
Programming classes N° 1

Numerical schemes for Cauchy problems.

- ⋆ Important information is available on the Moodle page.
- ⋆ In `Python`, we will avoid, as much as possible, to use the `list` type. Instead, we will make use of the `array` type as defined in the `NumPy` module.

```
import numpy as np
```

The objective of these exercises is to study different numerical schemes to solve initial value problems (Cauchy problems). After this class, it will be mandatory to be able to:

- implement a simple Euler scheme,
- graphically compare the solution of a numerical scheme with the one of the continuous problem,
- study the convergence of the scheme.

We are interested in approximating the solution of an initial value problem (Cauchy problem): let $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, find a real valued function u defined on $[0,T]$ such that,

$$(P) \qquad \begin{cases} u'(t) = F(t, u(t)) \quad \text{for} \quad t \in]0,T[, \\ u(0) = u_D \in \mathbb{R}^d. \end{cases}$$

We are going to study different numerical schemes to approximate the solution of the problem (P). Let $N \in N^*$, a non-zero integer. Let us introduce the time discretisation grid: $t_n = nh$, $n = 0, 1, \ldots, N$ with $h = T/N$. The numerical methods will compute approximations $\{u_n\}_{0 \le n \le N}$ of the solution values $\{u(t_n)\}_{0 \le n \le N}$.
We give, hereafter, three examples of such numerical methods:

$$\text{Explicit Euler} \qquad \begin{cases} u_0 = u_D \\ u_{n+1} = u_n + hF(t_n, u_n) \ , \ n = 0, 1, \ldots, N-1 \end{cases}$$

$$\text{Runge–Kutta Heun (RK2)} \qquad \begin{cases} u_0 = u_D \\ k_n^{(1)} = F(t_n, u_n) \\ k_n^{(2)} = F(t_n + h/2, u_n + k_n^{(1)} h/2) \\ u_{n+1} = u_n + hk_n^{(2)} \ , \ n = 0, 1, \ldots, N-1 \end{cases}$$

$$\text{Crank-Nicolson (CN)} \qquad \begin{cases} u_0 = u_D \\ u_{n+1} = u_n + \dfrac{h}{2} \left(F(t_{n+1}, u_{n+1}) + F(t_n, u_n) \right), n = 0, 1, \ldots, N-1 \end{cases}$$

The RK2 Heun method can be seen as a composition of two Explicit Euler scheme when introducing the midpoints of the intervals (t_n, t_{n+1}). The CN method can be seen as the average between an Explicit and Implicit Euler. When studying the scheme, assessing the convergence, we will change the discretisation parameter N. We will make vary h accordingly.

Exercise 1 Let us consider the following problem: let $a \in \mathbb{R}$, find an approximation of the real valued function u defined on $[0, T]$ such that:

$$(P_1) \qquad \begin{cases} u'(t) = au(t) \ , \ t > 0 \ , \\ u(0) = u_D . \end{cases}$$

The problem (P_1) admits the exact solution: $u(t) = u_D e^{at}$. From now on we will take $T = 1$, $a = -1$ and $u_D = 1$.

1. Find F such that the problem P_1 can be rewritten in the form (P).
2. Implement the Explicit Euler scheme, take $N = 25$ and compare graphically the so obtained approximation with the exact solution.
3. Do the same with RK2 Heun scheme.
4. For these schemes, let us make N vary ($N = 25$, $N = 50$, $N = 100$, $N = 200$). Plot the results. What do you observe?
5. For the two implemented schemes, let us compute the ℓ^∞ norm of the error as function of N ($N = 25$, $N = 50$, $N = 100$, $N = 200$):
$$e_\infty := \max_{0 \le n \le N} |u(t_n) - u_n| \ .$$
6. Represent the errors as function of h in loglog scale. In order to highlight the order of the scheme we will plot (on the same figure) the functions $h \mapsto h$ et $h \mapsto h^2$.
7. Do the same study for the Crank-Nicolson scheme.

Exercise 2 The pendulum problem reads as follows: find the real valued function φ defined on $[0, T]$ such that:

$$(P_2) \qquad \begin{cases} \varphi''(t) + \frac{g}{L} \sin \varphi(t) = 0 \ , \ t > 0 \ , \\ \varphi(0) = \varphi_0 \ , \\ \varphi'(0) = \psi_0 \ , \end{cases}$$

where $g > 0$ is the gravity acceleration, L the pendulum length. The function φ represents the pendulum angle with respect to the vertical axis. We will take: $T = 50$, $g = 10$, $L = 1$ and the initial condition: $\varphi_0 = 1$ et $\psi_0 = 0$.

1. Find u and F such that the problem (P_2) can be rewritten as (P). Let us remark that, by assuming the notation of (P), $d = 2$ in this example.
2. Implement the Explicit Euler and the RK2 Heun schemes. Choose different values of N and test these schemes.
3. Plot the angle as function of time. Plot the angular velocity as function of the angle. What do you observe? Take $N = 2500$, $N = 10000$ et $N = 40000$ and repeat the computation.
4. Implement the CN scheme and repeat these computations.