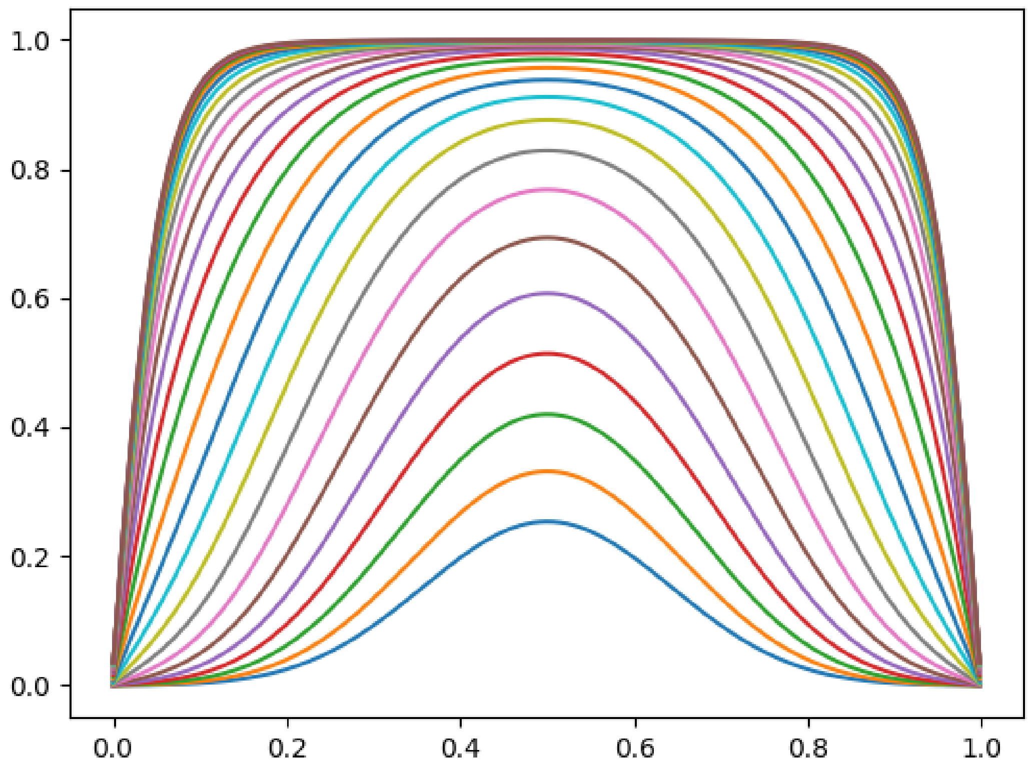
Sorbonne Université

4MA106 Foundations of numerical methodst

Fisher-Kolmogorov-Petrovski-Piskunov equation



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1 Introduction

$\bar{\Omega} = [0, 1]$ is the spatial domain and $t \in [0, 1]$ the time. The diffusion coefficient is ν = 0.001 and the reaction rate is $\gamma = 10$. We consider the following equation :

$$\begin{cases} \partial_t u = \nu \Delta u + \gamma u(1-u) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0,25sin(\pi x)exp(-20(x-0,5)^2) \end{cases} \tag{1}$$

And the following scheme :

$$\frac{u^{(n+1)} - u^{(n)}}{\Delta t} = \nu \partial_x^2 u^{(n+1)} + \gamma u^{(n+1)} - \gamma u^{(n)} u^{(n+1)} \tag{2}$$

2 Exercice 1

2.1 First scheme

We know that

$$\nu \partial_x^2 u_j^{(n+1)} = \nu \frac{u_{j+1}^{(n)} + u_{j-1}^{(n)} - 2u_j^{(n)}}{\Delta x^2} \tag{3}$$

So with (2) and (3) we have

$$u_j^{(n+1)} = \Delta t \nu \frac{u_{j+1}^{(n)} + u_{j-1}^{(n)} - 2u_j^{(n)}}{\Delta x^2} - \Delta t \gamma u_j^{(n+1)} - \Delta t \gamma u^{(n)} u^{(n+1)}$$

Therefore, we can write the following matrices :

A = $\begin{pmatrix} 1 + \frac{2\nu\Delta t}{\Delta x^2} - \gamma\Delta t & -\frac{\nu\Delta t}{\Delta x^2} & \cdots & 0 \\ -\frac{\nu\Delta t}{\Delta x^2} & 1 + \frac{2\nu\Delta t}{\Delta x^2} - \gamma\Delta t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + \frac{2\nu\Delta t}{\Delta x^2} - \gamma\Delta t \end{pmatrix}$

$$B = \begin{pmatrix} \Delta t \gamma Uh & 0 & \cdots & 0 \\ 0 & \Delta t \gamma Uh & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta t \gamma Uh \end{pmatrix}$$

F = $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Such that :

$$\left[A + B(U_h^{(n)})\right] U_h^{(n+1)} = U_h^{(n)} + F$$

Then, we can implement it in python and plot the curve for different values of time (FIGURE 1)

