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Survey in operations research and management science

Knapsack problems — An overview of recent advances. Part II: Multiple, multidimensional, and quadratic knapsack problems

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A R T I C L E I N F O

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A B S T R A C T

After the seminal books by Martello and Toth (1990) and Kellerer, Pferschy, and Pisinger (2004), knapsack problems became a classical and rich research area in combinatorial optimization. The purpose of this survey, structured in two parts, is to cover the developments appeared in this field after the publication of the latter volume. *Part I* treats the classical single knapsack problems and their variants. The present *Part II* covers multiple, multidimensional, and quadratic knapsack problems, as well as other relevant variants, such as, e.g., multiobjective and online versions.

1. Introduction

This is the second part of a survey aimed to review the developments appeared on knapsack problems after the publication of the books by Martello and Toth (1990) and Kellerer et al. (2004), until Summer 2021. We recall here the main definitions, referring the reader to the first section of Part I (Cacchiani et al., 2022) for a general introduction to this research area.

The problem which originated this field is the famous *0–1 Knapsack Problem* (KP01): given a set of n items, each associated with a *profit* p_j and a *weight* w_j ($j = 1, \dots, n$), and a container (*knapsack*) of *capacity* c, find a subset of items with maximum total profit having total weight not exceeding the capacity.

A systematic research on the KP01 and its many variants started in the Fifties. It produced, over the next fifty years, an impressive number of scientific results, making this field a very relevant area of combinatorial optimization. The two mentioned monographs include in total about 700 bibliographic entries. The purpose of this survey is to review the subsequent developments, by mostly concentrating on the problems treated in the main chapters of Kellerer et al. (2004) (which also correspond to the main chapters in Martello and Toth, 1990) and on recent "hot" topics. We privilege problems with a clear combinatorial aspect, with a partial exception for non-linear knapsack problems: these are briefly described in order to introduce the *quadratic knapsack problems*, which are usually handled through combinatorial optimization tools. Other problems, closer to the computer science community (online knapsack problems) or belonging to the area of multiple criteria decision aiding (multi-objective knapsack problems), are succinctly treated in Section 6.

In Part I, we have reviewed single knapsack problems. The present part is devoted to multiple, multidimensional, and quadratic knapsack problems, with a final section on the other knapsack problems mentioned above. *Multiple knapsack problems* are the natural generalization of the KP01: the items are packed into different knapsacks, each having its own capacity. For the *multidimensional knapsack problems*, the literature is instead ambiguous. Most of the papers use this term to refer to the case in which each item is characterized by two or more independent weighting functions (such as, for example, weight, volume, level of toxicity, etc.) and a capacity limit is imposed on each function. Other papers adopt the same name for the case in which both items and knapsack(s) are multidimensional rectangular boxes, and items have to be packed without overlapping. We refer to the former problem as the *multidimensional vector knapsack problem* (or the *multidimensional knapsack problem*, when no confusion arises), and to the latter problem as the *multidimensional geometric knapsack problem*.

Section 2 of Part I contains a thorough review of books, surveys, and special issues dedicated to knapsack problems. We only remind here the surveys specifically dealing with the arguments treated in Part II:

- Bretthauer and Shetty (2002), Li and Sun (2006): non-linear knapsack problems (continuous, integer, convex, nonconvex, separable, and nonseparable). We also refer to Ibaraki and Katoh (1988) and Lin (1998) for previously appeared surveys;
- Fréville (2004), Fréville and Hanafi (2005): *multidimensional knapsack problem* (exact, heuristic, approximation, and metaheuristic algorithms);

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- Pisinger (2007): *quadratic knapsack problem* (with special emphasis on upper bounds computations);
- Lust and Teghem (2012): *multiobjective* version of single and multidimensional knapsack problems (exact, approximation, heuristic and metaheuristic algorithms);
- Kellerer and Strusevich (2012): *symmetric quadratic knapsack problem* (exact and approximation algorithms and their application on various scheduling problems);
- Christensen et al. (2017): *multidimensional geometric knapsack problem* (approximation and online algorithms);
- Laabadi et al. (2018): variants of the *multidimensional knapsack problem* (heuristic algorithms);
- Silva et al. (2019): *three-dimensional geometric knapsack problem* (exact methods and extensive comparative experiments);
- Leao et al. (2020): *multidimensional geometric knapsack problem* (devoted to the case of irregular shapes);
- Iori et al. (2021): *two-dimensional geometric knapsack problem* (exact algorithms and mathematical models).

The contents of Part II are the following:

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2. Multiple knapsack problems

Given m knapsacks with capacities c_i ($i = 1, \dots, m$) and n items with profits p_j and weight w_j ($j = 1, \dots, n$), in the *Multiple Knapsack Problem* (MKP) the goal is to select m disjoint subsets of items so that the total profit of the selected items is a maximum, and each subset is assigned to a knapsack whose capacity is no less than the total weight of the items in the subset. Let x_{ij} be a binary variable that takes the value one iff item j is assigned to knapsack i. The MKP can be modeled as:

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} p_j x_{ij} \tag{1}$$

$$\text{s.t.} \sum_{j=1}^{n} w_j x_{ij} \le c_i \qquad (i = 1, \dots, m) \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} \le 1 \qquad (j = 1, \dots, n) \tag{3}$$

$$x_{ij} \in \{0, 1\} \qquad (i = 1, \dots, m, j = 1, \dots, n). \tag{4}$$

The MKP is $\mathcal{NP}$-hard in the strong sense, as it can be shown by reduction from the *3-partition problem* (see, e.g., Martello and Toth (1990), Section 1.3).

Exact solution. Exact *Branch-and-Bound* (B&B) algorithms for the MKP were proposed by Fukunaga and Korf (2007), who developed a method for pruning nodes through dominance criteria between assignments of items to knapsacks. Fukunaga (2011) presented improved B&B approaches making use of dominance criteria and symmetry breaking strategies, integrated with the B&B technique proposed in the Eighties by Martello and Toth (1981). The resulting algorithms were successfully tested on instances with up to 10 knapsacks and 100 items. Sitarz (2014) developed an exact method based on a multiple criteria *Dynamic Programming* (DP). Computational tests showed, however, that the developed method was slower than the direct solution of (1)–(4) by a commercial software. Hickman and Easton (2015) introduced a new class of valid inequalities obtained by merging two lower-dimension inequalities over the MKP polyhedron and presented conditions to check if the produced inequalities are facet defining. Dell'Amico et al. (2019) solved the MKP by means of a pseudo-polynomial arc-flow model (inspired by Valério de Carvalho, 1999). In this model, the m item subsets assigned to the knapsacks are represented as m paths in a graph where nodes are partial knapsack fillings and arcs are items. The model was improved by the inclusion of additional optimization techniques, such as partial B&B, primal decomposition, Benders cuts, and a graph reduction procedure. Computational tests proved the effectiveness of the resulting algorithm on instances with up to 500 items and 50 knapsacks, or 300 items and 150 knapsacks. A new upper bound for the MKP was recently developed by Detti (2021).

Approximation. In contrast with the KP01, the MKP does not admit a *Fully Polynomial Time Approximation Scheme* (FPTAS) unless $\mathcal{P} = \mathcal{NP}$. Chekuri and Khanna (2006) presented a *Polynomial Time Approximation Scheme* (PTAS) running in $n^{O(\log(1/\varepsilon)/\varepsilon^8)}$ time. Jansen (2009) improved this result by proposing an *Efficient Polynomial Time Approximation Scheme* (EPTAS) with running time $2^{O(\log(1/\varepsilon)/\varepsilon^5)} \cdot poly(n) + O(m)$. Wang and Xing (2009) proposed an approximation algorithm that iteratively fills the knapsacks, according to nondecreasing capacity, by selecting for each knapsack the subset of items with highest profit by means of an exact KP01 procedure, and provided a worst-case analysis for the cases $m = 2$ and $m = 3$. Khutoretskii et al. (2018) developed a 0.5-approximation algorithm with time complexity $O(mn)$ (excluding an initial sorting of items and knapsacks), based on specific lexicographic orderings of both knapsacks and items.

