Heat Equation With Backward Euler (Fixed Point Iteration)

We will solve heat equation

$$u' = \alpha \Delta u \tag{1}$$

using backward Euler.

We use the discrete Laplacian operator for triangle meshes:

$$\Delta u_i = \frac{1}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i) \tag{2}$$

Applying backward Euler to Eq(1), we get for each vertex i:

$$u_i^{k+1} = u_i^k + \tau \alpha \Delta u_i^{k+1} \tag{3}$$

Substituting Eq(2) into Eq(3) and using fixed point method, we have

$$u_i^{k+1} = u_i^k \quad \forall i \tag{4}$$

$$u_i^{k+1} = u_i^k + \frac{\tau \alpha}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j^{k+1} - u_i^{k+1}) \tag{5}$$

where α_{ij} and β_{ij} are computed based on offsets at timestep $k+1$.

For each call to backward Euler, after initialization by Eq(4), we perform several iterations of Eq(3) on the mesh.