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An approximation scheme for the two-stage, two-dimensional knapsack problem

Alberto Caprara [a], Andrea Lodi [a,*], Michele Monaci [b]

[a] DEIS, Università di Bologna, Viale Risorgimento 2, I-40136 Bologna, Italy

[b] DEI, Università di Padova, Via Gradenigo 6/A, I-35131 Padova, Italy

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ABSTRACT

We present an approximation scheme for the two-dimensional version of the knapsack problem which requires packing a maximum-area set of rectangles in a unit square bin, with the further restrictions that packing must be orthogonal without rotations and done in two stages. Achieving a solution which is close to the optimum modulo a small *additive* constant can be done by taking wide inspiration from an existing asymptotic approximation scheme for two-stage two-dimensional bin packing. On the other hand, getting rid of the additive constant to achieve a canonical approximation scheme appears to be widely nontrivial.

1. Introduction

The classical *Knapsack Problem* (KP) is a central problem in combinatorial optimization. In this paper, we will consider the two-dimensional (geometric) generalization of KP in which items correspond to rectangles of specified size and have to be packed in a bin, corresponding to another (larger) rectangle. The objective is the maximization of the area of the rectangles packed; in other words, the profit of each item is equal to its area. In this paper, we will restrict our attention to the *two-stage* case, referred to as the *Two-Dimensional Shelf Knapsack Problem* (2SKP) and formally defined below. We show that a suitable adaptation of the asymptotic approximation scheme in [1] for the analogous two-dimensional generalization of the bin packing problem leads to an "asymptotic" approximation scheme for 2SKP, "asymptotic" meaning that there is an arbitrarily small additive constant in the guarantee. Moreover, we show how to turn this into an (absolute) approximation scheme by a careful handling of pathological cases, which is the main contribution of the paper.

To the best of our knowledge, this is the first nontrivial approximation result for the 2SKP. For the version without the two-stage packing requirement (but with the restriction that edges of the items are packed parallel to the edges of the bin) and arbitrary item profits, a basic result of Steinberg [2] easily leads to an approximation guarantee arbitrarily close to 3 [3]. The best known approximation algorithm for the problem, due to Jansen and Zhang [4], has an approximation guarantee arbitrarily close to 2. To the best of our knowledge, no inapproximability result is known. Recently, [5] proposed an approximation scheme for the case in which, as in this paper, the item profits are equal to their areas. However, this result has no direct implication on the two-stage case.

Formal problem definition and notation

In the 2SKP, we are given n items, the j-th having *width* $w_j \leq 1$ and *height* $h_j \leq 1$; we also say that the j-th item has *size* (w_j, h_j). We will denote by $p_j := w_j \cdot h_j$ the *profit* of the j-th item. With an abuse of notation, we will identify items with

* Corresponding author. Tel.: +39 051 2093029; fax: +39 051 2093073.
E-mail addresses: alberto.caprara@unibo.it (A. Caprara), andrea.lodi@unibo.it (A. Lodi), monaci@dei.unipd.it (M. Monaci).

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their indices. Accordingly, the set of items given on input will be denoted by $N = \{1, \dots, n\}$. Given a set $S \subseteq N$, we let $w(S) := \sum_{j \in S} w_j$, $h(S) := \max_{j \in S} h_j$, and $p(S) := \sum_{j \in S} p_j$. A *shelf* is a set $S \subseteq N$ whose *width* $w(S)$ does not exceed 1. The *height* of the shelf is given by $h(S)$ and its *profit* by $p(S)$. We let $\mathbb{N}$ denote the set of integer nonnegative numbers (including 0).

Formally, the 2SKP requires that the items be packed in shelves of overall height at most 1 (that fit in a unit bin) so as to maximize the overall profit of the shelves. Note that, when $h_j = 1/m$ ($j \in N$) for some integer m, the 2SKP coincides with the variant of the well-known bin packing problem in which one must pack a maximum-size subset of the items in m bins [6], and therefore it is strongly NP-hard.

Given a set of items N, let $\text{opt}(N)$ denote the value of the optimal solution for N. A *Polynomial Time Approximation Scheme* (PTAS) is an algorithm that receives on input also a required *accuracy* $\varepsilon > 0$, runs in time polynomial in the size of the item set N, and produces a solution of value at least $(1 - \varepsilon) \cdot \text{opt}(N)$, for every N; see e.g. [7].

In our algorithm, we will often pack some of the items by *Next Fit Decreasing Height* (NFDH) [8], in which the items are considered in *decreasing order of height* and packed in shelves by a next fit policy, closing the current shelf and starting a new one when the current item does not fit in the shelf. The packing stops when the next shelf would not fit on top of those already created.

2. An "Asymptotic" PTAS

Noting that the optimal 2SKP solution value is not larger than 1, in this section we will present an algorithm that finds a solution of value at least $(1 - 8\varepsilon)\text{opt}(N) - 2\varepsilon$ for every $\varepsilon > 0$. The method, widely based on the ideas in [1], is presented in the following and analyzed in Section 2.3. In Section 3 we discuss how to get rid of the additive constant -2ε. For the sake of simplicity, we will assume that ε is sufficiently small and that $1/\varepsilon$ is an integer.

The general structure of the algorithm is the following. First, we partition the items into sets A_i having similar heights, and remove some items, so as to ensure that there is a gap between the item heights in distinct sets A_i and A_j. This allows us to pack in each shelf only items from the same set A_i, i.e. to avoid shelves with items having significantly different heights. This simplifies things a lot but unavoidably introduces an (arbitrarily small) additive constant in the performance guarantee.

The main difficulty that has to be faced for the 2SKP with respect to the bin packing counterpart [1] is that only a subset of the items will be packed, and it is not clear how to pack the items of each set A_i in shelves without knowing the space in the final solution that will be occupied by these shelves. This forces us to perform some additional "brute force" (but still polynomial) enumeration. Specifically, for the tallest item set A_0, by enumeration we can guess the heights of the shelves in the optimal solution and the packing of the items in these shelves. The nontrivial part is to guess how the residual height of the bin is subdivided for the shelves with items in the other sets A_i. To this aim, for each set A_i we first find the (approximately) best packing for (approximately) each of the possible total heights of the associated shelves. This is done by suitably reducing the heights and widths of the items to a constant number and then by solving a constant-size (mixed-)integer linear program. Finally, once we know the profit that we would gain by reserving in the bin a given height for the shelves from each set A_i, we find the (approximately) best subdivision by (approximately) solving a multiple-choice KP.

The formal structure of our APTAS for the 2SKP is shown in Fig. 1, in which Steps (a), (b) and (e) are defined formally. Steps (c) and (d) will be described in detail in Sections 2.1 and 2.2, showing that the running time is polynomial for every fixed value ε.

2.1. Step (c)

Of course, we restrict attention to the $O(n)$ sets A_i, $i \in \mathbb{N} \setminus \{0\}$, such that $A_i \neq \emptyset$. For each such A_i, rounding the heights to the nearest multiple of ε^{f+it} from above in Step (c.1) leads to up to $g_i \leq 1/\varepsilon^t$ different heights in the set, denoted by $\overline{h}_1, \dots, \overline{h}_{g_i}$. Let $\underline{h}_j$ denote the *rounded-down* height of an item j, i.e. the nearest multiple of ε^{f+it} from below.