Heuristics. The MKP has been a playground for a variety of heuristics, including: population-based (Shah-Hosseini, 2008), recursive constructive procedures (Lalami et al., 2012), artificial bee colony (Wei and Zhang, 2012 and Sabet et al., 2013) and artificial fish swarm (Liu et al., 2014).

2.1. Multiple subset sum problem

The *Multiple Subset Sum Problem* (MSSP) is a relevant special case of the MKP in which all knapsacks have the same capacity and the item profits are equal to their weights. The problem is $\mathcal{NP}$-hard in the strong sense, as it can be proved through the same reduction mentioned for the MKP.

A PTAS for the MSSP was presented by Caprara et al. (2000a), who also described a $\frac{2}{3}$-approximation algorithm for the bottleneck version of the problem (where the minimum total weight contained in any bin is to be maximized). They also showed that this is the best possible performance ratio achievable for the problem in polynomial time (unless $\mathcal{P} = \mathcal{NP}$). The PTAS was then generalized by Caprara et al. (2000b) to the case of different knapsack capacities. Later, Caprara et al. (2003) introduced a polynomial time $\frac{3}{4}$-approximation algorithm for the MSSP, with running time that is linear in the number of items and quadratic in the number of knapsacks.

Kellerer et al. (2011) studied the MSSP with inclusive assignment set restrictions. In this problem, the assignment set of an item (i.e., the set of knapsacks that the item may be assigned to) is either a subset or a superset of the assignment set of another item. They proposed a PTAS and an efficient 0.6492-approximation algorithm. Pan and Zhang (2016) showed how to solve MSSP instances with low density with an oracle for the *Shortest Vector Problem* (SVP), where the density d is measured by $d = n/(m \log_2 \bar{w})$, with $\bar{w} = \max_j\{w_j\}$, and they focused on instances having $d \leq 0.9408$.

2.2. Multiple knapsack assignment problem

Kataoka and Yamada (2014) formulated the *Multiple Knapsack Assignment Problem* (MKAP) as an extension of the MKP in which the items are partitioned into disjoint sets and each knapsack may only be assigned items from one of the sets in the partition. Having the MKP as a special case, the problem is obviously strongly $\mathcal{NP}$-hard. The authors provided upper and lower bounds and used them to develop a heuristic that was computationally tested on randomly generated instances. More effective approaches were later presented by Martello and Monaci (2020), who developed Lagrangian and surrogate relaxations, a constructive heuristic and a metaheuristic refinement procedure. Computational tests on benchmark MKAP instances proved the effectiveness of the algorithms.

Related problems having applications in real-world contexts were studied by Dimitrov et al. (2017), who focused on variants arising in emergency relocation, and by Homsi et al. (2021), who tackled a variant, arising in military and humanitarian situations, which involves loading constraints. The former article provides a heuristic algorithm, whereas the latter presents a mathematical model, Lagrangian and surrogate relaxations, and heuristic and matheuristic algorithms. Other MKP variants arising in military contexts were studied by Simon et al. (2017), who extended the MKP in several ways so as to model constraints on self-sufficiency (i.e., capacity of maintaining operations with external aid) of a squad of Marines. They developed mathematical models and computationally tested them on a variety of randomly generated instances.

2.3. Multiple knapsack problems with special constraints

The *Multiple Knapsack Problem with Conflicts* (MKPC), in which pairs of items cannot be placed together in the same knapsack, was addressed by Basnet (2018). He developed constructive heuristic algorithms and tested them on instances with up to 500 items and 15 knapsacks.

The *Multiple Knapsack Problem with Setup* (MKPS) is an extension of the *Knapsack Problem with Setup* (KPS) treated in Part I (Cacchiani et al., 2022, Section 6) to the case of multiple knapsacks: the items are characterized by a knapsack-dependent profit and belong to disjoint families, each one associated with a knapsack-dependent setup cost. Lahyani et al. (2019) proposed a two-phase matheuristic algorithm, and a decomposition-based Tabu search matheuristic approach that extends the former. Amiri and Barkhi (2021) recently studied a Lagrangian relaxation that decomposes the problem into a set of m independent single problems, and developed a greedy heuristic to produce feasible solutions. Both methods were evaluated through extensive computational experiments, showing that good quality solutions can be obtained in reasonable CPU times.

Motivated by a real application in the steel industry, Forrest et al. (2006), formulated the *Multiple Knapsack Problem with Color Constraints* (MKPCC), in which a color is associated with each item and the number of colors in any knapsack is restricted. They presented computational results for two very difficult MKPCC instances, solved using a column-generation approach.

Yamada and Takeoka (2009) formulated the *Fixed-Charge Multiple Knapsack Problem* (FCMKP) as an extension of the MKP in which a fixed cost f_i must be paid if knapsack i is used in the solution. The problem is to decide the set of knapsacks to use, and to assign items to them, so that the total net profit (item profits minus knapsack costs) is maximized. A B&B algorithm presented in Yamada and Takeoka (2009) was able to solve within 10 s almost all FCMKP instances with up to 32 000 items and 50 knapsacks. You and Yamada (2011) introduced the *Budget-Constrained Multiple Knapsack Problem* (BCMKP), where again knapsack i costs f_i, but there is a prefixed budget to buy the knapsacks, and the objective is to maximize the total profit of the selected items. The problem was solved with a B&B algorithm making use of tools similar to those employed in Yamada and Takeoka (2009) for the FCMKP.

Chen and Zhang (2018) studied a generalization of the MKP in which the set of items is partitioned into groups, and, while each item has its own weight, the profit of a group is obtained only if every item of the group is packed. They derived both approximation and inapproximability results for a parameterized version of the problem where the total weight of the items in each group is bounded by a factor $\delta \in (0, 1)$ of the total capacity of all knapsacks.

Laalaoui and M'Hallah (2016) proposed a variable neighborhood search algorithm for a particular MKP arising in a single machine scheduling problem where jobs must be processed in multiple time-windows and machine unavailability periods must be taken into account.

Nip and Wang (2019) presented three approximation algorithms for the *two-phase knapsack problem*: given an MKP instance with an additional container of given capacity, the items have to be packed into the knapsacks and, in a second phase, the knapsacks have to be packed into the container so that the profit of the selected items is maximized.

3. Multidimensional (vector) knapsack problems

Given a knapsack having d different capacities c_i $(i = 1, \dots, d)$ and n items having profits p_j $(j = 1, \dots, n)$ and weight w_{ij} for the ith capacity $(i = 1, \dots, d; j = 1, \dots, n)$, the *Multidimensional Knapsack Problem* (MdKP) consists in determining a subset of items such that its total ith weight does not exceed the ith capacity $(i = 1, \dots, d)$ and its total profit is a maximum. The MdKP can be modeled as:

$$\max \sum_{j=1}^{n} p_j x_j \tag{5}$$

$$\text{s.t.} \sum_{j=1}^{n} w_{ij} x_j \leq c_i \qquad (i = 1, \dots, d) \tag{6}$$

$$x_j \in \{0, 1\} \qquad (j = 1, \dots, n). \tag{7}$$

As already reported in the Introduction, surveys devoted to the MdKP have been published by Fréville (2004), Fréville and Hanafi (2005), and Laabadi et al. (2018). The MdKP is in practice an *Integer Linear Programming* (ILP) problem with binary variables and non-negative coefficients. It is strongly $\mathcal{NP}$-hard, although some authors classify it as weakly $\mathcal{NP}$-hard. The confusion probably arises from the fact that: (i) the general problem (5)–(7), in which n and d are input values, is indeed strongly $\mathcal{NP}$-hard (see, e.g., Garey and Johnson (1979), Problem [MP1]); (ii) if instead d is a constant, DP solves the problem in pseudo-polynomial time $O(n\bar{c}^d)$ (where $\bar{c} = \max_{i \in \{1,\dots,d\}} c_i$). (A similar situation occurs for the MKP, although, to the best of our knowledge, all authors classify it as strongly $\mathcal{NP}$-hard.)

