

# Manifold Matching Complexes

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## Abstract

The matching complex of a graph is the simplicial complex whose vertex set is the set of edges of the graph with a face for each independent set of edges. In this paper we completely characterize the pairs (graph, matching complex) for which the matching complex is a combinatorial manifold, with or without boundary.

## 1 Introduction

The matching complex  $M(G)$  of a graph  $G$  is a simplicial complex representing the matchings (sets of independent edges) of the graph. There is an extensive literature describing the matching complexes of certain types of graphs.

There are many results on the topology of the matching complexes of interesting classes of graphs. For example, there has been much study of chessboard complexes,  $\Delta_{m,n} = M(K_{m,n})$ . Björner, et al., [2] prove that  $M(K_{m,n})$  is  $\nu$ -connected, where  $\nu = \min\{m, n, \lfloor \frac{m+n+1}{3} \rfloor\} - 2$ . Ziegler [18] shows that for  $m \geq 2n - 1$ ,  $M(K_{m,n})$  is shellable, and Jojić [8] uses this to give a recursion for the  $h$ -vectors of these chessboard complexes. Athanasiadis [1] studies vertex decomposability of skeleta of hypergraph matching complexes and chessboard complexes. Wachs [17] surveys results on the homology of chessboard complexes and matching complexes of the complete graph. Jonsson's dissertation (published as [9]) studies various complexes associated with graphs, including matching complexes. Kozlov [11] proves that, for  $\nu_n = \lfloor \frac{n-2}{3} \rfloor$ , the matching complex  $M(P_{n+1})$  of the length  $n$  path is homotopy equivalent to the sphere  $S^{\nu_n}$  when  $n \bmod 3 \neq 1$ , and the matching complex  $M(C_n)$  of the

$n$ -cycle is homotopy equivalent to the sphere  $S^{n-1}$  when  $n \bmod 3 \neq 0$ . (As is standard in graph theory, the subscript on a graph name indicates the number of vertices; for paths this is one more than the length.) Matching complexes of grid graphs have been studied by Braun and Hough [3] and by Matsushita [14]. Marietti and Testa [13] proved that matching complexes of forests are contractible or homotopy equivalent to a wedge of spheres. For caterpillar graphs, Jelić Milutinović, et al. [7] give explicit formulas for the number of spheres in each dimension. They also study the connectivity of matching complexes of honeycomb graphs.

We are interested in the reverse question: which simplicial complexes are matching complexes of graphs? In this paper we will classify combinatorial manifolds, with and without boundary, that are matching complexes. In Section 2, we review definitions and introduce several tools that we will rely on in later sections. In Section 3, we describe all graphs whose matching complexes are 1- and 2-dimensional spheres. In Section 4, we describe all combinatorial manifolds without boundary that arise as matching complexes. All of these matching complexes are spheres, except in dimension two, where the torus is also a matching complex. In Section 5, we finish the story with a complete description of the matching complexes that are combinatorial manifolds with boundary. In dimension two, a variety of manifolds with boundary arise as matching complexes. In dimensions three and higher, these matching complexes are all balls. Moreover, the graphs that produce manifold matching complexes are all constructed from the disjoint union of copies of a finite set of graphs, which we explicitly specify.

## 2 Preliminaries

### 2.1 General properties of matching complexes

**Definition 2.1** A *matching* of a graph  $G$  is a set of edges of  $G$ , no two of which share a vertex.

**Definition 2.2** The *matching complex* of a graph  $G$  is the simplicial complex  $M(G)$  with vertex set the set  $E$  of edges of  $G$  and simplices every subset  $\sigma \subseteq E$  that forms a matching of  $G$ .

In what follows we will use the notational convention: if  $v$  is a vertex of  $M(G)$ , then the corresponding edge of  $G$  is denoted  $\bar{v}$ . We extend that to sets: if  $\sigma$  is a face of  $M(G)$ , then  $\bar{\sigma}$  is the corresponding matching of  $G$ .

Note that an isolated vertex of  $G$  would contribute nothing to  $M(G)$ ; we avoid them to simplify statements. Allowing a multiple edge in a graph would duplicate a subcomplex in the matching complex. A loop is not considered to be in any matching. So from now on we will assume the following:

All graphs have no isolated vertices, loops, or multiple edges.

Note that a matching complex  $M(G)$  does not determine  $G$  uniquely. For example  $G_1 = K_{1,3}$  and  $G_2 = K_3$  have the same matching complex,  $M(G_i) = 3P_1$ .

**Definition 2.3** A *missing face*  $\sigma$  of a simplicial complex  $\Delta$  is a subset of vertices of  $\Delta$  such that  $\sigma$  is not a face of  $\Delta$ , but all proper subsets of  $\sigma$  are faces of  $\Delta$ . A simplicial complex  $\Delta$  is a *flag complex* if and only if every missing face of  $\Delta$  is of size 2.

**Proposition 2.4** If  $M(G)$  is the matching complex of a graph  $G$ , then  $M(G)$  is a flag complex.

**Proof:** Let  $M(G)$  be the matching complex of a graph  $G$ , and let  $S$  be a subset of the vertex set of  $M(G)$  of size at least 3. Assume that each proper subset of  $S$  is a face of  $M(G)$ . Since  $|S| \geq 3$ , every pair of elements  $\{v, w\}$  of  $S$  is a face of  $M(G)$ . So the corresponding edges  $\bar{v}$  and  $\bar{w}$  of  $G$  do not share a vertex. This is true for every pair of elements of  $S$ , so the set  $\bar{S}$  of edges of  $G$  form a matching of  $G$ . Thus  $S$  is a face of  $M(G)$ . So  $M(G)$  is a flag complex.  $\square$

**Observation.** In analogy with graph terminology, we say that a subcomplex  $N$  is an *induced subcomplex* of a simplicial complex  $M$ , if  $N$  is the restriction of  $M$  to some subset of the vertices of  $M$ , that is,  $N$  is a subcomplex of  $M$  with vertex set  $V(N) \subseteq V(M)$ , and the faces of  $N$  are all the faces of  $M$  with vertices contained in  $V(N)$ . If  $M$  is the matching complex of a graph  $G$  and  $N$  is an induced subcomplex of  $M$ , then  $N$  is the matching complex of a subgraph of  $G$ , namely the subgraph spanned by the edges corresponding to vertices of  $N$ .

**Definition 2.5** For a face  $\sigma$  of a simplicial complex  $\Delta$ , the *link* of  $\sigma$  in  $\Delta$  is  $\text{link}_\Delta \sigma = \{\tau \in \Delta : \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \emptyset\}$ .

**Lemma 2.6 (Link Lemma)** *Let  $\sigma \in M(G)$ . Then  $\text{link}_{M(G)} \sigma = M(G_{\bar{\sigma}})$ , where  $G_{\bar{\sigma}}$  is the subgraph of  $G$  spanned by all edges of  $G$  that are not incident to any edge in  $\bar{\sigma}$ .*

**Proof:** If  $\tau \in \text{link}_{M(G)} \sigma$  then each vertex of  $\tau$  forms an edge with each vertex of  $\sigma$ . So the corresponding edges form a matching of  $G$ , so  $\bar{\tau} \subseteq E(G_{\bar{\sigma}})$ . Thus  $\text{link}_{M(G)} \sigma \subseteq M(G_{\bar{\sigma}})$ . Similarly, given a face  $\tau \in M(G_{\bar{\sigma}})$ , we see that  $\bar{\tau} \cup \bar{\sigma}$  is a matching, and thus  $\tau \in \text{link}_{M(G)} \sigma$ .  $\square$

We will often blur the distinction between the subgraph  $G_{\bar{\sigma}}$  and the set of its edges.

**Definition 2.7** The *join* of two disjoint simplicial complexes  $\Delta$  and  $\Sigma$  is the simplicial complex  $\Delta * \Sigma = \{\tau \cup \sigma : \tau \in \Delta \text{ and } \sigma \in \Sigma\}$

Joins arise in matching complexes of disconnected graphs.

**Lemma 2.8 (Join Lemma)** *Let  $M(G)$  be the matching complex of a graph  $G$ . Then  $M(G) = M_1 * M_2$  for some disjoint simplicial complexes  $M_1$  and  $M_2$  (neither equal to  $\{\emptyset\}$ ) if and only if there exist nonempty graphs  $G_1$  and  $G_2$  such that  $G$  is the disjoint union of  $G_1$  and  $G_2$ ,  $M_1 = M(G_1)$ , and  $M_2 = M(G_2)$ .*

**Proof:** If  $G$  is the disjoint union of two nonempty graphs  $G_1$  and  $G_2$ , then the maximal matchings of  $G$  are exactly the unions of maximal matchings of  $G_1$  with maximal matchings of  $G_2$ , so  $M(G) = M(G_1) * M(G_2)$ . Conversely, suppose  $M(G) = M_1 * M_2$ . Let  $G_i$  be the graph induced by the edges  $\bar{v}$  where  $v$  is a vertex of  $M_i$ . Since the vertices of  $M_i$  are vertices of  $M(G)$ ,  $G_i$  is a subgraph of  $G$ . Furthermore, since each vertex  $v$  of  $M_1$  forms an edge in  $M(G)$  with each vertex  $w$  in  $M_2$ , every pair of graph edges, one in  $G_1$  and one in  $G_2$ , share no vertex. So  $G_1$  and  $G_2$  are disjoint. Each face  $\sigma$  of  $M_i$  is a face of  $M(G)$ , and hence corresponds to a matching of  $G$ , which is necessarily a matching of  $G_i$ . So  $M(G_i) = M_i$ .  $\square$

Thus the matching complex of any disconnected graph is connected. For what graphs are the matching complexes disconnected? Here we consider a matching complex to be connected if and only if the subcomplex consisting of all its vertices and edges (its “1-skeleton”) is connected. A path between

two vertices in a matching complex  $M(G)$  corresponds to a sequence of size two matchings in  $G$  such that consecutive matchings share an edge.

The following characterization of disconnected matching complexes is due to Jelić Milutinović, et al.

**Theorem 2.9** [7, Proposition 3.1] *The matching complex  $M(G)$  of a graph  $G$  is not connected if and only if there exists a subset  $I$  of  $E(G)$  such that  $1 \leq |I| < |E(G)|$ , and every edge of  $E(G) \setminus I$  is incident to every edge of  $I$ .*

We use a corollary.

**Corollary 2.10** *A graph  $G$  has a disconnected matching complex if and only if  $G = C_4$ ,  $G = K_4$ , or  $G$  has one edge incident to all other edges.*

The disconnected matching complexes are all of dimension 0 or 1, and as follows. Note that  $G \sqcup H$  denotes the disjoint union of graphs  $G$  and  $H$ , and  $nG$  denotes  $n$  disjoint copies of the graph  $G$ .

- $nP_1$  ( $n$  isolated vertices) is the matching complex of  $K_{1,n}$ .
- $2P_2$  is the matching complex of  $C_4$ .
- $3P_2$  is the matching complex of  $K_4$ .
- $P_1 \sqcup K_{m,n}$  is the matching complex of the graph obtained from an edge by adding  $m$  pendant edges to one of its vertices and  $n$  pendant edges to its other vertex.
- $P_1 \sqcup (K_{m,n} \setminus H)$ , where  $H$  is a nonempty set of  $r$  independent edges of  $K_{m,n}$ , is the matching complex of the graph consisting of  $r$  triangles sharing an edge  $\{x, y\}$ , along with  $m - r$  pendant edges on vertex  $x$  and  $n - r$  pendant edges on vertex  $y$ .

As noted in [7], Theorem 2.9 implies that every connected matching complex has diameter at most 4. We use a result in a similar vein.

**Proposition 2.11** *If  $M(G)$  is the matching complex of some graph, then  $M(G)$  has no induced path of length 5.*

**Proof:** Suppose  $M(G)$  contains an induced  $P_6$  subgraph. Label the vertices of this subgraph in order 1 through 6. Then  $G$  includes six edges  $\bar{1}$  through  $\bar{6}$ , with incidences the ten pairs of edges  $\bar{i}$  and  $\bar{j}$ , where  $|j - i| \geq 2$ . Note that three edges in  $G$  are pairwise incident if and only if the three form a triangle or the three share a single vertex (forming a star,  $K_{1,3}$ ). Consider the incidences among edges  $\bar{1}$ ,  $\bar{3}$ , and  $\bar{5}$ . Suppose they form a triangle. Note that edge  $\bar{2}$  is incident to edge  $\bar{5}$ , but not to edges  $\bar{1}$  or  $\bar{3}$ . This is not possible, since each vertex of  $\bar{5}$  is shared with either  $\bar{1}$  or  $\bar{3}$ . Thus the three edges  $\bar{1}$ ,  $\bar{3}$ , and  $\bar{5}$  all meet at a single vertex  $a$ . Now  $\bar{4}$  is incident to  $\bar{1}$ , but not to  $\bar{3}$  or  $\bar{5}$ , and  $\bar{2}$  is incident to  $\bar{4}$  and  $\bar{5}$ , but not to  $\bar{1}$  and  $\bar{3}$ , so the induced subgraph of  $G$  on the vertices contained in the edges  $\bar{1}$  through  $\bar{5}$  is a 4-cycle with a pendant edge. This graph is called the *banner graph*. (See Figure 1.)

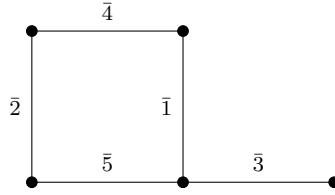


Figure 1: Banner graph  $\Gamma$  whose matching complex is  $P_5$

Now edge  $\bar{6}$  would need to be incident to all of these edges except  $\bar{5}$ , which is not possible. So there is no graph  $G$  whose matching complex has an induced  $P_6$ .  $\square$

Note that every path of length at most 4 is a matching complex, as shown in Table 1. These are all the matching complexes that are 1-dimensional manifolds with boundary.

