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Ch:5 => Ex: 5.1 => Q 3, 5, 17, 5.2 => Q: 9, 11, 19
Ch: 6 => Ex: 6.1 => Q: 2, 5, 7, Ex: 6.2 => Q: 18, 17, 19, Ex: 6.3: Q 5, 10, 20.
Ch: 8 => Ex: 8.1 => Q: 7, 12, 18, Ex: 8.2 => Q: 3, 12, 19, Ex: 8.3 Q: 11, 19, 16.

EXERCISE 5.1

Q#3)

$$\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} ; \quad X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$A_x = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+0+1 \\ 2+6+2 \\ 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 5x$$

x is eigenvector of A corresponding to the eigen~~vector~~ value 5.

QNO.5) i) Characterstic Equation
ii) Eigen Value iii) bases

(a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

(i) Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda - 3) - (-2)(-4)$$

$$= (\lambda^2 - 3\lambda - \lambda + 3 - 8)$$

$$= \lambda^2 - 4\lambda - 5$$

$$\boxed{\text{Characterstic Equation of } A = \lambda^2 - 4\lambda - 5 = 0}$$

(ii) $\lambda^2 - 4\lambda - 5 = (\lambda + 1)(\lambda - 5)$

(iii) $\lambda_1 = -1$ & $\lambda_2 = 5$ are eigen values of A

$$-I - A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1-1 & 0-4 \\ 0-2 & -1-3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix}$$

$$5I - A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$$

Applying $(\lambda I - A)x = 0$, for 'x' to be an eigenvector of A corresponding to eigenvalue λ

Taking $Z = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x - 4y \\ -2x - 4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2x - 4y = 0$$

$$x + 2y = 0$$

$$\boxed{x = -2y}$$

$$Z = \begin{bmatrix} -2y \\ y \end{bmatrix} \Rightarrow y\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The eigenvectors of A for $\lambda = -1$ are non-zero vectors $\begin{bmatrix} -2t \\ t \end{bmatrix}$ and $\{(-2, 1)\}$ is bases for $\lambda = -1$

Now,

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \text{ for } \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4x - 4y \\ -2x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 4x - 4y &= 0 \Rightarrow \boxed{x = y} \\ -2x + 2y &= 0 \end{aligned}$$

Thus,

$$v = \begin{bmatrix} x \\ x \end{bmatrix} \Rightarrow x\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

eigenvectors for A for $\lambda = 5$ are non-zero vectors $\begin{bmatrix} t \\ t \end{bmatrix}$ and $\{(1,1)\}$ is bases for eigenspaces for $\lambda = 5$.

- Characterstic Equation $= \lambda^2 - 4\lambda - 5 = 0$
- λ is eigenvalue of A
- $\lambda = -1$, bases for eigenspace is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
- $\lambda = 5$, bases for eigenspace is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

(i) $A = \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda + 2 & 7 \\ -1 & \lambda - 2 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+2 & 7 \\ -1 & \lambda-2 \end{vmatrix}$$

$$= (\lambda+2)(\lambda-2) - (7)(-1)$$

$$= \lambda^2 - \cancel{2\lambda} + \cancel{2\lambda} - 4 + 7$$

$$= \lambda^2 + 3$$

Characterstic Equation of A $\Rightarrow$ $\lambda^2 + 3 = 0$

(ii)- As $\lambda^2 + 3 = 0$ has no real roots, hence A has no real eigenvalues

$$\lambda^2 = -3$$

$$\lambda = \pm\sqrt{3}i$$

$$\lambda_1 = \sqrt{3}i \quad , \quad \lambda_2 = -\sqrt{3}i$$

(iii)- Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigenvector for $\lambda = \sqrt{3}i$

$$\begin{bmatrix} -\lambda-2 & -7 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = \sqrt{3}i$

$$\begin{bmatrix} -\sqrt{3}i-2 & -7 \\ 1 & 2-\sqrt{3}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A'x = 0$$

By row reducing the matrix A', we get

$$\begin{bmatrix} 1 & -7/_{-2-\sqrt{3}i} \\ 0 & 0 \end{bmatrix}$$

$$x_1 + \frac{-7}{-2+(-\sqrt{3}i)} x_2 = 0$$

$$x_1 = \frac{7}{-2-\sqrt{3}i} x_2$$

$$x_1 = (-2+\sqrt{3}i)x_2$$

Putting $x_2 = t$, $t \in R$

$$x_1 = (-2+\sqrt{3}i)t \quad , x_2 = t$$

The eigenvector for $\lambda = \sqrt{3}i$ is

$$x = \begin{bmatrix} (-2+\sqrt{3}i)t \\ t \end{bmatrix} = t\begin{bmatrix} -2+\sqrt{3}i \\ 1 \end{bmatrix}$$

Thus, the basis of the eigenspace to the eigenvalue $\lambda = \sqrt{3}i$ is $\begin{bmatrix} -2+\sqrt{3}i \\ 1 \end{bmatrix}$

Similarly, for $\lambda = -\sqrt{3}i$ is $\begin{bmatrix} -2-\sqrt{3}i \\ 1 \end{bmatrix}$