Letting n_j be the number of items having height $\overline{h}_j$, in the solution there may be up to n_j shelves of this height. Correspondingly, we enumerate in Step (c.2) each possibility for the overall height H of the shelves with items in A_i in the solution by considering each vector $(m_1, \dots, m_{g_i})$ such that $\sum_{j=1}^{g_i} m_j \overline{h}_j \leq 1$ and $0 \leq m_j \leq n_j$ for $j = 1, \dots, g_i$, setting $H := \sum_{j=1}^{g_i} m_j \overline{h}_j$. The number of these vectors is at most $\prod_{j=1}^{g_i} (n_j + 1)$; i.e., by a very rough estimation, $O((|A_i| + 1)^{g_i}) = O((n + 1)^{g_i})$.

For each value of H, we find a near optimal packing of the items in A_i in shelves with total height at most H as follows. First, we pack the (wide and thin) items in A_i in shelves of total height at most H by NFDH (allowing thin items to be cut, even if this is inessential). If all the items in A_i can be packed, we clearly have the optimal packing of these items in shelves of total height at most H and nothing else has to be done. Otherwise, knowing that the profit of this optimal packing is at least $\varepsilon^{t-1}H/4$ (as shown in Lemma 5), we consider each subset S_{ij} defined by the wide items in A_i with the *same height* $\overline{h}_j$ after Step (c.1) and apply Steps (c.2.1), (c.2.2), and (c.2.3), illustrated in detail in the following.

2.1.1. Step (c.2.1): preprocessing

In the preprocessing part, we define a subset $R_{ij} \subseteq S_{ij}$ as follows. For each $k = 1/\varepsilon, \dots, 1/\varepsilon^{t+1}$, we consider each set $T_{ijk} := \{\ell \in S_{ij} : w_\ell \in (k\varepsilon^{t+1}, (k+1)\varepsilon^{t+1}]\}$, letting for convenience $T_{ij1/\varepsilon}$ contain also the items $\ell \in S_{ij}$ with $w_\ell = \varepsilon^t$.

Algorithm APTAS$_{2SKP}$:

(a) **Decomposition:**

(a.1) Guess the integer f $(1 \le f \le t := 1/\varepsilon)$ and for $i \in \mathbb{N}$ *remove* all the items with height in $(\varepsilon^{f+it}, \varepsilon^{f+it-1})$. **Comment:** See Lemma 2.

(a.2) Let A_i be the set of the items with heights in $[\varepsilon^{f+it-1}, \varepsilon^{f+(i-1)t}]$ for each $i \in \mathbb{N}$. Form the shelves in the final solution by considering *separately* each set A_i. **Comment:** See Lemma 3.

(b) **Item Subdivision and Vertical Melting:**
For each set A_i, $i \in \mathbb{N}$:

(b.1) Separate *wide* items (having width at least ε^t) from *thin* items (having width smaller than ε^t).

(b.2) Allow thin items to be cut *vertically*.

(c) **Definition of Low Shelves:**
For each set A_i, $i \in \mathbb{N} \setminus \{0\}$:

(c.1) Round-up all *heights* to the nearest multiple of ε^{f+it}, letting $\overline{h}_1, \dots, \overline{h}_{g_i}$ be the distinct heights after rounding and $n_1, \dots, n_{g_i}$ the numbers of items of each height. For the profit (area) computation, consider *rounded-down* heights. **Comment:** See Lemma 4.

(c.2) For each vector $(m_1, \dots, m_{g_i})$ such that $0 \le m_j \le n_j$ $(j = 1, \dots, g_i)$ and $H := \sum_{j=1}^{g_i} m_j \overline{h}_j \le 1$, if not all items in A_i can be packed in shelves of total height at most H by NFDH:

(c.2.1) Preprocess the wide items in A_i. **Comment:** See Section 2.1.1 and Lemma 6.

(c.2.2) Round the *widths* of the preprocessed items with the *same height* after Step (c.1) by *linear grouping*. For the profit (area) computation, consider *rounded-down* widths. **Comment:** See Section 2.1.2 and Lemma 7.

(c.2.3) Pack some items in A_i in shelves of overall height at most H and overall maximum profit by enumeration of the packings of wide items in A_i combined with the solution of (1)-(6). **Comment:** See Section 2.1.3.

(d) **Definition of Tall Shelves:**
For each subset $\{j_1, \dots, j_s\} \subseteq A_0$ such that $\sum_{\ell=1}^{s} h_{j_\ell} \le 1$, and for each packing of the wide items in A_0 in s shelves of height $h_{j_1}, \dots, h_{j_s}$:

(d.1) Pack the thin items in A_0 in these shelves by a variant of NFDH. **Comment:** See Section 2.2.1.

(d.2) Find the optimal subdivision of the residual bin height $1 - \sum_{\ell=1}^{s} h_{j_\ell}$ to the shelves found in Step (c) by applying a PTAS to the associated *multiple-choice KP*. **Comment:** See Section 2.2.2.

(e) **Vertical Solidification:**
Remove the thin items that are cut vertically in the solution, if any. **Comment:** See Lemma 8.

Fig. 1. General structure of the APTAS for the 2SKP.

Noting that at most H/ε^{f+it-1} shelves with items in S_{ij} fit in a vertical space of H, and that each shelf contains at most $1/\varepsilon^t$ wide items, for $k = 1/\varepsilon, \dots, 1/\varepsilon^{t+1}$ we assign to R_{ij} the up to $H/\varepsilon^{f+(i+1)t-1}$ items in T_{ijk} with smallest width (possibly, all the items in T_{ijk} are assigned to R_{ij}). Afterwards, only the items in R_{ij} are considered in the packing (for each $j = 1, \dots, g_i$). Letting $r_{ij} := |R_{ij}|$, note that $r_{ij} \le H/\varepsilon^{f+(i+2)t}$.

2.1.2. Step (c.2.2): width grouping

The grouping operation is the classical one proposed first in [9]: for $j = 1, \dots, g_i$, if $H/\varepsilon^{f+(i+2)t} < 8/\varepsilon^{5t}$ we leave the widths of all the items in R_{ij} unchanged, defining for convenience $\overline{w}_\ell := \underline{w}_\ell := w_\ell$ for $\ell \in R_{ij}$. Otherwise, letting $w_1 \ge w_2 \ge \dots \ge w_{r_{ij}}$ be the widths of the items in R_{ij} after sorting, and $p_i := \lfloor H/4\varepsilon^{f+(i-3)t} \rfloor$, we form $d_{ij} := \lceil r_{ij}/p_i \rceil$ groups of p_i consecutive items, starting from the p_i widest ones and letting the last group contain $r_{ij} - (d_{ij} - 1)p_i$ items (possibly forming only one group). For each group, the rounded-up width $\overline{w}_\ell$ of each item ℓ is set to the original width of the first (widest) item. Moreover, for the profit computation, we use the *rounded-down* width $\underline{w}_\ell$, defined as the largest width in the group following that of ℓ ($\underline{w}_\ell := 0$ if ℓ is in the last group).

A simple calculation shows that, after grouping, the number d_{ij} of different widths in R_{ij} is $O(1/\varepsilon^{5t})$. Therefore, the number of different sizes for the wide items in A_i after preprocessing is $s_i = O(1/\varepsilon^{6t})$, recalling that $g_i = O(1/\varepsilon^t)$.