Over the years, the MdKP has been addressed with some exact techniques and many (meta)heuristic methods.

Exact solution. Martello and Toth (2003) proposed a B&B algorithm for the case $d = 2$, making use of heuristics, reduction techniques, and multiple relaxations. Vimont et al. (2008) developed an implicit enumeration scheme which uses reduced cost constraints to fix non-basic variables and to prune nodes of the search tree. Puchinger et al. (2010) presented different exact and heuristic algorithms, all based on adapting to the MdKP the classical *core* concept for the KP01 (limit the search, at least initially, to a small set of core items, while setting all variables corresponding to items outside the core to their presumably 0–1 optimal values). The core problem was also used by Della Croce and Grosso (2011) in a reduction method, and by Mansini and Speranza (2012) in an exact approach based on a recursive variable-fixing process. Other B&B methods were proposed by Boussier et al. (2010), who improved the algorithm in Vimont et al. (2008) through a multi-level search strategy, and by Boyer et al. (2010), who developed a method combining B&B and DP. Bektas and Oğuz (2007) proposed a simple separation procedure to identify cover inequalities.

With the aim of producing good upper bounds, Kaparis and Letchford (2008) developed a cutting plane method based on lifted cover inequalities. Balev et al. (2008) presented a preprocessing procedure based on the iterated use of DP and *Linear Programming* (LP) relaxations. Gu (2016) studied reduction criteria based on the concept of core problem.

Very recently, Setzer and Blanc (2020) studied the geometric aspect of the MdKP solution space, and proposed an empirical orthogonal constraint generation method aimed to reduce the number of capacity constraints, producing some new best-known upper bounds and proving optimality for an open instance. They computationally tested this procedure on benchmarks proposed by Chu and Beasley (1998) (90 instances with $d = 30$ and $n \in \{100, 250, 500\}$) and compared it with other approaches from the literature, especially with Vimont et al. (2008). Their computational results showed that, for small instances ($n = 100$), the original MdKP formulation can be easily solved by means of a commercial solver (Gurobi). Instead, the more complex approaches proposed in Vimont et al. (2008) and Setzer and Blanc (2020) payed off in cases of large instances ($n \in \{200, 500\}$) for which the optimal solution is known by now just for 9 out of the 60 instances.

Heuristics. Already in 2004, Kellerer et al. (2004) mentioned that: *"In recent years (MdKP) turned out to be one of the favorite playgrounds for experiments with metaheuristics, in particular Tabu search and genetic algorithms".* In the last years, this trend has increased enormously, and a large variety of metaheuristics (sometimes fanciful, sometimes equipped with strong mathematical background) has been developed to tackle the MdKP. In the following, we provide some details on the methods obtained from relevant mathematical frameworks. For the sake of completeness, we give in the next two paragraphs a concise list of other heuristics and nature inspired approaches. Moraga et al. (2005) proposed a metaheuristic to construct and improve feasible solutions by means of randomized priority rules and local search techniques. Vasquez and Vimont (2005) introduced a hybrid method which combines a limited B&B variable fixing heuristic with Tabu search. Thiongane et al. (2006) proposed a hybrid Lagrangian heuristic for the case $d = 2$. Boyer et al. (2009) presented a heuristic approach based on surrogate relaxation, DP and *Branch-and-Cut* (B&C). Wilbaut et al. (2009) proposed an iterative scheme based on a dynamic fixing of the variables through LP relaxations. Al-Shihabi and Ólafsson (2010) designed a hybrid algorithm that combines a nested partition method with a binary ant system and LP. Angelelli et al. (2010) proposed a heuristic based on a kernel search framework. Hanafi and Wilbaut (2011) described a heuristic based on an iterative generation of upper and lower bounds. Della Croce and Grosso (2012) proposed a heuristic that combines core problem approaches and a branching scheme. Yoon et al. (2012) and Hill et al. (2012) studied Lagrangian-based heuristics.

Various kinds of metaheuristics for the MdKP have been studied: harmony search (Kong et al., 2015), scatter search (Hanafi and Wilbaut, 2008), multi-verse optimization (Abdel-Basset et al., 2019), local search (Wang et al., 2012b), genetic and randomized approaches (Jalali Varnamkhasti and Lee, 2012; Lai et al., 2014; Ünal and Kayakutlu, 2016; García et al., 2020; Martins and Ribas, 2021). The main nature inspired paradigms adopted for the MdKP are particle swarm (Chih et al., 2014; Chih, 2015, 2018; Haddar et al., 2016; Mingo López et al., 2018) and ant colony optimization (Kong et al., 2008, Ke et al., 2010).

Variants and generalizations

A number of MdKP variants and generalizations have been investigated in the recent literature.

In the *Unbounded Multidimensional Knapsack Problem* (UMdKP), modeled by the MdKP with (7) replaced by

$$x_j \geq 0 \text{ and integer} \qquad (j = 1 \dots, n),$$

the number of copies that can be selected for each item is unlimited. The problem was tackled by He et al. (2016) through an extension of their approach to the *unbounded knapsack problem*, see Part I (Cacchiani et al., 2022), Section 5.2.

Quadri et al. (2009) developed a B&B algorithm for a quadratic variant of the MdKP which calls for the maximization of a concave separable quadratic objective function.

The *Multidemand Multidimensional Knapsack Problem* (MDMdKP) extends the MdKP by including q additional dimensions, for each of which a demand constraint must be satisfied. This implies adding

$$\sum_{j=1}^{n} w_{ij} x_j \geq c_i \qquad (i = d + 1, \dots, d + q)$$

to model (5)–(7). The problem has been attacked through metaheuristics: Tabu search algorithms were developed by Cappanera and Trubian (2005), Arntzen et al. (2006), and Lai et al. (2019), while a scatter search approach was proposed by Hvattum and Løkketangen (2007).

The *Multiple Multidimensional Knapsack Problem* (MMdKP) generalizes both the MdKP and the MKP (Section 2) by considering multiple knapsacks having multiple dimensions. Ahuja and Cunha (2005) solved it by means of a very large-scale neighborhood search. Ang et al. (2007) formulated a multi-period sea cargo mix problem as an MMdKP and solved it by means of two heuristic algorithms. A further generalization of the problem, the *Multiple Multidimensional Knapsack with Family-Split Penalties* (MMdKPF), was proposed by Mancini et al. (2021) who solved it through a combinatorial Benders decomposition approach.

3.1. Multidimensional multiple-choice knapsack problem

The *Multidimensional Multiple-Choice Knapsack Problem* (MdMCKP) is a generalization of the MdKP in which, as for the *multiple-choice knapsack problem* (see Part I (Cacchiani et al., 2022), Section 7), the set of items is partitioned into ℓ classes $N_1, \dots, N_\ell$, and exactly one item of each class must be selected. This implies adding

$$\sum_{j \in N_i} x_j = 1 \qquad (i = 1, \dots, \ell) \tag{8}$$

to model (5)–(7).

