'' + 16x = 0, \quad x(0) = 0, \quad x\left(\frac{\pi}{2}\right) = 0$ (3)

$$\left.\begin{array}{l} x'' + 16x = 0, \quad x(0) = 0 \\ x\left(\frac{\pi}{8}\right) = 0 \end{array}\right] (4)$$

$$\left.\begin{array}{l} x'' + 16x = 0, \\ x(0) = 0 \\ x\left(\frac{\pi}{2}\right) = 1 \end{array}\right] \to (5)$$

*** Example 4**

The function $y_1 = x^2$ and $y_2 = x^2 \ln x$

$y = c_1 x^2 + c_2 x^2 \ln x$

is also a solution of the equation on the interval.

The function $y = e^{7x}$ is a

solution of the equation $y'' - 9y' + 14y = 0$ because it's linear and Homogenous.

* Example 5

The set of functions $f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, $f_3(x) = \sec^2 x$, $f_4(x) = \tan^2 x$

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

when $c_1 = c_2 = 1$, $c_3 = -1$ and $c_4 = 1$. we should use $\cos^2 x + \sin^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$

* Example 6

The set of functions $f_1(x) = \sqrt{x} + 5$, $f_2(x) = \sqrt{x} + 5x$, $f_3(x) = x - 1$, $f_4(x) = x^2$ is linearly dependent on the interval $(0, \infty)$

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* Example 9

The functions $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$ satisfy the third-order equation

$$y''' - 6y'' + 11y' - 6y = 0$$

Since

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 2e^{6x} \neq 0$$

* Example 10

By substitution the function $y_p = \frac{-11}{12} - \frac{1}{2}x$ is readily shown

$$y''' - 6y'' + 11y' - 6y = 3x$$

$$y''' - 6y'' + 16y' - 6y = 0$$

The interval $(-\infty, \infty)$

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

$$y = y_c + y_p = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x$$

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* Example 11

$y_{p_1} = -4x^2$ is solution of

$y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{p_2} = e^{2x}$ is solution of

$y'' - 3y' + 4y = 2e^{2x}$

$y_{p_3} = xe^{x}$ is solution of

$y'' - 3y' + 4y = 2xe^{x} - e^{x}$

$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x}$

is a solution of

$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^{x} - e^{x}$

1

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* Exercise 4.1

*** Q2.** $C_1e^{4x} + C_2e^{-x}$, $(-\infty, \infty)$;

$y'' - 3y' - 4y = 0 \quad y(0) = 1, \; y'(0) = 2$

$y(0) = 1, \; y'(0) = 2$ — (i)

$1 = C_1e^0 + C_2e^0$

$C_1 + C_2 = 1$ —— (a)

$y' = 4C_1e^{4x} - C_2e^{-x}$ — (2)

$2 = 4C_1e^0 - C_2e^0$

$4C_1 - C_2 = 2$ (b)

$C_1 = 3/5$ & $C_2 = 2/5$

$y = 3/5\,e^{4x} - 2/5\,e^{-x}$ is the solution of the given initial condition.

*** Q3:** $y = C_1x + C_2x\ln x$, $(0, \infty)$;

$x^2y'' - xy' + y = 0, \quad y(1) = 3, \; y'(1) = -1$

$3 = C_1 \times (1) + C_2(1) \times \ln(1)$

$3 = C_1 + C_2 \times (0)$

$C_1 = 3$ (a)

$y' = C_1 + C_2\ln x + C_2x\,1/x$

$C_1 + C_2(\ln x + 1)$ (2)

$-1 = C_1 + C_2 \times (\ln(1) + 1)$

$C_1 + C_2 = -1$ (b)

$y = 3x - 4x\ln x$

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Q10: $y'' + (\tan x)y = e^x$, $y(0) = 1$, $y'(0) = 0$

y'' has no coefficient in front of it, thus by theorem, we are safe continue looking for discontinuation.

we find that $\tan(x)$ is continuous on the interval $(-\pi/2, \pi/2)$

Q19: $y = c_1x^2 + c_2x^4 + 3$, $x^2y'' - 5xy' + 8y = 24$

a) $y(-1) = 0$, $y(1) = 4$ b) $(y(0)) = 1$, $y(1) = 2$

c) $y(0) = 3$, $y(1) = 0$ d) $y(1) = 3$, $y(2) = 15$

(1) $y(-1) = 0 \rightarrow c_1 + c_2 + 3 = 0$

$y(1) = 4 \rightarrow c_1 + c_2 + 3 = 4$ possible

(2)

$y(0) = 1 \rightarrow 3 = 1$ possible

c) $y(0) = 3 \rightarrow 3 = 3$

$y(1) = 0 \rightarrow c_1 + c_2 + 3 = 0$

d) $y(1) = 3 \rightarrow c_1 + c_2 + 3 = 3$

$y(2) = 15 \rightarrow 4c_1 + 16c_2 + 3 = 15$

$c_1 = -1$, $c_2 = 1$

$y = -x^2 + x^4 + 3$

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Q: $f_1(x) = x,\ f_2(x) = x-1,\ f_3(x) = x+3.$

Firstly,

$$W(F_1(f_1, f_2, f_3)) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x & x-1 & x+3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$W(F_1(F_1, f_2, f_3)) = x \times \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} - (x-1) \times \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$+ (x+3) \times \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= x[(1)(0) - (0)(1)] - (x-1)[(1)(0) - 0(1)] + (x+3)[0-1]$$

$$= 0$$

Q21: $f_1(x) = 1+x,\ f_2(x) = x,\ f_3(x) = x^2$

Firstly,

$$W(F_1(f_1, f_2, f_3)) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} 1+x & x & x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$$= (1+x)[(1)(2) - (0)(2x)] - x[(1)(2) - 0(2x)]$$

$$+ x^2[(1)(0) - (0)(1)]$$

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$2 + 2x - 2x = 2$

★ Q30: $y^{(4)} + y'' = 0$; $1, x, \cos x, \sin x, (-\infty, \infty)$

1) $y = 1$, $y' = 0$, $y'' = 0$, $y''' = 0$, $y'''' = 0$

2) x, $y' = 1$, $y'' = 0$, $y''' = 0$, $y'''' = 0$

3) $y = \cos x$, $y' = -\sin x$, $y'' = -\cos x$, $y''' \sin x$

$y'''' = \cos x$

4) $y = \sin x$, $y' = \cos x$, $y'' = -\sin x$, $y''' = -\cos x$

$y'''' = \sin x$

★ Q31: $y'' - 7y' + 10y = 24e^x$;

$y = C_1 e^{2x} + C_2 e^{5x} + 6e^x$, $(-\infty, \infty)$

$y'' - 7y' + 10y = 24e^x$

$y = C_1 e^{2x} + C_2 e^{5x} + 6e^x$

For the first function $y_1 = e^{2x}$

$y_1' = 2e^{2x}$, $y_1'' = 4e^{2x}$

$4e^{2x} - 7 \times 2e^{2x} + 10 \times e^{2x} = 0$

$14e^{2x} - 14e^{2x} = 0$

$0 = 0$

For the second function

$y_2 = e^{5x}$:

$y_1' = 5e^{5x}$; $y_2'' = 25e^{5x}$

4.2

Example 1: Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.

if $y = u(x)y_1(x) = u(x)e^x$, then

$y' = ue^x + e^x u'$, $y'' = ue^x + 2e^x u'$

$y'' - y = e^x(u'' + 2u') = 0$

$y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2 e^x$

Example 2:

$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$.

$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4}\,dx$

$x^2 \int \frac{dx}{x} = x^2 \ln x$.

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$25e^{5x} - 7 \times 5e^{5x} + 10 \times e^{5x} = 0$

$0 = 0$

Then the two functions of the homo part satisfy the homogeneous D.E $y'' - 7y' + 10y = 0$

* ~~Given $f_1(x) = x$, $f_2(x) = x-1$, $f_3(x) = x+3$~~

* **Exercise 4.2**

Q: $y'' + 2y' + y = 0$, $y_1 = xe^{-x}$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$$

$$= xe^{-x} \int \frac{e^{-\int 2dx}}{(xe^{-x})} dx$$

$$= xe^{-x} \int \frac{1}{x^2} dx$$

$$= xe^{-x} \int x^{-2} dx$$

$$x = -e^{-x}$$

* **Q:** $9y'' - 12y' + 4y = 0$; $y_1 = e^{2x/3}$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$$

$$\cos 4x \int \frac{e^{-\int 0 dx}}{(\cos 4x)^2} dx$$

$$= \cos 4x \times x \times \frac{1}{4} \tan 4x$$

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$\frac{K}{4} \cos 4x \quad x \frac{\sin \alpha x}{\cos \alpha x}$

$= \sin 4x$

Q14: $x^2 y'' - 3xy' + 5y = 0; \; y_1 = x^2 \cos(\ln x)$

$y'' + P(x) y' + q(x) y = 0$

$y_2(x) = y_1(x) \int \frac{e^{-\int P(x) dx}}{[y_2(x)]^2} dx$

$= x^2 \cos(\ln x) \int \frac{e^{-\int \frac{3}{x} dx}}{e^2 \cos(\ln x)^2} dx$

$= x^2 \cos(\ln x) \int \frac{e^3 \ln x}{x^2 \cos(\ln x)^2} dx$

$= x^2 \cos(\ln x) \int \frac{1}{x \cos^2(\ln x)} dx$

Q15: $(1 - 2x - x^2) y'' + 2(1+x) y' - 2y = 0 \quad y_1 = x+1$

$y_2(x) = y_1(x) \int \frac{e^{-+(P(x)dx}dx}{(y_1)^2} dx$

$= \ln x \int \frac{e^{-\int \frac{1}{x} dx}}{(\ln x)^2} dx$

$= \ln x \int \frac{x^{-1}}{\ln^2 x} dx$

$= \ln x \int \frac{1}{x \ln^2 x} dx$

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* Q17: $y'' - 4y = 2$; $y_1 = e^{-2x}$

let $y(x) = u(x) y_1(x)$.

