Chapter 1

Reminders on matrix topology

On multiple occasions, we will have to examine the convergence of a sequence of vectors or matrices. This is why this chapter will

1.1 Vector and matrix norms

Given an integer $n \geq 0$, we recall that a norm over $\mathbb{C}^n$ is an application $|\cdot|_* : \mathbb{C}^n \to \mathbb{R}_+$ satisfying: for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^n$ and all $\lambda \in \mathbb{C}$

- $|\boldsymbol{v}|_* = 0 \;\Rightarrow\; \boldsymbol{v} = 0$
- $|\boldsymbol{v} + \boldsymbol{w}|_* \leq |\boldsymbol{v}|_* + |\boldsymbol{w}|_*$
- $|\lambda \boldsymbol{v}|_* = |\lambda|\, |\boldsymbol{v}|_*$

In the case of the space $\mathrm{V} = \mathbb{C}^n$, let us mention three classical norms. Given a vector $\boldsymbol{u} = (u_j)_{j=1}^n \in \mathbb{C}^n$, we define

$$
\begin{aligned}
|\boldsymbol{u}|_1 &:= \textstyle\sum_{j=1}^n |u_j| \\
|\boldsymbol{u}|_2 &:= (\textstyle\sum_{j=1}^n |u_j|^2)^{1/2} \\
|\boldsymbol{u}|_\infty &:= \sup_{j=1\ldots n} |u_j|
\end{aligned}
$$

It can be easily verified that each of these three applications is indeed a norm. Besides, let us point that $|\cdot|_2$ is the norm naturally attached to the scalar product $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{u}^\top \overline{\boldsymbol{v}}$ over $\mathbb{C}^n$. Let us recall that these norms are equivalent as $\mathbb{C}^n$ is a finite dimensional space: we have $|\boldsymbol{u}|_1 \leq \sqrt{n}|\boldsymbol{u}|_2 \leq n|\boldsymbol{u}|_\infty \leq n|\boldsymbol{u}|_1$ for all $\boldsymbol{u} \in \mathbb{C}^n$. The choice of a norm $|\cdot|_*$ over $\mathbb{C}^n$ induces a norm over $\mathbb{C}^{n\times n}$ defined by

$$
\|\mathrm{A}\|_* := \sup_{\boldsymbol{u} \in \mathbb{C}^n \setminus \{0\}} \frac{|\mathrm{A}\boldsymbol{u}|_*}{|\boldsymbol{u}|_*} \quad \text{for } \mathrm{A} \in \mathbb{C}^{n\times n}. \tag{1.1}
$$

A norm over $\mathbb{C}^{n\times n}$ taking the form above is said to be subordinated. Subordinated norms possess the following elementary property.

Lemme 1.1.
If $\|\,\|_*$ *is a subordinated norm over* $\mathbb{C}^{n\times n}$ *then* $\|\mathrm{A}^k\|_* \leq \|\mathrm{A}\|_*^k$ *for all* $\mathrm{A} \in \mathbb{C}^{n\times n}, k \geq 0$.

In the following we will denote $\|\cdot\|_1$ the norm subordinated to $|\cdot|_1$. Similarly we will denote $\|\cdot\|_2$ and $\|\cdot\|_\infty$. There exist matrix norms that are not subordinated. Here is an example. Given a matrix $\mathrm{A} = (a_{j,k}) \in \mathbb{C}^{n\times n}$, let $\mathrm{tr}(\mathrm{A}) := \sum_{j=1}^n a_{j,j}$ refer to its trace. In addition, we shall denote $\mathrm{A}^* := (\overline{\mathrm{A}})^\top$ its adjoint i.e. its hermitian transpose. Then the application $(\mathrm{A},\mathrm{B}) \mapsto \mathrm{tr}(\mathrm{B}^*\mathrm{A})$ provides a scalar product over $\mathbb{C}^{n\times n}$. The norm associated to this scalar product, called the Frobenius norm,

$$\|\mathrm{A}\|_{\mathrm{F}} := \sqrt{\mathrm{tr}(\mathrm{A}^*\mathrm{A})} = \sum_{j=1}^n \sum_{k=1}^n |a_{j,k}|^2$$

is not subordinated. Indeed if it was, one would necessarily have $\|\mathrm{Id}\|_* = 1$. However in the case of the Frobenius norm, a direct calculation shows that $\|\mathrm{Id}\|_{\mathrm{F}} = \sqrt{n}$.