Exact solution. Sbihi (2007) proposed a B&B algorithm that generates an initial lower bound and determines an upper bound for each level of exploration according to a best-first strategy. Extensive computational experiments showed that the algorithm can solve instances with up to 50 classes, 20 items per class, and 7 capacity constraints. Han et al. (2010) proposed new methods to generate hard benchmark instances of the MdMCKP, and provided an extensive experimental evaluation. Ghasemi and Razzazi (2011) extended the core concept (see Section 3) to the MdMCKP, and used it in a B&B algorithm. Extensive computational experiments showed that the algorithm can solve large size instances when there is no correlation between profits and weights (up to 100 classes, 100 items per class, and 15 capacity constraints), but it can only solve small-size instances when such correlation exists. Hifi and Wu (2012) proposed an alternative model obtained by using information collected from the optimal Lagrangian solution. Computational experiments showed that the CPLEX solver is more efficient when the new model is used. Gokce and Wilhelm (2015) provided valid inequalities for the minimization version of the MdMCKP, and two lifting procedures to strengthen them. The usefulness of these inequalities was evaluated by embedding them in a branch-and-cut algorithm. Voss and Lalla-Ruiz (2016) compared the above model (5)–(8) to an alternative formulation which uses binary variables x_{ij} taking the value 1 iff item j from class i is selected: computational experiments showed that the former obtains slightly better results. Mansini and Zanotti (2020) proposed an algorithm based on the concept of core (see Section 3), which iteratively constructs and solves a sequence of sub-problems by means of a recursive variable-fixing procedure, until an optimality condition is satisfied. Computational tests showed that the algorithm could solve to proven optimality ten open benchmark instances (proposed by Hifi et al., 2006, Shojaei et al., 2013, and Mansi et al., 2013) and improve the best-known values for many other instances.

Heuristics. Hifi et al. (2004) presented a guided local search metaheuristic based on a penalization strategy, and tested it on 13 instances with up to 10 capacity constraints and 4000 items. These results were improved by Hifi et al. (2006), who proposed a reactive local search algorithm and introduced twenty new instances with up to 30 capacity constraints and 7000 items. Akbar et al. (2006) presented a polynomial time heuristic based on the iterative construction of convex hulls, and tested it on randomly generated instances. Hiremath and Hill (2007) provided an analysis of the 13 MdMCKP benchmark instances, and additionally proposed a new test set of 270 instances, which, however, were rarely used in subsequent papers. Shahriar et al. (2008) proposed a parallel version of a heuristic algorithm previously developed by Akbar et al. (2001). Cherfi and Hifi (2010) developed a rounding procedure combined with a truncated B&B algorithm applied, at selected nodes, to a restricted model handled in a column generation fashion. Their approach improved the computational results in Hifi et al. (2006). Cherfi and Hifi (2009) developed an approach that combines local branching with a truncated B&B algorithm, based on their column generation procedure (Cherfi and Hifi, 2010). Feng et al. (2012) presented a method merging ant colony optimization and Lagrangian relaxation, and compared it with the algorithm in Hifi et al. (2006), obtaining comparable results. Crévits et al. (2012) proposed four variants of a heuristic method based on the iterated solution of semi-continuous relaxations where only a subset of variables is restricted to binary values. The algorithms were enhanced by a local search procedure and by reduction rules, improving the results in Cherfi and Hifi (2009). Mansi et al. (2013) improved the results in Cherfi and Hifi (2010) and Crévits et al. (2012) with a hybrid heuristic that iteratively refines the global upper bound value through LP relaxations, and the lower bound value by solving reduced problems. Heuristics based on the principles of Pareto algebra were proposed by Shojaei et al. (2013) and Zennaki (2013). Htiouech et al. (2013) developed and computationally tested an algorithm that alternates between constructive and destructive phases. Chen and Hao (2014b) presented two variants of a “reduce and solve” heuristic that combines problem reduction with CPLEX solver and favorably compared it with the methods in Mansi et al. (2013), Shojaei et al. (2013), Cherfi and Hifi (2009). Hifi and Wu (2015) developed a heuristic based on a Lagrangian relaxation used to define a neighborhood search and successfully compared it with the methods in Crévits et al. (2012), Mansi et al. (2013). Gao et al. (2017) proposed an algorithm based on the concept of *pseudo-gap*, an hypothesized gap between upper and lower bounds. A pseudo-gap enumeration was combined with cuts based on reduced cost constraints. The current state-of-the-art is an effective kernel search heuristic recently presented by Lamanna et al. (2022).

Variants. Caserta and Voss (2019) addressed the MdMCKP with uncertainty on parameters w_{ij}, by adopting a robust optimization approach. They developed a matheuristic algorithm to tackle both the nominal and the robust versions of the problem, and tested it on benchmark instances.

In the *Multidimensional Knapsack Problem with Generalized Upper Bound Constraints* (MdKP-GUB), items belong to disjoint families and it is required that at most one item per family is chosen. The MdKP-GUB was invented and attacked through various metaheuristics in a series of papers by Li and Curry (2005), Li (2005), Li et al. (2012).

4. Multidimensional geometric knapsack problems

In *Cutting and Packing* (C&P) problems one is given a set of geometric objects in multidimensional real space which have to be packed, without overlapping, into a given set of multidimensional containers (knapsacks). Equivalently, the containers have to be cut in order to produce the items, which explains why these two application domains, despite being quite different in practice, are studied together and lead to similar mathematical models and solution algorithms.

The C&P literature addresses several multidimensional geometric knapsack problems. They have in common the fact that each item is associated with a profit and there is a single knapsack where to pack/cut the items so as to maximize the total profit, but they differ in a number of characteristics, such as shape and number of dimensions of items and knapsacks, as well as additional geometric constraints.

Revising the complete literature on this area is beyond the scope of this survey, so we limit our review to the case of *orthogonal* packing, where items and knapsack are rectangular shapes (rectangles or boxes) and the items have to be packed/cut with their edges parallel to those of the knapsack. This represents by far the most studied field in the C&P area. For the case of irregular shapes, we refer the reader to the very recent survey by Leao et al. (2020), who presented a complete review of mathematical models and solution techniques.

4.1. Two-dimensional knapsack problem

The most famous multidimensional geometric knapsack problem is known as the *Two-Dimensional Knapsack Problem* (2D-KP), and consists in finding a maximum profit subset of a set of rectangular items that can be orthogonally packed into a single rectangular knapsack. The 2D-KP is strongly $\mathcal{NP}$-hard, since in the special case in which all item heights are equal, the problem of testing whether the whole set of items fits in the knapsack is equivalent to the well-known (one-dimensional) *bin packing problem* (pack n items of weight w_j ($j = 1, \dots, n$) into the minimum number of knapsacks of capacity c).

Exact solution. For exact algorithms and mathematical models we refer the reader to the very recent survey by Iori et al. (2021), who also implemented a library, `2DPackLIB`, of benchmarks and links for two-dimensional orthogonal cutting and packing problems (see http://or.dei.unibo.it/library/2dpacklib-2-dimensional-cutting-and-packing-library). To the best of our knowledge, the only relevant work published on the 2D-KP after this survey is the one by Cunha et al. (2020), who presented different reduction procedures based on several types of grids of points, and used them to improve the performance of a mathematical model.

Approximation. Caprara and Monaci (2004) presented an approximation algorithm that guarantees a worst-case ratio of $3+\varepsilon$. Jansen and Zhang (2007) improved the result with an algorithm having a worst-case ratio at most $2+\varepsilon$. For the special 2D-KP case in which items and knapsack are squares, Harren (2009) introduced a $(5/4+\varepsilon)$-approximation algorithm, whereas Heydrich and Wiese (2019) introduced an EPTAS with running time $O_\varepsilon(1)n^{O(1)}$, where $O_\varepsilon(1)$ denotes a value that is constant for constant ε. For the case in which the weight of each item is equal to its area, Lan et al. (2013) presented a EPTAS having running time $O(n\log n + 4^{\varepsilon^{-6}})$. For a more detailed overview on approximation and online algorithms for the 2D-KP, we refer the reader to the survey by Christensen et al. (2017) (mainly devoted to the *bin packing problem* but including a section on geometric knapsack problems).

Heuristics. A number of heuristic and metaheuristic algorithms was introduced in recent years for the 2D-KP. Beasley (2004) proposed a successful population heuristic based upon a non-linear formulation and tested it on several benchmarks involving up to 4000 items. Improved computational results were obtained by Alvarez-Valdés et al. (2007) (Tabu search), Hadjiconstantinou and Iori (2007) (genetic algorithm), Leung et al. (2012) (simulated annealing), Kierkosz and Luczak (2014) (hybrid evolutionary algorithm), and Shiangjen et al. (2018) (iterative bidirectional heuristic).