2.1.3. Step (c.2.3): solution by constant-size mixed-integer linear programs

After Step (c.2.2), the number of different sizes for the wide items in A_i is bounded by a constant s_i, and each of the associated widths is at least ε^t. Let $(\overline{w}_\ell, \overline{h}_\ell)$ be the ℓ-th size and n_ℓ the corresponding number of items ($\ell = 1, \ldots, s_i$), and note that the number of different heights for the wide items is $O(1/\varepsilon^t)$. Let also $\underline{w}_\ell$ and $\underline{h}_\ell$ be the rounded-down width and height for the ℓ-th size. Moreover, after rounding the heights in Step (c.1), we have a constant number $q \le g_i = O(1/\varepsilon^t)$ of *groups* of thin items, each containing all thin items with the same height. Let $(w_j, \overline{h}_j)$ $(j = 1, \ldots, q)$ denote the size of a single item equivalent to all thin items of height $\overline{h}_j$ (recalling that the widths of thin items are not changed and that they may be cut vertically), with $\overline{h}_1 > \overline{h}_2 > \cdots > \overline{h}_q$, and $\underline{h}_j$ the associated rounded-down height.

Using the same notation as in [1], for A_i we let a *shelf configuration* S be defined by a *height* $\overline{h}_S$, chosen among the $O(1/\varepsilon^t)$ possible heights of the wide and thin items, and by a number $n_{\ell S}$ of wide items of size $(\overline{w}_\ell, \overline{h}_\ell)$ contained in S for $\ell = 1, \ldots, s_i$. A shelf configuration S must satisfy $n_{\ell S} = 0$ if $\overline{h}_\ell > \overline{h}_S$ ($\ell = 1, \ldots, s_i$) and $\overline{w}_S := \sum_{\ell=1}^{s_i} n_{\ell S}\overline{w}_\ell \le 1$, where $\overline{w}_S$ denotes the width of S. Note that $\overline{h}_S$ may be larger than all heights of the wide items in S, to allow thin items with larger heights to fit. Since the width of each wide item is at least ε^t, we have $n_{\ell S} \le 1/\varepsilon^t$ and therefore a bound of $O(1/\varepsilon^{t\, s_i})$ on the number of possibilities for the $n_{\ell S}$ vectors. Let $p_S := \sum_{\ell=1}^{s_i} n_{\ell S}\underline{w}_\ell\underline{h}_\ell$ denote the profit of S, noting that it considers only the wide items in S. Let $\mathcal{S}$ be the collection of all possible shelf configurations, noting that $|\mathcal{S}| = O(1/\varepsilon^t \cdot 1/\varepsilon^{t\, s_i})$. Note that, if a shelf associated with $S \in \mathcal{S}$ is formed in the solution, the residual horizontal space $1 - w_S$ can be used to pack thin items whose height does not exceed $\overline{h}_S$. For $j = 1, \ldots, q$, let $\mathcal{S}_j := \{S \in \mathcal{S} : \overline{h}_S \ge \overline{h}_j\}$ denote the shelf configurations in which thin items of height $\overline{h}_j$ may fit, noting that $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots \subseteq \mathcal{S}_q \subseteq \mathcal{S}$.

We introduce the integer variables x_S, $S \in \mathcal{S}$, which represent the number of shelves associated with each shelf configuration S in the solution, and the continuous variables z_j, $j = 1, \ldots, q$, which represent the total width of thin items of height $\overline{h}_j$ $(j = 1, \ldots, q)$ that are packed. The problem to optimally pack the items associated with A_i in shelves of total height at most H reads

$$\text{maximize} \quad \sum_{S\in\mathcal{S}} p_S x_S + \sum_{j=1}^{q} \underline{h}_j z_j \tag{1}$$

$$\text{subject to} \quad \sum_{S\in\mathcal{S}} \overline{h}_S x_S \le H, \tag{2}$$

$$\sum_{S\in\mathcal{S}} n_{\ell S} x_S \le n_\ell, \quad \ell = 1, \ldots, s_i, \tag{3}$$

$$\sum_{S\in\mathcal{S}_j} (1 - \overline{w}_S) x_S \ge \sum_{k=1}^{j} z_k, \quad j = 1, \ldots, q, \tag{4}$$

$$0 \le z_j \le w_j, \quad j = 1, \ldots, q, \tag{5}$$

$$x_S \ge 0 \text{ integer}, \quad S \in \mathcal{S}. \tag{6}$$

Objective function (1) and constraint (2) require the maximization of the profit of the items in the shelves guaranteeing that the overall height of the shelves is not larger than H. Constraints (3) ensure the availability of the wide items that are packed. Finally, recalling that thin items can be cut vertically, constraints (4) guarantee that thin items of height $\overline{h}_j$ are packed only in shelves associated with shelf configurations in $\mathcal{S}_j$, whereas constraints (5) impose that no more thin items than those available are packed. Since the number of variables is constant, this model can be solved in polynomial time by the algorithm of [10]. Note that the number of constraints is $O(1/\varepsilon^{6t})$.

2.2. Step (d)

For set A_0, we do not perform any height and/or width rounding, but just allow the thin items to be cut vertically. Noting that each height is at least ε^{t-1}, we have that at most $1/\varepsilon^{t-1}$ shelves with these items can be packed in the bin. This yields $O(|A_0|^{1/\varepsilon^{t-1}})$ possibilities for the heights of the shelves with items in A_0 in the solution. For each shelf, at most $1/\varepsilon^t$ wide items may fit in the shelf, yielding $O(|A_0|^{1/\varepsilon^t})$ possibilities for the single shelf and, overall, $O((|A_0|^{1/\varepsilon^t})^{1/\varepsilon^{t-1}}) = O(n^{1/\varepsilon^{2t-1}})$ choices for the packing of the wide items in A_0 in shelves. For each choice, having fixed the height of each shelf along with the set of wide items it contains, we apply Steps (d.1) and (d.2), illustrated in detail in the following.

2.2.1. Step (d.1): packing the thin items in A_0

We pack the thin items by the following variant of NFDH. Note that the height of a shelf may be *strictly larger* than the maximum height of the wide items it contains.

Initially, all shelves are *open* and all thin items *unpacked*. We consider the tallest open shelf, whose height is (say) h, and the tallest unpacked thin item j that fits in the shelf (i.e. h_j is maximum subject to $h_j \le h$). If j completely fits in the shelf (i.e. there is enough horizontal residual space), we pack j in the shelf and iterate. Otherwise, we pack the fraction of j that fits in the shelf, *close* the latter, define a new thin item that represents the unpacked fraction of j, and iterate. The procedure terminates when either all shelves are closed, or all thin items are packed, or the minimum height of an unpacked thin item is larger than the maximum height of an open shelf. It is easy to check that the packing produced is best possible given the choice above.