Proposition 1.2.
For any matrix $\mathrm{A} = (a_{j,k}) \in \mathbb{C}^{n\times n}$*, we have*

$$\begin{aligned} \|\mathrm{A}\|_1 &= \sup_{k=1\dots n} \sum_{j=1}^n |a_{j,k}| \\ \|\mathrm{A}\|_\infty &= \sup_{j=1\dots n} \sum_{k=1}^n |a_{j,k}| \end{aligned}$$

Proof:
We start by proving the identity for $\|\mathrm{A}\|_1$. Given a vector $\boldsymbol{u} = (u_k)_{k=1}^n \in \mathbb{C}^n \setminus \{0\}$, applying a simple triangular inequality yields

$$\begin{aligned} |\mathrm{A}\boldsymbol{u}|_1 = \sum_{j=1}^n \Big| \sum_{k=1}^n a_{j,k} u_k \Big| &\leq \sum_{j=1}^n \sum_{k=1}^n |a_{j,k}|\,|u_k| \\ &\leq \sum_{k=1}^n |u_k| \Big(\sum_{j=1}^n |a_{j,k}| \Big) \leq |\boldsymbol{u}|_1 \sup_{k=1\dots n} \sum_{j=1}^n |a_{j,k}| \end{aligned}$$

Since this holds for all $\boldsymbol{u} \in \mathbb{C}^n \setminus \{0\}$, dividing the last inequality by $|\boldsymbol{u}|_1$ and taking the supremum with respect to $\boldsymbol{u}$, we obtain that $\|\mathrm{A}\|_1 \leq \sup_{k=1\dots n} \sum_{j=1}^n |a_{j,k}|$. To conclude we can construct a $\boldsymbol{u}_\star \in \mathbb{C}^n$ such that $|\mathrm{A}\boldsymbol{u}_\star|_1/|\boldsymbol{u}_\star| = \sup_{k=1\dots n} \sum_{j=1}^n |a_{j,k}|$. Pick $k_\star \in \{1\dots n\}$ such that $\sup_{k=1\dots n} \sum_{j=1}^n |a_{j,k}| = \sum_{j=1}^n |a_{j,k_\star}|$. It suffices then to define $\boldsymbol{u}_\star = (u_k)$ by $u_k = 0$ if $k \neq k_\star$ and $u_{k_\star} = 1$.

Let us now examine the case of $\|\mathrm{A}\|_\infty$. Similarly to what precedes, for an arbitrary $\boldsymbol{u} \in \mathbb{C}^n$ we have

$$|\mathrm{A}\boldsymbol{u}|_\infty = \sup_{j=1\dots n} \Big| \sum_{k=1}^n a_{j,k} u_k \Big| \leq \sup_{j=1\dots n} \sum_{k=1}^n |a_{j,k}|\,|\boldsymbol{u}|_\infty.$$

Again dividing by $|\boldsymbol{u}|_\infty$ and taking the supremum of the left hand side with respect to $\boldsymbol{u}$, we obtain $\|\mathrm{A}\|_\infty \leq \sup_{j=1\dots n} \sum_{k=1}^n |a_{j,k}|$. Let us prove that this upper bound is reached. Choose $j_\star$ such that $\sum_{k=1}^n |a_{j_\star,k}| = \sup_{j=1\dots n} \sum_{k=1}^n |a_{j,k}|$. We can take $\boldsymbol{u}_\star = (u_k) \in \mathbb{C}^n$ with entries given by $u_k = \overline{a}_{j_\star,k}/|a_{j_\star,k}|$, and we then obtain $|\mathrm{A}\boldsymbol{u}|_\infty/|\boldsymbol{u}|_\infty = \sup_{j=1\dots n} \sum_{k=1}^n |a_{j,k}|$. □

Given a matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$, we denote $\sigma(\mathrm{A})$ the spectrum of A defined as the set of its eigenvalues $\sigma(\mathrm{A}) = \{\lambda \in \mathbb{C},\ \mathrm{Ker}(\lambda \mathrm{Id} - \mathrm{A}) \neq \{0\}\}$. On the other hand, we will denote $\varrho(\mathrm{A}) = \sup\{|\lambda|,\ \lambda \in \sigma(\mathrm{A})\}$, the spectral radius of the matrix A.