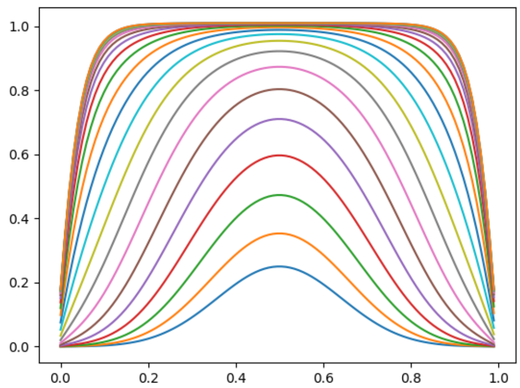

FIGURE 1 – numerical approximation with Nx = 100 and Nt until 100

We observe a curve between 0 and 1 in time and space so seems good.

2.2 Convergence analysis

We have to compute :

$\varepsilon^2_{\Delta t,\Delta x} = \max_{n\geq 0} \|U^n_{\Delta x} - \bar{U}^n_{\Delta x}\|_{2,\Delta}$

and $\varepsilon^\infty_{\Delta t,\Delta x} = \max_{n\geq 0} \|U^n_{\Delta x} - \bar{U}^n_{\Delta x}\|_{\infty,\Delta}$

We can do this numerically considering $\bar{U}$ as U with Nt = 1000 and Nx = 1000 and we except that the error is decreasing when Nt andd Nx are increasing

Remarks that I write in my code function to plot only with variation of Nx or Nt (where the evolution of the error is more visible, but I don't want to overload this report with too much figures

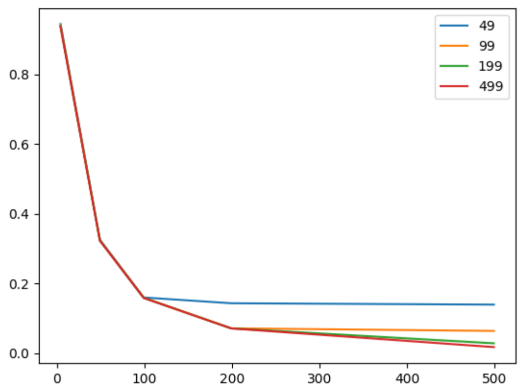

FIGURE 2 – $\varepsilon^2_{\Delta t,\Delta x}$ with Nx = 100 and some value of Nt

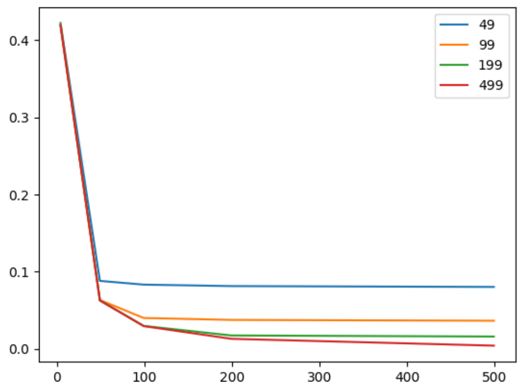

FIGURE 3 – $\varepsilon^{\infty}_{\Delta t,\Delta x}$with Nx = 100 and some value of Nt

We see that the error decrease as an negative exponential, so the convergence of the scheme is really good

3 Exercice 2

3.1 Second scheme : Strang splitting

We will write the numerical scheme with strang splitting.
So, the equation is divided in 3 steps, a half diffusion step, a reaction step and another half diffusion step.

Firstly, the diffusion step is given by :

$$\frac{u_j^{n_{tmp}} - u_j^n}{\frac{\Delta t}{2}} = \nu \partial_x^2 u_j^{n_{tmp}} \tag{4}$$

And with (3) we have :

$$u_j^n = \frac{-\Delta t \nu}{2}\left(\frac{u_{j+1}^{n_{tmp}} + u_{j-1}^{n_{tmp}} - 2u_j^{n_{tmp}}}{2\Delta x^2}\right) + u_j^{n_{tmp}} (5)$$

With (5) we can write the matrix A such that :

$\mathrm{U}_h^n = AU_h^{n_{tmp}}(6)$

Let $\lambda = \frac{\nu \Delta t}{4\Delta x^2}$

So A = $\begin{pmatrix} 1+2\lambda & -\lambda & \cdots & 0 \\ -\lambda & 1+2\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+2\lambda \end{pmatrix}$

So we have to solve the linear system (6) for the diffusion step.
Secondly, the reaction step is given by :

$$\frac{u^{n_{tmp}} - u^n}{\Delta t} = \gamma u^{n_{tmp}}(1 - u^{n_{tmp}}) \tag{7}$$

we will use the notation $x = u^{n_{tmp}}$ and $y = u^n$ for more clarity

solve (7) is equivalent to solve :

$$y = -\Delta t \gamma x(1-x) + x \tag{8}$$

$\Leftrightarrow 0 = \Delta t \gamma x^2 + x(-\Delta t \gamma + 1) - y$

To solve this, we can comput the discriminant :

