
Programming classes N° 2
Finite differences methods

Exercise 1 Laplacian discretisation. In this exercise, we will compute a numerical approximation of the solution $u : [0,1] \longrightarrow \mathbb{R}$ of the following problem:

$$\left\{ \begin{array}{rcl} -u''(x) & = & f(x)\ , \\ u(0) & = & \alpha\ , \\ u(1) & = & \beta\ . \end{array} \right. \tag{1}$$

The function f is a problem datum. In the following, we will assume that the function u has the regularity which is needed for the computation to make sense.
Problem discretisation (1) : Let $N \in \mathbb{N}^*$, we set $h = \frac{1}{N+1}$, and define the space discretisation points

$$x_i = ih, \quad i = 0, \ldots, N+1.$$

We set $u_0 = u(0) = \alpha$, $u_{N+1} = u(1) = \beta$ and, for $i = 1, \ldots, N$, we denote u_i the numerical approximation of $u(x_i)$. A finite differences method (centred finite differences) for the problem (1) is given by:

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i), \quad i = 1, \ldots, N. \tag{2}$$

In what follows, we will denote $U_h = (u_1, \ldots, u_N)^T$ and $\overline{U}_h = (u(x_1), \ldots, u(x_N))^T$.

1. Show, by making an analytical computation (by hand), that the above written equations can be written in the form:
$$A_h\, U_h = F_h, \tag{3}$$
where $A_h \in \mathcal{M}_N(\mathbb{R})$ and $F_h \in \mathbb{R}^N$ have to be determined in relation to the problem.

2. Implement a function which takes as inputs N, f, α et β and gives, as output A_h and F_h.

3. **Validation**. Let us assume that $f = 1$, the constant function, equal to 1 in all points. Let the boundary conditions be homogenous, *i.e.* $\alpha = \beta = 0$.

   (a) Verify, by hand, that the solution of the problem reads: $u(x) = \dfrac{x(1-x)}{2}$.

   (b) Implement a function which takes as input the value of x and returns, as output $u(x)$.

   (c) Solve numerically the system (3) and compare the vectors $\overline{u}_h$ and u_h. To do so, compute the ℓ^∞ norm of their differece.

4. **Convergence**.

   (a) determine by hand a function", such that the function $u(x) = \exp(-4x)\sin(\pi x)$ is the solution of the problem (1). What are the values of α anf β?

(b) We want to study the convergence of the finite difference method as function of the discretisation parameter h. Let us recall that we say that a method converges at the order $p \in \mathbb{R}^+$, if there exists a real positive number $C > 0$, which does not depend on h, such that, asymptotically, we have the relation: $\|\overline{U}_h - U_h\| = \mathcal{O}(h^p)$. In general, the order p might depend on the norm we chose to measure the error magnitude. In the following, we will make use of the discrete L^q norms. For different values of h (which is equivalent to make N vary), compute the errors $\|\overline{U}_h - U_h\|_{1,\triangle}$, $\|\overline{U}_h - U_h\|_{2,\triangle}$, $\|\overline{U}_h - U_h\|_{\infty,\triangle}$. Plot the three of them as function of h, in log–log scale and determine p.

We recall, hereafter, the expressions of the norms. For their definition and more details, have a look to Chapter 2, Section 2 of the lecture notes.

$$
\begin{aligned}
\|\overline{U}_h - U_h\|_{\infty,\triangle} &= \max_{1\le i\le N} \mid u(x_i) - u_i \mid \\
\|\overline{U}_h - U_h\|_{2,\triangle}^2 &= \sum_{i=1}^{N} h \; (\; u(x_i) - u_i \;)^2 \\
\|\overline{U}_h - U_h\|_{1,\triangle} &= \sum_{i=1}^{N} h \mid u(x_i) - u_i \mid
\end{aligned}
$$

Exercise 2 An ill-posed problem. In this exercise we look for an approximation of the solution $u : [0,1] \longrightarrow \mathbb{R}$ to the following problem:

$$
\left\{
\begin{array}{rcl}
-u''(x) - \pi^2 u(x) &=& 1 \; , \\
u(0) &=& 0 \; , \\
u(1) &=& 0 \; .
\end{array}
\right.
\tag{4}
$$

1. Adapt the method (2) in order to approximate the problem (4) and deduce the expression of the linear system to be solved. As done in the previous exercise, implement a function which takes as inputs the discretisation parameter and the boundary conditions and returns, as output, the system matrix and the right hand side vector.

2. For different values of N, solve the linear system and store the lowest value the numerical solution attains. What do you observe? Compute the determinant and the condition number of the matrix A_h. To get more insight on what observed, read Section 1.4 of the Chapter 1 of the lecture notes.