- Characteristic equation $= \lambda^2 + 3 = 0$

  λ is eigenvalue of A

- $\lambda = \sqrt{3}i$, basis of eigenspace $\begin{bmatrix} -2+\sqrt{3}i \\ 1 \end{bmatrix}$

- $\lambda = -\sqrt{3}i$, basis of eigenspace $\begin{bmatrix} -2-\sqrt{3}i \\ 1 \end{bmatrix}$

(c)- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(i)- $\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 1 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda - 1)^2 - 0$

$= \lambda^2 - 2\lambda + 1$

characterstic Equation of $A \Rightarrow \lambda^2 - 2\lambda + 1 = 0$

(ii)- As $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \Rightarrow \lambda_1 = 1$ is the eigenvalue of A.

(iii)- $I - A = I - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

For 'x' to be an eigenvector of A to the eigenvalue λ, we must have $(\lambda I - A)x = 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ for all } \begin{bmatrix} x \\ y \end{bmatrix} \in R^2$$

R^2 is eigenspace of A for $\lambda = 1$

Thus, $\{(1,0), (0,1)\}$ is basis for eigenspace of $\lambda = 1$

- Characterstic Equation : $\lambda^2 - 2\lambda + 1 = 0$
- $\lambda = 1$, basis for eigenspace are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(a) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

(i) $\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \lambda-1 & 2 \\ 0 & \lambda-1 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-1)^2 - 0$

Characteristic Equation $= \lambda^2 - 2\lambda + 1$

(ii) $\lambda^2 - 2\lambda + 1 = (\lambda-1)^2$

$\lambda_1 = 1$ is the eigenvalue of A.

(iii) We have $I - A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

For x to be an eigenvector of A to the eigenvalue λ, we must have $(\lambda I - A)x = 0$

$V = \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 + 2y \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2y = 0$ & $y = 0$

$V = \begin{bmatrix} x \\ 0 \end{bmatrix} \Rightarrow x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

eigenvector for $\lambda = 1$ are non-zero vectors of the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$ and therefore $\{(1, 0)\}$ is a basis for eigenspace for $\lambda = 1$

characteristic Equation : $\lambda^2 - 2\lambda + 1 = 0$

$\lambda = 1$, basis for eigenspace $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Q NO: 17

(a)- D^2 is linear

If we have function $f(x)$ & $g(x)$

$$D^2(f(x) + g(x)) = (f(x) + g(x))''$$
$$= f''(x) + g''(x)$$

$$D^2(cf(x)) = (cf(x))''$$
$$= cf''(x)$$

D^2 is a linear transformation

(b)- Eigenvalues

In order to prove that ω is a positive constant then $\sin(\sqrt{\omega}x)$ and $\cos(\sqrt{\omega}x)$ are eigenvectors of D^2

(ii)- finding D^2 for $\sin(\sqrt{\omega}x)$

$$f(x) = \sin(\sqrt{\omega}x)$$
$$f'(x) = (\sqrt{\omega}x)'\cos(\sqrt{\omega}x)$$
$$= (\sqrt{\omega})\cos(\sqrt{\omega}x)$$
$$f''(x) = (f'(x))'$$
$$= (\sqrt{\omega})'\cos(\sqrt{\omega}x)'$$
$$= -\omega\sin\sqrt{\omega}x$$

so,

$$f''(x) = -\omega f(x) \Rightarrow \lambda_1 = -\omega$$

(iii)- Finding D^2 for $\cos(\sqrt{\omega}x)$

$$f(x) = \cos(\sqrt{\omega}x)$$

$$f'(x) = (\sqrt{\omega}x)' \cdot (-\sin\sqrt{\omega}x)$$
$$= -\sqrt{\omega}\sin(\sqrt{\omega}x)$$
$$f''(x) = (f'(x))'$$
$$= (-\sqrt{\omega})'(\sin\sqrt{\omega}x)'$$
$$= -\omega\cos(\sqrt{\omega}x)$$

so, $f''(x) = -\omega f(x)$ means $\lambda_2 = -\omega$

$$\lambda_2 = \lambda_1$$

EXERSICE # 5.2

QNO-9) $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

(a) Eigenvalues

Solving $\det(\lambda I - A) = 0$ to find eigenvalues of A

$$\det(\lambda I - A) = \det\left(\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} \lambda-4 & 0 & -1 \\ -2 & \lambda-3 & -2 \\ -1 & 0 & \lambda-4 \end{bmatrix}\right)$$

$$= (\lambda-3)\det\left(\begin{bmatrix} \lambda-4 & -1 \\ -1 & \lambda-4 \end{bmatrix}\right)$$

$= (\lambda-3)((\lambda-4)^2-1)$

$= (\lambda-3)(\lambda^2-8\lambda+15)$

$= (\lambda-3)(\lambda-3)(\lambda-5) = 0$

$(\lambda-3)(\lambda-5) = 0$

$\lambda = 3, \quad \lambda = 5$

Eigenvalues of matrix A are '3' and '5'

(b) Rank of $\lambda I - A$

Let $\lambda = 3$, then

$$3I - A = \begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix}$$

By Elementary Row Operation

$$\sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It can be seen that there is only one row i.e non-zero the rank of $3I-A$ is 1.

