Calibration of Stochastic Volatility Models using Particle Markov Chain Monte Carlo Methods

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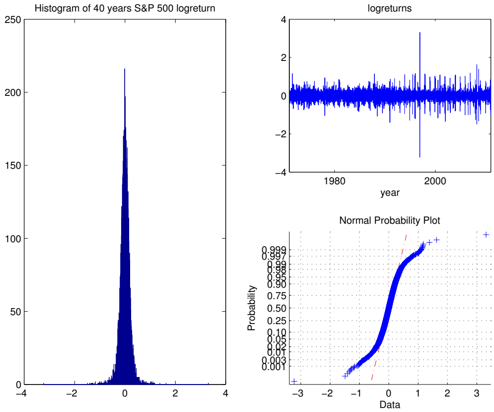
LOGRETURNS



$$Y_k = \log\left(\frac{S_k}{S_{k-1}}\right)$$

Model Proposal

$$Y_k = \beta e^{\frac{1}{2}X_k} u_k, \tag{1}$$

$$X_k = \alpha X_{k-1} + \sigma w_k, \tag{2}$$

$$(u_k, w_k) \sim \mathcal{N}(0, \Sigma), \tag{3}$$

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{4}$$

Bayesian Inference

Bayesian inference, view the parameter as a random variable

Observation:

$$Y \mid \theta^* = \theta \sim p(y \mid \theta^* = \theta), \quad \theta^* \sim p(\theta) \tag{5}$$

Parameter posterior distribution:

$$p(\theta \mid y) = \frac{p(y \mid \theta) p(\theta)}{\int_\Theta p(y \mid \theta') p(d\theta')} \propto p(y \mid \theta) p(\theta) \tag{6}$$

Example:

$$p(\beta \mid \alpha, \sigma, \rho, x_{0:n}, y_{0:n}) \propto p(\beta, \alpha, \sigma, \rho, x_{0:n}, y_{0:n}) \tag{7}$$

POSTERIOR FOR β

$$
\begin{aligned}
p(\beta|\alpha,\sigma,\rho,x_{0:n},y_{0:n}) &\propto p(\beta,\alpha,\sigma,\rho,x_{0:n},y_{0:n}) && (8)\\
&= p(x_{0:n},y_{0:n}|\beta,\alpha,\rho,\sigma)p(\beta)p(\alpha,\rho,\sigma) && (9)\\
&\propto p(x_{0:n},y_{0:n}|\beta,\alpha,\rho,\sigma)p(\beta) && (10)\\
&= p(x_n,y_n|\beta,\alpha,\rho,\sigma,x_{0:n-1},y_{0:n-1}) && (11)\\
&\times p(x_{0:n-1},y_{0:n-1}|\beta,\alpha,\rho,\sigma)p(\beta) && \\
&= p(\beta)p(y_0,x_0|\beta,\alpha,\rho,\sigma) && (12)\\
&\times \prod_{k=1}^{n} p(x_k,y_k|\beta,\alpha,\rho,\sigma,x_{k-1}) &&
\end{aligned}
$$

Posterior for β

$$p(x,y) = \frac{1}{|\beta|\sigma 2\pi\sqrt{1-\rho^2}} \times \exp\left(-\frac{\left[\left(\frac{y}{\beta e^{\frac{1}{2}x}}\right)^2 + \left(\frac{x-\alpha x_{k-1}}{\sigma}\right)^2 - 2\rho\frac{y(x-\alpha x_{k-1})}{\sigma\beta e^{\frac{1}{2}x}}\right]}{2(1-\rho^2)} - \frac{1}{2}x\right) \tag{13}$$

PRIOR SELECTION

$$p(\beta) = \frac{1}{\beta^2} \tag{14}$$

$$p(\alpha) = (\alpha + 1)^{\delta - 1}(1 - \alpha)^{\gamma - 1} \tag{15}$$

$$p(\rho) = \frac{1}{2} \tag{16}$$

$$p(\sigma) = \frac{1}{\sigma^2 \sigma^{2(t/2-1)}} e^{-\frac{1}{2\sigma^2}\tilde{S}_0} \tag{17}$$

Example:

$$p(\beta \mid \alpha, \sigma, \rho, x_{0:n}, y_{0:n}) \propto p(x_{0:n}, y_{0:n} \mid \alpha, \beta, \rho, \sigma) p(\beta) \tag{18}$$

IDEA

- Simulate the parameters from the posterior distributions!

