DATA STRUCTURES AND ALGORITHMS

LECTURE 13

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In Lecture 12...

- Binary Trees
- Binary Search Trees

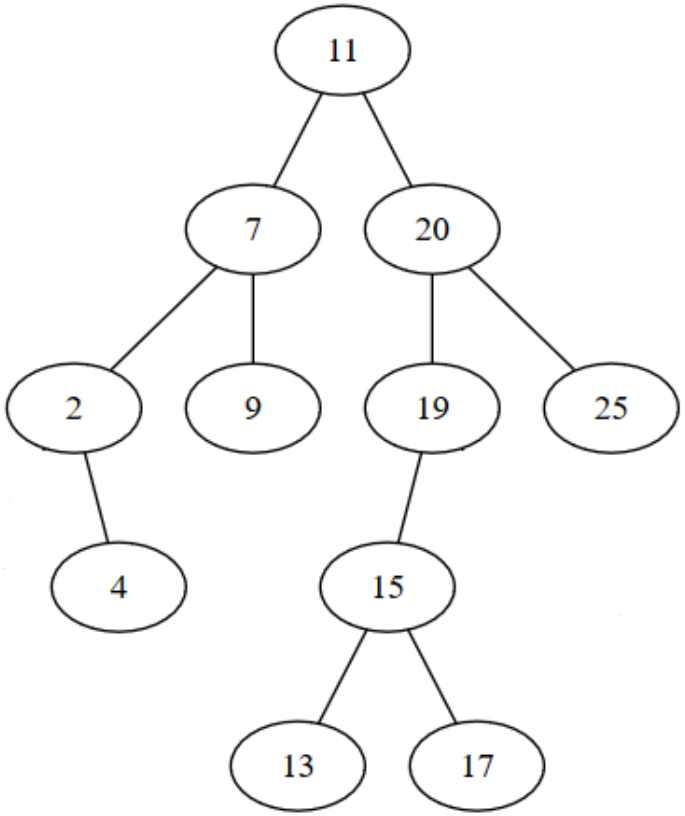
Today

1. Binary Search Trees
2. AVL Trees
3. Problems with stacks and queues

Binary search trees

- A *Binary Search Tree* is a binary tree that satisfies the following property:
  - if x is a node of the binary search tree then:
    - For every node y from the left subtree of x, the information from y is less than or equal to the information from x
    - For every node y from the right subtree of x, the information from y is greater than or equal to the information from x
- Obviously, the relation used to order the nodes can be considered in an abstract way (instead of having "$\leq$" as in the definition).

Binary Search Tree Example



- If we do an inorder traversal of a binary search tree, we will get the elements in increasing order (according to the relation used).

Think about it

- For a Binary Tree we have discussed that in some situations, if we have two tree traversals, we can rebuild the tree based on them.
- Can we do the same for a Binary Search Tree from one single traversal? Which one(s)?

Binary Search Tree

- Think about it:
  - Can we define a Sorted List on a Binary Search Tree? If not, why not? If yes, how exactly? What would be the most complicated part?

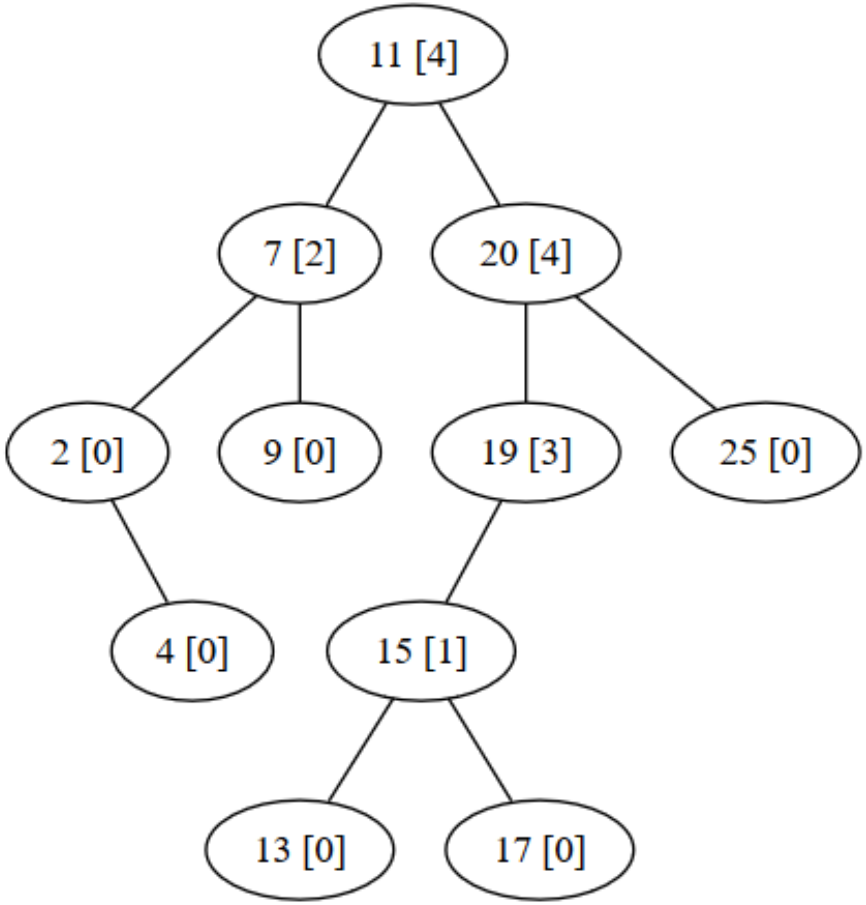
Binary Search Tree

- Think about it:
  - Can we define a Sorted List on a Binary Search Tree? If not, why not? If yes, how exactly? What would be the most complicated part?
  - Lists have positions and operations based on positions. In case of a SortedList we have an operation to remove an element from a position and to return an element from a position (but no insert to position). How can we keep track of positions in a binary search tree?

Binary Search Tree

- Think about it:
  - Can we define a Sorted List on a Binary Search Tree? If not, why not? If yes, how exactly? What would be the most complicated part?
  - Lists have positions and operations based on positions. In case of a SortedList we have an operation to remove an element from a position and to return an element from a position (but no insert to position). How can we keep track of positions in a binary search tree?
  - We can keep in every node, besides the information, the number of nodes in the left subtree. This gives us automatically the "position" of the root in the SortedList. When we have operations that are based on positions, we use these values to decide if we go left or right.

Binary Search Tree



- Find the node from position 8 (first node is at position 1)

Binary Search Tree with duplicate values

- Starting from an initially empty Binary Search Tree and the relation $\leq$, insert into it, in the given order, the following values: 10, 20, 5, 7, 15, 5, 30, 3, 5, 5, 1, 9, 29, 2.

Binary Search Tree with duplicate values

- Starting from an initially empty Binary Search Tree and the relation $\leq$, insert into it, in the given order, the following values: 10, 20, 5, 7, 15, 5, 30, 3, 5, 5, 1, 9, 29, 2.
- How would you count how many times the value 5 is in the tree?

Binary Search Tree with duplicate values

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- How would you count how many times the value 5 is in the tree?
- Remove 3 (show both options)

Binary Search Tree with duplicate values

- Starting from an initially empty Binary Search Tree and the relation $\leq$, insert into it, in the given order, the following values: 10, 20, 5, 7, 15, 5, 30, 3, 5, 5, 1, 9, 29, 2.
- How would you count how many times the value 5 is in the tree?
- Remove 3 (show both options)
- How would you count now how many times the value 5 is in the tree now?

