Wave Equation With Improved Euler

We will solve wave equation

$$u'' = \Delta u - \lambda u' \tag{1}$$

using improved Euler:

$$u^{k+1} = u^k + \frac{1}{2}\tau(u'^k + u'^{k+1}) \tag{2}$$

First we break Eq(1) into two equation:

$$u' = v \tag{3}$$

$$v' = \Delta u - \lambda v \tag{4}$$

We use the discrete Laplacian operator for triangle meshes:

$$\Delta u_i = \frac{1}{2}\sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i) \tag{5}$$

Applying improved Euler to Eq(3, 4) and using Eq(5), we get update rules for each vertex i:

$$v_i^{k+1} = v_i^k + \tau \left[\frac{1}{2}\sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j^k - u_i^k) - \lambda v_i^k \right] \tag{6}$$

$$u_i^{k+1} = u_i^k + \frac{\tau}{2}(v_i^k + v_i^{k+1}) \tag{7}$$