# Getting started with matching complexes

1. (a) A **simplicial complex** on $[n]$ is a set $\Delta \subseteq 2^{[n]}$ with the following property:

   If $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. (In other words, $\Delta$ is "closed under taking subsets.")

   (b) Elements of $\Delta$ are **faces**, and *maximal* elements are **facets**.

   (c) The dimension of $\Delta$ is $\max\{|\sigma| - 1 \mid \sigma \in \Delta\}$. (This is often lower than the dimension that $\Delta$ can be "embedded" into.)

   (d) Given a graph $G$, a **matching** is a collection of edges such that no two share an endpoint. If $G$ is a graph with edges $1, \ldots, n$, then its **matching complex**, denoted $M(G)$, is the set of all matchings of $G$. From our discussion, we saw that $M(G)$ is a simplicial complex whose vertices were the *edges* of $G$.

2. Given a simplicial complex $\Delta$, its **geometric realization** $||\Delta||$ is created by drawing the following:
   - A *vertex* for each face containing only a single element,
   - an *edge* for each face containing exactly two elements,
   - a (filled in) *triangle* for each face containing exactly three elements,
   - a (filled in) *tetrahedron* for each face containing exactly four elements,
   - and so on...

3. Find the matching complexes for $C_3$ (the cycle graph with 3 edges) and for the graph with three edges all meeting in a single vertex (a star graph with three edges). What do you notice?

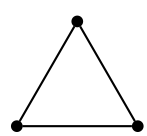
4. Here are some more examples to consider:

   (a) $C_5$, $C_6$, and $C_7$. (Cycle graphs)

   (b) $K_4$ (The complete graph on 4 vertices)

   (c) $K_{3,2}$ and $K_{3,4}$. (Complete bipartite graphs — this last one is a bit complicated, but it has a nice result.)

5. The following is a geometric realization of the complex $\Delta = \langle 12, 13, 23 \rangle$ (i.e., $\Delta$ is just $C_3$). Can this be the matching complex for some graph $G$?



**To discuss:** *Flag* complexes

6. How do the following affect $M(G)$?
   (a) An isolated vertex?
   (b) A loop edge?
   (c) Multiple edges between the same two vertices?

7. In general, we want to investigate the following sort of questions:
   (a) Given a graph $G$ (or class of graphs, like complete graphs, cycles, paths, trees, etc.), what can we say about $M(G)$?
   (b) Given a simplicial complex $\Delta$, is it the matching complex of some graph? Given some properties of the matching complex, can we say something about $G$?
   (c) The above questions are *super vague*. We'd like to narrow these down into questions that are more specific. Here are some to start thinking about:
      (i) When is $M(G)$ also a graph?
      (ii) When is $M(G)$ connected? (We can talk more specifically about what this means next week.)
   (d) In general, keep track of any questions you have while thinking about this or working through examples, and then we can discuss them. **Generating and attempting to answer questions like these is the heart of mathematics research.** I have lots of questions that I can pose for us, but we'll probably be more successful if we don't limit ourselves to the questions that I ask.