Several variants of the 2D-KP have been also considered in the literature. da Silveira et al. (2013) presented a $(4+\varepsilon)$-approximation algorithm for the 2D-KP with unloading constraints and a $(3+\varepsilon)$-approximation algorithms for two other special cases. Zhou et al. (2019) presented mathematical models and computational experiments for the 2D-KP with block packing constraints, in which the knapsack is divided into disjoint blocks and each packed item has to be allocated inside a block. de Queiroz et al. (2017) proposed ILP models, a B&C algorithm, and metaheuristics for the 2D-KP with conflicts, in which some pairs of items cannot both be packed into the knapsack. For the special case in which the knapsack is a square and the objective is to pack all the items into the smallest possible square, lower bounds, mathematical models, and an exact algorithm have been presented in Caprara et al. (2006) and Martello and Monaci (2015).

4.2. Two-dimensional knapsack problems with guillotine constraints

The most widely studied variant of the 2D-KP is the *Two-Dimensional Knapsack Problem with Guillotine Constraints* (2D-KPGC), in which the items must be obtained from the knapsack through a series of *guillotine cuts* (i.e., edge-to-edge cuts parallel to the edges of the knapsack). Each series of parallel cuts is called a *stage*, and when a limit k (frequently encountered in real-world applications) is imposed on the maximum number of stages, the problem is called the *k-staged* 2D-KPGC. In the constrained version of the 2D-KPGC the number of items that can be selected for each item type is bounded. While this version is strongly NP-hard, the *unconstrained* version (here referred to as the 2D-UKPGC) can be solved to optimality in pseudo-polynomial time through DP (see Iori et al., 2021).

Exact solution. For what concerns exact algorithms and mathematical models, we again refer the reader to the thorough recent survey by Iori et al. (2021).

Approximation. As regards approximation algorithms, Caprara et al. (2010) considered the special case of the 2-staged 2D-KPGC in which the profit of each item coincides with its area and generalized the approximation scheme presented in Caprara et al. (2000a), obtaining an (absolute) approximation scheme.

Heuristics. Alvarez-Valdés et al. (2002) presented a *Greedy Randomized Adaptive Search Procedure* (GRASP) and a Tabu search algorithm for the 2D-KPGC. Hifi (2004) introduced a hybrid approach which combines depth-first search using hill-climbing strategies and DP. Bortfeldt and Winter (2009) proposed a genetic algorithm for the 2D-KP and the 2D-KPGC, also able to handle both the 2D-UKPGC and the case where the items can be rotated by 90°. Borgulya (2019) introduced, for the 2D-KPGC, an evolutionary method based on an estimation of a probability distribution from a set of solutions, and computationally compared it with the approach in Bortfeldt and Winter (2009).

4.3. Geometric knapsack problems in higher dimensions

The *Three-Dimensional Geometric Orthogonal Knapsack Problem* (3D-KP) is the extension of the 2D-KP to the case in which knapsack and items are three-dimensional boxes. The literature on this problem and on related variants is quite scarce compared to the one on the 2D-KP.

For what concerns exact methods, a recent review, also providing the outcome of extensive comparative experiments, was presented by Silva et al. (2019).

Diedrich et al. (2008) proposed a $(7+\varepsilon)$-approximation algorithm.

Egeblad and Pisinger (2009) generalized to the 3D-KP an iterative heuristic for 2D-KP. They adopted the sequence pair representation by Murata et al. (1996), which represents a solution by two item permutations, and obtained feasible packings through the algorithm by Pisinger (2007).

de Queiroz et al. (2012) developed a DP algorithm based on reduced raster points for a k-staged three-dimensional knapsack problem with guillotine constraints.

Baldi et al. (2012) introduced the 3D-KP with balancing constraints, where the packing center of mass is forced to lie into a specific boxed domain inside the knapsack. They presented a mixed-integer linear programming model and a heuristic algorithm.

We finally mention the *d-Dimensional Orthogonal Knapsack Problem* (D-KP), a further extension of the 2D-KP to the case in which items and knapsack are d-dimensional rectangles. Harren (2009) provided a $(1+1/2^d+\varepsilon)$-approximation algorithm for the D-KP.

5. Quadratic knapsack problems

The quadratic knapsack problem is a special case in the area of non-linear knapsack problems, characterized by a combinatorial structure. Before analyzing the problem in detail, we briefly introduce this general area in the next section.

5.1. Nonlinear knapsack problems

As previously stated, this survey is mainly addressed to the combinatorial optimization community, so we just mention in a succinct way arguments with a clear continuous, non-linear structure like those treated in this section. Consider a general optimization problem of the form

$$\max\ f(x) \tag{9}$$

$$\text{s.t. } g(x) \le c \tag{10}$$

$$x \in D \tag{11}$$

where $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $f(x)$ and $g(x)$ are continuous differentiable functions, c is a non-negative value, and $D\subseteq\mathbb{R}^n$. Now consider the special case where both $f(x)$ and $g(x)$ are separable functions, and D includes bounds and integrality requirements on (part of) the variables. Using the knapsack terminology, the resulting problem can be defined as follows. Given n items, with item j having a *profit function* $f_j(x_j)$ and a *weight function* $g_j(x_j)$, associated with n variables x_j limited by lower bounds l_j and upper bounds u_j $(j=1,\dots,n)$, determine non-negative x_j values such that the total weight does not exceed a given *capacity* c and the total produced profit is a maximum. A subset $\overline{N}$ of the x_j variables can be restricted to take integer values. Formally, the *Non-Linear Knapsack Problem* (NLKP) is:

$$\max \sum_{j\in N} f_j(x_j) \tag{12}$$

$$\text{s.t. } \sum_{j\in N} g_j(x_j) \le c \tag{13}$$

$$l_j \le x_j \le u_j \qquad (j=1,\dots,n) \tag{14}$$

$$x_j \text{ integer} \qquad (j\in\overline{N}\subseteq\{1,\dots,n\}), \tag{15}$$

where $f_j(x_j)$ and $g_j(x_j)$ are nonlinear, non-negative, non-decreasing functions. Note that, in general, there is no further assumption on functions $f_j(x_j)$ and $g_j(x_j)$, i.e., they can be nonconvex and nonconcave.

The NLKP has many applications in various fields such as portfolio selection, stratified sampling, resource-allocation, production planning, and resource distribution. Nonlinear knapsack problems have been studied in the books by Ibaraki and Katoh (1988), and Li and Sun (2006). For a general introduction to these problems, the reader is referred to the classical survey by Bretthauer and Shetty (2002) as well as to Lin (1998) who reviewed a number of knapsack problem variants, among them the MKP, the MdKP, and the *quadratic knapsack problem* discussed in the next section. D'Ambrosio and Martello (2011) proposed a fast and effective heuristic algorithm enriched by a local search post-optimization procedure for the NLKP. Relaxations and heuristics for the natural extension of the NLKP to the case of multiple knapsacks (*Non-Linear Multiple Knapsack Problem*, NLMKP) were proposed by D'Ambrosio et al. (2018). Other variants and generalizations of the NLKP and the NLMKP have been studied by Stefanov (2015), Elbassioni et al. (2019), Goos et al. (2020), and D'Ambrosio et al. (2020).