2.2.2. Step (d.2): packing the low shelves

The problem to be solved in Step (d.2) goes under the name of (continuous) *multiple-choice KP*; see for instance [11]. In this problem, we have a set of (say) objects, each having a profit and a weight, partitioned into m sets $S_1, \dots, S_m$. The objective is to pack a maximum-profit subset of the objects in a knapsack of given capacity ensuring that *at most* one object for each subset is packed. (Actually, in the original version, one requires that *exactly* one object for each subset is packed, which can easily be achieved by adding a dummy object of null profit and weight for each subset S_i.) In our case, the knapsack capacity is $1-\sum_{\ell=1}^{s} h_{j_\ell}$, each subset S_i corresponds to item set A_i, $i \in \mathbb{N}\setminus\{0\}$, and each object in S_i corresponds to the optimal packing of the items in A_i in shelves of overall height at most H, for some relevant H, found in Step (c). The weight of the object is H and its profit is the profit of the associated shelves. As discussed above, the number of objects, and hence the size of the resulting multiple-choice KP, is polynomially bounded. We find a near-optimal solution of this problem by applying a (fully) PTAS such as, e.g., the one in [11].

The whole discussion in Sections 2.1 and 2.2 is summarized in the following:

Lemma 1. APTAS$_{\text{2SKP}}$ *runs in polynomial time for every fixed* ε.

2.3. Proof of approximation guarantee

In this section, we prove the approximation guarantee of the method, by considering all the steps that decrease the optimal solution value and letting z and z' denote the optimal solution value before and after each step.

Lemma 2. *There exists a value* $f \in \{1, \dots, t\}$ *such that, for Step* (a.1), $z' \geq (1-\varepsilon)z$.

Proof. Consider an optimal solution and the contribution p_f to the overall profit of the items in $S_f := \bigcup_{i\in\mathbb{N}}\{j \in N : h_j \in (\varepsilon^{f+it}, \varepsilon^{f+it-1})\}$ for $f = 1, \dots, t$. By an obvious average argument, since $t \geq 1/\varepsilon$, there exists a value of f such that $p_f \leq \varepsilon z$. Correspondingly, the optimal solution value z' after removing the items in S_f is at least $(1-\varepsilon)z$. □

Lemma 3. *For Step* (a.2), $z' \geq z - \varepsilon$.

Proof. Consider an optimal solution of value z and a shelf in the optimal solution with items $S \subseteq N$, letting $P := \{i \in \mathbb{N} : S \cap A_i \neq \emptyset\}$. If $|P| = 1$, the shelf is feasible also after decomposition. Otherwise, letting k be the minimum index in P, remove from the shelf all the items in A_i, $i > k$. If h is the height of the shelf, the heights of all the items removed are at most εh, and therefore their profit is also at most εh. Since the total height of the shelves in the optimal solution is at most 1, after performing the above for all shelves we have a feasible solution for the problem after decomposition of value at least $z - \varepsilon$. □

Note that in Lemma 3 we need a gap between the minimum height of an item in A_i and the maximum height of an item in A_{i+1}, ensured by decomposition. Note also that, for the instances with two items, one of size $(\varepsilon^t, 1)$ and the other of size $(1-\varepsilon^t, \varepsilon^t)$, the optimal value is $\varepsilon^t(2-\varepsilon^t)$ and the optimal value after decomposition is ε^t. This shows that the *absolute* worst-case ratio between the optimal solution values before and after decomposition cannot be better than 2. In Section 3, we will show how to get rid of the constant $-\varepsilon$ in the approximation guarantee.

Lemma 4. *For Step* (c.1), $z' \geq (1-2\varepsilon)z - \varepsilon$.

Proof. In step (c.1), each height not larger than ε (low height) is increased by a factor at most $(1+\varepsilon)$. Consider an optimal solution of value z, letting p be the overall profit for the shelves with low height and $h \geq p$ be the associated height occupied, with an average profit/height ratio of p/h. After increasing the heights (but leaving the item profits equal to their original areas), the average profit/height ratio for these shelves is at least $p/((1+\varepsilon)h)$. If these shelves are repacked in the vertical space h by decreasing profit/height ratios, stopping as soon as the next shelf does not fit, for the repacked shelves the overall height occupied is at least $h - \varepsilon$, and the average profit/height ratio at least $p/((1+\varepsilon)h)$. This implies that the profit of the repacked shelves is at least $(h-\varepsilon) \cdot p/((1+\varepsilon)h) \geq (1-\varepsilon)p - \varepsilon$. Moreover, by using rounded-down heights in the profit computation, the profit (area) of each item is changed by a factor at least $(1-\varepsilon)$. Summarizing, we have $z' \geq (1-\varepsilon)^2 z - \varepsilon \geq (1-2\varepsilon)z - \varepsilon$. □

In the remainder of the analysis of Step (c), we assume that H is the height actually occupied in the optimal solution by the items in A_i and compare the optimal profits z and z' that can be obtained by packing the items in A_i in shelves of overall height H before and after each step. The following lemma proves a useful lower bound on the optimal solution value in case not all the items can be packed by NFDH.

Lemma 5. *Consider a set of items with heights in $[\delta h, h]$ for some $\delta \in (0, 1]$. Pack these items in shelves of total height at most H (with $h \le H$) by* NFDH. *Replace the last shelf by the first item that does not fit (if any) if this increases the profit. If not all the items fit, the overall profit of the items packed is at least $\delta H/4$.*

Proof. Consider two consecutive shelves S_1, S_2 that are packed. Since the first item j in the second shelf does not fit in the first, $w(S_1 \cup \{j\}) > 1$. Moreover, since the height $h(S_1)$ of the first shelf is at most h and the height of each item is at least δh, we have $p(S_1 \cup \{j\}) \ge \delta h$, whereas $h(S_1) + h(S_2) \le 2h$. The same reasoning applies to the last shelf alone if the one formed by next fit decreasing height is possibly replaced by the first item that does not fit: the shelf profit is at least $\delta h/2$ and its height at most h. This means that, if the overall height of the shelves is H', their profit is at least $\delta H'/2$. The proof is concluded by noting that $H' \ge H/2$, since the items are packed in decreasing order of height. □

Applying Lemma 5 to our case, in which $\delta = \varepsilon^{t-1}$, yields an overall profit of at least $\varepsilon^{t-1}H/4$.

Lemma 6. *For Step* (c.2.1)*, $z' \ge (1-\varepsilon)z$.*

Proof. Consider an optimal solution before preprocessing. Consider also a shelf S in this solution containing at least one item $s \in S$ removed by preprocessing. Since the height of each shelf is at least ε^{f+it-1} and the width of each wide item is at least ε^t, no more than $H/\varepsilon^{f+(i+1)t-1}$ wide items are packed. Hence, recalling the definition of R_{ij}, if $s \in T_{ijk} \setminus R_{ij}$ is packed in the optimal solution, then at least one item $s' \in T_{ijk} \cap R_{ij}$ is not packed. Let h denote the height of both s and s'. The profit of s is at most $(k+1)\varepsilon^{t+1}h$, and the profit of s' is at least $k\varepsilon^{t+1}h$, for some $k \ge 1/\varepsilon$ (both items being wide). Therefore, the profit of s' is at least $(1-\varepsilon)$ times that of s. This implies that replacing all the items removed in the preprocessing phase by items of the same height and almost the same width yields a solution of value at least $(1-\varepsilon)z$. □

