Advanced Numerical Algorithms (MU4IN920)

Tutorial-Practical 2 - Iterative methods for solving linear systems

1. Tutorial

Exercise 1 (Diagonally dominant matrix). Let $A = (a_{ij})$ be a matrix of size $n \times n$ with complex coefficients.

1. The matrix A is said to be *strictly diagonally dominant* if for all $i = 1 : n$, we have

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|.$$

Show that if A is strictly diagonally dominant then it is nonsingular.

2. What can you say about the following matrix

$$\begin{pmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{pmatrix}?$$

Exercise 2 (Convergence of Jacobi method). Let $A = (a_{ij})$ be a matrix of size $n \times n$ with real coefficients and b a vector of size n. Let D be the matrix consisting of the diagonal of A.

1. Show that if A is strictly diagonally dominant matrix, then Jacobi method converges whatever the initial vector x_0 is.

2. Let A be a symmetric positive definite matrix with $A = M - N$ (with M a nonsingular matrix). We suppose also that $M^T + N$ is symmetric positive definite. Show that the iterative method

$$Mx_{k+1} = Nx_k + b$$

converges to the solution of the system $Ax = b$.

3. Deduce that if A and $2D - A$ are symmetric positive definite, then Jacobi method converges.

Exercise 3 (Successive over-relaxation). Show that the SOR method can converge only if $\omega \in]0, 2[$.

Exercise 4 (Conjugate gradient algorithm). We use the notations seen in class for the conjugate gradient algorithm. Show that we have

1. $p_k^T r_0 = p_k^T r_k$;
2. $p_i^T r_j = 0$ pour $i < j$;
3. $p_k^T r_k = r_k^T r_k$;
4. $r_i^T r_j = 0$ pour $i \neq j$.

2. Practical

Exercise 5. Implement the Jacobi, Gauss-Seidel, SOR and conjugate gradient methods. Compare the performances of these algorithms.