Proposition 1.3.
For any matrix $\mathrm{A} = (a_{j,k}) \in \mathbb{C}^{n\times n}$, *we have* $\|\mathrm{A}\|_2 = \sqrt{\varrho(\mathrm{A}^*\mathrm{A})}$

Proof:
Remind that a matrix $\mathrm{U} \in \mathbb{C}^n$ is called unitary if and only if $|\mathrm{U}\boldsymbol{x}|_2 = |\boldsymbol{x}|_2$ for all $\boldsymbol{x} \in \mathbb{C}^n$. We also recall that $\mathrm{U}^{-1} = \mathrm{U}^*$ when U is unitary. As the matrix $\mathrm{A}^*\mathrm{A}$ is symteirc positive, we know that there exists a unitary matrix $\mathrm{U} \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_j \in \sigma(\mathrm{A}) \subset \mathbb{R}_+$ such that $\mathrm{A}^*\mathrm{A} = \mathrm{U}^*\Lambda\mathrm{U}$. Renumbering if necessary, we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, so that $\lambda_1 = \varrho(\mathrm{A}^*\mathrm{A})$. Given a vector $\boldsymbol{x} \in \mathbb{C}^n$, set $\boldsymbol{y} = \mathrm{U}\boldsymbol{x}$ and denote $\boldsymbol{y} = (y_j)_{j=1}^n$ the entries of this vector. Then we have

$$\begin{aligned}|\mathrm{A}\boldsymbol{x}|_2^2 = \boldsymbol{x}^*\mathrm{A}^*\mathrm{A}\boldsymbol{x} = (\mathrm{U}\boldsymbol{x})^*\mathrm{D}(\mathrm{U}\boldsymbol{x}) = \boldsymbol{y}^*\Lambda\boldsymbol{y} = \sum_{j=1}^n \lambda_j |y_j|^2 \\ \leq \lambda_1 \sum_{j=1}^n |y_j|^2 = \lambda_1 |\boldsymbol{y}|_2^2 = \lambda_1 |\mathrm{U}\boldsymbol{x}|_2^2 = \varrho(\mathrm{A}^*\mathrm{A})|\boldsymbol{x}|_2^2.\end{aligned}$$

Since this holds for any $\boldsymbol{x} \in \mathbb{C}^n \setminus \{0\}$, we can divide by $|\boldsymbol{x}|_2^2$ and take the supremum. We finally obtain that $\|\mathrm{A}\|_2^2 \leq \varrho(\mathrm{A}^*\mathrm{A})$.

To conclude, let us choose $\boldsymbol{v} \in \mathbb{C}^n \setminus \{0\}$ as an eigenvector of $\mathrm{A}^*\mathrm{A}$ attached to the eigenvalue λ_1. Then we have $|\mathrm{A}\boldsymbol{v}|_2^2 = \boldsymbol{v}^*\mathrm{A}^*\mathrm{A}\boldsymbol{v} = \lambda_1|\boldsymbol{v}|_2^2$, which implies $|\mathrm{A}\boldsymbol{v}|_2^2/|\boldsymbol{v}|_2^2 = \varrho(\mathrm{A}^*\mathrm{A})$. This ends the proof. □

In the particular case where A is hermitian $\mathrm{A}^* = \mathrm{A}$ we have $\sqrt{\varrho(\mathrm{A}^*\mathrm{A})} = \sqrt{\varrho(\mathrm{A}^2)} = \sqrt{\varrho(\mathrm{A})^2} = \varrho(\mathrm{A})$ as $\sigma(\mathrm{A}^2) = \{\lambda^2,\ \lambda \in \sigma(\mathrm{A})\}$. We then conclude that $\|\mathrm{A}\|_2 = \varrho(\mathrm{A})$ for the particular case of an hermitian matrix.

