

# Sat Solver Optimisations

Anatoly Weinstein

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This document contains research into optimisations of CNF-SAT problems.

## 1 Definitions

**Boolean Function.** A boolean function  $f : V \subseteq Variables \rightarrow \{\perp, \top\}$  with truthy sets  $P \subseteq \mathcal{P}(Variables)$  yields  $\top$  (truthy) if  $V \in P$ , else yields  $\perp$  (falsy).

**Boolean Satisfiability Equation.** We define a boolean formula by the following context-free grammar with start variable  $S$ , a given variable set  $Variable = \{x_i : i \in [n - 1]\}$  of  $n$  variables and a set of binary operations  $Operation = \{\cdot, +, \oplus, \dots\}$  and an assigned boolean function  $f_{Operation} : \mathcal{P} \subseteq \{L, R\} \rightarrow \{\perp, \top\}$ , where  $L$  and  $R$  is the truthiness of left and right respectively.

$$\begin{array}{ll} S & \rightarrow (S) \\ S & \rightarrow S \text{ Operation } S \\ S & \rightarrow \neg S \\ S & \rightarrow Variable \end{array}$$

The *boolean equation* is satisfied with interpretation  $P$ , if there is an interpretation  $X$  with  $f_{SAT}(X) = \top$ .

**CNF** *Unmixed conjunctive normal form.* We define a boolean formula in conjunctive normal form by the following context-free grammar with start variable  $S$  and a given variable set  $Variable = \{x_i : i \in [n - 1]\}$  of  $n$  variables.

|               |               |                         |                             |
|---------------|---------------|-------------------------|-----------------------------|
| $S$           | $\rightarrow$ | $(Disjunction)$         | Short: $\mathbf{S} : \{D\}$ |
| $S$           | $\rightarrow$ | $S \cdot (Disjunction)$ |                             |
| $Disjunction$ | $\rightarrow$ | $Literal$               | Short: $\mathbf{D} : \{X\}$ |
| $Disjunction$ | $\rightarrow$ | $Disjunction + Literal$ |                             |
| $Literal$     | $\rightarrow$ | $Variable$              | Short: $\mathbf{L} : x_i$   |
| $Literal$     | $\rightarrow$ | $\neg Variable$         |                             |

The notation  $(a + \bar{b}) \cdot (b + \bar{c} + \bar{d}) \cdot (\bar{a} + \bar{c})$  will be used as type “set of literal sets”  $\{\{a, \neg b\}, \{b, \neg c, \neg d\}, \{\neg a, \neg b\}\}$  in this paper.

## 2 Transformation Techniques

**U-SAT** *Unmixed conjunctive normal form.* Given a *CNF-SAT* formula, we call it *unmixed*, if every disjunction consists of either positive or negated literals.

A polynomial time reduction  $SAT \leq_p U-SAT$  for an equation  $S_E \rightarrow U_E$  would be:

$$\forall D \in S :$$

TODO De Morgan

**3-CNF-SAT** *Three-literals conjunctive normal form.* Given a *CNF-SAT* formula, we call it *n-SAT*, if every disjunction consists of exactly *n*-literals.

TODO Proof

**U3-SAT** *Three-literals unmixed conjunctive normal form.* Given a *CNF-SAT* formula, we call it *U3-SAT*, if it is *unmixed* and every disjunction consists of exactly 3-literals.

TODO

## 3 Optimisation Techniques

**Zugzwang.** *Common Literals.* If a disjunction consists of one literal, the value of this literal is fixed.

$$S = X \cdot A(X) \rightarrow (f_{SAT}(Y) \Rightarrow X \in Y)$$

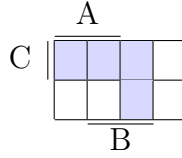
**Contradiction.** *Contradictional Disjunctions.*

A formula with contradictional disjunctions is not satisfiable.

$$\forall A : X \cdot \neg X \cdot A(X) = \perp$$

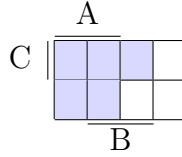
**Common Contradicting Literals.** *Disjunction with common contradictional part.*

$$(A + B) \cdot (\neg A + C) \cdot X = (AC + B\bar{A}) \cdot X$$



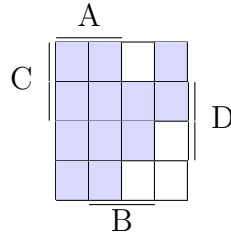
**Common Literals.** *Disjunction with common positive part.*

$$(A + B) \cdot (A + C) \cdot X = (A + BC) \cdot X$$



**Common Literals and Contradictional Part.** *Disjunction with positive and negative common parts.*

$$(A + B + C) \cdot (A + \neg B + D) \cdot X = (A + BD + C\bar{B}) \cdot X$$



**Connected Component.** *Two independent sets of variables.* If there exists two sets of variables  $A$  and  $B$ , such that  $\forall a \in A, b \in B \nexists D : a \in D \wedge b \in D$ , then you can create two equations as following and solve them independently with reduced variable count.

$$\forall V \in \{A, B\} : S_V = \{d \in D \mid \forall l \in d : l \in V\}$$