$y' = u' e^{-2x} - 2u e^{-2x}$, $y'' = u'' e^{-2x} - 4u' e^{-2x}$

$y'' - 4y = e^{-2x}(u'' - 4u') = 2 = 1$

$u'' - 4u' = 2e^{2x}$

$w' - 4w = 2e^{2x}$, $\mu(x) = e^{-4x}$

(4)

$$w(x) = u' = \frac{1}{e^{-4x}}\left(\int 2e^{-2x} dx + c\right)$$

$$= ce^{4x} - e^{2x}$$

$$= u(x) = k_2 e^{4x} - \frac{1}{2} e^{2x} + k_1$$

$y_2(x) = e^{2x}$, $y_p = -\frac{1}{2}$

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Example 1:

Second order DEs

a) $2y'' - 5y' - 3y = 0$

a) $2m^2 - 5m - 3 = (2m+1)(m-3) = 0$

$m_1 = -\frac{1}{2}, m_2 = 3$

$y = c_1 e^{-x/2} + c_2 e^{3x}$

b) $m^2 - 10m + 25 = (m-5)^2 = 0, \quad m_1 = m_2 =$

c) $m^2 + 4m + 7 = 0, \quad m_1 = -2 + \sqrt{3}i$

$m_2 = -2 - \sqrt{3}i$

Ex # 2: An initial-value Prob

solve $4y'' + 4y' + 17y = 0$

$y(0) = -1, \; y'(0) = 2$

$4m^2 + 4m + 17 = 0$

$m_1 = -1/2 + 2i$ and $m_2 = -\frac{1}{2} - 2i$

Differentiating

$y = e^{-x/2}(-\cos 2x + c_2 \sin 2x)$

gives $2c_1 + 1/2 = 2$

Example 3: Third order DE

$m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4)$

$= (m-1)(m+2)^2.$

order roots are $m_2 = m_3 = -2$

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* Example 2:

$$y = C_1 e^{ix} + C_2 e^{-ix} + C_3 x e^{ix} + C_4 x e^{-ix}$$

$$C_1 \cos x + C_2 \sin x$$

$$y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 \sin x$$

$$e^{(\alpha+i\beta)x}, \quad x e^{(\alpha+i\beta)x}, \quad x^2 e^{(\alpha+i\beta)x}$$

$$e^{\alpha x} \sin\beta x \quad e^{\alpha x} \sin\beta x \quad k\beta e^{-x}$$

It is not reasonable to expect students in this course to computer.

$$4.317 \frac{d^4 y}{dx^4} + 2.179 \frac{d^3 y}{dx^3} + 1.416 \frac{d^2 y}{dx^2}$$

$$y = C_1 e^{-0.7} \cos(0.618) + C_2 e^{-0.7} \sin(0.618)$$

S ②

S ①

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* Exercise 4.3

* Q7 $12y'' - 5y' - 2y = 0$

Assume that $y = e^{mx}$ is a solution for homogeneous differential equation.

$$y' = me^{mx}$$

$$y'' = m^2 e^{mx}$$

$$12m^2 e^{mx} - 5me^{mx} - 2e^{mx} = 0$$

$$e^{mx}(12m^2 - 5m - 2) = 0$$

Since e^{mx} can not be equal to zero

$$12m^2 - 5m - 2 = 0$$

$$(3m-2)(4m+1) = 0$$

Then the roots are

$$m_1 = \frac{2}{3}, \text{ and } m_2 = -1/4$$

which are real and distinct.

General differential equation $12y'' - 5y' - 2y = 0$ as

$$y = C_1 e^{2/3 x} + C_2 e^{-1/4 x}$$

* Question 23

* $y^{(4)} + y''' + y'' = 0$

Let's assume that $y = m$. Now

$m^4 + m^3 + m^2 = 0$

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$m^2(m^2+m+1)$

$m=0$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$m = \frac{-1 \pm \sqrt{1^2-4(1)(1)}}{2(1)}$

$m = -1/2 \pm i\frac{\sqrt{3}}{2}$

Therefore $\alpha = -1/2$ and $\beta = \sqrt{3}/2$

values for m, α and β.

$y = c_1e^{mx} + c_2xe^{mx} + c_3e^{\alpha x}\cos\beta x + c_4e^{\alpha x}\sin\beta x$

$y = c_1 + c_2x + c_3e^{-x/2}\cos\frac{\sqrt{3}}{2}x + c_4e^{-x/2}\sin\frac{\sqrt{3}}{2}x$

* Question 27.

* $\frac{d^5u}{dr^5} + 5\frac{d^4u}{dr^4} - 2\frac{d^3u}{dr^3} - 10\frac{d^2u}{dr^2} + \frac{du}{dr} + 5u = 0$

let us assume that:

$u = e^{\lambda x}$

$\frac{du}{dr} = \lambda e^{\lambda r} \rightarrow ①$

$\frac{d^2u}{dr^2} = \lambda^2 e^{\lambda r} \rightarrow ②$

$\frac{d^3u}{dr^3} = \lambda^3 e^{\lambda r} \rightarrow ③$

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$$\frac{d^4u}{dr^4} = \lambda^4 e^{\lambda r} \rightarrow ④$$

and $$\frac{d^5u}{dr^5} = \lambda^4 e^{\lambda r} \rightarrow ⑤$$

Substitute $u = e^{\lambda r}$ in equations 1, 2, 3, 4, and 5 into differential equation 1

$$\lambda^5 e^{\lambda r} + 5\lambda^4 e^{\lambda r} - 2\lambda^3 e^{\lambda r} - 10\lambda^2 e^{\lambda r} + \lambda e^{\lambda r} + 5e^{\lambda r} = 0$$

$$e^{\lambda r}(\lambda^5 + 5\lambda^4 - 2\lambda^3 - 10\lambda^2 + \lambda + 5) = 0$$

Since $e^{\lambda r}$ can not be equal to zero

$$\lambda^5 + 5\lambda^4 - 2\lambda^3 - 10\lambda^2 + \lambda + 5$$

$$(\lambda+5)(\lambda^4 - 2\lambda^2 + 1) = 0$$

using long devision on $(\lambda+5)$

$$(\lambda+5)(\lambda^2-1)^2 = 0$$

$$\lambda_1 = -5,\ \lambda_2 = 1,\ \lambda_3 = 1\ \ \lambda_4 = -1,\ \lambda_5 = -1$$

$$u = c_1 e^{-5r} + c_2 e^{r} + c_3 r e^{r} + c_4 e^{-r} + c_5 r e^{-r}$$

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* Question 30

* $\frac{d^2y}{d\theta^2} + y = 0;\ y(\frac{\pi}{3}) = 0;\ y'(\frac{\pi}{3}) = 2$

Second order diffirential eq.

$$\frac{d^2y}{d\theta^2} + y = 0$$

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with initial conditions:

$y(\pi/3)=0$, and $y'(\pi/3)=2$

let assume that $y=e^{\lambda\theta}$

$$y'=\lambda e^{\lambda\theta}$$

and $y''=\lambda^2 e^{\lambda\theta}$

Substitute with $y=e^{\lambda\theta}$

$$\lambda^2 e^{\lambda\theta}+e^{\lambda\theta}=0$$

$$e^{\lambda\theta}(\lambda^2+1)=0$$

Since $e^{\lambda\theta}$ cannot be equal to zero.

$$\lambda^2+1=0$$

$$\lambda^2=-1$$

Then roots are

$$\lambda_{1,2}=\sqrt{-1}=\pm i$$

$$y=c_1e^{ix}+c_2e^{-ix}$$

$$=c_1e^0(\cos x+\sin x)+c_2e^0(\cos x-\sin x)$$

$$=k_1\cos x+k_2\sin x$$

where $k_1=c_1+c_2$ and $k_2=c_1-c_2$ are arbitrary constants.

we have to apply with the Point

$$(\theta,y)=(\frac{\pi}{3},0)$$

$$0=k_1\cos(\pi/3)+k_2\sin(\pi/3)$$

$$\frac{1}{2}k_1+\frac{\sqrt{3}}{2}k_2$$

$$k_1+\sqrt{3}k_2=0$$

5 ②

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$y' = -k_1 \sin x + k_2 \cos x$

Then apply with the point $(o, y') = (\frac{\pi}{3}, 2)$ as

$2 = -k_1 \sin(\pi/3) + k_2 \cos(\pi/3)$

$-\sqrt{3}/2\, k_1 + 1/2\, k_2 = 2$

$-\sqrt{3} k_1 + k_2 = 4$

$k_1 = -\sqrt{3}$ and $k_2 = 1$

$y = -\sqrt{3} \cos x + \sin x$

* Question 36

* $y''' + 2y'' - 5y' - 6y = 0 \quad y(0) = y'(0) = 0, y''(0) = 1$

$y' = me^{mx} \rightarrow ①$

and $y'' = m^2 e^{mx} \rightarrow ②$

and $y''' = m^3 e^{mx} \rightarrow ③$

Substitute $y = e^{mx}$, equation ①, ② and ③ into differential equation.

$m^3 e^{mx} + 2m^2 e^{mx} - 5m e^{mx} - 6e^{mx} = 0$

$e^{mx}(m^3 + 2m^2 - 5m - 6) = 0$

Since e^{mx} cannot be equal to zero.

$m^3 + 2m^2 - 5m - 6 = 0$

$(m+1)(m^2 + m - 6) = 0$

$(m+1)(m-2)(m+3) = 0$

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$m_1 = -1,\ m_2 = 2,\ m_3 = -3$

$y = k_1 e^{-x} + k_2 e^{2x} + k_3 e^{-3x}$

k_1, k_2, k_3 are arbitrary constants.