## 2.2 Combinatorial manifolds

The main results of this paper concern matching complexes that are combinatorial spheres and manifolds. We wish to be clear about what these are. We base our treatment of these definitions on [10].

<i>Graph <math>G</math></i>	<i>Matching Complex <math>M(G)</math></i>
$P_2$	$P_1$
$2P_2$	$P_2$
$P_3 \sqcup P_2$	$P_3$
$P_5$	$P_4$
$\Gamma$	$P_5$

Table 1: Matching complexes that are paths

**Definition 2.12**

A *combinatorial  $d$ -ball* is a simplicial complex that is PL-homeomorphic to a  $d$ -simplex. A *combinatorial  $d$ -sphere* is a simplicial complex that is PL-homeomorphic to the boundary of a  $(d + 1)$ -simplex.

A closed *combinatorial  $d$ -manifold* (without boundary) is a simplicial complex  $\Delta$  such that for every nonempty face  $\sigma$  of  $\Delta$ ,  $\text{link}_\Delta(\sigma)$  is a combinatorial  $(d - |\sigma|)$ -sphere.

A *combinatorial  $d$ -manifold with boundary* is a simplicial complex  $\Delta$  such that for every nonempty face  $\sigma$  of  $\Delta$ ,  $\text{link}_\Delta(\sigma)$  is either a combinatorial  $(d - |\sigma|)$ -sphere or a combinatorial  $(d - |\sigma|)$ -ball, with at least one such link being a ball.

We say that  $\sigma$  is in the *interior* of  $\Delta$  if  $\text{link}_\Delta \sigma$  is a sphere and on the *boundary* if  $\text{link}_\Delta \sigma$  is a ball. For  $d \geq 1$ , the boundary of a  $d$ -manifold is the subcomplex generated by all  $(d - 1)$ -faces that are contained in exactly one  $d$ -face.

Note that combinatorial spheres are combinatorial manifolds without boundary, and combinatorial balls are combinatorial manifolds with boundary. Hereafter, when we refer to spheres or balls, we are always assuming that they are combinatorial.

A single vertex is a combinatorial 0-ball, and the two-vertex complex,  $2P_1$ , is a combinatorial 0-sphere. We will not consider 0-complexes with a larger number of vertices in the context of manifolds, but we have already observed that  $nP_1$  is the matching complex of  $K_{1,n}$ .

In what follows we will need to recognize combinatorial manifolds with and without boundary that are joins of lower dimensional manifolds. We use Theorem 2 from [12], applied in our context.

**Proposition 2.13**

1. Let  $X$  be a combinatorial  $d$ -manifold without boundary such that  $X = \Delta * \Sigma$ , where  $\Delta$  and  $\Sigma$  are nonempty simplicial complexes and the simplicial complex structure of  $X$  is induced from  $\Delta$  and  $\Sigma$ . Then  $X$  is homeomorphic to  $S^d$ , and  $\Delta$  and  $\Sigma$  are combinatorial spheres of lower dimension.
2. Let  $X$  be a combinatorial  $d$ -manifold with boundary such that  $X = \Delta * \Sigma$ , where  $\Delta$  and  $\Sigma$  are nonempty simplicial complexes and the simplicial complex structure of  $X$  is induced from  $\Delta$  and  $\Sigma$ . Then  $X$  is homeomorphic to the ball  $B^d$ , and  $\Delta$  and  $\Sigma$  are spheres or balls, such that at least one of  $\Delta$  or  $\Sigma$  is a ball.

With Proposition 2.13 in mind, we define two important sets. The first is the set of *basic sphere graphs*:

$$\mathcal{SG} = \{P_3, C_5, K_{3,2}\}. \quad (1)$$

Basic sphere graphs are so named because their matching complexes are spheres. (These matching complexes are the 0-sphere,  $C_5$ , and  $C_6$ , respectively.) The disjoint unions of these graphs give matching complexes that are higher dimensional spheres.

**Proposition 2.14** *Let  $G$  be a disjoint union of graphs from  $\mathcal{SG}$ . Then  $M(G)$  is a sphere.*

*In particular, let  $G = \ell P_3 \sqcup m C_5 \sqcup n K_{3,2}$ . Then the matching complex  $M(G)$  is a triangulation of the  $(\ell + 2m + 2n - 1)$ -sphere.*

**Proof:** The matching complex of this graph is the join of  $\ell$  copies of  $S^0$  and  $m + n$  copies of  $S^1$ . This join is a sphere of dimension  $\ell + 2m + 2n - 1$ .  $\square$

Note, in particular, that the matching complex of  $\ell P_3$  is the boundary complex of an  $\ell$ -dimensional crosspolytope (generalized octahedron).

Before defining the second set, we will note a particular sequence of graphs, generalizing  $P_5$ , and their matching complexes.

**Definition 2.15** The *spider*  $\text{Sp}_k$  is the graph obtained by identifying one end vertex of each of  $k$  copies of  $P_3$ .



(In the literature, “spider” usually refers to a more general class of graphs, allowing legs of different lengths.)

We can visualize  $\text{Sp}_k$  as having one central vertex  $x$  with  $k$  “legs” of length 2 emanating from that vertex. (The graph  $\text{Sp}_2$  is just  $P_5$ .) Label the edges of the  $i$ th leg  $\bar{u}_i$  and  $\bar{v}_i$ , with  $\bar{u}_i$  containing the central vertex  $x$ . The set  $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k\}$  is a maximal matching of  $\text{Sp}_k$ . Since no two  $\bar{u}_i$ s are in any matching, the other maximal matchings are obtained from  $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k\}$  by replacing a single  $\bar{v}_i$  by the neighboring  $\bar{u}_i$ . The matching complex  $M = M(\text{Sp}_k)$  thus has a central  $(k-1)$ -simplex  $C = \{v_1, v_2, \dots, v_k\}$ , and  $k$  other facets containing the vertex  $u_i$  and intersecting  $C$  at  $C \setminus \{v_i\}$ . This is a  $(k-1)$ -ball, thus a manifold with boundary.

We now can define the set of *basic ball graphs*

$$\mathcal{BG} = \{P_2, \Gamma, \text{Sp}_k\}, \quad (2)$$

where  $\Gamma$  is the banner graph pictured in Figure 1 and  $\text{Sp}_k$  is as defined above, for all  $k \geq 2$ . We note again that  $P_5 = \text{Sp}_2$ , so this graph is contained in  $\mathcal{BG}$  as well.

We have already seen that disjoint unions of graphs from  $\mathcal{SG}$  produce matching complexes that are spheres. Here is the analogous result for balls.

**Proposition 2.16** *Let  $G$  be a disjoint union of graphs from  $\mathcal{BG}$  and  $\mathcal{SG}$  with at least one component from  $\mathcal{BG}$ . Then  $M(G)$  is a ball.*

*In particular, let  $G = iP_2 \sqcup j\Gamma \sqcup \bigsqcup_{d \geq 2} k_d \text{Sp}_d \sqcup \ell P_3 \sqcup mC_5 \sqcup nK_{3,2}$ , with  $i + j + \sum k_d \geq 1$ . Then the matching complex  $M(G)$  is a triangulation of the  $(i + 2j + \sum_d dk_d + \ell + 2m + 2n - 1)$ -ball.*

**Proof:** The matching complex of this graph is the join of  $i$  copies of  $B^0$ ,  $j + k_2$  copies of  $B^1$ ,  $k_d$  copies of  $B^{d-1}$  (for  $d > 2$ ),  $\ell$  copies of  $S^0$ , and  $m + n$  copies of  $S^1$ . This join is a ball of dimension  $i + 2j + \sum_d dk_d + \ell + 2m + 2n - 1$ .  $\square$

In the following sections, we will show that Propositions 2.14 and 2.16 provide the only way to construct spheres and balls as matching complexes. Moreover, outside of dimension 2, we will show that these propositions produce all possible manifold matching complexes.

### 3 Low-dimensional spheres

We now focus on the following question:

For which graphs  $G$  is the matching complex  $M(G)$  a combinatorial sphere?

We give the complete answer when  $0 \leq d \leq 2$  in this section and finish the story for higher dimensions in Section 4. Throughout we assume our graphs are simple (having no loops or multiple edges) and have no isolated vertices.

The 0-dimensional sphere is simply the complex consisting of two isolated points. As a matching complex, this represents two edges of the graph that together do not form a matching. In other words,  $G = P_3$ .

A triangulated 1-dimensional sphere is  $C_n$  for some  $n \geq 3$ . We obtain the complete classification in this case.

**Theorem 3.1** *The matching complexes that are 1-spheres are  $C_4$ ,  $C_5$  and  $C_6$ . The graphs giving these matching complexes are  $2P_3$ ,  $C_5$  and  $K_{3,2}$ , respectively.*

**Proof:** We consider 1-dimensional matching complexes that are  $n$ -cycles  $C_n$  for  $n \geq 3$ .

**n = 3.** By Proposition 2.4, a matching complex cannot be the one-dimensional complex  $C_3$ .

**n = 4.** Label the vertices of  $C_4$  in cyclic order 1 through 4. If  $M(G) = C_4$ , then  $G$  has four edges  $\bar{1}$  through  $\bar{4}$ , with edges  $\bar{1}$  and  $\bar{3}$  incident and edges  $\bar{2}$  and  $\bar{4}$  incident (and no other incidences among edges). Thus  $G$  is the disjoint union of two paths, one with edges  $\bar{1}$  and  $\bar{3}$ , and the other with edges  $\bar{2}$  and  $\bar{4}$ . Thus,  $M(G) = C_4$  if and only if  $G = 2P_3$ .

**n = 5.** Label the vertices of  $C_5$  in cyclic order 1 through 5. If  $M(G) = C_5$ , then  $G$  has five edges  $\bar{1}$  through  $\bar{5}$ , with incidences of exactly five pairs of edges: edges  $\bar{1}$  and  $\bar{3}$ ; edges  $\bar{3}$  and  $\bar{5}$ ; edges  $\bar{5}$  and  $\bar{2}$ ; edges  $\bar{2}$  and  $\bar{4}$ ; and edges  $\bar{4}$  and  $\bar{1}$ . Thus,  $M(G) = C_5$  if and only if  $G = C_5$ .

**n = 6.** Label the vertices of  $C_6$  in cyclic order 1 through 6. If  $M(G) = C_6$ , then  $G$  has six edges  $\bar{1}$  through  $\bar{6}$ , with incidences of exactly nine pairs of edges: edges  $\bar{1}$  and  $\bar{3}$ ; edges  $\bar{3}$  and  $\bar{5}$ ; edges  $\bar{5}$  and  $\bar{1}$ ; edges  $\bar{2}$  and  $\bar{4}$ ; edges  $\bar{4}$  and  $\bar{6}$ ; edges  $\bar{6}$  and  $\bar{2}$ ; edges  $\bar{1}$  and  $\bar{4}$ ; edges  $\bar{2}$  and  $\bar{5}$ ; and edges  $\bar{3}$  and  $\bar{6}$ . Recall that three edges in  $G$  are pairwise incident if and only if the three form a triangle or the three share a single vertex (forming a star,  $K_{1,3}$ ). As

in the proof of Proposition 2.11, the edges  $\bar{1}$ ,  $\bar{3}$  and  $\bar{5}$  cannot form a triangle, so must meet at a single vertex  $a$ . Similarly for the edges  $\bar{2}$ ,  $\bar{4}$  and  $\bar{6}$ , which must meet at a single vertex  $b$ . The three remaining incidences identify the endpoints (other than  $a$  and  $b$ ) of edges  $\bar{1}$  and  $\bar{4}$ , of edges  $\bar{2}$  and  $\bar{5}$ , and of edges  $\bar{3}$  and  $\bar{6}$ . The resulting graph is  $K_{3,2}$  (Figure 2). Thus,  $M(G) = C_6$  if and only if  $G = K_{3,2}$ .

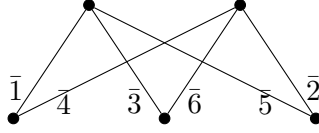


Figure 2: Graph  $K_{3,2}$  whose matching complex is  $C_6$

$n \geq 7$ . Every cycle  $C_n$  with  $n \geq 7$  has an induced  $P_6$  subgraph, and so is not the matching complex of a graph by Proposition 2.11.

Therefore, the cycles that are matching complexes are  $C_4$ ,  $C_5$ , and  $C_6$ , and the corresponding graphs are  $2P_3$ ,  $C_5$ , and  $K_{3,2}$ .  $\square$

We note that the graphs that appear in Theorem 3.1 are disjoint unions of graphs from  $\mathcal{SG}$ , i.e., the basic sphere graphs. In particular, we see that all 1-spheres are constructed using Proposition 2.14.

Similarly, we can use Proposition 2.14 to produce 2-spheres. That Proposition gives exactly three 2-spheres, the matching complexes of  $3P_3$ ,  $P_3 \sqcup C_5$  and  $P_3 \sqcup K_{3,2}$ . These graphs are the disjoint union of  $P_3$  with the graphs of Theorem 3.1, so the matching complexes are the bipyramids over  $C_4$ ,  $C_5$  and  $C_6$ . The following theorem shows that this is in fact the only way to realize the 2-sphere as a matching complex.