Lemma 7. *For Step* (c.2.2)*, $z' \ge (1-2\varepsilon)z$.*

Proof. For each height $\bar{h}_j$, recall that r_{ij} is the total number of items in R_{ij} and $p_i := \lfloor H/4\varepsilon^{f+(i-3)t} \rfloor$ is the number of items in one group. Letting $\overline{w}_\ell$ denote the (increased) width of item ℓ after grouping, since $\overline{w}_{\ell+p_i} \le w_\ell$ for $\ell = 1, \dots, r_{ij} - p_i$, we can define a solution after grouping in which each item ℓ is replaced by $\ell + p_i$ if $\ell \le r_{ij} - p_i$, and removed if $\ell > r_{ij} - p_i$. Moreover, $\underline{w}_\ell \ge w_{\ell+p_i}$ for $\ell = 1, \dots, r_{ij} - p_i$ implies that $\underline{w}_{\ell+p_i} \ge w_{\ell+2p_i}$ if $\ell \le r_{ij} - 2p_i$. Combining the two inequalities above, we get that each item ℓ, where $\ell \le r_{ij} - 2p_i$, in the solution before grouping is replaced in the solution after grouping by an item whose profit is at least equal to the profit of item $\ell + 2p_i$ (for the instance before grouping). This implies that the total profit decrease after the replacement is not larger than the sum of the profits of the $2p_i$ largest items in R_{ij}, i.e. at most

$$2p_i \cdot h_j \cdot w_1 \le \frac{2H}{4 \cdot \varepsilon^{f+(i-3)t}} \cdot \varepsilon^{f+(i-1)t} \cdot 1 = \frac{\varepsilon^{2t}H}{2}.$$

Having performed the replacement for all the $g_i \le 1/\varepsilon^t$ heights, the overall loss is at most

$$\frac{1}{\varepsilon^t} \cdot \frac{\varepsilon^{2t}H}{2} = \frac{\varepsilon^t H}{2} \le 2\varepsilon z,$$

recalling that $z \ge \varepsilon^{t-1}H/4$ from Lemma 5. □

Lemma 8. *For Step* (e)*, $z' \ge (1-\varepsilon)z$.*

Proof. Consider an optimal solution of value z (after decomposition) where thin items are cut. It is easy to check that, by removing the thin items from their shelves and then repacking the whole set of thin items by the variant of NFDH in Section 2.2.1, considering separately the shelves for each set A_i according to decomposition, we obtain another optimal solution. In the construction of this solution, a thin item may be cut only when a shelf is closed, when it is completely filled horizontally (i.e. its width is 1). We associate with each thin item j cut the first shelf of width 1 in which j is cut. In this way, distinct shelves are associated with distinct items. Given this, note that after decomposition the overall profit of each shelf of height h and width 1 is at least $\varepsilon^{t-1}h$, whereas the profit of the associated thin item cut cannot exceed $\varepsilon^t h$. Therefore, by removing from the solution the thin items that are cut, the profit decreases by a factor at most $(1-\varepsilon)$. □

According to Lemmas 2–8, and considering the additional $(1-\varepsilon)$ factor due to the fact that in Step (d.2) we apply a PTAS for multiple-choice KP with accuracy ε, the final solution produced by $\text{APTAS}_{\text{2SKP}}$ for item set N has value at least

$$(1-\varepsilon)^4(1-2\varepsilon)^2\text{opt}(N) - 2\varepsilon \ge (1-8\varepsilon)\text{opt}(N) - 2\varepsilon.$$

Theorem 1. $\text{APTAS}_{\text{2SKP}}$ *is an APTAS for 2SKP.*

3. Getting an absolute PTAS

The removal of the additive constant $-\varepsilon$ in the performance guarantee is possible but apparently far from trivial. Actually, a few bad examples starting from the straightforward one given after Lemma 3 seem to force a complex solution.

The (obvious) starting observation leading towards an absolute approximation scheme is that, if the optimal solution value is at least equal to a constant (once the required accuracy ε has been specified), say $\text{opt}(N) \ge \varepsilon^k$ for some absolute constant k, then running the APTAS of the previous section with accuracy ε^{k+1} provides a final heuristic solution of value at least

$$(1-\varepsilon^{k+1})\mathrm{opt}(N)-\varepsilon^{k+1}\geq(1-\varepsilon)\mathrm{opt}(N)-\varepsilon\mathrm{opt}(N)=(1-2\varepsilon)\mathrm{opt}(N),$$

i.e. we already have an absolute PTAS. Unfortunately, if the optimal solution value is not lower bounded by a constant, the APTAS is not sufficient and still the 2SKP can be shown to be strongly NP-hard.

We first prove in Section 3.1 the structural results leading to our method, and then present the method formally in Section 3.2. For convenience (and without loss of generality) we will work with an internal accuracy $\varepsilon \leq 1/12$.

3.1. Structural results

Let $T := \{j \in N : h_j \geq \varepsilon\}$ be the set of the *tall items* and $W := \{j \in N : w_j \geq \varepsilon^4\}$ be the set of the *wide items*. Items in $N \setminus W$ are referred to as *thin items*.

Lemma 9. *If not all the items in $N \setminus T$ can be packed by* NFDH *in the bin, calling* $\mathrm{APTAS}_{\mathrm{2SKP}}$ *with accuracy ε yields a solution of value at least $(1-4\varepsilon)\mathrm{opt}(N)$.*

Proof. We show that, in case some items remain unpacked, $\mathrm{opt}(N) \geq (1-\varepsilon)/2 \geq 1/3$ for $\varepsilon \leq 1/3$. The lemma then follows since $\mathrm{APTAS}_{\mathrm{2SKP}}$ with accuracy ε yields a solution of value at least $(1-\varepsilon)\mathrm{opt}(N) - \varepsilon \geq (1-4\varepsilon)\mathrm{opt}(N)$ for $\mathrm{opt}(N) \geq 1/3$. We complete the proof by showing the above lower bound on $\mathrm{opt}(N)$. Let $S_1, S_2, \ldots, S_m$ be the shelves packed in the bin by NFDH and, for convenience, S_{m+1} contains only the first item that does not fit, noting that $\sum_{i=1}^{m+1} h(S_i) > 1$ and hence $\sum_{i=2}^{m+1} h(S_i) > 1-\varepsilon$ since $h(S_1) < \varepsilon$. For each pair of consecutive shelves S_i, S_{i+1}, reasoning as in the proof of Lemma 5 and letting j be the first (tallest) item in S_{m+1}, we have $p(S_i \cup S_{i+1}) \geq p(S_i \cup \{j\}) \geq w(S_i \cup \{j\}) \cdot h_j > h_j = h(S_{i+1})$. Consider the solution obtained by replacing S_1 by S_{m+1} if this increases the profit. The profit of this solution is

$$\sum_{i=2}^{m} p(S_i) + \max\{p(S_1), p(S_{m+1})\} \geq \frac{1}{2}\sum_{i=1}^{m} p(S_i \cup S_{i+1}) \geq \frac{1}{2}\sum_{i=2}^{m+1} h(S_i) \geq (1-\varepsilon)/2. \quad \square$$

Lemma 10. *If $w(T) \geq \varepsilon^3$, calling* $\mathrm{APTAS}_{\mathrm{2SKP}}$ *with accuracy $\varepsilon^2 \cdot \min\{1, w(T)\}$ yields a solution of value at least $(1-3\varepsilon)\mathrm{opt}(N)$.*

Proof. The proof follows from $\mathrm{opt}(N) \geq \varepsilon \cdot \min\{1/2, w(T)\}$, as we now show. By packing the items in T in arbitrary order in a unique shelf S, either all items fit, in which case $p(S) \geq \varepsilon w(T)$, or an item j does not fit, in which case $w(S) + w_j > 1$, hence $\mathrm{opt}(N) \geq \max\{p(S), p_j\}$ and $p(S) + p_j > \varepsilon$, i.e. $\mathrm{opt}(N) \geq \varepsilon/2$. $\square$

In the rest of Section 3.1 we will consider the case in which all the items in $N \setminus T$ are packed by NFDH and $w(T) < \varepsilon^3$. A *tall shelf* is a shelf S such that $S \cap T \neq \emptyset$ (or, equivalently, $h(S) \geq \varepsilon$).

