Sat Solver Optimisations

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This document contains research into optimisations of CNF-SAT problems.

1 Definitions

Boolean Function. A boolean function $f : V \subseteq Variables \to \{\bot, \top\}$ with truthy sets $P \subseteq \mathcal{P}(Variables)$ yields $\top$ (truthy) if $V \in P$, else yields $\bot$ (falsy).

Boolean Satisfyability Equation. We define a boolean formula by the following context-free grammar with start variable S, a given variable set $Variable = \{x_i : i \in [n-1]\}$ of n variables and a set of binary operations $Operation = \{\cdot, +, \oplus, \dots\}$ and an assigned boolean function $f_{Operation} : \mathcal{P} \subseteq \{L, R\} \to \{\bot, \top\}$, where L and R is the truthiness of left and right respectively.

$$\begin{array}{ccc} S & \to & (S) \\ S & \to & S\ Operation\ S \\ S & \to & \neg S \\ S & \to & Variable \end{array}$$

The *boolean equation* is satisfied with interpretation P, if there is an interpretation X with $f_{SAT}(X) = \top$.

CNF *Unmixed conjunctive normal form.* We define a boolean formula in conjunctive normal form by the following context- free grammar with start variable S and a given variable set $Variable = \{x_i : i \in [n-1]\}$ of n variables.

$$
\begin{array}{ccccc}
S & \rightarrow & (Disjunction) & \text{Short: } \mathbf{S}:\{D\} \\
S & \rightarrow & S \cdot (Disjunction) & \\
Disjunction & \rightarrow & Literal & \text{Short: } \mathbf{D}:\{X\} \\
Disjunction & \rightarrow & Disjunction + Literal & \\
Literal & \rightarrow & Variable & \text{Short: } \mathbf{L}:x_i \\
Literal & \rightarrow & \neg Variable &
\end{array}
$$

The notation $(a+\overline{b})\cdot(b+\overline{c}+\overline{d})\cdot(\overline{a}+\overline{c})$ will be used as type "set of literal sets" $\{\{a,\neg b\},\{b,\neg c,\neg d\},\{\neg a,\neg b\}\}$ in this paper.

2 Transformation Techniques

U-SAT *Unmixed conjunctive normal form.* Given a *CNF-SAT* formula, we call it *unmixed*, if every disjunction consists of either positive or negated literals.

A polynomial time reduction $SAT \leq_p U{-}SAT$ for an equation $S_E \rightarrow U_E$ would be:

$$\forall D \in S:$$

TODO De Morgan

3-CNF-SAT *Three-literals conjunctive normal form.* Given a *CNF-SAT* formula, we call it $n - SAT$, if every disjunction consists of exactly n-literals.

TODO Proof

U3-SAT *Three-literals unmixed conjunctive normal form.* Given a *CNF-SAT* formula, we call it $U3{-}SAT$, if it is *unmixed* and every disjunction consists of exactly 3-literals.

TODO

3 Optimisation Techniques

Zugzwang. *Common Literals.* If a disjunction consists of one literal, the value of this literal is fixed.

$$S = X \cdot A(X) \rightarrow (f_{SAT}(Y) \Rightarrow X \in Y)$$

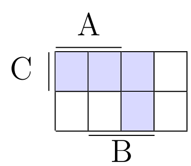
Contradiction. *Contradictional Disjunctions.*

A formula with contradictional disjunctions is not satisfiable.

$$\forall A : X \cdot \neg X \cdot A(X) = \bot$$

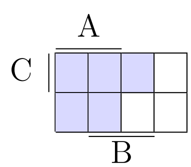
Common Contradicting Literals. *Disjunction with common contradictional part.*

$$(A + B) \cdot (\neg A + C) \cdot X = (AC + B\overline{A}) \cdot X$$



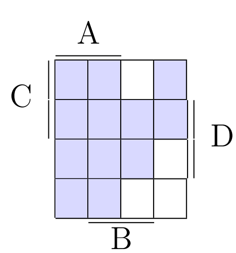
Common Literals. *Disjunction with common positive part.*

$$(A + B) \cdot (A + C) \cdot X = (A + BC) \cdot X$$



Common Literals and Contradictional Part. *Disjunction with positive and negative common parts.*

$$(A + B + C) \cdot (A + \neg B + D) \cdot X = (A + BD + C\overline{B}) \cdot X$$



Connected Component. *Two independent sets of variables.* If there exists two sets of variables A and B, such that $\forall a \in A, b \in B \nexists D : a \in D \land b \in D$, then you can create two equations as following and solve them independently with reduced variable count.

$$\forall V \in \{A, B\} : S_V = \{d \in D \ \forall \ l \in d : \ l \in V\}$$

3×3 **Combination.** *Brute-Force.*

$$\begin{aligned} &(a + b + c) \cdot (a + d + e) \\ =&a + bd + be + cd + ce \\ =&a + (b + c) \cdot (d + e) \end{aligned}$$

4 CNF-SAT Optimisation

TODO 3-CNF-SAT equations

All equal. $D \cdot D = D$

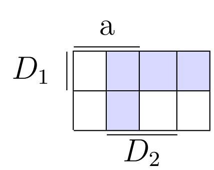
If two disjunctions are equal, remove the second one.

One off. $(a + b + c) \cdot (a + b + d)$

If two disjunctions differ in only one letter, the following can be said:

$$\begin{aligned} (a + b + c) \cdot (a + b + d) \\ = a + b + c \cdot d \end{aligned}$$

Opposing Literal. $(a + D_1) \cdot (\overline{a} + D_2)$



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