$0 = k_1 e^0 + k_2 e^0 + k_3 e^0$

$k_1 + k_2 + k_3 = 0$

$y' = -k_1 e^{-x} + 2k_2 e^{2x} - 3k_3 e^{-3x}$

$0 = -k_1 e^0 + 2k_2 e^0 + 3k_3 e^0$

$-k_1 + 2k_2 - 3k_3 = 0$

$y'' = k_1 e^{-x} + 4k_2 e^0 + 9k_3 e^0$

$k_1 + 4k_2 + 9k_3 = 1$

$k_1 = -1/6\, e^{-x} + 1/15\, e^{2x} + 1/10\, e^{-3x}$

$y = -1/6\, e^{-x} + 1/15\, e^{+2x} + 1/10\, e^{-3x}$

* Question 37
* $y'' - 10y' + 25y = 0 \quad y(0) = 1,\ y(1) = 1$

Let us assume that $y = e^{mx}$

$y' = me^{mx} \rightarrow ①$

$y'' = m^2 e^{mx} \rightarrow ②$

Substitute $y = e^{mx}$ in equations ① and ② in orignal Functions.

$m^2e^{mx} - 10me^{mx} + 25e^{mx} = 0$

$e^{mx}(m^2 - 10m + 25) = 0$

Since e^{mx} cannot be equal to zero

$m^2 - 10m + 25 = 0$

$(m-5)^2 = 0$

$m_{1,2} = 5$

$y = e^{5x} - xe^{5x}$

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* Question - 41
* $y'' - 3y = 0 \quad y(0) = 1,\ y'(0) = 5$

$y'' - 3y = 0$, $m^2 - 3 = 0$, $m \pm \sqrt{3}$

$y = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}$

$1 = c_1 e^{\sqrt{3}(0)} + c_2 e^{-\sqrt{3}(0)}$

$1 = c_1 + c_2$

$y' = c_1\sqrt{3}e^{\sqrt{3}x} - c_2\sqrt{3}e^{-\sqrt{3}x}$

$5 = c_1\sqrt{3(0)} - c_2\sqrt{3(0)}e^{-\sqrt{3}(0)}$

$5 = c_1\sqrt{3} - c_2\sqrt{3}$

$1 = c_1 + c_2$, $c_1 = 1 - c_2$

$5 = c_1\sqrt{3} - c_2\sqrt{3} \Rightarrow \frac{5\sqrt{3}}{3} = c_1 - c_2$

$\Rightarrow \frac{5\sqrt{3}}{3} - 1 = -2c_2 \Rightarrow -\frac{5\sqrt{3}}{6} + \frac{1}{2} = c_2$

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$1 = c_1 + \frac{1}{2} - \frac{5\sqrt{3}}{6} \Rightarrow c_1 = \frac{1}{2} + \frac{5\sqrt{3}}{6}$

$\Rightarrow 1 = \frac{1}{2} + \frac{5\sqrt{3}}{6} + c_2 \Rightarrow \frac{1}{2} - \frac{5\sqrt{3}}{6}$

$y = \frac{1}{2}\left(1 + \frac{5\sqrt{3}}{3}\right)e^{\sqrt{3}x} + \frac{1}{2}\left(1 - \frac{5\sqrt{3}}{3}\right)e^{-\sqrt{3}x}$

$\because \cosh(0) = 1,\ \sinh(0) = 0$

$y = c_1 \cosh\sqrt{3}x + c_2 \sinh\sqrt{3}x$

$1 + c_1 \cosh(0) + c_2 \sinh(0) \Rightarrow c_1 = 1$

$y' = c_1\sqrt{3}\sinh\sqrt{3}x + c_2\sqrt{3}\cosh\sqrt{3}x$

$5 = c_1\sqrt{3}\sinh(0) + c_2\sqrt{3}\cosh(0)$

$5 = c_2\sqrt{5}$

$c_1 = 1,\ c_2 = \frac{5\sqrt{3}}{3}$

$y = \frac{1}{2}\left(1 + \frac{5\sqrt{3}}{3}\right)e^{\sqrt{3}x} + \frac{1}{2}\left(1 - \frac{5\sqrt{3}}{3}\right)e^{-\sqrt{3}x}$

$y = \cosh\sqrt{3}x + \frac{5\sqrt{3}}{3}\sinh\sqrt{3}x$

* Example 1

$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$.

$y_p = Ax^2 + Bx + C$.

y_p is a solution of (2)

where (2) is $y'' + 4y' - 2y = 2x^2 - 3x + 6$

$y_p' = 2Ax + B$ and $y_p'' = 2A$

$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx$
$- 2C = 2x^2 - 3x + 6$

that is $-2A = 2$, $8A - 2B = -3$,
$2A + 4B - 2C = 6$

Solving this system of equations leads to values $A = -1$,

$B = \frac{-5}{2}$

and $C = -9$

$y_p = -x^2 - \frac{5}{2}x - 9$.

* Example 2

find particular solution of $y'' - y' + y = 2\sin 3x$

$y_p = A\cos 3x + B\sin 3x$

Differentiating y_p and substituting the results into differential equation

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after regrouping

$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x +$

$(3A - 8B)\sin 3x = 2\sin 3x$

from the resulting system of equations

$-8A - 3B = 0, \quad 3A - 8B = 2$

$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$

*** Example 3**

$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$

$y_p = \frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$

$y = c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}$

substituting

$y_{p1} = Ax + B$ into $y'' - 2y' - 3y = 4x - 5$

$y_{p2} = Cxe^{2x} + Ee^{2x}$ into $y'' - 2y' - 3y = 6xe^{2x}$

$y_{p1} = -\frac{4}{3}x + \frac{23}{9}$

$y_{p2} = Cxe^{2x} + Ee^{2x}$

*** Example 4**

$y'' - 5y' + 4y = 8e^{x}$

$y_p = Axe^{x}$

$y_p'' - 5y_p' + 4y_p = -3Ae^{x} = 8e^{x}$

* Example 5

(a) $y'' - 8y' + 25y = 5x^3 e^{-x} - 7e^{-x}$

(b) $y'' + 4y = x\cos x$

$$y_p = (Ax^3 + Bx^2 + Cx + E)e^{-x}$$

$$y_c = e^{4x}(c_1\cos 3x + c_2\sin 3x)$$

$$y_p = (Ax + B)\cos x + (Cx + E)\sin x$$

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_n}$$

* Example 6

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}$$

$3x^2$ we assume $y_{p_1} = Ax^2 + Bx + C$

$-5\sin 2x$ we assume $y_{p_1} = E\cos 2x + F\sin 2x$

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}$$

No term in this assumption duplicates a term in $y_c = c_1e^{2x} + c_2e^{7x}$

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* Example 7

$y'' - 2y' + y = e^x \rightarrow (1)$

$y_p = Ax^2e^x$

substitute into equation (1)

$y_p = Ax^2e^x$ that is

$y_p = \frac{1}{2}x^2e^x$

$g(x)$ consists of m terms

$y_p = y_{p_1} + y_{p_2} + \dots y_{p_m}$

* Example 8

$y'' + y = 4x + 10\sin x,\ y(\pi) = 0,\ y'(\pi) = 2$

$y_p = Ax + B + C\cos x + E\sin x$

$y_p = Ax + B + Cx\cos x + Ex\sin x$

$y = y_c + y_p = c_1\cos x + c_2\sin x + 4x - 5x\cos x$

$y' = -9\pi\sin x + c_2\cos x + 4 + 5x\sin x - 5\cos x$

$y'(\pi) = -9\pi\sin\pi + c_2\cos\pi + 4 + 5\pi\sin\pi - 5\cos\pi = 2$

$y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x \rightarrow (A)$

we find $c_2 = 7$ and solution is therefore given in above equation (A).

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* Example 9

$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$

complementery function is

$y_c = c_1 e^{3x} + c_2 x e^{3x}$

$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{Ee^{3x}}_{y_{p_2}}$

$y_p = Ax^2 + Bx + C + Ex^2 e^{3x}$

It follows that $A = \frac{2}{3}$, $B = \frac{8}{9}$, $C = \frac{2}{3}$, $E = -6$.

general solution is

$y = y_c + y_p$ is $y = c_1 e^{3x} + c_2 x e^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x - 6x^2 e^{3x}$

* Example 10

$y''' + y'' = e^{x}\cos x$

From equation $m^3 + m^2 = 0$

we find $m_1 = m_2 = 0$ and $m_3 = -1$

complementery equation is

$y_c = c_1 + c_2 x + c_3 e^{-x}$

$y_p = Ae^{x}\cos x + Be^{x}\sin x$

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$y_p''' + y_p'' = (-2A+4B)e^{x}\cos x + (-4A-2B)e^{x}\sin x = e^{x}\cos x$

we get $-2A+4B=1$ and $-4A-2B=0$.

final equation becomes

$$y = y_c + y_p = c_1 + c_2 x + c_3 e^{-x} - \frac{1}{10} e^{x}\cos x + \frac{1}{5} e^{x}\sin x$$

*** Example 11**

$y^{(4)} + y''' = 1 - x^2 e^{-x}$

Comparing $y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$

with

$$y_p = \underbrace{A}_{y_{p_1}} + \underbrace{Bx^2 e^{-x} + Cxe^{-x} + Ee^{-x}}_{y_{p_2}}$$

Thus we see duplication between y_c and y_p are eliminated and particular solution is

$$y_p = Ax^3 + Bx^3 e^{-x} + Cx^2 e^{-x} + Exe^{-x}$$

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* Exercise 4.4

Solve the differential equation by undetermined coefficients.