**Theorem 3.2** *Let  $M$  be a combinatorial 2-sphere. Suppose there exists a simple graph with  $M(G) = M$ . Then  $G \in \{3P_3, P_3 \sqcup C_5, P_3 \sqcup K_{3,2}\}$  and  $M(G)$  is the boundary of the bipyramid over  $C_n$  for  $n \in \{4, 5, 6\}$ .*

*In particular, if the matching complex of a simple graph  $G$  is a triangulation of a 2-sphere, then  $G$  is not connected.*

**Proof:** Assume  $G$  is a simple graph with no isolated vertices, and its matching complex  $M = M(G)$  is a triangulation of the 2-sphere. We say that a vertex of  $M$  has degree  $k$  in  $M$  if it is contained in exactly  $k$  edges of  $M$ ; in

this case the link of the vertex in  $M$  is a  $k$ -cycle. By Lemma 2.6, the link is itself a matching complex, and so by Theorem 3.1,  $k$  must be 4, 5, or 6. By Eberhard's Theorem (1891; see [6, Theorem 1 in Section 13.3]) a triangulated 2-sphere must have some vertex of degree at most 5. We consider cases, based on the degrees of vertices.

**Case I.**  $M$  has a vertex, all of whose neighbors have degree 4.

**Case II.**  $M$  has a pair of adjacent vertices, each of degree 5.

**Case III.**  $M$  has a pair of adjacent vertices, one of degree 5, one of degree 6.

**Case IV.**  $M$  has no vertex of degree 5.

**Case I.** Let  $v$  be a vertex of  $M$  such that all neighbors of  $v$  have degree 4. We show that  $M$  is the boundary of a bipyramid over  $C_4$ ,  $C_5$ , or  $C_6$ .

Suppose the neighbors of  $v$  are (in cyclic order)  $i$ ,  $1 \leq i \leq k$ , where  $4 \leq k \leq 6$ . The edges  $\{i, i+1\}$  (and  $\{k, 1\}$ ) are ridges of the 2-sphere, and hence are in a unique triangle not containing  $v$ . Say that  $\{k, 1\}$  is in triangle  $\{k, 1, w\}$  ( $w \neq v$ ). The four edges containing vertex 1 are  $\{v, 1\}$ ,  $\{k, 1\}$ ,  $\{w, 1\}$ , and  $\{2, 1\}$ . Each pair of consecutive edges spans a triangle of the complex. So  $\{1, 2, w\}$  is a triangle in  $M$ . Repeating this argument, we see that for all  $i$ , the edge  $\{i, i+1\}$  forms a triangle with this vertex  $w$ . So  $M$  contains the boundary of the bipyramid with base  $C_n$  (vertices  $i$ ) and apices  $v$  and  $w$ . A 2-sphere cannot properly contain a 2-sphere as a subcomplex, so  $M$  is in fact the boundary of the bipyramid.

**Case II.** Let  $u$  and  $v$  be adjacent vertices of  $M$  of degree 5. The links of  $u$  and  $v$  are induced  $C_5$ 's; the corresponding subgraphs in  $G$  are copies of  $C_5$ , which share edges  $\bar{1}$  and  $\bar{4}$ . See Figure 3.

We will show that  $M$  must contain the edges 36 and 25. If  $M$  did not contain 36, then edges  $\bar{3}$  and  $\bar{6}$  would be incident in  $G$ . If the edge  $\bar{6}$  were incident with  $\bar{3}$ , then it would also be incident with either  $\bar{1}$  or  $\bar{u}$ , but this is not possible, because 16 and  $6u$  are edges of  $M$ . Thus  $\bar{3}$  and  $\bar{6}$  form a matching in  $G$ , and so 36 is an edge of  $M$ . Similarly 25 is an edge of  $M$ . But then the vertices and edges of  $M$  form a nonplanar graph. Thus Case II cannot happen.

**Case III.** This case is similar to Case II. A subcomplex of  $M$  and its corresponding subgraph are shown in Figure 4.

The link of  $v$  is an induced  $C_5$ , with corresponding graph  $C_5$ ; the link of  $u$  is an induced  $C_6$ , with corresponding graph  $K_{3,2}$ . These subgraphs in  $G$  share edges  $\bar{1}$  and  $\bar{4}$ .

Just as in Case II, 25 and 36 must be edges of  $M$ , and the vertices and

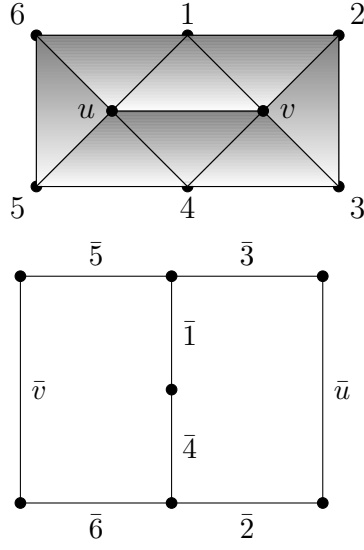


Figure 3: Subcomplex of  $M$  and Subgraph of  $G$  for Case II

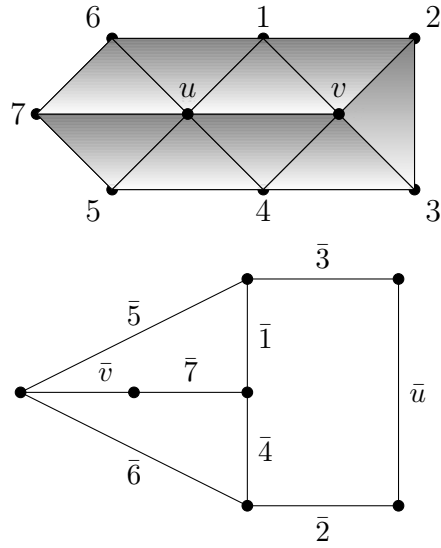


Figure 4: Subcomplex of  $M$  and Subgraph of  $G$  for Case III

edges of  $M$  then form a nonplanar graph. So Case III cannot happen.

**Case IV.** By Eberhard's Theorem  $M$ , having no vertices of degree 3 or 5, must have at least one vertex  $v$  of degree 4. By Case I, we need only consider the subcase where  $v$  has at least one neighbor  $u$  of degree 6. A subcomplex of  $M$  and its corresponding subgraph are shown in Figure 5. (The vertices are labeled  $u, v$ , and 1 through 7, except 3, to most closely match Case II.)

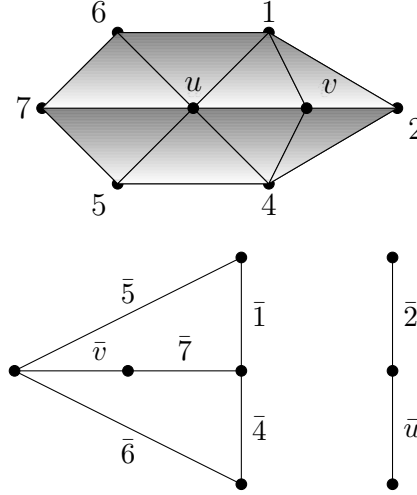


Figure 5: Subcomplex of  $M$  and Subgraph of  $G$  for Case IV

The link of  $v$  is an induced  $C_4$ , with corresponding graph  $2P_3$ ; the link of  $u$  is an induced  $C_6$ , with corresponding graph  $K_{3,2}$ . These subgraphs in  $G$  share edges  $\bar{1}$  and  $\bar{4}$ .

In  $M$ , vertex 2 is adjacent to vertices 1, 4 and  $v$ , so in  $G$  edge  $\bar{2}$  is not incident to edges  $\bar{1}$ ,  $\bar{4}$  and  $\bar{v}$ . But then it cannot be incident to any of the edges  $\bar{5}$ ,  $\bar{6}$  and  $\bar{7}$ . So  $M$  must have edges 25, 26 and 27. Then  $M$  contains vertices and edges forming the graph of a bipyramid over the hexagon 16754 $v$ . Since  $M$  is flag, it contains the whole boundary of the bipyramid as a subcomplex. As noted before, this implies that all of  $M$  is the boundary of the bipyramid.

So in all cases, if  $M$  is the matching complex of a simple graph (with no isolated vertices) and  $M$  is the triangulation of a 2-sphere,  $M$  must be the boundary of a bipyramid over a  $k$ -gon,  $k \in \{4, 5, 6\}$ . We have already seen that the graphs that give these matching complexes are  $3P_3$ ,  $P_3 \sqcup C_5$ , and  $P_3 \sqcup K_{3,2}$ .  $\square$

Therefore, Proposition 2.14 gives the only way to realize the 2-sphere as a matching complex, since the graphs in Theorem 3.2 are disjoint unions of elements of  $\mathcal{SG}$ .

Determining which combinatorial spheres are matching complexes in higher dimension (aside from those of Proposition 2.14) is more complicated. We approach this problem by considering the more general question of which matching complexes are combinatorial manifolds.

## 4 Manifolds without boundary

In this section, we want to answer the following questions:

For which graphs  $G$  is the matching complex  $M(G)$  a combinatorial manifold? Given a combinatorial manifold, can it be the matching complex of a graph?

As before, all graphs are simple (without loops or multiple edges) and do not have any isolated vertices.

We will use the following standard observation: If  $X$  and  $Y$  are combinatorial  $d$ -manifolds without boundary, and  $Y \subseteq X$ , then  $X = Y$ . That is, no proper, full-dimensional subcomplex of a manifold without boundary is a manifold without boundary.

Disconnected matching complexes were classified in Corollary 2.10. None of these matching complexes (of dimension greater than 1) are manifolds without boundary, so we can restrict ourselves to connected combinatorial manifolds.

Since the only closed combinatorial  $d$ -manifolds without boundary for  $d < 2$  are spheres (and disjoint unions of spheres), cases  $d = 0$  and  $d = 1$  are answered in Section 3. We summarize the results below. Throughout,  $M(G)$  is a combinatorial manifold without boundary and  $G$  is a simple graph.

- Let  $\dim M(G) = 0$ . Then  $G = P_2$ ,  $G = C_3$  or  $G = K_{1,n}$  for any  $n \geq 2$ .
- Let  $\dim M(G) = 1$ . Then  $G \in \{2P_3, C_5, K_{3,2}\}$ .

However, when  $\dim M = 2$ , the situation becomes more complex.

**Proposition 4.1** [2, Page 30] *If  $G = K_{4,3}$ , then  $M(G) = T^2$ , the (two-dimensional) torus.*

Figure 6 shows the labeled graph  $K_{4,3}$  and its matching complex. We can see that the top and bottom edges of the object are identified with the same orientation; the same is true for the left and right sides. Therefore the matching complex of  $K_{4,3}$  is a triangulation of  $T^2$ , the torus.

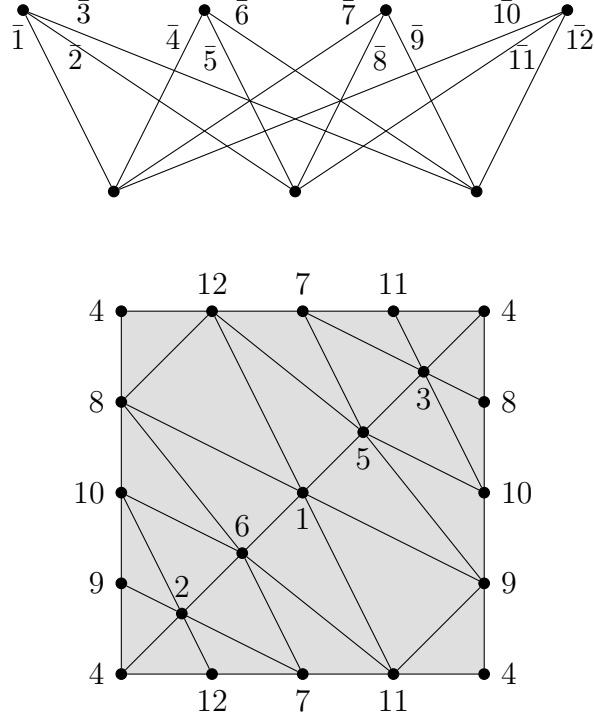


Figure 6: The graph  $K_{4,3}$  and its matching complex. Edges with identical vertex labels are identified.

When  $M(G)$  is two dimensional, the next theorem shows that there are two options:  $M(G)$  is either the torus  $T^2$  or the sphere  $S^2$ .

**Theorem 4.2** *Let  $M$  be a two-dimensional combinatorial manifold. Suppose there exists a simple graph  $G$  with  $M(G) = M$ . Then either*

1.  $G = K_{4,3}$  and  $M = T^2$ , or
2.  $G \in \{3P_3, P_3 \sqcup C_5, P_3 \sqcup K_{3,2}\}$  and  $M = S^2$ .



**Proof:** If  $G$  is disconnected, then  $G = G_1 \sqcup G_2$  and thus  $M(G) = M(G_1) * M(G_2)$  by Lemma 2.8. Therefore by Proposition 2.13,  $M(G)$  is a sphere. This case is covered by Theorem 3.2.

Otherwise assume that  $G$  is connected. We will show that the only possibility in this case is that  $G = K_{4,3}$ . Since  $M$  is a combinatorial manifold, if  $v$  is a vertex of  $M$ , then  $\text{link}_M v$  is a combinatorial 1-sphere. Furthermore, for any face  $\sigma \in M$ ,  $\text{link}_M \sigma = M(G_{\bar{\sigma}})$  by Lemma 2.6. Therefore,  $\text{link}_M v$  is either  $C_4$ ,  $C_5$ , or  $C_6$  by Theorem 3.1. We will use this to consider three different cases.

**Case I.** All vertices  $v \in M$  have  $\text{link}_M v = C_4$ .

**Case II.** There exists a vertex  $v \in M$  such that  $\text{link}_M v = C_5$ .

**Case III.** There exists a vertex  $v \in M$  such that  $\text{link}_M v = C_6$ .

**Case I.** In this case,  $G_{\bar{v}} = 2P_3$  for all vertices  $v \in M$ . Let  $v$  be some vertex of  $M$ . Then  $G$  must contain a subgraph as in Figure 7.

