Properties of The Friendship and Spider Graphs

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Abstract

Graph theory is a fundamental area of combinatorics with diverse applications ranging from computer science and operations research to sociology and biology. This paper explores two special classes of graphs: the Friendship graph F_n, also known as the windmill graph, and the Spider graph $F(a, b)$, characterized by their distinct structural properties. We rigorously analyze these graphs' key attributes, including their chromatic numbers, chromatic edge numbers, domination numbers, diameters, radii, and independence numbers. Through detailed mathematical proofs and discussions, we highlight how these properties reflect their underlying combinational structures and implications in theoretical and practical contexts.

1 Introduction

Graph theory investigates the properties of discrete structures composed of vertices and edges, providing powerful tools to model complex relationships across various disciplines. Among numerous interesting graph families, Friendship and Spider graphs stand out due to their unique topological structures and theoretical significance.

The Friendship graph, F_n, illustrates a centralized structure consisting of n triangles joined at a single common vertex, termed the "politician". Originating from the well-known Friendship theorem, this graph models scenarios such as social networks where a central node shares direct connections to multiple distinct groups. Conversely, the Spider graph $F(a, b)$ consists of a single central vertex from which extend a disjoint paths, or "legs," each of length b. Spider graphs epitomize hierarchical or radial networks frequently encountered in communication and transportation systems.

This paper systematically examines these graphs' inherent characteristics, providing comprehensive analyses of parameters such as chromatic numbers, chromatic edge numbers, domination numbers, diameters, radii, and independence numbers. By doing so, we offer deeper insight into their structural complexity, demonstrating both similarities and differences in their combinational and algebraic properties. This study not only enriches the theoretical understanding of these special graph families but also lays the groundwork for their application in network analysis and combinational optimization problems.

2 Friendship Graph F_n

2.1 Definition and Background

The friendship graph F_n (also called the windmill graph) consists of n triangles all sharing a single common vertex. Figure 1 below shows such a graph for F_4 [2].

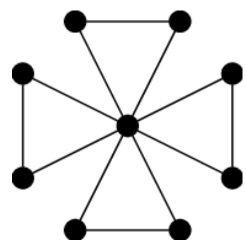


Figure 1: Friendship Graph for F_4

The friendship graph's vertex set $V(F_n)$ and edge set $E(F_n)$ are defined by:

$$\begin{aligned} V(F_n) &= \{v_0\} \cup V_1 \\ V_1 &= \{v_1, v_2, v_3, \ldots, v_{2n}\} \end{aligned}$$

$$\begin{aligned} E(F_n) &= E_1 \cup E_2 \\ E_1 &= \{\{v_0, v\} : v \in V(F_n) - \{v_0\}\} \\ E_2 &= \{\{v_{2n+1}, v_{2n+2}\} : 0 \leq i \leq n-1\} \end{aligned}$$

References [1] Mehdi Behzad. Here, v_0 acts as the center vertex, which we will call the "politician". The politician is adjacent to all the vertices in V_1, so its degree is $2n$, since V_1 contains $2n$ vertices.

The vertices in V_1 come in n disjoint pairs. Each pair is joined to the politician and to each other, forming a triangle. Thus, each non-politician vertex has degree 2.

To determine the order of the graph, observe that V_1 has $2n$ vertices and one additional vertex v_0, making the total:

$$|V(F_n)| = |V_1| + 1 = 2n + 1.$$

To find the size of the graph, add the number of edges from E_1 and E_2:

$$|E(F_n)| = |E_1| + |E_2| = 2n + n = 3n.$$

The Friendship graph also relates to the Friendship theorem, which states that "In a party of n persons, if every pair of persons has exactly one common friend, then there is someone in the party (the politician) who is everyone else's friend" [3]. From this, we infer that as the politician is adjacent to all the vertices, the dominating number $\gamma(F_n) = 1$.

2.2 Results

2.2.1 Chromatic Number

Theorem 1. *For a Friendship graph F_n, the chromatic number $\chi(F_n) = 3$.*

Proof. Recall that the chromatic number $\chi(G)$ of a graph G is the smallest number of colors required to color the vertices of G such that no two adjacent vertices share the same color [4].

In the vertex set of $F(n)$, the politician v_0 is connected to every other vertex along with n disjoint pairs each forming a complete graph K_3. The chromatic number of the complete graph K_3 is exactly 3, because each vertex is connected to the other, each must have a distinct color. Thus, each K_3 sub graph of triangles in F_n requires exactly 3 colors.

However, each triangle in F_n shares only the common vertex v_0, with no other vertices in common. Thus, it is possible to color each pair of vertices in V_1 using two colors and one color to the politician.

This coloring is valid because no two adjacent vertices share the same color. Specifically, v_0 differs in color from all vertices in V_1, and within each pair in V_1, vertices differ in color as well. Therefore, exactly three colors suffice, and since fewer than three colors cannot properly color even a single triangle, the chromatic number must be exactly 3.

$$\chi(F_n) = 3.$$

□

2.2.2 Chromatic Edge Number

Theorem 2. *For a Friendship graph F_n, the chromatic edge number $\chi'(F_n) = 2n$.*

Proof. Recall that the chromatic edge number $\chi'(G)$ of a graph G is the smallest number of colors required to color the edges of G such that no two incident edges share the same color [1].