*** Question 7**

* $y'' + 3y = -48x^2e^{3x}$

Firstly we find the general solution of the corresponding homogeneous eq

$$y'' + 3y = 0$$

Let $y = e^{mx}$

$$y' = me^{mx} \rightarrow ①$$

$$y'' = m^2e^{mx} \rightarrow ②$$

Substitute $y = e^{mx}$ eq ① and ② into the differential equation.

$$m^2e^{mx} + 3e^{mx} = 0$$

$$e^{mx}(m^2+3) = 0, \quad m^2 = -3$$

$$\lambda_1 = \sqrt{3}i, \quad \lambda_2 = -\sqrt{3}i$$

Corresponding homogeneous eq

$$y'' + 3y = 0$$

$$y_h = c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)$$

Now we have to find the solution of non homogeneous equation.

$$y_p = Ax^2e^{3x} + Bxe^{3x} + Ce^{3x} \text{ is}$$

the solution of non-homogeneous equation

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where $q(x) = -48x^2e^{3x}$

$y'p = 2Axe^{3x} + 3Ax^2e^{3x} + 3Bxe^{3x} + 3Ce^{3x}$

$= 3Ax^2e^{3x} + (2A+3B)xe^{3x} + (B+3C)e^{3x}$

and

$y''p = 6Axe^{3x} + 9Ax^2e^{3x} + (2A+3B)xe^{3x} + (B+3C)e^{3x}$

$A = -4$

and $12A + 12B = 0$ then $B = 4$

and $2A + 6B + 12C = 0$, $C = \frac{-4}{3}$.

$yp = -4x^2e^{3x} + 4xe^{3x} - \frac{4}{3}e^{3x}$

$y = yh + yp$

$= C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x) - 4x^2e^{3x} + 4xe^{3x} - \frac{4}{3}e^{3x}$

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* Question - 21

* $y''' - 6y'' = 3 - \cos x$

Let assume that $y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2e^{mx}$

$y''' = m^3e^{mx}$

$m^3e^{mx} - 6m^2e^{mx} = 0$

$e^{mx}(m^3 - 6m^2) = 0$

$m_{1,2} = 0$; $m_3 = 6$

$y_h = ne^0(k + l_x) + C_3e^{6x}$

$C_1 + C_2x + C_3e^{6x}$

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$y'p = -A\sin x + B\cos x + 2Cx$

$y''p = -A\cos x - B\sin x + 2C$

$y'''p = A\sin x - B\cos x$

$A = -6/37,\ B = 1/37,\ C = -1/4$

$yp = -6/37\cos x + 1/37\sin x - 1/4\, x^4$

$y = yh + yp$

$= C_1 + C_2 x + C_3 e^{6x} - 6/37\cos x + 1/37\sin x - 1/4\, x^2$

★ Question 32

★ $y'' - y = \cosh x,\ y(0) = 2,\ y'(0) = 12$

using hyperbolic - Trignometric identity

$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$y'' - y' = \dfrac{e^x + e^{-x}}{2} \rightarrow ①$

from auxilliary equation

$r^2 - 1 = 0,\ r = \pm 1$

$yh = C_1 e^x + C_2 e^{-x}$

$yp = Axe^x + Bxe^{-x}$

Substitute $y''p = Axe^x + 2Ae^x + Bxe^{-x}$

$- 2Be^{-x}$ and yp into ①

$A = 1/4,\ B = -1/4$

The general Solution.

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$y = \frac{1}{4}xe^{x} - \frac{1}{4}xe^{-x} + c_1e^{x} + c_2e^{-x} \rightarrow (2)$

using $y(0) = 2$, $y'(0) = 12$ in (2)

$c_1 + c_2 = 2$

$c_1 - c_2 = 12$ by ellimination.

$c_1 = 7$, $c_2 = -5$

The Solution $y = \frac{1}{2}x\left(\frac{e^{x} + e^{-x}}{2}\right) + 2e^{x} + 5e^{x} - 5e^{-x}$

$y = \frac{1}{2}x\sinh(x) + 2e^{x} + 10\sinh(x)$

$y = \frac{1}{2}x\sinh(x) + 10\sinh(x) + 2e^{x}$

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* $y'' + y = x^2 + 1$, $y(0) = 5$, $y(1) = 0$

* Question - 37

from the Auxiliary eq: $\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y_h = c_1\cos(x) + c_2\sin(x)$

$y_p = Ax^2 + Bx + C$

$y_p = Ax^2 + Bx + C$ and $y''_p = 2A$

into the d.e.

$2A + Ax^2 + Bx + C = x^2 + 1 \Rightarrow A = 1, B = 0, C = -1$

General solution

$y = x^2 - 1 + c_1\cos(x) + c_2\sin(x)$

Apply $y(0) = 5$, $c_1 - 1 = 5$, $c_1 = 6$

$y(1) = 0$, $6\cos(1) + c_2\sin(1) = 0$

$c_2 = 6\cos(1)$

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$y = 6\cos(x) - 6\cot(1)\sin(x) + x^2 - 1$

*** Q4.** $y'' + 4y = g(x), \quad y(0) = 1$

$y'(0) = 2$ where

$$g(x) = \begin{cases} \sin x & 0 \le x \le \pi/2 \\ 0 & , x > \pi/2 \end{cases}$$

$$y'' + 4y = \begin{cases} \sin(x) & , 0 \le x \le \pi/2 \\ 0 & , x > \pi/2 \end{cases}$$

$g(x) = \sin(x)$

$y_p = A\cos(x) + B\sin(x)$

$y'' = -A\cos(x) - B\sin(x).$

By substituetion, we get

$-A\cos(x) - B\sin(x) + 4A\cos(x) + 4B\sin(x) =$

$3A\cos(x) + 3B(\sin)(x) = \sin x$

$y = \cos(2x) + \frac{5}{6}\sin(2x) + \frac{1}{3}\sin(x)$

$y = \frac{2}{3}\cos(2x) + \frac{5}{6}\sin(2x)$

$$y(x) = \begin{cases} \cos(2x) + \frac{5}{6}\sin(2x) + \frac{1}{3}\sin(x), & 0 \le x \le \pi/2 \\ \frac{2}{3}\cos(2x) + \frac{5}{6}\sin(2x), & x > \pi/2 \end{cases}$$

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4.5

* Example 1

(a) $1-5x^2+8x^3$ (b) e^{-3x} (c) $4e^{2x}-10xe^{2x}$

From (a) we know that

$D^4 x^3 = 0$

$D^4(1-5x^2+8x^3)=0$

(b) with $\alpha=-3$ and $n=1$

$(D+3)e^{-3x}=0$

(c) From (5) and (6) with $\alpha=2$

and $n=2$

$(D-2)^3(4e^{2x}-10xe^{2x})=0$

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* Example 2

$5e^{-x}\cos 2x - 9e^{-x}\sin 2x$

when $\alpha=0$ and $n=1$

a special case of (7) is

$(D^2+\beta^2)\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}=0$

$7-x+6\sin 4x$.

K (constant) x^m, $x^m e^{\alpha x}$, $x^m e^{\alpha x}\cos\beta x$

and $x^m e^{\alpha x}\sin\beta x$

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* Example 3

$$y'' + 3y' + 2y = 4x^2$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$D^3(D^2 + 3D + 2)y = 0 \quad \to (10)$$

$$m^3(m^2 + 3m + 2) = 0 \quad \text{or}$$

$$m^3(m+1)(m+2) = 0$$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + c_5 e^{-2x} \to (11)$$

$$y_p = A + Bx + Cx^2 \quad (12)$$

$$y_p' = B + 2Cx, \quad y_p'' = 2C$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2$$

* Example 4

$$y'' - 3y' = 8e^{3x} + 4\sin x$$

auxiliary equation $y'' - 3y' = 0$

is $m^2 - 3m = m(m-3) = 0$

So $y_c = c_1 + c_2 e^{3x}$

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$(D-3)(D^2+1)(D^2-3D)y = 0$

auxiliary equation is

$(m-3)(m^2+1)(m^2-3m) = 0$

or

$m(m-3)^2(m^2+1) = 0$

Thus $y = c_1 + c_2 e^{3x} + c_3 x e^{3x} + c_4 \cos x + c_5 \sin x.$

$y_p = Axe^{3x} + B\cos x + C\sin x$

$y = c_1 + c_2 e^{3x} + \frac{8}{3} x e^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$

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* Example 5

$y'' + y = x\cos x - \cos x$

complementary function is

$y_c = c_1 \cos x + c_2 \sin x.$

we see $\alpha = 0$ and $n = 1$

$(D^2+1)^2 (D^2+1) y = 0$

$(D^2+1)^3 y = 0$

Since i and $-i$ are both complex roots of multiplicity 3

$y = c_1 \cos x + c_2 \sin x + \frac{1}{4} x \cos x - \frac{1}{2} x \sin x + \frac{1}{4} x^2 \sin x$

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* Example 6

$$y'' - 2y' + y = 10e^{-2x}\cos x$$

Complementery function of equation is

$$y_c = c_1 e^{x} + c_2 x e^{x}$$

now from ⊕ with $\alpha = -2, \beta = 1$ and $n = 1$

$$(D^2 + 4D + 5)e^{-2x}\cos x = 0$$

applying the operator $(D^2 + 4D + 5)$

$$e^{-2x}\cos x = 0$$

and finally it gives

$$y_p = Ae^{-2x}\cos x + Be^{-2x}\sin x$$

* Example 7

$$y''' - 4y'' + 4y' = 5x^2 - 6x + 4x^2 e^{2x} + 3e^{5x} \quad (19)$$

$D^3(5x^2 - 6x) = 0$, $(D-2)^3 x^2 e^{2x} = 0$

and $(D-5)e^{5x} = 0$

Therefore $D^3(D-2)^3(D-5)$ applied to (19) gives

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{2x} + c_6 x e^{2x} + c_7 x^2 e^{2x} + c_8 x^3 e^{2x} + c_9 x^4 e^{2x} + c_{10} e^{5x} \quad (20)$$

equation (20) gives

$$y_p = Ax + Bx^2 + Cx^3 + Ex^2 e^{2x} + Fx^3 e^{2x} + Gx^4 e^{2x} + He^{5x}$$

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* Exercise 4.5

*** Q7:** $y''' + 2y'' - 13y' + 10y = xe^{-x}$

$D^3y + 2D^2y - 13Dy + 10y = xe^{-x}$

$(D^3 + 2D^2 - 13D + 10)y = xe^{-x}$

$1^3 + 2(1)^2 - 13(1) + 10 = 0$ is true

$(D-1)(D^2 + 3D - 10)y = xe^{-x}$

$(D-1)(D+5)(D-2)y = xe^{-x}$

$(D-1)(D+5)(D-2)y = xe^{-x}$

*** Q13:** $(D-2)(D+5)$; $y = e^{2x} + 3e^{-5x}$

$(D-2)(D+5)y = (D-2)(D+5)(e^{2x} + 3e^{-5x})$

$= (D^2 + 5D - 2D - 10)(e^{2x} + 3e^{-5x})$

$= (D^2 + 3D - 10)(e^{2x} + 3e^{-5x})$

$= D^2((e^{2x}) + 3e^{-5x}) + 3D(e^{2x}) + 3D(e^{-5x}) - 10(e^{2x} + 3e^{-5x})$

$= D^2(e^{2x}) + 3D^2e^{-5x} + 3D(e^{2x}) + 9D(e^{-5x}) - 10(e^{2x} + 3e^{-5x})$

$= 2D(e^{2x}) - 15De^{-5x} + 6e^{2x} - 45e^{-5x} - 10e^{2x} - 30e^{-5x}$

$= (4 + 6 - 10)e^{2x} + (75 - 45 - 30)e^{-5x}$

$= 0 + 0$

$= 0$

The differential operator $(D-2)(D+5)$ annihilates the indicated function $y = e^{2x} + 3e^{-5x}$