Balanced Binary Search Trees

- Specific operations for binary trees run in $O(h)$ time, which can be $\theta(n)$ in worst case
- Best case is a balanced tree, where height of the tree is $\Theta(log_2 n)$

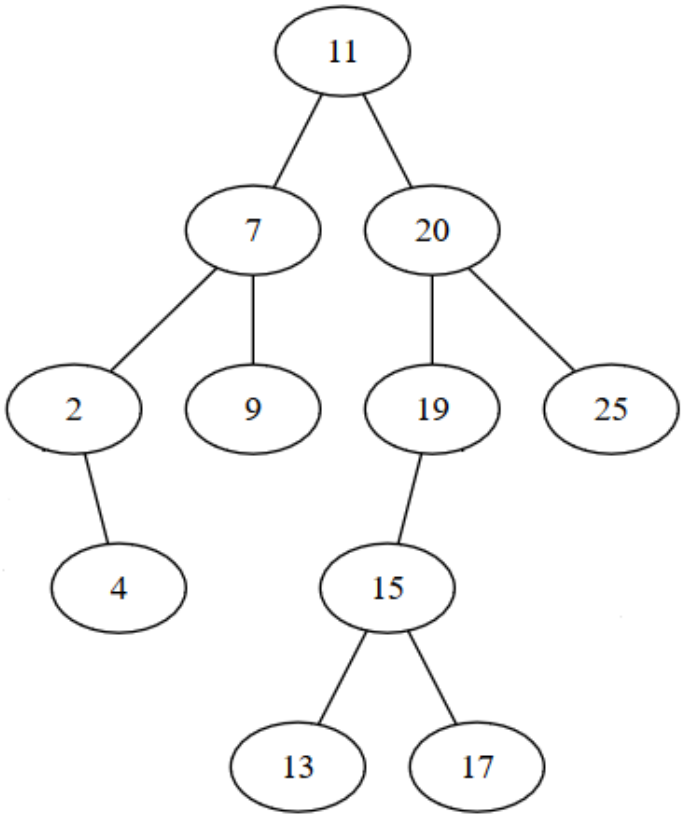
Balanced Binary Search Trees

- Specific operations for binary trees run in $O(h)$ time, which can be $\theta(n)$ in worst case
- Best case is a balanced tree, where height of the tree is $\Theta(log_2 n)$
- To reduce the complexity of algorithms, we want to keep the tree balanced. In order to do this, we want every node to be balanced.
- When a node loses its balance, we will perform some operations (called rotations) to make it balanced again.

AVL Trees

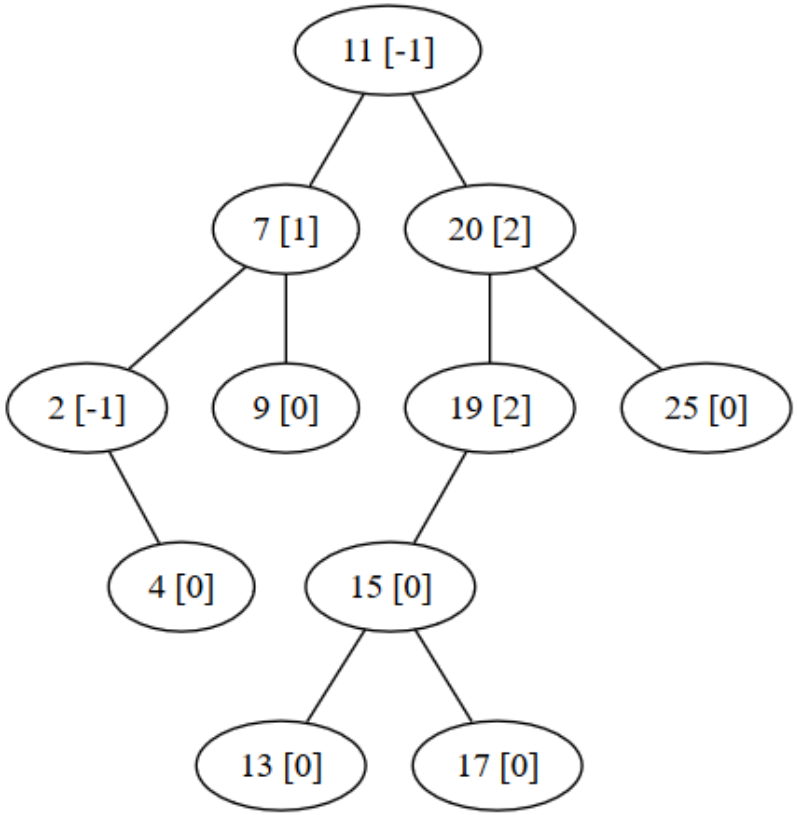
- Definition: An AVL (Adelson-Velskii Landis) tree is a binary tree which satisfies the following property (AVL tree property):
  - If x is a node of the AVL tree:
    - the difference between the height of the left and right subtree of x is 0, 1 or -1 (balancing information)
- Observations:
- Height of an empty tree is -1
- Height of a single node is 0

AVL Trees



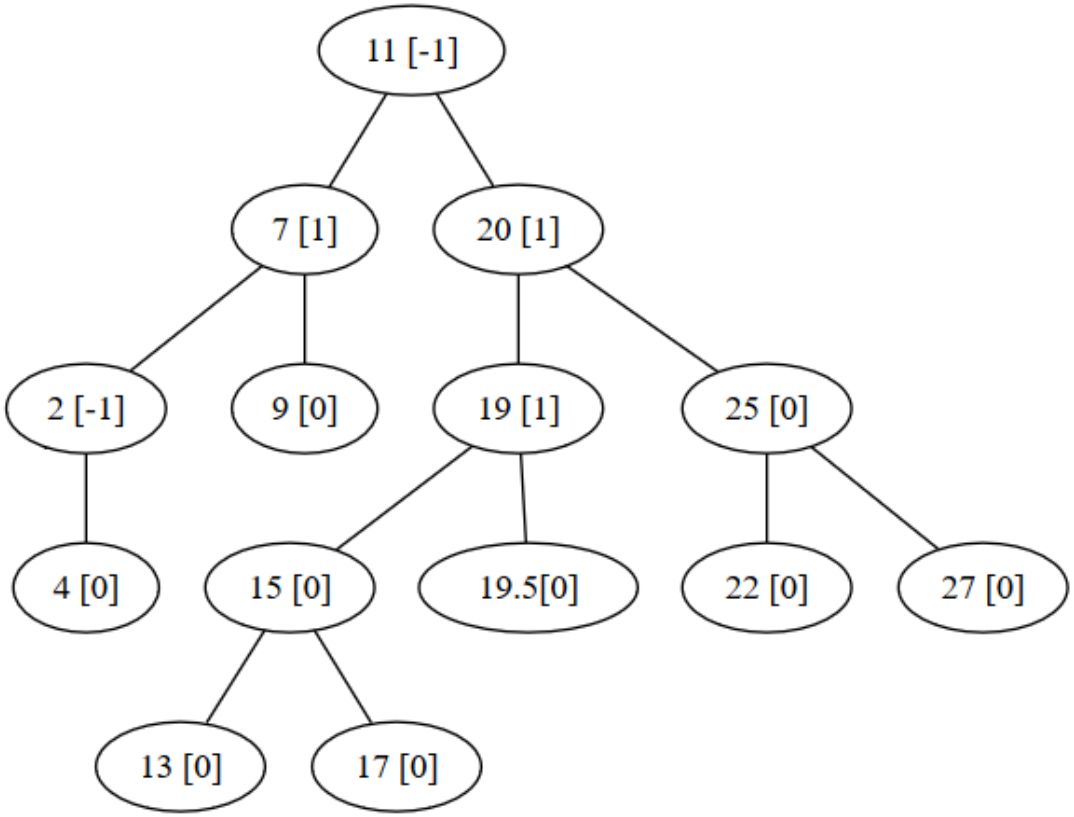
- Is this an AVL tree?

AVL Trees



- Values in square brackets show the balancing information of a node. The tree is not an AVL tree, because the balancing information for nodes 19 and 20 is 2.

AVL Trees



- This is an AVL tree.

