Sorbonne Université, Master 1 Mathématiques et applications 2022–2023
4MA006 Fondements des méthodes numériques

TP0 : Python getting started

Warning:
Important information and documents concerning the TP are available here:

`https://moodle-sciences.upmc.fr/moodle-2022/`

Please, regularly check this page!

Objectives :
The goal of these exercises is to get familiar or to revise the basics of programming in view of more complex and numerical methods oriented exercises. After this set of exercises, it is mandatory that you know how to:

- manipulate correctly `for` loops, `if` conditional statements,
- define and properly test functions,
- represent functions in 2D plots,
- basic linear algebra routines: norm, linear solver calling,
- use functions such as `fsolve` in order to get an approximation of a simple non-linear problem,
- solve a Cauchy problem.

In view of helping you in these first exercises, some keywords are reported. These can help you in searching useful information (syntax, calling, function prototype) about them.

Remark about `Python`**:**
For algorithmic reasons (efficiency) we won't use `list` type to store and manipulate vectors and matrices. Instead, we will make use of the type `array` as is it defined in `NumPy`. From now on, we will make use on this module: `import numpy as np`.

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1 Fundamental Exercises

Exercise 1 [*Loop.*] By using a `for` loop, print the integers between 1 and 15. Then, by modifying the loop step, print the odd integers between 1 and 20.
Keywords: `for, range, print // for`.

Exercise 2 [*Function writing and testing.*] Write a function which computes arithmetic and geometric means of two real numbers. Test it for $(3, 2)$ et $(5, -6)$. *Remember:* We call arithmetic mean of a and b: $(a + b)/2$. The geometric mean, when it exists (!), is defined as: $\sqrt{ab}$.
Keywords `def, np.sqrt, if`.

Exercise 3 [*Simple plot.*] Represent the function $\sin(2\pi x)$ on $[0, 1]$.
Keywords: `plt.plot [import matplotlib.pyplot as plt]`.

Exercise 4 [*Matrices.*] Find and test functions making it possible to build a matrix whose elements are zero, a matrix whose elements are all equal to 1, the identity matrix and a diagonal matrix.
Keywords: `np.zeros, np.ones, np.eye, np.diag`.

Exercise 5 [*Vector norms.*] We denote u, v et w the following vectors:

$$u = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \ , \ v = \begin{pmatrix} 10 \\ -1 \\ 3 \end{pmatrix} \ , \ w = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} \ .$$

Compute $3u$, $\|u\|_2$, $2u - v + 5w$, $\|2u - v + 5w\|_1$, $\|w - 4v\|_\infty$ and $\langle u, v \rangle$, the scalar product between u and v.
Keywords: `np.linalg.norm, np.dot`.

Exercise 6 [*Linear algebra.*] We set:

$$A = \begin{pmatrix} 1 & -1 & 7 \\ -4 & 2 & 11 \\ 8 & 0 & 3 \end{pmatrix} \ , \ B = \begin{pmatrix} 3 & -2 & -1 \\ 7 & 8 & 6 \\ 5 & 1 & 3 \end{pmatrix} .$$

Find the functions to:

1. Multiply each entry of A by π.
2. Perform the pointwise multiplication between A and B.
3. Find the number of rows and columns of A and the total number of elements of B.
4. Compute AB^T, $Id - BB^T$.
5. Right multiply A by a matrix whose all entries are 1.
6. Extract the diagonal entries of A, put them into a vector.
7. Set the diagonal of B to zero.
8. Concatenate the matrices $\begin{pmatrix} A & B \end{pmatrix}$ and $\begin{pmatrix} A \\ B \end{pmatrix}$.
9. Extract from $B = (b_{ij})_{1 \leq i,j \leq 3}$ the elements whose rows and columns indices are odd.

Keywords: `np.shape, np.size, .T, np.concatenate`.

Exercise 7 [*Finding the zero of a function.*] We would like to compute an approximation of the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$. We recall that φ is the unique positive solution of $x^2 - x - 1 = 0$.

1. Define a funtion $f(x) = x^2 - x - 1$.
2. Use a Scipy built-in solver to get an approximation of x such that $f(x) = 0$.

Keywords: `from scipy.optimize import fsolve`.

Exercise 8 [*Linear solver.*] We consider the Hilbert matrix $H_n = (h_{i,j})_{1 \leq i,j \leq n}$ of size $n \times n$ defined as:

$$h_{i,j} = \frac{1}{i + j - 1} \quad \text{pour} \quad 1 \leqslant i, j \leqslant n.$$

Write a function to build H_n. We denote e the vector of size n whose entries are all equal to 1 and $b = H_n e$. Solve, by calling a Numpy built-in solver, the linear sytem $H_n x = b$ (Mandatory: do not use matrix inverse!). Plot the relative error of the solution with respect to n. Study the conditioning (2-norm) as function of n.
Keywords: `np.linalg.solve, np.linalg.cond`.

Exercise 9 [*Cauchy problem.*] Let us consider the time evolution of the temperature of a cup of coffee in a room at constant temperature. It can be modelled as follows.

$$\begin{cases} T'(t) &= 0.1(T_A - T(t)), \\ T(0) &= T_{ini}, \end{cases}$$

where T_A is the temperature of the room and T_{ini} the initial temperature of the cup of coffee. Solve the initial value problem (called Cauchy problem) by using a built-in solver. Plot $T(t)$ on the time interval $[0, 60]$ with $T_A = 20$ and $T_{ini} = 100$. What do you observe?
Keywords: `from scipy.integrate import odeint (Python) // ode (Scilab).`

2 Supplementary exercises

Exercise 10 We define the vectors u_1, u_2, u_3 et u_4, elements of $\mathbb{R}^5$:

$$u_1 = \begin{pmatrix} 1 \\ -3 \\ 3 \\ 5 \\ 4 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 2 \\ -5 \\ -1 \\ -6 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ -2 \\ 0 \end{pmatrix}$$

Let A be the matrix whose columns are given by the vectors $u_1, \dots, u_4$. What is the rank of A? What about if we replace u_4 by the vector $w = (-3, 11, 4, 13, 4)^T$?

Exercise 11 Write a function to replace the $i-$th row of a matrix A, denoted by L_i, by the line αL_j and the $j-$th line by L_i. In the case in which $i = j$ the function returns the matrix A, unchanged. Propose some examples to test this function.

Exercise 12 Write and test a function which replaces the entries $A_{ij} > 0$ by 0 and let the other entries unchanged.

Exercise 13 We want to estimate the golden ratio. We define the following sequence: $(F_n)_{n\geqslant 1}$, $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.
On s'intéresse de nouveau à l'estimation du nombre d'or, $\varphi = \frac{1+\sqrt{5}}{2}$. On définit la suite $(F_n)_{n\geqslant 1}$ par $F_1 = F_2 = 1$ et $F_n = F_{n-1} + F_{n-2}$ pour $n > 2$.

1. Write a function whose input is $N \geqslant 1$ and outputs F_N.
2. Verify numerically that: $\varphi = \lim\limits_{n\to+\infty} \dfrac{F_{n+1}}{F_n}$.

Exercise 14 Build a matrix of size 9×9 whose all elements are 0 except the boundaries $i \in \{1, 9\}$ where $j \in \{1, 9\}$ and the centre elements $(i, j) \in \{4, 5, 6\} \times \{4, 5, 6\}$ which are equal to 1.