Let $\lambda = 5$

$$\lambda I - A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

By Row Operations:

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Due to two linearly Independent rows, rank of $5I-A$ is 2.

(c)- Diagonalizable:

Rank of $3I-A$ is 1, the nullity is 2, there are 2 ~~eigenvalues~~ eigenvectors corresponding to eigenvalue 3. There is 1 eigenvector corresponding to eigenvalue 5. There are 3 eigenvectors, therefore the given 3×3 matrix is diagnalizable.

Q NO-11)- $P^{-1}AP$

$$\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}\right) = 0$$

$$(-1-\lambda)\left[(4-\lambda)(3-\lambda)-0\right] - 4\left[-3(3-\lambda)\right] + 2\left[-3+3(4-\lambda)\right] = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

By dividing $(\lambda-1)$ from equation

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

eigenvalues are $1, 2, 3$

$(\lambda I - A)x = 0$

$$\begin{bmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 1$

$$\begin{bmatrix} 2 & -4 & 2 \\ 3 & -3 & 0 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 + x_3 = 0$$
$$3x_1 - 3x_2 = 0$$
$$3x_1 - x_2 - 2x_3 = 0$$

Let $x_1 = t$, then $x_2 = t$ and $x_3 = t$

so, $x = \begin{bmatrix} t \\ t \\ t \end{bmatrix} \Rightarrow t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

take $t = 1$, then

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the corresponding eignvector to $\lambda = 1$

basis for eigenspace to $\lambda = 1$

for $\underline{\lambda = 2}$

$$\begin{bmatrix} 3 & -4 & 2 \\ 3 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 - 4x_2 + 2x_3 = 0$$
$$3x_1 - 2x_2 = 0$$
$$3x_1 - x_2 - x_3 = 0$$

Let $x_1 = t$

Then we have $x_2 = 3/2\, t$ and $x_3 = 3/2\, t$

So, $x = \begin{bmatrix} t \\ 3/2\, t \\ 3/2\, t \end{bmatrix}$ take $t=1$

$x_1 = \begin{bmatrix} 1 \\ 3/2 \\ 3/2 \end{bmatrix}$ is the corresponding eigenvector to $\lambda = 2$

As, it is linearly independent, it forms a basis for eigenspace to $\lambda = 2$

For $\underline{\lambda = 3}$

$$\begin{bmatrix} 4 & -4 & 2 \\ 3 & -1 & 0 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 4x_2 + 2x_3 = 0$$
$$3x_1 - x_2 = 0$$
$$3x_1 - x_2 = 0$$

Take $x_1 = t$, then $x_2 = 3t$ & $x_3 = 4t$

$x = \begin{bmatrix} t \\ 3t \\ 4t \end{bmatrix} \Rightarrow t\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$t = 1$

$$x = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$ is the corresponding eigenvector to $\lambda = 3$

As it is linearly independent, it forms a basis for eigenspace to $\lambda = 3$

As, there are 3 basis vectors.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3/2 \\ 3/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

the matrix A is diagnolizable. For all these eigenvalues, geometric and algebraic multiplication is 1. We will form the matrix P.

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3/2 & 3 \\ 1 & 3/2 & 4 \end{bmatrix}$$

We have to compute $(P^{-1})^T AP$. To find P^{-1}, we have to find det(P) and adj(P)

$$|P| = 1(6 - 9/2) - 1(4-3) + 1(3/2 - 3/2) = 1/2$$

Now,

$$\text{adj } P = \begin{bmatrix} 3/2 & -1 & 0 \\ -5/2 & 3 & -1/2 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \text{adj} P = \frac{1}{1/2} \begin{bmatrix} 3/2 & -1 & 0 \\ -5/2 & 3 & -1/2 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

so, $$P^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ -5 & 6 & -1 \\ 3 & -4 & 1 \end{bmatrix}$$

$$(P^{-1})^T = \begin{bmatrix} 3 & -5 & 3 \\ -2 & 6 & -4 \\ 0 & -1 & 1 \end{bmatrix}$$

Finally,

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

QNO-19) Find A^{11}

$$A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$

Find P^{-1} first by

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P$$

$$\det(P) = 1\cdot(0-0) - 1(0-1) + 1(0-0) = 1$$

$$\operatorname{adj}(P) = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

To confirm that P diagnalize A, we have

$$M = P^{-1}AP = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

P diagnolize matrix A

As, $M = P^{-1}AP$, Then $A = PMP^{-1}$

So, we have $A^{11} = PM^{11}P^{-1}$

$$A^{11} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} (-2)^{11} & 0 & 0 \\ 0 & (-1)^{11} & 0 \\ 0 & 0 & (-1)^{11} \end{bmatrix} \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^{11} = \begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$$

EXERSICE # 6.1

QNO-2) $\langle \underline{u}, \underline{v} \rangle = \frac{1}{2} u_1 v_1 + 5 u_2 v_2$

