Sorbonne Université, Master 1 Mathématiques et applications 2022–2023
4MA106 Foundations of mathematical methods

Programming classes N° 3
Approximation of the heat equation solution

Let $L > 0$ the length of a segment in space, $T > 0$ a final time, $\omega > 0$ the diffusivity, we look for an approximation of

$$\bar{u} : [0, L] \times [0, T] \longrightarrow \mathbb{R} ,$$

solution of:

$$\begin{cases} \partial_t \bar{u}(x,t) \;-\; \omega\; \partial_{xx}\bar{u}(x,t) \;=\; 0, & \forall (x,t) \in]0,L[\times]0,T] \;, \\ \bar{u}(0,t) \;=\; \bar{u}(L,t) \;=\; 0, & \forall t \in [0,T] \;, \\ \bar{u}(x,0) \;=\; u_{\text{ini}}(x), & \forall x \in [0,L] \;. \end{cases} \tag{1}$$

The initial datum satisfies the boundary conditions: $u_{\text{ini}}(0) = u_{\text{ini}}(L) = 0$.

The numerical solution is defined as a sequence of values, denoted by u_j^n, which are an approximation of the values the solution attains in $x_j = j\Delta x$ et $t_n = n\Delta t$: $u_j^n \approx \bar{u}(x_j, t_n)$.

First, we study the scheme obtained by discretising in time by means of Explicit Euler and using centred finite differences in space. This leads to:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} \;+\; \omega\; \frac{-u_{j-1}^n + 2u_j^n - u_{j+1}^n}{\Delta x^2} \;=\; 0 \;. \tag{2}$$

Let us recall that this scheme has a consistency error of order $\mathcal{O}(\;\Delta t + \Delta x^2\;)$ and it is stable for the ℓ^2 et ℓ^∞ norms under the CFL condition (referred to as parabolic CFL in this case): $2\omega\Delta t \leq \Delta x^2$. In what follows, let us set $\lambda = \omega\frac{\Delta t}{\Delta x^2}$, so that the CFL condition reads: $\lambda \leq \frac{1}{2}$.

Q0. Use (2) and write u_j^{n+1} as a linear combination of $(u_j^n)_j$, by making λ appearing explicitly. Let $J \in \mathbb{N}^*$, we set $\Delta x = \frac{L}{J+1}$. Propose a space discretisation of $[0, L]$. We have then $\Delta t = \frac{\lambda}{\omega}\Delta x^2$. We introduce $M \in \mathbb{N}^*$ such that $M\Delta t \leq T < (M+1)\Delta t$. Write down the numerical scheme by introducing, at discrete level, initial and boundary conditions.

Q1. Code the explicit scheme derived above:
`Input :` ω, J, L, T, u_{ini}, λ
`Output :` $(x_j)_j$, t_M, $(u_j^M)_j$
The advantage of an explicit scheme consists also in the fact that it needs a limited amount of storage. In this exercise try to deal with the memory in an efficient way.

Q2. In order to validate the code, we consider a test case with analytic solution:

$$\bar{u}(x,t) \;=\; \sin\left(\frac{\pi x}{L}\right) \exp\left(-\frac{\pi^2 \omega t}{L^2}\right) \;. \tag{3}$$

(a) Show that the function in Eq.(3) is a solution of (1) for a function u_{ini} to be determined.

(b) Compare graphically the exact solution at final time T with its approximation obtained by applying (2). We will take: $\omega = 0.15$, $L = 5$, $T = 2$ and different values of J and λ which you have to choose in order to provide a meaningful validation.

(c) We have to study the convergence of the method. To this end, we define two errors:

$$\varepsilon^{(2)}_{\Delta t,\Delta x} = \max_{n\geq 0} \left(\|U^n_{\Delta x} - \bar{U}^n_{\Delta x}\|_{2,\Delta} \right) , \tag{4}$$

$$\varepsilon^{(\infty)}_{\Delta t,\Delta x} = \max_{n\geq 0} \left(\|U^n_{\Delta x} - \bar{U}^n_{\Delta x}\|_{\infty,\Delta} \right) , \tag{5}$$

where $U^n_{\Delta x} = (u^n_1, \dots, u^n_J)^T$ et $\bar{U}^n_{\Delta x} = (\bar{u}(x_1, t_n), \dots, \bar{u}(x_J, t_n))^T$. Let the CFL be fixed, study the error behaviours (4)-(5) as function of Δx. What do you observe?

(d) Choose $\lambda = \frac{1}{6}$ and repeat this study. You should observe a phenomenon called super-convergence. To get some theoretical insight, derive the expression of the consistency error by keeping a higher order, when the exact solution is (3).

Q3. (Bonus). Verify, by hand, that the scheme written in Eq.(2) is equivalent to the following finite volumes scheme:

$$\Delta x \frac{u^{n+1}_j - u^n_j}{\Delta t} + f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}} = 0 \ , \ \text{avec} \ f^n_{j+\frac{1}{2}} = -\omega \frac{u^n_{j+1} - u^n_j}{\Delta x} \ . \tag{6}$$

Implement the scheme (6) by following the pseudo-code written hereafter and verify that you find the same results as in **Q2**.

For $t^n \to t^{n+1}$:

  For to compute $f^n_{j+\frac{1}{2}}$

  For to update $u^{n+1}_j \leftarrow u^n_j - \frac{\Delta t}{\Delta x}\left(f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}}\right)$