Wave Equation With Backward Euler (Newton's Method)

We will solve wave equation

$$u'' = \Delta u - \lambda u' \tag{1}$$

using backward Euler with Newton Method.

First we break Eq(1) into two equation:

$$u' = v \tag{2}$$

$$v' = \Delta u - \lambda v \tag{3}$$

We use the discrete Laplacian operator for triangle meshes:

$$\Delta u_i = \frac{1}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i) \tag{4}$$

Applying backward Euler to Eq(2, 3) and using Eq(4), we get fixed point update rules for each vertex i:

$$v_i^{k+1} = v_i^k \tag{5}$$

$$u_i^{k+1} = u_i^k \tag{6}$$

$$v_i^{k+1} = v_i^k + \tau(\Delta u_i^{k+1} - \lambda v_i^{k+1}) \tag{7}$$

$$u_i^{k+1} = u_i^k + \tau v_i^{k+1} \tag{8}$$

$$v_i^{k+1} = v_i^k + \tau \left[\frac{1}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j^{k+1} - u_i^{k+1}) - \lambda v_i^{k+1} \right] \tag{9}$$

where α_{ij} and β_{ij} are computed based on offsets at timestep $k+1$.

After initializing velocity and offset for each vertex using Eqs(5, 6), we run Eqs(7, 8) for each vertex. In Eq(9), we are going to use Newton Method to compute v_i^{k+1}. So we rewrite it as

$$F(v_i^{k+1}) = (1+\lambda)v_i^{k+1} - v_i^k - \tau \left[\frac{1}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j^{k+1} - u_i^{k+1}) \right] \tag{10}$$

And using the following Newton Iteration to get v_i^{k+1}

$$v_i^{k+1[n+1]} = v_i^{k+1[n]} - \frac{F(v_i^{k+1[n]})}{F'(v_i^{k+1[n]})} \tag{11}$$