(a) $\langle \underline{u}, \underline{v} \rangle = \frac{1}{2} \cdot 1 \cdot 3 + 5 \cdot 1 \cdot 2$

$= 3/2 + 10 = 23/2$

(b) $\langle k\underline{v}, \underline{w} \rangle = \frac{1}{2} \langle kv_1 \rangle w_1 + 5 \langle kv_2 \rangle w_2$

$= \frac{1}{2}(3 \cdot 3) \cdot 0 + 5(3 \cdot 2) - 1$

$= 0 + (-30)$

$= -30$

(c)- $\langle \underline{u}+\underline{v}, \underline{w} \rangle = 1/2 \langle u_1 + v_1 \rangle \underline{w}_1$

$= 1/2 (u_1 + v_1) w_1 + 5(u_2 + v_2) w_2$

$= 1/2 (1+3) \cdot 0 + 5(1+2)(-1)$

$= -15$

(d)- $\|\underline{v}\| = \sqrt{1/2 (v_1 v_1) + 5(v_2 v_2)}$

$= \sqrt{1/2 (3 \cdot 3) + 5(2 \cdot 2)}$

$\|v\| = \sqrt{49/2} = \frac{7}{\sqrt{2}}$ Ans

(e)- $d(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|$

$d(\underline{u}, \underline{v}) = \|(-2, -1)\|$

$d(\underline{u}, \underline{v}) = \sqrt{1/2 (-2(-2) + 5(-1)(-1)}$

$= \sqrt{2+5} = \sqrt{7}$

(f)- $\underline{u} - k\underline{v} = (1,1) - 3(3,2)$

$= (1,1) - (9,6)$

$= (-8, -5)$

$d(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|$

$= \|(-8, -5)\|$

$d(\underline{u}, \underline{v}) = \sqrt{1/2 (-8(-8) + 5(-5)(-5)}$

$= \sqrt{157}$

Q NO-5) Inner Product on R^2

$$\langle \underline{u}, \underline{v} \rangle = 2u_1v_1 + 3u_2v_2$$

The weighted Euclidean inner product for vectors $\underline{u} = (u_1, u_2)$ and $\underline{v} = (v_1, v_2)$ is:

$$\langle \underline{u}, \underline{v} \rangle = \omega_1 u_1 v_1 + \omega_2 u_2 v_2 + \dots + \omega_n v_n u_n$$

Matrix:
$$\begin{bmatrix} \sqrt{\omega_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\omega_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sqrt{\omega_n} \end{bmatrix}$$

$\langle \underline{u}, \underline{v} \rangle = 2u_1v_1 + 3u_2v_2$ matrix generates the Euclidean inner product space is:

$$A = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

Q NO-7) $A = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}$

find $\langle \underline{u}, \underline{v} \rangle$ for $\underline{u} = (0, -3)$, $\underline{v} = (6, 2)$

Suppose $\underline{u}$ and $\underline{v}$ are vectors in R^n which can be written in column form and consider the invertible matrix of size n. It is known that Euclidean inner product is defined as:

$$\langle \underline{u}, \underline{v} \rangle = A\underline{u} \cdot A\underline{v}$$

$$\langle \underline{u}, \underline{v} \rangle = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 9 \end{bmatrix} \begin{bmatrix} 26 \\ 6 \end{bmatrix}$$

$\langle u, v \rangle = 26(-3) + 9(6) = -78 + 54 = -24$

EXERSICE # 6.2

QNO-17) - Orthogonal w.r.t standard inner product on P_2

$P_1 = 2 + kx + 6x^2$, $P_2 = l + 5x + 3x^2$

$P_3 = 1 + 2x + 3x^2$

Consider P_1, P_2, P_3 given in P_2 with its standard inner product.

$$\langle P_1, P_2 \rangle = 2l + 5k + 18$$

$$\langle P_1, P_3 \rangle = 2 + 2k + 18 = 2k + 18$$

$$\langle P_2, P_3 \rangle = l + 10 + 9 = l + 19$$

Thus, in order for P_1, P_2, P_3 to be mutually orthogonal w.r.t the standard inner product on P_2, we must have

$$2l + 5k + 18 = 0 \quad -①$$

$$2k + 20 = 0 \quad -②$$

$$l + 19 = 0 \quad -③$$

Equation ② & ③ gives $k = -10$ and $l = -19$, since $2(-19) + 5(-10) + 18 = -38 - 50 + 18 = -70 \neq 0$

$$2l + 5k + 18 = 0$$

$$2k + 20 = 0$$

$$l + 19 = 0$$

has no solution.

There are no scalars k and l such that the vectors P_1, P_2, P_3 are mutually orthogonal w.r.t standard inner product on P_2

Q NO-18) $\underline{u} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

we know that the inner Product on R^2 generated by a matrix A is given by

$$\langle u, v \rangle = \langle A\underline{u} \rangle \cdot (A\underline{v})$$

$$= \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 9 \\ 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= 18 - 18 = 0$$

Orthogonal w.r.t inner product on R^2.