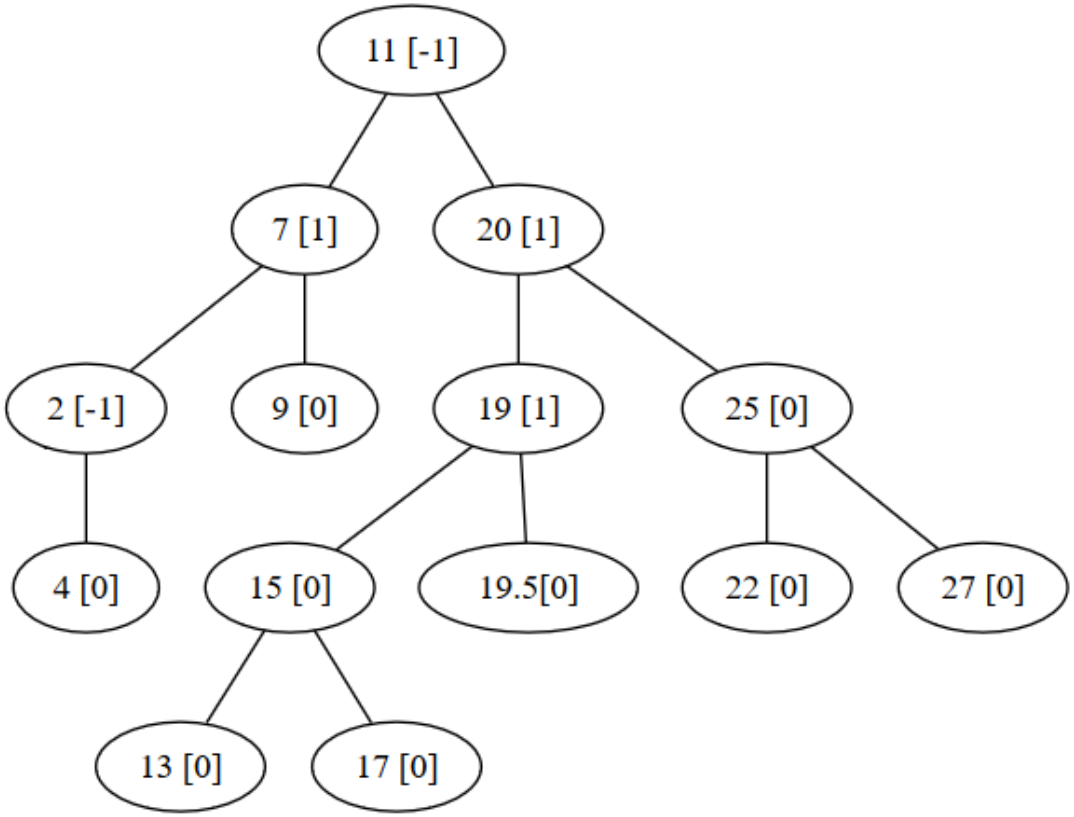
AVL Trees - rotations

- Adding or removing a node might result in a binary tree that violates the AVL tree property.
- In such cases, the property has to be restored and only after the property holds again is the operation (add or remove) considered finished.
- The AVL tree property can be restored with operations called **rotations**.

AVL Trees - rotations

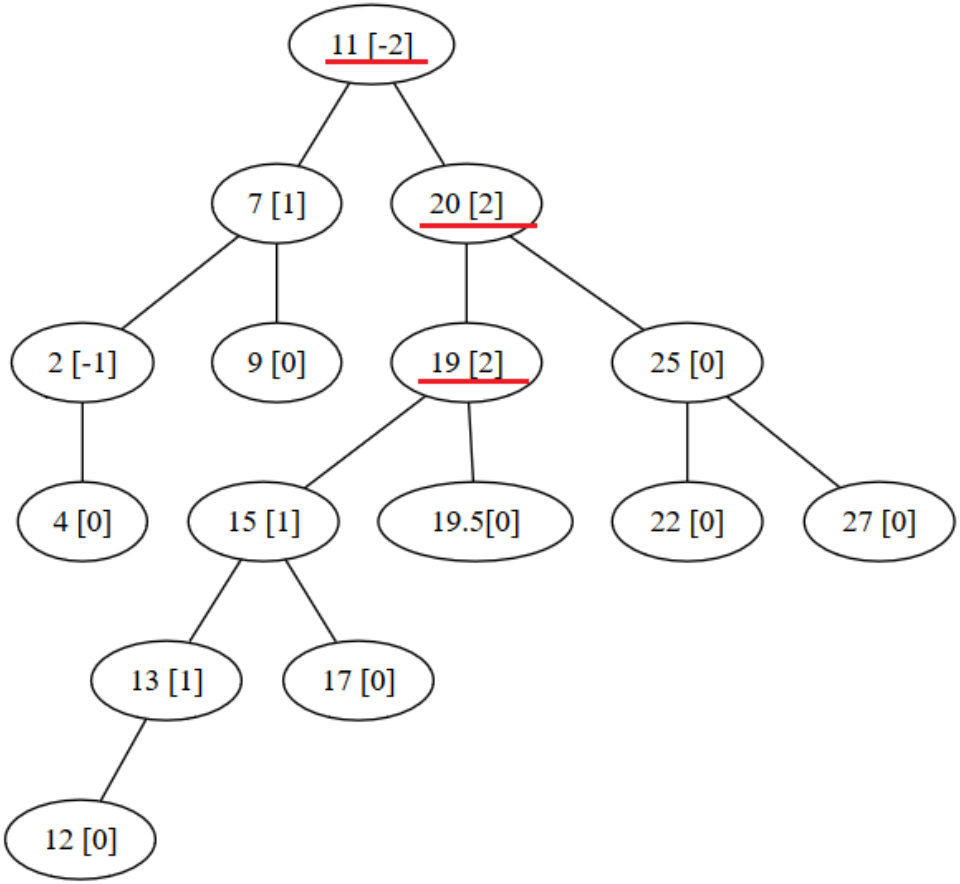
- After an insertion, only the nodes on the path to the modified node can change their height.
- We check the balancing information on the path from the modified node to the root. When we find a node that does not respect the AVL tree property, we perform a suitable *rotation* to rebalance the (sub)tree.

AVL Tress - rotations



- What if we insert element 12?

AVL Trees - rotations

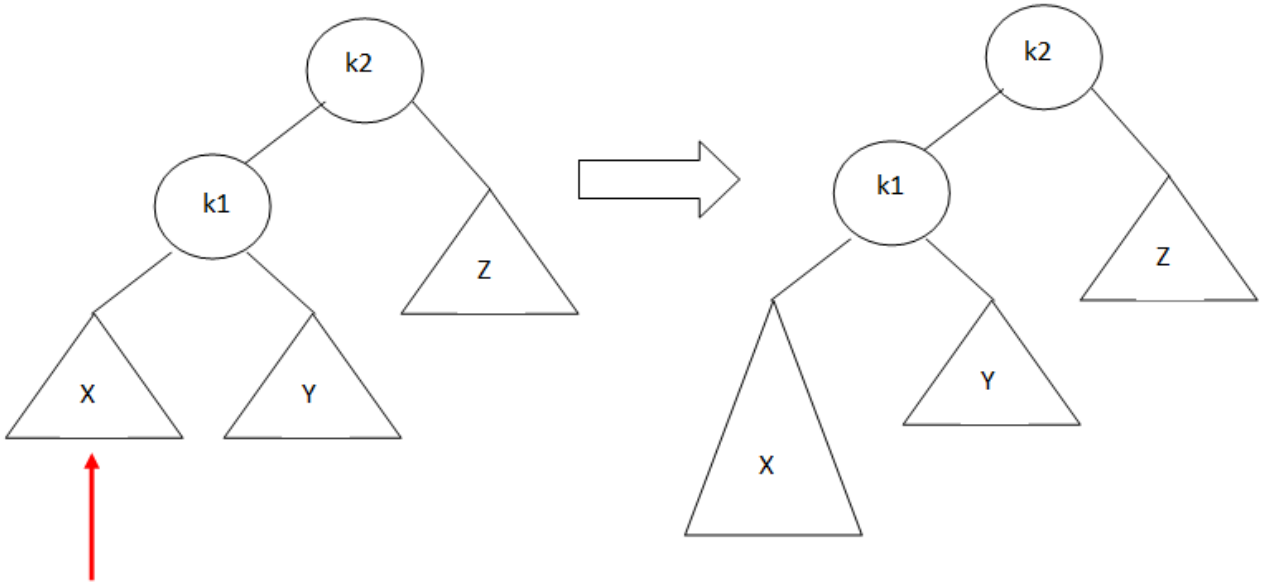


- Red lines show the unbalanced nodes. We will rebalance node 19.

