

Wave Equation With Backward Euler (Newton Method)

We will solve wave equation

$$u'' = \Delta u - \lambda u' \quad (1)$$

using backward Euler with Newton Method.

First we break Eq(1) into two equation:

$$u' = v \quad (2)$$

$$v' = \Delta u - \lambda v \quad (3)$$

We use the discrete Laplacian operator for triangle meshes:

$$\Delta u_i = \frac{1}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i) \quad (4)$$

Applying backward Euler to Eq(2, 3) and using Eq(4), we get fixed point update rules for each vertex i :

$$v_i^{k+1} = v_i^k \quad (5)$$

$$u_i^{k+1} = u_i^k \quad (6)$$

$$v_i^{k+1} = v_i^k + \tau(\Delta u_i^{k+1} - \lambda v_i^{k+1}) \quad (7)$$

$$u_i^{k+1} = u_i^k + \tau v_i^{k+1} \quad (8)$$

$$v_i^{k+1} = v_i^k + \tau \left[\frac{1}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j^{k+1} - u_i^{k+1}) - \lambda v_i^{k+1} \right] \quad (9)$$

where α_{ij} and β_{ij} are computed based on offsets at timestep $k + 1$.

After initializing velocity and offset for each vertex using Eqs(5, 6), we run Eqs(7, 8) for each vertex. In Eq(9), we are going to use Newton Method to compute v_i^{k+1} . So we rewrite it as

$$F(v_i^{k+1}) = (1 + \lambda)v_i^{k+1} - v_i^k - \tau \left[\frac{1}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j^{k+1} - u_i^{k+1}) \right] \quad (10)$$

And using the following Newton Iteration to get v_i^{k+1}

$$v_i^{k+1[n+1]} = v_i^{k+1[n]} - \frac{F(v_i^{k+1[n]})}{F'(v_i^{k+1[n]})} \quad (11)$$