Q NO-19) $P(x_0) = P(-2) = -2$

$P(x_1) = P(0) = 0$

and $P(x_2) = P(2) = 2$

$q(x_0) = q(-2) = 4$

$q(x_1) = q(0) = 0$

$q(x_2) = q(2) = 4$

Using evaluation inner product at the points x_0, x_1, x_2

$$\langle P, q \rangle = P(x_0)q(x_0) + p(x_1)q(x_1) + P(x_2)q(x_2)$$

$$= (-2)(4) + (0)(0) + (2)(4)$$

$$= -8 + 0 + 8$$

$$= 0$$

$\underline{P}$ and $\underline{q}$ are orthogonal w.r.t the evaluation inner product at the points x_0, x_1, x_2

EXERSICE # 6.3

QNO-5) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$

Consider the column space set $S = \{(1,0,-1), (2,0,2), (0,5,0)\}$ in R^3 with its standard inner product. Taking $w_1 = (1,0,-1)$, $w_2 = (2,0,2)$, $w_3 = (0,5,0)$. Now, we have

$$\langle w_1, w_2 \rangle = 2 - 2 = 0$$

$$\langle w_1, w_3 \rangle = 0 + 0 + 0 = 0$$

$$\langle w_3, w_2 \rangle = 0 + 0 + 0 = 0$$

from theorem 6.3.1 that these vectors are linearly independent and hence form a basis for R^3 by theorem 4.5.4. Therefore, S is an orthogonal set in R^3. Now we calculate the norms of w_1, w_2 and w_3 and then obtain the orthogonal basis:

$$\|w_1\| = \sqrt{(1)^2 + (-1)^2 + (0)^2} = \sqrt{2}$$

$$\|w_2\| = \sqrt{(2)^2 + (0)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\|w_3\| = \sqrt{(0)^2 + (5)^2 + (0)^2} = 5$$

$$\underline{v}_1 = \frac{\underline{w}_1}{\|w_1\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$\underline{v}_2 = \frac{\underline{w}_2}{\|w_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\underline{v}_3 = \frac{\underline{w}_3}{\|w_3\|} = (0, 1, 0)$$

Hence, $S' = \left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), (0,1,0) \right\}$

is the orthonormal basis for the set S.

Q No: 10) $\underline{v}_1 = (1,-1,2,-1)$, $\underline{v}_2 = (-2,2,3,2)$
$\underline{v}_3 = (1,2,0,-1)$, $\underline{v}_4 = (1,0,0,1)$
$\underline{u} = (1,1,1,1)$.

$$\langle v_1, v_2 \rangle = (1)(-2) + (-1)(2) + (2)(3) + (-1)(2)$$
$$= -2 - 2 + 6 - 2 = 0$$
$$\langle v_1, v_3 \rangle = (1)(1) + (-1)(2) + (2)(0) + (-1)(-1)$$
$$= 1 - 2 + 0 + 1 = 0$$
$$\langle v_1, v_4 \rangle = (1)(1) + (-1)(0) + (2)(0) + (-1)(1)$$
$$= 1 - 1 = 0$$
$$\langle v_2, v_3 \rangle = (-2)(1) + (2)(2) + (3)(0) + (2)(-1)$$
$$= -2 + 4 - 2 = 0$$
$$\langle v_2, v_4 \rangle = (-2)(1) + (2)(0) + (3)(0) + (2)(1)$$
$$= -2 + 2 = 0$$
$$\langle v_3, v_4 \rangle = (1)(1) + (2)(0) + (0)(0) + (-1)(1)$$
$$= 1 - 1 = 0$$

v_1, v_2, v_3, v_4 are non-zero vectors, hence, it is an orthogonal set of non-zero vectors.
From 6.3.1, set $\{v_1, v_2, v_3, v_4\}$ is linearly independent in R_4

$R^4 = 4$, from Theorem 46.4, form a basis for R^4 w.r.t Euclidean inner product.

Now, by using theorem 6.3.2(a)

$$\underline{u} = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2 +$$

$$\frac{\langle u, v_3 \rangle}{\|v_3\|^2} v_3 + \frac{\langle u, v_4 \rangle}{\|v_4\|^2} v_4$$

$\|v_1\|^2 = (1)(1) + (-1)(-1) + (2)(2) + (-1)(-1)$
$= 1 + 1 + 4 + 1 = 7$

$\|v_2\|^2 = (-2)(-2) + (2)(2) + (3)(3) + (2)(2)$
$= 4 + 4 + 9 + 4 = 21$

$\|v_3\|^3 = (1)(1) + (2)(2) + (0)(0) + (-1)(-1)$
$= 1 + 4 + 1 = 6$

$\|v_4\|^2 = (1)(1) + (0)(0) + (0)(0) + (1)(1)$
$= 1 + 1 = 2$

Now,

$\langle \underline{u}, v_1 \rangle = (1)(1) + (1)(-1) + (1)(2) + (1)(-1)$
$= 1$

$\langle u, v_2 \rangle = (1)(-2) + (1)(2) + (1)(3) + (1)(2)$
$= -2 + 2 + 3 + 2 = 5$

