Selective inference is easier with p-values

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Abstract

Selective inference is a subfield of statistics that enables valid inferences after selection of a data-dependent question. In this paper, we introduce selectively dominant p-values, a class of p-values that allow practitioners to easily perform selective inference after arbitrary selection procedures. Unlike a traditional p-value, whose distribution must stochastically dominate the uniform distribution under the null, a selectively dominant p-value must have a post-selection distribution that stochastically dominates that of a uniform having undergone the same selection process; moreover, this property must hold simultaneously for all possible selection processes. Despite the strength of this condition, we show that all commonly used p-values (e.g., p-values from two-sided testing in parametric families, one-sided testing in monotone likelihood ratio and exponential families, and permutation tests) are selectively dominant. By recasting two canonical selective inference problems—inference on winners and rank verification—in our selective dominance framework, we provide simpler derivations, a deeper conceptual understanding, and new generalizations and variations of these methods. Additionally, we use our insights to introduce selective variants of methods that combine p-values, such as Fisher's combination test.

1 Introduction

Selective inference is a subfield of statistics that allows practioners to make valid inferences even when statistical question at hand was chosen by a data-driven selection process. Many selective methods however, which operate by conditioning on the selection event, can be difficult to derive, hard to implement, and exhibit counterintuitive behaviors. To statisticians outside of the field, each selective procedure may seem to come from a different argument or approach.

In this paper, we provide a unifying framework for selective inference centered around p-values. So long as a statistician knows how to construct a p-value for their inferential question at hand, our framework provides an algorithmic approach for delivering hypothesis testing procedures and confidence intervals that are valid even after selection. Our framework (1) can greatly simplify the process of designing new selective methods and (2) results in more natural and general derivations of some existing selective methods, allowing for a deeper understanding of their behavior as well as new variations and extensions.

1.1 Motivation

As motivation, we consider the setting of independent Gaussian data $X \sim N(\mu, I_n)$ with unknown mean μ and recall the problem of doing inference on the winner. Since the largest observation $W = \text{argmax}_{i\in[n]} X_i$ is likely to correspond to the largest mean, a natural way to verify the existence of a large effect (i.e., a large μ_i) is to give a lower confidence bound (LCB) for μ_W, the mean of the winning value.

Normally we provide such an LCB via Sidak's simultaneous approach. Letting $z_{1-\alpha}$ denote the $1-alpha$ quantile of the standard normal distribution and defining $\alpha_n = 1-(1-\alpha)^{\frac{1}{n}}$, the lower bounds $\mu_i > X_i - z_{1-\alpha_n}$ hold simultaneously with probability $1-\alpha$. Therefore, $\hat{\mu}_{simul} = X_W - z_{1-\alpha_n}$ is a valid lower bound for μ_W that holds with probability at least $1-\alpha$. Performing simultaneous inference on n means, however, comes at a cost. As n grows, the quantile $z_{1-\alpha_n}$ grows as well (depicted in the left panel of Figure 1), and the distance from the winning observation to the LCB correspondingly increases.

A more modern approach is to provide a LCB that is valid conditionally on W. This approach provides inferences for only the winning mean μ_W and no other means, so we may hope that it avoids the simultaneous

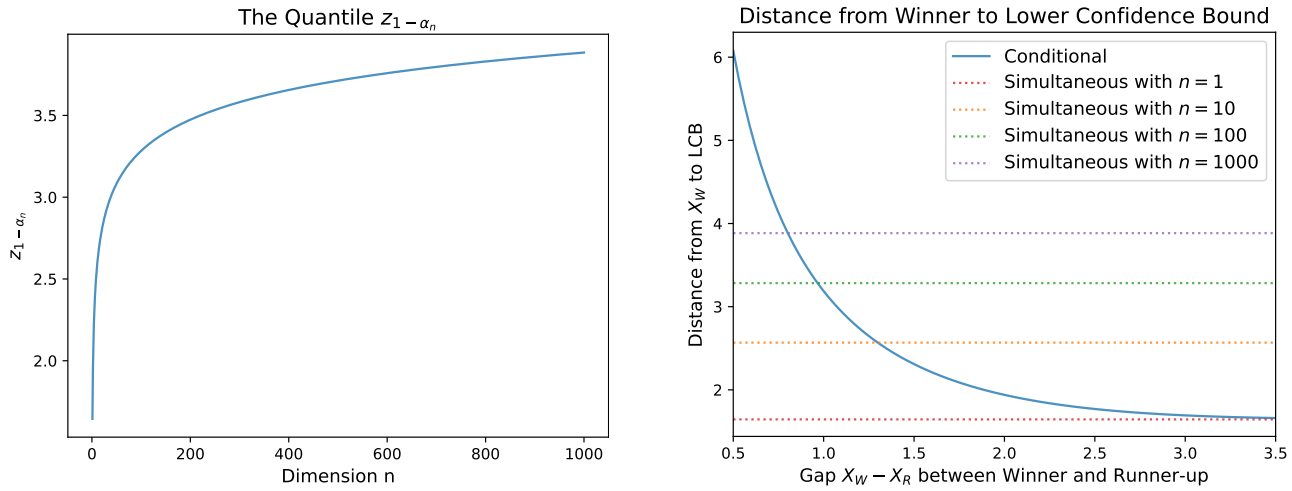

Figure 1: The first panel (left) shows the growth of the quantile $z_{1-\alpha_n}$ as a function of the dimension n. The second panel (right) gives the distance between the level $\alpha = 0.05$ conditional LCB and the winner X_W as a function of the gap $X_W - X_R$ between winner and the runner-up. For dimensions $n = 1, 10, 100, 1000$, it also gives the distance from X_W to the level $\alpha = 0.05$ simultaneous LCB.

inference's curse of dimensionality. Following the recipe of Fithian et al. [2017], one finds the conditional LCB to be

$$\hat{\mu}_{cond} = \inf\{\mu_0 : \mu_0 > X_W - \text{Quantile}(1-\alpha, TN(0,1,X_R-\mu_0,\infty))\}, \tag{1}$$

where X_R is the runner-up (second largest) observation and $TN(\mu,\sigma^2,a,b)$ is a $N(\mu,\sigma^2)$ distribution truncated to lie in the interval $[a,b]$.

The conditional LCB (1) is near impossible to parse, but it turns out to have some very interesting behavior. As plotted in right panel of Figure 1, the distance $X_W - \hat{\mu}_{cond}$ between the winner and the conditional LCB is purely a function of the gap $X_W - X_R$ between the winner and runner-up. If the gap between X_W and X_R is large, then the conditional LCB for μ_W will be roughly $X_W - z_{1-\alpha}$, i.e, what we expect in a one-dimensional inference problem. But as the runner-up gets close to the winner, the conditional LCB explodes quickly to $-\infty$, and can give much worse inferences than the classical approach. In summary, the conditional method appears to avoid the curse of dimensionality in some situations, but when it fails to, the consequences can be tremendous.

The motivation for this article comes from the following fact: the conditional LCB becomes shockingly simple to parse once written in terms of p-values. Imagine using each LCB to verify the existence of a positive mean, which we can do by ensuring that the LCB is non-negative. Normally, we verify the existence of positive means by testing the nulls $H_{0,i} : \mu_i \leq 0$ with the p-values $p_i = 1 - \Phi(X_i)$. It turns out that the simultaneous LCB verifies the existence of a positive mean when smallest p-value $p_{(1)}$ is at most α_n. In contrast, the conditional LCB does so when the ratio of the top two smallest p-values $p_{(1)}/p_{(2)}$ is at most α.

At least when the p-values p_i are uniform under the null, the smallest p-value $p_{(1)}$ should be uniform on $[0, p_{(2)}]$, and we should control error when we reject when $p_{(1)} \leq \alpha p_{(2)}$. But does this work when the p-values are not exactly uniform? Why does it work in the Gaussian case?

The framework we develop in this article allows us to easily prove the validity of the above procedure, where we reject $H_{0,(1)}$ when $p_{(1)} \leq \alpha p_{(2)}$. Moreover, t

1.2 Our Contributions

In this paper we introduce the **selective dominance** framework, which we summarize here. In the framework, we imagine using a p-value p to test the null hypothesis H_0.

The sufficient statisics can vary quite heavily from problem to problem, making many selective inference methods appear quite different. By focusing on the p-value...

The remainder of the paper illustrates our framework's utility via a number of applications.

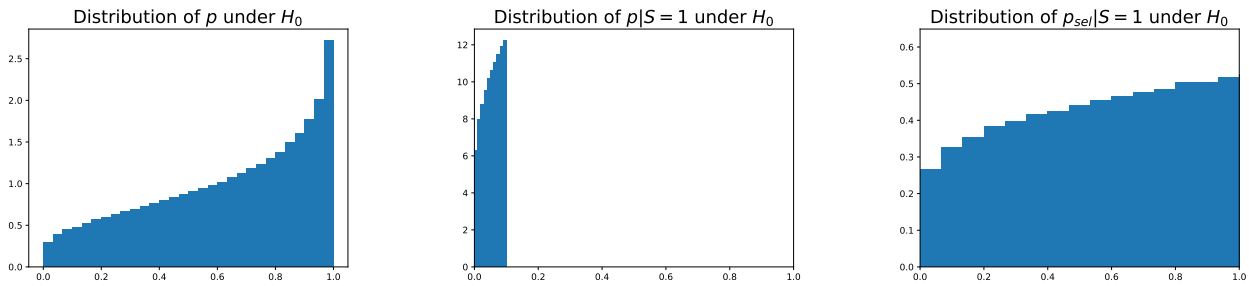

Figure 2: The first panel (left) depicts an example distribution of a p-value p under the null. The distribution is stochastically dominanted by the uniform distribution. The second panel (middle) depicts the distribution $p|S = 1$ of the same null p-value p given that it was selected for being most $\alpha = 0.1$. This distribution is not stochastically dominated by the uniform distribution. The third panel depicts the null distribution $p_{sel}|S = 1$ of Theorem 1's selective p-value p/α given selection. Thanks to Theorem 1's correction, this distribution again is stochastically dominanted by the uniform distribution.

1.3 Related Work

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Related work (Andrews and Fithian on selection). (Adapt for increasing p-values). Other works on selecting the winner (Tijana, Benjamini)

Sequential selective hypothesis testing.

2 Selectively Dominant p-Values

In this section we define selectively dominant p-values, a class of p-values that enable us to easily do inference after selection. We give a precise characterization of when p-values are selectively dominant and illustrate that the most commonly used p-values are all selectively dominant. Finally, we provide examples of how to apply our selective dominance framework.

2.1 Selective Dominance

In classical statistics, a p-value is a random variable p supported on $[0, 1]$ that stochastically dominates the uniform distribution $U \sim \text{Unif}([0, 1])$ under the null, i.e., $p \succeq_{H_0} U$. The left-most panel of Figure 2 gives an example of what a null p-value distribution may look like. When working with p-values, we maintain Type I error control if we reject the null H_0 when p is small:

$$P_{H_0}(\text{reject null}) = P_{H_0}(p \leq \alpha) \leq P(U \leq \alpha) = \alpha.$$

In essence, stochastic dominance allows us to use the uniform distribution as a reference distribution. We control the probability of p being small under the null by comparing it to the probability of a uniform being small.

In the problems we consider, we use p to test the null H_0 only after it has been "selected". In full generality, we consider a p-value p for testing the null H_0 that is conditionally valid given some random vector Z, i.e., $p \mid Z = z \succeq_{H_0} U$:

$$P_{H_0}(p \leq \alpha \mid Z = z) \leq P(U \leq \alpha) = \alpha. \tag{2}$$

Additionally, we consider a binary selection random variable $S \in \{0, 1\}$ that takes value one when p is selected and zero otherwise. The relationship between p, Z, and S is governed by a **selection function**,

$$s(x, z) = P(S = 1 \mid p = x, Z = z).$$

Intuitively, we imagine observing $p = x$ and $Z = z$ and then flipping a biased coin that comes up heads with probability $s(x, z)$. We only use p to test the null H_0 when this coin comes up heads, and otherwise do not perform inference. This process turns out to capture what happens in a wide range of selective inference problems. We decide the selection procedure in the problems we consider, so $s(x, z)$ is known. In cases where no Z is present, we can imagine $Z = 0$ and write the selection function $s(x)$ purely in terms of x.

Because we only perform inference when the coin comes up heads, our goal should be to design a procedure that controls Type I error conditional on this selection event:

$$P_{H_0}(\text{reject } H_0 \mid S = 1) \leq \alpha \tag{3}$$

As illustrated by our next example, the classical approach of rejecting when $p \leq \alpha$ does not maintain selective Type I error control as in (3).

Example 1 (Publication bias and the failure of classical inference)**.** *Consider a p-value p that is uniform on $[0, 1]$ under the null H_0. If we use the selection function $s(x) = I_{p\leq\alpha}$, i.e., we select p when it is at most α, then $p \mid S = 1 \sim Unif([0, \alpha])$. Our classical procedure will clearly fail to control Type I error conditional on selection:*

$$P_{H_0}(\textit{reject } H_0 \mid S = 1) = P_{H_0}(p \leq \alpha \mid S = 1) = 1 > \alpha.$$

Example 1 is a standard example in the literature on publication bias. If researchers publish studies testing the null H_0 only when their p-values are below α, the reader only gets to observe inferences made after this selection. As a consequence, the reader's observed type I error rate can be as high as one. Figure 2's middle panel displays the distribution of our earlier example p-value (left panel), but after it has been selected via Example 1's selection function. It is clear from the picture that, after selection, the null p-value distribution is no longer stochastically dominanted by the uniform distribution.

Essentially, after selection, the uniform distribution no longer suffices as a reference distribution. Naturally, we may instead try and use the distribution of a uniform after it has been selected by the same selection function. Formally, suppose that U has uniform distribution conditional on Z, i..e., $U \mid Z = z \sim \text{Unif}([0, 1])$, and let $S' \in \{0, 1\}$ be a different binary selection random variable whose joint distribution with U and Z is governed by the same selection function

$$P(S' = 1 \mid U = x, Z = z) = s(x, z).$$

Then, we could use the conditional distribution $U \mid Z, S' = 1$ of U given selection as our reference distribution. This approach is valid exactly when our p-value is selectively dominant, as we define below [1].

Definition 1 (Selective dominance)**.** *Considering a p-value p for the null H_0 that is valid given Z as in (2), we say that p is **selectively dominant given** Z if, under the null H_0, it has a conditional probability density function (PDF) given $Z = z$ and satisfies*

$$p \mid Z = z, S = 1 \succeq_{H_0} U \mid Z = z, S' = 1 \tag{4}$$

for every selection function $s(x, z)$ under which p and U both have a positive probability of being selected given $Z = z$.

As we will soon see, the majority of p-values that practitioners use are selectively dominant as described in (4). In Definition 1, we restrict to p-values with conditional PDFs under the null because it makes our theory and methods simpler to state. Because we can always make a p-value both have a conditional PDF and be more powerful via randomization, this restriction is never a practical issue. Also, after applying our machinery with randomized p-values, the user can always derandomize the resulting method if they would like.

To perform valid post-selection inference using a selectively dominant p-value, we can transform it so that it remains a p-value after selection. As Theorem 1 explains, we can "undo" the effects of selection by applying the conditional cumulative distribution function (CDF) $F_{U|Z,S'=1}(\cdot)$ of U given selection to p. In line with prior literature, we refer to this transformed p-value as a **selective p-value**. For simple selection functions, this selective p-value is often computable in closed form.

[1] For the sake of simplicity, we require selective dominance to hold point-wise for *every* z rather than almost all, and our later results similarly hold point-wise. Our definition and results, however, can be modified to accomodate more general case.

Theorem 1. *Let $F_{U|Z,S'=1}(u)$ denote the CDF of U conditional on selection. Then, under the null, the selective p-value*

$$p_{sel} = F_{U|Z,S'=1}(p) = \frac{\int_0^p s(x,Z)dx}{\int_0^1 s(x,Z)dx} \tag{5}$$

stochastically dominates the uniform distribution conditional on Z and selection,

$$P_{H_0}(p_{sel} \leq \alpha \mid Z = z, S = 1) \leq \alpha, \tag{6}$$

for any selection function $s(x,z)$ under which p and U both have a positive probability of being selected given $Z = z$. Further, for any distribution in H_0 under which p has an exact uniform distribution given $Z = z$, (6) holds with equality.

Essentially, Theorem 1 tells us that if we want selective Type I error control as in (3), then we should reject H_0 when p is less than the α quantile of $U \mid Z = z, S' = 1$ rather than the α quantile of U.

The right-most panel of Figure 2 depicts what happens when we apply Theorem 1's correction to our example null p-value. Unlike the null distribution of p given selection (middle panel), the null distribution of p_{sel} (which we derive explicitly later in Example 6) given selection is stochastically dominated by the uniform distribution.

2.2 Characterizing Selectively Dominant p-Values and Examples

Theorem 2 tells us that p-values are selectively dominant precisely when their conditional PDF is non-decreasing under the null.

Theorem 2 (Selective dominance and increasing density)**.** *If the conditional PDF of the p-value p given $Z = z$ is always non-decreasing under the null, then it is selectively dominant given Z as described in Definition 1. Conversely, if ever under the null, the conditional PDF of p given $Z = z$ is everywhere continuous and not non-decreasing, then p is not selectively dominant given Z.*

In what follows, we give a number of examples of selectively dominant p-values. Our examples include all the common p-values that practitioners use in real life. We recommend that the unfamiliar reader review uniformly most powerful (UMP) and uniformly most powerful unbiased (UMPU) testing [Lehmann et al., 1986, Chapter 3 and Chapter 4] prior to proceeding.

Example 2 (Two-sided testing in parametric families)**.** *Consider observing data from a parametric family P_θ and testing the null $H_0 : \theta = \theta_0$. Because the null is a point null, most p-values we construct will have an exact Unif$([0,1])$ distribution under the null and are therefore trivially selectively dominant.*

Example 3 (One-sided testing in monotone likelihood ratio families)**.** *Consider observing one-dimensional data from a parametric family $X \sim P_\theta$ that admits density $p_\theta(x)$ with respect to some carrier measure μ. We say that P_θ has a monotone likeliihood ratio (MLR) in the real valued function $T(x)$ if, the densities $p_\theta(x)$ share a common support and, for any $\theta < \theta'$, the ratio $p_{\theta'}(x)/p_\theta(x)$ is a non-decreasing function of $T(x)$. In this case, the UMP test for the null $H_0 : \theta \leq \theta_0$ rejects when $T(X)$ is large. The associated randomized p-value for this test (see Appendix B) is selectively dominant.*

Example 4 (Testing in in exponential families)**.** *Suppose we observe data $X \in \mathbb{R}^m$ from an exponential family P_θ parameterized by $\theta \in \mathbb{R}^n$ i.e., under P_θ the data X has density*

$$g_\theta(x) = \exp(\theta_1 T_1(X) + \cdots + \theta_n T_n(X) - \psi(\theta))g(x)$$

with respect to some carrier measure μ. In both the case of testing the two-sided null $H_0 : \theta_1 \neq \theta_{0,i}$ or one-sided null $H_0 : \theta_i \leq \theta_{0,i}$, the UMPU test conditions on the nuisance statistics $T_{-i}(X)$. The p-value associated with the UMPU test for $H_0 : \theta_1 \neq \theta_{0,i}$ has an exact Unif$([0,1])$ distribution conditional on $T_{-i}(X)$, so it is trivially selectively dominant given $Z = T_{-i}(X)$. For testing $H_0 : \theta_1 \leq \theta_{0,i}$, we are in the setting of an MLR family once we condition on $T_{-i}(X)$, so Example 3 implies that the p-value associated with the UMPU test is also selectively dominant given $Z = T_{-i}(X)$.

Example 5 (Permutation testing). *In a permutation test we observe data $X \in \mathcal{X}$ and compute a test statistic $T(X)$ that, under the null H_0, has a distribution that is invariant under a finite group of transformations $G : \mathcal{X} \to \mathcal{X}$. That is, $T(X) \overset{d}{=}_{H_0} T(g(X))$ for all $g \in G$. To run the test, we consider a collection of group elements $g_1, g_2, \ldots, g_w$ where $g_1 = id$ is fixed to be the identity transformation and $g_2, \ldots, g_w$ are either a random sample from G with replacement or a random sample from $G \setminus \{id\}$ without replacement. The test then rejects when $T(X)$ is large compared to the $T(g_j(X))$. Specifically, the randomized permutation test from Proposition 3 of Hemerik and Goeman [2018] uses the p-value*

$$p = \frac{\#\{1 \leq j \leq w : T(g_j(X)) > T(X)\}}{w} + U_{aux} \frac{\#\{1 \leq j \leq w : T(g_j(X)) = T(X)\}}{w},$$

where $U_{aux} \sim$ Unif$([0,1])$ adds auxiliary randomness that is independent of X. This p-value always has an exact Unif$([0,1])$ distribution under H_0 and is therefore is trivially selectively dominant.

Etablishing Example 3 and Example 4 is non-trivial, and the bulk of Appendix B is spent doing so.