The edge set for F_n is the union of E_1 and E_2. The set E_1 consists of all edges connecting the politician vertex v_0 to every vertex in V_1. Since all edges in E_1 are incident on the politician, they must all receive distinct colors. Thus, at least $|E_1| = 2n$ colors are needed to color the edges incident to v_0.

The set E_2 consists of n edges, where each edge connects a distinct pair of vertices in V_1. Each vertex in V_1 has degree 2 and is adjacent to the politician via an edge in E_1 and to its paired vertex via an edge in E_2.

Since each vertex in V_1 is already incident to exactly one edge from E_1, and since its degree is only 2, the edge from E_2 incident to it can be colored using an already existing color from the set used for E_1 without causing a conflict. Therefore, no additional new colors are required to color the edges in E_2.

Thus, the total number of colors required to color all edges of F_n is exactly $2n$. Hence,

$$\chi'(F_n) = 2n$$

□

2.2.3 Independence Number

Theorem 3. *For a Friendship graph F_n, the independence number $\alpha(n) = n$*

Proof. The independence number of a graph refers to the maximum number of vertices from the vertex set of a graph such that no two vertices are adjacent [5].

F_n consists of $2n$ vertices in the set V_1, which are partitioned into n disjoint pairs, each of which forms a K_3 triangle subgraph with the central vertex v_0. Each of the triangle shares the vertex v_0, so every vertex in V_1 is adjacent to v_0, and every pair of adjacent vertices in V_1 is connected by an edge. Hence, we exclude the politician v_0 from the independence set because v_0 is adjacent to all the vertices in V_1, making it impossible for v_0 to be a part of any independence set with more than one vertex. Therefore, the independent set must be chosen entirely from the vertices in V_1.

Likewise, within each triangle formed by a pair $\{v_{2n+1}, v_{2n+2}\}$, both vertices are adjacent to each other and to v_0. Hence, from each triangle, only one vertex can be included in the independence set. This means that from the set V_1, we can select at most one vertex from each of the n disjoint triangles. For a total of the remaining $2n$ vertices then, the independence number becomes

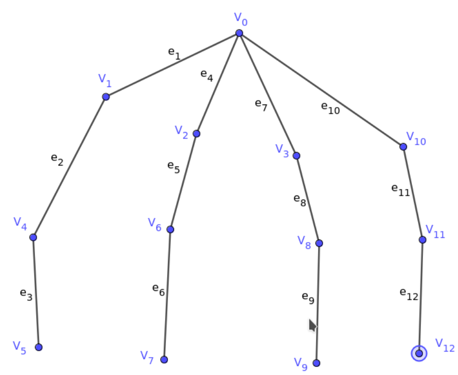
$$\alpha(G) = \frac{|V_1|}{2} = \frac{2n}{2} = n$$

Therefore, the maximum number of independent vertices that can be chosen is n, corresponding to selecting one vertex from each of the n disjoint pairs in V_1. □

3 Spider Graph $F(a, b)$

3.1 Definition and Background

The spider graph $F(a, b)$ is a tree consisting of a disjoint paths, also known as "legs" of length b, all emanating from a single central vertex $\{v_0\}$, or the "body" of the graph. Figure 2 below shows such a graph for $F(3, 3)$ [2].



x

Figure 2: Spider Graph for $F(3, 3)$

The spider graph's vertex set $V(F_{a,b})$ and edge set $E(F_{a,b})$ are defined by:

$$\begin{aligned} V(F_{a,b}) &= \{v_0\} \cup \bigcup_{i=1}^{a} V_i \\ V_i &= \{v_{i,1}, v_{i,2}, \ldots, v_{i,b}\}, \quad \text{for } 1 \leq i \leq a \end{aligned}$$

$$\begin{aligned} E(F_{a,b}) &= E_0 \cup E_1 \\ E_0 &= \{\{v_0, v_{i,1}\} : 1 \leq i \leq a\} \\ E_1 &= \{\{v_{i,j}, v_{i,j+1}\} : 1 \leq i \leq a,\ 1 \leq j \leq b-1\} \end{aligned}$$

Each leg V_i is a path of b vertices, beginning from the body and extending outward, which $b-1$ edges internally, plus one additional edge connecting the leg to the body. Hence, each leg contributes b edges to the graph.

