CNF · SAT

Anatoly Weinstein

January 26, 2024

TODO A CNF-SAT is an equation ...

Conjunctive normalform A satisfyability equation in conjunctive normalform is a set of literal sets. We denote it with + as the logical operation *or* and · (can be omitted) as *and*.

$$\{\{a, \overline{b}\}, \{b, \overline{c}, \overline{d}\}, \{\overline{a}, \overline{c}\}\}$$

$$(a + \overline{b}) \cdot (b + \overline{c} + \overline{d}) \cdot (\overline{a} + \overline{c})$$

Disjunctive normalform

$$ab + \overline{b}cd + ac$$

Formulas

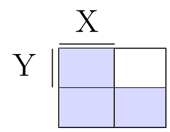
Zugzwang. If a disjunction consists of one literal, the value of this literal is fixed.

$$\{X\} \in CNF \Rightarrow X = \top$$

Absorption (CNF). If a disjunction is a superset of another disjunction, it can be ignored.

$$\forall X \subseteq Y : X \cdot Y \equiv_{\text{CNF}} X$$

Proof: $(X \Rightarrow Y) \vee \neg X$ makes Y irrelevant.



The Absorption Rule allows to freely create new disjunctions, as long it's consisting of an already existing disjunction.

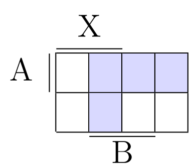
Absorption (DNF).

$$\forall X \subseteq Y : X + Y \equiv_{\text{DNF}} X$$

Proof: $(X \Rightarrow \top) \wedge (\neg X \Rightarrow \neg Y)$

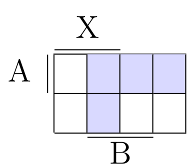
Common Contradiction.

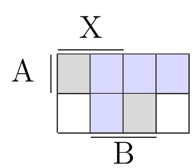
$$(A + X) \cdot (B + \overline{X}) \equiv_{\text{CNF}} (BX + A\overline{X})$$



Resolution.

$$(A + X)(B + \overline{X}) \equiv_{\text{CNF}} (A + X)(B + \overline{X})(A + B)$$



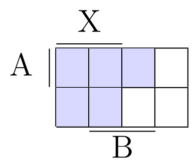


Proof:

$$(A + X)(B + \overline{X}) \text{ is stronger than } (A + B)$$

Common Part.

$$(A + X) \cdot (B + X) \equiv_{\text{CNF}} (X + AB)$$



Semi-Common Part.

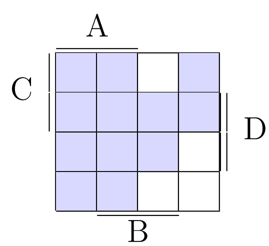
$$(AX + B)(A + C) \equiv_{\text{CNF}} A(X + B) + BC$$

Proof:

$$\begin{aligned} & (AX + B)(A + C) \\ \equiv_{\text{CNF}} & \ AX + AXC + AB + BC \\ \equiv_{\text{CNF}}^{\text{Absorption}} & \ AX + AB + BC \end{aligned}$$

Common and Contradicting Part

$$(A + B + C) \cdot (A + \neg B + D) \equiv_{\text{CNF}} (A + BD + C\overline{B})$$



Proof:

$$\begin{aligned} & (A + B + C) \cdot (A + \neg B + D) \\ \equiv_{\text{CNF}} & \ (A + (B + C)(\neg B + D)) \\ \equiv_{\text{CNF}} & \ (A + BD + C\overline{B}) \end{aligned}$$

Algorithms

`bf0` · Brute Force

1. Try out every combination of variable interpretations.

`da0` · **Disjunctive Absorption**

1. Use trivial optimisations.
2. Count the number of occurences of every literals (positive and negated) and find the literal with most occurences.
3. Branch: Assume the variable and accordingly clean up the equation using trivial transformations.
4. Absorption: Absorb as many disjunctions to the two- literals clauses.
5. If the equation is in 2-CNF-SAT, proceed with the next step, otherwise jump to the first step in a recursive step.
6. Solve 2-SAT the equation in polynomial time.