$\langle u, v_3 \rangle = (1)(1) + (1)(2) + (1)(0) + (1)(-1)$
$= 1 + 2 - 1 = 2$

$\langle u, v_4 \rangle = (1)(1) + (1)(0) + (1)(0) + (1)(1)$
$= 1 + 1 = 2$

$$\underline{u} = \frac{1}{7} v_1 + \frac{5}{21} v_2 + \frac{2}{6} v_3 + \frac{2}{2} v_4$$

$$\Rightarrow \underline{u} = \frac{1}{7} v_1 + \frac{5}{21} v_2 + \frac{1}{3} v_3 + v_4$$

QNO: 20) $\underline{u} = (3, -1, 2)$; $v_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

$$v_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

(a) Orthogonal Projection

Orthogonal Projection of $\underline{u}$ on $\underline{w}$ (plane formed by v_1 & v_2)

$$\underline{u}' = \text{proj}_{\underline{w}}\,\underline{u} = \frac{\langle \underline{u}, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle \underline{u}, v_2 \rangle}{\|v_2\|^2} v_2 \quad \text{—(1)}$$

Thus, $\langle \underline{u}, v_1 \rangle = \frac{3}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{4}{\sqrt{6}} = \frac{-2}{\sqrt{6}}$

$$\|v_1\|^2 = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-2}{\sqrt{6}}\right)^2 = \frac{1+1+4}{6} = 1$$

$$\langle u, v_2 \rangle = 3/\sqrt{3} - 1/\sqrt{3} + 2/\sqrt{3} = 4/\sqrt{3}$$

$$\|v_2\|^2 = \left(1/\sqrt{3}\right)^2 + \left(1/\sqrt{3}\right)^2 + \left(1/\sqrt{3}\right)^2$$

$$= \frac{1+1+1}{\sqrt{3}} = 1$$

Putting values in (1)

$$\underline{u}' = \text{proj}_{\omega}\,\underline{u} = \frac{-2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right) +$$

$$\frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \left(\frac{-2}{6}, \frac{-2}{6}, \frac{4}{6}\right) + \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$$

$$\underline{u}' = \left(\frac{-1+4}{3}, \frac{-1+4}{3}, \frac{2+4}{3}\right) = (1, 1, 2)$$

(b) Component of $\underline{u}$

As the component of $\underline{u}$, orthogonal to $\underline{v}$ is

$$\underline{u}'' = Proj_{w} \underline{v} = \underline{u} - Proj_{w} \underline{v} = \underline{u} - \underline{v}' \quad — ②$$

from (a)

$$\underline{u}' = (1, 1, 2)$$

Putting values in ②

$$\underline{u}'' = (3, -1, 2) - (1, 1, 2)$$
$$= (2, -2, 0)$$

$$\langle \underline{u}'', v_1 \rangle = 2(1/\sqrt{6}) - 2(1/\sqrt{6}) + 0$$
$$\langle u'', v_1 \rangle = 0$$
$$\langle u'', v_2 \rangle = 2(1/\sqrt{3}) . 2(1/\sqrt{3}) + 0 = 0$$

∴ Component of $\underline{u}$ orthogonal to the plane spanned by vectors v_1 & v_2 is $(2, -2, 0)$

EXERSICE # 8.1

QNO: 7) - (a) $T(a_0 + a_1x + a_2x^2)$

$$= a_0 + a_1(x+1) + a_2(x+1)^2$$

If T is linear transformation then for any $p, q \in P_2$ and any $K \in R$, it satisfies:

- $T(K(p(x)) = K\,T((p)(x))$ homogenity.
- $T(p(x) + q(x)) = T(p(x) + T(q(x)))$ additivity.

Let $p, q, \in P_2$ arbitrary polynomials of second degree

$$p(x) = a_0 + a_1x + a_2x^2.$$
$$q(x) = b_0 + b_1x + b_2x^2$$

(i) homogenity:

$$T(kp(x)) = ka_0 + ka_1(x+1) + ka_2(x+1)^2$$
$$= k(a_0 + a_1(x+1) + a_2(x+1)^2)$$
$$= kT(p(x))$$

(ii) Additivity:

$$T(p(x)+q(x)) = (a_0+b_0)+(a_1+b_1)(x+1) + (a_2+b_2)(x+1)^2$$
$$= (a_0 + a_1(x+1) + a_2(x+1)^2) + (b_0 + b_1(x+1) + b_2(x+1)^2)$$
$$= T(p(x)) + T(q(x))$$

T is a linear transformation.

Finding Kernel of linear operator.

$$p \in Ker(T) \iff T(p(x)) = 0$$
$$\iff a_0 + a_1(x+1) + a_2(x+1)^2 = 0$$
$$\iff (a_0 + a_1 + a_2) + (a_1 + a_2)x + a_2 x^2 = 0$$
$$\iff a_0 = a_1 = a_2 = 0$$

$$Ker(T) = \{0\}$$

(b) $T(a_0 + a_1x + a_2x^2)$

$$= (a_0+1) + (a_1+1)x + (a_2+1)x^2$$

$$T(a_0 + a_1x + a_2x^2) = (a_0+1) + (a_1+1)x + (a_2+1)x^2$$

$$p(x) = 1 = 1 + 0(x) + 0 + 0\cdot x^2$$

$$T(2p(x)) = T(2 + 0\cdot x + 0\cdot x^2)$$
$$= 3 + x + x^2 \neq 4 + 2x + 2x^2$$
$$= 2T(p(x))$$

T is not linear transformation since homogenity is not satisfied.