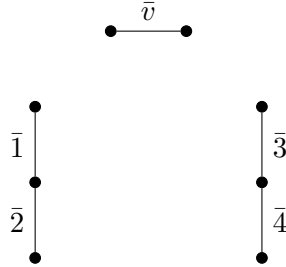


Figure 7: A subgraph for Case I of Theorem 4.2. All remaining edges of  $G$  must share an endpoint with  $\bar{v}$ .

$G$  is connected, and all remaining edges of  $G$  must have a vertex in common with  $\bar{v}$ . Therefore there must be an edge  $\bar{a}$  connecting  $\bar{v}$  and the component containing the edges  $\bar{3}$  and  $\bar{4}$ . Thus  $G_{\bar{1}}$  contains the edges  $\bar{a}$ ,  $\bar{v}$ ,  $\bar{3}$ , and  $\bar{4}$  and therefore contains a path of length three. But  $G_{\bar{1}}$  is  $2P_3$  by assumption, so this is a contradiction. Therefore Case I is not possible.

**Case II.** In this case,  $G$  must contain an edge  $\bar{v}$  and  $C_5$  that is disjoint from  $\bar{v}$ . Since  $G$  is connected and every other edge of  $G$  shares a vertex with  $\bar{v}$ , we assume without loss of generality that there is an edge  $\bar{a}$  as in Figure 8.

Now consider  $G_{\bar{3}}$ . Since it already contains  $\bar{v}$ ,  $\bar{a}$ ,  $\bar{1}$ , and  $\bar{5}$ , the only possibility is that  $G_{\bar{3}} = K_{3,2}$  by Theorem 3.1. Thus there are also the edges  $\bar{b}$  and  $\bar{c}$  in  $G$  as shown in Figure 9.

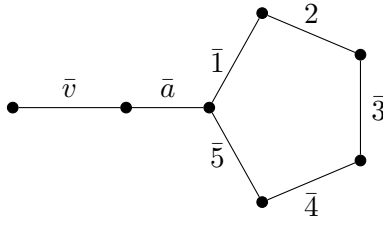


Figure 8: A subgraph for Case II of Theorem 4.2. All remaining edges of  $G$  must share an endpoint with  $\bar{v}$ .

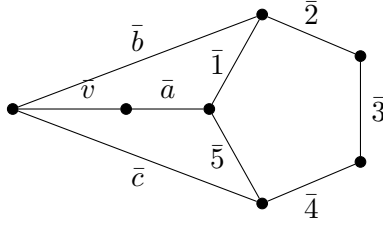


Figure 9: A subgraph for Case II of Theorem 4.2, with edges added from  $G_{\bar{3}}$ .

Similarly, we now consider  $G_{\bar{2}}$  and  $G_{\bar{4}}$  separately. By the same reasoning as for  $G_{\bar{3}}$  above, both of these subgraphs must be  $K_{3,2}$ , which gives us the new edges  $\bar{d}$  and  $\bar{e}$  in Figure 10.

Now we can see that the subgraph  $G_{\bar{1}}$  must contain the triangle  $\bar{d}\bar{e}\bar{3}$ , but this is a contradiction by Theorem 3.1. Therefore Case II is also impossible.

**Case III.** In this case,  $G$  must contain an edge  $\bar{v}$  and  $K_{3,2}$  that is disjoint from  $\bar{v}$ . Recall that  $G$  is connected and every other edge of  $G$  shares a vertex with  $\bar{v}$ . There must exist a subgraph of  $G$  as in Figure 11, and all other edges of  $G$  must share an endpoint with  $\bar{v}$ . We will split this case up into two subcases.

**Case III.1.** There are no edges between the endpoints of  $\bar{v}$  and vertices  $x$  and  $y$ .

Since  $G$  is connected, there must be an edge  $\bar{a}$  connecting one of the endpoints of  $\bar{v}$  with one of the middle vertices in the copy of  $K_{3,2}$  in Figure 11. Without loss of generality, we assume it is as in Figure 12.

Therefore  $G_{\bar{a}}$  contains a four cycle. By Theorem 3.1, this implies that

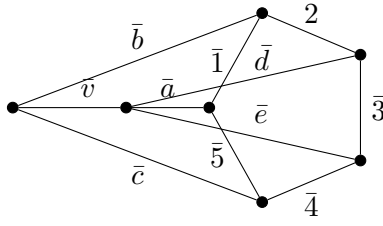


Figure 10: A subgraph for Case II of Theorem 4.2, with edges from  $G_{\bar{2}}$  and  $G_{\bar{4}}$ .

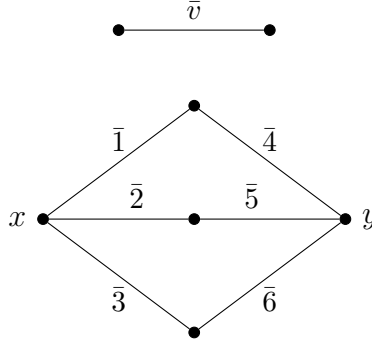


Figure 11: A subgraph for Case III of Theorem 4.2. All remaining edges of  $G$  must share an endpoint with  $\bar{v}$ .

$G_{\bar{a}} = K_{3,2}$ . Since all remaining edges in  $G$  must share an endpoint with  $\bar{v}$ , the only possibility in this case is to add the edges  $\bar{b}$  and  $\bar{c}$  as in Figure 13.

Now considering  $G_{\bar{1}}$ , we again see that this subgraph must be  $K_{3,2}$  by Theorem 3.1. Therefore there must be an edge connecting  $y$  and the left endpoint of  $\bar{v}$ . But this contradicts our assumption, so this case is not possible.

**Case III.2.** Edges between the endpoints of  $\bar{v}$  and  $x$  and  $y$  are allowed.

Assume without loss of generality that the edge  $\bar{a}$  is in  $G$  as depicted in Figure 14. Considering  $G_{\bar{6}}$ , we see that we must have edges  $\bar{b}$  and  $\bar{c}$  depicted in Figure 15, as  $G_{\bar{6}} = K_{3,2}$  by Theorem 3.1. Similarly, Theorem 3.1 shows that  $G_{\bar{5}}$  and then  $G_{\bar{1}}$  must also be  $K_{3,2}$ , which gives us edges  $\bar{d}$  and  $\bar{e}$ , respectively, as in Figure 16.

Observing the subgraph of  $G$  in Figure 16, we see that  $G$  must contain

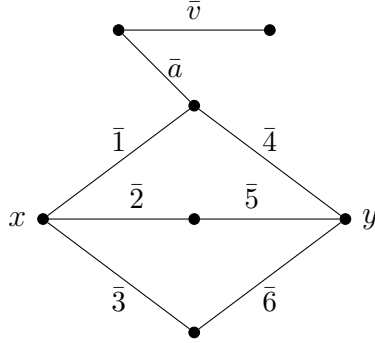


Figure 12: A subgraph for Case III.1 of Theorem 4.2. All remaining edges of  $G$  must share an endpoint with  $\bar{v}$ .

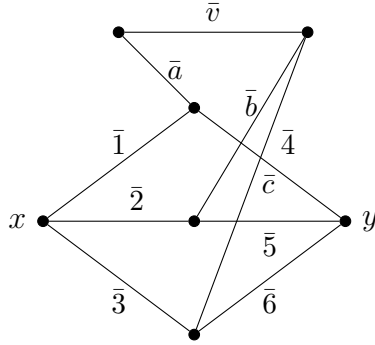


Figure 13: A subgraph for Case III.1 of Theorem 4.2.

a copy of  $K_{4,3}$ . Since  $M(K_{4,3}) = T^2$  by Proposition 4.1, and  $T^2$  cannot be a proper submanifold of a connected combinatorial manifold without boundary,  $G = K_{4,3}$ . This completes the proof.  $\square$

Although the appearance of the torus in Theorem 4.2 might lead us to believe that manifold matching complexes are more plentiful in higher dimension, the reverse is actually true. If  $d \geq 3$ , then the following theorem shows that the only manifold matching complexes without boundary are spheres. The graphs are constructed from the set of basic sphere graphs  $\mathcal{SG}$  listed in Equation (1).

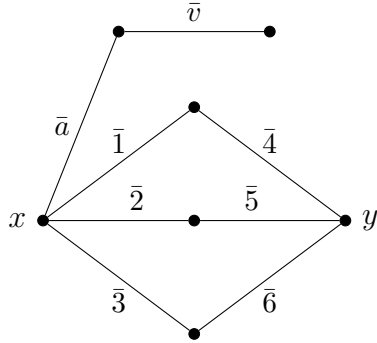


Figure 14: A subgraph for Case III.2 of Theorem 4.2.

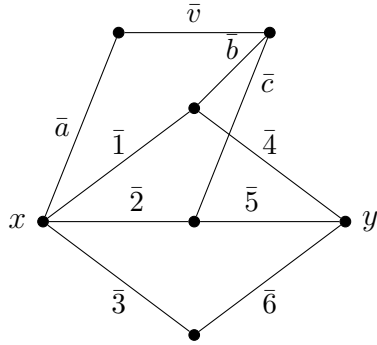


Figure 15: A subgraph for Case III.2 of Theorem 4.2.

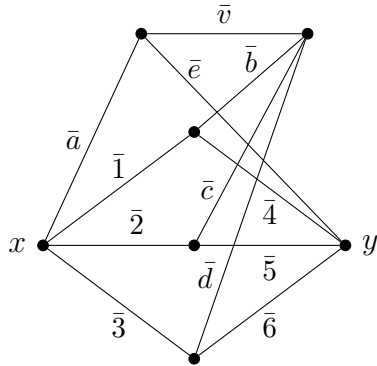


Figure 16: A subgraph for Case III.2 of Theorem 4.2.

**Theorem 4.3** *Let  $M$  be a  $d$ -dimensional manifold without boundary for some  $d \geq 3$ . Suppose there exists a simple graph with  $M(G) = M$ . Then  $G$  is the disjoint union of copies of  $P_3$ ,  $C_5$ , and  $K_{3,2}$ , and therefore  $M = S^d$ .*

**Proof:** We will prove that  $G$  is disconnected and therefore  $M(G) = M(G_1) * M(G_2)$  by Lemma 2.8. This will imply that  $M$  is a sphere by Proposition 2.13.

Let  $\dim M = d \geq 3$ . Assume, by way of induction, that if  $M(G') \cong S^{d-1}$  for some simple graph  $G'$ , then  $G'$  is the disjoint union of copies of the basic sphere graphs in (1). This is already known to be true when  $d = 3$  by Theorem 3.2.

Let  $v$  be a vertex of  $M$ . Since  $M$  is a combinatorial manifold,  $M(G_{\bar{v}}) = \text{link}_M v$  is a  $(d - 1)$ -sphere. By assumption,  $G_{\bar{v}} = H \sqcup J$ , where  $H$  is one of the basic sphere graphs and  $J$  contains at least one of  $2P_3$ ,  $C_5$  or  $K_{3,2}$ . We will consider each case and show that  $G$  must be disconnected in each case. Therefore  $M$  is a combinatorial sphere.

Now assume that  $G$  is connected. Notice that  $G_{\bar{v}}$  cannot contain a connected component that has more than 6 edges, since  $G_{\bar{v}}$  is the disjoint union of copies of elements of  $\mathcal{SG}$ . We will use this fact to reach a contradiction in each of the following cases.

**Case I.**  $G_{\bar{v}}$  contains three disjoint copies of  $P_3$ . We will label the edges as in Figure 17.

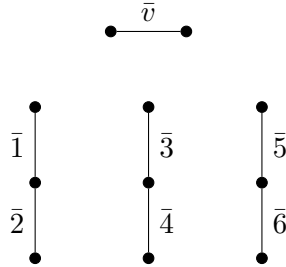


Figure 17: A subgraph for Case I of Theorem 4.3.

Since  $G$  is connected, there must be edges  $\bar{a}$  and  $\bar{b}$  connecting  $\bar{v}$  with the middle and right  $P_3$ , respectively, in Figure 17. Then  $G_{\bar{1}}$  will contain a connected subgraph with edges  $\bar{3}$ ,  $\bar{4}$ ,  $\bar{5}$ ,  $\bar{6}$ ,  $\bar{v}$ ,  $\bar{a}$ , and  $\bar{b}$ . But  $G_{\bar{1}}$  cannot contain a connected subgraph with more than 6 edges, so this is a contradiction.

**Case II.**  $G_{\bar{v}} = H \sqcup J$ , where  $J$  contains at least one of  $C_5$  or  $K_{3,2}$ . Note that  $J$  has at least 5 edges. Let  $\bar{1}$  be an edge of  $H$ . Then  $G_{\bar{1}}$  contains all edges of  $J$ ,  $\bar{v}$ , and at least one edge connecting  $\bar{v}$  with  $C_5$  or  $K_{3,2}$  from  $J$ . Again, we have a contradiction, since  $G_{\bar{1}}$  cannot contain a connected subgraph with more than 6 edges.

Thus, in both cases a contradiction shows the graph  $G$  is disconnected; write  $G = G_1 \cup G_2$  for nonempty subgraphs  $G_1$  and  $G_2$ . Then  $M(G) = M(G_1) * M(G_2)$ , and  $M(G_1)$ ,  $M(G_2)$  and  $M(G)$  are all spheres by Proposition 2.13. Then by the induction assumption  $G_1$  and  $G_2$ , and hence  $G$ , are each disjoint copies of the basic sphere graphs.  $\square$

Thus we see that the only combinatorial manifolds without boundary that occur as matching complexes are the 2-dimensional torus and spheres of all dimensions, given by Propositions 4.1 and 2.14 respectively.