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Q23: $e^{-x}+2xe^{x}-x^2e^{x}$

$e^{-x}+(2x-x^2)e^{x}$

e^{-x} is annihiliated by linear differential operator $(D+1)$, where $\alpha=-1$

$(D+1)(D-1)^3(e^{-x}+2xe^{x}-x^2e^{x})=0$

The used linear differential operator is $(D+1)(D-1)^3$.

Q29: $(D-6)(2D+3)$

The linear differential operator $(D-6)(2D+3)$ is in term $(D-\alpha)(D+\beta)$

$e^{(-6)x}$ and $e^{-\frac{3}{2}x}$

which is

e^{6x} and $e^{-\frac{3}{2}x}$

Q63: $y^{(4)}-2y'''+y''=e^{x}+1$

$y^{(4)}-2y'''+y''=0$

Differentiating w.r.t x,

$y'=me^{mx}$

$y''=m^2e^{mx}$ ②

Eq $y'''=m^3e^{mx}$ ③

Eq $y^{(4)}=m^4e^{mx}$ ④

2

$m_{1,2}=0$ and $m_{3,4}=1$

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$y_h = k_1 + k_2 x + k_3 e^x + k_4 x e^x$

$D^4 y - 2D^3 y + D^2 y = e^x + 1$

$D(D-1)(D^4 - 2D^3 + D^2)y = D(D-1)(e^x + 1)$

$D(D-1)(D^4 - 2D^3 + D^2)y = 0$

$m(m-1)(m^4 - 2m^3 + m^2)^3 = 0$

$\cancel{m^2(m-1)^3 m}$

$m^2(m-1)^2 m(m+1) = 0$

Roots are

$m_{1,2} = 0,\ m_{3,4} = 1,\ m_5 = 1$ & $m_6 = 0$

3

$y = k_1 + k_2 x + k_3 e^x + k_4 x e^x + k_5 x^2 e^x + k_6 x^2$

$y'_p = A x^2 e^x + 2A x e^x + 2Bx$

$y''_p = A x^2 e^x + 4A x e^x + 2A e^x + 2B$ (5)

$y'''_p = A x^2 e^x + 6A x e^x + 6A e^x$ (6)

$y_p^{(4)} = A x^2 e^x + 8A x e^x + 12A e^x$ (7)

$2A = 1$ then: $A = \frac{1}{2}$

and $2B = 1$ then: $B = \frac{1}{2}$

4

$y_p = \frac{1}{2} x^2 e^x + \frac{1}{2} x^2$

$k_5 = \frac{1}{2}$ and $k_6 = \frac{1}{2}$

$y = k_1 + k_2 x + k_3 e^x + k_4 x e^x + \frac{1}{2} x^2 e^x + \frac{1}{2} x^2$

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Q 67: $y'' - 5y' = x - 2,\ y(0) = 0\ y'(0) = 2$

It's 2nd ordered diffential equation

$y'' - 5y' = x - 2$

and initial conditions are

$y(0) = 0$ & $y'(0) = 2$

$y'' - 5y' = 0$

$y' = me^{mx} \to ②$

$y'' = m^2 e^{mx} \to ③$

$m^2 - 5m = 0$

$m(m-5) = 0$

$m^2(m^2 - 5m) = 0$

$(m^{\cancel{2}} - 5)m^2 \cdot m = 0$

$y'_p = 2Ax + B$

$(2A) - 5 \times (2Ax + B) = x - 2$

$y_p = -\frac{1}{10}x^2 + \frac{9}{25}x$

$k_3 = \frac{9}{25}$ and $k_4 = \frac{1}{10}$

$y = k_1 e^{5x} + k_2 - \frac{1}{10}x^2 + \frac{9}{25}x$

$0 = k_1 e^0 + k_2 + 0$

$k_1 + k_2 = 0$

$2 = 5k_1 e^0 + 0 + \frac{9}{25}$

$5k_1 = \frac{41}{25}$

[redacted]

4.6

* Example 1

$y'' - 4y' + 4y = (x+1)e^{2x}$

From auxilliary equation

$m^2 - 4m + 4 = (m-2)^2 = 0$ we have

$y_c = c_1 e^{2x} + c_2 x e^{2x}$ with

$y_1 = e^{2x}$ and $y_2 = x e^{2x}$

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$$

$$u_1' = \frac{-(x+1)xe^{4x}}{e^{4x}} = -x^2 - x,$$

$$u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x+1$$

It follows u_1 and u_2

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$
$$= \frac{1}{6}x^3 e^{2x} + \frac{1}{2}x^2 e^{2x}$$

$$y = y_c + y_p = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{6}x^3 e^{2x} + \frac{1}{2}x^2 e^{2x}$$

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* Example 2

$4y'' + 36y = \csc 3x$

we put equation by dividing 4.

$y'' + 9y = \frac{1}{4}\csc 3x$

$f(x) = \frac{1}{4}\csc 3x$, we obtain

$W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$

$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}$

$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$

$u_1' = \frac{W_1}{W} = -\frac{1}{12}$

$u_2' = \frac{W_2}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$

general solution of equation is

$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|.$

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* Example 3

$y'' - y = \frac{1}{x}$

The auxiliary equation $m^2 - 1 = 0$ yields $m_1 = -1$ and $m_2 = 1$

$y_c = c_1 e^{x} + c_2 e^{-x}$.

Now $W(e^x, e^{-x}) = -2$ and

$$u_1' = \frac{e^{-x}(1/x)}{2}$$

$$u_1 = \frac{1}{2}\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt$$

$$u_2' = \frac{e^{x}(1/x)}{-2},$$

$$u_2 = -\frac{1}{2}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt$$

$$y_p = \frac{1}{2}e^{x}\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt$$

and so solution is

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2}e^{x}\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt$$

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$k_1 = \frac{41}{125}$ and $k_2 = \frac{-41}{125}$

$y = \frac{-41}{125} + \frac{41}{125} e^{-x} - \frac{1}{10}x^2 + \frac{9}{25}x$

* **Exercise 4.6**

* **Q15:** $y'' + 2y' + y = e^{-t} \ln t$

The characteristic equation is

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r_1 = r_2 = 1$

$y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$

$W(t) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t}$

$u_1(t) = -\int \frac{y_2(t) \cdot g(t)}{W(t)} dt = -\int \frac{(te^{-t})(e^{-t} \ln t)}{e^{-2t}} dt$

$= -\int t \ln t \, dt$

$= -\frac{t^2 \ln t}{2} + \frac{t^2}{4}$

$u_2(t) = \int \frac{y_1(t) \cdot g(t)}{W(t)} dt = \int \frac{e^{-t} \cdot e^{-t} \ln t}{e^{-2t}} dt$

$= \int \ln t \, dt$

$= t \ln t - t$

$y_p(t) = \left(-\frac{t^2 \ln t}{2} + \frac{t^2}{4}\right) \cdot e^{-t} + (t \ln t - t) \cdot (te^{-t})$

$y_p(t) = \left(\frac{t^2 \ln t}{2} - \frac{3t^2}{4}\right) \cdot e^{-t}$

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$$y(t) = c_1 e^{-t} + c_2 t e^{-t} + \left(\frac{t^2 \ln t}{2} - \frac{3t^2}{4}\right) \cdot (e^{-t})$$

Q18. $4y'' - 4y' + y = e^{x/2}\sqrt{1-x^2}$

$$y'' - y' + \frac{1}{4}y = \frac{e^{\frac{1}{2}x}\sqrt{1-x^2}}{4}$$

$y'' + p(x)y' + q(x)\,y = f(x)$

$y'' - y' + \frac{1}{4}y = 0$

$y' = me^{mx}$

and $y'' = m^2 e^{mx}$

$e^{mx}(m^2 - m + \frac{1}{4}) = 0$

$(m - \frac{1}{2})^2 = 0$

$m_{1,2} = \frac{1}{2}$

$y_h = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{\frac{1}{2}x} & x e^{\frac{1}{2}x} \\ \frac{1}{2}e^{\frac{1}{2}x} & \frac{1}{2}x e^{\frac{1}{2}x} + e^{\frac{1}{2}x} \end{vmatrix}$$

$$= \frac{1}{2}xe^x + e^x - \frac{1}{2}xe^x$$

$$= e^x$$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$= \frac{-xe^x\sqrt{1-x^2}}{4}$$

