Programming classes N° 4

Upwind scheme for the transport equation at constant velocity

Let the final time be $T > 0)$ and $a \in \mathbb{R}$. We are looking for an approximation of $\bar{u} : \mathbb{R} \times (0,T) \to \mathbb{R}$, solution of the following problem:

$$\begin{cases} \partial_t \bar{u}(x,t) + \ a \ \partial_x \bar{u}(x,t) = 0, & \forall (x,t) \in \mathbb{R} \times (0,T) \ , \\ \bar{u}(x,0) = u_{\text{ini}}(x), & \forall x \in \mathbb{R} \ . \end{cases} \tag{1}$$

We recall that the solution of Eq.(1) is given by $\bar{u}(x,t) = u_{\text{ini}}(x - at)$, $\forall (x,t) \in \mathbb{R} \times (0,T)$.

We define the numerical solution as a sequence of values u_j^n, which are an approximation of $u_j^n \approx \bar{u}(x_j, t_n)$. Here we take $x_j = j\Delta x$ and $t_n = n\Delta t$ for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. We denote Δx the space step and Δt the time step.

We study the **upwind** scheme, $a > 0$:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} \ + \ a\frac{u_j^n - u_{j-1}^n}{\Delta x} \ = \ 0 \ . \tag{2}$$

This scheme is stable for the 2 and ∞ norms, under the CFL condition $a\Delta t \leq \Delta x$. Its consistency error is or order $\mathcal{O}(\ \Delta t + \Delta x \ )$. Let us set $\lambda = a\frac{\Delta t}{\Delta x}$.

In order to work on a space domain of finite measure, we will take an initial condition u_{ini} a $1-$ periodic function.

Scheme implementation, taking into account the periodicity.

1. Show, first, that u_{ini} being $1-$periodic implies that the solution is $1-$periodic: $\bar{u}(x + 1, t) = \bar{u}(x,t)$, $\forall (x,t) \in \mathbb{R} \times (0,T)$.
2. Let us take the initial condition: $u_{\text{ini}}(x) = \sin(2\pi x)$, $\forall x \in \mathbb{R}$. Cerify that it is a $1-$periodic function and code a function whose input is x and whose output is the value $u_{\text{ini}}(x)$.
3. We have to implement the scheme in such a way that the $1-$periodicity of the solution is respected. The space domain has to be of the form $[L, 1 + L[$. A natural choice is $L = 0$, which implies $[0, 1[$.
   (a) Let us set $J \in \mathbb{N}^*$, $\Delta x = \frac{1}{J}$. Define a sequence of points $x_j = j\Delta x$.
   (b) Determine α and β such that the scheme (2) can be written as,
   $$u_j^{n+1} = \alpha u_j^n + \beta u_{j-1}^n \ .$$
   Think about the scheme discretisation in order to properly take the periodicity into account.

(c) Code a function:
`Input :` a, T, u_{ini}, J, λ
`Output :` $(x_j)_j$, t_M, $(u_j^M)_j$, $(\bar{u}_j^M)_j$
where $t_M \leq T$ is the final time and $\bar{u}_j^M = \bar{u}(x_j, t_M)$, $\forall j$.

Validation of the scheme.

4. Show that if $\lambda = 1$ and the scheme is well initialised, it is exact in the points (x_j, t_n), i.e. $u_j^n = \bar{u}(x_j, t_n)$, $\forall j$ and $n \geq 0$.
5. Verify that the implementation satisfy this result, for $a > 0$ and different values of J, T.

Convergence study in the case of a regular initial datum.

6. Modify the function coded in 3.(c) to compute and output the following errors:

$$\varepsilon_{\Delta t, \Delta x}^{(2)} = \max_{n \geq 0} \left(\|U_{\Delta x}^n - \bar{U}_{\Delta x}^n\|_{2,\Delta} \right) , \tag{3}$$

$$\varepsilon_{\Delta t, \Delta x}^{(\infty)} = \max_{n \geq 0} \left(\|U_{\Delta x}^n - \bar{U}_{\Delta x}^n\|_{\infty,\Delta} \right) , \tag{4}$$

where $U_{\Delta x}^n = (u_j^n)_j$ and $\bar{U}_{\Delta x}^n = (\bar{u}(x_j, t_n))_j$.

7. Fix the CFL value, such that $\lambda \neq 1$. Study graphically the behaviour of the errors given by (3)-(4) as functions of Δx. We can take $T = 0.75$ and $a = 1$. Verify that our numerical observations agree with the theoretical results.

Bonus

Code a 2D animation to graphically compare the analytic solution and its numerical approximation in time.