1.2 Condition number

Although to any norm is attached a condition number, most of the time one considers the so-called "quadratic" condition number attached to the norm $\|\cdot\|_2$. For a matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$, it is defined by

$$\mathrm{cond}_2(\mathrm{A}) := \|\mathrm{A}\|_2\|\mathrm{A}^{-1}\|_2. \tag{1.2}$$

This quantity is only meaningful for an invertible matrix i.e. whose kernel is trivial $\mathrm{Ker}(\mathrm{A}) = \{0\}$. The condition number is systematically greater than 1. Indeed $1 = \|\mathrm{Id}\|_2 = \|\mathrm{A}\cdot\mathrm{A}^{-1}\|_2 \leq \|\mathrm{A}\|_2\|\mathrm{A}^{-1}\|_2 = \mathrm{cond}_2(\mathrm{A})$. On the other hand, applying the remark after Proposition 1.3, we deduce that $\mathrm{cond}_2(\mathrm{A}) = |\lambda_1|/|\lambda_n|$ for an hermitian matrix A, assuming that the eigenvalues are arranged by ascending order $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. The next proposition shows that the condition number quantifies the sensibility of a linear system with respect to perturbations.

Théorème 1.4.
Consider $\mathrm{A} \in \mathbb{C}^{n\times n}$ *invertible and* $\boldsymbol{b}, \delta\boldsymbol{b} \in \mathbb{C}^n$ *with* $\boldsymbol{b} \neq 0$. *Let* $\boldsymbol{x}, \delta\boldsymbol{x} \in \mathbb{C}^n$ *refer to vectors such that* $\mathrm{A}\boldsymbol{x} = \boldsymbol{b}$ *and* $\mathrm{A}(\boldsymbol{x} + \delta\boldsymbol{x}) = \boldsymbol{b} + \delta\boldsymbol{b}$. *Then we have*

$$\frac{|\delta\boldsymbol{x}|_2}{|\boldsymbol{x}|_2} \leq \mathrm{cond}_2(\mathrm{A})\frac{|\delta\boldsymbol{b}|_2}{|\boldsymbol{b}|_2}.$$

Proof:

We have $\mathrm{A}(\boldsymbol{x}+\delta\boldsymbol{x}) = \boldsymbol{b}+\mathrm{A}\delta\boldsymbol{x} = \boldsymbol{b}+\delta\boldsymbol{b}$ hence $\mathrm{A}\delta\boldsymbol{x} = \delta\boldsymbol{b}$ and thus $\boldsymbol{x} = \mathrm{A}^{-1}\delta\boldsymbol{b}$.We then deduce $|\boldsymbol{b}|_2 \leq \|\mathrm{A}\|_2|\boldsymbol{x}|_2$ on the one hand, and $|\delta\boldsymbol{x}|_2 \leq \|\mathrm{A}^{-1}\|_2|\delta\boldsymbol{b}|_2$ on the other hand. Multiplying the last two inequalities, and dividing by $|\boldsymbol{x}|_2|\boldsymbol{b}|_2$, we finally obtain the required inequality. □

1.3 Spectral radius

Proposition 1.5.
For any square matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$, *there exists an invertible matrix* $\mathrm{P} \in \mathbb{C}^{n\times n}$ *such that* PAP^{-1} *is upper triangular.*

Proof:

We prove this result by recurrence over the dimension n. It obviously holds for $n = 1$. Suppose the result holds for $n-1$ and let us prove that it still holds for n. Pick an arbitrary square matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$. There exists $\lambda \in \mathbb{C}$ and $\boldsymbol{e}_1 \in \mathbb{C}^n \setminus \{0\}$ such that $\mathrm{A}\boldsymbol{e}_1 = \lambda\boldsymbol{e}_1$. We can find linearly independent vectors $\boldsymbol{e}_2,\dots,\boldsymbol{e}_n$ such that $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ forms a basis of $\mathbb{C}^n$. For each $k = 2\dots n$, there are coefficients α_k and $\beta_{j,k} \in \mathbb{C}$, $j = 2\dots n$ such that

$$\mathrm{A}\boldsymbol{e}_k = \alpha_k\boldsymbol{e}_1 + \sum_{j=2}^{n}\beta_{j,k}\boldsymbol{e}_j \quad k = 2\dots n \tag{1.3}$$