2.3 Example Applications of Selective Dominance

Having developed our machinery, we provide a few examples that illustrate how to use it.

As an introductory example, we show how to correct for Example 1's publication bias. Using our selective dominance machinery, we can provide a one-line derivation of the p-value adjustment from Hung and Fithian [2020]. Hung and Fithian [2020] derive this correction specifically for p-values coming from z- and t-tests, but our machinery applies for all selectively dominant p-values.

Example 6 (Correcting for publication bias). *Suppose we have a selectively dominant p-value p for the null hypothesis H_0, and we choose to test H_0 only after observing that $p \leq \alpha$. We can apply our framework with $p = p$, $Z = 0$, and $s(x, z) = I_{x \leq \alpha}$. The selective p-value from (5) is p/α, so Theorem 1 tells us that rejecting when $p \leq \alpha^2$ controls selective Type I error:*

$$P_{H_0}(p \leq \alpha^2 | S = 1) = P_{H_0}(p/\alpha \leq \alpha | S = 1) \leq \alpha$$

As we have learned that essentially all the p-values researchers typically use are selectively dominant, Example 6 gives a simple way for readers to make valid inferences in the presence of publication bias: declare a studies' result significant when the associated p-value is at most α^2.

Our rule of thumb of rejecting when $p \leq \alpha^2$ should also deliver valid inferences in the presence of p-hacking. Rather discarding an experiment after observing a p-value larger than α, researches more typically tweak their analysis until the p-value crosses the significance threshold. This process, known as p-hacking, is difficult to study theoretically (hence Hung and Fithian [2020] do not study it). But it has been emprirically well established that, under the null, p-values resulting from p-hacking have left-skewed distributions, i.e., null p-hacked p-values can be reasonably modeled as having an increasing density on $[0, \alpha]$ [Simonsohn et al., 2013]. The transformed p-value p/α then has a density that is increasing on $[0, 1]$ under the null, so Theorem 2 guarantees that it is indeed a valid p-value.

Our second example shows how to use Theorem 1 to perform inference using the "winning" p-value. It illustrates how our selective dominance machinery enables us to test data dependent hypotheses, the core problem of selective inference.

Example 7 (Inference on the winning p-value). *Suppose we have n independent and selectively dominant p-values p_i for the null hypotheses $H_{0,i}$, and we choose to test only the jth null $H_{0,j}$ after observing that p_j is the smallest of the p_i. We will assume that under $H_{0,i}$ each p_i has density that is positive on all of $(0, 1)$. Applying our framework with $p = p_j$, $Z = p_{-j}$, and the selection function $s(x, z) = I_{x < \min_k z_k}$, it is straightforward to compute that the adjusted p-value p_{adj} from (5) is $p_j / \min_{i \neq j} p_i$, so Theorem 1 tells us that rejecting when $p_j \leq \alpha \min_{i \neq j} p_i$ controls selective Type I error:*

$$P_{H_{0,j}}(p_j \leq \alpha \min_{i \neq j} p_i \mid p_{-j}, S = 1) \leq \alpha. \tag{7}$$

If we let W be the index of the smallest p-value, it is now easy to see that rejecting the data-dependent "winning" null $H_{0,W}$ when $p_{(1)} \leq \alpha p_{(2)}$ is controls Type I error both conditionally on W and marginally.

Consider only the set of indices $j \in \mathcal{J}$ for which p_j has a positive probability of being the smallest. Conditional error control is immediate: If $H_{0,j}$ is not true, then trivially $P(\text{falsely reject } H_{0,W} \mid W = j) = 0 \leq \alpha$. For the case that $H_{0,j}$ is true, the event $W = j$ is the same event as selecting p_j for inference in (7), so

$$\begin{aligned} P(\text{falsely reject } H_{0,W} \mid W = j) &= P(p_{(1)} \leq \alpha p_{(2)} \mid W = j) \\ &= P(p_j \leq \alpha \min_{i \neq j} p_i \mid W = j) \\ &\leq \alpha. \end{aligned}$$

Marginal error control follows from the law of total probability.

$$\begin{aligned} P(\text{falsely reject } H_{0,W}) &= \sum_{j \in \mathcal{J}} P(\text{falsely reject } H_{0,j} \mid W = j) P(W = j) \\ &\leq \alpha \sum_{j \in \mathcal{J}} P(W = j) \\ &\leq \alpha. \end{aligned}$$

If the nulls are all true and the p_i are exactly uniform, then the inequalities become equalities and our error control is tight.

Rejecting the null $H_{0,W}$ when $p_{(1)} \leq \alpha p_{(2)}$ may seem like a strange procedure, but we will see that doing so is central to the conditional inference for winners method that arises from Fithian et al. [2017]. In fact, Fithian et al. [2017] implies that, in many settings, this test is UMP amongst tests that are valid conditional on W.

Lastly, we show how our framework also applies to data-carving. Specifically we consider the file-drawer problem from Fithian et al. [2017]. Fithian et al. [2017] argue that data-splitting, which involves using a chunk of the data for selection and an independent chunk of data for inference, is often an inadmissible approach in selective inference problems. In such settings, data-carving, as we describe below, results in strictly more powerful procedures. Although it initially appears that data-carving's selection procedure does not involve selecting a p-value as we describe in Section 2, we show via a coupling argument that it can be viewed in this way. This both serves to illustrate the breadth of our framework's applicability, as well as provide a new perspective on data carving.

Example 8 (Data carving and the file-drawer problem)**.** *In the file-drawer problem we observe two independent samples $X_1, X_2 \sim N(\mu, 2)$ (e.g., X_1 comes from the first half of the data and X_2 from the second). We test the null $H_0 : \mu \leq 0$, but only when we observe that $X_1 > t$ for some $t \in \mathbb{R}$.*

Data splitting ignores the first observation, which was used for selection, and simply tests the null with the p-value $p_{split} = 1 - \Phi(X_2/\sqrt{2})$. Intuitively, because this p-value is independent of the selection process, we should maintain Type I error control without performing any correction. Applying our framework with $p = p_{split}$, $Z = X_1$, and the selection function $s(x, z) = I(z > t)$, it is not hard to see that the selective p-value indeed offers no correction.

Data-carving, however, attempts to still use the more powerful p-value $p_{full} = 1 - \Phi((X_1 + X_2)/2)$, which leverages information from both samples. How can we apply our framework to this data-carving problem? In how we have stated the problem, it is <u>not</u> the case that we observe p_{full} and then decide whether or not to use it for inference. Instead, we decide based on X_1, and unlike in data-splitting, p is not a valid p-value given X_1. Noting that

$$X_1 | \frac{X_1 + X_2}{2} = y \sim N(y, 1),$$

however, we can compute the probability that selection happened given that p took a particular value:

$$P(X_1 > t | p_{full} = x) = 1 - \Phi(t - \Phi^{-1}(1 - x)).$$

We may as well therefore imagine that we observed that $p_{full} = x$, and then decided to use it to test the null H_0 with probability $1 - \Phi(t - \Phi^{-1}(1 - x))$. Although this is not what happens in the original problem (in our new characterization we may test H_0 even when $X_1 \leq 3$), the conditional distribution of p given selection is the

same in both cases. Hence we can apply our framework with $p = p_{full}$, $Z = 0$ *and* $s(x) = 1-\Phi(t-\Phi^{-1}(1-x))$. *Fithian et al. [2017] argue that the resulting selective p-value*

$$p_{carve} = \frac{\int_0^{p_{full}} 1 - \Phi(t - \Phi^{-1}(1-x))dx}{\int_0^1 1 - \Phi(t - \Phi^{-1}(1-x))dx} = \frac{\int_0^{p_{full}} 1 - \Phi(t - \Phi^{-1}(1-x))dx}{1 - \Phi(t/\sqrt{2})}$$

will result in strictly more rejections than p_{split}. *Appendix A.11 provides more details on the specific calculations done in this example.*

Crucially, in Example 8, the conditional distribution of the random variable X_1 used for selection given the p-value p_{full} did not depend on the unknown parameter μ. Hence, the selection function $s(x)$ had no dependence on μ, and we were able to correct our p-value without any issues. This did not happen by accident, and is actually a consequence of more general and interesting fact regarding the relationship between between data splitting, data carving, data fission [Leiner et al., 2023], and data thinning [Dharamshi et al., 2024, Neufeld et al., 2024].

In the most basic version of data fission, we add and subtract independent normal noise $Z \sim N(0, 1)$ to a normal sample $X \sim N(\mu, 1)$ to get two independent samples $X_1, X_2 \sim N(\mu, 2)$ centered at the same mean. This is meant to mimic data splitting: the first sample can be used for selection and the second for inference. Data thinning generalizes this idea to the setting where X is a random vector from a parametric family, and we add noise make k new random vectors $X_1, \ldots, X_k$ that (1) are independent and (2) can be used to recover X via a deterministic function $X = T(X_1, \ldots, X_k)$. Vanilla data thinning would involve using some of the X_i to perform selection and then the rest to do inference. Data carving, however, suggests using a p-value for inference that is a function of all the data $X = T(X_1, \ldots, X_k)$, despite some of the X_i being used for selection. Because the noise we add to X has no dependence on the unknown parameter, the joint distribution of the X_i given $T(X_1, \ldots, X_k)$ also has no dependence on the unknown paramter. Therefore, contrary to the what the language in Leiner et al. [2023] suggests, the selection function $s(x)$ is always known, and we can always apply our framework to data carve and get more power. If the selection process is highly complicated, it is true that $s(x)$ may be very difficult to compute, but in theory it is always accessible to us via extensive simulations.

Our framework also applies to regression problems, including Lee et al. [2016]'s foundational problem of doing inference on LASSO selected regression coefficients. For sake of brevity, we have moved this discussion to Appendix A.10.

Example 6, Example 7, Example 8 all share a common theme. In all three examples, the practioner cheats. They peak at the p-value and, to varying degrees, they only test the null when the p-value looks promising. The purpose of selective procedures is to adjust the p-value in a way that accounts for this cheating. The harsher the cheating is, the more this adjustment inflates the original p-value.

3 Inference on Winners

In this section we use our framework to study the inference on winners problem. Along with providing further discussion about Example 7, we also discuss how hybrid inference [Andrews et al., 2023], which offers a solution to the exploding interval problem, also naturally arises in our framework. For both appproaches, our discussion results in novel interpretations, generalizations, and methods. To be concrete, we will imagine performing inference at level $\alpha = 0.1$ throughout the section.

For now, we focus on the independent data setting. The tools we develop in the next section, however, can be applied to do inference on winners when data is generated from a multi-parameter exponential family, which encompasses many correlated data settings.

3.1 Conditional Inference

In this sub-section we discuss Fithian et al. [2017]'s conditional approach for performing inference on winners. This conditional approach turns out to be highly related to the testing procedure that we derived in Example 7.

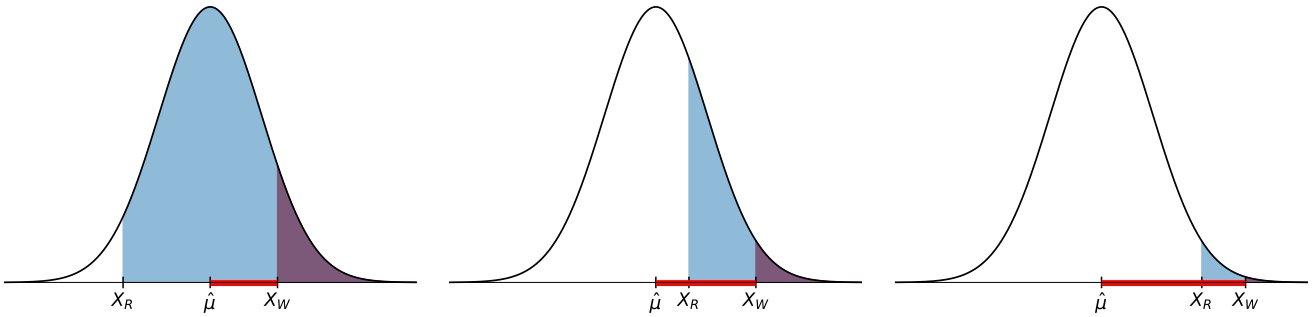


Figure 3: We plot the level $\alpha = 0.1$ conditional LCB $\hat{\mu}$ for different gaps between the winning value X_W and the runner up value X_R and highlight the distance between $\hat{\mu}$ and X_W in red. The LCB $\hat{\mu}$ is chosen exactly so that the tail probability $P(N(\hat{\mu}, 1) > X_R)$, shaded in blue, is $1/\alpha = 10$ times the tail probability $P(N(\hat{\mu}, 1) > X_W)$, shaded in red (the overlap appears purple). As X_W and X_R get closer, we need to take $\hat{\mu}$ further and further back for this condition to be satisfied.

Corollary 1 (Testing the winner)**.** *Suppose that p_i are n independent and selectively dominant p-values under the nulls $H_{0,i}$, and let W be the index of the smallest p-value. Rejecting $H_{0,W}$ when $p_{(1)} \leq \alpha p_{(2)}$ controls Type I error at level α conditionally on W, and therefore also marginally.*

Unlike Sidak's simultaneous approach, which rejects the winning null when the smallest p-value is small in absolute terms, the conditional approach rejects the winning null when the smallest p-value is small relative to the second smallest p-value. This procedure is strange, but fairly easy to interpret: we reject the winning null when the most extreme observation is $1/\alpha = 10$ more extreme under its null than the second most extreme observation. This can be quite a stringent requirement!

Once written in terms of p-values, its easy to mathematically see the merits and pitfalls of the conditional approach. If all the p-values except the smallest provide essentially no evidence against the null, then $p_{(2)} \approx 1$ and we reject when $p_{(1)} < \alpha$, the same p-value cutoff as a one-dimensional problem. On the other hand, if even just one other p-value provides a similar amount of evidence against the null as the smallest p-value, then then $p_{(2)} \approx p_{(1)}$ and we will never reject because $p_{(1)} \not\leq \alpha p_{(1)} \approx \alpha p_{(2)}$.

The p-value viewpoint also makes it clear that the conditional approach can only outperform the simultaneous approach when some null p-values are conservative (i.e., they are super-uniform). We give a heuristic argument here, and a more formal statement in Appendix A.1. Suppose one of our p-values p_1 is a very strong signal (so it is very small with high probability) but the remaining p-values $p_2, \ldots, p_n$ are null p-values that are uniform (i.e., they are not conservative). Our conditional procedure will reject when our smallest p-value, likely p_1, is less than α times the smallest of $p_2, \ldots, p_n$. The minimum of these $n-1$ uniform p-values is $1/n$ on average. Hence, roughly speaking, the conditional approach also rejects when $p_{(1)}$ is less than α/n, which is essentially the same as Sidak's simultaneous approach when n is large.

3.1.1 Confidence regions for the winner

In parametric problems, we can invert Corollary 1's test to get selective confidence regions for the winning parameter. Consider observing independent data $X_i \sim P_{\theta_i}$ from an MLR family P_θ parametrized by $\theta \in \mathbb{R}$. Let $p_i^{\theta_0}$ (which is a function of X_i) be the UMP p-value for testing the null $H_0 : \theta_i \leq \theta_0$. Details regarding these p-values can be found in Appendix B.1. We can define the winner $W = \text{argmin}_{j \in [n]} p_j^{\theta_0}$ to be the index of the smallest and most promising p-value. This winning index will be the same irrespsective of θ_0 [2]. By inverting Corollary 1's test, we get an LCB

$$\{\theta_0 : p_{(1)}^{\theta_0} / p_{(2)}^{\theta_0} > \alpha\} \tag{8}$$

[2]If we use the same auxiliary randomness to compute $p_i^{\theta_0}$ for every θ_0, then one index will result in the smallest p-value for every θ_0 and W is well-defined. The discussion in Appendix B.1 implies that this will be the case.

for the winning parameter θ_W that holds conditionally on W with probability exactly $1-\alpha$:

$$P(\theta_W \in \{\theta_0 : p^{\theta_0}_{(1)}/p^{\theta_0}_{(2)} > \alpha\}|W) = 1-\alpha.$$

The fact that the confidence region (8) is actually an LCB is a consequence of the selective p-value $p^{\theta_0}_{(1)}/p^{\theta^0}_{(2)}$ being monotone non-decreasing in null parameter θ_0. Appendix B.3 provides general conditions under which selective p-values like $p^{\theta_0}_{(1)}/p^{\theta^0}_{(2)}$ are monotone in the null parameter. We show these conditions apply to the winner problem, and also argue that (8) has exact $1-\alpha$ coverage in **??**. We also show in **??** how to invert Corollary 1's test to get a CI (rather than LCB), and that both our CI and LCB match Fithian et al. [2017]'s approach in the Gaussian case.

Writing the conditional inference in terms of p-values helps us better understand Fithian et al. [2017]'s conditional LCB (8). In particular, we learn that **??**'s LCB stretches back exactly to the $\hat{\theta}$ under which it is $1/\alpha = 10$ times less likely to see something as extreme as the winner than something as extreme as the runner-up. Figure 3 provides an illustration for the Gaussian case. It demonstrates why why the LCB diverges to $-\infty$ as the winner and runner-up get closer. If the winner and runner-up are very close, we will need the LCB $\hat{\mu}$ to be very far back for the winner to be ten times more extreme than the runner-up. Thanks to the decay of the Gaussian tail, however, we can always find such a mean if we go far back enough.

For non-Gaussian data, the amount the conditional LCB (8) stretches back depends on the tail decay of P_θ. The faster the tail decays, the larger the first $\hat{\theta}$ for which the winner is $1/\alpha = 10$ times as extreme as the runner-up. Seeing as the Gaussian distribution, whose tail shrinks as e^{-x^2}, still often results in very low lower bounds, we should expect that the distance from the winning observation to the lower bound will often be quite large in many settings.

As an example, consider observing independent exponential random variables $X_i \sim \text{Exp}(\lambda_i)$. The exponential distribution has a tail e^{-x} that decays fast, but not as fast as the Gaussian tail. Crucially, it also has a parameter space $\lambda \in (0, \infty)$ that is bounded below. The exponential distribution has an MLR in $T(x) = 1/x$, so the UMP test for $H^{\lambda_0}_{0,i} : \lambda_i \leq \lambda_0$ uses a p-value $p^{\lambda_0}_i$ that rejects when X_i is small. It turns out that

$$\lim_{\lambda^0 \downarrow 0} p^{\lambda_0}_{(1)}/p^{\lambda_0}_{(2)} = X_{(1)}/X_{(2)},$$

so the conditional LCB for the winning parameter λ_W,

$$\{\lambda_0 : p^{\lambda_0}_{(1)}/p^{\lambda_0}_{(2)} > \alpha\}, \tag{9}$$

is vacuous whenever $X_{(1)}/X_{(2)} > \alpha$, i.e., with positive probability the confidence region (8) spans the whole parameter space $(0, \infty)$. A careful derviation of this test and result can be found in Appendix A.5.

The failure of the conditional LCB (9) manifests in real data examples. On a dataset of car engine failure times [Molotaliev, 2024], we find that the conditional LCB (9) is always vacuous. The dataset has the failure times of one-hundred car engines, which we model as independent exponential random variables. Over many subsamples of just $n = 2$ failure times, the LCB (8), which does inference on the worse of the two engines, always gives a vacuous lower bound of zero. In contrast, the simultaneous approach always gives a non-vacuous lower bound. Figure 4 depicts the results. The result is concerning. Despite conditional LCB having exact $1-\alpha = 0.9$ coverage, our empirical coverage is perfect (the vacuous LCB must always cover the parameter). This suggests that the exponential distribution is likely not an appropriate model for this dataset, even though it is often a natural choice for modeling failure times.

3.1.2 More discoveries via the closure principle

Once written in terms of p-values, it is natural to treat Corollary 1's test as a test of the global null and try to close it (as in Marcus et al. [1976]). Closing a global null test precludes us from making confidence regions, but it allows us to make more individual discoveries. Closed global null testing procedures are often computationally intractable to implement, so it is interesting that Corollary 1's global null test admits a tractable closure.

Corollary 2 (Closed testing for winners)**.** *Suppose that p_i are n independent and selectively dominant p-values under the nulls $H_{0,i}$. As shorthand, let $H_{0,(j)}$ denote the null corresponding to the jth smallest p-value*

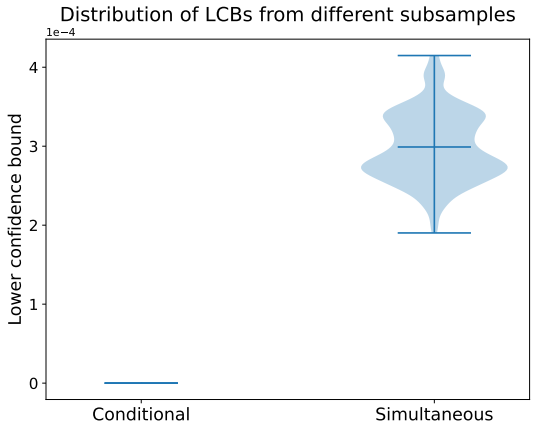


Figure 4: Over $B = 1000$ different subsamples of $n = 2$ failure times from the dataset Molotaliev [2024], the distribution of the LCB for the "winning" parameter resulting from the conditional and simulatneous approaches. The conditional LCB is always vacuous.