AVL Trees - rotations

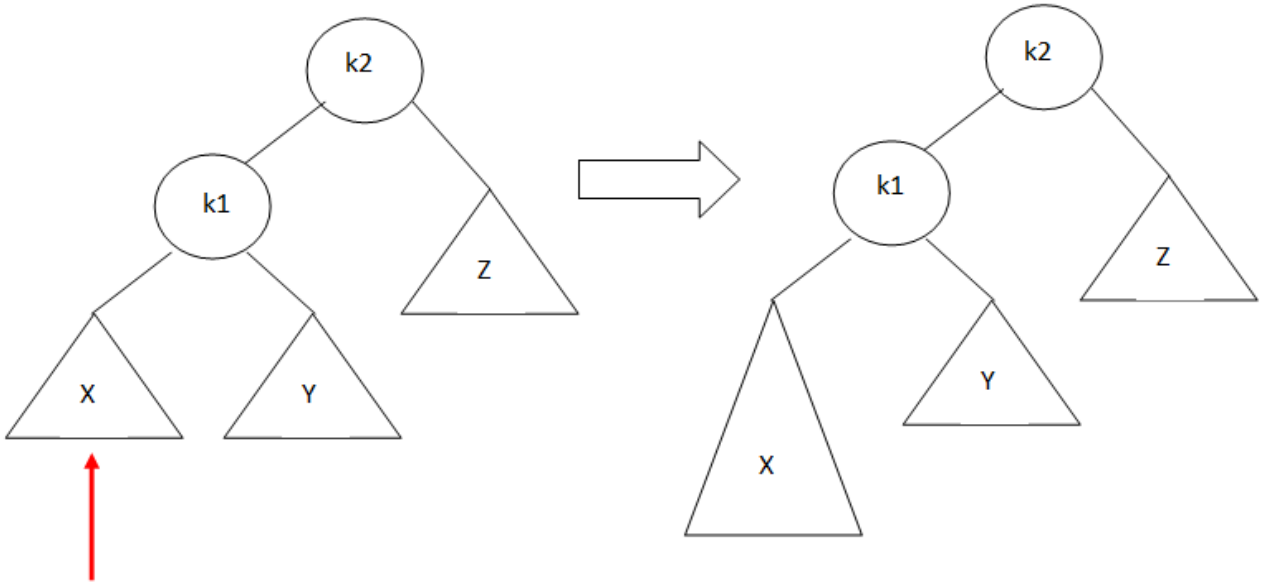
- Assume that at a given point α is the node that needs to be rebalanced.
- Since α was balanced before the insertion, and is not after the insertion, we can identify four cases in which a violation might occur:
  - Insertion into the left subtree of the left child of α
  - Insertion into the right subtree of the left child of α
  - Insertion into the left subtree of the right child of α
  - Insertion into the right subtree of the right child of α

AVL Trees - rotations - case 1



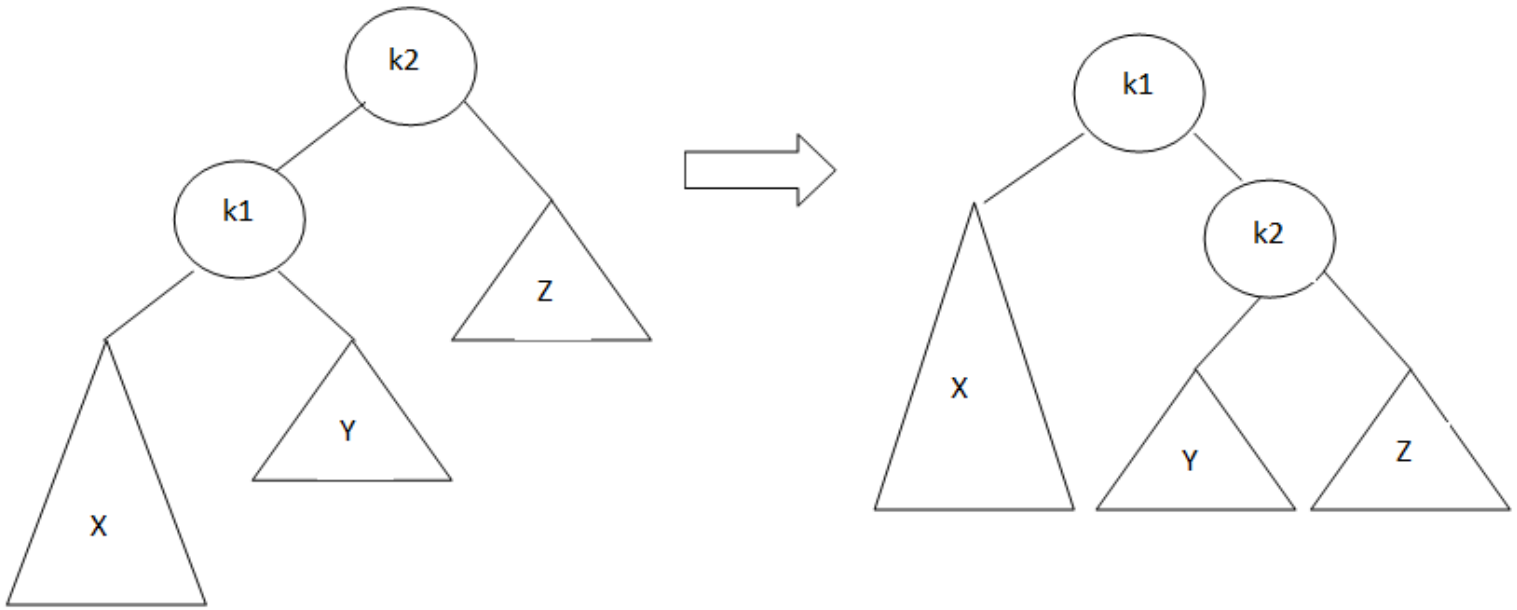
- Obs: X, Y and Z represent subtrees with the same height.

AVL Trees - rotations - case 1

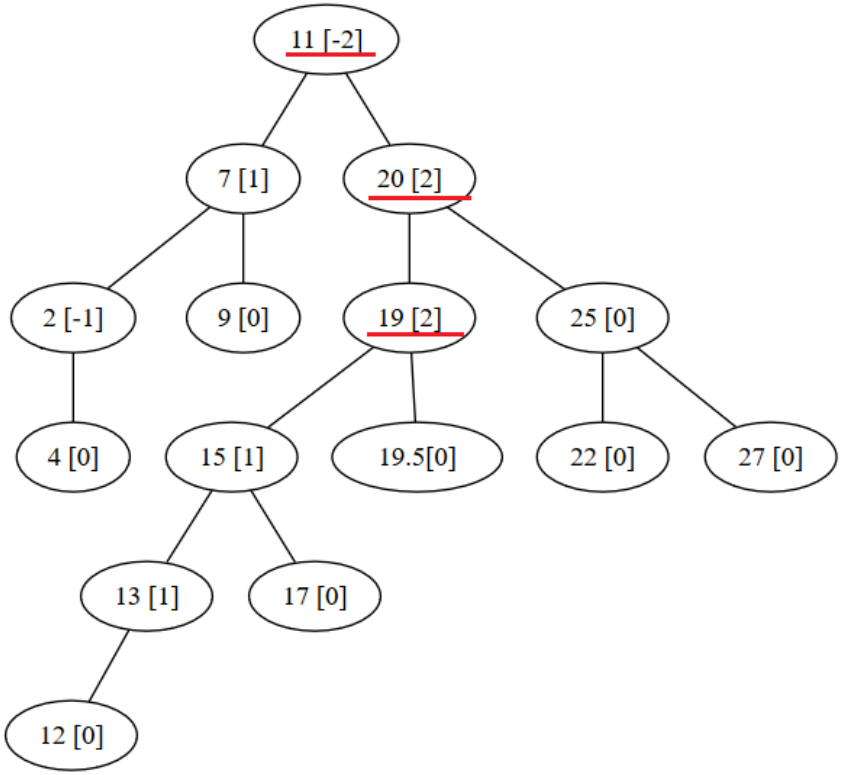


- Obs: X, Y and Z represent subtrees with the same height.
- Solution: single rotation to right