THE GIBBS SAMPLER [1]

1. For the first iteration choose $\xi^{(0)} = \{x_{0:n}^{(0)}, \theta^{(0)}\}$ arbitrarily
2. For $k = 1, 2, \ldots, N$, draw random samples
   2.1 $x_{0:n}^{(k)} \sim p_X(\cdot \mid \theta^{(k-1)}, y_{0:n})$
   2.2 $\theta_1^{(k)} \sim p_X(\cdot \mid x_{0:n}^{(k)}, \theta^{(k-1)}, y_{0:n})$

$$\vdots$$

$$\theta_D^{(k)} \sim p_X(\cdot \mid x_{0:n}^{(k)}, \theta_1^{(k)}, \theta_2^{(k)}, \ldots, \theta_D^{(k-1)}, y_{0:n})$$

Now as N tend to infinity, the sequence $\{\xi^{(k)}\}_{k=0}^{N}$ will have p_X as its stationary distribution.
New problem: How do we sample θ and x?

[1] Geman (1984)

METROPOLIS-HASTINGS SAMPLER [2]

Choose θ_0 arbitrarily, then for $k = 1, \ldots, N$

1. Sample $\theta^* \sim q(\cdot \mid \theta^{(k)})$
2. With probability

$$1 \wedge \frac{p(\theta^*)q(\theta^{(k)} \mid \theta^*)}{p(\theta^{(k)})q(\theta^* \mid \theta^{(k)})} \tag{19}$$

set $\theta^{(k+1)} = \theta^*$, otherwise set $\theta^{(k+1)} = \theta^{(k)}$

[2] Metropolis et. al. (1953), Hastings (1970)

SEQUENTIAL MONTE CARLO (PARTICLE FILTER)

$$\phi_k \triangleq p(x_k \mid y_{0:k}) \tag{20}$$

Propose

$$\phi_{k+1}(\tilde{\xi}) = \frac{\int l_k(\xi, \tilde{\xi}) \phi_k(\xi)\, d\xi}{\int \phi_k(\xi) \int l_k(\xi, \tilde{\xi})\, d\tilde{\xi}\, d\xi} \tag{21}$$

Our Model

In our setting:

$$\begin{aligned}
\phi_{k+1} &= p(x_{k+1}, y_{0:k+1}) / p(y_{0:k+1}) && (22)\\
&\propto \int p(y_{k+1} \mid x_{k+1}, x_k, y_{0:k}) && (23)\\
&\times p(x_{k+1} \mid x_k, y_{0:k}) p(x_k, y_{0:k}) \, dx_k \\
&= \int p(y_{k+1} \mid x_{k:k+1}) p(x_{k+1} \mid x_k) p(x_k \mid y_{0:k}) p(y_{0:k}) \, dx_k && (24)\\
&= \int p(y_{k+1} \mid x_{k:k+1}) p(x_{k+1} \mid x_k) \phi_k p(y_{0:k}) \, dx_k && (25)\\
&= \int G(y_{k+1} \mid x_{k:k+1}) Q(x_{k+1} \mid x_k) \phi_k p(y_{0:k}) \, dx_k && (26)
\end{aligned}$$

SUMMARIZED

Filter:

$$\phi_{k+1} = \frac{\int G(y_{k+1} \mid x_{k:k+1})Q(x_{k+1} \mid x_k)\phi_k \, dx_k}{\int\int G(y_{k+1} \mid x_{k:k+1})Q(x_{k+1} \mid x_k)\phi_k \, dx_k \, dx_{k+1}} \tag{27}$$

Smoother:

$$\begin{aligned} \phi_{0:k+1|k+1} &= p(x_{0:k+1} \mid y_{0:k+1}) \\ &= \frac{\int G(y_{k+1} \mid x_{k:k+1})Q(x_{k+1} \mid x_k)\phi_{0:k|k} \, dx_{0:k}}{\int\int G(y_{k+1} \mid x_{k:k+1})Q(x_{k+1} \mid x_k)\phi_{0:k|k} \, dx_{0:k} \, dx_{0:k+1}} \end{aligned} \tag{28}$$

Monte Carlo Integration

We want to evaluate:

$$\mu(f) = \int f(x) \frac{d\mu}{d\nu}(x)\,\nu(dx) \tag{29}$$

We use the estimate:

$$N^{-1} \sum_{j=1}^{N} f(\xi^j) \frac{d\mu}{d\nu}(\xi^j) \quad \xrightarrow[N\to\infty]{a.s.} \quad \mu(f) \tag{30}$$

Sequential Importance Sampling

1. Sampling: for $k = 0, 1, \ldots$
   Draw $\tilde{\xi}_{k+1}^{1}, \ldots, \tilde{\xi}_{k+1}^{N} \mid \tilde{\xi}_{0:k}^{1}, \ldots, \tilde{\xi}_{0:k}^{N}$
   1.1 Compute the importance weights