Q NO # 12)

(a) Kernel of T

Let V be any vector space and let us consider the following linear transformation.

$T: V \to V$ given by $T(\underline{v}) = 3\underline{v}$

$\underline{v} \in \ker(T) \iff T(\underline{v}) = 0$

$3\underline{v} = 0 \iff \underline{v} = 0$

$\ker(T) = \{0\}$

(b) Range of T

we have $\forall\ \underline{v} \in V \Rightarrow T\left(\frac{1}{3}\underline{v}\right) = 3 \cdot \frac{1}{3}\underline{v} = \underline{v}$

$R(T) = V$

Q NO = 18) a) $T(1 + \sin x + \cos x)$

Given V subspace of $C[0, 2\pi]$ spanned by the vectors 1, $\sin x$ and $\cos x$

$T: V \to R^3$ at sequence of points $0, \pi, 2\pi$

$T(1 + \sin x + \cos x) = (1 + \sin 0 + \cos 0, 1 + \sin \pi + \cos \pi,$
$1 + \sin 2\pi + \cos 2\pi)$

$= (1 + 0 + 1, 1 + 0 - 1, 1 + 0 + 1)$

$= (2, 0, 2)$

(b) $\ker(T)$

$f(x) = a + b\sin x + c\cos x \quad ; a, b, c \in R$

$f(x) \in \ker(T) \iff T(f(x)) = 0$

$\iff (a + b\sin 0 + c\cos 0, a + b\sin \pi +$

$c\cos\pi,\ a + b\sin 2\pi + c\cos 2\pi) = 0$

$(a+c,\ a-c,\ a+c) = 0$

$a = c = 0,\quad b \in R$

$Ker(T) = (\{\sin x\})$

(c) R(T)

According to theorem 8.14

$\dim V = rank(T) + nullity(T)$

$rank(T) = 3-1 = 2$

$\dim(V) = 3$, as $1, \sin x, \cos x$ are linearly independent functions that span subspace V.

$T(1) = (1,1,1)$

$T(\sin x) = (\sin 0, \sin\pi, \sin 2\pi) = (0,0,0)$

$T(\cos x) = (\cos 0, \cos\pi, \cos 2\pi) = (1,0,1)$

$$R(T) = \left(\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}\right)$$

EXERSICE # 8.2

QNO:3) a) $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$

$\underline{v} = (x,y) \in R^2$

$$A\underline{v} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ 2x-4y \\ -3x+6y \end{bmatrix}$$

if $\underline{v}$ is in the null space of A

$(x-2y, 2x-4y, -3x+6y) = (0,0,0)$

which gives us that $x-2y=0$, $2x-4y=0$
$-3x+6y=0$ and $x=2y$

$\underline{v} = (2y, y) = y(2,1) \in \text{span}\{(2,1)\}$

and then null space of A is a subset of span$\{(2,1)\}$

$\underline{v} = (2a, a) \in \text{span}\{(2,1)\}$ we have $A\underline{v} = (2a-2a,$
$2(2a)-4a, -3(2a)+6a) = (0, 4a-4a, -6a+6a) = (0,0,0)$

which gives us that $\underline{v}$ is in the null space of A
and span $\{(2,1)\}$ is a subset of null space of A.

By 8.2.1, multiplication by A is not one-to-one

(b) $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix}$

Given $\underline{v} = (x, y, z, w) \in R^4$

$$Av = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$= \begin{bmatrix} x+3y+z+7w \\ 2x+7y+2z+4w \\ -x-3y \end{bmatrix}$$

if $\underline{v}$ is in null space of A, we have that
$(x+3y+z+7w, 2x+7y+2z+4w, -x-3y) = (0,0,0)$

which gives:

$x+3y+z+7w=0$, $2x+7y+2z+4w=0$,
$-x-3y=0$

$x+3y+z+7w-x-3y=0+0$

$2(x+3y+z+7w)-(2x+7y+2z+4w)=2\cdot 0-0$

which gives $z=-7w$ & $y=10w$

$-x-3y=0 \Rightarrow x=-3y=-30w$

$\underline{v}=(-30w, 10w, -7w, w)=w(-30,10,-7,1)\in$ span $\{(-30,10,-7,1)\}$ and then the null space of A is a subset of span $\{(-30,10,-7,1)\}$

$v=(-30a, 10a, -7a, a)\in$ span

$Av=(-30a+3(10a)-7a+7a,\ 2(-30a)+7(10a)+2(-7a)+4a,\ -(-30a)-3(10a))=(0,0,0)$

which gives us that $\underline{v}$ is in the null space of A and span $\{(-30,10,7,1)\}$ is a subset of the null space of A. Hence, null space of A is span $\{-30,10,-7,1\}$ and then by theorem 8.2.1, multiplication by A is not one-to-one. Transformation is not one-to-one.