AVL Trees - rotation - Single Rotation to Right

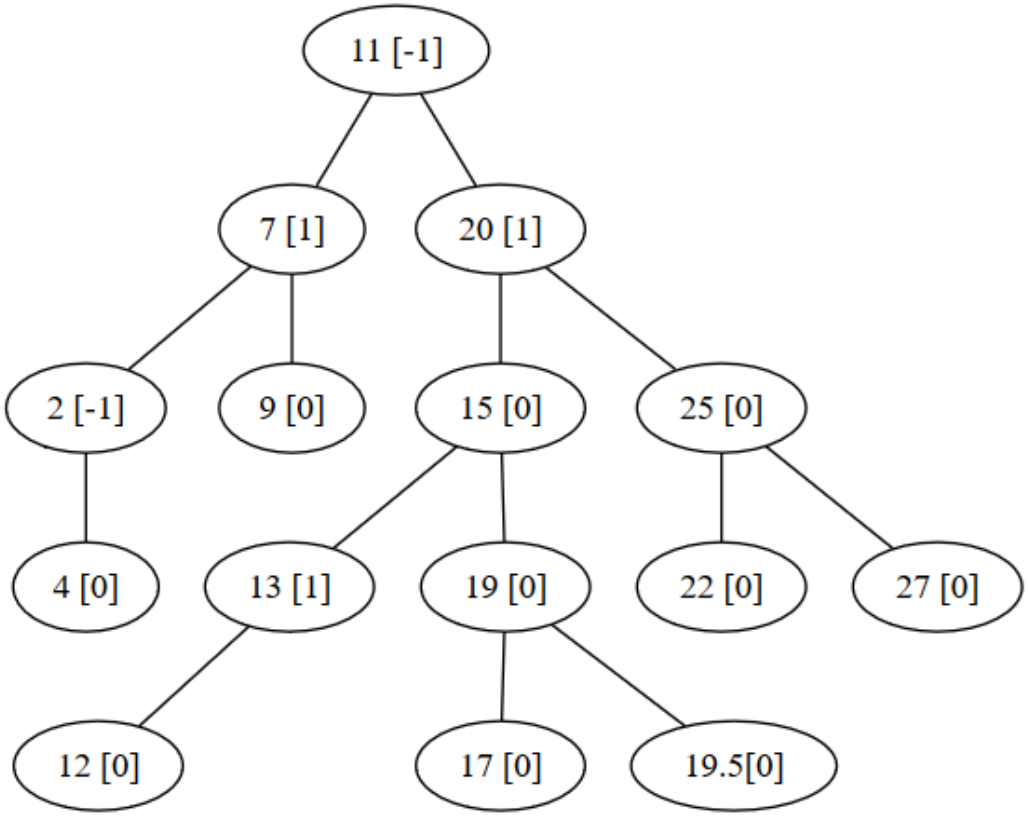


AVL Trees - rotations - case 1 example

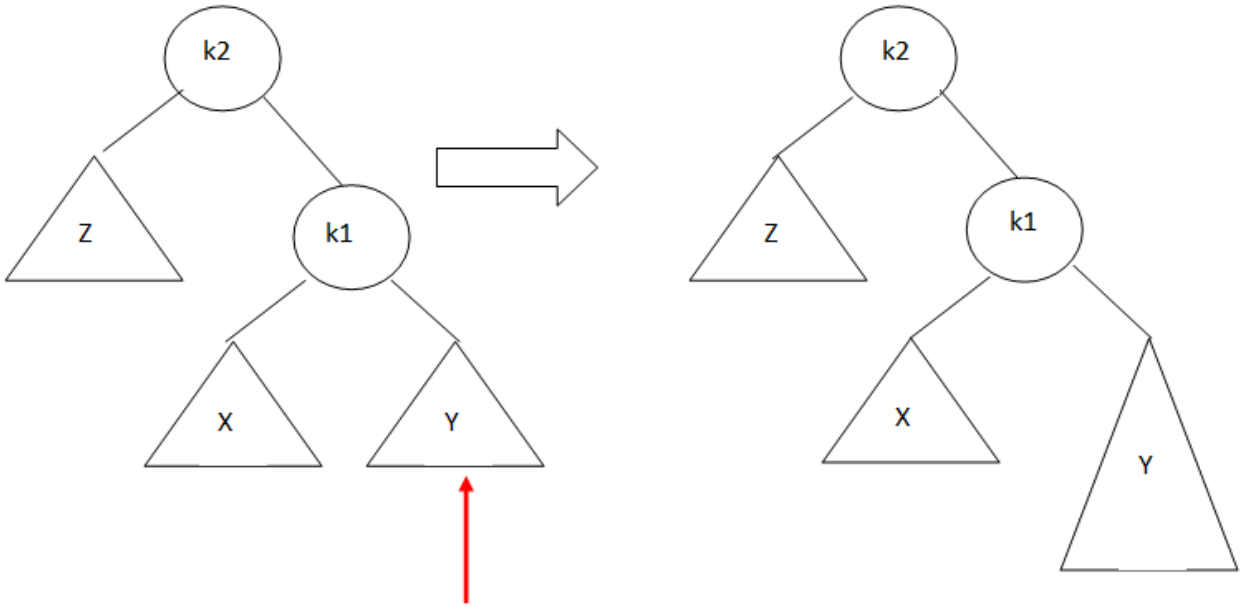


- Node 19 is imbalanced, because we inserted a new node (12) in the left subtree of the left child.
- Solution: **single rotation to right**

