

1. Tutorial

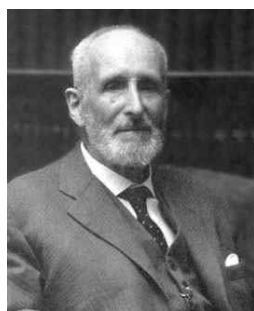
Exercise 1 (Some algebraic properties of the SVD). Let $A = U\Sigma V^T$ be the SVD of the $m \times n$ matrix A , where $m \geq n$.

1. Show that if A has full rank, then the solution of $\min_x \|Ax - b\|_2$ is $x = V\Sigma^{-1}U^Tb$.
2. Show that $\|A\|_2 = \sigma_1$ and that if A is also square and nonsingular then $\|A^{-1}\|_2^{-1} = \sigma_n$ and $\|A\|_2 \cdot \|A^{-1}\|_2 = \sigma_1/\sigma_n$.
3. Write $V = [v_1, v_2, \dots, v_n]^t$ and $U = [u_1, u_2, \dots, u_n]^t$ so $A = U\Sigma V^T = \sum_{i=1}^n \sigma_i u_i v_i^T$. Show that a matrix of rank $k < n$ closest to A (measured with $\|\cdot\|_2$) is $A_k = \sum_{i=1}^k \sigma_i u_i v_i^T$ and $\|A - A_k\|_2 = \sigma_{k+1}$.

Exercise 2 (Perron-Frobenius theorem). A vector $x \in \mathbb{R}^n$ is *nonnegative*, and we write $x \geq 0$ if its coordinates are nonnegative. It is *positive*, and we write $x > 0$, if its coordinates are (strictly) positive. Furthermore, a matrix $A \in \mathbb{R}^{n \times m}$ (not necessarily square) is nonnegative (respectively, positive) if its entries are nonnegative (respectively, positive); we again write $A \geq 0$ (respectively, $A > 0$). More generally, we define an order relation $x \leq y$ whose meaning is $y - x \geq 0$. Given $x \in \mathbb{R}^n$, we let $|x|$ denote the nonnegative vector whose coordinates are the numbers $|x_j|$. Likewise, if $A \in \mathbb{R}^{n \times n}$, the matrix $|A|$ has entries $|a_{ij}|$.

1. Show that $|Ax| \leq |A||x|$.
2. Show that a matrix A is nonnegative if and only if $x \geq 0$ implies $Ax \geq 0$.
3. Let A be a nonnegative matrix. Show that $\rho(A)$ is an eigenvalue of A associated with a nonnegative eigenvector.

Hint: One can use the Brouwer theorem: a continuous function from a compact convex subset of \mathbb{R}^N into itself has a fixed point.



Oskar Perron (1880-1975)



Georg Frobenius (1849-1917)

Exercise 3 (Stochastic matrices). A matrix $P \in \mathbb{R}^{n \times n}$ is said to be *stochastic* if $P \geq 0$ and one has:

$$\sum_{j=1}^n P_{ij} = 1 \quad \text{all } i = 1, 2, \dots, n.$$

1. Show that 1 is an eigenvalue of P . Give an eigenvector associated to this eigenvalue.
2. Show that $\rho(P) = 1$.

Exercise 4 (Power method). The aim of this exercise is to study an algorithm for computing the largest eigenvalue of a matrix in absolute value and the corresponding eigenvector.

```

i = 0
repeat
    yi+1 = Axi
    xi+1 = yi+1 / ||yi+1||
     $\tilde{\lambda}_{i+1} = x_{i+1}^T A x_{i+1}$ 
    i = i + 1
until convergence

```

1. Show that if $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ with $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$ then x_i converges to $\pm e_1$ and $\tilde{\lambda}_i$ converges to λ_1 .
2. We now assume that $A = S\Lambda S^{-1}$ is diagonalizable with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$. Show that x_i converges to $\pm s_1$ (where s_1 is the first column of S) and $\tilde{\lambda}_i$ converges to λ_1 .