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

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$$= \frac{e^x \sqrt{1-x^2}}{4}$$

$$u_1' = \frac{W_1}{W(y_1, y_2)}$$

$$= \frac{-x\sqrt{1-x^2}}{4}$$

$$u_2' = \frac{W_2}{W(y_1, y_2)}$$

$$= \frac{\sqrt{1-x^2}}{4}$$

$$u_1 = \int u_1' \, dx$$

$$= \int \frac{-x\sqrt{1-x^2}}{4} \, dx$$

$$= \frac{1}{8}\int -2x\sqrt{1-x^2} \, dx$$

$$u_1 = \frac{1}{8}\int \sqrt{v} \, dv$$

$$= \frac{1}{8} \times \cancel{\frac{8}{4}} \frac{2}{3} v^{\frac{3}{2}}$$

$$= \frac{1}{12}(1-x^2)^{\frac{3}{2}}$$

$$= \frac{\sqrt{(1-x^2)^3}}{12}$$

$$u_2 = \int u_2' \, dx$$

$$= \frac{1}{8}(y + \sin y \cos y)$$

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$$u_2 = \frac{1}{8}\left(\sin^{-1} x + x\sqrt{1-x^2}\right)$$

$y_p = u_1 y_1 + u_2 y_2$

$y = y_h + y_p$

$y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x} + \frac{1}{12} e^{\frac{1}{2}x}\sqrt{(1-x^2)^3} + \frac{1}{8} x$
$e^{\frac{1}{2}x}\left(\sin^{-1} x + x\sqrt{1-x^2}\right)$

*** Q22:** $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

$y'' + P(x)y' + q(x)y = f(x)$

$y(0) = 1$ and $y'(0) = 0$

$y'' - 4y' + 4y = 0$ ①

$y' = me^{mx}$ ②

$y'' = m^2 e^{mx}$ ③

$c_1 = 1$ and $c_2 = -2$

$y = e^{2x} - 2x e^{2x} + x^4 e^{2x} - x^3 e^{2x}$

***Q23:** $x^2 y'' + xy' + (x^2 - \frac{1}{4})y = x^{3/2}$;
$y_1 = x^{-1/2}\cos x$, $y_2 = x^{-1/2}\sin x$

$y'' + \frac{1}{x}y' + (1 - \frac{1}{4}x^{-2}) = x^{-\frac{1}{2}}$

$y'' + P(x)y' + q(x)y = f(x)$

$y_1 = x^{-\frac{1}{2}}\cos x$ and $y_2 = x^{-\frac{1}{2}}\sin x$

$y_h = c_1 y_1 + c_2 y_2$
$= c_1 x^{-\frac{1}{2}}\cos x + c_2 x^{-\frac{1}{2}}\sin x$

$y = y_h = y_p$

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4.7

* Example 1

Solve $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$

Differentiate twice

$$\frac{dy}{dx} = mx^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

and substitute back into differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 . m(m-1)x^{m-2} - 2x . mx^{m-1} - 4x^m$$

$$= x^m(m(m-1) - 2m - 4)$$

$$= x^m(m^2 - 3m - 4) = 0$$

If $m^2 - 3m - 4 = 0$.

Now $(m+1)(m-4) = 0$ implies

$m_1 = -1$, $m_2 = 4$ so

$$y = c_1 x^{-1} + c_2 x^4$$

* Example 2

Solve $4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0$

The substitution $y = x^m$ yields

$$4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = x^m(4m(m-1) + 8m + 1)$$

$$= x^m(4m^2 + 4m + 1) = 0$$

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when $4m^2+4m+1=0$ or

$(2m+1)^2=0$ since $m_1=-\frac{1}{2}$

$y=c_1x^{-1/2}+c_2x^{-1/2}\ln x$

For higher order equations, if m_1 is a root of multiplicity k, then it can be shown

$x^{m_1}, x^{m_1}\ln x, x^{m_1}(\ln x)^2 \dots x^{m_1}(\ln x)^{k-1}$

*** Example 3**

Solve $4x^2y''+17y=0$, $y(1)=-1$

$y'(1)=-\frac{1}{2}$

Substitute $y=x^m$ yields

$4x^2y''+17y=x^m(4m(m-1)+17)$

$=x^m(4m^2-4m+17)=0$

$y=x^{1/2}[c_1\cos(2\ln x)+c_2\sin(2\ln x)]$

By applying the initial condition

$y(1)=-1$, $y'(1)=\frac{1}{2}$

$c_1=-1$ and $c_2=0$

Hence solution is $y=x^{1/2}\cos(2\ln x)$.

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* Example 4

solve $x^3\frac{d^3y}{dx^3}+5x^2\frac{d^2y}{dx^2}+7x\frac{dy}{dx}+8y=0$

The three derivatives of $y=x^m$ are

$\frac{dy}{dx}=mx^{m-1};\ \frac{d^2y}{dx^2}=m(m-1)x^{m-2}$

$\frac{d^3y}{dx^3}=m(m-1)(m-2)x^{m-3}$

So equation becomes

$x^3\frac{d^3y}{dx^3}+5x^2\frac{d^2y}{dx^2}+7x\frac{dy}{dx}+8y=$

$x^3m(m-1)(m-2)x^{m-3}+5x^2m(m-1)x^{m-2}$

$+\ 7xmx^{m-1}+8x^m$

$=\ x^m(m^3+2m^2+4m+8)$

$=\ x^m(m+2)(m^2+4)=0$

In this case we see that

$y=x^m$ will be solution

if $m_1=-2,\ m_2=2i$ and $m_3=-2i$

Hence general solution

of differential equation

becomes

$y=c_1x^{-2}+c_2\cos(2\ln x)+c_3\sin(2\ln x)$

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* Example 5

Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$

From auxiliary equation $(m-1)(m-3) = 0$. we find $y_c = c_1x + c_2x^3$.

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3,$$

$$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x$$

we find

$$u_1' = \frac{2x^5e^x}{2x^3} = -x^2e^x$$

$$u_2' = \frac{2x^3e^x}{2x^3} = e^x$$

finally we get general solution as

$$y = y_c + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$$

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* Example 6

Solve $x^2y'' - xy' + y = \ln x$

with the substitution $x = e^t$ or $t = \ln x$ it follows that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$ → chain rule

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{1}{x}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Substituting in given differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t$$

$$y = c_1 e^t + c_2 t e^t + 2 + t$$

So the general solution of equation is

$$y = c_1 x + c_2 x \ln x + 2 + \ln x$$

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$y = c_1 x^{-\frac{1}{2}}\cos x + c_2 x^{-\frac{1}{2}}\sin x + x^{-\frac{1}{2}}$

*** Q 26:** $y''' + 4y' = \sec 2x$

$y''' + g(x)y'' + P(x)y' = q(x)y = f(x)$

$y''' + 4y' = 0$ ①

$y' = me^{mx}$ ②

$y'' = m^2 e^{mx}$

$y''' = m^3 e^{mx}$ ③

$y_h = c_1 + c_2\cos 2x + c_3 \sin 2x$

$y = y_h + y_p$

$y = c_1 + c_2\cos 2x + c_3 \sin 2x + \frac{1}{8}\ln(\sec 2x + \tan 2x) - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x \ln(\cos 2x)$

*** Exercise 4.7**

*** Q16:** $x^3 y''' + xy' - y = 0$

$y = y^m$

$y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$y''' = m(m-1)(m-2)x^{m-3}$

so the equation becomes

$x^3 m(m-1)(m-2)x^{m-3} + xmx^{m-1} - x^m = 0$

$x^m(m^3 - 3m^2 + 3m - 1) = 0$

$m^3 - 3m^2 + 3m - 1 = 0$

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$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0$$

$$m = 1, 1, 1$$

$$y = c_1 x + c_2(\ln x)x + c_3(\ln x)^2 x$$

*** Q17:** $xy^{(4)} + 6y''' = 0$

$$xy^{(4)} + 6y''' = 0$$

$$x^4 y^4 + 6x^3 y''' = 0$$

$$y' = m x^{(m-1)}$$

$$y^n = m(m-1)x^{(m-2)}$$

$$y^m = m(m-1)(m-2)\, x^{(m-3)} \quad (2)$$

$$y^{(4)} = m(m-1)(m-2)(m-3)\, x^{(m-4)} \quad (3)$$

x^m can't be equal 0, then

$$m(m-1)(m-2)(m+3) = 0$$

$$m_1 = 0,\ m_2 = 1,\ m_3 = 3,\ m_4 = -3$$

$$y_h = c_1 x^0 + c_2 x^1 + c_3 x^2 + c_4 x^{-3}$$

$$= c_1 + c_2 x + c_3 x^2 + c_4 x^{-3}$$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^{-3}$$

*** Q21:** $x^2 y'' - xy' + y = 2x$

$$x^2 y'' - xy' + y = 2x$$

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = \frac{2}{x}$$

$$y'' + p(x) y' + q(x) y = f(x)$$

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$x^2y'' - xy' + y = 0$

$y' = mx^{(m-1)}$ ②

$y'' = m(m-1)x^{(m-2)}$ ③

$(m-1)^2 = 0$

$m_{1,2} = 1$

$y_h = c_1x + c_2x\ln x$

$y = y_h + y_p$

$y = c_1x + c_2x\ln x + x(\ln x)^2$

Q29: $xy'' + y' = x, \quad y(1) = 1, \quad y'(1) = \frac{-1}{2}$

$y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

Substituting in equation

$m(m-1)x^{m-1} + mx^{m-1} = 0 \quad \div \quad x^{m-1}$

$m(m-1) + m = 0$

$m^2 = 0$

$m = 0, 0$

$y_1 = 1, \quad y_2 = \ln x$

$y = \frac{3}{4} - \ln x + \frac{1}{4}x^2$

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Q35. $x^2y'' - 3xy' + 13y = 4 + 3x$

According to Euler equation

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 13y = 4 + 3x$$

$x = e^t$

$t = \ln x$

$dt = \frac{1}{x} dx$

$\frac{dt}{dx} = \frac{1}{x}$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$y_p = \frac{3}{10}e^t + \frac{4}{13}$

$y(x) = y_c + y_p$

$y(x) = K_1x^2\cos(6\ln x) + K_2x^2\sin(6\ln x) +$

$\frac{3}{10}x + \frac{4}{13}$

Q 37: $4x^2y'' + y = 0, \; y(-1) = 2, \; y'(-1) = 4$

Let $x = -t$, $dx = -dt \Rightarrow 4t^2y'' + y = 0$

$\therefore y(1) = 2, \; y'(2) = 4$

$4m^2 - 4m + 1 = 0 \Rightarrow m = \frac{1}{2}$

$\therefore y_1 = c_1t^{\frac{1}{2}} + c_2t^{\frac{1}{2}}\ln t$

$y(1) = 2 \Rightarrow c_1 = 2$

$y'(1) = 4 \Rightarrow \frac{2}{2t^{\frac{1}{2}}} + c_2\left[\frac{7}{2t^{\frac{1}{2}}} + \frac{1}{t^{\frac{1}{2}}}\right] = 4$

$y = 2t^{\frac{1}{2}} + 2t^{\frac{1}{2}}\ln t$

Chapter 11.