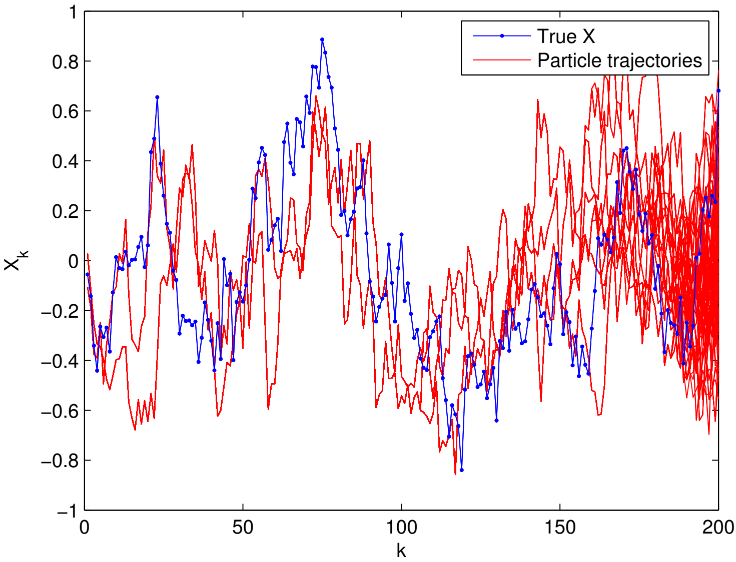
$$\omega_{k+1}^{j} = \omega_{k}^{j} g_{k+1}(\tilde{\xi}_{k+1}^{j}) \tag{31}$$

2. Resampling: Draw N particles from the N-sized population where the probability of selecting particle j is

$$\frac{\omega_{k+1}^{j}}{\sum_{s}^{N} \omega_{k+1}^{s}} \tag{32}$$

3. Update the trajectory: Copy the resampled particles trajectories and replace the ones we discarded.

EXAMPLE



True X
Particle trajectories
X_k
k
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
50
100
150
200

Recap

- Object: Model the price
- Need parameters
  - Need X trajectories

Which we now have!

$$Y_k = \beta e^{\frac{1}{2}X_k} u_k, \tag{33}$$

$$X_k = \alpha X_{k-1} + \sigma w_k, \tag{34}$$

$$(u_k, w_k) \sim \mathcal{N}(0, \Sigma), \tag{35}$$

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{36}$$

The Gibbs Sampler

1. For the first iteration choose $\xi^{(0)} = \{x_{0:n}^{(0)}, \theta^{(0)}\}$ arbitrarily
2. For $k = 1, 2, \ldots, N$, draw random samples

   2.1 $x_{0:n}^{(k)} \sim p_X(\cdot \mid \theta^{(k-1)}, y_{0:n})$

   2.2 $\theta_1^{(k)} \sim p_X(\cdot \mid x_{0:n}^{(k)}, \theta^{(k-1)}, y_{0:n})$

   $\vdots$

   $\theta_D^{(k)} \sim p_X(\cdot \mid x_{0:n}^{(k)}, \theta_1^{(k)}, \theta_2^{(k)}, \ldots, \theta_D^{(k-1)}, y_{0:n})$

Now as N tend to infinity, the sequence $\{\xi^{(k)}\}_{k=0}^{N}$ will have p_X as its stationary distribution

METROPOLIS-HASTINGS SAMPLER [2]

Choose θ_0 arbitrarily, then for $k = 1, \ldots, N$

1. Sample $\theta^* \sim q(\cdot \mid \theta^{(k)})$
2. With probability

$$1 \wedge \frac{p(\theta^*)q(\theta^{(k)} \mid \theta^*)}{p(\theta^{(k)})q(\theta^* \mid \theta^{(k)})} \tag{19}$$

set $\theta^{(k+1)} = \theta^*$, otherwise set $\theta^{(k+1)} = \theta^{(k)}$

[2] Metropolis et. al. (1953), Hastings (1970)

PARTICLE MARGINAL METROPOLIS HASTINGS [3]

1. Initialization, $k = 0$
   1.1 Set θ_0 arbitrarily
   1.2 Run an SMC algorithm targeting $p_{\theta^{(0)}}(x_{1:T}|y_{1:T})$, sample our first trajectory of particles $\tilde{\xi}^{(0)}_{1:T} \sim \hat{p}_{\theta^{(0)}}(\cdot|y_{1:T})$ and denote the marginal likelihood by $\hat{p}_{\theta_0}$
2. For iteration $k \geq 1$
   2.1 Sample $\theta^* \sim q(\cdot \mid \theta_{k-1})$
   2.2 Run an SMC algorithm targeting $p_{\theta^*}(x_{1:T} \mid y_{1:T})$, sample the trajectory of particles as in 1.2
   2.3 With probability

$$1 \wedge \frac{\hat{p}_{\theta^*}(y_{1:T})p(\theta^*)q(\theta_{k-1} \mid \theta^*)}{\hat{p}_{\theta_{k-1}}(y_{1:T})p(\theta_{k-1})q(\theta^* \mid \theta_{k-1})} \tag{37}$$

put $\theta_k = \theta^*, \xi^{(k)}_{1:T} = \xi^*_{1:T}$, and $p_{\theta_k}(y_{1:T}) = p_{\theta^*}$

