1. Tutorial

Exercise 1 (Cards!). Suppose we have a box with 8 cards numbered 1 through 8, and 2 cards with the number 10 written on them. Compute the mean and variance of the distribution corresponding to drawing a card from the box at random and recording the resulting number.

Exercise 2 (Density).

1. Verify that $f(x) = 3x^2$ is a probability density function on the domain $[0,1]$.
2. Find the mean and the variance.

Exercise 3 (Generator). Write MATLAB statements using the function `rand`, which generates a sample uniformly distributed in $[0,1]$, to generate a random number from the following distribution:

- The probability that the number is 0 is 0.6.
- The probability that the number is 1 is 0.4.

Exercise 4 (Buffon's needle). George Louis Leclerc, comte de Buffon, proposed the following problem: if a needle of length l is thrown at random in the middle of a surface on which are drawn parallel lines spaced by $d > l$, what is the probability that the needle crosses one of the lines?

1. Answer the previous question.
2. If we throw the needle n times, how many crossings should we observe?
3. Using this result, propose a Monte Carlo algorithm for the calculation of the constant π.

Exercise 5 (Birthday Paradox). How many people do we need in a room so that there is a probability 0.5 that two people were born on the same day of the year?

2. Practical

Exercise 6 (Approximate calculation of π). We want to estimate the value of π using a set of random points in a unit square.

1. Find a condition that a random point satisfies with probability $\pi/4$.
2. Write an algorithm that calculates the value of π.

Exercise 7 (Volume of the unit sphere). Write a MATLAB program to estimate the volume of the unit sphere $\mathbb{B}_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leqslant 1\}$ using a Monte Carlo method. Estimate the volume of the unit sphere $\mathbb{B}_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leqslant 1\}$.

Exercise 8 (Option pricing with Monte Carlo method)**.** An *option* is a contract that gives the right (but not the obligation) to buy (or sell) at maturity T a unit of an asset (e.g. a stock), at a price K fixed in advance. The purpose of this contract is to hedge the buyer against upward (or downward) fluctuations of this asset, between the present time $t = 0$ and the expiration date $t = T$. The purchaser of this option must pay a premium C at the time of signing the contract.
There are generally two types of options: the *call* and the *put*. The call is a call option. At time 0, the buyer of a call buys at a price C, the right to buy at time T, a given stock at a fixed price K, regardless of the price of the stock at time T.
Let (S_t) be the (random) price of the stock considered at time t. If $S_T > K$, then the call holder exercises his right, and buys the stock at price K and resells it immediately at price S_T; his gain is then $(S_T - K) - C = (S_T - K)_+ - C$ if we note $(X)_+ = \max(0, X)$.
If, on the other hand, $S_T < K$, then the holder does not exercise his right (he does not buy the stock), and his payoff is then $-C = (S_T - K)_+ - C$. In both cases, the payoff is on average $E[(S_T - K)_+] - C$ where $E[\cdot]$ denotes the expectation of the random variable. For the problem to be fair, the expectation of the payoff must be zero, so $C = E[(S_T - K)_+]$. It remains to model the price of the shares. We use the following model: between instants t_i and t_{i+1} the price of the stock varies by

$$S_{t_{i+1}} - S_{t_i} = Y_i S_{t_i},$$

where Y_i are iid variables of centered normal distribution $\mathcal{N}(0,1)$.
The central limit theorem allows us to deduce that $S_T \approx e^Z$ with Z of law $\mathcal{N}(0,1)$. The previous formula was introduced by Black and Scholes which earned them the Nobel Prize in Economics.
The *put* is a put option. At time $t = 0$, the buyer buys at a price P the right to sell a given stock at time T at a fixed price K, regardless of the price of the stock at time T. If $S_T \geqslant K$, the put holder does not exercise his right (he does not sell the stock) and his gain is therefore $-P = (K - S_T) - P$. If, on the other hand, if $S_T < K$, he exercises his right, and sells the stock at price K and buys it back at the current price S_T. His gain is then $(K - S_T) - P = (K - S_T)_+ - P$. On average, the payoff will therefore be $E[(K - S_T)_+] - P$. For the game to be fair, $P = E[(K - S_T)_+]$. We will assume in the following that in the following that $K = 1$.

1. Compute by a Monte Carlo method the quantities $C = E[(e^G - 1)_+]$ and $P = E[(1 - e^G)_+]$, where G follows the normal distribution $\mathcal{N}(0,1)$.
2. Let use denote $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt$. We can show that $C = e^{1/2}\Phi(1) - 1/2$ and $P = 1/2 - e^{1/2}\Phi(-1)$. Compare with the values found previously. One can use the function `erfc`.