(ties broken randomly) and define $p_{(n+1)} = 1$. *Rejecting the null hypotheses* $H_{0,(k)}$ *for which* $p_{(j)} \leq \alpha p_{(j-1)}$ *for every* $j \leq k$ *controls FWER error at level* α.

As is often the case for closed procedures, Corollary 2 procedure is best understood sequentially. We reject $H_{0,(1)}$ if $p_{(1)} \leq \alpha p_{(2)}$. Then, if we rejected $H_{0,(1)}$, we reject $H_{0,(2)}$ if $p_{(2)} \leq \alpha p_{(3)}$, so on and so forth until we fail to reject.

3.2 Hybrid Inference

Hybrid inference, originally proposed by Andrews et al. [2023], is an inference on winners procedure that attempts to balance the benefits of the simultaneous and conditional approaches. It is a very elegant idea, but it currently only applies to Gaussian data and can be difficult to parse and implement. Using our selective dominance framework, we give a simpler exposition of hybrid inference that enables its application in more general settings, provided that the data is independent. As a bonus, our new procedure is very easy to undestand and implement.

Corollary 3 presents our hybrid testing procedure. We give the sketch of a proof and defer a detailed proof to Appendix D.3.

Corollary 3 (Hybrid test for the winner)**.** *Suppose that* p_i *are* n *independent and selectively dominant p-values under the nulls* $H_{0,i}$*, let* W *be the index of the smallest p-value, and fix some* $\beta < \alpha$. *Rejecting* $H_{0,W}$ *when*

$$p_{(1)} \leq \frac{\alpha - \beta}{1 - \beta} p_{(2)} + \left(1 - \frac{\alpha - \beta}{1 - \beta}\right) \beta_n \tag{10}$$

controls Type I error at level α.

Proof sketch. Let B be the event that the smallest p-value comes from a null and is at most β_n. We know from Sidak's procedure that $P(B) \leq \beta$. Hence, on the complementary event B^c, which has probability $\geq 1 - \beta$, it suffices to ensure that we fail to falsely reject $H_{0,W}$ with probability at least $(1 - \alpha)/(1 - \beta)$. Supposing $H_{0,j}$ is true, imagine testing $H_{0,j}$ using p_j only when B^c happens and $W = j$. This is exactly like selecting p_j to use for inference when it is between β_n and $\max_{i \neq j} p_i$. For this selection, Theorem 1's selective p-value is given by $(p_j - \beta_n)/(\max_{i \neq j} p_i - \beta_n)$, so we can ensure that we fail to reject $H_{0,j}$ when B^c and $W = j$ happen with probability at least $(1 - \alpha)/(1 - \beta)$ if we fail to reject whenever

$$\frac{p_j - \beta_n}{\max_{i \neq j} p_i - \beta_n} > 1 - \frac{1 - \alpha}{1 - \beta} \iff p_j > \frac{\alpha - \beta}{1 - \beta} \max_{i \neq j} p_i + \left(1 - \frac{\alpha - \beta}{1 - \beta}\right) \beta_n.$$

Our hybrid inference procedure fails to reject in this case. □

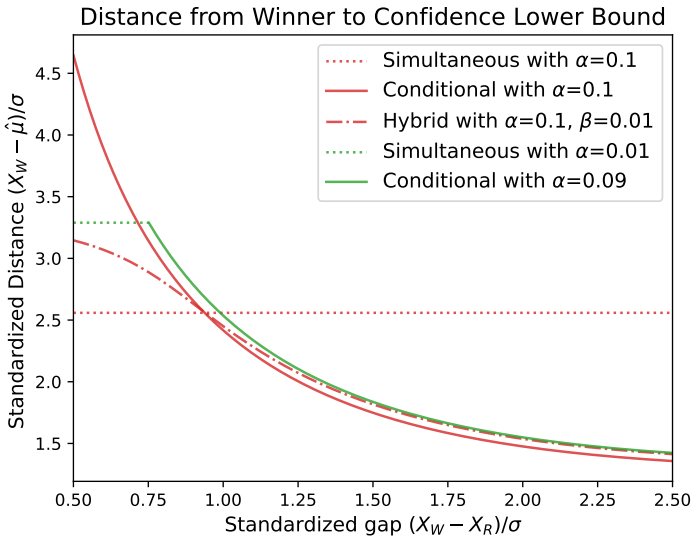


Figure 5: For the $n = 20$ dimensional Gaussian problem $X_i \sim N(\mu_i, \sigma^2)$ with largest observation X_W and second largest observation X_R, the standardized distance $(X_W - \hat{\mu})/\sigma$ from X_W to the level $\alpha = 0.1$ hybrid, conditional, and simultaneous LCB $\hat{\mu}$ as a function of the standardized gap $(X_W - X_R)/\sigma$ between the winner and runner-up. For hybrid we take $\beta = 0.01$. The larger of the level $\alpha = 0.09$ conditional LCB and level $\alpha = 0.01$ simultaneous LCB are shown in green (i.e., the union bound LCB).

Written in terms of p-values, it is easy to see how the hybrid approach balances the benefits of the simultaneous and conditional approaches. It will reject both when the smallest p-value is small in absolute terms or when it small relative to the second smallest p-value. When the other p-values provide essentially no evidence against the null (i.e., $p_{(2)} \approx 1$), hybrid always rejects when $p_{(1)}$ is less than $(\alpha - \beta)/(1 - \beta)$, a cutoff that has no dependence on the problem dimension n. In this situation, it performs at least on par with the conditional procedure run at level $(1 - \alpha)/(1 - \beta)$. On the other hand, even if some other p-value provides as much evidence against the null as the smallest, hybrid still rejects whenever the level β simultaneous approach does. This is because when $p_{(1)} \leq \beta_n$, the hybrid cutoff is a mixture of two things that are larger than $p_{(1)}$, and we will reject.

The parameter β allows hybrid inference to interpolate between the simultaneous and conditional approaches. When we set $\beta = 0$ then the hybrid cutoff (10) becomes $\alpha p_{(2)}$ and we recover the conditional method, and if we set $\beta = \alpha$ it becomes α_n and we recover the simultaneous method.

3.2.1 Confidence Regions

In parametric settings, we can get hybrid confidence regions for the winning parameter by inverting Corollary 3's test. Again suppose we have independent data $X_i \sim P_{\theta_i}$ from an MLR family P_θ parametrized by $\theta \in \mathbb{R}$, and let $p_i^{\theta_0}$ be the UMP p-value for testing the null $H_{0,i}^{\theta_0} : \theta_i \leq \theta_0$. By inverting Corollary 3's test we get a hybrid LCB for the winning parameter θ_W:

$$\left\{\theta_0 \in \mathbb{R} : p_{(1)}^{\theta_0} > \frac{\alpha - \beta}{1 - \beta} p_{(2)}^{\theta_0} + \left(1 - \frac{\alpha - \beta}{1 - \beta}\right) \beta_n\right\}. \tag{11}$$

We argue in Appendix A.3 that the confidence region (11) indeed gives a LCB. We also show in Appendix A.3 how to invert Corollary 3's test to get a CI (rather than an LCB), and that both our CI and LCB match the original construction from Andrews et al. [2023] in the Gaussian case.

3.2.2 Comparison to the Union Bound

Another way to balance the benefits of the conditional and simultaneous approaches is to apply a union bound. Naively, we can reject the winning null whenever the level β simultaneous approach rejects or the

level $\alpha - \beta$ conditional approach rejects, i.e., whenever

$$p_{(1)} \leq \max\{(\alpha - \beta)p_{(2)}, \beta_n\}. \tag{12}$$

The union bound harshly switches between the simultaneous and conditional approaches, whereas the hybrid approach smoothly interpolates between them. This is illustrated in Figure 5, which compares the LCBs resulting from the hybrid versus union bound approaches in the $n = 20$ dimensional Gaussian problem.

Written in terms of p-values, we easily see that the hybrid approach dominates the union bound approach, which is affirmed by Figure 5. Both methods reject when $p_{(1)} \leq \beta_n$. But, when $p_{(1)} > \beta_n$, it is easy to verify that the hybrid cutoff (10) will be strictly larger than the union bound cutoff (12), meaning hybrid will reject whenever the union bound does and more [3].

Practically speaking, however, hybrid inference does not result in much improvement over the union bound, especially as it pertains to making discoveries. This is already somewhat evident in Figure 5, where we see that the hybrid LCB, although always larger than the union bound LCB, is still always very close to it. As the variance σ^2 gets large, the absolute difference $\hat{\mu}_{hyb} - \hat{\mu}_{union}$ between the hybrid and union bound LCBs grows with σ, but the relative difference $(\mu_{hyb} - \hat{\mu}_{union})/\sigma$ (in units of standard deviation) remains the same (see Appendix A.3.3). Accordingly, even when the hybrid cutoff (10) is larger than that of the union bound (12), it is provably not much larger. We detail why in Appendix A.4, where we also run a number of simulations comparing the power of the hybrid and union bound approaches. In our simulations, we are unable to find a setting where the hybrid approach results in a appreciable power gain.

Overall, we suggest viewing hybrid inference as a procedure that squeezes the remaining power out of the union bound approach. As it is not computationally more expensive and our p-value viewpoint makes it equally easy to implement, it is always worth using in place of the union bound.

3.2.3 Applying the Closure Principal

Like was true in the conditional case, treating Corollary 3's test as a global null test and closing it allows us to make more discoveries. As we allow β to range from 0 to α, this closed procedure interpolates between Corollary 2's closed procedure and the Holm-Sidak procedure, which is the closure of Sidak's global null test.

Corollary 4 (Closed hybrid testing for winners)**.** *Suppose that p_i are n independent and selectively dominant p-values under the nulls $H_{0,i}$. As shorthand, let $H_{0,(j)}$ denote the null corresponding to the jth smallest p-value and define $p_{(n+1)} = 1$. Fixing some $\beta < \alpha$, rejecting the null hypotheses $H_{0,(k)}$ for which*

$$p_{(j)} \leq \frac{\alpha - \beta}{1 - \beta} p_{(j-1)} + \left(1 - \frac{\alpha - \beta}{1 - \beta}\right) \beta_{n-j+1}$$

for every $j \leq k$ controls FWER error at level α.

This closed procedure is also best understood sequentially. We reject $H_{0,(1)}$ if $p_{(1)} \leq \frac{\alpha-\beta}{1-\beta} p_{(2)} + (1 - \frac{\alpha-\beta}{1-\beta})\beta_n$. Then, if we rejected $H_{0,(1)}$, we reject $H_{0,(2)}$ if $p_{(2)} \leq \frac{\alpha-\beta}{1-\beta} p_{(3)} + (1 - \frac{\alpha-\beta}{1-\beta})\beta_{n-1}$, so on and so forth until we fail to reject.

4 Rank Verification in Exponential Families

In this section we consider the problem of verifying that that the winning parameter is actually larger than the other parameters, i.e., rather than doing inference on the winning parameter, we do inference on the gap between the winning and remaining parameters. Mainly, we give an account of the seminal work Hung and Fithian [2019] in our selective dominance framework. We show, however, that Hung and Fithian [2019] do not correctly handle cases where there can be ties for the winner. This is a subtle point, but it is a mistake that is easy to avoid when using our selective dominance framework.

Overall, the section serves to illustrate how our selective dominance machinery provides a straightforward way to correctly design intricate and counter-intuitive selective procedures. For examples of how one may apply these methods, we refer the reader to the original article Hung and Fithian [2019], where there are many.

[3] the authors Andrews et al. [2023] only point out that hybrid dominates the level β classical approach, which is weaker than our statement

4.1 Warm-up: Rank Verification and Type III Error Control

To motivate the rank verification problem and shed some light on its relationship with selective dominance, we consider a seemingly unrelated classical statistical question about Type III errors.

A researcher wants to test if the unknown means of two univariate Gaussian samples, $X_1 \sim N(\mu_1, 1/\sqrt{2})$ and $X_2 \sim N(\mu_2, 1/\sqrt{2})$, are different. They end up rejecting the null hypothesis $H_0 : \mu_1 = \mu_2$ because the two-sided p-value $2(1 - \Phi(|X_1 - X_2|))$ they learned from introductory statistics is at most α. After rejecting, they note $X_1 > X_2$, and claim "not only are the two means are different, but they must be different because μ_1 is bigger than μ_2". Your friend, however, only rejected the null that the means are equal. Can they make a claim about the direction of inequality? This is a quesiton of Type III error, and we can use our selective dominance framework to show that the researcher's claim is actually statistically valid.

Based on the claim, it seems that what the researcher really wants to do is test the one-sided null $H_{0,12} : \mu_1 \leq \mu_2$ whenever they observe that $X_1 > X_2$, and test the complementary one-sided null $H_{0,21} : \mu_2 \leq \mu_1$ whenever they observe that $X_2 < X_1$. To test the null $H_{0,ij} : \mu_i \leq \mu_j$ we normally use the UMP p-value $p_{ij} = 1 - \Phi(X_i - X_j)$. In the researcher's case, however, they only select this p-value to use for inference when they observe that $X_i > X_j$, or equivalently that $p_{ij} < 1/2$. Since the p_{ij} are selectively dominant, Theorem 1 tells us that, when using p_{ij} to test $H_{0,ij}$, the researcher should correct for this selection and reject when $2p_{ij} \leq \alpha \iff p_{ij} \leq \alpha/2$. Letting W be the index of the winner and R of the runner-up, the final procedure is to reject $H_{0,WR}$ when $p_{WR} \leq \alpha/2$.

The approach described above verifies the rank of the winning mean with Type I error control, i.e., it affirms not just that the means are different, but that the mean of the winning observation is the larger of the two. The procedure, which rejects when the smaller of the two one-sided p-values is at most $\alpha/2$, is identical to the procedure that rejects when the above two-sided p-value is at most α. Hence, for reasons likely unbeknowst to them, the researcher's original claim is indeed statistically valid. We walk through deriving this procedure much more carefully (with as much detail as we did in Example 7) in Appendix A.8.

It turns out that, for the n-dimensional Gaussian problem (we considered the 2-dimensional problem up to this point), it has long been known that verifying the winner's rank by running a one-sided test comparing the winner and runner-up at level $\alpha/2$ controls Type I error [Gutmann and Maymin, 1987]. Hung and Fithian [2019] claim further that the test can be run at level $n/(n-1) \cdot \alpha/2$, however this claim is false. By taking $\mu_1 = \mu_2$ and $\mu_3 = \cdots = \mu_n = -\infty$, it is not hard to verify that the Type I error one gets by running the test at level $\alpha/2$ is exactly α. Hence, the error cannot be inflated any further.

Interestingly, Hung and Fithian [2019]'s claim becomes true if, rather than wanting to verify that the winning mean is strictly bigger than the other mean, we want to verify that it is at least as big. Again let us focus on the 2-dimensional case. If, instead of testing the null $H_{0,WR} : \mu_W \leq \mu_R$, we test the null $H_{0,WR} : \mu_W < \mu_R$, then we can indeed run the one-sided test comparing the winner and runner-up at level α instead of level $\alpha/2$. Assuming without loss of generality that $\mu_1 \geq \mu_2$, the proof is straightforward:

$$
\begin{aligned}
&P_{\mu_1,\mu_2}(\text{falsely reject } H_{0,WR}) \\
&= P_{\mu_1,\mu_2}(p_{21} < \alpha | W = 2) P(W = 2) I_{\mu_2 < \mu_1} && \text{(no false rejection when } W = 1 \text{ or } \mu_2 = \mu_1) \\
&\leq P_{\mu_1,\mu_2}(2p_{21} < 2\alpha | p_{21} \leq 1/2) \cdot \frac{1}{2} \cdot I_{\mu_2 < \mu_1} && (X_1 \text{ wins w.p. at least } 1/2\ ) \\
&\leq 2\alpha \cdot \frac{1}{2} \cdot I_{\mu_2 < \mu_1} \leq \alpha. && \text{(selective dominantce)}
\end{aligned}
$$

As rejection probabilities are typically continuous in the parameter space, it is counter-intuitive that excluding the boundary of the null makes a tangible difference. But, since the null hypothesis is data-dependent, the false rejection region is a highly discontinuous function of the parameters, which elicits this behavior. For example, under the data-dependent null $H_{0,WR} : \mu_W < \mu_R$, the false rejection region is empty when $\mu_1 = \mu_2$, but whenever μ_1 is even ϵ larger than μ_2, the false rejection region $X_2 - X_1 > z_{1-\alpha}$ becomes highly non-trivial. Under the null $H_{0,WR} : \mu_W \leq \mu_R$ the false rejection $|X_1 - X_2| > z_{1-\alpha}$ is as large as possible when $\mu_1 = \mu_2$, which prevents us from inflating the level of the test. Fixing $\mu_2 = 0$, Figure 6 considers both nulls $H_{0,WR} : \mu_W < \mu_R$ and $H_{0,WR} : \mu_W \leq \mu_R$ and plots the Type I error of the level α one-sided test comparing the winner and runner up for different values of μ_1. The behavior at the discontinuity $\mu_1 = 0$ illustrates why we have Type I error control for $H_{0,WR} : \mu_W < \mu_R$ but not for $H_{0,WR} : \mu_W \leq \mu_R$.

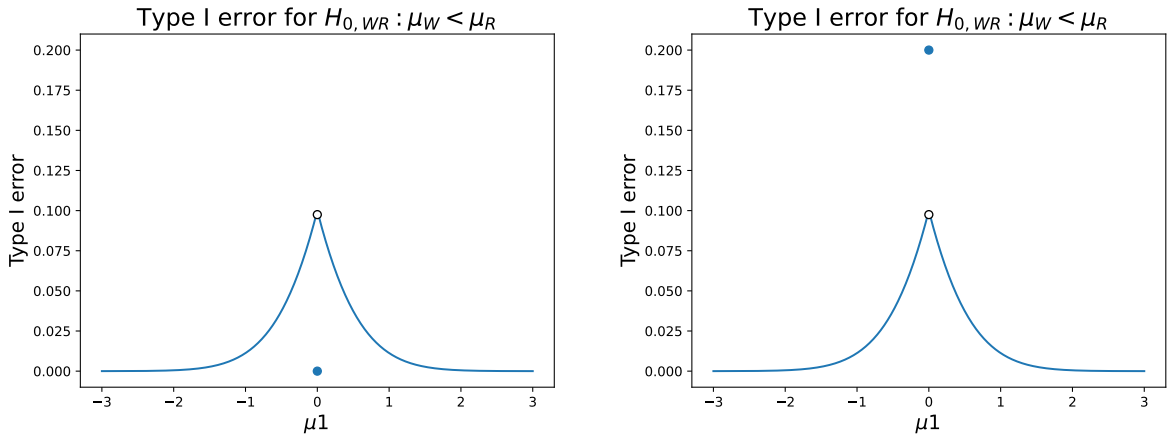


Figure 6: For $\mu_2 = 0$ and different μ_1, the type I error of rejecting $H_{0,WR} : \mu_W - \mu_R < 0$ and $H_{0,WR} : \mu_W - \mu_R \leq 0$ when the level α one-sided test comparing the winner of $X_1 \sim N(\mu_1, 1/\sqrt{2})$ and $X_2 \sim N(\mu_2, 1/\sqrt{2})$ to the runner-up rejects.

A nice implication of our above discussion is the following surprising fact: if we want to to verify that the winning mean amongst $X_1 \sim N(\mu_1, 1/\sqrt{2})$ and $X_2 \sim N(\mu_2, 1/\sqrt{2})$ is at least as large as the other mean, we can run a one-sided test comparing the winner to the runner up at level α, i.e., with no correction, and still mantain Type I error control.

4.2 Rank Verification in Exponential Families

In this sub-section we illustrate how to do rank verification when we observe data $X \in \mathbb{R}^n$ from a natural multiparameter exponential family P_θ with density

$$g_\theta(x) = \exp(\theta_1 x_1 + \cdots + \theta_n x_n - \psi(\theta))g(x), \tag{13}$$

with repsect to some base measure. [4]. Like Hung and Fithian [2019], we want to verify that θ_W is larger than the remaining θ_j, where W is the index of the largest X_i. In case of ties, we follow Hung and Fithian [2019]'s lead, and set W to randomly be one of the winning indices. Since $E_\theta[X_i] = \theta_i$, the winning parameter θ_W may reasonably be the largest of the θ_i.

Considering the nulls $H_{0,ij} : \theta_i \leq \theta_j$, we want to reject the data-dependent null $\cup_{j \neq W} H_{0,Wj}$ and affirm that θ_W larger than any other parameter [5]. Our strategy will be to come up with a valid test for $H_{0,Wj}$ for all $j \neq W$, and then reject $\cup_{j \neq W} H_{0,Wj}$ whenever we reject the tests $H_{0,Wj}$ for all $j \neq W$. Fixing $i \neq j$, we start by constructing the UMPU p-value p_{ij} for testing $H_{0,ij} : \theta_i - \theta_j \leq \delta$. Ultimately, we will only use this p-value to test $H_{0,ij}$ when i is selected as our winning index, and we will correspondingly adjust the p-value to account for this selection.