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Exercise 11.1

*** Example 1**

The function $f_1(x) = x^2$ and $f_2(x) = x^3$ are orthogonal on the interval $[-1,1]$ since.

$$(f_1, f_2) = \int_{-1}^{1} f_1(x) f_2(x)\, dx = \int_{-1}^{1} x^2 \cdot x^3\, dx = 0$$

*** Exemple 2**

Show that the set $\{1, \cos x, \cos 2x, \ldots\}$ is orthogonal on the interval $\{-\pi, \pi\}$

$\phi_0(x) = 1 \qquad \phi_n(x) = \cos nx$

$$\int_{-\pi}^{\pi} \phi_0(x)\, \phi_n(x)\, dx = 0 , \quad n \neq 0$$

$$\int_{-\pi}^{\pi} \cos nx\, dx = \frac{1}{n} \sin nx \Big|_{-\pi}^{\pi} = 0$$

Hen $n \neq 0$

$$(\phi_m, \phi_n) = \int_{-\pi}^{\pi} \phi_m(x)\, \phi_n(x)\, dx$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx$$

$$\frac{1}{2}\left[\frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n}\right] = 0$$

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Example 3

$\phi_0(x) = 1$

$\|\phi_0(x)\|^2 = \int_{-\pi}^{\pi} dx = 2\pi$

$|\phi_n(x)|^2 = \int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi}(1+\cos 2nx)dx$

$n > 0, \ \|\phi_n(x)\| = \sqrt{\pi}$

Hence, it can be normalized for set $\{1, 2, 3, \dots\}$ by deviding each function by its norm.

Example 4

$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots\right\}$

$u = c_1 v_1 + c_2 v_2 + c_3 v_3$

$c_1 = \frac{(u \cdot v_1)}{\|v_1^2\|}, \quad c_2 = \frac{u \cdot v_2}{|v_2|}$

$c_3 = \frac{(u \cdot v_3)}{|v_3|}$

$\sum_{n=1}^{3} \left(\frac{u \cdot v_n}{|v_n|}\right) v_n$

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Exercise 11.1

Q 3. $f_1(x) = e^{x}$

$f_2(x) = xe^{-x} - e^{-x}$, $[0, 2]$

$\int_0^2 e^{x}(xe^{-x} - e^{-x})\,dx$

$\int_0^2 (xe^{-x+x} - e^{0})\,dx = \int_0^2 (-1+x)\,dx$

$\left[\frac{1}{2}x^2 - x\right]_0^2 = \left(\frac{4}{2} - 2\right) - (0-0)$

$= (2-2) - (0)$

$= 0$

Hence, orthogonal.

Q 5. $f_1(x) = x$

$f_2(x) = \cos 2x$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$(f_1, f_2) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_1(x)\, f_2(x)\,dx = 0$

To Show

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos 2x\,dx$

Since, $f(x)$ is an odd function

$f(-x) = (-x)\cos(-2x)$

$-x\cos 2x = -f(x)$

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$$(f_1, f_2) = \int_0^{\frac{\pi}{2}} x \cos 2x \, dx = 0$$

Hence, it is orthogonal function.

Q 7 :-

$\{\sin x, \sin 3x, \sin 5x \dots\} \; [0, \frac{\pi}{2}]$

$$(\phi_n, \phi_m) = \int_a^b \phi_n(x) \phi_m(x) \, dx = 0$$

$\phi_n(x) = (\sin 2x + 1) \qquad n = 1, 2 \dots n$

$$\int_0^{\frac{\pi}{2}} \sin(2x+1)x \sin(2m+1)x \, dx \quad m \neq n$$

using Trig function's identity

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$(\phi_m, \phi_n) = \frac{1}{2} \int_0^{\frac{\pi}{2}} (\cos(2n-2m)x -$$

$$\cos(2n+2m+2)x \, dx$$

$$= \phi \; |\phi_m(x)| = \sqrt{\int_0^{\frac{\pi}{2}} \sin^2(2x+1)x \, dx}$$

$$= \sqrt{\frac{\pi}{2}}$$

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Q 12.

$\left\{1, \cos\frac{n\pi}{p}x, \sin\frac{n\pi}{p}x\right\}$

$n = 1, 2, 3 \ldots$

$m = 1, 2, 3 \ldots \quad [-p, p]$

To Shown

$\int_{-p}^{p} \phi_n(x)\,\phi_m(x)\,dx = 0$

whenever $m \neq n$

$\phi_0(x) = 1$

$\phi_n(x) = \cos\frac{n\pi}{p}x$

$\phi_m(x) = \cos\frac{m\pi}{p}x$

$(1, \phi_n) = \int_{-p}^{p} \cos\frac{n\pi}{p}x\,dx$

$= \left[\frac{p}{n\pi}\sin\frac{n\pi x}{p}\right]_{-p}^{p} = \frac{p\sin n\pi}{n\pi} - \frac{p\sin(-n\pi)}{n\pi}$

$0 - 0 = 0$

$\|\phi_n(x)\| = \sqrt{\int_{-p}^{p}\phi_n^2 x\,dx}$

$= \sqrt{\int_{-p}^{p}dx} = \sqrt{[x]_{-p}^{p}}$

$= \sqrt{2p}$

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$\Phi_n(x) = \cos \frac{n\pi}{p} x$

$= \int_{-p}^{p} \cos^2 x \frac{\pi n}{p} dx$

$= \left| \frac{1}{2}\left(x + \frac{p \sin 2x n\pi}{2n\pi} \right) \right|_{-p}^{p}$

$= \frac{1}{2}\left(p + \frac{p \sin 2n\pi}{2n\pi} \right) - \frac{1}{2}\left(-p + \frac{p \sin -2n\pi}{2n\pi} \right)$

$= p$

= Taking Square root

$= \sqrt{p}$

$||1|| = p$ and $\cos \frac{n\pi}{p} = \sqrt{p}$
and finally $\left| \sin \frac{n\pi}{p} \right| = \sqrt{p}$
all are same magnitude.

Q 13.

$H_0(x) = 1 \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2$
$(-\infty, \infty) \quad w(x) = e^{-x^2}$

$\int_{-\infty}^{\infty} w(x) \cdot H_0(x), H_1(x) \cdot H_2(x) \, dx = 0$
To Show.

$1. \int_{-\infty}^{\infty} e^{-x^2} 2x \, dx = \int_{-\infty}^{\infty} 2x e^{-x^2}$

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$$\int_{-\infty}^{\infty} 2x e^{-x^2} = \lim_{a\to-\infty}\int_a^0 -2x e^{-x^2}\,dx$$

$$+ \lim_{b\to\infty}\int_0^b -2x e^{-x^2}\,dx$$

$$\lim_{a\to\infty}\left[e^{-x^2}\right]_a^0 + \lim_{b\to\infty}\left[e^{-x^2}\right]_{x=0}^{x=b}$$

$$\left(-1 - e^{-a^2}\right) + \left(e^{-b^2} - 1\right)$$

$\lim_{a\to\infty}$ and $\lim_{b\to\infty}$ $= 0$

Q 18. $f_1(x) = x$, $f_2(x) = x^2$ are orthogonal on int $[-2, 2]$

find c_1, c_2, ~~c_3~~

$f_3(x) = x + c_1 x^2 + c_2 x^3$ is orthogonal to both f_1, f_2.

$$\int_{-2}^{2} f_1(x)\left[f_1(x) + c_1 f_2(x) + c_2 x^3\right] dx$$

$$= \int_{-2}^{2} f_1(x)\, f_1(x)\,dx + \int_{-2}^{2} f(x)\, f_2(x)\,dx$$

$$+ \int_{-2}^{2} f_1(x)\, x^3\,dx$$

given $c_1 \int_{-2}^{2} f_1(x)\, f_2(x)\,dx = 0$

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$$(f_1, f_3) = \int_{-2}^{2} f_1(x)\, f_1(x)\, dx + C_2 \int_{-2}^{2} f_1(x)\, x^3\, dx$$

$$= \left[\frac{x^3}{3}\right]_{-2}^{2} + C_2 \left[\frac{x^5}{5}\right]_{x=-2}^{x=2}$$

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* **Example 1:**

Expand $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)dx\right]$$

$$= \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)dx\right]$$

$$= -\frac{1}{n\pi}\left.\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{-\cos n\pi + 1}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx = \frac{1}{n}$$

Therefore $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\}$

---

* **Example 2:** The function in example 1 satisfies the conditions of therom 11.1.

Thus for every x in the interval $(-\pi, \pi)$ except at $x=0$ the series will covereqe to $f(x)$.