5.2. Quadratic knapsack problem

In the *Quadratic Knapsack Problem* (QKP), one is given a knapsack with capacity c and n items having profit p_j and weight w_j $(j = 1, \dots, n)$. An extra non-negative profit p_{ij} is earned if both items i and j are selected $(i, j = 1, \dots, n;\ i \neq j)$. The objective is to find a subset of items of total weight not exceeding the capacity, which maximizes the overall profit, calculated as the sum of the profits of the selected items and of their pairwise profits. Formally

$$\max \sum_{j=1}^{n} p_j x_j + \sum_{j=1}^{n-1} \sum_{i=1+j}^{n} p_{ij} x_i x_j \tag{16}$$

$$\text{s.t.} \sum_{j=1}^{n} w_j x_j \leq c \tag{17}$$

$$x_j \in \{0, 1\} \qquad (j = 1, \dots, n). \tag{18}$$

The problem is strongly $\mathcal{NP}$-hard, as it can be shown (see, e.g., Kellerer et al., 2004) by reduction from the *clique problem* (given an undirected graph, find a maximal complete subgraph). A thorough review of the QKP was provided by Pisinger (2007). The `C` code of an effective algorithm (`Quadknap`) for the exact solution of the problem, developed by Caprara et al. (1999), is available at http://hjemmesider.diku.dk/~pisinger/codes.html.

Exact solution. Billionnet and Soutif (2004a) presented a B&B algorithm based on the computation of an upper bound by means of a Lagrangian decomposition previously studied in Billionnet et al. (1999). The method, evaluated on a large set of randomly generated benchmarks, solved almost all instances with up to 150 variables, and with up to 300 variables for medium and low density profit matrices. Billionnet and Soutif (2004b) proposed three ways of linearizing the QKP into an equivalent *Mixed-Integer Program* and solved it with a commercial software. Pisinger et al. (2007) proposed an exact algorithm based on a variable fixing procedure, called *aggressive reduction*: it employs the upper bound proposed in Billionnet and Soutif (2004a) and another upper bound introduced in Caprara et al. (1999), as well as several heuristic algorithms to compute lower bounds. The procedure is followed by a B&B algorithm. The algorithm was able to solve large-size instances with up to 1500 variables. Wang et al. (2010) compared the performance of the CPLEX optimizer for quadratic integer programs and that of the method in Billionnet and Soutif (2004a), showing that the former outperforms the latter. Létocart et al. (2012) introduced a reoptimization method that improves the efficiency of the solution of the numerous continuous linear knapsack problems generated by the Lagrangian approaches proposed in Billionnet et al. (1999) and Caprara et al. (1999). Rodrigues et al. (2012) presented a linearization method which replaces the quadratic terms of the objective function with a set of linear constraints, and applied it within a B&B algorithm. Computational experiments showed that the algorithm is able to provide better results than that of Pisinger et al. (2007) for all the tested instances with low density and up to 200 variables. Fampa et al. (2020) introduced a parametric convex quadratic relaxation of the problem, developed a primal–dual interior point method to obtain the best possible bound, and proposed valid inequalities for the convex quadratic model. Worth is mentioning however that Schauer (2016) criticized the randomly generated instances normally adopted for computational experiments on algorithms for the QKP: he showed that, for a large family of classical test instances used in the literature, a basic greedy algorithm (consecutively insert the items sorted by non-increasing weight as long as they fit) produces solutions of value asymptotically very close to the optimum as the instance size tends to infinity. In Schauer (2016) an additional class of instances, for which finding a good solution is much harder, was introduced. Fomeni et al. (2022) presented a Cut-and-Branch algorithm that embeds a cutting-plane phase, primal heuristics, reduction rules, and a B&B phase, and tested it also on instances introduced in Schauer (2016).

Approximation. Taylor (2016) presented an approximation algorithm that guarantees, for any given $\varepsilon > 0$, an approximation ratio within $O(n^{2/5+\varepsilon})$ and a run time within $O(n^{9/\varepsilon})$. The quadratic profit p_{ij} can be seen as the cost of the edge between vertices i and j of an undirected weighted graph whose vertices represent the knapsack items. Pferschy and Schauer (2016) studied the QKP on special graph classes. They provided an FPTAS for the QKP on graphs of bounded treewidth, a PTAS for the QKP on planar graphs, and they showed that the QKP is strongly $\mathcal{NP}$-hard on 3-book embeddable graphs (generalizations of planar graphs). Wu et al. (2020) studied the convergency property of a logarithmic descent direction approximation algorithm.

Heuristics. Patvardhan et al. (2012) proposed a quantum inspired evolutionary algorithm, later improved and parallelized in Patvardhan et al. (2016). Yang et al. (2013) presented a GRASP and a Tabu search algorithm and successfully compared them with a heuristic approach developed by Xie and Liu (2007). Azad et al. (2014) proposed an artificial fish swarm algorithm having good computational performance, although not outperforming the method in Xie and Liu (2007). Fomeni and Letchford (2014) presented an effective DP heuristic obtained by modifying the classical DP approach for the KP01. The algorithm was enhanced by considering specific item ordering and a tie-breaking rule, giving solutions very close to the optimal ones. Toumi et al. (2015) introduced two variable neighborhood search heuristics and successfully compared them with the method in Yang et al. (2013). Cunha et al. (2016) investigated two Lagrangian heuristics and compared them with the method in Fomeni and Letchford (2014), showing that all these approaches provide good quality solutions. Chen and Hao (2017) proposed a method which introduces an additional cardinality constraint to decompose the problem into several disjoint sub-problems. The most promising sub-problems are then solved through reduction procedures and Tabu search. The resulting algorithm was compared with the algorithms in Xie and Liu (2007), Yang et al. (2013), Fomeni and Letchford (2014).

Variants and generalizations

Kellerer and Strusevich (2010) presented an FPTAS for a variant of the QKP, the *Symmetric Quadratic Knapsack Problem* (SQKP), which has several applications in machine scheduling. In the SQKP, the extra profit p_{ij} is obtained by multiplying the weight w_i by a non-negative integer coefficient, and it is earned if both i and j are selected or neither i nor j are selected. This FPTAS was later improved by Xu (2012). A generalization and extension of these works, also relevant for the SQKP, appeared in Kellerer and Strusevich (2016). In their survey, Kellerer and Strusevich (2012) reviewed the main results on the SQKP and reformulated a number of scheduling problems through it.

Schulze et al. (2020) introduced the *Rectangular Knapsack Problem* (RKP) as a special case of the QKP in which the items have profit $p_j = 0$, weight $w_j = 1$ (so that the capacity constraint corresponds to a cardinality constraint), and the extra profit has the form $p_{ij} = a_i b_j$, with $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ positive integer vectors. They designed a polynomial time algorithm that provides a constant approximation ratio of 4.5, and performed computational experiments on randomly generated instances with different correlation structures.

The *Quadratic Knapsack Problem with Conflict Graphs* (QKPCG) is a generalization of the QKP in which, as for the *knapsack problem with* *conflict graph* (see Part I (Cacchiani et al., 2022), Section 9), a given undirected graph $G = (V, E)$ defines the pairs of incompatible items that cannot be simultaneously selected. Shi et al. (2017) presented a large neighborhood search algorithm for the QKPCG. Dahmani and Hifi (2021) proposed a descent-based heuristic that combines local search and reactive search for diversification. The approach obtained better results than those in Shi et al. (2017). Dahmani et al. (2020) developed a method based on particle swarm combined with local search, and further improved the results in Dahmani and Hifi (2021).

The *Quadratic Knapsack Problem with Multiple Knapsack Constraints* (MdQKP) is a generalization of the QKP in which, as for the MdKP (see Section 3), the knapsack has d different capacities. The corresponding quadratic mathematical model is thus defined by (16), (6), and (7). The MdQKP was introduced by Wang et al. (2012a), who compared the quadratic model with three alternative linearizations, solved with CPLEX on test instances with up to 800 variables. The computational experiments showed that, for large instances, the quadratic model outperforms by a large margin the linearized ones both in terms of solution quality and solution time.