Defining the transformed sufficient statistics $Y \in \mathbb{R}^n$ by

$$Y_i = \frac{X_i - X_j}{2}, \qquad Y_j = \frac{X_i + X_j}{2}, \qquad Y_\ell = X_\ell \text{ for } \ell \neq i, j, \tag{14}$$

the random vector Y has an exponential family density given by (15) as

$$\tilde{g}_\theta(y) = \exp\left((\theta_i - \theta_j)y_i + (\theta_i + \theta_j)y_j + \sum_{\ell \neq i,j} \theta_\ell y_\ell - \psi(\theta)\right) \tilde{g}(y) \tag{15}$$

[4] We assume that $g(x)$ is symmetric and $T_i(x) = x_i$ so that our discussion more closely mirrors Hung and Fithian [2019], although we do not need to. Hung and Fithian [2019] also assume that $g(x)$ is Schur concave for other purposes, but we leave this assumption out.

[5] If we performed the same analysis for the nulls $H^\delta_{0,ij} : \theta_i - \theta_j \leq \delta$ then we could verify that θ_W is more than δ larger than any other θ_j. Inverting these tests would result in a LCB for the difference $\theta_W - \max_{j \neq W} \theta_j$ between the winning and next largest parameter.

with respect to some other base measure. It is then well established (see Appendix B.1) that if we denote the conditional left-continuous survival function of Y_i and its righthand limit as

$$G_{ij}(y_i|y_{-i}) = P_{\theta_i=\theta_j}(Y_i \geq y_i|Y_{-i} = y_{-i}) \qquad G^+_{ij}(y_i|y_{-i}) = \lim_{u \downarrow y_i} G_{ij}(u|y_{-i}), \tag{16}$$

then the UMPU p-value p_{ij} for testing $H_{0,ij} : \theta_i \leq \theta_j$ is given by

$$p_{ij} = G^+_{ij}(Y_i|Y_{-i}) + U_{ij,aux}(G_{ij}(Y_i|Y_{-i}) - G^+_{ij}(Y_i|Y_{-i})), \tag{17}$$

where $U_{ij,aux}$ are Unif($[0, 1]$) random variables that are independent from each other and the data. By Example 4, the p-value p_{ij} is selectively dominant given Y_{-i}.

Crucially, we can tell if X_i is a winner by exmaining the p-value p_{ij}. It is straightforward to confirm that X_i is the sole winner exactly when $Y_i > \max_{k \neq i} Y_k - Y_j$. Equivalently, this happens when p_{ij} is strictly smaller than

$$q^+_{ij}(Y_{-i}) = G^+_{ij}(\max_{k \neq i} Y_k - Y_j|Y_{-i}). \tag{18}$$

Likewise, one can confirm that X_i is one of multiple winners exactly when $Y_i = \max_{k \neq i} Y_k - Y_j$, or equivalently when p_{ij} is at least q^+_{ij} but at most

$$q_{ij}(Y_{-i}) = G_{ij}(\max_{k \neq i} Y_k - Y_j|Y_{-i}). \tag{19}$$

Moreover, in the case that there are multiple winners, the number of winners is also a deterministic function of Y_{-i}:

$$N_i(Y_{-i}) = 1 + |\{\ell \neq i : Y_\ell = \max_{k \neq i} Y_k\}|. \tag{20}$$

Note that $N_i(Y_{-i})$, which is always at least two, is <u>not</u> the same as the number of winners, which can be one. Rather, it is the number of winners there will be if X_i is a winner and at least one other X_k is as well (see Appendix A.6 for details).

Leveraging these facts, we use selective dominance framework to come up with a valid test for $H_{0,Wj}$. Essentially, we use p_{ij} to test $H_{0,ij}$ with probability one when it is less than q^+_{ij}, and with probability $1/N_i$ (we randomly select one of the N_i winners) when it is between q^+_{ij} and q_{ij}. Explicitly, letting $p = p_{ij}$ and $Z = Y_{-j}$, we can apply our framework with the selection function

$$s(x, z) = \begin{cases} 1 & \text{if } x < q^+_{ij}(z), \\ \frac{1}{N_i(z)} & \text{if } x \in [q^+_{ij}(z), q_{ij}(z)] \\ 0 & \text{otherwise} \end{cases}.$$

This is a piece-wise linear function that is easy to integrate, and computations detailed in Appendix A.7 show that the corresponding selective p-value from Theorem 1 is given by

$$\frac{p_{ij} - \left(1 - \frac{1}{N_i}\right)(p_{ij} - q^+_{ij})_+}{q^+_{ij} + \frac{1}{N_i}(q_{ij} - q^+_{ij})}. \tag{21}$$

The crucial difference between our derivation and Hung and Fithian [2019]'s is that Hung and Fithian [2019] use the same selective p-value when there can and cannot be ties amongst the X_k. If there cannot be ties amongst the X_k, then $q_{ij} = q^+_{ij}$ always, and the selective p-value (21) simplifies to p_{ij}/q_{ij}. If we use p_{ij}/q_{ij} when ties are possible, however, we will not achieve conditional error control (as is claimed in Hung and Fithian [2019]). We give an example in Figure 7, where $X \in \mathbb{R}^3$ is composed of three independent binomials. The left panel of Figure 7 depicts the conditional distribution of p_{12} given $W = 1$ for a specific setting of the nusiances statistics Y_{-j}. It makes it clear that rejecting when $p_{ij}/q_{ij} \leq \alpha$ does not maintain conditional error control, as is affirmed in Figure 7's right panel.

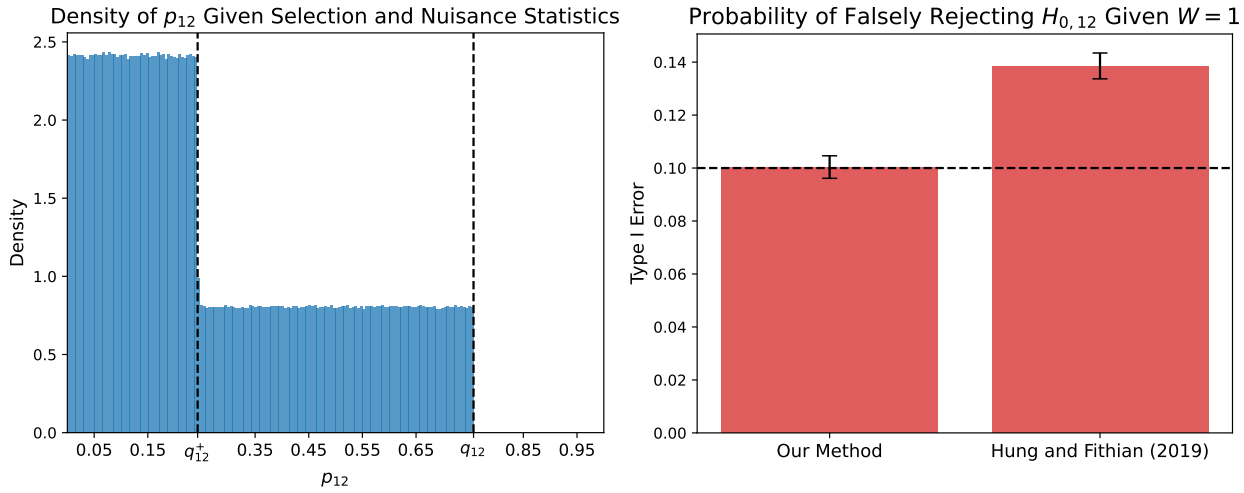


Figure 7: Considering three independent binomials $X_i \sim \text{Bin}(b, s_i)$ with $b = 4$ and $s_i = 1/2$, the first panel (left) depicts $N = 10^6$ draws from the conditional distribution of the p-value p_{12} (used for testing $H_{0,12} : s_1 \leq s_2$) given $W = 1$ and the nuisance statistics $(X_1 + X_2)/2 = 2$, $X_3 = 2$. When $p_{12} < q_{12}^+$, then X_1 is the sole winner and the p-value is selected for inference with probability one, but when $p_{12}^0 \in [q_{12}^+, q_{12}]$ there is a three-way tie and it is selected for inference with probability $1/3$. Hence, the p-value's conditional distribution is not uniform on $[0, q_{12}]$ as Hung and Fithian [2019] implicitly assume. The next panel (right) displays the consequence. Conditional on $W = 1$, Hung and Fithian [2019] do not maintain Type I error control when testing H_{W2}^0 at level $\alpha = 0.1$ (denoted by horizontal dashed line), whereas our method does. Error bars denote a 99% confidence interval.

5 Combining selective p-values

In this section we illustrate how our selective dominance allows us to combine inferences across many p-values, even post-selection. Rather than just selecting just one p-value as in Section 2, some of this section's methods select multiple p-values to use for inference. As such, we need to slightly generalize our framework from Section 2. The generalization is intuitive, and for sake of brevity we have deferred a formal account of it to Appendix C. The validity of the methods we propose in this section are a direct consequence of the discussion in Appendix C's.

5.1 Publication bias corrected meta-analysis

Performing replication studies is a crucial part of the scientific process, especially when one is wary that the original study may suffer from publication bias. To judge the prevalence of publication bias in pyschology, the open science collaboration conducted a mass replication analysis of pyschology studies [Collaboration, 2015]. Via their efforts, we have access to p-values from 92 pairs of original and replication pyschology studies [6], depicted in Figure 8. We refer to p-values from the original study as p_O and p-values from the replication study as p_R. The p-values p_O from all the original studies are significant at the $\alpha = 0.05$ level, while only 34 of the replication p-values p_R are significant.

Although the original study p-values suffer from publication bias, they still contain valuable and usable information. By Example 6 and its subsequent discussion, p_O/α should still be a valid p-value despite any publication bias (or p-hacking). Via Fisher's combination test, we can use both the corrected original p-value p_O/α and uncorrected replication p-value p_R for inference. As a refresher, Fisher's test considers n independent p-values p_i for the nulls $H_{0,i}$ and rejects the global null $\cap_{i=1}^n H_{i,0}$ when the test statistic $-2\sum_{i=1}^n \log(p_i)$ is at least as large as the $1-\alpha$ quantile of the χ^2_{2n} distribution. In our case, we have two independent p-values p_O/α and p_R that test the same null hypothesis, and we can reject this null hypothesis

[6] We exclude seven studies whose original p-value p_O is larger than $\alpha = 0.05$.

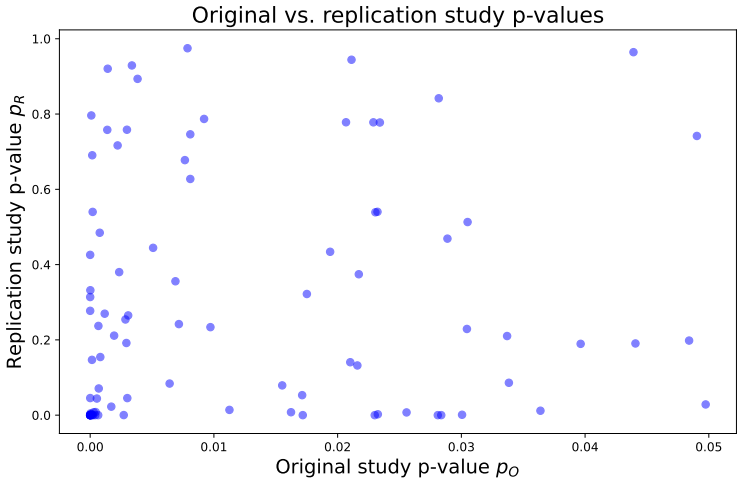


Figure 8: Scatter plot of the original p_O and replication p_R p-values for 92 pyschology studies from the open science collaboration's replication analysis Collaboration [2015]. Note that the x-axis, which ranges from $[0, 0.05]$ is on a different scale than the y-axis, which ranges from $[0, 1]$.

when

$$-2(\log(p_O/\alpha) + \log(p_R)) \geq \text{Quantile}(1-\alpha, \chi^2_4).$$

This approach is analogous to data-carving. We use part of our data for selection (the original study) and part for inference (the replication study), but we still use the information remaining in the the first part after selection for inference as well. Unlike existing approaches to data-carving, which are often complex and problem specific, Fisher's combination test in junction with our selective dominance framework provides a general and simple way to data-carve.

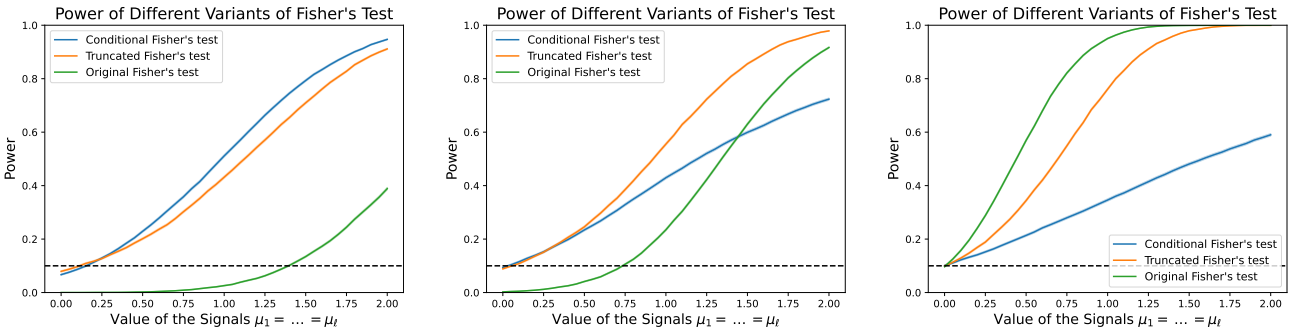
On the open science dataset, our combination approach allows us to make more powerful inferences from the replication studies. Our approach finds that 47 study pairs have signicant findings, whereas solely using the original study p-value p_O/α or the replication study p-value p_R results in only 39 or 34 significant findings respectively. It is suprising that, even after a harsh adjustment for publication bias, the corrected original study p-values result in more discoveries than replication study p-values. It is hard to gauge if this is due to chance, differences between the original and replication studies (e.g., minor differences in population demographics, devices used for measurement, sample size), or because there is somehow even harsher selection bias in the original studies than what we have accounted for. It may even be possible that some replicators felt incentivized to induce bias in the opposite direction, and tried to ensure that the replication studies were not significant.

5.2 Adaptive Versions of Fisher's Combination Test

By employing similar ideas to the previous sub-section, we can come up with variants of Fisher's combination test that are more powerful when some null p-values are conservative (i.e., super-uniform). Corollary 5, which is of similar flavor to the conditional inference on winners procedure, gives a conditional version of Fisher's combination test that only uses the top k p-values for inference. For this procedure, k must be fixed beforehand.

Corollary 5 (Fisher's top-k combination test)**.** *Suppose that p_i are n independent and selectively dominant p-values under the nulls $H_{0,i}$ and let $H_{0,(j)}$ denote the null corresponding to the jth smallest p-value. For some fixed $1 \leq k \leq n$, rejecting the data-dependent global null $\cap_{j=1}^{k} H_{0,(j)}$ (and therefore also the global null $\cap_{i=1}^{n} H_{0,i}$) when*

$$-2\sum_{j=1}^{k} \log(p_{(j)}/p_{(k+1)}) \geq \textit{Quantile}(1-\alpha, \chi^2_{2k}) \tag{22}$$



(a) $\ell = 3$ (b) $\ell = 5$ (c) $\ell = 10$

Figure 9: For $\ell = 3$ (left), $\ell = 5$ (middle), and $\ell = 10$ (right), power of the top $k = 3$ conditional, $\tau = 0.5$ truncated, and original Fisher's combination test for data drawn from $N(\mu, I_{10})$ with $\mu_1 = \cdots = \mu_\ell$ varying according to the x-axis and $\mu_{\ell+1} = \cdots = \mu_n = -2$. Power results from an average over $N = 10^4$ trials, bands denote one standard error (barely visible), and the level $\alpha = 0.1$ is denoted by the dashed line.

controls Type I error at level α conditional on the indicies of the smallest k p-values (and therefore also marginally).

Examining (22), we see that when the $(k+1)$st p-value provides essentially no evidence against the null (so $p_{(k+1)} \approx 1$), running Corollary 5's test is like running Fisher's test using just the top k p-values and ignoring that any selection took place. In this case, our test statistic will have an essentially identical value to Fisher's original test statistic, but the critical value required for rejection will be much smaller. On the flipside, if many of the $p_{(j)}$ for $j \leq k$ are not sufficiently smaller than $p_{(k+1)}$, then Corollary 5's test statistic will be small and the test will fail to reject.

Another approach to improving Fisher's combination test is truncation: only use a p-value for inference if it is below some fixed threshold $\tau \in \mathbb{R}$ [Zaykin et al., 2002]. Past and previous work, however, only establishes validity of this test when the null p-values have exact Unif[0, 1] distributions Zaykin et al. [2002], Zhang et al. [2020]. Corollary 6, however, gives a version of Fisher's truncated combination test that is still valid whenever the p-values are independent and selectively dominant [7]

Corollary 6. *Suppose that p_i are n independent and selectively dominant p-values under the nulls $H_{0,i}$ and fix n thresholds $\tau_i \in [0, 1]$. Letting $j \in J$ denote the random set of indices for which $p_j \leq \tau_j$, rejecting the data-dependent global null $\cap_{j \in J} H_{0,j}$ (and therefore also the global null $\cap_{i=1}^n H_{0,i}$) when*

$$-2 \sum_{j=1}^{k} \log(p_j/\tau_j) \geq \mathit{Quantile}(1 - \alpha, \chi^2_{2|J|})$$

controls type I error at level α conditional on J (and therefore also marginally).

If some p_j are substantially lower than their truncation point τ_j but most are above it, then Corollary 6 will be powerful. In this case, Corollary 6's test will have a slightly smaller statistic compared to Fisher's original combination test but a much smaller critical value. Hence, the truncated Fisher test is most powerful when some p-values come from strong alternatives but many come from conservative nulls. As such, Corollary 6 actually generalizes the truncated Fisher test to the settings where it is most applicable. On the flipside, if most of the p_j are below τ_j, the truncated test statistic pays a penalty due to selection while its critical value remains essentially unchanged compared to Fisher's original test.

To illustrate the benefits and drawbacks of these methods, we display their power alongside that of Fisher's original test for a simple $n = 10$ dimensional Gaussian problem, where we sample $X \sim N(\mu, I_n)$ and use the p-values $p_i = 1 - \Phi(X_i)$ try and detect the existence of a positive mean. For $\ell \in \{3, 5, 10\}$, we vary

[7] We give a variant that is valid conditional on which p-values are selected. If we just care about rejecting the global null $\cap_{i=1}^n H_{0,i}$ we can give a more powerful test via marginalization, as Zaykin et al. [2002] do.

the strength $\mu_1 = \cdots = \mu_\ell > 0$ of our signals and set $\mu_{\ell+1} = \cdots = \mu_n = -2$ to be conservative nulls. We do inference on the top $k = 3$ p-values using the conditional version of Fisher's method and set the truncation $\tau = 0.5$ for the truncated version (i.e., we include p_i for which $X_i > 0$). The results are displayed in Figure 9.

As expected, the new methods outperform Fisher's original method when conservative nulls are present. When $\ell = 3$ and the bottom three p-values are much smaller than the rest, the conditional method does incredibly well. But its performance quickly degrades when $\ell = 5, 10$ and the fourth smallest p-value becomes close to the bottom three. The truncated method is more robust, and still considerably improves power when $\ell = 5$. Unsurprisingly, both selective methods perform worse than Fisher's original method when $\ell = 10$ and every μ_i is a signal.

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A Additional Derivations, Details, and Comments

A.1 Comparing Conditional and Sidak Global Null Testing

We consider a setting where we have n independent and selectively dominant p-values $p_1, \dots, p_n$ that are all anti-conservative, i.e., $p_j \preceq \text{Unif}([0,1])$. At worst, these p-values are exact uniforms (e.g., they come from the boundary of the null).

We will show that, on an event with probability at least $1-\epsilon$, the conditional procedure, which rejects when $p_{(1)} \leq \alpha p_{(2)}$, can only reject if $p_{(1)} \leq C_\epsilon/n$ for some constant $C_\epsilon > 0$. Hence, without conservative nulls, the conditional approach behaves roughly on the same order as the classical approach (Sidak).

Letting $U_1, \dots, U_n$ be independent Unif$([0,1])$ random variables, two facts are clear. First that $U_{(2)} \sim \text{Beta}(2, n-1)$ has mean $\frac{2}{n+1} < \frac{2}{n}$ and standard deviation $\sqrt{\frac{2(n-1)}{(n+1)^2(n+2)}} \leq \frac{2}{n}$, and second that $p_{(2)} \preceq U_{(2)}$.