[3] Andrieu et. al. (2010)

Unbiased Parallel Metropolis Hastings

Choose θ_0^m arbitrarily, then for $k = 1, \ldots, N$

1. For each of the C cores (where $\theta^{(k)} = \theta_k^m$):
   1.1 Sample $\theta^* \sim q(\cdot \mid \theta^{(k)})$
   1.2 With probability

$$1 \wedge \frac{p(\theta^*)q(\theta^{(k)} \mid \theta^*)}{p(\theta^{(k)})q(\theta^* \mid \theta^{(k)})} \tag{38}$$

   set $\theta^{(k+1)} = \theta^*$, otherwise set $\theta^{(k+1)} = \theta^{(k)}$

2. Iterate through the C cores, take the first accepted sample and put $\theta_{k+\gamma}^m$ equal to it. Put $\theta_{k:k+\gamma-1}^m = \theta_k^m$. Throw away all samples after that. If no sample is accepted, $\theta_{k:C}^m = \theta_k^m$

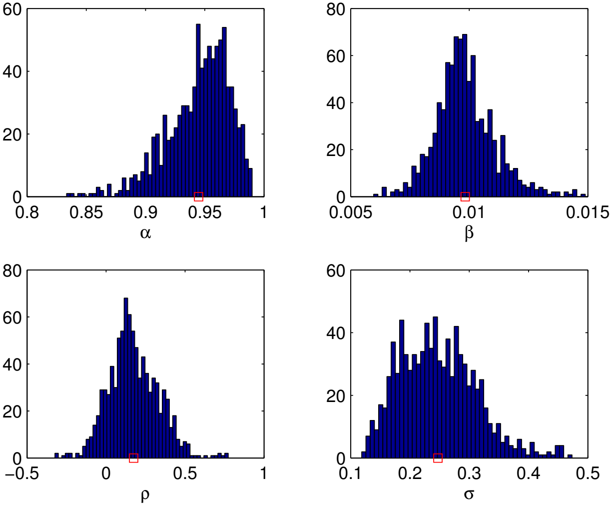
UNBIASED PARALLEL METROPOLIS HASTINGS

- Roughly $(\text{Acceptance rate})^{-1}$ times faster as numbers of cores grow large
- Easy to implement

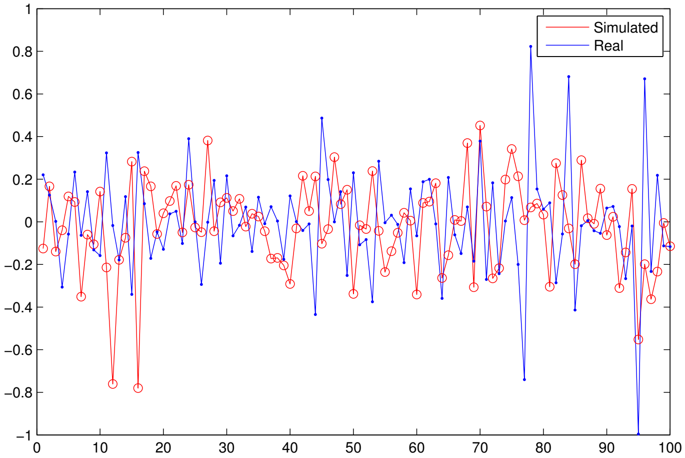
Model Proposal

$$Y_k = \beta e^{\frac{1}{2}X_k} u_k, \tag{1}$$
$$X_k = \alpha X_{k-1} + \sigma w_k, \tag{2}$$
$$(u_k, w_k) \sim \mathcal{N}(0, \Sigma), \tag{3}$$
$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{4}$$

Estimates for GBP/USD Data



Simulation of S&P500



Prediction

Dataset	Model	RMSE (10^{-3})
GBP/USD	*SVOL*	**11.706**
GBP/USD	$SVOL_{\rho=0}$	11.714
BIDU	*SVOL*	**20.188**
BIDU	$SVOL_{\rho=0}$	20.232
S&P500	*SVOL*	252.94
S&P500	$SVOL_{\rho=0}$	**252.93**
XBC/USD	*SVOL*	**5.5762**
XBC/USD	$SVOL_{\rho=0}$	5.5920

Any remark, question or suggestion is welcomed!

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