Lemma 11. *If all the items in $N \setminus T$ can be packed by* NFDH *in the bin and $w(T) < \varepsilon^3$, either $p(N \setminus T) \geq (1-\varepsilon)\mathrm{opt}(N)$, or there exists an optimal solution with at most one tall shelf.*

Proof. Consider an optimal solution. If it contains at most one tall shelf we are done. Supposing that this is not the case, let R be the set of items j with $h_j \geq \varepsilon^2$ packed by the solution. (R contains all the items in T that are packed.) If $w(R) > 1$, the overall profit of the items in R is at least ε^2, whereas the overall profit of the items in $R \cap T$ is at most $p(T) < \varepsilon^3$; therefore

$$p(N \setminus T) \geq \mathrm{opt}(N) - p(R \cap T) \geq (1-\varepsilon)\mathrm{opt}(N).$$

Finally, if $w(R) \leq 1$, let m be the number of shelves with items in R and $h_1 \geq h_2 \geq \cdots \geq h_m$ be the associated heights, in decreasing order, with $h_2 \geq \varepsilon \geq 2\varepsilon^2$, and hence $\sum_{i=2}^{m} h_i \geq m\varepsilon^2$. The proof follows by considering the new optimal solution obtained by removing all the items in R from their shelves and packing them in a unique new shelf, which is the unique tall shelf of the new solution. In the new solution, we have one new shelf of height h_1 and m new shelves of height smaller than ε^2, and $h_1 + m\varepsilon^2 \leq h_1 + \sum_{i=2}^{m} h_i$. This implies that the new solution is feasible, since for the new shelves the overall height is smaller than for the old ones. $\square$

Therefore, we restrict attention to the case in which there exists an optimal solution with a unique tall shelf. In fact, we need additional properties of a (near-)optimal solution in order to analyze our absolute PTAS. First of all, we state a general result about the structure of a near-optimal 2SKP solution, showing that at most one shelf can be "almost empty". Note that for each shelf S we have $p(S) \leq h(S)$, equality holding if all the area occupied by the shelf is filled with items.

Lemma 12. *There exists a 2SKP solution whose value is at least $(1-6\varepsilon)\mathrm{opt}(N)$ in which $p(S) \leq \varepsilon h(S)$ for at most one shelf S, which is the tallest shelf in the solution.*

The above general lemma, whose proof is deferred, can be used to prove the following technical lemma, whose proof is also deferred.

Lemma 13. *If there exists an optimal solution with a unique tall shelf, then there exists a solution Σ such that:*

- $p(\Sigma) \geq (1-9\varepsilon)\text{opt}(N)$*;*
- Σ *contains a unique tall shelf* Σ_T*;*
- *for all the shelves S different from* Σ_T, $p(S) > \varepsilon h(S)$*;*
- *if* $w(\Sigma_T \cap W) < 1-\varepsilon^3$*, then the set of thin items* $\Sigma_T \setminus W$ *is obtained by packing the thin items with height at most* $h(\Sigma_T)$ *by decreasing heights in the residual horizontal space of width* $1 - w(\Sigma_T \cap W)$*, stopping as soon as the current item does not fit.*

In the following, we will refer to a solution Σ as in Lemma 13, letting Σ also denote the associated items and $\Sigma_L := \Sigma \setminus \Sigma_T$ denote the set of items packed in the shelves different from the tall one in this solution.

We next illustrate how we compute a near-optimal solution Σ'. By enumeration, we can guess in polynomial time a tall shelf Σ'_T in Σ' having the same height $H := h(\Sigma_T)$ and the same set of wide items as Σ_T. How to pack the thin items in this tall shelf depends on the total width $w(\Sigma_T \cap W)$, which forces us to consider two cases (see below). Assuming we have decided the thin items packed in the tall shelf by Σ', let $D := \Sigma'_T \cap \Sigma_L$ denote the set of thin items packed in the tall shelf in Σ' and in the remaining shelves in Σ. We construct the remaining shelves for Σ' as follows. Let $N' := N \setminus \Sigma'_T$ be the set of remaining items and $T' := \{j \in N' : h_j \geq \varepsilon(1-H)\}$ be the set of tall items with respect to the residual height $1-H$.

Lemma 14. *Consider the packing of the items in N' in a bin of width 1 and height $1-H$ obtained as follows:*

- *if not all the items in $N' \setminus T'$ can be packed in the bin by* NFDH*, by calling* $\text{APTAS}_{\text{2SKP}}$ *with accuracy* ε*;*
- *otherwise, if* $w(T') \geq \varepsilon^3$*, by calling* $\text{APTAS}_{\text{2SKP}}$ *with accuracy* $\varepsilon^2 \cdot \min\{1, w(T')\}$*;*
- *otherwise, by packing all the items in* $N' \setminus T'$ *by* NFDH.

The resulting set $\Sigma'_L \subseteq N'$ *of items packed satisfies*

$$p(\Sigma'_L) \geq (1-4\varepsilon)p(\Sigma_L \setminus D).$$

Proof. If not all the items in $N' \setminus T'$ can be packed in the bin by NFDH or $w(T') \geq \varepsilon^3$, Lemmas 9 and 10 imply that $p(\Sigma'_L)$ is at least $(1-4\varepsilon)$ times the profit that can be achieved by packing the items in N' in a bin of width 1 and height $1-H$. This implies that $p(\Sigma'_L) \geq (1-4\varepsilon)p(\Sigma_L \setminus D)$. Accordingly, in the rest of the proof, we will consider the remaining case, in which $p(\Sigma'_L) = p(N' \setminus T')$. If $\Sigma_L \cap T' = \emptyset$, we have $p(\Sigma'_L) \geq p(\Sigma_L \setminus D)$ and we are done. Otherwise, consider a shelf S in Σ which is not the tall one (i.e. $S \subseteq \Sigma_L$) and contains an item in T'. By Lemma 13, we must have $p(S) > \varepsilon h(S)$. Then, since $w(T') < \varepsilon^3$ implies that $p(S \cap T') < \varepsilon^3 h(S)$, we must have $p(S \setminus T') > p(S) - \varepsilon^3 h(S) > (1-\varepsilon^2)p(S)$. Summing over all the shelves, this implies that $p(\Sigma_L \setminus T') > (1-\varepsilon^2)p(\Sigma_L)$, i.e.

$$p(\Sigma'_L) = p(N' \setminus T') \geq p((\Sigma_L \setminus T') \setminus D) \geq (1-\varepsilon^2)p(\Sigma_L \setminus D) \geq (1-2\varepsilon)p(\Sigma_L \setminus D). \quad \square$$

As anticipated, we still have to specify how we pack the thin items in the tall shelf in Σ', depending on the horizontal space for these items left by the wide items.

