Proof of Conjecture

Anay Aggarwal

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Conjecture

The $n \times n$ matrix $\mathbb{A}$ can be row-reduced to the $n \times n$ identity matrix $\mathbb{I}$ in n^2 moves.

Construction

Partition $\mathbb{A}$ into n column vectors. Define the index of each vector as the column number of that vector. Starting with the vector with the smallest index, go down the vector, then move on to the next vector. At each $\mathbb{A}_{i,k}$ perform

$$R_i \to \begin{cases} R_i - \frac{\mathbb{A}_{i,k}}{\mathbb{A}_{i+1,k}} R_{i+1} & \text{if } i < k \\ R_i \cdot \frac{1}{\mathbb{A}_{i,k}} & \text{if } i = k \\ R_i - \mathbb{A}_{i,k} R_k & \text{if } i > k \end{cases}$$

Proof that the elements match those of the identity matrix

Case 1: $i < k$.

We need $\mathbb{A}_{i,k} = 0$. Notice that in the specific column, $R_{i+1} = A_{i+1,k}$. And since $\frac{\mathbb{A}_{i,k}}{\mathbb{A}_{i+1,k}} \cdot \mathbb{A}_{i+1,k} = -\mathbb{A}_{i,k}$, we're done.

Case 2: $i = k$

Clearly, $\mathbb{A}_{i,k} \cdot \frac{1}{\mathbb{A}_{i,k}} = 1$, as desired.

Case 3: $i > k$

Notice that $\mathbb{A}_{k,k} = 1$, since we have already worked on the elements in row k. So

$$\mathbb{A}_{i,k} - \mathbb{A}_{i,k}\mathbb{A}_{k,k} = \mathbb{A}_{i,k} - \mathbb{A}_{i,k} = 0,$$

as desired.

Proof that each move preserves all of the other elements

By Induction, it suffices to prove that all $\mathbb{A}_{i,j} | j < k$ are preserved.

Case 1: $i < k$.

Notice that $\mathbb{A}_{a_1,a_2} = 0 \forall a_2 < k, a_1 \neq a_2$. Therefore, unless $i + 1 = k$, we are subtracting $c \cdot 0 = 0$, for constant c, from each element, thus preserving them.

So it suffices to show that this works for $i + 1 = k$. Notice that the matrix is now

$$\begin{vmatrix} 1 & 0 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & 1 & \mathbb{A}_{i,k} & ... & 0 \\ \boxed{0} & ... & \boxed{0} & 1 & ... & 0 \end{vmatrix}.$$

Since all boxed elements are 0, we're done with this case.

Case 2: $i = k$

Notice that all previous elements are 0, so multiplying them by a constant will preserve them.

Case 2: $i > k$

Clearly, it suffices to show that $\mathbb{A}_{k,j} = 0 \forall j \in \mathbb{Z}_{<k}$. This follows immediately from our inductive hypothesis.