5.2.1. Quadratic multiple knapsack problem

The *Quadratic Multiple Knapsack Problem* (QMKP) is a generalization of the QKP in which, as for the MKP (see Section 2), one is given m knapsacks with capacities c_i ($i = 1, \dots, m$) and n items having profit p_j and weight w_j ($j = 1, \dots, n$). In addition, as for the QKP, each pair of distinct items i and j gives an extra profit p_{ij} if both items are assigned to the same knapsack. The objective is to select m disjoint subsets of items so that each subset is assigned to a knapsack whose capacity is no less than the total weight of its items, and the total profit (sum of the profits of the selected items and of the pairwise profits of items assigned to the same knapsack) is maximized. Formally

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} p_j x_{ij} + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \sum_{i=1}^{m} p_{kj} x_{ij} x_{ik} \tag{19}$$

s.t. (2)–(4).

Exact solution. The first exact solution approach to the QMKP is due to Bergman (2019). He introduced an exponential-size ILP model solved through a *Branch-and-Price* (B&P) algorithm. Computational tests showed that the method outperforms commercial optimization solvers. The B&P algorithm was also applied to the automated table event seating problem (a variant of the QMKP in which each item is required to be placed in some knapsack). Galli et al. (2021) studied polynomial size formulations and upper bounds for the QMKP. They proposed surrogate and Lagrangian relaxations, providing theoretical properties and dominances, as well as extensive computational experiments. An effective B&B algorithm for the QMKP was recently presented by Fleszar (2022).

Heuristics. García-Martínez et al. (2014a,b) introduced two metaheuristics for the QMKP, one based on strategic oscillation, one consisting of a Tabu-enhanced greedy approach. They showed that their methods are competitive with the state-of-the-art algorithms. Chen and Hao (2014a) presented an algorithm that combines a threshold based exploration with a descent based improvement and successfully compared it with the methods in García-Martínez et al. (2014a,b). Chen et al. (2016) proposed an evolutionary path relinking approach and showed that it successfully competes with the algorithms in García-Martínez et al. (2014a,b), and Chen and Hao (2014a). Peng et al. (2016) proposed an ejection chain method with an adaptive perturbation mechanism and positively compared it with the algorithms in García-Martínez et al. (2014a,b). Qin et al. (2016) presented a Tabu search based on local search algorithms that consider both feasible and infeasible solutions, and favorably compared it with the previous literature on the QMKP.

Variants and generalizations

Olivier et al. (2021) introduced the *Quadratic Multiknapsack Problem with Conflicts and Balance Constraints* (QMPCBC) as a generalization of the QMKP which includes pairwise conflicts between items and asks for packing all the items into the knapsacks in such a way that the load of every knapsack is within a lower and an upper bound. They proposed a constraint programming model and two quadratic integer programming formulations. The best computational results were obtained by a column generation algorithm that solves one of the quadratic formulations and adopts a constraint program for the pricing subproblem.

Saraç and Sipahioglu (2014) introduced the *Generalized Quadratic Multiple Knapsack Problem* (GQMKP), which extends the QMKP as follows: (i) the items are partitioned into classes, and a setup cost has to be paid when at least one item from a class is assigned to a knapsack; (ii) additional assignment constraints impose that some items and classes cannot be assigned to certain knapsacks; (iii) the profits depend on both items and knapsacks. The authors introduced a mathematical model and proposed two heuristic approaches: a genetic algorithm and a hybrid method which combines the former with a subgradient approach. Chen and Hao (2016) developed a memetic algorithm which combines a crossover operator with multi-neighborhood simulated annealing, and successfully compared it with the methods in Saraç and Sipahioglu (2014). Avci and Topaloglu (2017) developed a multi-start iterated local search algorithm embedding an adaptive perturbation mechanism and a Tabu list. The proposed approach was favorably compared with the previous literature. Adouani et al. (2022) introduced a new ILP model and a matheuristic approach which integrates variable local search techniques, an adaptive perturbation mechanism, and quadratic integer programming. A hybrid evolutionary search approach that embeds a number of metaheuristic techniques was recently presented by Zhou et al. (2022). Experimental results show that this is the algorithm providing so far the best results in the literature for the GQMKP.

6. Other knapsack problems

6.1. Online knapsack problems

In the *Online Knapsack Problem* (OKP), the information on the items is given one by one during the packing process. More specifically, only after a decision is made on a given item, the information on the next one is then available and it has to be decided immediately whether to pack this item (and keep it in the knapsack forever) or discard it (without the possibility of packing it again). As for the offline version (KP01), the objective of the OKP is to maximize the profit under the capacity constraint.

An online algorithm is said to be *r-competitive* if there exists a constant r (the *competitive ratio*) such that, for any possible incoming sequence of items, the ratio between its performance and the performance achieved by an exact offline algorithm is bounded by r. Han et al. (2015) provided a simple randomized 2-competitive algorithm for the online version of the *Subset Sum Problem* (SSP, see Part I (Cacchiani et al., 2022), Section 4), referred to in the literature as the *unweighted* case, and showed that it is the best possible one. Böckenhauer et al. (2014) analyzed complexity issues of the OKP.

Other minor variants of the OKP have been studied. Han and Makino (2016) showed that no algorithm has a bounded competitive ratio for the minimization version of the OKP, in which the goal is to find a subset of items such that the sum of their weights is at least equal to the knapsack capacity and the sum of their profits is minimized. Cygan et al. (2016) investigated the online variant of the MKP. Thielen et al. (2016) considered the OKP *with incremental capacity*, in which the knapsack capacity increases by a constant amount in each time period. Navarra and Pinotti (2017) addressed the *knapsack of unknown capacity problem*, an online variant of the KP01 where the capacity c of the knapsack is unknown and it is revealed during the packing process. When the knapsack capacity is revealed, no other item can be inserted and the first item in the sequence of packed items that does not completely fit in the knapsack is removed, as well as all the subsequent inserted items.

Iwama and Taketomi (2002) introduced the *Online Removable Knapsack Problem* (ORKP), a version of the OKP in which the already stored items can be removed during the packing process in order to accept the current one. They provided an r-competitive algorithm for the unweighted case, where $r = (1+\sqrt{5})/2 \approx 1.618$. For the general case, they showed that no algorithm can have a bounded competitive ratio. They also provided some tight bounds for the ORKP with resource augmentation (see Iwarna and Zhang, 2010), which allows online algorithms to use more capacity than offline algorithms. Han et al. (2014b) addressed the ORKP under a convex function $f(\cdot)$ such that $p_j = f(w_j)$ $(j = 1,\ldots,n)$. They proposed a greedy online algorithm with competitive ratio 2 and an improved online algorithm with competitive ratio $5/3$. Han et al. (2015) introduced a simple randomized 2-competitive algorithm and provided a lower bound $1+1/e$ for the competitive ratio. For the unweighted case, they proposed a randomized $10/7$-competitive algorithm and proved that there exists no randomized online algorithm with competitive ratio less than $5/4$. Han et al. (2019) studied the ORKP under a concave function $f(\cdot)$ such that $p_j = f(w_j)$ $(j = 1,\ldots,n)$, provided a lower bound $\max\{q, \frac{f'(0)}{f(1)}\}$, and showed that, for the concave case, the competitive ratio of the online algorithm proposed in Iwama and Taketomi (2002) is $\frac{f'(0)}{f(1/q)}$. In addition, they proposed another online algorithm with a competitive ratio $\frac{f'(0)}{f(1)}+1$.

Several variants of the ORKP have been also considered in the literature. Han and Makino (2010) studied the ORKP with limited cuts, in which items are allowed to be cut/packed at most $k \geq 1$ times. Han et al. (2014a) addressed the unweighted case of the ORKP with removal cost. In this variant, if an item is packed into the knapsack making the current total weight to exceed the knapsack capacity, then some items in the knapsack have to be removed, paying some additional removal costs. The aim is to maximize the sum of the total weight of the packed items minus the total removal cost. Han et al. (2014a) provided competitive ratios for the cases of proportional (to the weights) or unit removal costs. Han and Makino (2016) addressed the minimization version of the ORKP and obtained the following results: they provided an 8-competitive deterministic algorithm, proposed a $2e$-competitive randomized algorithm, derived a lower bound of 2 for deterministic algorithms, and proposed a 1.618-competitive deterministic algorithm for the unweighted case, showing that this is the best possible.