Exercise 5 (Householder transformation). In this exercise, we consider a third QR algorithm, based on Householder transformations.

1. Suppose we are given an $n \times 1$ vector z . Represent its first coordinate in polar notation as $z_1 = \zeta e^{i\theta}$, where ζ is real. Define $v = z - \alpha e_1$ where $\alpha = -e^{i\theta} \|z\|$ and let $u = v / \|v\|$. Define

$$Q = I - 2uu^*.$$

Verify that Q is a unitary matrix and that $Qz = \alpha e_1$. The matrix Q is one version of a Householder transformation.

2. Given an $m \times n$ matrix A , we can determine a Householder transformation Q_1 so that $A_1 = Q_1 A$ has zeros in its first column below the main diagonal. Show how to determine Q_2 in the form

$$Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & I - 2u_2 u_2^* \end{pmatrix},$$

so that $A_2 = Q_2 A_1$ has zeros in its first and second column below the main diagonal.

3. Continuing this process, write an algorithm to reduce a matrix A to upper-triangular form by multiplying by a series of Householder transformations.
4. How many floating-point multiplications does your algorithm take?

2. Practical

Exercise 6 (Image compression using the SVD). Figure 1 represents a 320×200 pixel image corresponding to a 320×200 matrix X .

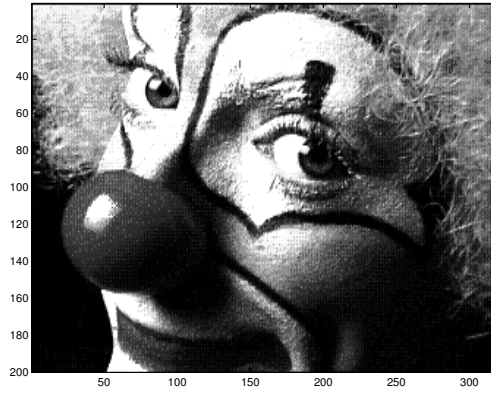


Figure 1: 320×200 pixel image

These images were produced by the following commands in MATLAB:

```
load clown.mat;
imagesc(X);
colormap gray;
```

1. Using question 3 of exercise 1, define an algorithm for image compression. Test your algorithm on figure 1.
2. Define a compression ratio to measure the quality of the compression.

Exercise 7 (Deblurring images using the SVD). The aim of this exercise is to deblur image 2. It is an example of *linear inverse problem*. Given a blurred image and a linear model for the blurring, we want to reconstruct the original image. Figure 2 represents a blurred image.

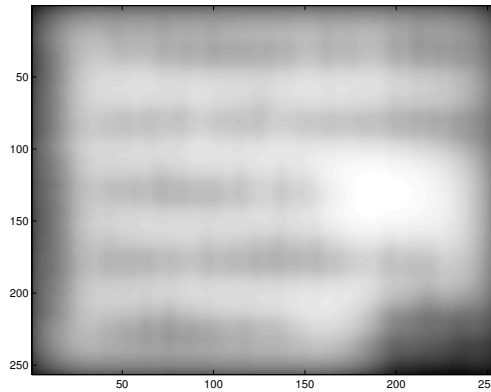


Figure 2: 250×250 pixel blurred image

Suppose we have a blurred, noisy image G , as in Figure 2, and some knowledge of the blurring operator, and we want to reconstruct the true original image F . We will use the following operator vec that transforms a matrix X into a column vector $x = \text{vec}(X)$ by stacking the columns of X . Let us denote $g = \text{vec}(G)$ and $f = \text{vec}(F)$. The blurring model is defined by $g = Kf + \eta$ where K is a matrix and η is a vector representing (unknown) noise or measurement errors. We know K and g . To find f , we want to solve the following problem:

$$\min_f \|g - Kf\|_2^2. \quad (1)$$

The Kronecker product $A \otimes B$ where A is an $m \times m$ matrix is defined to be

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix}.$$

We assume that the matrix K can be factorized as $K = A \otimes B$.