Fix any $\epsilon > 0$. We have by Chebyshev's inequality that

$$\begin{aligned} P\left(p_{(2)} \leq \frac{2}{n}(1+\epsilon^{-\frac{1}{2}})\right) &\geq P(p_{(2)} \leq E[U_{(2)}] + \sqrt{\text{Var}(U_{(2)})}/\sqrt{\epsilon}) \\ &\geq P(U_{(2)} \leq E[U_{(2)}] + \sqrt{\text{Var}(U_{(2)})}/\sqrt{\epsilon}) \\ &\geq P(|U_{(2)} - E[U_{(2)}]| < \sqrt{\text{Var}(U_{(2)})}/\sqrt{\epsilon}) \\ &\geq 1-\epsilon \end{aligned}$$

Define $C_\epsilon = 2(1+\epsilon^{-\frac{1}{2}})\alpha$ and the event $A_\epsilon = \{p_{(2)} \leq \frac{2}{n}(1+\epsilon^{-\frac{1}{2}})\}$. The event A_ϵ has probability at least $1-\epsilon$, and on this event a the conditional procedure never rejects when the procedure $p_{(1)} > C_\epsilon/n$, completing our claim.

In other words, the conditional procedure can only reject when $p_{(1)} \leq C_\epsilon/n$ (except for a small probability event) and hence suffers the same curse of dimensionality as the classical method:

$$P(p_{(1)} \leq \alpha p_2 \text{ and } p_{(1)} > C_\epsilon/n) \leq P(A_\epsilon^c) \leq \epsilon.$$

A.2 Conditional Inference on Winners

We first provide the standard derivation for the LCB and CI for the winning mean in the setting of independent Gaussian data. Then we show that it matches our p-value viewpoint. We then argue that the selective p-value we use for this winner problem is monotone in the null parameter, which allows us to prove the some facts about the more general conditional LCB and CI that applies for all MLR families. Finally, we use our p-value viewpoint to argue that, in the independent Gaussian case, the standardized distance between the winner X_W and conditional LCB is purely a function of the standardized distance between the winner X_W and runner-up X_R.

A.2.1 Standard Derivation

Suppose we observe independent Gaussian data $X \sim N(\mu, I_n)$ and want to make a LCB for the mean μ_W of the winner $W = \text{argmax}_{i\in[n]} X_i$.

To construct the conditional LCB, we follow the framework of Fithian et al. [2017]. Letting R be the index of the runner-up (second largest observation), we note that the deviation of X_W from μ_W has a truncated normal distribution once we condition on W and the nusiance statistics X_{-W}:

$$X_W - \mu_W \mid W, X_{-W} \sim TN(0, 1, X_R - \mu_W, \infty). \tag{23}$$

Let

$$q_{1-\alpha}(x_r, \mu_w) = \text{Quantile}_\mu(1-\alpha, X_W - \mu_W \mid W = w, X_{-w} = x_{-w}) \tag{24}$$

denote the $1-\alpha$ quantile of this conditional distribution(23), which is a function of largest entry x_r of x_{-w} and the mean μ_w at the winning index. It is straightforward to show that

$$LCB_{cond}(X) = \{\eta : \eta > X_W - q_{1-\alpha}(X_R, \eta)\} \tag{25}$$

is a $1-\alpha$ confidence region for μ_W that has exact coverage conditional on W and X_{-W}:

$$P_\mu(\mu_W \in LCB_{cond}(X)|W, X_{-W}) = P_\mu(X_W - \mu_W < q_{1-\alpha}(X_R, \mu_W)|W, X_{-W}) = 1-\alpha.$$

Later, via our p-value viewpoint, we will argue that this confidence region is indeed an LCB.

Fixing some $0 < \alpha_1, \alpha_2 < \alpha$ such that $\alpha_1 + \alpha_2 = \alpha$, we also see that

$$CI_{cond}(X) = \{\eta : X_W - q_{\alpha_2}(X_R, \eta) > \eta > X_W - q_{1-\alpha_1}(X_R, \eta)\} \tag{26}$$

is a $1-\alpha$ confidence region that has exact coverage conditional on W and X_{-W}. Again, later via our p-value viewpoint we will argue that this region is a CI.

A.2.2 p-value viewpoint

For the same setting as above, we want to use the p-values $p_i^{\mu_0} = 1 - \Phi(X_i - \mu_0)$ to characterize when the conditional LCB is at least $\mu_0 \in \mathbb{R}$. This happens exactly when μ_0 is not included in the set (25). Examining (23), (24), and (25), this happens when $X_W - \mu_0$ is at least as large as the $1-\alpha$ quantile Q of a standard normal truncated to be larger than $X_R - \mu_0$. This quantile satisfies

$$\alpha = \frac{1 - \Phi(Q)}{1 - \Phi(X_R - \mu_0)}.$$

Solving for Q gives $Q = \Phi^{-1}(1 - \alpha(1 - \Phi(X_R - \mu_0)))$, meaning we reject exactly when

$$\begin{aligned} X_W - \mu_0 \geq \Phi^{-1}(1 - \alpha(1 - \Phi(X_R - \mu_0))) &\iff 1 - \Phi(X_W - \mu_0) \leq \alpha(1 - \Phi(X_R - \mu_0)) \\ &\iff p_{(1)}^{\mu_0} \leq \alpha p_{(2)}^{\mu_0}. \end{aligned}$$

Now we do the same for the conditional CI (26). We want to characterize when μ_0 is not included in the set (26). Examining (23), (24), and (26), this happens either when $X_W - \mu_0$ is at least as large as the $1-\alpha_1$ quantile or at most as small as the α_2 quantile of the same truncated normal distribution. That is, either

$$X_W - \mu_0 \geq \Phi^{-1}(1 - \alpha_1(1 - \Phi(X_R - \mu_0))) \iff p_{(1)}^{\mu_0} \leq \alpha_1 p_{(2)}^{\mu_0},$$

or

$$X_W - \mu_0 \leq \Phi^{-1}(1-(1-\alpha_1)(1-\Phi(X_R-\mu_0))) \iff p^{\mu_0}_{(1)} \geq (1-\alpha_2)p^{\mu_0}_{(2)},$$

This will match the conditional CI we give later, which we will write in terms of p-values.

A.2.3 Monotonicity of the selective p-value

Recall the setting where $X_i \sim P_{\theta_i}$ are independent samples from some parametric family P_θ with MLR in $T(x)$, and let $p_i^{\theta_0}$ be the UMP p-values for testing $H_0 : \theta \leq \theta_0$.

We want to show that the selective p-value $p_j^{\theta_0} / \min_{i \neq j} p_j^{\theta_0}$ is monotone non-decreasing in θ_0. This will imply that $p^{\theta_0}_{(1)}/p^{\theta_0}_{(2)}$ is monotone non-decreasing in θ_0. So long as we can write our selection function in terms of the data with no dependence on θ_0, Appendix B.3 guarantees that this will be the case. Recall that each p-value $p_i^{\theta_0}$ is a function of $T(X_i)$ and auxiliary uniform random variables $U_{i,aux}$ that are independent of the data and each other. Imagining using our framework with $p = p_j^{\theta_0}$ and $Z = (T(X_{-j}), U_{-j,aux})$, we can write our selection function in terms of the data $T(X_i)$ and $U_{i,aux}$ as

$$\tilde{s}(t_j, u_j, t_{-j}, u_{-j}) = \begin{cases} 1, & \text{if } t_j > \max_{i \neq j} t_i, \\ 1, & \text{if } t_j = \max_{i \neq j} t_i \text{ and } u_j \leq \max_{i \neq j : t_i = t_j} u_i, \\ 0, & \text{otherwise.} \end{cases}$$

where t_{-j} and u_{-j} jointly represent z.

Because this selection function does not depend on θ_0, Appendix B.3 gaurantees that the selective p-value resulting from it will be monotone. Note that, to apply Appendix B.3, we also needed to ensure that Z did not depend on θ_0, which is was not the case in Example 7's original treatment, but is the case in our current treatment.

A.2.4 The MLR LCB and CI

Again suppose that $X_i \sim P_{\theta_i}$ are independent samples from some MLR family P_θ, and let $p_i^{\theta_0}$ be the UMP p-values for testing $H_0 : \theta \leq \theta_0$. Let $W = \text{argmin}_{i \in [n]} p_i^{\theta_0}$ be the index of the smallest p-value, and define the parameter vector Θ.

First we claim that

$$\{\theta_0 : p^{\theta_0}_{(1)}/p^{\theta_0}_{(2)} > \alpha\}$$

is a LCB for the winning parameter θ_W that has exact $1-\alpha$ coverage conditional on W. The region being an LCB follows from what we showed earlier: $p^{\theta_0}_{(1)}/p^{\theta_0}_{(2)}$ is monotone non-decreasing in θ_0. We get exact $1-\alpha$ coverage because $p_j^{\theta_j}$ has an exact uniform distribution given $p_{-j}^{\theta_j}$ under P_Θ (see Lemma B.3). Thus conditional on $W = j$ and $p_{-j}^{\theta_j}$ (i.e., conditional on selection and Z), Theorem 1 tells us that the selective p-value $p_j^{\theta_j} / \min_{i \neq j} p_i^{\theta_j}$ has an exact uniform distribution conditional on selection:

$$\begin{aligned} P_\Theta\left(\theta_W \in \left\{\theta_0 : \frac{p^{\theta_0}_{(1)}}{p^{\theta_0}_{(2)}} > \alpha\right\} | W = j\right) &= P_\Theta\left(\theta_j \in \left\{\theta_0 : \frac{p_j^{\theta_0}}{\min_{i \neq j} p_i^{\theta_0}} > \alpha\right\} | W = j\right) \\ &= P_\Theta\left(\frac{p_j^{\theta_j}}{\min_{i \neq j} p_i^{\theta_j}} > \alpha | W = j\right) \\ &= 1 - \alpha, \end{aligned}$$

Likewise, we see that

$$\{\theta_0 : \alpha_1 < p^{\theta_0}_{(1)}/p^{\theta_0}_{(2)} < 1-\alpha_2\}$$

is a CI for θ_W that has exact $1-\alpha$ coverage conditional on W. The fact that it is a CI again follows from the monotonicity of $p^{\theta_0}_{(1)}/p^{\theta_0}_{(2)}$, and exact coverage follows from an identical argument to the one above.

A.2.5 Distance Between X_W and Conditional LCB

Consider observing Gaussian data $X \sim N(\mu, \sigma^2 I_n)$ and let W and R be the indicies of the winner and runner up respectively. We will use our p-value viewpoint to show that that the standardized distance $D = (X_W - \hat{\mu})/\sigma$ between the winner $\hat{\mu}$ and the conditional LCB $\hat{\mu}$ for μ_W depends only on the standardized gap $(X_W - X_R)/\sigma$ between the winner and runner-up. Per our earlier discussions, the conditional LCB $\hat{\mu}$ satisfies

$$\frac{p^{\hat{\mu}}(X_W)}{p^{\hat{\mu}}(X_R)} = \alpha \iff \frac{1 - \Phi((X_W - \hat{\mu})/\sigma)}{1 - \Phi((X_R - \hat{\mu})/\sigma)} = \alpha \iff \frac{1 - \Phi(D)}{1 - \Phi(D - (X_W - X_R)/\sigma)} = \alpha.$$

Clearly then, D is a function of $(X_W - X_R)/\sigma$.

A.3 Hybrid Inference on Winners

A.3.1 Standard Derivation

Like before, we want to make a LCB for the mean μ_W of the winner $W = \text{argmax}_{i \in [n]} X_i$ in the case of independent Gaussian data $X \sim N(\mu, I_n)$ with unknown mean $\mu \in \mathbb{R}^n$.

The core idea behind hybrid inference is giving a confidence region $C_{hyb}(X)$ that has a very high probability of containing μ_W on a "good" event G_μ. Oddly, this good event depends on the unknown parameter. For some $\beta < \alpha$, we need G_μ to happen with probability at least $1 - \beta$. Then, if we ensure that $C_{hyb}(X)$ has at least $(1 - \alpha)/(1 - \beta)$ coverage on the G_μ, it will achieve $1 - \alpha$ coverage overall:

$$\begin{aligned} P_\mu(\mu_W \in C_{hyb}(X)) &= P_\mu(G_\mu) P_\mu(\mu_W \in C_{hyb}(X) | G_\mu) + P_\mu(G_\mu^c) P_\mu(\mu_W \in C_{hyb}(X) | G_\mu^c) \\ &\geq (1 - \beta) \left(\frac{1 - \alpha}{1 - \beta} \right) \\ &= 1 - \alpha. \end{aligned}$$

Considering some $\beta < \alpha$ and defining $\beta_n = 1 - (1 - \beta)^{\frac{1}{n}}$ as in (**??**), our good event is that the confidence lower bounds $X_i - z_{1-\beta_n}$ for the means μ_i all simultaneously hold:

$$G_\mu = \{X_i < \mu_i + z_{1-\beta_n} \text{ for all } i \in [n]\}.$$

From our earlier reasoning, we know that this good event happens with probability exactly $1 - \beta$.

Now, we can make a confidence region that contains the mean with probability at least $(1 - \alpha)/(1 - \beta)$ on this good event. If we condition on G_μ along with W and X_{-W}, the deviation of X_W from μ_W has a truncated normal distribution like (23) that is further truncated from above:

$$X_W - \mu_W \mid W, X_{-W}, G_\mu \sim TN(0, 1, X_R - \mu_W, z_{1-\beta_n}). \tag{27}$$

On the good event G_μ, we always have that $X_R - \mu_W < X_W - \mu_W < z_{1-\beta_n}$, so the lower truncation is indeed below the upper one. Let

$$q^h_{\frac{1-\alpha}{1-\beta}}(w, x_{-w}, \mu_w) = \text{Quantile}_\mu \left(\frac{1 - \alpha}{1 - \beta}, X_W - \mu_W \mid W = w, X_{-w} = x_{-w}, G_\mu \right) \tag{28}$$

denote the $(1-\alpha)/(1-\beta)$ quantile of the conditional distribution (27). Per the prior discussion, the function (28) only makes sense if the largest value in x_{-w} at most $z_{1-\beta_n}$, and we will take the quantile (28) to be $-\infty$ if it is not. It is then straightforward to show that

$$C_{hyb}(X) = \{\eta : \eta > X_W - q^h_{\frac{1-\alpha}{1-\beta}}(W, X_{-W}, \eta)\} \tag{29}$$

contains μ_W with high probability conditional on W, X_{-W}, and the event G_μ:

$$P_\mu(\mu_W \in C_{hyb}(X) | W, X_{-W}, G_\mu) = P_\mu(X_W - \mu_W < q^h_{\frac{1-\alpha}{1-\beta}}(W, X_{-W}, \mu_W) | W, X_W, G_\mu) = \frac{1 - \alpha}{1 - \beta}.$$

Based on our earlier discussions, this is sufficient to imply that $C_{hyb}(X)$ from (29) will contain μ_W with probability at least $1-\alpha$.

As was the case for our conditional confidence region (**??**), the hybrid confidence region (29) is a little hard to interpret at first. We will get a clearer sense of its benefits when we write it in terms of p-values. As a teaser, the rightmost panel of **??** plots distance between the winner X_W and the hybrid LCB for a $n = 10$ dimensional problem. We argue (using our upcoming p-value viewpoint) in Appendix A.3.3 that, once we fix α and β, this distance is a function of just the gap $X_W - X_R$ between the winner and runner up and the problem dimension n.

A.3.2 p-Value Viewpoint

Considering data $X \sim N(\mu, I_n)$ with unknown mean μ and the p-values $p_i^{\mu_0} = 1 - \Phi(X_i - \mu_0)$, we want to characterize when the hybrid LCB is at least $\mu_0 \in \mathbb{R}$. Examining (27), (28), and (29), we can consider two cases to figure out when this happens.

Case One - $X_R - \mu_0 \geq z_{1-\beta_n}$: If $X_R - \mu_0 \geq z_{1-\beta_n}$, then $q^h_{\frac{1-\alpha}{1-\beta}}(W, X_{-W}, \mu_0) = -\infty$, so μ_0 cannot be in (29). This case happens precisely when

$$X_R - \mu_0 \geq z_{1-\beta_n} \iff 1 - \Phi(X_R - \mu_0) \leq 1 - \Phi(z_{1-\beta_n}) \iff p^{\mu_0}_{(2)} \leq \beta_n.$$

Case Two - $X_R - \mu_0 < z_{1-\beta_n}$: If $X_R - \mu_0 < z_{1-\beta_n}$, then μ_0 is not in (29) exactly when $X_W - \mu_0$ is at least as large as the $\frac{1-\alpha}{1-\beta}$ quantile Q of a standard normal truncated to be larger than $X_R - \mu_0$ but smaller than $z_{1-\beta_n}$. This quantile satisfies

$$\frac{\alpha-\beta}{1-\beta} = \frac{\Phi(z_{1-\beta_n}) - \Phi(Q)}{\Phi(z_{1-\beta_n}) - \Phi(X_R - \mu_0)} = \frac{1 - \beta_n - \Phi(Q)}{1 - \beta_n - \Phi(X_R - \mu_0)}$$

Solving for Q gives

$$Q = \Phi^{-1}\left(1 - \beta_n - \frac{\alpha-\beta}{1-\beta}\left(1 - \beta_n - \Phi(X_R - \mu_0)\right)\right) = \Phi^{-1}\left(\left(1 - \frac{\alpha-\beta}{1-\beta}\right)(1-\beta_n) + \frac{\alpha-\beta}{1-\beta}\Phi(X_R - \mu_0)\right),$$

so we reject exactly when

$$\begin{aligned} X_W - \mu_0 &\geq \Phi^{-1}\left(\left(1 - \frac{\alpha-\beta}{1-\beta}\right)(1-\beta_n) + \frac{\alpha-\beta}{1-\beta}\Phi(X_R - \mu_0)\right) \\ \iff 1 - \Phi(X_W - \mu_0) &\leq \left(1 - \frac{\alpha-\beta}{1-\beta}\right)\beta_n + \frac{\alpha-\beta}{1-\beta}(1 - \Phi(X_R - \mu_0)) \\ &\iff p^{\mu_0}_{(1)} \leq \frac{\alpha-\beta}{1-\beta}p^{\mu_0}_{(2)} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right)\beta_n \end{aligned}$$

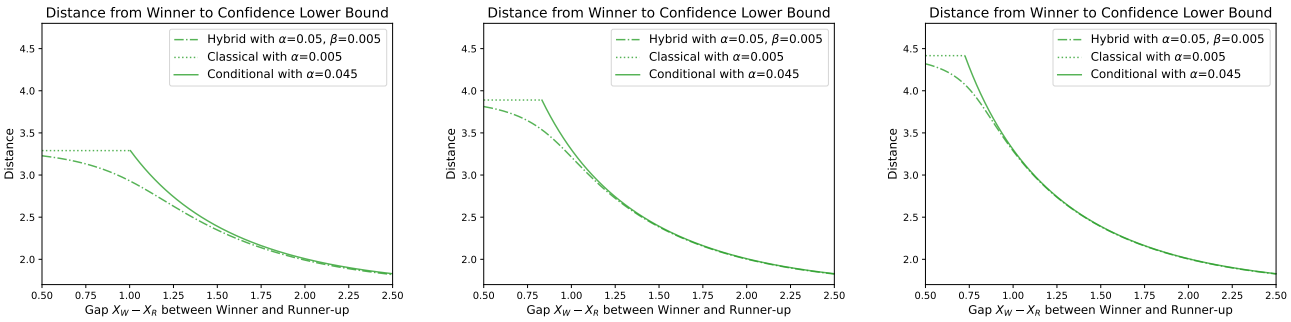
It turns out we can combine these two cases. Because $p^{\mu_0}_{(1)} \leq p^{\mu_0}_{(2)}$, the fact that $p^{\mu_0}_{(2)} \leq \beta_n$ in Case One implies that $p^{\mu_0}_{(1)} \leq \beta_n$ also. Therefore in Case One, $p^{\mu_0}_{(1)}$ must be strictly smaller than a mixture of $p^{\mu_0}_{(2)}$ and β_n:

$$p^{\mu_0}_{(1)} \leq \frac{\alpha-\beta}{1-\beta}p^{\mu_0}_{(2)} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right)\beta_n. \tag{30}$$

Therefore, if we reject according to (30), we will always reject in Case One, and we will reject at the appropriate times in Case Two.

A.3.3 Distance Between X_W and Hybrid LCB

Consider observing Gaussian data $X \sim N(\mu, I_n)$ and let W and R be the indicies of the winner and runner up respectively. We briefly justify that the distance between X_W and the chybrid LCB for μ_W



(a) $n = 10$ (b) $n = 100$ (c) $n = 1000$

Figure 10: For $n = 10$ (left), $n = 100$ (middle), and $n = 1000$ (right) the distance between the hybrid LCB to the winner (dash-dot line) and union bound LCB to the winner (dotted and solid line) with $\alpha = 0.05$ and $\beta = 0.005$ plotted as a function of the gap between the winning and runner-up observation.

depends only on the gap between $X_W - X_R$ and the dimension n. Define $D = X_W - \mu_0$ to be distance between X_W and the hybrid LCB $\hat{\mu}$. The hybrid LCB $\hat{\mu}$ satisfies

$$\begin{aligned} p^{\hat{\mu}}(X_W) &= \frac{\alpha-\beta}{1-\beta} p^{\hat{\mu}}(X_R) + \left(1 - \frac{\alpha-\beta}{1-\beta}\right)\beta_n \\ 1 - \Phi(X_W - \hat{\mu}) &= \frac{\alpha-\beta}{1-\beta}(1 - \Phi(X_R - \hat{\mu})) + \left(1 - \frac{\alpha-\beta}{1-\beta}\right)\beta_n \\ \iff 1 - \Phi(D) &= \frac{\alpha-\beta}{1-\beta}(1 - \Phi(D - (X_W - X_R))) + \left(1 - \frac{\alpha-\beta}{1-\beta}\right)\beta_n. \end{aligned}$$

Clearly then D is a function of $X_W - X_R$ and n.