$\Delta = (-\Delta t\gamma + 1)^2 + 4y\Delta t\gamma$

Assume that $y \in [0, 1]$, $\Delta t > 0$and $\gamma > 0$
Then $\Delta > 0$ exept if $y = 0$ and $-\Delta t\gamma + 1 = 0$
$\Leftrightarrow \Delta t = \frac{1}{\gamma}$
For us, $\gamma = 10$ so this case will append if Nt = 10 We have to be careful about this special Nt value

So there is two roots :

$x = \frac{\Delta t\gamma - 1 \pm \sqrt{(-\Delta t\gamma+1)^2+4y\Delta t\gamma}}{2\Delta t\gamma}$ (9)

Assume that $\Delta t \leq \frac{1}{\gamma}$ so 1 - $\Delta t\gamma \geq 0$
In our case, $Nt \geq 10$
x must be non negative because it must be in [0,1]

claim : if we choose the + root and if $y \in [0, 1]$ then $x \in [0, 1]$
Proof :
Consider $\phi : y \longrightarrow \frac{\Delta t\gamma - 1 + \sqrt{(-\Delta t\gamma+1)^2+4y\Delta t\gamma}}{2\Delta t\gamma}$
We compute the derivative of and it is positive, so ϕ is croissant. And $\phi(0) = 0$ and $\phi(1) = 1$ Because ϕ is continuous, then it takes value in [0,1]
Therefore, the sign in the discriminant must be a plus

So, to solve the reaction step, we just have to compute $x = \frac{\Delta t\gamma - 1 + \sqrt{(-\Delta t\gamma+1)^2+4y\Delta t\gamma}}{2\Delta t\gamma}$

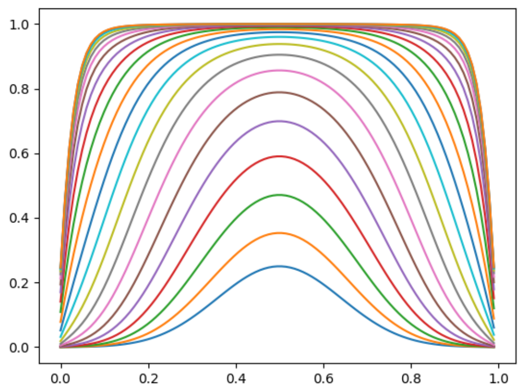

Figure 4 – numerical approximation with Nx = 100 and Nt until 100

We observe a very similar curve that the one of the precedent exercise, so the scheme is certainly good.

3.2 Convergence analysis

As done on the exercice 1, we compute the different error, and we also except a decreasing curve

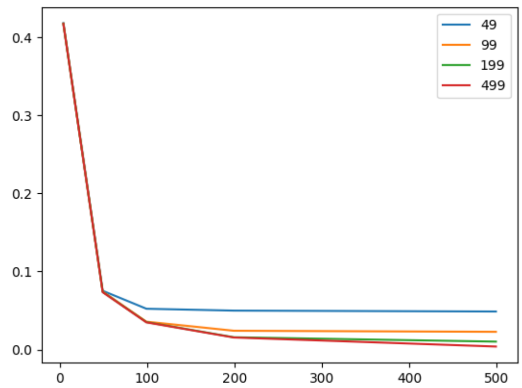

FIGURE 5 – $\varepsilon^2_{\Delta t, \Delta x}$ with Nx = 100 and some value of Nt

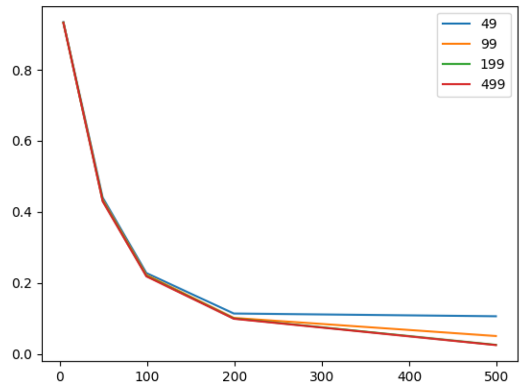

FIGURE 6 – $\varepsilon^{\infty}_{\Delta t, \Delta x}$ with Nx = 100 and some value of Nt

We have also a negative exponential curve when Nt and Nx are increasing. So the convergence is really good.

4 Conclusion

We can observe that for each scheme, we have a good convergence. It's difficult to compare them because we don't take the same "exact solution". And if we do, the one of the same scheme that the solution could be better. However, we can assume that the error are comparable because of the large size of the parameters of the "exact solution". And we can observe that the scheme with strang splitting seems better than the first one. However, strang splitting is a bit longer than the scheme of the exercise 1 (because of solving two linear system instead of one, even if their are lighter, it takes more time). Remarks that the infinity-norm error is worst for strang splitting, it is a max over all the time step. Whereas the 2-norm is better, and is more an average on all the time step. So, the choice of the scheme depend probably of the application and how should the error be. If we prefer an error good in average but with some worst value, or something worst but more safety.