## 5 Manifolds with boundary

We turn now to the question of which manifolds with boundary (of dimension at least 1) are matching complexes. Here we find once again that the dimension 2 case has the most complicated answer. By Corollary 2.10, the only disconnected manifolds with boundary (of dimension at least 1) that are matching complexes are the matching complex of  $C_4$ , which is  $2P_3$ , and the matching complex of  $K_4$ , which is  $3P_3$ . So in what follows we assume the manifold is connected. A connected 1-dimensional combinatorial manifold with boundary is just a path. The nontrivial paths that arise as matching complexes have 2 to 5 vertices; see Table 1.

Next we look at matching complexes of disconnected graphs. We start by applying Proposition 2.13 to matching complexes of disconnected graphs.

**Corollary 5.1** *Let  $M$  be a  $d$ -dimensional manifold with boundary. Suppose there exists a disconnected graph  $G$  with matching complex  $M(G) = M$ . Then for some graphs  $G_1$  and  $G_2$ ,  $G = G_1 \sqcup G_2$ ,  $M(G) = M(G_1) * M(G_2)$ , and for some  $k$ ,  $M(G_1)$  is isomorphic to either  $S^k$  or  $B^k$ , and  $M(G_2) \cong B^{d-k-1}$ . Thus  $M(G)$  is a combinatorial ball.*

From this and the classification of 0- and 1-dimensional matching complexes that are spheres and balls, we can find all disconnected graphs that

have as their matching complex a 2-dimensional combinatorial manifold with boundary.

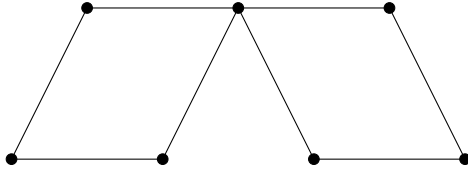
**Theorem 5.2** *Let  $M$  be a 2-dimensional manifold with boundary. Suppose there exists a disconnected graph  $G$  with matching complex  $M(G) = M$ . Then  $M$  is a ball, and  $M \cong B^0 * B^1$ ,  $M \cong B^0 * S^1$ , or  $M \cong S^0 * B^1$ . Furthermore the pair  $G, M(G)$  is one of the following:*

Graph $G$	Manifold $M(G)$	Description of $M(G)$
$3P_2$	$P_1 * P_2$	Triangle
$2P_2 \sqcup P_3$	$P_1 * P_3$	Two triangles sharing an edge
$P_2 \sqcup P_5$	$P_1 * P_4$	Chain of three triangles sharing a vertex
$P_2 \sqcup \Gamma$	$P_1 * P_5$	Chain of four triangles sharing a vertex
$P_2 \sqcup 2P_3$	$P_1 * C_4$	Triangulated square
$P_2 \sqcup C_5$	$P_1 * C_5$	Triangulated pentagon
$P_2 \sqcup K_{3,2}$	$P_1 * C_6$	Triangulated hexagon
$P_3 \sqcup P_5$	$2P_1 * P_4$	Six triangles: suspension over path of three edges
$P_3 \sqcup \Gamma$	$2P_1 * P_5$	Eight triangles: suspension over path of four edges

A surprising variety of 2-dimensional manifolds arise as matching complexes of connected graphs.

**Theorem 5.3** *Let  $M$  be a 2-dimensional combinatorial manifold with boundary. Suppose there exists a connected graph  $G$  with matching complex  $M(G) = M$ . Then  $M$  is one of the following four topological types, arising from the following graphs.*

1.  $M$  is a ball.  $G$  is the spider graph  $Sp_3$ .
2.  $M$  is a triangulated annulus.  $G$  is the graph with 7 vertices and 8 edges:

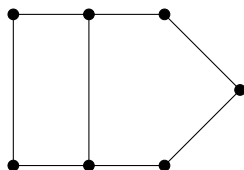




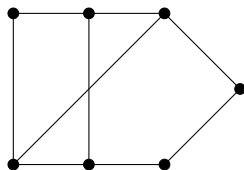
3.  $M$  is a triangulated Möbius strip.

(a)  $G = C_7$ , the 7-cycle

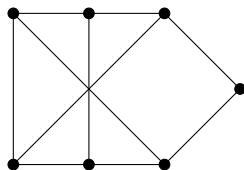
(b)  $G$  is the graph with 7 vertices and 8 edges:



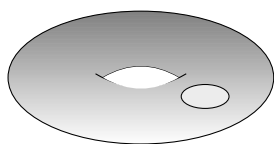
(c)  $G$  is the graph with 7 vertices and 9 edges:



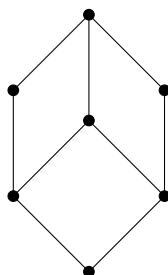
(d)  $G$  is the graph with 7 vertices and 10 edges:



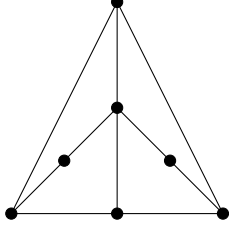
4.  $M$  is a triangulated torus with a 2-ball removed.



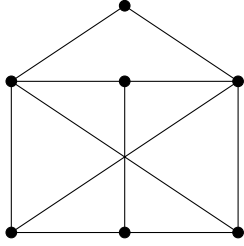
(a)  $G$  is the graph with 7 vertices and 9 edges:



(b)  $G$  is the graph with 7 vertices and 10 edges:



(c)  $G$  is the graph with 7 vertices and 11 edges:



**Proof:** The proof relies heavily on analysis of the links of vertices in the manifold. We start by reviewing the possible structures of these links. We showed previously (Lemma 2.6) that the link of a vertex  $v$  of  $M$  is an induced subcomplex of  $M$ , which is then the matching complex of the subgraph  $G_{\bar{v}}$  of  $G$ . We write  $x_v$  and  $y_v$  for the vertices of the edge  $\bar{v}$  of  $G$ .

Assume  $G$  is a connected graph and  $M = M(G)$  is a 2-dimensional manifold with boundary.

If  $v$  is a boundary vertex of  $M$ , then the link of  $v$  is a 1-dimensional ball, that is, a path  $P_j$ ,  $2 \leq j \leq 5$ , so the corresponding subgraph  $G_{\bar{v}}$  of  $G$  is in the set  $\{2P_2, P_3 \sqcup P_2, P_5, \Gamma\}$ . Also, the endpoints of  $P_j$  are also boundary vertices of  $M$ .

If  $v$  is an interior vertex of  $M$ , then the link of  $v$  is a 1-dimensional sphere, that is, a cycle  $C_j$ ,  $4 \leq j \leq 6$ , so the corresponding subgraph  $G_{\bar{v}}$  of  $G$  is in the set  $\{2P_3, C_5, K_{3,2}\}$ .

Note that in all cases  $G_{\bar{v}}$  does not contain a triangle. In fact, it is easy to see that this implies that the entire graph  $G$  does not contain a triangle: if an edge  $\bar{v}$  were disjoint from the triangle,  $G_{\bar{v}}$  would contain a triangle; otherwise all edges of  $G$  would contain a vertex of the triangle, and for any such edge  $\bar{v}$ ,  $G_{\bar{v}}$  would not be in either of the sets above.

We are assuming the matching complex has nonempty boundary, and we split the proof into cases based on the structure of the links of boundary vertices.

**Case I.** For some boundary vertex  $v$ ,  $\text{link}_M(v) = P_2$ , so  $G_{\bar{v}} = 2P_2$ . Let 1 and 2 be the neighbors of  $v$  in  $M$ . Thus  $G_{\bar{v}}$  has the two nonincident edges  $\bar{1}$  and  $\bar{2}$ , which are also boundary vertices, and all other edges of  $G$  are incident to  $\bar{v}$ . Since  $G$  is connected,  $G$  must have an edge connecting  $\bar{v}$  and  $\bar{1}$ , say (without loss of generality)  $\bar{3} = \{x_v, x_1\}$ . Since  $G_{\bar{2}}$  does not contain a triangle,  $G$  does not contain edges  $\{x_v, y_1\}$  and  $\{y_v, x_1\}$ . We split Case I into two subcases, depending on whether the edge  $\{y_v, y_1\}$  is in  $G$ .

**Case I.1.** If  $\{y_v, y_1\}$  is in  $G$ , then  $G_{\bar{2}}$  contains a 4-cycle, so must be  $\Gamma$  (since 2 is also a boundary vertex), and all edges of  $G$  besides  $\bar{2}$  and the edges of  $\Gamma$  are edges with one endpoint in  $\bar{v}$  and one endpoint in  $\bar{2}$ . By connectivity, there must be at least one such edge. Say without loss of generality it is the edge  $\{y_2, y_v\}$ . The pendant edge of  $\Gamma$  is either  $\{x_v, z\}$  or  $\{y_v, z\}$ . Since  $G_{\bar{1}}$  does not contain a triangle,  $G$  does not contain edges  $\{y_2, x_v\}$  or  $\{x_2, y_v\}$ . The graph  $G$  may or may not contain the edge  $\{x_2, x_v\}$ . In all cases  $G_{\bar{3}}$  is either  $P_4$  or a  $P_4$  with an additional edge on one of its interior vertices; these are not possible. See Figure 18.

**Case I.2.** If  $\{y_v, y_1\}$  is not in  $G$ , then all remaining edges of  $G$  are incident to  $\bar{v}$ , but not to  $\bar{1}$ . The subgraph  $G_{\bar{2}}$  contains the path  $P_4 : y_1, x_1, x_v, y_v$ . Since 2 is a boundary vertex of  $M$ , its link is a ball, so  $G_{\bar{2}}$  is  $P_5$  or  $\Gamma$ . With no more edges incident to  $\bar{1}$ ,  $\Gamma$  is not possible, so  $G_{\bar{2}}$  is  $P_5$ . That is, there is exactly one more edge in  $G$  not incident to  $\bar{2}$ , call it  $\bar{4}$ , and it contains the vertex  $y_v$ . Now turn again to  $G_{\bar{1}}$ . It contains edges  $\bar{v}$ ,  $\bar{4}$ ,  $\bar{2}$ , and any other edges (at least one) incident to  $\bar{v}$  and  $\bar{2}$ . Since 1 is a boundary vertex of  $M$ ,  $G_{\bar{v}}$  must also be  $P_5$  or  $\Gamma$ . If it is  $\Gamma$ , then  $G$  contains exactly one edge  $\bar{5}$  containing a vertex of  $\bar{2}$  and the vertex  $y_{\bar{v}}$ . But then the graph  $G_{\bar{3}}$  is  $P_4$ , which is not possible. This leaves only the possibility that  $G_{\bar{1}}$  is  $P_5$ , and the graph  $G$  is the spider graph  $\text{Sp}_3$ . See Figure 19. This is part 1 of the Theorem.

**Case II.** No boundary vertex has  $\text{link}_M(v) = P_2$ , and for some boundary vertex  $v$ ,  $\text{link}_M(v) = P_3$ , so  $G_{\bar{v}} = P_3 \sqcup P_2$ . Let the link of  $v$  have vertices 1, 2, 3 (forming a path in that order). Then 1 and 3 are boundary vertices, and  $G_{\bar{v}}$  consists of the path with two edges  $\bar{1}$  and  $\bar{3}$  and the disjoint edge  $\bar{2}$ . Write the common vertex in  $G$  of  $\bar{1}$  and  $\bar{3}$  as  $y_1$ , with  $x_1$  and  $x_3$  their other vertices. Besides  $\bar{v}$ ,  $\bar{1}$ ,  $\bar{2}$  and  $\bar{3}$ , every other edge of  $G$  must contain a vertex of  $\bar{v}$ . Also,  $G$  is connected, so there must be at least one edge between a vertex of  $\bar{v}$  and a vertex of  $\bar{2}$ . Without loss of generality,  $G$  contains the edge  $\bar{4} = \{x_v, x_2\}$ . Since  $G$  contains no triangles, the only other possible edge between  $\bar{v}$  and  $\bar{2}$  is  $\{y_v, y_2\}$ . We split Case II into two cases, depending on whether that edge is in  $G$ .

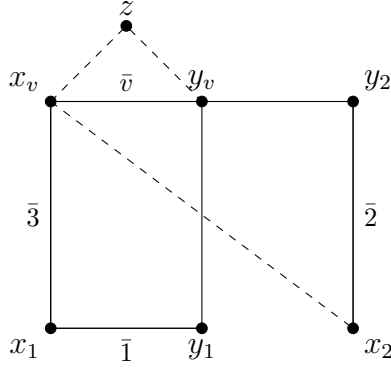


Figure 18: Graph for Case I.1;  $G$  contains exactly one of the edges from  $z$  and may or may not contain the edge  $\{x_v, x_2\}$ .

