Sat Solver Optimisations

Anatoly Weinstein

January 9, 2024

This document contains research into optimisations of CNF-SAT problems.

1 Definitions

Boolean Function. A boolean function $f : V \subseteq Variables \to \{\bot, \top\}$ with truthy sets $P \subseteq \mathcal{P}(Variables)$ yields $\top$ (truthy) if $V \in P$, else yields $\bot$ (falsy).

Boolean Satisfyability Equation. We define a boolean formula by the following context-free grammar with start variable S, a given variable set $Variable = \{x_i : i \in [n-1]\}$ of n variables and a set of binary operations $Operation = \{\cdot, +, \oplus, \dots\}$ and an assigned boolean function $f_{Operation} : \mathcal{P} \subseteq \{L, R\} \to \{\bot, \top\}$, where L and R is the truthiness of left and right respectively.

$$\begin{array}{ccc} S & \to & (S) \\ S & \to & S\ Operation\ S \\ S & \to & \neg S \\ S & \to & Variable \end{array}$$

The *boolean equation* is satisfied with interpretation P, if there is an interpretation X with $f_{SAT}(X) = \top$.

CNF *Unmixed conjunctive normal form.* We define a boolean formula in conjunctive normal form by the following context- free grammar with start variable S and a given variable set $Variable = \{x_i : i \in [n-1]\}$ of n variables.

$$
\begin{array}{ccccc}
S & \to & (Disjunction) & \text{Short: } \mathbf{S} : \{D\} \\
S & \to & S \cdot (Disjunction) & \\
Disjunction & \to & Literal & \text{Short: } \mathbf{D} : \{X\} \\
Disjunction & \to & Disjunction + Literal & \\
Literal & \to & Variable & \text{Short: } \mathbf{L} : x_i \\
Literal & \to & \neg Variable &
\end{array}
$$

The notation $(a+\overline{b}) \cdot (b+\overline{c}+\overline{d}) \cdot (\overline{a}+\overline{c})$ will be used as type "set of literal sets" $\{\{a, \neg b\}, \{b, \neg c, \neg d\}, \{\neg a, \neg b\}\}$ in this paper.

2 Transformation Techniques

U-SAT *Unmixed conjunctive normal form.* Given a *CNF-SAT* formula, we call it *unmixed*, if every disjunction consists of either positive or negated literals.

A polynomial time reduction $SAT \leq_p U{-}SAT$ for an equation $S_E \to U_E$ would be:

$$\forall D \in S :$$

TODO De Morgan

3-CNF-SAT *Three-literals conjunctive normal form.* Given a *CNF-SAT* formula, we call it $n - SAT$, if every disjunction consists of exactly n-literals.

TODO Proof

U3-SAT *Three-literals unmixed conjunctive normal form.* Given a *CNF-SAT* formula, we call it $U3{-}SAT$, if it is *unmixed* and every disjunction consists of exactly 3-literals.

TODO

3 Optimisation Techniques

Zugzwang. *Common Literals.* If a disjunction consists of one literal, the value of this literal is fixed.

$$S = X \cdot A(X) \to (f_{SAT}(Y) \Rightarrow X \in Y)$$

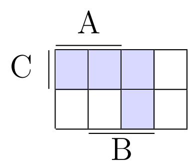
Contradiction. *Contradictional Disjunctions.*

A formula with contradictional disjunctions is not satisfiable.

$$\forall A : X \cdot \neg X \cdot A(X) = \bot$$

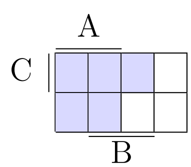
Common Contradicting Literals. *Disjunction with common contradictional part.*

$$(A + B) \cdot (\neg A + C) \cdot X = (AC + B\overline{A}) \cdot X$$



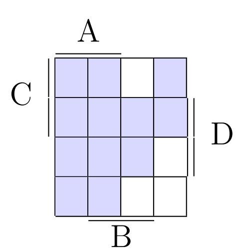
Common Literals. *Disjunction with common positive part.*

$$(A + B) \cdot (A + C) \cdot X = (A + BC) \cdot X$$



Common Literals and Contradictional Part. *Disjunction with positive and negative common parts.*

$$(A + B + C) \cdot (A + \neg B + D) \cdot X = (A + BD + C\overline{B}) \cdot X$$



Connected Component. *Two independent sets of variables.* If there exists two sets of variables A and B, such that $\forall a \in A, b \in B \nexists D : a \in D \wedge b \in D$, then you can create two equations as following and solve them independently with reduced variable count.

$$\forall V \in \{A, B\} : S_V = \{d \in D \;\forall\; l \in d : \; l \in V\}$$

3×3 **Combination.** *Brute-Force.*

$$\begin{aligned} &(a+b+c)\cdot(a+d+e) \\ =&a+bd+be+cd+ce \\ =&a+(b+c)\cdot(d+e) \end{aligned}$$

4 CNF-SAT Optimisation

TODO 3-CNF-SAT equations

All equal. $D \cdot D = D$

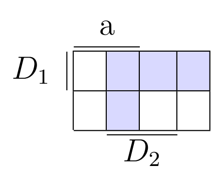
If two disjunctions are equal, remove the second one.

One off. $(a+b+c)\cdot(a+b+d)$

If two disjunctions differ in only one letter, the following can be said:

$$\begin{aligned} (a+b+c)\cdot(a+b+d) \\ = a+b+c\cdot d \end{aligned}$$

Opposing Literal. $(a + D_1) \cdot (\overline{a} + D_2)$



Add the following