Note $\mathrm{B} = (\mathrm{B}_{j,k}) \in \mathbb{C}^{(n-1)\times(n-1)}$ defined by the coefficients $\mathrm{B}_{j,k} = \beta_{j+1,k+1}$, as well as the vector $\boldsymbol{\alpha} := (\alpha_2,\dots,\alpha_n)^\top \in \mathbb{C}^{n-1}$. Let us also define $\mathrm{P} := [\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ the matrix associated to basis we have just defined. Writing the matrix representation of (1.3) then leads to

$$\mathrm{P}^{-1}\cdot\mathrm{A}\cdot\mathrm{P} = \begin{bmatrix} \lambda & \boldsymbol{\alpha}^\top \\ 0 & \mathrm{B} \end{bmatrix} \tag{1.4}$$

By recurrence hypothesis, there exists $\mathrm{Q} \in \mathbb{C}^{(n-1)\times(n-1)}$ invertible such that $\mathrm{T} := \mathrm{Q}^{-1}\mathrm{BQ} \in \mathbb{C}^{(n-1)\times(n-1)}$ is upper tirangular. Setting $\mathrm{R} = \mathrm{diag}(1,\mathrm{Q}) \in \mathbb{C}^{n\times n}$, we then obtain

$$(\mathrm{PR})^{-1}\cdot\mathrm{A}\cdot(\mathrm{PR}) = \mathrm{R}^{-1}\cdot(\mathrm{P}^{-1}\mathrm{AP})\cdot\mathrm{R} = \begin{bmatrix} \lambda & \boldsymbol{\alpha}^\top\mathrm{Q} \\ 0 & \mathrm{T} \end{bmatrix}$$

Note that PR is invertible since both P and R are invertible. As the matrix in the right hand side above is uppoer triangular, the preceding identity exhibits a trigonalisation of the matrix A which concludes the recurrence, and hence the proof. □

Proposition 1.6.
For any subordinated matrix norm $\|\ \|_*$ *over* $\mathbb{C}^{n\times n}$ *we have* $\varrho(\mathrm{A}) \leq \|\mathrm{A}\|_*$, $\forall\mathrm{A} \in \mathbb{C}^{n\times n}$. *Reciprocally, for any matrix* $\mathrm{A} \in \mathbb{C}^{n\times n}$ *and any* $\epsilon > 0$, *there exists a subordinated norm* $\|\ \|_*$ *over* $\mathbb{C}^{n\times n}$ *such that*

$$\|\mathrm{A}\|_* \leq \varrho(\mathrm{A}) + \epsilon. \tag{1.5}$$

Proof:

First consider a matrix norm $\| \ \|_*$ over $\mathbb{C}^{n\times n}$ subordinated to the vector norm $| \ |_*$. For a matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$, take $\lambda \in \sigma(\mathrm{A})$ such that $|\lambda| = \varrho(\mathrm{A})$, and let $\boldsymbol{x}_0 \in \mathbb{C}^n \setminus \{0\}$ be an eigenvector of A attached to this eigenvalue. Then we have

$$\varrho(\mathrm{A}) = \frac{|\mathrm{A}\boldsymbol{x}_0|_*}{|\boldsymbol{x}_0|_*} \leq \sup_{\boldsymbol{x}\in\mathbb{C}^n\setminus\{0\}} \frac{|\mathrm{A}\boldsymbol{x}|_*}{|\boldsymbol{x}|_*} \leq \|\mathrm{A}\|_* \tag{1.6}$$

This proves the first part of the proposition. Next consider a matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$. We have to propose a subordinated norm satisfying (1.5). According to Proposition 1.5, there exists an invertible matrix $\mathrm{P} \in \mathbb{C}^{n\times n}$ such that $\mathrm{T} := \mathrm{P}^{-1}\mathrm{AP}$ is upper triangular,

$$\mathrm{T} = \begin{bmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{n,n} \end{bmatrix}.$$