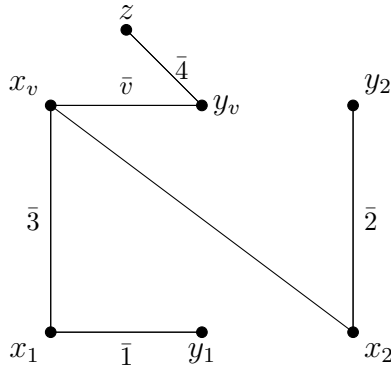


Figure 19: Graph for Case I.2

**Case II.1.** Assume  $\{y_v, y_2\}$  is not an edge of  $G$ . Consider  $G_{\bar{3}}$ . It contains the path  $P_4 : y_v, x_v, x_2, y_2$ , so it is either  $P_5$  or  $\Gamma$ . Without the edge  $\{y_v, y_2\}$ ,  $G_{\bar{3}}$  cannot be  $\Gamma$ , so  $G_{\bar{3}}$  is  $P_5 : z, y_v, x_v, x_2, y_2$ . All remaining edges of  $G$  must connect the edge  $\bar{v}$  with the edge  $\bar{3}$ . If  $z = x_1$ , then  $G_{\bar{1}}$  contains the path  $P_4 : y_v, x_v, x_2, y_2$  and at most one other edge connecting  $\bar{v}$  and  $x_3$ . The subgraph  $G_{\bar{1}}$  must then be  $P_5 : x_3, y_v, x_v, x_2, y_2$ . Then  $G_{\bar{4}}$  contains a cycle of length 4, and no edge of  $G$  can complete it to  $\Gamma$ . If  $z$  is not in the edge  $\bar{1}$ , then  $G_{\bar{2}}$  contains  $2P_3$  (edges  $\bar{1}$ ,  $\bar{3}$ ,  $\bar{v}$  and  $\{z, y_v\}$ ). But then there can be no edges between vertices of  $\bar{v}$  and vertices of  $\bar{1}$  and  $\bar{3}$ , contradicting the connectedness of  $G$ .

**Case II.2.** Assume  $\bar{5} = \{y_v, y_2\}$  is an edge of  $G$ . Since  $G$  does not contain a triangle, all remaining edges of  $G$  contain a vertex of  $\bar{v}$ , but no vertex of  $\bar{2}$ . Also, the subgraphs  $G_{\bar{1}}$  and  $G_{\bar{3}}$  each contain the  $C_4 : x_v, y_v, y_2, x_2, x_v$ , so must be copies of  $\Gamma$ . We consider what additional edge or edges are needed for this.

**Case II.2.a.** Suppose an additional edge contains (without loss of generality) the vertex  $x_v$  and a vertex  $z$  not in  $\bar{1}$  or  $\bar{3}$ . Then there must be one more edge to make  $G$  connected, and it must contain the vertex  $y_1$  common to edges  $\bar{1}$  and  $\bar{3}$  (so as not to change  $G_{\bar{1}}$  or  $G_{\bar{3}}$ ). But this would result in an invalid subgraph for  $G_{\bar{2}}$ , one having a degree 3 vertex that is not  $K_{3,2}$ .

**Case II.2.b.** Suppose there are edges  $\bar{6}$  and  $\bar{7}$  from the same vertex  $x_v$  (without loss of generality) to vertices  $x_1$  and  $x_3$ . The result is the union of two 4-cycles with a common vertex, with matching complex an annulus, a manifold with boundary. Any other edge of  $G$  must be incident to  $\bar{v}$ . In addition, the graph so far contains copies of  $\Gamma$  for  $G_{\bar{1}}$  and  $G_{\bar{3}}$ , so any other edge of  $G$  must be incident to the edges  $\bar{v}$ ,  $\bar{1}$ , and  $\bar{3}$ . Since  $G$  does not contain a triangle, edge  $y_1x_v$  does not exist. Also,  $G$  does not contain edge  $y_1y_v$  because of  $G_{\bar{4}}$ . So Case II.2.b must be the graph and its matching complex shown in Figure 20. This is part 2 of the Theorem.

**Case II.2.c.** Otherwise, the graphs  $G_{\bar{1}}$  and  $G_{\bar{3}}$  are copies of  $\Gamma$  with pendant edges containing different vertices of  $\bar{v}$ . Without loss of generality, the graph  $G$  contains edges  $\bar{6} = \{x_v, x_1\}$  and  $\bar{7} = \{y_v, x_3\}$ . Any other edge of  $G$  would create a triangle. So the graph  $G$  and its matching complex are shown in Figure 21. In this, and in subsequent drawings of matching complexes arrows show the identification of edges. This is part 3(b) of the Theorem.

**Case III.** No boundary vertex has  $\text{link}_M(v) = P_2$  or  $P_3$ , and for some boundary vertex  $v$ ,  $\text{link}_M(v) = P_4$ , so  $G_{\bar{v}} = P_5$ . Let the link of  $v$  have vertices

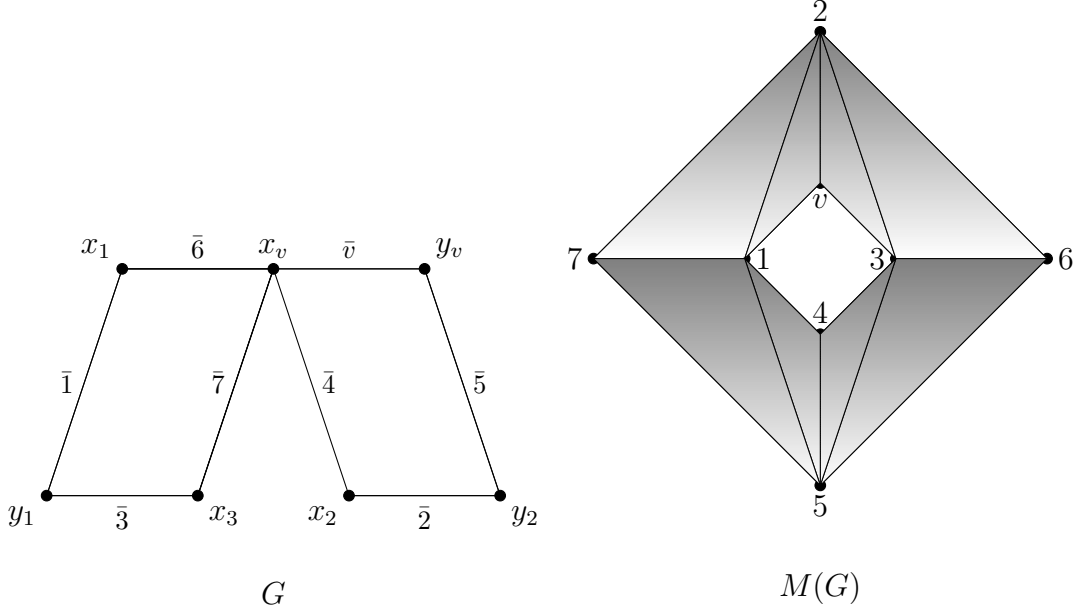


Figure 20: Graph and Matching Complex for Case II.2.b

1, 2, 3, 4 (forming a path in that order). Then 1 and 4 are boundary vertices of  $M$ ,  $G_{\bar{1}}$  and  $G_{\bar{4}}$  are each either  $P_5$  or  $\Gamma$ , and  $G_{\bar{v}}$  consists of the path with consecutive edges  $\bar{2}$ ,  $\bar{4}$ ,  $\bar{1}$ ,  $\bar{3}$  and vertices  $x_2, y_2, x_1, y_1, y_3$ . Besides the edges  $\bar{v}$ ,  $\bar{1}$ ,  $\bar{2}$ ,  $\bar{3}$ , and  $\bar{4}$ , every other edge of  $G$  must contain a vertex of  $\bar{v}$ . The vertex 2 (and similarly 3) could be either a boundary vertex or an interior vertex of  $M$ .

**Case III.1.** Assume 2 is a boundary vertex of  $M$ . Then  $G_{\bar{2}} = P_5$  or  $\Gamma$ .

**Case III.1.a.** Assume  $G_{\bar{2}} = P_5$ . Since  $G_{\bar{2}}$  contains edges  $\bar{v}$ ,  $\bar{1}$  and  $\bar{3}$ , it contains either  $\{x_v, x_1\}$  or  $\{x_v, y_3\}$  ( $x_v$  being either vertex of  $\bar{v}$ ), and all other edges of  $G$  contain a vertex of  $\bar{v}$  and a vertex of  $\bar{2}$ . If  $G_{\bar{2}}$  contained  $\{x_v, x_1\}$ , then  $G_{\bar{4}}$  would contain edge  $\bar{3}$  and no other edges incident to  $\bar{3}$ ; this cannot happen in  $P_5$  or  $\Gamma$ . Thus  $G_{\bar{2}}$  contains  $\bar{5} = \{x_v, y_3\}$ . Now consider  $G_{\bar{4}}$ ; it contains the path  $P_4 : \bar{v}, \bar{5}, \bar{3}$ , and it must be  $P_5$  or  $\Gamma$ . Since all other edges of  $G$  contain a vertex of  $\bar{v}$  and a vertex of  $\bar{2}$ ,  $G_{\bar{4}}$  must contain the edge  $\bar{6} = \{y_v, x_2\}$ . Thus  $G$  contains the 7-cycle  $C_7 : y_v, x_v, y_3, y_1, x_1, y_2, x_2, y_v$ . Since  $G_{\bar{3}}$  cannot contain a triangle, the only other possible edge in  $G$  is  $\bar{7} = \{x_v, y_2\}$ . However if edge  $\bar{7}$  is in  $G$ , then the corresponding subgraph  $G_{\bar{7}}$  would be  $P_2 \sqcup P_3$ , so  $M$  would fall into Case II. So  $G$  must be  $C_7$ . Then  $M$  is

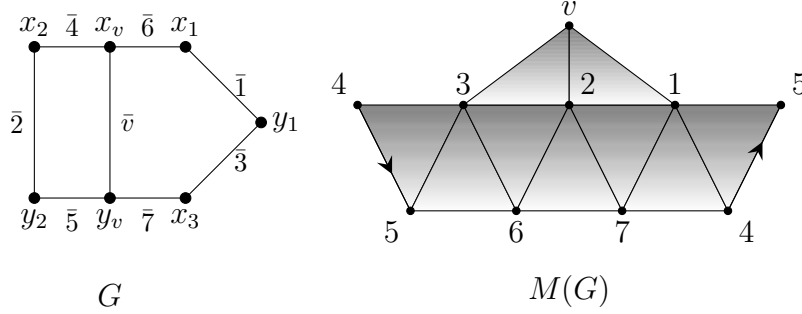


Figure 21: Graph and Matching Complex for Case II.2.c

a triangulated Möbius strip with triangles 215, 156, 564, 643, 43v, 3v2, v21. This is part 3(a) of the Theorem.

**Case III.1.b.** Assume  $G_2 = \Gamma$ . Since all remaining edges of  $G$  contain a vertex of  $\bar{v}$ ,  $G_2$  is either the 4-cycle  $C_4 : x_1, x_v, y_v, y_1, x_1$  with pendant edge  $\{y_1, y_3\}$  or the 4-cycle  $C_4 : y_1, x_v, y_v, y_3, y_1$  with pendant edge  $\{x_1, y_1\}$ . The first alternative is not possible, because it would leave  $G_{\bar{1}}$  with only four vertices  $(x_2, y_2, x_v, y_v)$ . So assume  $G$  contains edges  $\bar{5} = \{y_1, x_v\}$  and  $\bar{6} = \{y_3, y_v\}$ . All remaining edges of  $G$  must contain a vertex of  $\bar{v}$  and a vertex of  $\bar{2}$ . The 4-cycle with edges  $\bar{3}, \bar{5}, \bar{v}, \bar{6}$  is also in  $G_{\bar{4}}$ , so  $G_{\bar{4}}$  must also be  $\Gamma$ , and  $G$  must contain an edge connecting  $x_2$  to  $\bar{v}$ . Since  $G_{\bar{5}}$  contains edges  $\bar{2}, \bar{4}$ , and  $\bar{6}$  and, being in Case III,  $G_{\bar{5}}$  cannot be  $P_3 \sqcup P_2$ , there must be another edge containing  $y_v$  and completing a  $P_5$ , namely, the edge  $\bar{7} = \{x_2, y_v\}$ .

Finally,  $G_{\bar{1}}$  contains edges  $\bar{2}, \bar{6}, \bar{7}$ , and  $\bar{v}$  and must be  $\Gamma$  with edge  $\bar{8} = \{y_2, x_v\}$ . Every other edge of  $G$  must be incident to  $\bar{v}$  and  $\bar{2}$ , but such an edge would create a triangle in  $G$ . So the graph  $G$  has just the nine edges described. The matching complex  $M$  is a combinatorial manifold with triangles 745, 456, 562, 612, 12v, 2v3, v34, 34v, 618, 187, and 873. See Figure 22. We observe that after gluing together the identified edges, we obtain a 2-manifold with boundary homeomorphic to the torus with a 2-ball removed from the surface. (The 2-ball is bounded by a cycle 1, 7, 5, 2, 3, 8, 6, 4, v, 1.) This is part 4(a) of the Theorem.

**Case III.2.** Assume vertices 2 and 3 are interior vertices of  $M$ . Then  $G_2 \in \{2P_3, C_5, K_{3,2}\}$ . We consider each of those cases.