A.4 Comparing Hybrid Inference to the Union Bound

As discussed earlier the hybrid cutoff (10) is strictly larger than the union bound cutoff (12) when $p_{(1)} > \beta_n$. Thus, both procedures reject when $p_{(1)} \leq \beta_n$. When $p_{(1)} > \beta_n$, the hybrid procedure rejects but the union bound does not whenever

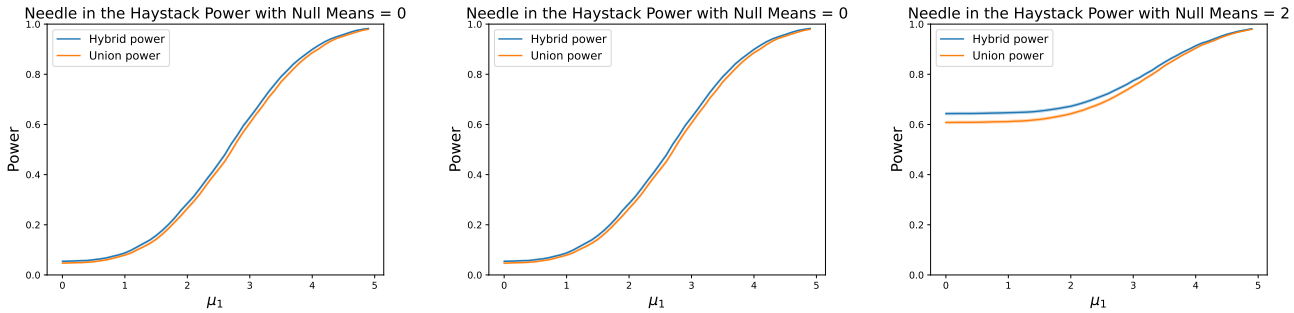
$$p_{(1)} \in \left((\alpha-\beta)p_{(2)}, \frac{(\alpha-\beta)}{1-\beta}p_{(2)} + \left(1 - \frac{(\alpha-\beta)}{1-\beta}\right)\beta_n\right]$$

When $p_{(2)} = 1$ and $n = 1$, the length of the interval is β, which is the largest it can possible be. Thus, in the case where we can have additional rejections, the hybrid cutoff is never more than β plus the union bound cutoff. Andrews et al. [2023] suggests taking $\beta = \alpha/10$, so when $\alpha = 0.05$, for example, $\beta = 0.005$ is quite small.

Still, this is not a precise statement about power gain. The computations required to compute the power gain analytically are messy, so instead we gauge the power gain via simulation. We sample data $X \sim N(\mu, I_n)$ for $n = 10$ and attempt to reject the winning null $H_W : \mu_W \leq 0$ where $W = \text{argmin}_{i\in[n]} 1 - \Phi(X_i)$ is the index of the smallest p-value $p_i = 1 - \Phi(X_i)$.

We choose $n = 10$ because it is a reasonably small dimension size where one may apply hybrid inference (e.g., the main example from Andrews et al. [2023] has $n = 13$). Let R denote the index of the runner-up (second smallest p-value). For the dimensions $n = 10, 100, 1000$, Figure 10 compares the distance from the winner X_W the hybrid LCB and the union bound LCB. As illustrated in the plot, the benefit of hybrid inference dissapates as the dimension of the problem increases. This is because conditioning on the good event has less and less of an effect as n grows.

Our simulation results indicate that hybrid inference typically results in a fairly small power gain. We consider two simulated settings:



(a) $\mu_2 = -2$ (b) $\mu_2 = 0$ (c) $\mu_2 = 2$

Figure 11: For $\mu_2 = 0$ (left), $\mu_2 = -0.5$ (middle), and $\mu_2 = -1$ (right) the empirical power over $N = 10^4$ trials of the hybrid inference approach versus the union bound approach for the needle in the haystack alternative. One standard error bands are also plotted. For the most part, they are so small that they are hardly visible.

Needle in a haystack: First, we consider a needle in the haystack setting where $\mu_1 > 0$ and all the other μ_i for $i \neq 1$ are set to μ_2. We try $\mu_2 = -2, 0, 2$. The power comparison is ploted in Figure 11. Whenever we truly have a needle in the haystack problem, i.e., $\mu_1 > \mu_2$, hybrid inference results in essentially no power gain. The only setting where we see some gain (up to around 0.05) is when $\mu_2 > \mu_1$. In this setting we actually have a dense alternative (many small signals). We expect the top two p-values to be close to each other, so conditional methods should perform poorly. The union bound approach indeed performs essentially identically to the level β classical test (not pictured). Hybrid, however, manages to eke out some additional power. Both methods pale in comparison to the level α classical test however, which would achieve power > 0.95 throughout the whole plot (not pictured). For various values of σ_1 and σ_2, which we assume are known, we also tried re-running the experiments with $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_i \sim N(\mu_2, \sigma_2^2)$ when $i > 1$. The results were not appreciably different.

Two possible signals: Seeing as the hybrid and union bound approaches both reject based on the winning and running-up p-value, we ran a simulation for all pairs $\mu_1, \mu_2 \in \{-3, -2.9, \ldots, 2.9, 3\}$ with $\mu_1 > \mu_2$ and $\mu_1 > 0$. We forcibly set $\mu_i = -\infty$ for $i > 2$. For each setting we ran $N = 10000$ to get an empirical estimate of power for each method. Across the 1492 simulated settings, the median emprical power increase from hybrid was ≈ 0.003, the 90th percentile empirical power increase was ≈ 0.023 and the maximum empirical power increase was ≈ 0.042. As the results indicate, the power increases from hybrid were negligble for most settings. We also re-ran the same simulations but sampled $X_1 \sim N(\mu, \sigma_1^2)$ and $X_2 \sim N(\mu, \sigma_2^2)$ for various values of σ_1 and σ_2, which we assume are known. The results were not appreciably different, and, if anything, the difference in power was notably smaller for some values of σ_1 and σ_2.

A.5 Conditional inference on winners for exponentials

Recall the exponential distribution $X \sim \text{Exp}(\lambda_i)$ which has PDF

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \\ 0 & x \leq 0 \end{cases}.$$

Defining $T(x) = 1/x$, we see for $x > 0$ and $\lambda_2 \geq \lambda_1$, the ratio

$$f_{\lambda_2}(x)/f_{\lambda_1}(x) = \frac{\lambda_2}{\lambda_1} \exp\left(-\frac{\lambda_2 - \lambda_1}{T(x)}\right)$$

is monotone non-decreasing in $T(x)$. Thus this family has an MLR in $T(x)$, and the UMP test for $H_0 : \lambda \leq \lambda_0$ thus rejects when $T(X)$ is large, or correspondingly when X is small. In particular, noting that the CDF of

X is given by

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0, \\ 0 & x \leq 0 \end{cases}$$

it rejects according to the p-value $p^{\lambda^0} = 1 - e^{-\lambda_0 X}$ (see Appendix B.1 for details).

Now consider observing some independent data $X_i \sim \text{Exp}(\lambda_i)$. We know from Appendix A.2 that the selective p-value $p^{\lambda_0}_{(1)}/p^{\lambda_0}_{(2)}$ is monotone non-decreasing in λ_0. Using L'hopital's rule, we can compute that

$$\begin{aligned} \lim_{\lambda_0 \downarrow 0} \frac{p^{\lambda_0}_{(1)}}{p^{\lambda_0}_{(2)}} &= \lim_{\lambda_0 \downarrow 0} \frac{1 - e^{-\lambda_0 X_{(1)}}}{1 - e^{-\lambda_0 X_{(2)}}} \\ &= \lim_{\lambda_0 \downarrow 0} \frac{X_{(1)} e^{-\lambda_0 X_{(1)}}}{X_{(2)} e^{-\lambda_0 X_{(2)}}} \\ &= \frac{X_{(1)}}{X_{(2)}}, \end{aligned}$$

which suffices to imply our claims in the main text.

A.6 Number of ties in rank verification

Considering a random vector $X \in \mathbb{R}^n$ let

$$Y_i = \frac{X_i - X_j}{2}, \qquad Y_j = \frac{X_i + X_j}{2}, \qquad Y_\ell = X_\ell \text{ for } \ell \neq i, j, \tag{31}$$

Suppose that $X_i \geq X_k$ for all $k \neq i$, and $X_i = X_\ell$ for some $\ell \neq i$ (i.e., there is at least one tie). We count the number of ties in for the winner in three different cases.

- Suppose $Y_j > \max_{k \neq i,j} Y_k$. If any of the X_k for $k \neq i, j$ were equal to X_i, then we would have

  $$Y_k = X_k = \frac{X_i + X_k}{2} \geq \frac{X_i + X_j}{2} = Y_j$$

  which would be a contradiction. Thus, if there is a tie, the only possible tie is X_j. Since,

  $$1 + |\{\ell \neq i : Y_\ell = \max_{k \neq i} Y_k\}| = 2$$

  in this case, we are done.

- Suppose that $Y_j < \max_{k \neq i,j} Y_k$. Then some X_k for $k \neq i, j$ must be strictly than larger X_j. Otherwise we would have for every $k \neq i, j$ that

  $$Y_j = \frac{X_i + X_j}{2} \geq \frac{X_i + X_k}{2} \geq X_k = Y_k,$$

  which is a contradiction. Thus, if there is a tie, the number of ties is one plus the number of X_k for $k \neq i, j$ that are equal to each other. In this case, this matches

  $$1 + |\{\ell \neq i : Y_\ell = \max_{k \neq i} Y_k\}|,$$

  so we are done.

- Now suppose that $Y_j = \max_{k \neq i,j} Y_k$. In this case we must have $X_j = X_i$. If not, then $X_j < X_i$ and there must be some $\ell \neq i, j$ such that $X_i = X_\ell$, so

  $$Y_j = \frac{X_i + X_j}{2} < \frac{X_i + X_\ell}{2} = X_\ell = Y_\ell,$$

  which is a contradiction. Thus $Y_j = X_i$ in this case, and the number of ties is therefore clearly

  $$1 + |\{\ell \neq i : Y_\ell = \max_{k \neq i} Y_k\}|.$$

A.7 p-value Adjustment for Exponential Families

Suppose p is a p-value for the null H_0 that is selectively dominant given Z and we select p to use for inference according to the selection function

$$s(x,z) = \begin{cases} 1 & \text{if } x < q^+(z), \\ \frac{1}{N(z)} & \text{if } x \in [q^+(z), q(z)], \\ 0 & \text{otherwise,} \end{cases},$$

Then the adjusted p-value (5) is given by

$$p_{adj} = \frac{\int_0^p s(x,Z)dx}{\int_0^1 s(x,Z)dx} = \begin{cases} \frac{p}{q^+(Z)+\frac{1}{N(Z)}(q(Z)-q^+(Z))} & \text{if } p < q^+(Z), \\ \frac{q^+(Z)+\frac{1}{N(Z)}(p-q^+(Z))}{q^+(Z)+\frac{1}{N(Z)}(q(Z)-q^+(Z))} & \text{if } p \in [q^+(Z), q(Z)], \end{cases},$$

which can be re-written as

$$p_{adj} = \frac{p - (1-\frac{1}{N(Z)})(p-q^+(Z))_+}{q^+(Z)+\frac{1}{N(Z)}(q^+(Z)-q(Z))}.$$

This is sufficient to imply the claim from the main text.

A.8 Rank verification warm-up additional details

Example 9 (Rank verification in a simple case)**.** *Suppose that p is a selectively dominant p-value for testing the null H_0, but we only choose to test H_0 when $p < 1/2$. Applying our framework with the p-value p and selection function $s(x) = I_{x<1/2}$, Theorem 1 tells us that we control selective Type I error if we reject according to the adjusted p-value from (5) is $p_{adj} = 2p$:*

$$P_{H_0}\left(p_j \leq \frac{\alpha}{2} | S = 1\right) \leq \alpha. \tag{32}$$

Now, consider data $X_1 \sim N(\mu_1, 1/\sqrt{2})$ and $X_2 \sim N(\mu_2, 1/\sqrt{2})$, the one-sided nulls $H_{0,j} : \mu_j \leq \mu_{-j}$, and their corresponding selectively dominant p-values $p_j = 1 - \Phi(X_j - X_{-j})$ (selective dominance follows from Example 3). Denoting the winner $W = \text{argmax}_{j=1,2} X_j$, it is now clear rejecting the data-dependent null $H_{0,W} : \mu_W \leq \mu_{-W}$ when $p_W \leq \alpha/2$ maintains Type I error control both conditionally on W and marginally. If $H_{0,j}$ is not true, then trivially $P(\text{falsely reject } H_{0,W} \mid W = j) = 0 \leq \alpha$. For the case that $H_{0,j}$ is true, the event $W = j$ is identical to the event $p_j < 1/2$, and hence is the same event as selecting p_j for inference in (32). Therefore,

$$\begin{aligned} P(\text{falsely reject } H_{0,W} \mid W = j) &= P\left(p_W \leq \frac{\alpha}{2} \mid W = j\right) \\ &= P\left(p_j \leq \frac{\alpha}{2} \mid W = j\right) \\ &\leq \alpha, \end{aligned}$$

implying error control conditional on W. Marginal error control then follows from the law of total probability:

$$\begin{aligned} P(\text{falsely reject } H_{0,W}) &= \sum_{j=1,2} P(\text{falsely reject } H_{0,j} \mid W = j) P(W = j) \\ &\leq \alpha \sum_{j=1,2} P(W = j) \\ &= \alpha. \end{aligned}$$

If $\mu_1 = \mu_2$ then the inequalities become equalities and our error control is tight.

A.9 Rank verification additional details

Example 10. *Suppose p is a p-value for the null H_0 that is selectively dominant given Z. If we select p to use for inference according to the selection function*

$$s(x,z) = \begin{cases} 1 & \text{if } x < q^+(z), \\ \frac{1}{N(z)} & \text{if } x \in [q^+(z), q(z)], \\ 0 & \text{otherwise,} \end{cases},$$

where $N(z) > 1$ and $0 \leq q^+(z) \leq q(z) \leq 1$ are known functions of z, then the adjusted p-value from (5) turns out to be (see Appendix A.7 for computations)

$$p_{adj} = f(p, q^+(Z), q(Z), N(Z)) \qquad f(a,b,c,d) = \frac{a - (1-\frac{1}{d})(a-b)_+}{b + \frac{1}{d}(c-b)}. \tag{33}$$

Therefore, Theorem 1 tells us that rejecting when (33) is at most α is a selective Type I error controlling procedure:

$$P_{H_0}\left(f(p, q^+(Z), q(Z), N(Z)) \leq \alpha \mid Z, S = 1\right) \leq \alpha. \tag{34}$$

Now, suppose we observe X drawn from the exponential family (15) and let W be the index $i \in \mathcal{S}$ of the largest sufficient satistic (with ties broken randomly). For $i \neq j$, Example 4 tells us that the UMPU p-value $p = p^\delta_{ij}$ from (17) for the null $H^\delta_{0,ij} : \theta_i - \theta_j \leq \delta$ is selectively dominant given the transformed nuisance statistics $Z = \widetilde{T}_{-i}$ from (14). Taking $q^+(Z) = q^{\delta,+}_{ij}(\widetilde{T}_{-i})$, $q(Z) = q^\delta_{ij}(\widetilde{T}_{-i})$, $N(Z) = N_i(\widetilde{T}_{-i})$ and f from (18), (19), (20), and (33), it is now easy to show that rejecting the data-dependent null $H^\delta_{0,Wj}$ when $f(p^\delta_{Wj}, q^{\delta,+}_{Wj}, q^\delta_{Wj}, N_W) \leq \alpha$ controls Type I error both conditional on W and marginally. Again it suffices to restrict our attention to indices $i \in \mathcal{I}$, $i \neq j$ that have a positive probability of being the winner (if $i = j$, then we know H^0_{ii} is true and we simply do not reject). If $H^\delta_{0,ij}$ is not true, then trivially $P(\text{falsely reject } H^\delta_{0,Wj} | W = i) = 0 \leq \alpha$. For the case that $H^\delta_{0,Wj}$ is true (and $i \neq j$), the event $W = i$ is the same event as selecting $p^\delta_{0,ij}$ for inference in (33), so

$$\begin{aligned} P(\text{falsely reject } H^\delta_{0,Wj} | W = i) &= P(f(p^\delta_{Wj}, q^{\delta,+}_{Wj}, q^\delta_{Wj}, N_{Wj}) \leq \alpha | W = i) \\ &= P(f(p^\delta_{ij}, q^{\delta,+}_{ij}, q^\delta_{ij}, N_i) \leq \alpha | W = i) \\ &\leq \alpha. \end{aligned}$$

Marginal error control follows from the usual law of total probability argument.

We can reject the data-dependent global null when $\cap_{j \in \mathcal{S}-W} H^\delta_{0,Wj}$ when we reject all of the individual nulls $H^\delta_{0,Wj}$ for $j \in \mathcal{S} - W$. It is straightforward to see that this will control Type I error both conditionally on W and marginally: If there is an $i \in \mathcal{I}$ for which $\cap_{j \in \mathcal{S}-i} H^\delta_{0,ij}$ is false, then trivially $P(\text{falsely reject } \cap_{j \in \mathcal{S}-W} H^\delta_{0,Wj} | W = i) = 0 \leq \alpha$ for this i. Otherwise, there exists a $k \in \mathcal{S} - i$ for which $\theta_i \leq \theta_k$, and

$$P(\text{falsely reject } \cap_{j \in \mathcal{S}-W} H^\delta_{0,Wj} | W = i) \leq P(\text{falsely reject } H^\delta_{0,Wk} | W = i) \leq \alpha.$$

Again, marginal error control follows from the usual law of total probability argument.

A.10 Post selection inference for the LASSO

Coming soon.

A.11 Data carving for Gaussian file-drawer

We have two data samples $X_1 \sim N(\mu, 2)$ and $X_2 \sim N(\mu, 2)$ that are independent and want to test $H_0 : \mu \leq 0$. Suppose we only do inference because we observed that $X_1 > t$ for some threshold t. If we consider the p-value $p_{full} = 1 - \Phi((X_1 + X_2)/2)$, then our selection function is given by

$$
\begin{aligned}
s(x) &= P(X_1 > t | p_{full} = x) \\
&= P(X_1 > t | \frac{X_1 + X_2}{2} = \Phi^{-1}(1-x)) \\
&= 1 - \Phi(t - \Phi^{-1}(1-x))
\end{aligned}
$$

where we have used that

$$
\begin{bmatrix} X_1 \\ \frac{X_1+X_2}{2} \end{bmatrix} \sim N\left(\begin{bmatrix} \mu \\ \mu \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right)
$$

so

$$
X_1 | \frac{X_1 + X_2}{2} = y \sim N(y, 1)
$$

Thus our corrected p-value is given by

$$
p_{carve} = \frac{\int_0^{p_{full}} 1 - \Phi(t - \Phi^{-1}(1-x))dx}{\int_0^1 1 - \Phi(t - \Phi^{-1}(1-x))dx}
$$

$$
p_{carve} = \frac{\int_{\bar{X}}^{\infty} \phi(z)(1 - \Phi(t-z))dz}{\int_{-\infty}^{\infty} \phi(z)(1 - \Phi(t-z))dz} = \frac{\int_{\bar{X}}^{\infty} \phi(z)(1 - \Phi(t-z))dz}{1 - \Phi(t/\sqrt{2})}
$$

We now show that p_{carve} is monotone non-decreasing in t. Letting Z and Y be independent standard normal random variables and fixing some constant a, the selective p-value is given by

$$
p_{carve} = \frac{P(Z + Y > t, Z > a)}{P(Z + Y > t)} = P(Z > a | Z + Y > t)
$$

for $a = \bar{X}$. Letting $W = Z + Y$ we can write $Z = \frac{1}{2}W + \epsilon$ where ϵ is independent of W. This gives us

$$
p_{carve} = P(\frac{1}{2}W + \epsilon > a | W > t) = E[P(W > 2(a-\epsilon) | W > t, \epsilon) | W > t] = E[P(W > 2(a-\epsilon) | W > t, \epsilon)]
$$

Then the fact that p_{carve} is monotone non-decreasing in t follows from the fact that $P(W > c | W > t)$ is monotone non-decreasing in t for every constant c:

$$
P(W > c | W > t) = \begin{cases} \frac{P(W>c)}{P(W>t)} & \text{if } t \leq c, \\ 1 & \text{if } t > c. \end{cases}
$$

B Selective Dominance and One-Sided Testing

In this appendix, we establish the selective dominance property for UMP p-values in MLR familes and UMPU p-values in exponential families. We also show that in these cases, the adjusted p-value from Theorem 1 is monotone in the parameter, under suitable conditions. Throughout the appendix, we draw from the discussion and proof strategy in Appendix B.1 of Lei and Fithian [2018].