Since, there is one central vertex and a legs with b vertices, the order of the graph becomes

$$|V(F_{a,b})| = 1 + ab$$

Likewise, using the definition of a tree graph, the size of the graph becomes,

$$|E(F_{a,b})| = a + a(b-1) = ab$$

The central vertex v_0, is connected to the first vertex $v_{i,1}$ of each leg V_i for $1 \leq i \leq a$. Therefore, the degree of v_0 is

$$\deg(v_0) = |\{\{v_0, v_{i,1}\} : 1 \leq i \leq a\}| = a.$$

Since all other vertices in the graph are part of simple paths and have degree at most 2, the maximum degree in the graph is attained at v_0, as

$$\Delta(F(a,b)) = \max_{v \in V(F_{a,b})} \deg(v) = a.$$

On the other hand, for each leg $V_i = \{v_{i,1}, \dots, v_{i,b}\}$, the vertex $v_{i,b}$ lies at the terminal end and is connected only to $v_{i,b-1}$. Thus,

$$\deg(v_{i,b}) = 1,$$

and since there are a such terminal vertices, the minimum degree in the graph is

$$\delta(F(a,b)) = \min_{v \in V(F_{a,b})} \deg(v) = 1.$$

3.2 Results

3.2.1 Diameter, Radius, and Center of $F(a,b)$

Theorem 4. *For a spider graph $F(a,b)$ with one central vertex $\{v_0\}$ and a number of legs of length b,*

$$\operatorname{diam}(F(a,b)) = 2b \quad \textit{and} \quad \operatorname{rad}(F(a,b)) = b,$$

Proof. For any graph G:

- The *distance* $d(u,v)$ is the length of the shortest path between u and v.
- The *eccentricity* of a vertex x is

$$\operatorname{ecc}(x) = \max_{y \in V(G)} d(x,y).$$

- The *diameter* of a graph G is

$$\operatorname{diam}(G) = \max_{x \in V(G)} \operatorname{ecc}(x).$$

- The *radius* of a graph G is

$$\operatorname{rad}(G) = \min_{x \in V(G)} \operatorname{ecc}(x).$$

In $F(a,b)$, the longest distance is between two leaves on separate legs. For instance, between $u_{i,b}$ and $u_{j,b}$, the shortest path goes:

$$u_{i,b} \to u_{i,b-1} \to \cdots \to u_{i,1} \to v_0 \to u_{j,1} \to \cdots \to u_{j,b},$$

which has length $2b$. Thus, $\mathrm{diam}(F(a,b)) = 2b$.

Next, consider the eccentricity of the hub:

$$\mathrm{ecc}(v_0) = \max_{x \in V} d(v_0, x) = b.$$

For any non-hub vertex $u_{i,j}$, its distance to a leaf on a different leg is:

$$d(u_{i,j}, u_{k,b}) = j + b > b,$$

so no non-hub vertex has eccentricity as small as b. Hence, the radius is b, and the unique center is $\{v_0\}$. □

3.2.2 Domination Number of $F(a, b)$

Theorem 5. *For the spider graph $F(a, b)$, the domination number is:*

$$\gamma(F(a,b)) = 1 + a \left\lceil \frac{b-1}{3} \right\rceil.$$

Proof. Let D be a minimum dominating set.

1. Necessity of including the hub v_0:

Suppose $v_0 \notin D$. Then each leg's vertex $v_{i,1}$ must be dominated separately, either by itself or by $v_{i,2}$. No single non-hub vertex can dominate more than one of them, and v_0 still needs to be dominated. So, at least $a + 1$ vertices are needed—more than if we include v_0. Therefore, $v_0 \in D$ in any minimal set.

2. Dominating the remaining path of each leg:

After v_0 covers each $v_{i,1}$, what's left on leg i is a path on $b - 1$ vertices:

$$v_{i,2}, v_{i,3}, \ldots, v_{i,b}.$$

The domination number of a path with n vertices is $\lceil \frac{n}{3} \rceil$, so each leg requires $\lceil \frac{b-1}{3} \rceil$ more vertices.

In total, this gives us,

$$\gamma(F(a,b)) = 1 + a \left\lceil \frac{b-1}{3} \right\rceil.$$

□

3.2.3 Independence Number

Theorem 6. *The independence number of the spider graph $F(a, b)$ is:*

$$\alpha(F(a, b)) = a \cdot \left\lceil \frac{b}{2} \right\rceil.$$

Proof. Each leg is a path P_b, and the independence number of such a path is $\lceil \frac{b}{2} \rceil$.

If we do not include v_0, we can independently choose those maximum sets on each leg. Including v_0 forces us to exclude all $v_{i,1}$, which decreases the count.

Therefore,

$$\alpha(F(a, b)) = a \cdot \left\lceil \frac{b}{2} \right\rceil.$$

□

4 Conclusion

In this paper, we have extensively examined two distinctive classes of graphs, the Friendship graph F_n and the Spider graph $F(a, b)$. Our analysis provided rigorous proofs of several essential graph properties, including chromatic numbers, edge chromatic numbers, domination numbers, independence numbers, diameters, and radii. These properties illuminate fundamental structural insights and highlight significant combinational characteristics unique to these graphs. Friendship graphs, due to their centralized connectivity, are particularly illustrative of social network dynamics, while spider graphs effectively represent hierarchical and radial structures found in various communication and transportation scenarios. The comprehensive examination presented here establishes a solid foundation for further exploration of these and related graph structures, emphasizing their theoretical importance and practical applications in network design and optimization.

References

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