QNO-12) - $T_1(x,y)=(2x,-3y,x+y)$,
$T_2(x,y,z)=(x-y,\ y+z)$

Find $(T_2 \circ T_1)(x,y)$

Given $(x,y)\in R^2$, we have that

$(T_2\circ T_1)(x,y)=T_2(T_1(x,y))=T_2(2x,-3y,x+y)$

$=(2x-(-3y),\ -3y+x+y)$

$=(2x+3y,\ x-2y)$

Thus,

$(T_2\circ T_1)(x,y)=T_2(T_1(x,y))=(2x+3y,\ x-2y)$

Q NO-19) $T: P_1 \to R^2$

$T(p(x)) = (p(0), p(1))$

(a) $T(1-2x)$

$T(1-2x) = (1-2\cdot 0, 1-2\cdot 1)$

$= (1-0, 1-2)$

$= (1, -1)$

(b) Show that T is a linear transformation

Let $p, q \in P_1$:

$T(p(x)+q(x)) = (p(0)+q(0), p(1)+q(1))$

$= (p(0), p(1)) + (q(0)+q(1))$

$= T(p(x)) + T(q(x))$

Let $p \in P_1$ and $K \in R$

$T(K(p(x)) = (Kp(0), Kp(1))$

$= K(p(0), p(1))$

$= KT(p(x))$

T is a linear transformation.

(c) Show that T is one-to-one

Let $p(x) = a + bx$

$T(p(x)) = (p(0), p(1))$

$= (a, a+b)$

$= (0, 0)$

$a = 0$

9

$b = 0 - a = 0$

$\mathrm{Ker}(T) = \{0\}$

T is one-to-one.

(d) $T^{-1}(2,3)$ and its graph

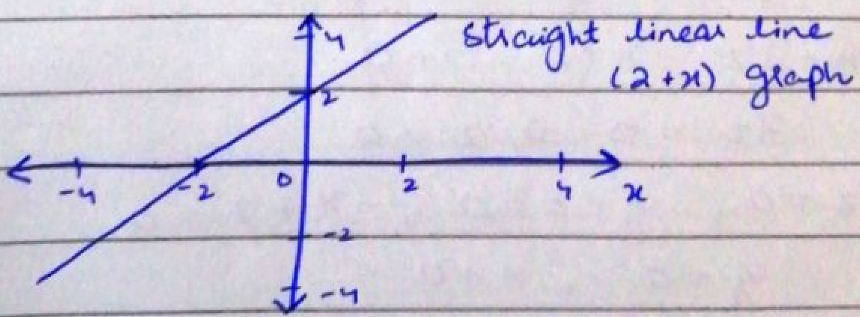
Let $S: R^2 \to P_1$ such that

$$S(a,b) = a + (b-a)x$$

$$(S \circ T)(a+bx) = S(T(a+bx))$$
$$= S(a, a+b)$$
$$= a + (a+b-a)x$$
$$= a + bx$$

$$\Rightarrow S = T^{-1}$$

$$T^{-1}(2,3) = 2 + x$$



EXERSICE # 8.3

QNO:11) $T_A : R^3 \to R^3$ is an isomorphism

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$T_A(\underline{v}) = A\underline{v} \quad \forall \ \underline{v} \in R^3$

$\underline{v} = (x, y, z) \in Ker(T_A)$

$T_A(\underline{v}) = (0,0,0)$

$$T_A(v) = A\underline{v} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y - z \\ x + 2z \\ -x + z \end{bmatrix}$$

$(y - z, x + 2z, -x + z) = (0,0,0)$

$y - z = 0, \ x + 2z = 0 \ \& \ -x + z = 0$

$x + 2z = 0 \ \& \ -x + z = 0$

$x + 2z - x + z = 0 + 0$

$3z = 0 \Rightarrow z = 0$

$z = 0, \ y - z = 0, \ -x + z = 0$

$y = 0, \ x = 0$

$\underline{v} = (0,0,0)$

$Ker(T_A) = \{(0,0,0)\}$

T_A is one-to-one

$\dim(R^3) = 3$, T_A is onto and

therefore isomorphism

Q NO: 19) $T(x - x^2) = \begin{bmatrix} 0-0^2 & 1-1^2 \\ 1-1^2 & 0-0^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$T(x^2 - x) = \begin{bmatrix} 0^2-0 & 1^2-1 \\ 1^2-1 & 0^2-0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$T(x - x^2) = T(x^2 - x)$

$x - x^2 \neq x^2 - x$

T is not one-to-one, hence it is not an isomorphism

Q NO: 16)

Consider linear transformation $T: M_{22} \to R^4$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a+b \\ a+b+c \\ a+b+c+d \end{bmatrix}$

$T\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1+(-1) \\ 1+(-1) \\ 1+(-1)+0 \\ 1+(-1)+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \in \mathrm{Ker}(T)$

$\mathrm{Ker}(T) \neq \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

T is not one-to-one and is not an isomorphism