1. Show that the solution of Equation (1) can be written as

$$f^* = V \Sigma^{-1} U^T g = \sum_{i=1}^n \frac{u_i^T g}{\sigma_i} v_i,$$

where $K = U \Sigma V^T$, u_i is the i th column of U and v_i the i th column of V . In fact, we can only look at a truncated expansion

$$f_p^* = \sum_{i=1}^p \frac{u_i^T g}{\sigma_i} v_i,$$

for some value of $p < n$.

Let $A = U_A \Sigma_A V_A^T$ and $B = U_B \Sigma_B V_B^T$ be the SVD of A and B . One can show that $A \otimes B = (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A \otimes V_B)$ and that one also has

$$F = B^{-1} G A^{-T} = V_B \Sigma_B^{-1} U_B^T G U_A \Sigma_A^{-1} V_A^T.$$

If we denote $\widehat{G} = U_B^T G U_A$, then

$$\Sigma_B^{-1} \widehat{G} \Sigma_A^{-1} = \widehat{G} ./ S,$$

where $S = \text{diag}(\Sigma_B) \text{diag}(\Sigma_A)^T$.

2. Write a MATLAB program that takes matrices A and B and an image G and computes an image F . Experiment to find the value of the parameter p that gives the clearest image. Sample data (i.e., a blurred image G , and the matrices A and B) can be found in the file `defloutage.mat`¹. One has to load the data using:

```
load defloutage.mat;
imshow(G);
```

For more information, see for example:

- Per Christian Hansen, James G. Nagy et Dianne P. O’Leary, *Deblurring Images: Matrices, Spectra, and Filtering*, SIAM, 2006

Exercise 8 (Google PageRank algorithm). Let W be the set of Web pages that can be reached by following a chain of hyperlinks starting at some root page, and let n be the number of pages in W . Let G be the $n \times n$ connectivity matrix of a portion of the Web, that is, $g_{ij} = 1$ if there is a hyperlink to page i from page j and $g_{ij} = 0$ otherwise. The matrix G can be huge, but it is very sparse. Its j th column shows the links on the j th page. The number of nonzeros in G is the total number of hyperlinks in W . Let c_j be the column sums of G :

$$c_j = \sum_{i=1}^n g_{ij}.$$

The quantity c_j is the *out-degree* of the j th page. Let p be the probability that the random walk follows a link. A typical value is $p = 0.85$. Then $1 - p$ is the probability that some arbitrary page is chosen and $\delta = (1 - p)/n$ is the probability that a particular random page is chosen.

Let $A \in \mathbb{R}^{n \times n}$ a matrix whose elements are

$$A_{ij} = \begin{cases} p g_{ij} / c_j + \delta & \text{if } c_j \neq 0, \\ 1/n & \text{if } c_j = 0. \end{cases}$$

¹available at <http://www-pequan.lip6.fr/~grailat/teach/anum/defloutage.mat>

1. Show that A^T is a stochastic matrix. What can we say about the largest eigenvalue of A ?
2. Using the Perron-Frobenius theorem, show there exists a nonnegative vector $x \in \mathbb{R}^n$ such that $Ax = x$ and $\sum_{i=1}^n x_i = 1$.
3. Compute x using the power method presented in the tutorial (exercise 4). The page-rank of the page i is x_i . Provide a ranking of the pages according to their page-rank.

You can use the file `surfer.m`² to generate the matrix G .

For more information, see for example:

- L. Page, S. Brin, R. Motwani et T. Winograd. “The PageRank citation ranking: Bringing order to the web”, technical report, Stanford University, 1998.
- A.M. Langville et C.D. Meyer. *Google’s PageRank and Beyond: The Science of Search Engine Rankings*, Princeton University Press, 2006.

²available at <http://www-pequan.lip6.fr/~graillat/teach/anum/surfer.m>