B.1 MLR Families

We consider a parametric family P_θ parameterized by a real parameter $\theta \in R$ such that each P_θ has density $p_\theta(x)$ with respect to some carrier measure μ. We will suppose that these densities share support (i.e., for any two θ and θ' we have $p_\theta(x) > 0 \iff p_{\theta'}(x) > 0$). Further, we suppose that for any $\theta \leq \theta'$, the likelihood ratio $p_{\theta'}(x)/p_\theta(x)$ is a monotone non-decreasing function of some real-valued function $T(x)$ on this support (i.e, for any $x_1 \leq x_2$ with $p_\theta(x_1), p_\theta(x_2) > 0$, we have $p_{\theta'}(x_1)/p_\theta(x_1) \leq p_{\theta'}(x_2)/p_\theta(x_2)$).

Recall that for a testing problem, the critical function $\phi(x)$ (see [Lehmann et al., 1986, Section 3.1]) tells us the probability of rejecting the null having observed data x, so $\phi(X) = P(\text{reject } H_0|X)$. From Theorem 3.4.1 of Lehmann et al. [1986] we know the test governed by the critical function

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) > C \\ \gamma & \text{if } T(x) = C \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

is UMP for testing $H_0 : \theta \leq \theta_0$ against the alternatives $H_a : \theta > \theta_0$ so long C and γ satisfy

$$P_{\theta_0}(T(X) > C) + \gamma P_{\theta_0}(T(X) = C) = \alpha. \tag{36}$$

Denote the left-continuous survival function of $T(X)$ and its righthand limit under P_{θ_0} as

$$G(t) = P_{\theta_0}(T(X) \geq t) \qquad G^+(t) = \lim_{u \downarrow t} G(u).$$

Since $G(t)$ is a monotone non-increasing function, we can also define its generalized inverse

$$G^{-1}(z) = \inf\{t : G(t) \leq z\},$$

Lemma B.1 gives a natural way to set C and γ in Equation (35) to get an UMP test.

Lemma B.1 (An UMP test)**.** *Adopting the convention that* $0/0 = 0$*, taking* $C = G^{-1}(\alpha)$ *and* $\gamma = (\alpha - G^+(C))/(G(C) - G^+(C))$ *in Equation* (35) *gives an UMP test.*

Proof. At continuity points t of $G(\cdot)$, we have $G(G^{-1}(t)) = t$ and also $P(T(X) = t) = 0$. Thus, if $C = G^{-1}(\alpha)$ is a continuity point of $G(\cdot)$, then $G^+(C) = P_{\theta_0}(T(X) > C) = P_{\theta_0}(T(X) \geq C) = G(C) = \alpha$ and $\gamma = 0$, so the constraint (36) is immediately satisfied.

If $C = G^{-1}(\alpha)$ is not a continuity point of $G(\cdot)$, then $G(C) - G^+(C) > 0$ and the constraint (36) is also immediately satisfied. To ensure that we still have a valid test, however, we need $\gamma \in [0, 1]$. This is true so long as $\alpha \in [G^+(C), G(C)]$. We know that $G(t) \leq \alpha$ for any $t > C$, so $G^+(C) \leq \alpha$. If $G(C) < \alpha$, then we could find some $t^- < C$ such that $G(C) < \alpha$ by left-continuity, but this would contradict that $C = G^{-1}(\alpha)$, finishing the proof. □

Letting $U_{aux} \sim \text{Unif}([0, 1])$ be auxiliary randomness that is independent of X, a simple way to instantiate the test from Lemma B.1 is to reject whenever $T(X) > C$ or when $T(X) = C$ and $U_{aux} \leq \gamma$. Lemma B.2 explains how this is the same as rejecting when the p-value, termed as a fuzzy p-value in Geyer and Meeden [2005],

$$p = G^+(T(X)) + U_{aux}(G(T(X)) - G^+(T(X))) \tag{37}$$

is at most α.

Lemma B.2 (Fuzzy p-value is UMP)**.** *Rejecting* $H_0 : \theta \leq \theta_0$ *when the fuzzy p-value* (37) *is at most* α *instantiates the test from Lemma B.1, and is therefore UMP.*

Proof. We rewrite

$$p = (1 - U_{aux})G^+(T(X)) + U_{aux}G(T(X))$$

and consider four cases.

- If $t < G^{-1}(\alpha)$ then we can find some $t^+ > t$ such that $G(t^+) > \alpha$. Thus $G(t) \geq G^+(t) > \alpha$. So $p > \alpha$ whenever $T(X) < G^{-1}(\alpha)$
- If $t = G^{-1}(\alpha)$ and $G^{-1}(\alpha)$ is a continuity point of $G(\cdot)$, then $G^+(t) = G(t) = \alpha$. Thus, in this case $p \leq \alpha$ whenever $T(X) = G^{-1}(\alpha)$ and $U_{aux} \leq \frac{\alpha - G^+(G^{-1}(\alpha))}{G(G^{-1}(\alpha)) - G^+(G^{-1}(\alpha))} = \infty$.
- If $t = G^{-1}(\alpha)$ and $G^{-1}(\alpha)$ is not a continuity point of $G(\cdot)$, then we must have that $G(t) - G^+(t) > 0$. Also by right continuity we have $G(t) \geq \alpha$ and by how $G^{-1}(\cdot)$ is defined we have $G^+(t) \leq \alpha$. In this case $p \leq \alpha$ also whenever $T(X) = G^{-1}(\alpha)$ and $U_{aux} \leq \frac{\alpha - G^+(G^{-1}(\alpha))}{G(G^{-1}(\alpha)) - G^+(G^{-1}(\alpha))}$.

- If $t > G^{-1}(\alpha)$ then $G^+(t) \leq G(t) \leq \alpha$. So $p \leq \alpha$ whenever $T(X) > G^{-1}(\alpha)$.

This implies the following set equalities:

$$\begin{aligned}\{p \leq \alpha\} &= \{T(X) > G^{-1}(\alpha)\} \cup \left\{T(X) = G^{-1}(\alpha), U_{aux} \leq \frac{\alpha - G^+(G^{-1}(\alpha))}{G(G^{-1}(\alpha)) - G^+(G^{-1}(\alpha))}\right\} \\ &= \{T(X) > C\} \cup \{T(X) = C, U_{aux} \leq \gamma\}\end{aligned}$$

□

Lemma B.3 shows that $p \sim \text{Unif}([0,1])$ under P_{θ_0}, which will be useful for us later.

Lemma B.3 (Fuzzy p-value is uniform at null boundary)**.** *Under P_{θ_0}, the p-value (37) has a Unif$([0,1])$ distribution.*

Proof. Using the set equality from Lemma B.2 but replacing α with $z \in (0,1)$, we find

$$\begin{aligned}&P_{\theta_0}(G^+(T(X)) + U_{aux}(G(T(X)) - G^+(T(X))) \leq z) \\ &= P_{\theta_0}(T(X) > G^{-1}(z)) + P_{\theta_0}\left(T(X) = G^{-1}(z), U \leq \frac{z - G^+(G^{-1}(z))}{G(G^{-1}(z)) - G^+(G^{-1}(z))}\right) \\ &= P_{\theta_0}(T(X) > G^{-1}(z)) + P_{\theta_0}(T(X) = G^{-1}(z)) P_{\theta_0}\left(U \leq \frac{z - G^+(G^{-1}(z))}{G(G^{-1}(z)) - G^+(G^{-1}(z))}\right) \\ &= G^+(G^{-1}(z)) + (G(G^{-1}(z)) - G^+(G^{-1}(z))) \cdot \frac{z - G^+(G^{-1}(z))}{G(G^{-1}(z)) - G^+(G^{-1}(z))} \\ &= z.\end{aligned}$$

□

Now we can show that p is a selectively dominant p-value for testing the null $H_0 : \theta \leq \theta_0$. In what follows, we consider some fixed $\theta \leq \theta_0$ and prove some facts that allow us to relate the distribution of $T(X)$ under P_{θ_0} to its distribution under P_θ.

Lemma B.4 (Distribution of $T(X)$)**.** *Let $g_\theta(T(x))$ be a non-increasing function that equals the likelihood ratio $p_\theta(x)/p_{\theta_0}(x)$ on the support and*

$$\nu(A) = \int I(T(x) \in A) p_{\theta_0}(x) \mu(dx)$$

be the meausre of $T(X)$ under $X \sim P_{\theta_0}$. Then

$$P_\theta(T(X) \in A) = \int_A g_\theta(t) \nu(dt)$$

Proof. We know that

$$P_\theta(T(X) \in A) = \int I(T(x) \in A) p_{\theta_0}(x) g_\theta(T(x)) \mu(dx),$$

so we need to show that

$$\int I(T(x) \in A) p_{\theta_0}(x) g_\theta(T(x)) \mu(dx) = \int_A g_\theta(t) \nu(dt) \tag{38}$$

If $g_\theta(T(x)) = I(T(x) \in A')$ happens to be an indicator then (38) holds. Therefore, we can apply the standard machine (see the discussion after Equation 42 in Lei and Fithian [2018]) to show that (38) holds for all non-negative functions $g_\theta(\cdot)$. □

Lemma B.5 (Distribution of $(T(X), U_{aux})$)**.** *If ω denotes the product measure of ν and Lesbesgue measure λ on $[0,1]$, i.e., the distribution of $(T(X), U_{aux})$ under P_{θ_0}, then*

$$P_\theta((T(X), U) \in B) = \int_B g_\theta(t)\omega(dt, du) \tag{39}$$

Proof. We will first argue that (39) holds for any B which is a product set $A_1 \times A_2$. We can futher reduce to the case that $g_\theta(T(x)) = I(T(x) \in A_1')$ is an indicator. Then we see using our previous lemma that

$$\begin{aligned}
P_\theta((T(X), U_{aux}) \in A_1 \times A_2) &= P_\theta(T(X) \in A_1)P(U_{aux} \in A_2) \\
&= \int_{A_1} g_\theta(t)\nu(dt) \cdot \int_{A_2} \lambda(du) \\
&= \int_{A_1 \cap A_1'} \nu(dt) \cdot \int_{A_2} \lambda(du) \\
&= \int_{A_1 \cap A_1' \times A_2} \omega(dt, du) \\
&= \int_{A_1 \times A_2} g_\theta(t)\omega(dt, du)
\end{aligned}$$

To handle the case of general $g_\theta(\cdot)$ we can again simply apply the standard machine.

The full result then follows from an application of the $\pi - \lambda$ theorem: the set of B for which (39) holds is a λ-system, and (39) holds for every set in the π system of all product sets $B = A_1 \times A_2$. □

Note that our p-value p is a deterministic function of $T(X)$ and U_{aux}:

$$p = m(T(X), U_{aux}) \qquad m(t, u) = G^+(t) + u(G(t) - G^+(t)).$$

As such, we sometimes write our selection function as a function of $T(X)$ and U_{aux}:

$$s(t, u) = s(m(t, u)).$$

We use this abuse of notation in our next lemma, which characterizes the conditional distribution of $T(X)$ given selection.

Lemma B.6 (Distribution of $(T(X), U_{aux})$ given selection)**.** *For any selection function $s(x)$ under which p has a positive probability of selection under P_θ,*

$$P_\theta((T(X), U_{aux}) \in B | S = 1) = \frac{\int_B g_\theta(t)s(t, u)\omega(dt, du)}{\int g_\theta(t)s(t, u)\omega(dt, du)}$$

Proof. First note that

$$P_\theta((T(X), U_{aux}) \in B | S = 1) = \frac{P_\theta((T(X), U_{aux}) \in B, S = 1)}{P_\theta(S = 1)}.$$

Thus it suffices to show for any set B that

$$P_\theta((T(X), U_{aux}) \in B, S = 1) = \int_B g_\theta(t)s(t, u)\omega(dt, du).$$

By the definition of conditional expectation

$$\begin{aligned}
P_\theta((T(X), U_{aux}) \in B, S = 1) &= E_\theta[E_\theta[I(S = 1) \mid T(X), U_{aux}]I((T(X), U_{aux}) \in B)] \\
&= E_\theta[s(T(X), U_{aux})I((T(X), U_{aux}) \in B)]
\end{aligned}$$

If $s(t, u) = I_{(t,u)\in B}$ is an indicator function, then the result is implied by our previous lemma. We again get the result for general selection functions $s(t, u)$ by applying the standard machine. □

With these lemmas under our belt, we can show Proposition B.1, the main result of this sub-section. Since $p \sim_{P_{\theta_0}} \text{Unif}([0,1])$ by Lemma B.3, this proposition is sufficient to imply selective dominance.

Proposition B.1. *For any selection function $s(x)$ for which p has positive probability of selection under both θ and θ_0,*

$$P_\theta(p \le z | S = 1) \le P_{\theta_0}(p \le z | S = 1).$$

Proof. Fix $z \in (0,1)$. If z is such that $P_\theta(p \le z|S=1) = 0$ then the desired inequality is trivial. To handle the non-trival case, we note three facts from the proof of Lemma B.2:

- If $(t,u) \in m^{-1}([0,z])$ then $t \ge G^{-1}(z)$,
- If $(t,u) \in m^{-1}((z,1])$ then $t \le G^{-1}(z)$,
- The sets $m^{-1}([0,z])$ and $m^{-1}((z,1])$ are disjoint.

Thus,

$$\begin{aligned}
\frac{1}{P_\theta(p \le z|S=1)} &= \frac{\int_{m^{-1}([0,1])} g_\theta(t)s(t,u)\omega(dt,du)}{\int_{m^{-1}([0,z])} g_\theta(t)s(t,u)\omega(dt,du)} \\
&= \frac{\int_{m^{-1}([0,z])} g_\theta(t)s(t,u)\omega(dt,du) + \int_{m^{-1}((z,1])} g_\theta(t)s(t,u)\omega(dt,du)}{\int_{m^{-1}([0,z])} g_\theta(t)s(t,u)\omega(dt,du)} \\
&= 1 + \frac{\int_{m^{-1}((z,1])} g_\theta(t)s(t,u)\omega(dt,du)}{\int_{m^{-1}([0,z])} g_\theta(t)s(t,u)\omega(dt,du)} \\
&\ge 1 + \frac{g_\theta(G^{-1}(z)) \int_{m^{-1}((z,1])} s(t,u)\omega(dt,du)}{g_\theta(G^{-1}(z)) \int_{m^{-1}([0,z])} s(t,u)\omega(dt,du)} \\
&= 1 + \frac{\int_{m^{-1}((z,1])} s(t,u)\omega(dt,du)}{\int_{m^{-1}([0,z])} s(t,u)\omega(dt,du)} \\
&= \frac{1}{P_{\theta_0}(p \le z|S=1)},
\end{aligned}$$

where to finish we have noted that $g_{\theta_0}(t) = 1$ almost eveywhere in the measure ω. □

B.2 Exponential Families

Suppose we observe data $X \in \mathbb{R}^m$ from an exponential family P_θ parameterized by $\theta \in \mathbb{R}^n$ i.e., under P_θ the data X has density

$$g_\theta(x) = \exp(\theta_1 T_1(x) + \cdots + \theta_n T_n(x) - \psi(\theta))g(x)$$

with respect to some carrier measure μ. We consider the problem of testing $H_0 : \theta_i \le \theta_{0,i}$.

The UMPU test for $H_0 : \theta_i \le \theta_{0,i}$ is valid conditional on $T_{-i}(X)$. More specifically, Theorem 4.4.1 of Lehmann et al. [1986] tells us that any test of the form

$$\phi(t_i, t_{-i}) = \begin{cases} 1 & \text{if } t_i > C_0(t_{-i}) \\ \gamma(t_{-i}) & \text{if } t_i = C_0(t_{-i}) \\ 0 & \text{otherwise} \end{cases}$$

where the functions $\gamma(\cdot)$ and $C_0(\cdot)$ satisfy

$$E_{\theta_{0,i}}[\phi(T_i(X), t_{-i}) | T_{-i}(X) = t_{-i}] = \alpha$$

is UMPU for testing $H_0 : \theta_i \le \theta_{0,i}$. Lemma 2.7.2 of Lehmann et al. [1986] tells us that the conditional distribuition of $T_i(X)$ given $T_{-i}(X) = t_{-i}$ admits a density

$$g_{\theta_i,t_{-i}}(t_i) = \exp(\theta_i t_i - \tilde{\psi}(\theta_i))$$

with respect to some base measure $\mu_{t_{-i}}$. This density has an MLR in t_i (to be specific, we are imagining observing $T_i(X)$ from its conditional distribution $T_{-i}(X)$, and the map $T(\cdot)$ from the previous sub-section is actually the identity). Hence, a concrete UMPU test is to just run our UMP test from the previous section using the conditional distribution given $T_{-i}(X) = t_{-i}$. In particular, our work from the previous section implies that it is UMPU to reject when the p-value

$$p = G^+(T_i(X)|T_{-i}(X)) + U_{aux}(G(T_i(X)|T_{-i}(X)) - G_i^+(T_i(X)|T_{-i}(X))), \tag{40}$$

where U_{aux} is an uniform random variable independent of the data and

$$G(t_i|t_{-i}) = P_{\theta_0}(T_i(X) \geq t|T_{-i}(X) = t_{-i}) \qquad G^+(t_i|t_{-i}) = \lim_{u \downarrow t_i} G(u|t_{-i}),$$

is at most α. Our work from the previous section also implies that this p-value is selectively dominant given $T_{-i}(X)$.

B.3 Montonicty of Selective MLR p-Values

In this sub-section, we consider data (X, Z) where the conditional distribution $X|Z = z \sim P_{\theta,z}$ is parameterized by $\theta \in \mathbb{R}$ and has an MLR in $T(x)$. Because we do everything conditional on $Z = z$, we can imagine without loss of generality there is no Z and just work with X. Letting

$$G^{\theta_0}(t) = P_{\theta_0}(T(X) \geq t) \qquad G^{\theta_0,+}(t) = \lim_{u \downarrow t} G^{\theta_0}(u)$$

we let

$$p^{\theta_0} = G^{\theta_0,+}(T(X)) + U_{aux}(G^{\theta_0}(T(X)) - G^{\theta_0,+}(T(X)))$$

be the UMP p-value for testing $H_0 : \theta \leq \theta_0$. Again, U_{aux} is a uniform random variable independent of the data. Let

$$m^{\theta_0}(t, u) = G^{\theta_0,+}(t) + u(G^{\theta_0}(t) - G^{\theta_0,+}(t))$$

be the map such that $m^{\theta_0}(T(X), U) = p^{\theta_0}$.

Considering a class of selection functions $s^{\theta_0}(x)$ under which p^{θ_0} and U both have a positive probability of selection given, we want to show that the selective p-values from Theorem 1,

$$p^{\theta_0}_{sel} = \frac{\int_0^{p^{\theta_0}} s^{\theta_0}(x)dx}{\int_0^1 s^{\theta_0}(x)dx},$$

are monotone non-decreasing in θ_0. Specifically, we will show that this is true when the selection function $s^{\theta_0}(x)$ is independent of θ_0 once written in terms of the data (i.e., selection can be stated in terms of the data without reference to the null parameter being tested). Formally, we establish monotonicity whenever there exists $\tilde{s}(\cdot, \cdot)$, independent of θ_0, such that

$$s^{\theta_0}(x) = \tilde{s}(t, u) \text{ for all } t, u \text{ with } x = m^{\theta_0}(t, u),$$

Like before, we now consider some fixed $\theta \leq \theta_0$ and let $g_\theta(T(x))$ be the non-increasing function that equals the likelihood ratio $p_\theta(x)/p_{\theta_0}(x)$ on the support. The next lemma allows us to rewrite the selective p-value in a useful way.

Lemma B.7. *Letting $E_{r,s} = \{(t, u) : t > r \text{ or } t = r \text{ and } u \leq s\}$. For any t and u such that $m^\theta(t, u) = y$,*

$$\int_0^y s^\theta(x)dx = \int_{E_{t,u}} g_\theta(t)\tilde{s}(t, u)\omega(dt, du).$$

Proof. First we handle the case that $s^\theta(x) = I(x \in B_\theta)$ is an indicator of membership to some set. Then the left-hand side of the equation is the Lesbesgue measure of the set $\{x : x \leq y\}$ intersected with B_θ. By Lemma B.3 this is the same as the probability under P_θ of the p-value p^θ being at most y and in B_θ. By our choice of t and u, the set difference between $E_{t,u}$ and the set $\{(t,u) : m^{\theta_0}(t,u) \leq y\}$ is measure zero under P_θ. Thus by Lemma B.5 and how $\tilde{s}$ is defined, the right-hand side of the equation is also the probability probability under P_θ of the p-value p^θ being at most y and in B_θ. We again get the result for general selection functions $s^\theta(x)$ by applying the standard machine. □

According the the previous lemma,

$$p^\theta_{sel} = \frac{\int_0^{p^\theta} s^\theta(x)dx}{\int_0^1 s^\theta(x)dx} = \frac{\int_{E_{T(X),U_{aux}}} g_\theta(t)\tilde{s}(t,u)\omega(dt,du)}{\int g_\theta(t)\tilde{s}(t,u)\omega(dt,du)}$$

Hence it would suffice to show that for all r and s,

$$\frac{\int_{E_{r,s}} g_\theta(t)\tilde{s}(t,u)\omega(dt,du)}{\int g_\theta(t)\tilde{s}(t,u)\omega(dt,du)}$$

is montone in non-decreasing in θ. This is the subject of our next lemma.