**Case III.2.a.** If  $G_2 = 2P_3$ , then  $G_2$  consists of the path with edges  $\bar{1}$  and  $\bar{3}$  and another  $P_3$  path with edges  $\bar{v}$  and  $\bar{5}$ , for some edge  $\bar{5}$  containing a

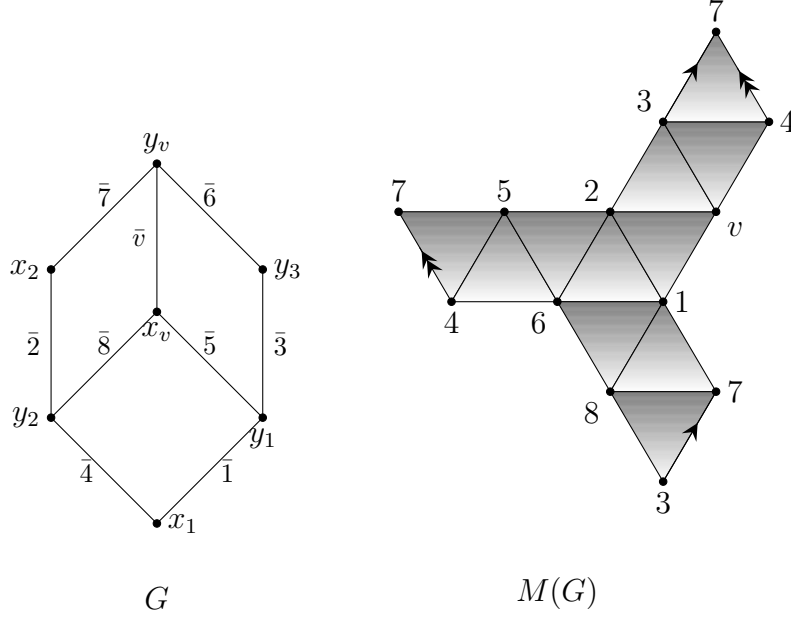


Figure 22: Graph and Matching Complex for Case III.1.b

vertex of  $\bar{v}$  and another vertex, not in the edges  $\bar{1}$ ,  $\bar{2}$ ,  $\bar{3}$ ,  $\bar{4}$ . All other edges of  $G$  must be incident to  $\bar{2}$  and  $\bar{v}$ . Then  $G_4$  is a disconnected subgraph of  $G$ , with the edge  $\bar{3}$  forming one component. However, 4 is a boundary vertex of  $M$ , so this is not possible (in case III).

**Case III.2.b.** If  $G_{\bar{2}} = C_5$ , then without loss of generality  $G$  contains the edges  $\bar{5} = \{x_1, x_v\}$  and  $\bar{6} = \{y_3, y_v\}$  (so  $G_{\bar{2}}$  is  $C_5 : x_1, y_1, y_3, y_v, x_v, x_1$ ), and all other edges of  $G$  contain a vertex of  $\bar{v}$  and a vertex of  $\bar{2}$ . Then  $G_4$ , which contains  $P_4 : y_1, y_3, y_v, x_v$ , must be  $P_5$ , ending in edge  $\bar{7} = \{x_v, x_2\}$ . Now  $G_{\bar{5}}$  contains  $P_3 \sqcup P_2$ , with vertices  $y_1, y_3, y_v; x_2, y_2$ . If  $G_{\bar{5}}$  is just  $P_3 \sqcup P_2$ , then it is covered in Case II. Otherwise, there is one more edge  $\bar{8} = \{y_v, y_2\}$ , making  $G_{\bar{5}} = P_5$ . Any other edge would form a triangle in  $G$ , so we have described all edges of  $G$ ;  $G$  and  $M$  are shown in Figure 23. This is part 3(c) of the Theorem.

**Case III.2.c.** If  $G_{\bar{2}} = K_{3,2}$ , then without loss of generality  $G$  contains the three edges  $\bar{5} = \{x_v, y_1\}$ ,  $\bar{6} = \{y_v, y_3\}$  and  $\bar{7} = \{y_v, x_1\}$ . Then  $G_4$  contains  $C_4 : y_1, x_v, y_v, y_3, y_1$ , and so must be  $\Gamma$ . To complete  $\Gamma$ , there must be one more edge from  $x_2$  to  $\bar{v}$ . If that edge is  $\{x_2, x_v\}$ , then  $G_{\bar{3}}$  is  $C_5$ , and that



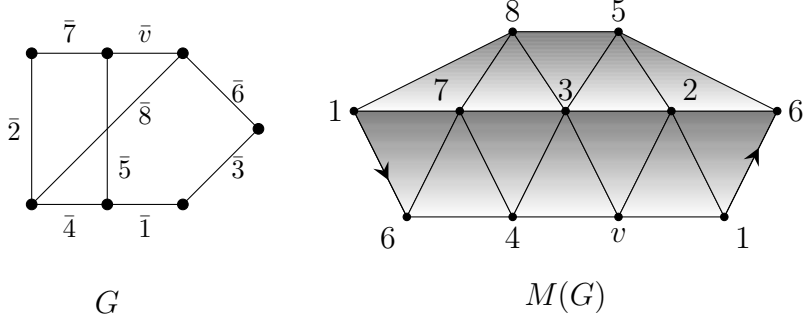


Figure 23: Graph and Matching Complex for Case III.2.b

case is covered in Case III.2.b. So let  $\bar{8} = \{x_2, y_v\}$ . Then  $G_{\bar{1}}$  contains three edges containing vertex  $y_v$ , so must be  $\Gamma$ . Therefore  $G$  also contains the edge  $\bar{9} = \{y_2, x_v\}$ . Any other edge would create a triangle or would disrupt  $G_{\bar{1}}$  or  $G_{\bar{4}}$ , so we have described all edges of  $G$ ;  $G$  and  $M(G)$  are shown in Figure 24. This is part 4(b) of the Theorem.

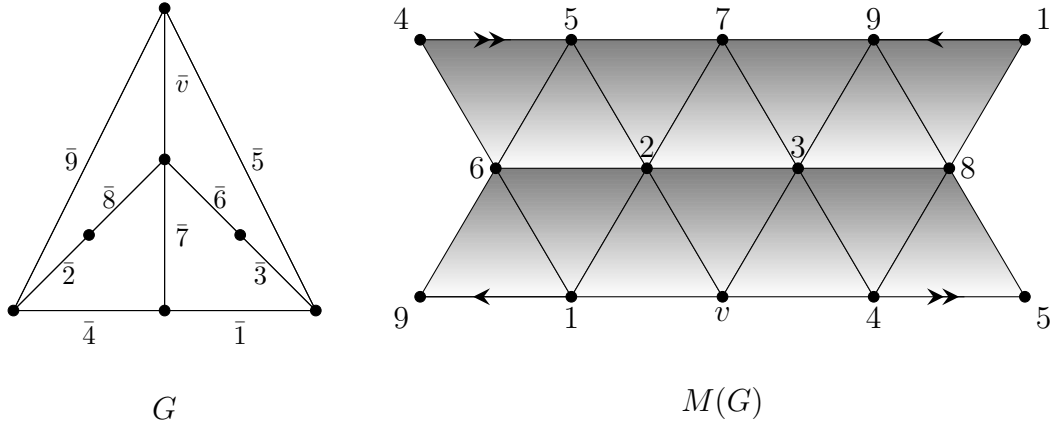


Figure 24: Graph and Matching Complex for Case III.2.c

**Case IV.** All boundary vertices  $v$  of  $M$  have  $\text{link}_M(v) = P_5$ , so  $G_{\bar{v}} = \Gamma$ . Fix such a boundary vertex  $v$ , and let the link of  $v$  have vertices 1, 2, 3, 4, 5 (forming a path in that order). Then 1 and 5 are boundary vertices of  $M$ , and  $G_{\bar{1}}$  and  $G_{\bar{5}}$  are both  $\Gamma$ . See Figure 25 for the subgraph of  $G$  containing  $\bar{v}$  and  $G_{\bar{v}}$ . Every other edge of  $G$  must contain a vertex of  $\bar{v}$ .

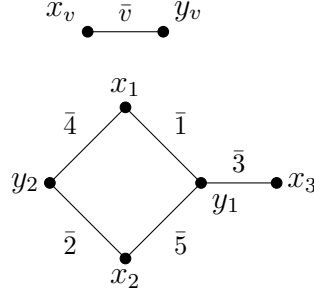


Figure 25: Subgraph  $G_{\bar{v}}$  for Case IV.

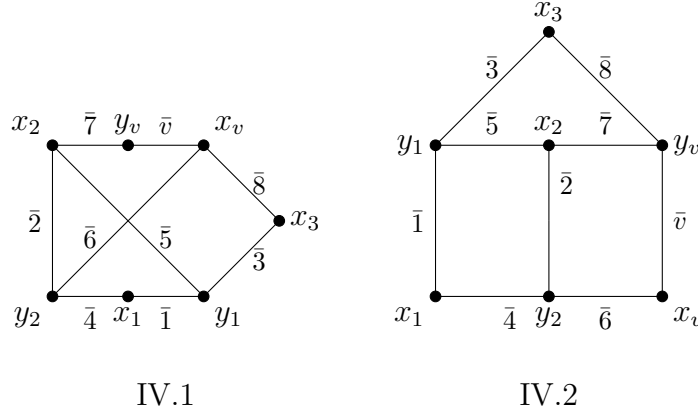


Figure 26: Two possible subgraphs of  $G$  containing  $G_{\bar{v}}$  and  $G_{\bar{1}}$  for Case IV.

Now consider  $G_{\bar{1}}$ , which is also  $\Gamma$  (since 1 is a boundary vertex of  $M$ ). The graph  $G_{\bar{1}}$  contains the edges  $\bar{2}$  and  $\bar{v}$ , but not the edges  $\bar{3}$ ,  $\bar{4}$ , or  $\bar{5}$ . To complete  $G_{\bar{1}}$  to  $\Gamma$ , there must be three more edges, all incident to  $\bar{v}$ . Without loss of generality, two of those edges are  $\bar{6} = \{x_v, y_2\}$  and  $\bar{7} = \{y_v, x_2\}$ . If the third edge is not incident to  $\bar{3}$ , then  $G_{\bar{3}}$  contains  $G_{\bar{1}}$  and the edge  $\bar{4}$ . With six edges,  $G_{\bar{3}}$  must be  $K_{3,2}$ , with vertex partition (without loss of generality)  $\{y_2, y_v\}$ ,  $\{x_v, x_1, x_2\}$ . Then  $G_{\bar{5}}$  contains the 4-cycle  $x_v, y_2, x_1, y_v$ , but this cannot be completed to a  $\Gamma$  using only edges incident to  $\bar{v}$  and  $\bar{1}$ . Thus, one of the edges of  $G_{\bar{1}}$  is incident to  $\bar{3}$ . There are two possibilities, depending on its vertex in edge  $\bar{v}$ . See Figure 26, where the vertices and edges are placed differently to illustrate the constructions to follow.

**Case IV.1** Assume the edge connecting  $x_3$  to  $\bar{v}$  is  $\bar{8} = \{x_3, x_v\}$ . Note that any other edges in  $G$  must be incident to edges  $\bar{v}$  and  $\bar{1}$ . Now consider  $G_{\bar{5}}$ , another  $\Gamma$ , since 5 is a boundary vertex of  $M$ . So far  $G_{\bar{5}}$  contains the edges  $\bar{4}$ ,  $\bar{6}$ ,  $\bar{v}$  and  $\bar{8}$ . To complete it to  $\Gamma$ , the edge  $\bar{9} = \{x_1, y_v\}$  is needed. Note that including any other edge in  $G$  would create a triangle, which is not allowed. So it remains to find the matching complex of the graph IV.1 of Figure 26 with the single edge  $\bar{9}$  added. The graph and its matching complex (a Möbius strip) are shown in Figure 27. This is part 3(d) of the theorem.

**Case IV.2** Assume the edge connecting  $x_3$  to  $\bar{v}$  is  $\bar{8} = \{x_3, y_v\}$ . It is still the case that any other edges in  $G$  must be incident to edges  $\bar{v}$  and  $\bar{1}$ . Again consider  $G_{\bar{5}}$ , which is  $\Gamma$ . This forces the edge  $\bar{9} = \{x_1, y_v\}$ . At this point, any other edge of  $G$  must be incident to  $\bar{v}$ ,  $\bar{1}$  and  $\bar{5}$ . Consider  $G_{\bar{7}}$ ; it contains the edges  $\bar{3}$ ,  $\bar{1}$ ,  $\bar{4}$ , and  $\bar{6}$ , which form a path  $P_5$ . If  $G_{\bar{7}}$  were just  $P_5$ , then  $G$  and  $M(G)$  would have been considered in Case III. So  $G_{\bar{7}}$  must contain another edge, and as it must be incident to  $\bar{v}$ ,  $\bar{1}$  and  $\bar{5}$ , it must be the edge  $\bar{10} = \{x_v, y_1\}$ . (This means that  $\bar{7}$  is a boundary vertex, with  $G_{\bar{7}} = \Gamma$ .) Any additional edge would create a triangle. So, we conclude that the graph for Case IV.2 must be that of Figure 28. The manifold is a torus with a 2-ball removed. This is part 4(c) of the Theorem.

This concludes the proof of Theorem 5.3.  $\square$

Just as in the case without boundary, while there are many 2-dimensional manifold matching complexes with boundary, it turns out that if  $d \geq 3$ , then manifold matching complexes are very simple. The following theorem classifies all of them.

**Theorem 5.4** *Let  $M$  be a  $d$ -dimensional manifold with boundary for some  $d \geq 3$ . Suppose there exists a simple graph  $G$  with  $M(G) = M$ . Then  $M$  is a  $d$ -ball. In particular,  $G$  is as described in Proposition 2.16.*

**Proof:** Let  $\dim M = d \geq 3$  and assume for lower dimensions that the only graphs which have spheres and balls as matching complexes are those graphs described in Propositions 2.14 and 2.16, respectively. We will use this to show that  $G$  is as described in Proposition 2.16 and therefore  $M$  is a  $d$ -ball.

If  $v \in M$  is an interior vertex then  $\text{link}_M v$  is a  $(d-1)$ -sphere, and if  $v$  is a boundary vertex then  $\text{link}_M v$  is a  $(d-1)$ -ball. We already know from Theorems 4.2 and 4.3 that if  $v$  is an interior vertex, then  $G_{\bar{v}}$  is disconnected. We will focus on boundary vertices and analyze which sort of links can appear.