AVL Trees - rotation - case 1 example

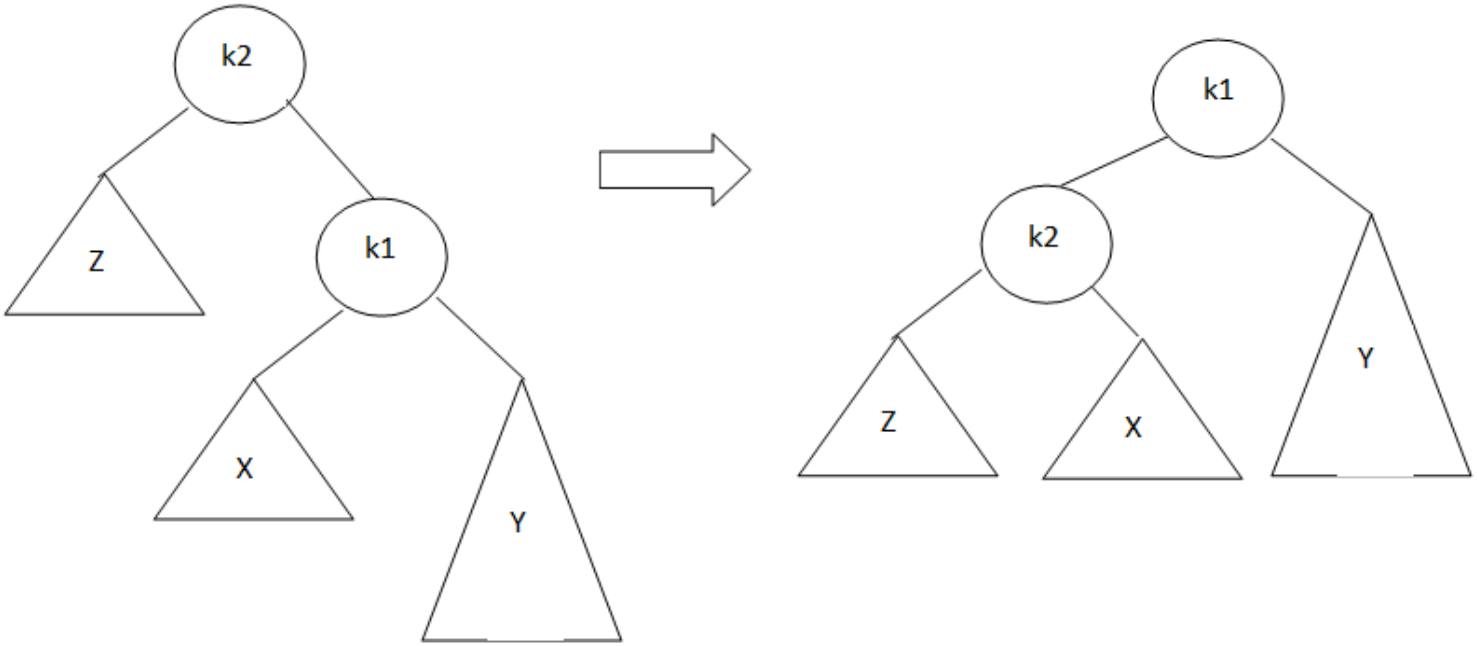


AVL Trees - rotations - case 4

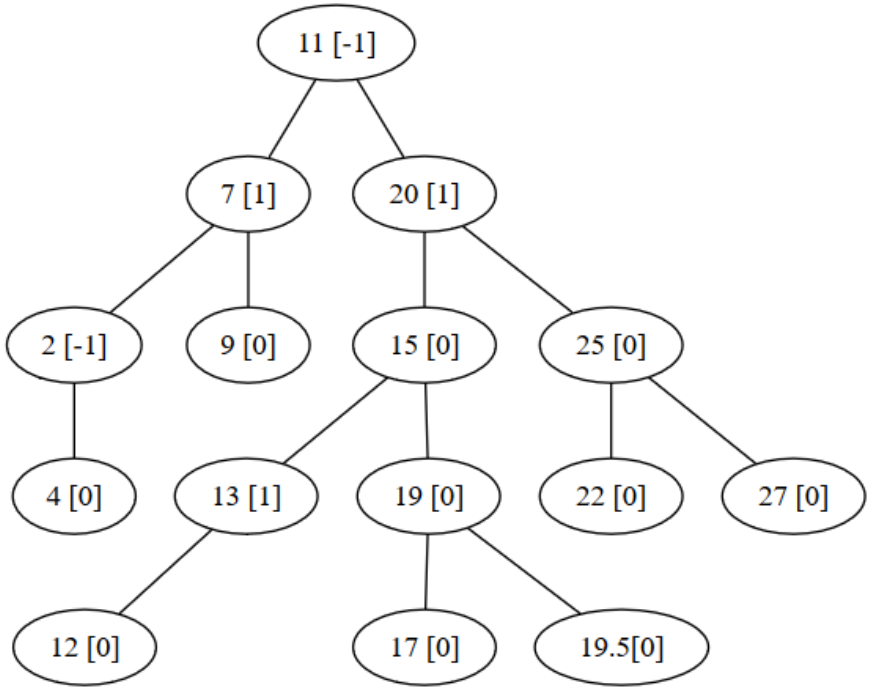


- Solution: **single rotation to left**

AVL Trees - rotation - Single Rotation to Left

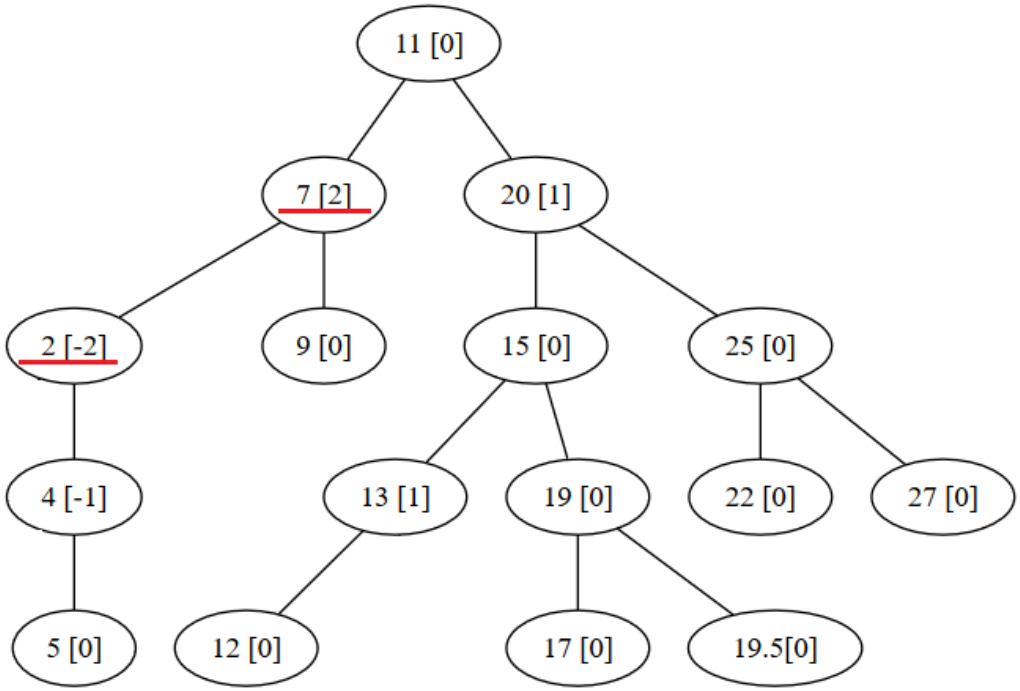


AVL Trees - rotations - case 4 example



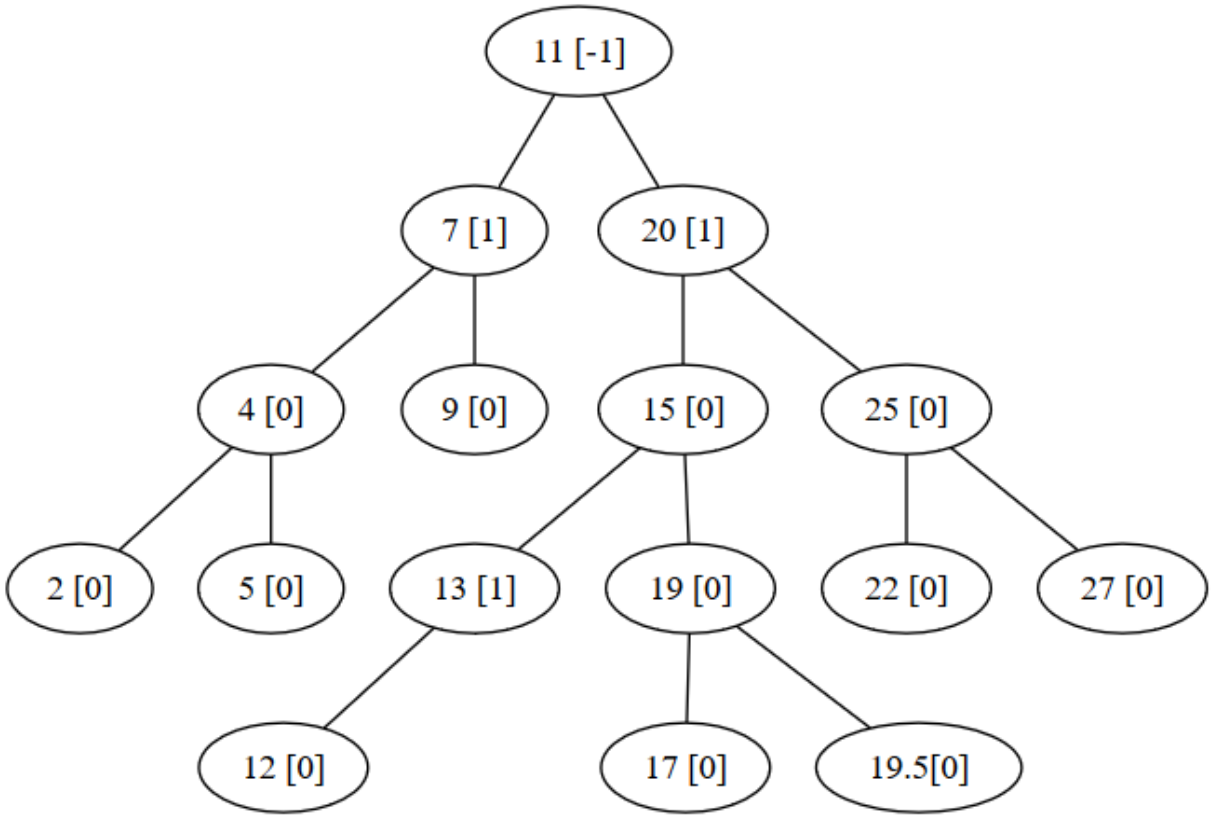
- Insert value 5

AVL Trees - rotations - case 4 example



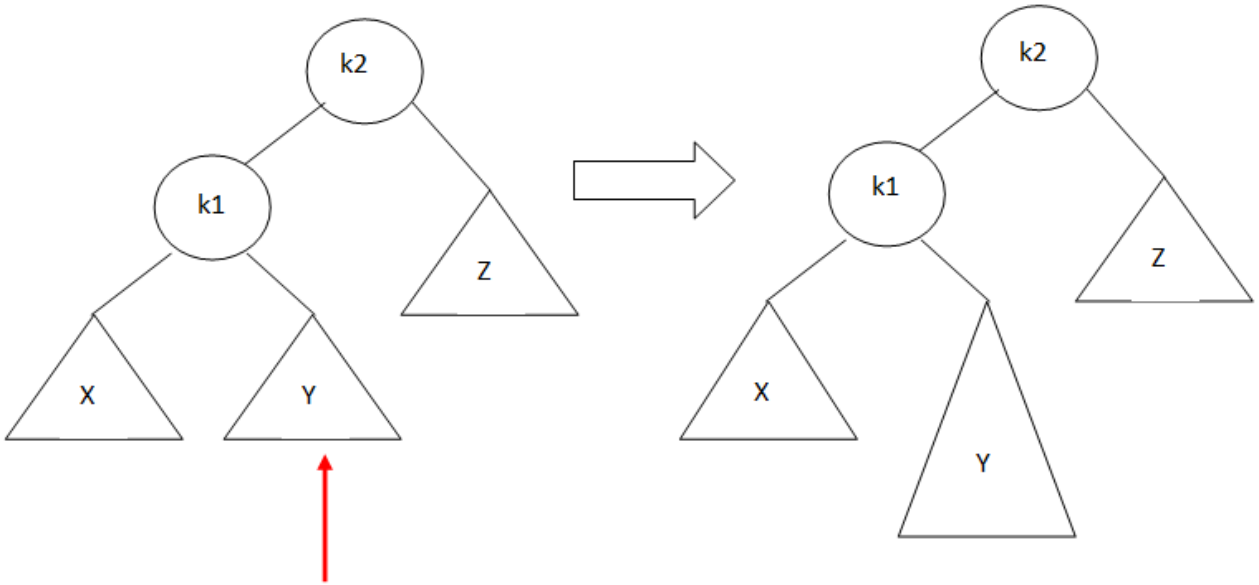
- Node 2 is imbalanced, because we inserted a new node (5) to the right subtree of the right child
- Solution: **single rotation to left**