Note that the entries $t_{j,j}, j = 1\dots n$ are the eigenvalues of the matrix A. Given some $\delta > 0$ that we shall choose a posteriori, set $\mathrm{D}_\delta := \mathrm{diag}(1, \delta, \dots, \delta^{n-1})$ and define the matrix $\mathrm{T}_\delta := \mathrm{D}_\delta^{-1}\mathrm{P}^{-1}\mathrm{APD}_\delta = (\mathrm{PD}_\delta)^{-1}\mathrm{A}(\mathrm{PD}_\delta) = \mathrm{D}_\delta^{-1}\mathrm{TD}_\delta$. Examining its values, we see that

$$\mathrm{T}_\delta = \begin{bmatrix} t_{1,1} & \delta t_{1,2} & \cdots & \delta^{n-1}t_{1,n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \delta t_{n-1,n} \\ 0 & \cdots & 0 & t_{n,n} \end{bmatrix}$$

The matrix above decomposes as $\mathrm{T}_\delta = \Lambda + \mathrm{R}_\delta$ where $\Lambda = \mathrm{diag}(t_{1,1}, \dots, t_{n,n})$ and R_δ is the upper triangular part located strictly above the diagonal. According to Proposition 1.2, we have $\lim_{\delta\to 0} \|\mathrm{R}_\delta\|_1 = 0$. Given some ϵ, we can choose δ small enought to garantee that $\|\mathrm{R}_\delta\|_1 \leq \epsilon$. In addition, one readily checks that $\|\Lambda\|_1 = \max_{j=1\dots n} |t_{j,j}| = \varrho(\mathrm{A})$. With δ chosen as indicated, we obtain $\|\mathrm{T}_\delta\|_1 \leq \varrho(\mathrm{A}) + \epsilon$. Now set $\mathrm{P}_\delta := \mathrm{PD}_\delta$ and consider the norm $|\boldsymbol{x}|_* := |\mathrm{P}_\delta^{-1}\boldsymbol{x}|_1$. With the matrix norm $\| \ \|_*$ subordinated to $| \ |_*$, the matrix A then satisfies

$$\|\mathrm{A}\|_* = \sup_{\boldsymbol{x}\in\mathbb{C}^n\setminus\{0\}} \frac{|\mathrm{A}\boldsymbol{x}|_*}{|\boldsymbol{x}|_*} = \sup_{\boldsymbol{x}\in\mathbb{C}^n\setminus\{0\}} \frac{|\mathrm{P}_\delta^{-1}\mathrm{AP}_\delta\boldsymbol{x}|_1}{|\boldsymbol{x}|_1} = \|\mathrm{T}_\delta\|_1 \leq \varrho(\mathrm{A}) + \epsilon$$

$\square$

Proposition 1.7.
Given a $\mathrm{A} \in \mathbb{C}^{n\times n}$, *the following conditions are equivalent:*

i) $\lim_{k\to\infty} \mathrm{A}^k = 0$

ii) $\lim_{k\to\infty} \mathrm{A}^k\boldsymbol{x} = 0$ *for all* $\boldsymbol{x} \in \mathbb{C}^n$

iii) $\varrho(\mathrm{A}) < 1$

iv) $\|\mathrm{A}\|_* < 1$ *for at least one subordinated norm* $\| \ \|_*$.

Proof:

The implication *i*) $\Rightarrow$ *ii*) is obvious. To prove *ii*) $\Rightarrow$ *iii*), let us establish the contrapositive, assuming that $\varrho(\mathrm{A}) \geq 1$. Then there exists $\boldsymbol{x} \in \mathbb{C}^n$ such that $|\boldsymbol{x}|_2 = 1$ and $\mathrm{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ for a certain $\lambda \in \mathbb{C}$ satisfying $|\lambda| \geq 1$, and we have $|\mathrm{A}^k \boldsymbol{x}|_2 = |\lambda|^k \geq 1$ so $\lim_{k\to\infty} \mathrm{A}^k \boldsymbol{x} \neq 0$ and *ii*) does not hold. The implication *iii*) $\Rightarrow$ *iv*) is a direct consequence of Proposition 1.6. The implication *iv*) $\Rightarrow$ *i*) is a direct consequence of Lemma 1.1. $\square$