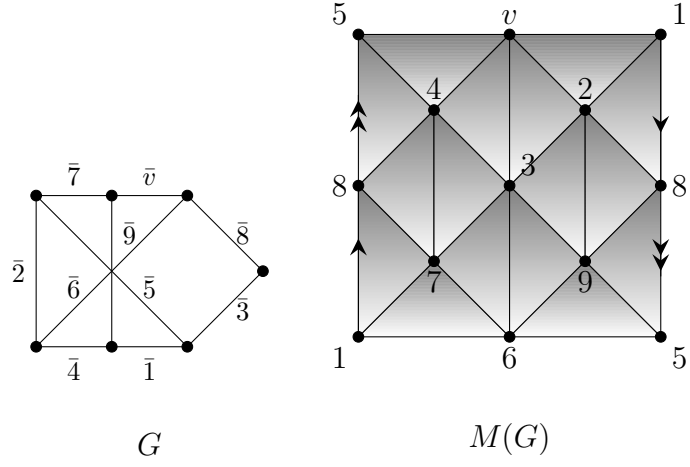


Figure 27: Graph and Manifold for Case IV.1

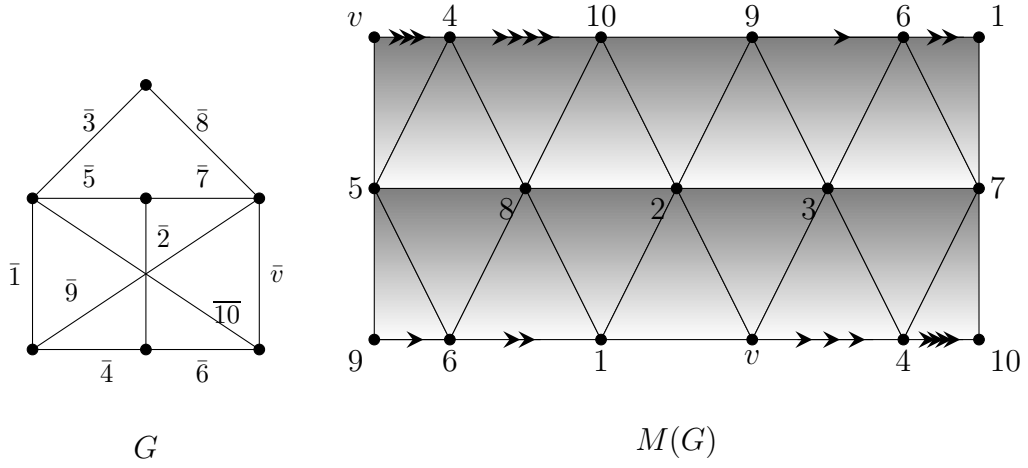


Figure 28: Graph and Manifold for Case IV.2

We will consider two cases:

**Case I.** For all boundary vertices  $v \in M$ ,  $G_{\bar{v}}$  is disconnected.

**Case II.** There exists a vertex  $v$  in the boundary of  $M$  such that  $G_{\bar{v}}$  is connected.

We will show that Case I implies that  $G$  must be disconnected and, due to Proposition 2.13,  $M$  must be a ball. In Case II, we will show that either  $G$  is disconnected or  $G$  is the spider graph  $\text{Sp}_{d+1}$ . In either case,  $M$  is a ball.

**Case I.** For all vertices  $v \in M$ ,  $G_{\bar{v}}$  is disconnected. In particular, for boundary  $v \in M$ ,  $G_{\bar{v}}$  contains a basic ball graph as a connected component. We show that  $G$  itself is disconnected.

**Case I.1.** There exists a boundary vertex  $v \in M$  such that  $G_{\bar{v}}$  contains  $P_2$  as a connected component. Write  $G_{\bar{v}} = P_2 \sqcup G'$ , and let  $\bar{a}$  be the edge of  $P_2$ . Consider  $G_{\bar{a}}$ , which is also disconnected. Since all other edges of  $G$  must contain a vertex of  $\bar{v}$ , all edges of  $G_{\bar{a}}$  are either incident to  $\bar{v}$  or contained in  $G'$ . Since  $G_{\bar{a}}$  is disconnected part (or all) of  $G'$  is not in the same component of  $G$  as  $\bar{v}$ . Thus  $G$  is not connected.

**Case I.2.** There are no boundary vertices  $v \in M$  such that  $G_{\bar{v}}$  contains  $P_2$  as a connected component. Let  $v$  be a boundary vertex of  $M$  and  $G^o$  be a component of  $G_{\bar{v}}$  that is either  $\Gamma$  or  $\text{Sp}_k$  ( $k \geq 2$ ), and  $G' = G_{\bar{v}} \setminus G^o$ . See Figures 29 and 30.

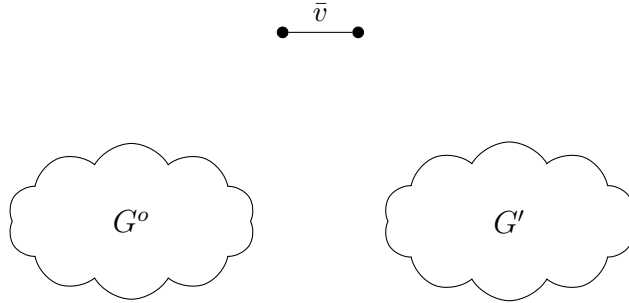


Figure 29: The subgraph  $G_{\bar{v}}$  for Case I.2 See Figure 30 for options for  $G^o$ .

For each possibility in Figure 30 a special edge  $\bar{a}$  of  $G^o$  and an edge  $\bar{b}$  of  $G^o$  not incident to  $\bar{a}$  have been identified. Since  $G_{\bar{a}}$  is assumed not to contain  $P_2$  as a connected component,  $\bar{b}$  must be connected to an edge of  $\bar{v}$ . So  $G^o$  is in the component of  $G$  containing  $\bar{v}$ . As in Case I.1, since all other edges of  $G$  must contain a vertex of  $\bar{v}$ , all edges of  $G_{\bar{a}}$  are either incident to  $\bar{v}$  or

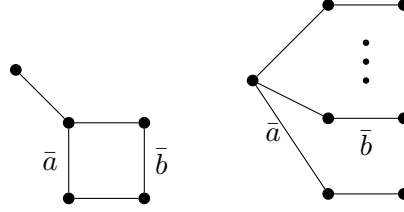


Figure 30: The options for  $G^o$  for Case I.2

contained in  $G'$ . Since  $G_{\bar{a}}$  is disconnected part (or all) of  $G'$  is not in the same component of  $G$  as  $\bar{v}$ . Thus  $G$  is not connected.

Therefore in Case I,  $G$  must be disconnected, and by Corollary 5.1  $M$  must be a  $d$ -ball.

**Case II.** There exists a vertex  $v$  in the boundary of  $M$  such that  $G_{\bar{v}}$  is connected.

By our induction hypothesis, Proposition 2.16 lists all balls that arise as matching complexes for dimensions less than  $d$ . Since  $\text{link}_M v$  is a  $(d-1)$ -ball, this implies that  $G_{\bar{v}} = \text{Sp}_d$ . We consider  $G_{\bar{a}_1}$  in Figure 31 (ignore the dotted edge  $\bar{y}$ ).

**Case II.1.** The vertex  $a_1$  is an interior vertex of  $M$ .

This implies that  $G_{\bar{a}_1}$  is a disjoint union of copies of elements from  $\mathcal{SG}$ . Since the path  $\bar{a}_2, \bar{b}_2, \bar{b}_d, \bar{a}_d$  is in  $G_{\bar{a}_1}$ , this path is part of either  $C_5$  or  $K_{3,2}$ . But the only edges of  $G$  not included in Figure 31 must be incident to  $\bar{v}$ , which is a contradiction. Therefore this subcase is impossible.

**Case II.2.** The vertex  $a_1$  is a boundary vertex of  $M$ .

Again considering Figure 31 (ignore the dotted edge  $\bar{y}$ ), we see that the subgraph  $G_{\bar{a}_1}$  contains  $\text{Sp}_{d-1}$  and  $\bar{v}$ . Since  $\text{link}_M a_1$  is a  $(d-1)$ -ball, our induction hypothesis says that there are only three options for the corresponding subgraph.

**Case II.2.a.**  $G_{\bar{a}_1} = \text{Sp}_{d-1} \sqcup P_2$  or  $G_{\bar{a}_1} = \text{Sp}_{d-1} \sqcup P_3$  The graph  $G_{\bar{a}_1}$  is shown in Figure 31, with a dotted edge  $\bar{y}$  for the extra edge in the case  $\text{Sp}_{d-1} \sqcup P_3$ . The only other possible edges of  $G$  connect one of the vertices of  $\bar{a}_1$  to one of the vertices of the edge  $\bar{v}$ . Assume any one of these four edges exists and call it  $\bar{x}$ .

We now consider  $\text{link}_M \bar{a}_2$ , which by induction must be as described in Proposition 2.16. However, we can see that  $G_{\bar{a}_2}$  must contain an induced path of length 5 (either  $\bar{x}, \bar{a}_1, \bar{b}_1, \bar{b}_d, \bar{a}_d$  or  $\bar{v}, \bar{x}, \bar{b}_1, \bar{b}_d, \bar{a}_d$  depending on whether  $\bar{x}$  is

incident to  $\bar{b}_1$  or not). But none of the graphs described in Proposition 2.16 have an induced  $P_6$ . Therefore no such edges  $\bar{x}$  exist and so  $G$  is exactly the graph pictured in Figure 31.

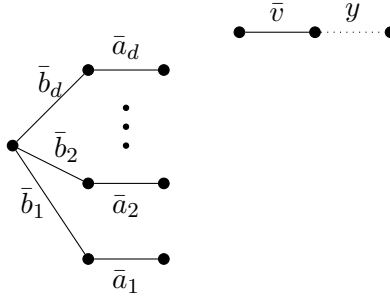


Figure 31: A subgraph for Case II.2.a. Any remaining edges of  $G$  must be incident to both  $\bar{v}$  and  $\bar{a}_1$ .

**Case II.2.b.**  $G_{\bar{a}_1} = \text{Sp}_d$

In this case,  $G$  contains the subgraph in Figure 32. Again, the only possible edges of  $G$  that do not appear in Figure 32 are edges that connect one of the vertices of  $\bar{a}_1$  to one of the vertices of the edge  $\bar{v}$ . Again we consider  $\text{link}_M \bar{a}_2$ . If any of these four possible edges exists, then  $G_{\bar{a}_2}$  is  $\text{Sp}_d$  with an additional edge connecting two of the legs of the spider. This again contradicts Proposition 2.16. Therefore  $G$  is exactly the graph pictured in Figure 32.

Therefore we have proved that either  $G = \text{Sp}_{d+1}$  or that  $G$  is disconnected. In the case that  $G$  is disconnected,  $M$  is a nontrivial join of some complexes  $M(G_1)$  and  $M(G_2)$  where  $G = G_1 \sqcup G_2$ . Therefore by Proposition 2.13, we know that  $M$  is a triangulated  $d$ -ball. By our induction hypothesis, this means that  $G_1$  and  $G_2$  are exactly as described in Proposition 2.16, and therefore so is  $G$ . This completes the proof of the theorem.  $\square$

Thus we see that the combinatorial manifolds without boundary that occur as matching complexes are balls, the 2-dimensional annulus, the Möbius strip, and the 2-dimensional torus with a ball removed.

Jelić Milutinović, et al. [7] show that (the 1-skeleton of) a connected matching complex has diameter at most 2 unless the graph has some pair of edges such that every other edge is incident to at least one edge of the





defined via similar link conditions, e.g., Buchsbaum, Cohen-Macaulay, and vertex decomposable complexes. Some of these have appeared in the context of matching complexes; for example, Ziegler showed in [18] that the  $\nu_{m,n}$ -skeleton of  $M(K_{m,n})$  is vertex decomposable (and hence shellable), where  $\nu_{m,n} = \min \left\{ m, \left\lfloor \frac{m+n+1}{3} \right\rfloor \right\} - 1$ . Similarly, Athanasiadis showed in [1] that the  $\nu_n$ -skeleton of  $M(K_n)$  is vertex decomposable, where  $\nu_n = \left\lfloor \frac{n+1}{3} \right\rfloor - 1$ . (Shellability in this case was originally shown by Shareshian and Wachs in [15].)

We might ask the opposite sort of question: which complexes with various link condition properties can be realized as matching complexes? Our approach would be extended most naturally to Buchsbaum complexes. A pure  $d$ -dimensional complex is said to be *Buchsbaum* if the link of any nonempty face  $\sigma$  has the homology of a wedge of  $(d - |\sigma|)$ -spheres. All manifolds are Buchsbaum, but there are many matching complexes which are Buchsbaum but not manifolds. One simple example is  $M(K_{1,3} \sqcup P_2)$ .

Recall that matching complexes are flag complexes. Frohmader [5] proved a conjecture of Kalai (see [16, Section III.4]) constructing, for every flag complex  $\Delta$ , a balanced complex  $\Delta'$  with the same number of faces of each dimension as  $\Delta$ . (Balanced means the 1-skeleton of the complex has chromatic number equal to the size of the largest face in the complex.) Many, but not all, of our manifold matching complexes are balanced. The basic sphere graph  $C_5$  has matching complex  $C_5$ , which is clearly not balanced. Matching complexes that are spheres and balls arise from graphs that are disjoint unions of basic sphere and ball graphs. Any such union that does not include  $C_5$  has a balanced matching complex. The two-dimensional torus, matching complex of  $K_{4,3}$ , is balanced. Among the matching complexes that are manifolds with boundary, the triangulated Möbius strips are not balanced. (They all come from graphs with odd cycles.) But the others, the triangulated annulus and the torus with 2-ball removed, are balanced. For those that are not balanced, it would be interesting to examine the corresponding balanced complexes Frohmader's method would construct. We could also ask what other matching complexes (that are not manifolds) are balanced.

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