AVL Trees - rotation - case 4 example



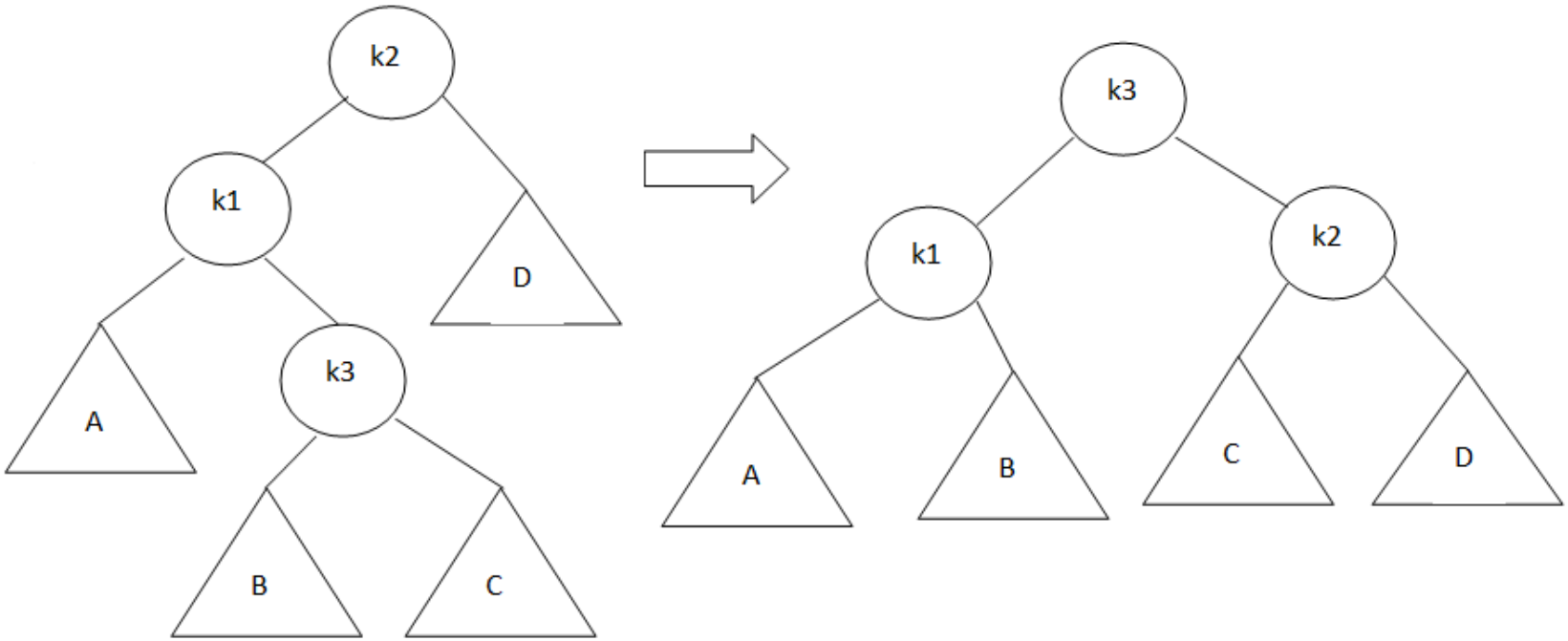
- After the rotation

AVL Trees - rotations - case 2

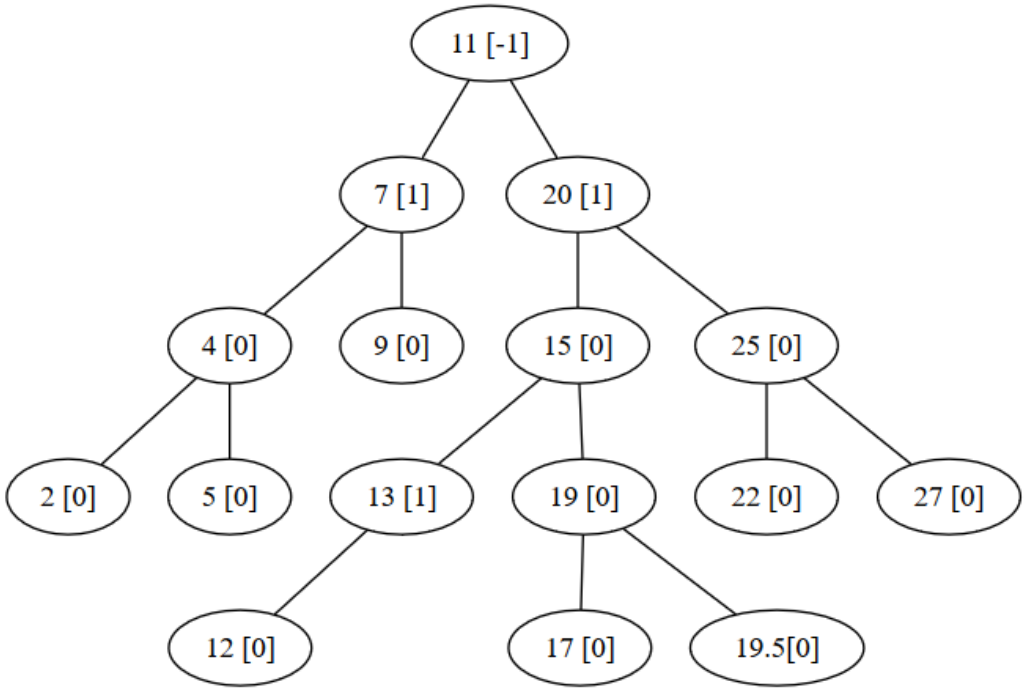


- Solution: **Double rotation to right**

AVL Trees - rotation - Double Rotation to Right

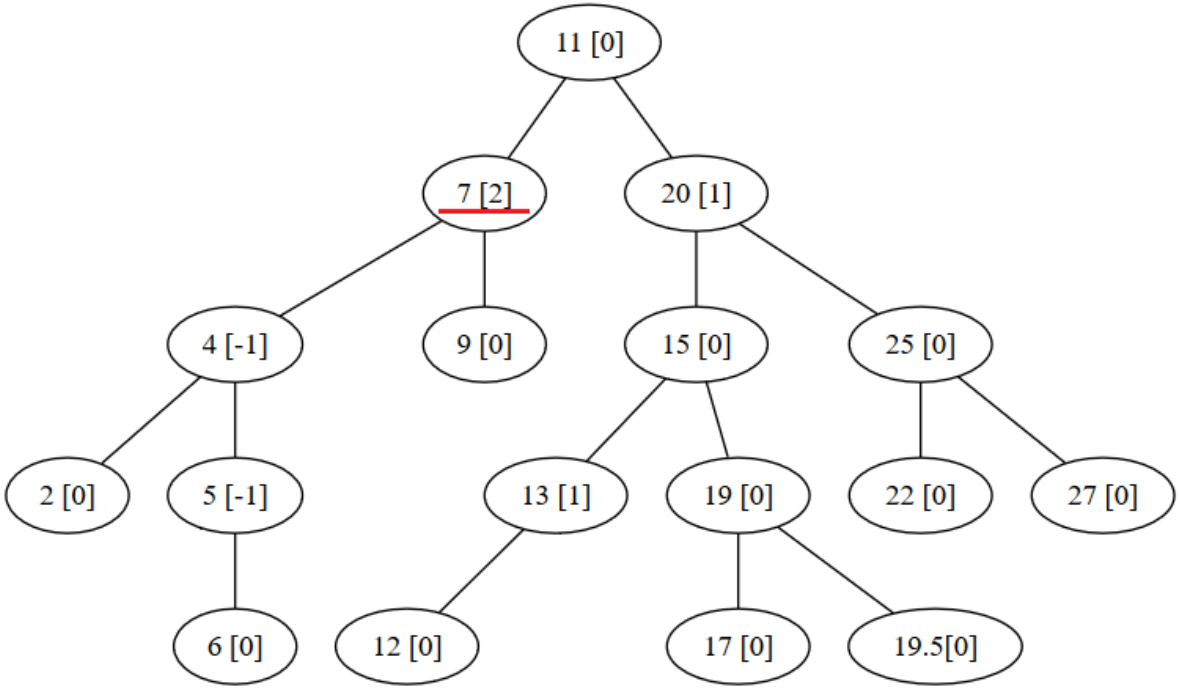


AVL Trees - rotations - case 2 example



