Introduction to Boolean Arithmetics

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In this document we will contain dry theory of boolean equations and arithmetics.

For the rest of the paper: $0 \in \mathbb{N}$ be truthy.

Axiom of Two Values. All unknowns x in a boolean system can take at least one of two values. $\forall x : x \in \mathbb{B} = \{ \top, \bot \}.$

Axiom of Falsy Contradictions. If a variable in a boolean system is known to contradict the first axiom, then the system is a paradox (\bot) .

We will call Σ the *environment*. Abstractly speaking, the environment of any *boolean system* is the set of all inputs. An element $x \in \Sigma$ is an *interpretation*. A *finite boolean function* $f : \Sigma \to \mathbb{B}$ with $\Sigma = \mathbb{B}^n$, a degree $n \in \mathbb{N}$ and the set of truthy interpretations $I \subseteq \Sigma$ is defined as following:

$$x \in \Sigma : f(x) = \begin{cases} \top, & x \in I \\ \bot, & x \notin I \end{cases}$$

We say the function is satisfiable if |I| > 0, unsatisfiable or a contradiction if |I| = 0 and a tautology if $I = \mathcal{P}(X) = \{Y : Y \subseteq X\}$. An interpretation x is truthy if f(x) = T, and falsy otherwise. Two boolean functions f_1, f_2 are equal if and only if their truthy interpretations sets I_1, I_2 are equal.

An infinite boolean function $f:[0,1]_2 \to \mathbb{B}$ is a boolean function of infinite degree. $[0,1]_2$ is the set of all irrational numbers between 0 and 1 in base 2.

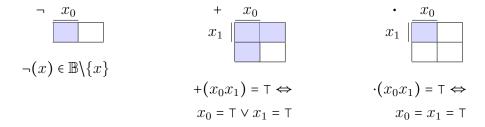
Proof of Truthy. \top (tautalogy) is *truthy*.

$$f:\varnothing\to\mathbb{B}\mapsto \top\Rightarrow I_f=\{\varnothing\}\Rightarrow |I_f|=1\Rightarrow \top \text{ is truthy}.$$

Proof of Falsy. \perp (contradiction) is falsy.

$$f: \varnothing \to \mathbb{B} \mapsto \bot \Rightarrow I_f = \varnothing \Rightarrow |I_f| = 0 \Rightarrow \bot \text{ is falsy.}$$

In this document we will mainly work with the 1st-degree function not (noted as \neg or overline), and the 2nd-degree functions and (noted as \cdot) and or (noted as +) defined as:



1 Boolean Equations

A boolean equation is a boolean system with tuple of variables X, the degree n = |X| and the set of truthy interpretations $I \subseteq \Sigma = \mathbb{B}^n$.

Boolean Equation. A boolean equation with environment Σ and binary functions $\mathcal{F} \subseteq \{f : \mathbb{B}^2 \to \mathbb{B}\}$ is a string defined by the following context-free grammar with macro \mathcal{M}_X creating productions for every element of X.

$$S \rightarrow (S) \mid S \mathcal{M}_{\mathcal{F}} S \mid \neg(S) \mid \mathcal{M}_{\mathbb{B}} \mid \mathcal{M}_{\Sigma}$$

The boolean function for this equation is defined by propagating the syntax tree, calling the binary functions while iterating over the branches. For every boolean equation S we assign $\chi_S(X)$

For the rest of the document we say that $\mathcal{F} = \{+,\cdot\}$.

Conjunctive normal form. A boolean equation is in conjunctive normal form, when the following grammar, too, applies.

$$S \rightarrow (D) \mid S \cdot (D)$$

$$D \rightarrow L \mid D + L$$

$$L \rightarrow \mathcal{M}_{\Sigma} \mid \neg \mathcal{M}_{\Sigma}$$

A boolean equation in CNF consists of disjunctions D, a disjunction consists of literals L. The notation $(a + \overline{b}) \cdot (b + \overline{c} + \overline{d}) \cdot (\overline{a} + \overline{c})$ will be used as type "set of literal sets" $\{\{a, \neg b\}, \{b, \neg c, \neg d\}, \{\neg a, \neg b\}\}$ in this paper.

Notation of CNF. We will now define some macros for working with CNF-SAT equations. Be S a boolean equation in CNF-SAT, consisting of disjunctions D, $d_i \in D$, $D_i \subseteq D$, over the environment Σ .

$$D_i =_S D_j \Leftrightarrow f_{D_i \cdot D} = f_{D_j \cdot D}$$

Two disjunctions are equivalent over the equation S if under union with every disjunction from S, the equations are equivalent.