Lemma B.8. *For any r and s, the quantity*

$$\frac{\int_{E_{r,s}} g_\theta(t)\tilde{s}(t,u)\omega(dt,du)}{\int g_\theta(t)\tilde{s}(t,u)\omega(dt,du)}$$

is monotone non-decreasing in θ.

Proof. The proof strategy is the same as our earlier results. We would like to show that

$$\frac{\int_{E_{r,s}} g_\theta(t)\tilde{s}(t,u)\omega(dt,du)}{\int g_\theta(t)\tilde{s}(t,u)\omega(dt,du)} \leq \frac{\int_{E_{r,s}} g_{\theta_0}(t)\tilde{s}(t,u)\omega(dt,du)}{\int g_{\theta_0}(t)\tilde{s}(t,u)\omega(dt,du)}$$

If the numerator of the left-hand side is zero then the inequality must hold. In the other case we see that

$$\begin{aligned}
\frac{\int g_\theta(t)\tilde{s}(t,u)\omega(dt,du)}{\int_{E_{r,s}} g_\theta(t)\tilde{s}(t,u)\omega(dt,du)} &= 1 + \frac{\int_{E^c_{r,s}} g_\theta(t)\tilde{s}(t,u)\omega(dt,du)}{\int_{E_{r,s}} g_\theta(t)\tilde{s}(t,u)\omega(dt,du)} \\
&\geq 1 + \frac{g_\theta(r)\int_{E^c_{r,s}} \tilde{s}(t,u)\omega(dt,du)}{g_\theta(r)\int_{E_{r,s}} \tilde{s}(t,u)\omega(dt,du)} \\
&= 1 + \frac{\int_{E^c_{r,s}} \tilde{s}(t,u)\omega(dt,du)}{\int_{E_{r,s}} \tilde{s}(t,u)\omega(dt,du)} \\
&= \frac{\int g_{\theta_0}(t)\tilde{s}(t,u)\omega(dt,du)}{\int_{E_{r,s}} g_{\theta_0}(t)\tilde{s}(t,u)\omega(dt,du)}
\end{aligned}$$

where we have noted that if $(t,u) \in E_{r,s}$ then $t \geq r$ and if $(t,u) \in E_{r,s}$ then $t \leq r$, and also that $g_{\theta_0}(t) = 1$ almost everywhere in the measure ω. □

C Selecting Multiple p-Values for Inference

In this appendix, we generalize our selective dominance framework to allow us to select multiple p-values for inference. As we applied the results from Section 2 in Example 7, one can apply the results from this section to establish validity of all the methods we propose in Section 5.

Suppose we have n independent and selectively dominant p-values for the nulls $H_{0,i}$. We imagine conditioning on some collection of them, which, without loss of generality, we can take to be $Z = (p_{k+1}, \ldots, p_n)$. Note that, due to independence, conditioning on $Z = z$ does not change the distribution of p_j. Thus, the p_j remain p-values under the nulls $H_{0,j}$ that are independent. Now, for $1 \leq j \leq k$, we consider k binary selection random variables $S_j \in \{0, 1\}$, where $S_j = 1$ when p_j is selected. The relationship between p_j, Z, and S_j is governed by the selection function

$$s_j(x, z) = p(S_j = 1 | p_j = x, Z = z)$$

Supposing that U_j are independent uniform random variables that are also independent of, we can imagine selecting the U_j using the same selection functions. That is, we can imagine that a binary selection variable $S'_j \in \{0, 1\}$ whose joint distribution with U_j is governed by

$$P(S'_j = 1 | U_j = x, Z = z) = s_j(x, z)$$

So long as we consider selection functions $s_j(x, z)$ under which p_j and U_j both have positive probability of being selected given $Z = z$, then the machinery from Section 2 tells us that

$$p_{sel,j} = F_{U_j | Z, S'_j = 1}(p_j) = \frac{\int_0^{p_j} s_j(x, Z) dx}{\int_0^1 s_j(x, Z) dx}$$

is p-value (it stochastically dominates the uniform distribution under the null) conditional on Z and selection $S_j = 1$.

We will further assume that the selection happens independently, i.e.,

$$P(S_1 = 1, \ldots, S_k = 1 \mid p_1, \ldots, p_k, Z) = \prod_{j=1}^{k} P(S_j = 1 \mid p_j, Z)$$

By taking expectations conditional on Z with respect to both sides, we find that the S_j are independent given Z:

$$\begin{aligned} P(S_1 = 1, \ldots, S_k = 1 \mid Z) &= E[P(S_1 = 1, \ldots, S_k = 1 | p_1, \ldots, p_k, Z) \mid Z] \\ &= E\left[\prod_{j=1}^{k} P(S_j = 1 | p_j, Z) \,\middle|\, Z \right] \\ &= \prod_{j=1}^{k} E[P(S_j = 1 | p_j, Z)] \\ &= \prod_{j=1}^{k} P(S_j = 1 | Z) \end{aligned}$$

where we have used that the p_j are conditionally independent given Z to move the expectation inside the product. Finally, conditional on Z and all the selections $S_j = 1$, the adjusted p-values $p_{sel,j}$ are all independent of one another.

$$P(p_{sel,1} \in A_1, \ldots, p_{sel,k} \in A_k \mid Z, S_1 = 1, \ldots, S_k = 1) = \prod_{j=1}^{k} P(p_{sel,j} \in A_j \mid Z, S_j = 1)$$

This can be confirmed via Bayes rule. Consider a collection of sets A_j. Conditional on $Z = z$, the event

$p_{sel,j} \in A_j$ can be written as $p_j \in B_{j,z}$ for some set $B_{j,z}$. Thus

$$\begin{aligned}
&P(p_{sel,1} \in A_1, \ldots, p_{sel,k} \in A_k \mid Z, S_1 = 1, \ldots, S_k = 1) \\
&= \frac{P(p_{sel,1} \in A_1, \ldots, p_{sel,k} \in A_k, |Z) P(S_1 = 1, \ldots, S_k = 1 \mid Z, p_{sel,1} \in A_1, \ldots, p_{sel,k} \in A_k)}{P(S_1 = 1, \ldots, S_k = 1|Z)} \\
&= \frac{P(p_1 \in B_{1,Z}, \ldots, p_k \in B_{k,Z}|Z) P(S_1 = 1, \ldots, S_k = 1 \mid Z, p_1 \in B_{1,Z}, \ldots, p_k \in B_{k,Z})}{P(S_1 = 1, \ldots, S_k = 1|Z)} \\
&= \prod_{j=1}^{k} \frac{P(p_j \in B_{j,Z}|Z) P(S_j = 1|Z, p_j \in B_{j,Z})}{P(S_j = 1|Z)} \\
&= \prod_{j=1}^{k} \frac{P(p_{sel,j} \in A_j|Z) P(S_j = 1|Z, p_{sel,j} \in A_j)}{P(S_j = 1|Z)} \\
&= \prod_{j=1}^{k} P(p_{sel,j} \in A_j|Z, S_j = 1),
\end{aligned}$$

where we have used that the p_j are independent conditional on Z.

D Additional Proofs

D.1 Proof of Theorem 1

Recall we are considering a selection function such that the probability that U is selected $P(S' = 1|Z = z) = \int_0^1 s(x, z)dx$ is positive. The CDF of U given selection is continuous because it cannot have any point masses:

$$P(U = x|Z = z, S' = 1) = \frac{P(U = x, S' = 1|Z = z)}{P(S' = 1|Z = z)} \leq \frac{P(U = x|Z = z)}{P(S' = 1|Z = z)} = 0.$$

Therefore, defining

$$F^{-1}_{U|Z=z,S'=1}(t) = \inf\{x : F_{U|Z=z,S'=1}(x) > t\}$$

continuity implies that $F_{U|Z=z,S=1}(F^{-1}_{U|Z=z,S=1}(t)) = t$ and $F_{U|Z=z,S=1}(x) \leq t \iff x \leq F^{-1}_{U|Z=z,S=1}(t)$. Then

$$\begin{aligned}
P_{H_0}(F_{U|Z=z,S'=1}(p) \leq t|Z = z, S = 1) &= P_{H_0}(p \leq F^{-1}_{U|Z=z,S'=1}(t)|Z = z, S = 1) \\
&\leq P(U \leq F^{-1}_{U|Z=z,S'=1}(t)|Z = z, S' = 1) \\
&= P(F_{U|Z=z,S'=1}(U) \leq t|Z = z, S' = 1)
\end{aligned}$$

where we have used that $p|Z = z, S = 1 \succeq_{H_0} U|Z = z, S' = 1$ to get the middle inequality. Finally

$$\begin{aligned}
P(F_{U|Z=z,S'=1}(U) \leq t|Z = z, S' = 1) &= P(U \leq F^{-1}_{U|S'=1}(t)|Z = z, S' = 1) \\
&= F_{U|Z=z,S'=1}(F^{-1}_{U|Z=z,S'=1}(t)) = t
\end{aligned}$$

so $F_{U|Z=z,S'=1}(U)|Z = z, S' = 1 \sim \text{Unif}([0, 1])$, which finishes the proof.

Further, if for some distribution under the null, p has an exact uniform distribution given $Z = z$, then the distributions $U|Z = z, S = 1'$ and $p|Z = z, S = 1$ are identical, so the fact that $F_{U|Z=z,S'=1}(U)|Z = z, S' = 1 \sim \text{Unif}([0, 1])$ also implies that $F_{U|Z=z,S'=1}(p)|Z = z, S = 1 \sim \text{Unif}([0, 1])$, and therefore (6) holds with equality in this case.

D.2 Proof of Theorem 2

Let f_z be the density of $p|Z = z$ under a distribution in the null H_0. We start by showing that, if f_z is non-decreasing, then $p|Z = z, S = 1$ dominates $U|Z = z, S' = 1$. Fixing a selection function $s(x, z)$, it

suffices to show that for any $t \in [0,1]$.

$$\begin{aligned} & P(p \leq t | Z = z, S = 1) \leq P(U \leq t | Z = z, S = 1) \\ \iff & \frac{\int_0^t s(x,z) f_z(x) dx}{\int_0^1 s(x,z) f_z(x) dx} \leq \frac{\int_0^t s(x,z) dx}{\int_0^1 s(x,z) dx} \end{aligned}$$

If $P(p \leq t | Z = z, S = 1)$ is zero then this trivially holds. Otherwise $P(p \leq t | Z = z, S = 1) = \int_0^t s(x,z) f_z(x) dx > 0$ and we see that,

$$\begin{aligned} \frac{\int_0^1 s(x,z) f_z(x) dx}{\int_0^t s(x,z) f_z(x) dx} &= 1 + \frac{\int_t^1 s(x,z) f_z(x) dx}{\int_0^t s(x,z) f_z(x) dx} \\ &\geq 1 + \frac{f_z(t) \int_t^1 s(x,z) dx}{f_z(t) \int_0^t s(x,z) dx} \\ &= 1 + \frac{\int_t^1 s(x,z) dx}{\int_0^t s(x,z) dx} \\ &= \frac{\int_0^1 s(x,z) dx}{\int_0^t s(x,z) dx}, \end{aligned}$$

which is sufficient to imply the claim.

Now assuming that $f_{z'}$ is continuous and not non-decreasing for some z', we can show that p is not selectively dominant. In general, it suffices for there to be two points $y_1 < y_2$ such that $f_{z'}$ is strictly larger in a neighborhood around y_1 than in a neighborhood around y_2, where these neighborhoods are disjoint. In particular for $\epsilon > 0$ let $N_\epsilon(y) = (y - \epsilon, y + \epsilon)$ be a ball around y. Then we need there to be y_1, y_2, $\epsilon > 0$, and some $\eta > 0$ such that, for all $w_1 \in N_\epsilon(y_1)$ and $w_2 \in N_\epsilon(y_2)$, $w_1 < w_2$ but $f_{z'}(w_2) + \eta < f_{z'}(w_1)$. If f is continuous and not non-decreasing, then this must be true. First define $B_{high} = \inf\{f_{z'}(w_1) : w_1 \in N_\epsilon(y_1)\}$ and $B_{low} = \sup\{f_{z'}(w_2) : w_2 \in N_\epsilon(y_2)\}$ so $B_{high} > B_{low}$. Then consider the selection function

$$s(x, z') = \begin{cases} 1 & \text{if } x \in N_\epsilon(y_1) \cup N_\epsilon(y_2) \text{ and } z = z' \\ 0 & \text{otherwise} \end{cases}$$

and let t be a value such that $t > w_1$ for all $w_1 \in N_\epsilon(y_1)$ and $t < w_2$ for all $w_2 \in N_\epsilon(y_2)$. Trivially,

$$P(U \leq t | Z = z', S' = 1) = \frac{1}{2}.$$

But the fact that

$$\begin{aligned} \frac{1}{P(p \leq t | Z = z', S = 1)} &= \frac{\int_{y_1 - \epsilon}^{y_1 + \epsilon} f_{z'}(x) dx + \int_{y_2 - \epsilon}^{y_2 + \epsilon} f_{z'}(x) dx}{\int_{y_1 - \epsilon}^{y_1 + \epsilon} f_{z'}(x) dx} \\ &= 1 + \frac{\int_{y_2 - \epsilon}^{y_2 + \epsilon} f_{z'}(x) dx}{\int_{y_1 - \epsilon}^{y_1 + \epsilon} f_{z'}(x) dx} \\ &\leq 1 + \frac{2\epsilon B_{low}}{2\epsilon B_{high}} \\ &< 2 \end{aligned}$$

implies that $P(p \leq t | Z = z', S = 1) > \frac{1}{2}$, which means that p is not selectively dominant.

D.3 Proof of Corollary 3

Suppose we have n independent and selectively dominant p-values p_i for the null hypotheses $H_{0,i}$. We restrict our attention to $j \in \mathcal{J}$ for which p_j has positive probability of being the smallest. Suppose we use p_j to test $H_{0,j}$ only when we observe that p_j is strictly larger than β_n but still the smallest of all the p-values. We can apply Section 2's framework with $p = p_j$, $Z = p_{-j}$ and the selection function $s(x, z) = I_{\beta_n < p_j < \min_{i \neq j} p_i}$. It is straightforward to see that our adjusted p-value p_{adj} is $(p_j - \beta_n)/(\min_{i \neq j} p_i - \beta_n)$, and Theorem 1 therefore tells us that

$$P_{H_{0,j}}\left(\frac{p_j - \beta_n}{\min_{i \neq j} p_i - \beta_n} \leq \frac{\alpha - \beta}{1 - \beta} \,\middle|\, p_{-j}, S = 1\right) \leq \frac{\alpha - \beta}{1 - \beta}.$$

Re-arranging things we get

$$P_{H_{0,j}}\left(p_j \leq \frac{\alpha - \beta}{1 - \beta} \min_{i \neq j} p_i + \left(1 - \frac{\alpha - \beta}{1 - \beta}\right) \beta_n \,\middle|\, p_{-j}, S = 1\right) \leq \frac{\alpha - \beta}{1 - \beta}. \tag{41}$$

Letting W be the index of the smallest p-value, we can now prove the claim that rejecting $H_{0,W}$ when

$$p_{(1)} \leq \frac{\alpha - \beta}{1 - \alpha} p_{(2)} + \left(1 - \frac{\alpha - \beta}{1 - \alpha}\right) \beta_n$$

controls Type I error at level α. Let $\widetilde{W}$ be the index of the smallest p-value if all the p-values are strictly larger than β_n. If some p-value is at most β_n, then force $\widetilde{W} = 0$. Let G_P be the event that a p-value corresponding to a true null is at most β_n (note the event G depends on the data generating process P). We know from Sidak's procedure (**??**) that $P(G_P) \leq \beta$. Three facts are immediate:

$$G_P \subseteq \{\widetilde{W} = 0\} \implies \{\widetilde{W} > 0\} \subseteq G_P^c \implies P(\widetilde{W} > 0) \leq 1 - \beta$$

$$P(\text{falsely reject } H_{0,W}, G_P, \widetilde{W} = 0) \leq P(G_P) \leq \beta,$$

$$P(\text{falsely reject } H_{0,W}, G_P^c, \widetilde{W} = 0) = 0.$$

If $H_{0,j}$ is true, the event $\widetilde{W} = j$ is the same event as selecting p_j for inference in (41), so

$$\begin{aligned} P(\text{falsely reject } H_{0,W} | \widetilde{W} = j) &= P\left(p_{(1)} \leq \frac{\alpha - \beta}{1 - \alpha} p_{(2)} + \left(1 - \frac{\alpha - \beta}{1 - \alpha}\right) \beta_n | \widetilde{W} = j\right) \\ &= P\left(p_j \leq \frac{\alpha - \beta}{1 - \alpha} \min_{i \neq j} p_i + \left(1 - \frac{\alpha - \beta}{1 - \alpha}\right) \beta_n | \widetilde{W} = j\right) \\ &\leq \frac{\alpha - \beta}{1 - \alpha}, \end{aligned}$$

and if $H_{0,j}$ is not true then trivially $P(\text{falsely reject } H_{0,W} | \widetilde{W} = j) = 0 \leq \frac{\alpha - \beta}{1 - \alpha}$. Our result then follows from law of total probability:

$$\begin{aligned} &P(\text{falsely reject} H_{0,W}) \\ &= P(\text{falsely reject } H_{0,W}, \widetilde{W} = 0, G_P) + P(\text{falsely reject } H_{0,W}, \widetilde{W} = 0, G_P^c) \\ &\quad + \sum_{j \in \mathcal{J}} P(\text{falsely reject } H_{0,W} | \widetilde{W} = j) P(\widetilde{W} = j) \\ &\leq \beta + \frac{\alpha - \beta}{1 - \beta} \sum_{j \in \mathcal{J}} P(\widetilde{W} = j) \\ &= \beta + \frac{\alpha - \beta}{1 - \beta} P(\widetilde{W} > 0) \\ &\leq \alpha. \end{aligned}$$

D.4 Proof of Corollary 4 and Corollary 2

It suffices to argue that closing our hybrid global null testing procedure rejects $H_{0,(k)}$ if and only if

$$p_{(j)} \leq \frac{\alpha-\beta}{1-\beta} p_{(j+1)} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right) \beta_{n-j+1}$$

for every $1 \leq j \leq k$. We will define $p_{(n+1)} = \alpha$ so that the right-hand side of the above equals α when $j = n$. Correspondingly, for subsets $I \subseteq [p]$ of size one, we define the hybrid procedure to reject the global null $H_{I,0}$ when the lone p-value is at most α. For subsets I of size strictly more than one, supposing that the smallest p-value in I is the ℓth smallest p-value and the second smallest p-value in I is the mth smallest p-value, the hybrid procedure rejects the global null $H_{I,0}$ when

$$p_{(\ell)} \leq \frac{\alpha-\beta}{1-\beta} p_{(m)} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right) \beta_{|I|}.$$

Necessity: For $1 \leq j \leq k$, let I_{n-j+1} be the size $n-j+1$ subset that excludes the $j-1$ smallest p-values (when $j = 1$ then $I = [p]$). This subset includes the index of the kth smallest p-value, so we must reject $H_{0,I}$ to reject $H_{0,(k)}$. It rejects exactly when

$$p_{(j)} \leq \frac{\alpha-\beta}{1-\beta} p_{(j+1)} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right) \beta_{n-j+1}$$

so our conditions are neccesary.

Sufficiency: Consider a subset I that contains the index of the kth smallest p-value. If it is size-one, then we reject because

$$p_{(k)} \leq \frac{\alpha-\beta}{1-\beta} p_{(k+1)} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right) \beta_{n-k+1} \leq \frac{\alpha-\beta}{1-\beta} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right) \beta = \alpha.$$

Now suppose that I is size $n-j+1$. Its smallest p-value is the ℓth smallest p-value for some $\ell \leq k$ and $\ell \leq j$, and its second smallest p-value is the mth smallest p-value for some $m > \ell$. We reject because

$$\begin{aligned} p_{(\ell)} &\leq \frac{\alpha-\beta}{1-\beta} p_{(\ell+1)} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right) \beta_{n-\ell+1} \\ &\leq \frac{\alpha-\beta}{1-\beta} p_{(m)} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right) \beta_{n-j+1} \\ &= \frac{\alpha-\beta}{1-\beta} p_{(m)} + \left(1 - \frac{\alpha-\beta}{1-\beta}\right) \beta_{|I|}. \end{aligned}$$