

Sat Solver Optimisations

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This document contains research into optimisations of CNF-SAT problems.

1 Definitions

Boolean Function. A boolean function $f : V \subseteq Variables \rightarrow \{\perp, \top\}$ with truthy sets $P \subseteq \mathcal{P}(Variables)$ yields \top (truthy) if $V \in P$, else yields \perp (falsy).

Boolean Satisfiability Equation. We define a boolean formula by the following context-free grammar with start variable S , a given variable set $Variable = \{x_i : i \in [n - 1]\}$ of n variables and a set of binary operations $Operation = \{\cdot, +, \oplus, \dots\}$ and an assigned boolean function $f_{Operation} : \mathcal{P} \subseteq \{L, R\} \rightarrow \{\perp, \top\}$, where L and R is the truthiness of left and right respectively.

$$\begin{aligned} S &\rightarrow (S) \\ S &\rightarrow S \text{ Operation } S \\ S &\rightarrow \neg S \\ S &\rightarrow Variable \end{aligned}$$

The *boolean equation* is satisfied with interpretation P , if there is an interpretation X with $f_{SAT}(X) = \top$.

CNF *Unmixed conjunctive normal form.* We define a boolean formula in conjunctive normal form by the following context-free grammar with start variable S and a given variable set $Variable = \{x_i : i \in [n - 1]\}$ of n variables.

S	\rightarrow	$(Disjunction)$	Short: $\mathbf{S} : \{D\}$
S	\rightarrow	$S \cdot (Disjunction)$	
$Disjunction$	\rightarrow	$Literal$	Short: $\mathbf{D} : \{X\}$
$Disjunction$	\rightarrow	$Disjunction + Literal$	
$Literal$	\rightarrow	$Variable$	Short: $\mathbf{L} : x_i$
$Literal$	\rightarrow	$\neg Variable$	

The notation $(a + \bar{b}) \cdot (b + \bar{c} + \bar{d}) \cdot (\bar{a} + \bar{c})$ will be used as type “set of literal sets” $\{\{a, \neg b\}, \{b, \neg c, \neg d\}, \{\neg a, \neg b\}\}$ in this paper.

2 Transformation Techniques

U-SAT *Unmixed conjunctive normal form.* Given a *CNF-SAT* formula, we call it *unmixed*, if every disjunction consists of either positive or negated literals.

A polynomial time reduction $SAT \leq_p U-SAT$ for an equation $S_E \rightarrow U_E$ would be:

$$\forall D \in S :$$

TODO De Morgan

3-CNF-SAT *Three-literals conjunctive normal form.* Given a *CNF-SAT* formula, we call it *n-SAT*, if every disjunction consists of exactly *n*-literals.

TODO Proof

U3-SAT *Three-literals unmixed conjunctive normal form.* Given a *CNF-SAT* formula, we call it *U3-SAT*, if it is *unmixed* and every disjunction consists of exactly 3-literals.

TODO

3 Optimisation Techniques

Zugzwang. *Common Literals.* If a disjunction consists of one literal, the value of this literal is fixed.

$$S = X \cdot A(X) \rightarrow (f_{SAT}(Y) \Rightarrow X \in Y)$$

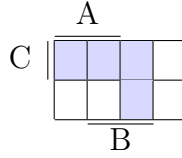
Contradiction. *Contradictional Disjunctions.*

A formula with contradictional disjunctions is not satisfiable.

$$\forall A : X \cdot \neg X \cdot A(X) = \perp$$

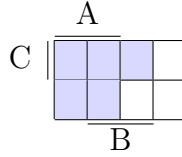
Common Contradicting Literals. *Disjunction with common contradictional part.*

$$(A + B) \cdot (\neg A + C) \cdot X = (AC + B\bar{A}) \cdot X$$



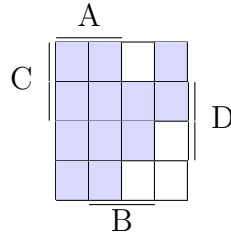
Common Literals. *Disjunction with common positive part.*

$$(A + B) \cdot (A + C) \cdot X = (A + BC) \cdot X$$



Common Literals and Contradictional Part. *Disjunction with positive and negative common parts.*

$$(A + B + C) \cdot (A + \neg B + D) \cdot X = (A + BD + C\bar{B}) \cdot X$$



Connected Component. *Two independent sets of variables.* If there exists two sets of variables A and B , such that $\forall a \in A, b \in B \nexists D : a \in D \wedge b \in D$, then you can create two equations as following and solve them independently with reduced variable count.

$$\forall V \in \{A, B\} : S_V = \{d \in D \mid \forall l \in d : l \in V\}$$

3×3 Combination. *Brute-Force.*

$$\begin{aligned} & (a + b + c) \cdot (a + d + e) \\ = & a + bd + be + cd + ce \\ = & a + (b + c) \cdot (d + e) \end{aligned}$$