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$\boxed{1}$ $\mathcal{P} = \{\Gamma(\alpha,\beta) : \alpha>0, \beta>0\}$. Асимп. мат. ков. для ОМП (α,β) —?

$$L(\alpha,\beta) = \prod_{i=1}^{n} \frac{\alpha^{\beta} x_i^{\beta-1}}{\Gamma(\beta)} e^{-\alpha x_i} = \frac{\alpha^{n\beta}}{\Gamma^n(\beta)} \left(\prod_{i=1}^{n} x_i\right)^{\beta-1} \cdot e^{-\alpha \sum_{i=1}^{n} x_i}, \quad \ell(\alpha,\beta) = \ln L(\alpha,\beta) = n\beta \ln\alpha - n\ln\Gamma(\beta) +$$

$$+ (\beta-1)\sum_{i=1}^{n} \ln x_i - \alpha \sum_{i=1}^{n} x_i, \quad \frac{\partial \ell}{\partial \alpha} = \frac{n\beta}{\alpha} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \frac{\beta}{\alpha} = \overline{X}, \quad \frac{\partial \ell}{\partial \beta} = n\ln\alpha - \frac{n}{\Gamma(\beta)}\Gamma'(\beta) + \sum_{i=1}^{n} \ln x_i =$$

$$= n\left(\ln\alpha - \psi(\beta) + \overline{\ln x}\right) = 0 \Rightarrow \psi(\beta) - \ln\alpha = \overline{\ln x} \quad (\psi\text{- дигамма-ф-ция}); \quad \frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n\beta}{\alpha^2}, \quad \frac{\partial^2 \ell}{\partial\alpha\partial\beta} =$$

$$= \frac{n}{\alpha}, \quad \frac{\partial^2 \ell}{\partial \beta^2} = -n\frac{\Gamma''(\beta)}{\Gamma(\beta)} + n\left(\frac{\Gamma'(\beta)}{\Gamma(\beta)}\right)^2 \Rightarrow \underset{\text{матрица Фишера}}{I(\alpha,\beta)} = -E\frac{\partial^2 \ell}{\partial\alpha\partial\beta} = -E\begin{pmatrix} -\frac{n\beta}{\alpha^2} & \frac{n}{\alpha} \\ \frac{n}{\alpha} & -n\frac{\Gamma''(\beta)}{\Gamma(\beta)} + n\left(\frac{\Gamma'(\beta)}{\Gamma(\beta)}\right)^2 \end{pmatrix} = n\begin{pmatrix} \beta/\alpha^2 & -1/\alpha \\ -1/\alpha & \frac{\Gamma''(\beta)}{\Gamma(\beta)} - \left(\frac{\Gamma'(\beta)}{\Gamma(\beta)}\right)^2 \end{pmatrix}$$

Асимп. мат. ков. $\operatorname{cov}(\hat\alpha,\hat\beta) = I^{-1} = \dfrac{n}{n^2\left(\frac{\beta}{\alpha^2}\left(\frac{\Gamma''(\beta)}{\Gamma(\beta)} - \left(\frac{\Gamma'(\beta)}{\Gamma(\beta)}\right)^2\right) - \frac{1}{\alpha^2}\right)} \begin{pmatrix} \frac{\Gamma''(\beta)}{\Gamma(\beta)} - \left(\frac{\Gamma'(\beta)}{\Gamma(\beta)}\right)^2 & 1/\alpha \\ 1/\alpha & \beta/\alpha^2 \end{pmatrix} = \dfrac{\alpha^2}{n\left(\beta\left(\frac{\Gamma''}{\Gamma} - \left(\frac{\Gamma'}{\Gamma}\right)^2\right) - 1\right)} \cdot$

$$\cdot \begin{pmatrix} \frac{\Gamma''}{\Gamma} - \left(\frac{\Gamma'}{\Gamma}\right)^2 & 1/\alpha \\ 1/\alpha & \beta/\alpha^2 \end{pmatrix} = \boxed{\frac{1}{n\left(\beta\left(\frac{\Gamma''}{\Gamma} - \left(\frac{\Gamma'}{\Gamma}\right)^2\right) - 1\right)} \begin{pmatrix} \alpha^2\left(\frac{\Gamma''}{\Gamma} - \left(\frac{\Gamma'}{\Gamma}\right)^2\right) & \alpha \\ \alpha & \beta \end{pmatrix}}$$

$\boxed{2}$ $X_1,\dots,X_n \sim \mathrm{Pois}(\theta)$; град., нат. град, метод Ньютона —?

$$L(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{X_i}}{X_i!} = \frac{e^{-n\theta}\theta^{\sum X_i}}{\prod X_i!}, \quad \ell(\theta) = \ln L(\theta) = -n\theta + \sum_{i=1}^{n} X_i \cdot \ln\theta - \sum_{i=1}^{n} \ln(X_i!), \quad \frac{\partial \ell}{\partial \theta} = \boxed{-n + \frac{\sum X_i}{\theta} = \nabla \ell}$$

$$I = -E\frac{\partial^2 \ell}{\partial\theta^2} = -E\left(-\frac{\sum X_i}{\theta^2}\right) = \frac{n\theta}{\theta^2} = \frac{n}{\theta}, \quad I^{-1} = \frac{\theta}{n}, \quad g^{\text{нат}} = I^{-1}\nabla\ell = \frac{\theta}{n}\left(-n + \frac{\sum X_i}{\theta}\right) = \frac{\sum X_i}{n} - \theta = \boxed{\overline{X} - \theta = g^{\text{нат}}}$$

$\nabla \ell$: не учитывает кривизну, может сходиться долго; $g^{\text{нат}}$: корректирует направление, может сойтись быстрее;

метод Ньютона: опирается на гессиан, способен быстро сойтись, но вычислительно затратен

$\boxed{3}$ $X_1,\dots,X_n \sim \mathrm{Bern}(\theta)$, $\hat\theta = X_1$, $T(X) = \sum_{i=1}^{n} X_i$ — дос.; улучшить $\hat\theta$ с помощью т. Колм.-Блэкуэлла-Рао

$T(X) \sim \mathrm{Bin}(n,\theta)$ (число успехов), $ET(X) = n\theta$; $\hat\theta_T = \dfrac{T(X)}{n}$, $E\hat\theta_T = \dfrac{ET(X)}{n} = \dfrac{n\theta}{n} = \theta \Rightarrow$ несмещенная.

$D\hat\theta = DX_i = \theta(1-\theta)$, $D\hat\theta_T = D\left(\dfrac{T(X)}{n}\right) = \dfrac{1}{n^2} D\,T(X) = \dfrac{1}{n^2} n\,\theta(1-\theta) = \dfrac{\theta(1-\theta)}{n} < D\hat\theta$ (улучшили)