Lemma 15. *If* $w(\Sigma_T \cap W) < 1-\varepsilon^3$*, packing the thin items in the tall shelf by decreasing height (stopping as soon as the current item does not fit) and then the items in N' in the remaining shelves as in Lemma 14 yields a solution of value at least* $(1-4\varepsilon)p(\Sigma)$*.*

Proof. By Lemma 13, in this case we have that the sets of thin items packed by Σ and Σ' in the tall shelf coincide, i.e. $D = \emptyset$. This implies that $p(\Sigma'_T) = p(\Sigma_T)$ and, by Lemma 14, $p(\Sigma'_L) \geq (1-4\varepsilon)p(\Sigma_L)$, yielding the proof. $\square$

Lemma 16. *If* $w(\Sigma_T \cap W) \geq 1-\varepsilon^3$*, packing the thin items in the tall shelf by a PTAS for the KP with accuracy ε and then the items in N' in the remaining shelves as in Lemma 14 yields a solution of value at least* $(1-6\varepsilon)p(\Sigma)$*.*

Proof. Since $\Sigma_T \cap W = \Sigma'_T \cap W$ and we apply a PTAS to pack the thin items in the tall shelf, we have $p(\Sigma'_T) \geq (1-\varepsilon)p(\Sigma_T)$.

Moreover, observe that $w(D) \leq \varepsilon^3$, since D is a subset of the thin items in the tall shelf in Σ', which are packed in a horizontal space of width at most ε^3. Since the items in D are packed in Σ in shelves different from the tall one, reasoning as in the proof of Lemma 14, this implies that $p(\Sigma_L \setminus D) > (1-\varepsilon^2)p(\Sigma_L)$, i.e. applying Lemma 14 itself,

$$p(\Sigma'_L) \geq (1-4\varepsilon)p(\Sigma_L \setminus D) \geq (1-4\varepsilon)(1-\varepsilon^2)p(\Sigma_L) \geq (1-6\varepsilon)p(\Sigma_L),$$

yielding the proof. $\square$

3.1.1. Proof of Lemma 12

We first show that, given a solution with two shelves S_1, S_2 such that $h(S_1) \geq h(S_2)$ and $p(S_2) \leq \varepsilon h(S_2)$, either the overall height of the shelves in the solution can be decreased, or the profit of S_1 is much larger than the profit of S_2.

Let $S_1 = \{j_1, j_2, \ldots\}$ with $h_{j_1} \geq h_{j_2} \geq \cdots$, and let ℓ be the index such that $\sum_{i=1}^{\ell} w_{j_i} \leq 1/2$, $\sum_{i=1}^{\ell+1} w_{j_i} > 1/2$, defining the partition of S_1 given by $L_1 := \{j_1, \ldots, j_\ell\}$ and $R_1 := \{j_{\ell+1}, \ldots\}$. If $w(S_1) \leq 1/2$, we have $R_1 = \emptyset$. Pictorially, if the items in S_1 are left justified and appear from left to right in decreasing height order, L_1 is the set of items in S_1 to the left of the horizontal midpoint of the shelf, and R_1 is the set of items to the right, including the possible item that overlaps this midpoint. We define analogously the partition of S_2 into L_2 and R_2. Note that $p(S_1) \geq h(R_1)/2$ and $p(S_2) \geq h(R_2)/2$, implying $h(R_2) \leq 2\varepsilon h(S_2)$.

If $h(R_1) + h(R_2) < h(S_2)$, replacing S_1, S_2 by the three shelves $L_1 \cup L_2, R_1, R_2$ yields another solution with the same value and strictly smaller overall height. Otherwise, since $h(R_2) \leq 2\varepsilon h(S_2)$, we have $h(R_1) \geq (1 - 2\varepsilon)h(S_2)$, and therefore

$$p(S_1) \geq \frac{h(R_1)}{2} \geq \frac{1-2\varepsilon}{2}\, h(S_2) \geq \frac{1-2\varepsilon}{2\varepsilon}\, p(S_2) = Mp(S_2),$$

where $M := (1 - 2\varepsilon)/(2\varepsilon)$.

The above discussion implies that, in case the optimal solution contains more than one shelf S with $p(S) \leq \varepsilon h(S)$, and its overall height cannot be decreased as above, the collection of shelves $S_1, S_2, S_3, \ldots$ with this property, with $h(S_1) \geq h(S_2) \geq h(S_3) \geq \cdots$ satisfies $p(S_1) \geq Mp(S_2) \geq M^2 p(S_3) \geq \cdots$. This implies that, by removing shelves $S_2, S_3, \ldots$ from the optimal solution, the total profit lost is at most

$$\sum_{i=1}^{\infty} \frac{1}{M^i}\, p(S_1) = \frac{1}{M-1}\, p(S_1),$$

i.e. that the overall profit of the resulting (near-optimal) solution is at least $(1-\frac{1}{M-1})\mathrm{opt}(N) \geq (1-3\varepsilon)\mathrm{opt}(N)$ for $\varepsilon \leq 1/12$.

Now consider the unique shelf S_1 in the resulting solution such that $p(S_1) \leq \varepsilon h(S_1)$. If S_1 is the tallest shelf, the proof is complete. Otherwise, letting S_0 be the tallest shelf, by reasoning as above one either gets a solution with strictly smaller overall height, the same profit, and at most one shelf S such that $p(S) \leq \varepsilon h(S)$, or $p(S_0) \geq Mp(S_1)$. In the latter case, removing S_1 yields a solution with no shelf S such that $p(S) \leq \varepsilon h(S)$ and profit at least $(1 - \frac{1}{M})(1 - \frac{1}{M-1})\mathrm{opt}(N) \geq (1 - 6\varepsilon)\mathrm{opt}(N)$.

3.1.2. *Proof of* Lemma 13

We consider an optimal solution with only one tall shelf, and apply the procedure in the proof of Lemma 12 to it, obtaining a solution Σ'' with the structure as in the statement of that lemma and a unique tall shelf. Note in particular that the procedure does not increase the number of tall shelves, since it either removes shelves, or it replaces two shelves S_1, S_2, out of which only S_1 can be tall, by three shelves of height, respectively, $h(S_1), h(R_1), h(R_2)$, where $h(R_1) + h(R_2) < h(S_2)$, and hence R_1, R_2 cannot be tall since S_2 is not. Let Σ''_T denote the set of items in the tall shelf in Σ''. If $w(\Sigma''_T \cap W) \geq 1-\varepsilon^3$, we let Σ coincide with Σ'' and the proof is complete. Otherwise, we define Σ from Σ'' as explained in the following, introducing, for the sake of the analysis, also another near-optimal solution Σ'.

In order to define Σ', rearrange the items in Σ'' so that, in each shelf, from left to right, the wide items are before the thin items, and the latter are sorted by decreasing height. Consider the portion P of the bin given by the rectangle of height 1 and width ε^4 whose right edge coincides with the right edge of the bin, and the twice-as-large portion $2P$ given by the rectangle of height 1 and width $2\varepsilon^4$ whose right edge coincides with the right edge of the bin. Remove permanently from Σ'' all the thin items *overlapping* $2P$. Before defining Σ', we evaluate the profit that remains after the removal. For the tall shelf, letting h denote the largest height of a thin item removed, the profit removed is at most $3\varepsilon^4 h$, and the profit of the thin items remaining is at least $(\varepsilon^3 - 3\varepsilon^4)h$, i.e. at least $(1 - 3\varepsilon)$ times the initial profit. For the remaining shelves, the profit of the thin items removed from each shelf S of height $h(S)$ is at most $3\varepsilon^4 h(S)$, whereas, by Lemma 12, the total profit of the shelf is $p(S) > \varepsilon h(S)$. This implies that the remaining profit in the shelf is at least $(1-3\varepsilon^3)p(S) \geq (1-3\varepsilon)p(S)$. This in turn implies that the overall profit after the removal is at least $(1 - 3\varepsilon)p(\Sigma'') \geq (1 - 9\varepsilon)\mathrm{opt}(N)$.