At $x=0$ the function is

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and so the series will converge to

$$\frac{f(0+) + F(0-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$

---

* **Example 3:** The Fourier series converges to the periodic extension of on the entire x-axis. The solid dots represents the value

$$\frac{F(0+) + F(0-)}{2} = \frac{\pi}{2}$$

at $0, \pm 2\pi, \pm 4\pi \ldots$

$$\frac{F(\pi-) + F(-\pi+)}{2} = 0$$

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* **Exercise 11.2**

* **Q7:** $f(x) = x+\pi, \quad -\pi < x < \pi$

we apply definitions and find the co-efficients as follows

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + x\pi\right]_{x=-\pi}^{x=\pi}$$

$$= \frac{\pi}{2} + \pi - \frac{\pi}{2} + \pi$$

$$= 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos\frac{n\pi x}{\pi}\,dx$$

From definition of the function, it converges, at the end points of interval, to

$$\frac{f(\pi^-) + f(-\pi^+)}{2} = \frac{(\pi+\pi)+(\pi-\pi)}{2}$$

$$\pi$$

$$f(x) = \pi + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n}\sin nx\right]$$

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* Q 9. $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx$$

$$= \frac{1}{\pi}[1+1]$$

$$= \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos \frac{n\pi x}{\pi}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x\,dx$$

$$\frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_{x=0}^{x=\pi}$$

$$f(x) = \frac{1}{x} + \frac{\sin x}{2} + \sum_{n=2}^{\infty}\left(\frac{(-1)^n+1}{\pi(1-n^2)}\cos nx\right)$$

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*Q 11. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$

$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$

$= \frac{1}{2}\left[\int_{-2}^{-1} 0\,dx + \int_{-1}^{0} -2\,dx + \int_{0}^{1} dx + \int_{1}^{2} 0\,dx\right]$

$= -1 + \frac{1}{2} = -\frac{1}{2}$

$a_n = \frac{1}{2}\left(-2\int_{-1}^{0} \cos\frac{n\pi x}{2}\,dx + \int_{0}^{1} \cos\frac{n\pi x}{2}\,dx\right]$

$= \frac{-\sin n\pi/2}{n\pi}$

$b_n = \frac{1}{2}\left[-2\int_{-1}^{0} \sin\frac{n\pi x}{2}\,dx + \int_{0}^{1} \sin\frac{n\pi x}{2}\,dx\right]$

$\frac{-2\cos n\pi/2}{n\pi} - \frac{\cos n\pi x/2}{n\pi}$

$= -\frac{3\cos n\pi/2}{n\pi}$

$f(x) = -\frac{1}{4} - \sum_{n=1}^{\infty}\left(\frac{\sin n\pi/2}{n\pi}\cos\frac{n\pi x}{2} + \frac{3\cos n\pi/2}{n\pi}\sin\frac{n\pi x}{2}\right]$

*Q19. Use result of problem 7 to show

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$f(x) = x + \pi$ where $-\pi < x < \pi$ is as follows

$$f(x) = \pi + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n}\sin nx\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

At point $x = \pi/2$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \pi$$

$$f\left(\frac{\pi}{2}\right) = \pi + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n}\sin n\frac{\pi}{2}\right]$$

$$\frac{\pi}{8} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

proved using theorem 11.2.1 for the point $x = \pi/2$

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* [redacted] Q 20. use result of problem 9 to show

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

From problem 9.

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ \sin x & \text{if } 0 \le x < \pi \end{cases}$$

$$f(x) = \frac{1}{\pi} + \frac{\sin x}{2} + \sum_{n=2}^{\infty} \left[\frac{(-1)^n + 1}{\pi(1-n^2)} \cos nx \right]$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

we use theorem 11.2.1. At point $x = \pi/2$, series converges to $f(\pi/2)$, implies that

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{\sin \pi/2}{2} + \sum_{n=2}^{\infty} \left[\frac{(-1)^n + 1}{\pi(1-n^2)} \cos\frac{n\pi}{2} \right]$$

$$\frac{\pi}{2} = \frac{1}{2} + \frac{\pi}{4} + \left[\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \cdots \right]$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7} + \cdots$$

proved using theorem 11.2.1 for the point $x = \pi/2$.

Chapter 12

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Exercise 12.1

Example 1

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u = 0$$ is homogenious

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2$$ is non homogenious equation.

$$\frac{\partial u}{\partial x} = x' y \qquad \frac{\partial u}{\partial y} = x y'$$

Example 3 (a)

(a) $$\frac{3\,\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$$

we can make the identifications $A = 3$, $B = 0$, and $C = 0$.

Since $B^2 - 4AC = 0$ The equation is parabolic.

(b) $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

we see that $A = 1$, $B = 0$, $C = -1$ and $B^2 - 4AC = -4(1)(-1) > 0$

The equation is hyperbolic

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(c) with $A=1, B=0, C=1$ and
$B^2 - 4AC = -4(1)(1) < 0$, The
equation is elliptic

*Q 13. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

first, assume that we have
particular solution
$u = X(x)Y(t) \to ②$
Second differentiate equation
(2) partially with respect to
x and y as

$\frac{\partial u}{\partial x} = X'Y$

then: $\frac{\partial u^2}{\partial x^2} = X''Y$ (a)

$\frac{\partial u^2}{\partial y^2} = XY''$ (b)

$\frac{X''}{X} = -h$ (c)

$-\frac{Y''}{Y} = -h$ (d)

If we have $h=0$ then

$\int X'' = 0$

$X = c_1 x + m_1 \to ①$

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$Y'' = 0$

$Y = c_2 y + m_2 \rightarrow$ (m)

Substitute from (l) and (m) into (2) we have then

$u = (c_1 x + m_1)(c_2 y + m_2)$

Substituting

$u = [a_3 e^{nx} + b_3 e^{-nx}][a_4 \cos(ny) + b_4 \sin(ny)]$

Q23. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{2\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} - \dfrac{6\partial u}{\partial y} = 0$

we can write

$A\dfrac{\partial^2 u}{\partial x^2} + B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} + D\dfrac{\partial u}{\partial x} +$

$E\dfrac{\partial u}{\partial y} + Fu = 0$

we have to classify equation is hypebolic, parabolic or elliptic

$A = 1$

$B = 2$

$C = 1$

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$D = 1$

$E = -6$

$F = 0$

we have to obtain $B^2 - 4AC$ as

$B^2 - 4AC = (2)^2 - 4(1)(1)$

$= 0$

Third form (a) since we have

$B^2 - 4AC = 0$

$B^2 - 4AC > 0$ hyperbolic

$B^2 - 4AC = 0$ parabolic

$B^2 - 4AC < 0$ elliptic

* **Q 25.** $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

we can write it as

$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial t} + C \frac{\partial^2 u}{\partial t^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial t}$

$+ Fu = 0$

First, we have to identify as

$A = a^2$

$B = 0$

$C = -1$

$D = 0$

$E = 0$

$F = 0$

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1st, we have to obtain value of $B^2 - 4AC$ as

$B^2 - 4AC = 0^2 - 4(a^2)(-1)$

$= 0 + 4a^2$

$= 4a^2 \rightarrow (a)$

Secondaly we have $B^2 - 4AC > 0$, then we consider equation is hyperbolic

$B^2 - 4AC > 0$ hyperbolic

$B^2 - 4AC = 0$ parabolic

$B^2 - 4AC < 0$ elliptic

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Exercise 12.2

Describe Briefly

This chapter deals with solving one dimensional differential equation, e.g., heat equation, wave equation,

equation's Solution of type

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad k>0 \quad (1)$$

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (2)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (3)$$

Laplacian equation, classical partial differential equation with modifications are examples

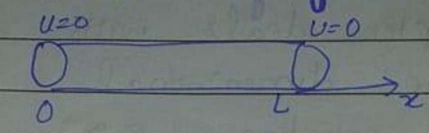
Solution to one dimensional equation is approached by laplace methode.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is two dimensional equation.

Exercise 12.3

Explain Briefly



$0 < x < L$

$t > 0$

$k \dfrac{\partial^2 U}{\partial x^2} = \dfrac{\partial U}{\partial t}$

$U(0,t) = 0 \qquad U(L,t) = 0$

Thin rod of length L, initial temprature $f(x)$ which is held at temprature zero for all tim $t \geq 0$ if rod satisfies The ~~the~~ given equation, then $U(x,t)$ then its solution is determined by boundary values.

using The product $U = X(x)\,T(t)$ and λ^2 as separation Constant

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$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda^2$$

$$X'' + \lambda^2 X = 0$$

$$T' + k\lambda^2 T = 0$$

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$u(x,t) = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

where $n = 1, 2, 3 \dots$

Exercise 12.4

Explain Briefly

the wave equation is important second order linear partial differential equation. it is applicable on all mechanical waves like sound, water etc.

It may contain scaler function $u = u(x_1, x_2, x_3 \dots)$ of a time variable t and one more spatial variable $x_1, x_2, x_n \dots$

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general form of equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots \right)$$

$$\nabla = \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \cdots$$

wave equation in One dimension

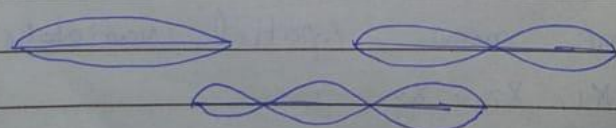
$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2 \partial^2 u}{\partial x^2}$$

It is derived from hooks laws.

general solution

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

Plane wave Eigenmode.
helps solve the complecated waves model.



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Chapter 12

Exercise 12.5

It is second order differential equation

$$\nabla^2 f = 0$$

OR

$$\Delta f = 0$$

In rectangular coordinates

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Model for fluid flow

$$U_x + V_y = 0$$

$$\nabla \times V = V_x - V_y = 0$$

Separation of variable leads

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda^2$$

$$X'' + \lambda^2 X = 0$$

$$Y'' - \lambda^2 Y = 0$$

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$Y = C_1 \cosh \lambda y + C_1 \sinh \lambda y$$

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Dirichlet problem

A boundary value problem for eliptic partial differential equation $\nabla^2 u = 0$ within a region that U takes on a prescribed values on entire boundary.

Supperposition principle.

Method of Separation does not apply on drechelet problem when boundary values of rectangle are non-homogeneous

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a$$
$$0 < y < b$$

$U(0,y) = F(y)$ $\quad$ $U(a,y) = G(y)$

$U(x,0) = f(x)$ $\quad$ $U(x,b) = g(x)$