6.2. Multiobjective knapsack problems

The *Multiobjective Knapsack Problem* (MOKP) is a generalization of the KP01 in which each item has $t \geq 2$ non-negative integer profits p_{jk} $(j = 1,\ldots,n; k = 1,\ldots,t)$ and the total profit objectives are maximized in the Pareto sense, i.e.,

$$\max\left(\sum_{j=1}^{n} p_{j1}x_j, \ \ldots, \ \sum_{j=1}^{n} p_{jt}x_j\right). \tag{20}$$

Most results in the literature concern the case in which $t = 2$, usually called the *Biobjective Knapsack Problem* (BOKP). Both the MOKP and the BOKP generalize the KP01 and hence are $\mathcal{NP}$-hard. We refer the reader to the fundamental book by Ehrgott (2005) on multicriteria optimization for a thorough discussion on the complexity status of the problems. More recently, a specific survey on the MOKP has been presented by Lust and Teghem (2012). In the following, we mostly discuss papers appeared after its publication.

Exact solution. A number of exact methods for the BOKP have been proposed. Rong et al. (2011), Rong and Figueira (2012) presented a DP algorithm and evaluated different reduction techniques aimed at decreasing the number of states explored by the algorithm. Algorithmic improvements of these methods were proposed by Figueira et al. (2013) and Rong and Figueira (2014). The use of special data structures for the BOKP was evaluated by Correia et al. (2018). Daoud and Chaabane (2018) proposed a reduction strategy for the BOKP and showed, through computational experiments, its effectiveness in reducing the search space.

Approximation and heuristics. Cerqueus et al. (2017) studied the effectiveness of various branching heuristics for the BOKP and proposed a dynamic adaptive branching strategy. Their computational experiments showed however that the proposed techniques are not useful in practice, as they require high CPU times. The MOKP has been also tackled through metaheuristics: we mention in particular a quantum-inspired evolutionary algorithm by Lu and Yu (2013), a Bayesian estimation of distribution algorithm by Martins et al. (2018), and a swarm intelligence approach by Zouache et al. (2018).

Variants and generalizations. Kozanidis (2009) proposed an algorithm for the biobjective version of the multiple choice knapsack problem studied in Kozanidis and Melachrinoudis (2004) (see Part I). Rong et al. (2013) proposed hybrid solution approaches for a variant of the MOKP which also includes so-called k-min objectives aiming at the maximization of the kth smallest objective coefficients. Castillo-Zunino and Keskinocak (2021) studied a multiple version of the MOKP, in which the items are partitioned into groups and either all or none of the items from a group are assigned.

6.2.1. Multidimensional multiobjective knapsack problems

Few results on the multidimensional version of multiobjective knapsack problems appeared after the publication of the survey by Lust and Teghem (2012). Cerqueus et al. (2015) studied the bidimensional version of the BOKP: they introduced a convex surrogate upper bound and presented two exact algorithms and a heuristic approach for its computation. Shah and Reed (2011), Mavrotas et al. (2015), and Luo et al. (2019) presented extensive computational evaluations of various metaheuristic approaches.

7. Conclusions

After over sixty years of intensive research, knapsack problems are a lively research area in a number of scientific fields. This work covers the recent developments in the richest of such fields, combinatorial optimization. The two parts of the survey list over 450 different papers, mostly appeared after 2004, the publication year of the latter of the two classical books specifically dedicated to these topics.

Part I (Cacchiani et al., 2022) covers the classical single knapsack problems and their many variants and generalizations: subset sum, item types, setup, multiple-choice, conflict graphs, precedences, sharing, bilevel, robust, among others. The section on extensions and generalizations provides pointers to the variants that are especially attractive from the point of view of possible future investigations.

Part II is mainly devoted to multiple, multidimensional (vector and geometric), and quadratic knapsack problems, but also contains a succinct treatment of online and multiobjective knapsack problems. We have mentioned, in the concluding remarks of Part I, a number of relevant real-world applications for knapsack problems. In this second part, we noticed other relevant applications, arising in emergency military and humanitarian contexts (the MKAP), steel production (the MKPCC), machine scheduling (the MKP), cutting of material (the 2D-KPGC), and container loading (the 3D-KP), among others. We believe that new applications will continue to appear in the next years.

Recent research has produced a good number of new exact algorithms, but there is still room for improvement on many problem variants. While some knapsack problems, as the KP01 and the SSP, are solved extremely well by the algorithms available in the literature, and difficult instances involve very large coefficients and a huge number of items, for other problems, as the 2D-KP, instances with just 30 items are still unsolved to proven optimality despite decades of research. We did not include here a section on hot topics as the publication dates of most reviewed articles (over 70% of the articles in the bibliography appeared in the last decade) indicate that all the main variants and generalizations of these problems still undergo intensive analysis and hence are attractive research areas to researchers interested in pursuing investigations in this fascinating area.

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Appendix. Acronyms

2D-UKPGC Two-Dimensional Unbounded Knapsack Problem with Guillotine Constraints.

2D-KPGC Two-Dimensional Knapsack Problem with Guillotine Constraints.

2D-KP Two-Dimensional Knapsack Problem.

3D-KP Three-Dimensional Geometric Orthogonal Knapsack Problem.

B&B Branch-and-Bound.

B&C Branch-and-Cut.

B&P Branch-and-Price.

BCMKP Budget-Constrained Multiple Knapsack Problem.

BOKP Biobjective Knapsack Problem.

C&P Cutting and Packing.

D-KP d-Dimensional Orthogonal Knapsack Problem.

DP Dynamic Programming.

EPTAS Efficient Polynomial Time Approximation Scheme.

FCMKP Fixed-Charge Multiple Knapsack Problem.

FPTAS Fully Polynomial Time Approximation Scheme.

GQMKP Generalized Quadratic Multiple Knapsack Problem.

GRASP Greedy Randomized Adaptive Search Procedure.

ILP Integer Linear Programming.

KP01 0–1 Knapsack Problem.

KPS Knapsack Problem with Setup.

LP Linear Programming.

MdKP Multidimensional Knapsack Problem.

MdKP-GUB Multidimensional Knapsack Problem with Generalized Upper Bound Constraints.

MdMCKP Multidimensional Multiple-Choice Knapsack Problem.

MDMdKP Multidemand Multidimensional Knapsack Problem.

MdQKP Quadratic Knapsack Problem with Multiple Knapsack Constraints.

MKAP Multiple Knapsack Assignment Problem.

MKP Multiple Knapsack Problem.

MKPC Multiple Knapsack Problem with Conflicts.

MKPCC Multiple Knapsack Problem with Color Constraints.

MKPS Multiple Knapsack Problem with Setup.

MMdKP Multiple Multidimensional Knapsack Problem.

MMdKPF Multiple Multidimensional Knapsack with Family-Split Penalties.

MOKP Multiobjective Knapsack Problem.

MSSP Multiple Subset Sum Problem.

NLKP Non-Linear Knapsack Problem.

OKP Online Knapsack Problem.

ORKP Online Removable Knapsack Problem.

PTAS Polynomial Time Approximation Scheme.

QKP Quadratic Knapsack Problem.

QKPCG Quadratic Knapsack Problem with Conflicts Graphs.

QMKP Quadratic Multiple Knapsack Problem.

QMPCBC Quadratic Multiknapsack Problem with Conflicts and Balance Constraints.

RKP Rectangular Knapsack Problem.

SQKP Symmetric Quadratic Knapsack Problem.

SSP Subset Sum Problem.

SVP Shortest Vector Problem.

UMdKP Unbounded Multidimensional Knapsack Problem.

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