Now consider the thin items remaining in the solution after removal, say R; temporarily remove them from their shelves, without changing the shelves' heights, and then define Σ' by repacking them in these shelves by considering the thin items in R and the shelves by decreasing height, packing the current thin item in the current shelf if it fits *without overlapping* P and switching to the next shelf if it does not. It is easy to see that all the thin items in R are repacked in this way, since, after removal of the thin items overlapping $2P$, $w(S) \leq 1 - 2\varepsilon^4$ for each shelf S, whereas in the repacking the width of the space available in each shelf is $1 - \varepsilon^4$ (recall that the thin items have widths smaller than ε^4). This implies that $p(\Sigma') \geq (1 - 9\varepsilon)\mathrm{opt}(N)$.

Finally, consider the solution Σ obtained from Σ' by (i) removing all thin items from their shelves, without changing the shelves' heights, and (ii) repacking the *overall set* $N \setminus W$ of the thin items in these shelves, as above, by considering the items and the shelves by decreasing heights, but without caring about overlaps with P, using all the horizontal space in the shelves. Although not all thin items in R are necessarily repacked, it is easy to check that the profit of this new solution Σ is at least equal to the profit of Σ'. Indeed, the thin items in Σ have in general larger heights than the items in Σ' and, if some thin items in Σ' remain unpacked in Σ, each shelf in Σ has a width at least $1 - \varepsilon^4$, whereas each shelf in Σ' has a width at most $1 - \varepsilon^4$.

3.2. *The overall algorithm*

The overall PTAS that follows from the results of Section 3.1, called $\mathrm{PTAS}_{\mathrm{2SKP}}$, is illustrated via a pseudo-code description given in Fig. 2. Procedure $\mathrm{PTAS}_{\mathrm{2SKP}}$ uses also a PTAS for the classical KP (actually fully PTASs are available for this case;

```
procedure PTAS_2SKP(N, ε);
begin
    T := {j ∈ N : h_j ≥ ε};
    W := {j ∈ N : w_j ≥ ε^4};
    if not all the items in N \ T can be packed in the bin by NFDH then
        z := APTAS_2SKP(N, ε);
        return z;
    else if w(T) ≥ ε^3 then
        z := APTAS_2SKP(N, ε^2 · min{1, w(T)});
        return z;
    else
        z_1 := p(N \ T); z_2 := 0;
        for each H ∈ {h_j : j ∈ T} do
            for each S ⊆ W : w(S) ≤ 1, h(S) ≤ H do
                define a tall shelf initially containing the wide items in S;
                if w(S) < 1 − ε^3 then
                    pack the thin items in N \ W with height at most H in the tall shelf
                    by decreasing heights (stopping when the current item does not fit);
                else
                    pack the thin items in N \ W with height at most H in the tall shelf
                    by a PTAS for KP with accuracy ε;
                end if;
                let Σ'_T be the set of items in the tall shelf and N' := N \ Σ'_T;
                for each j ∈ N' do h'_j := h_j/(1 − H);
                T' := {j ∈ N' : h'_j ≥ ε};
                if not all the items in N' \ T' can be packed in the bin by NFDH then
                    z'_2 := APTAS_2SKP(N', ε);
                else if w(T') ≥ ε^3 then
                    z'_2 := APTAS_2SKP(N', ε^2 · min{1, w(T')});
                else z'_2 := p(N' \ T');
                end if;
                if z'_2 > z_2 then z_2 := z'_2;
            end for each;
        end for each;
        return max{z_1, z_2};
    end if;
end.
```

Fig. 2. Pseudo-code description of the PTAS for the 2SKP.

see [12]). In this case, the knapsack *capacity* is given by $1 - w(S)$, which is the residual horizontal space in the tall shelf after packing the wide items, and the widths of the thin items play the role of the *weights* in the knapsack. We assume that all procedures return on output the value of the solution found and skip the (obvious) details about storing this solution.

Theorem 2. $\mathrm{PTAS}_{2\mathrm{SKP}}$ *is a PTAS for the 2SKP.*

Proof. There are $|T| = O(n)$ possibilities for the height H of the tall shelf and $O(n^{1/\varepsilon^4})$ possibilities for the wide items in S packed in the tall shelf. Hence, the method clearly runs in polynomial time. Lemmas 9–16 imply that the value of the final solution produced is at least $(1 - 15\varepsilon)\mathrm{opt}(N)$. □

4. Conclusions

We gave an (absolute) approximation scheme for the 2SKP. We note that our method generalizes the approximation scheme presented in [6]. The generalization of the 2SKP in which the profit of items is not equal to their area remains open. Note that many arguments used in our proofs heavily rely on having profits and areas coincident.

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References

[1] A. Caprara, A. Lodi, M. Monaci, Fast approximation schemes for two-stage, two-dimensional bin packing, Mathematics of Operations Research 30 (2005) 150–172.
[2] A. Steinberg, A strip-packing algorithm with absolute performance bound 2, SIAM Journal on Computing 26 (1997) 401–409.

[3] A. Caprara, M. Monaci, On the 2-dimensional knapsack problem, Operations Research Letters 32 (2004) 5–14.
[4] K. Jansen, G. Zhang, Maximizing the total profit of rectangles packed into a rectangle, Algoritmica 47 (2007) 323–342.
[5] N. Bansal, A. Caprara, K. Jansen, L. Prädel, M. Sviridenko, A structural lemma in 2-dimensional packing, and its implications on approximability, in: Proceedings of the 20th International Symposium on Algorithms and Computation, ISAAC 2009, in: Springer Lecture Notes in Computer Science, vol. 5878, 2009.
[6] A. Caprara, H. Kellerer, U. Pferschy, The multiple subset sum problem, SIAM Journal on Optimization 11 (2000) 308–319.
[7] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, M. Protasi, Complexity and Approximation—Combinatorial Optimization Problems and Their Approximability Properties, Springer, Berlin, Germany, 1999.
[8] E.G. Coffman Jr., M.R. Garey, D.S. Johnson, R.E. Tarjan, Performance bounds for level-oriented two-dimensional packing algorithms, SIAM Journal on Computing 9 (1980) 801–826.
[9] W. Fernandez de la Vega, G.S. Lueker, Bin packing can be solved within $1 + \varepsilon$ in linear time, Combinatorica 1 (1981) 349–355.
[10] H.W. Lenstra Jr., Integer programming with a fixed number of variables, Mathematics of Operations Research 8 (1983) 538–548.
[11] H. Kellerer, U. Pferschy, D. Pisinger, Knapsack Problems, Springer, Berlin, Germany, 2004.
[12] O.H. Ibarra, C.E. Kim, Fast approximation algorithms for the knapsack and subset sum problems, Journal of the ACM 22 (1975) 463–468.