Heat Equation With Backward Euler (Newton's Method)

We will solve heat equation

$$u' = \alpha \Delta u \tag{1}$$

using backward Euler.

We use the discrete Laplacian operator for triangle meshes:

$$\Delta u_i = \frac{1}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i) \tag{2}$$

Applying backward Euler to Eq(1), we get for each vertex i:

$$u_i^{k+1} = u_i^k + \tau \alpha \Delta u_i^{k+1} \tag{3}$$

Substituting Eq(2) into Eq(3), we have

$$F(u_i^{k+1}) = -\frac{\tau\alpha}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij}) u_j^{k+1} + \left[\frac{\tau\alpha}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij}) + 1 \right] u_i^{k+1} - u_i^k = 0 \tag{4}$$

We are going to use Newton Method to solve Eq(4) for u_i^{k+1}. And According to Newton Method, we can numerically solve Eq(4) by the following iteration

$$u_i^{k+1[n+1]} = u_i^{k+1[n]} - \frac{F(u_i^{k+1[n]})}{F'(u_i^{k+1[n]})} \tag{5}$$