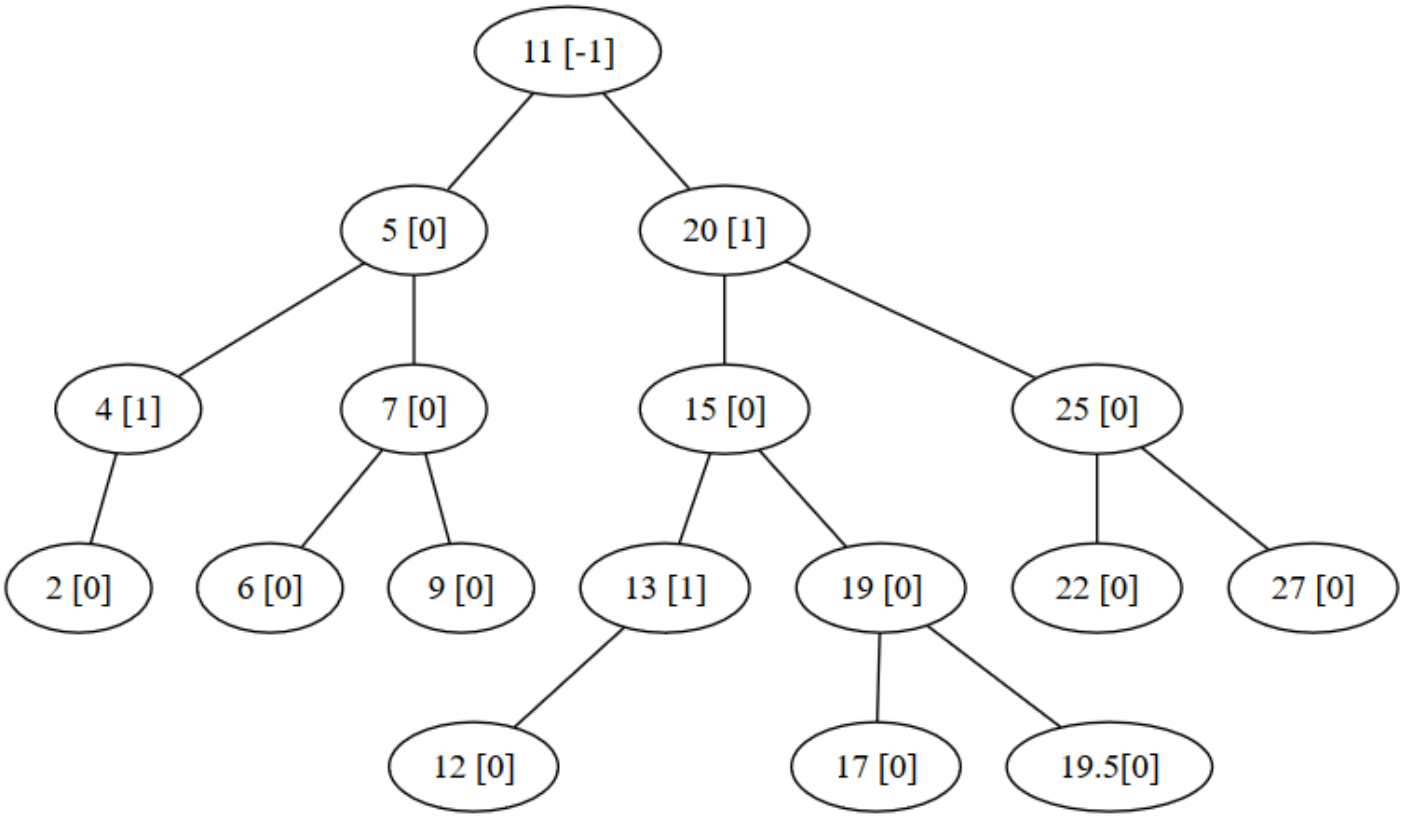
- Insert value 6

AVL Trees - rotations - case 2 example



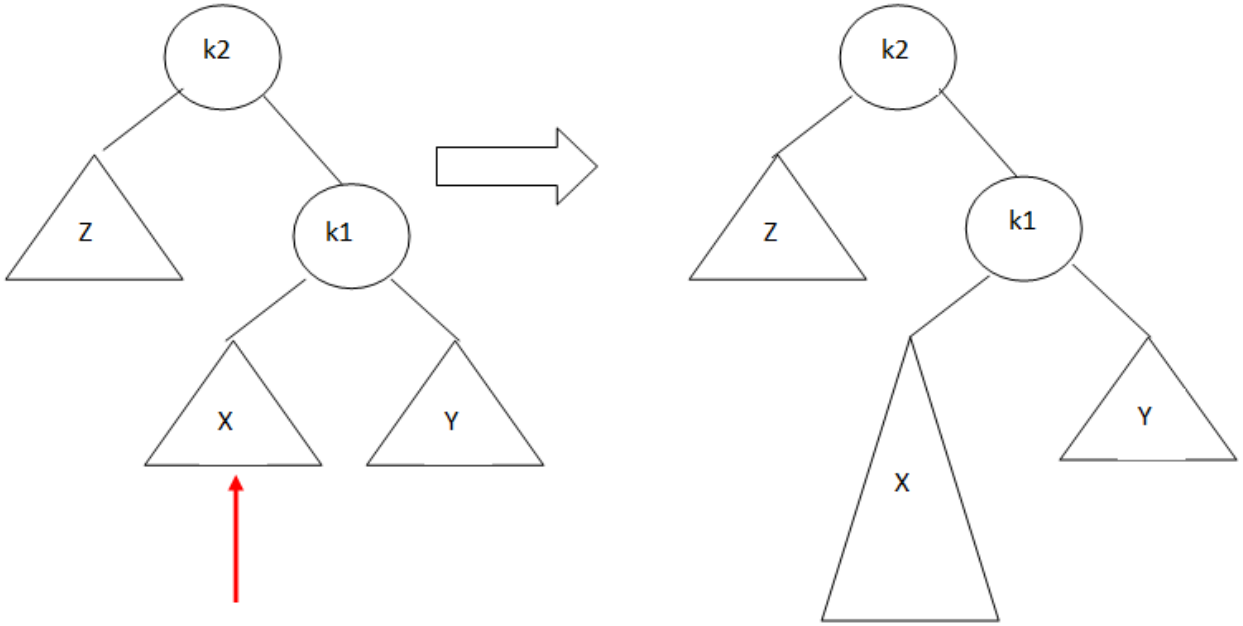
- Node 7 is imbalanced, because we inserted a new node (6) to the right subtree of the left child
- Solution: **double rotation to right**

AVL Trees - rotation - case 2 example



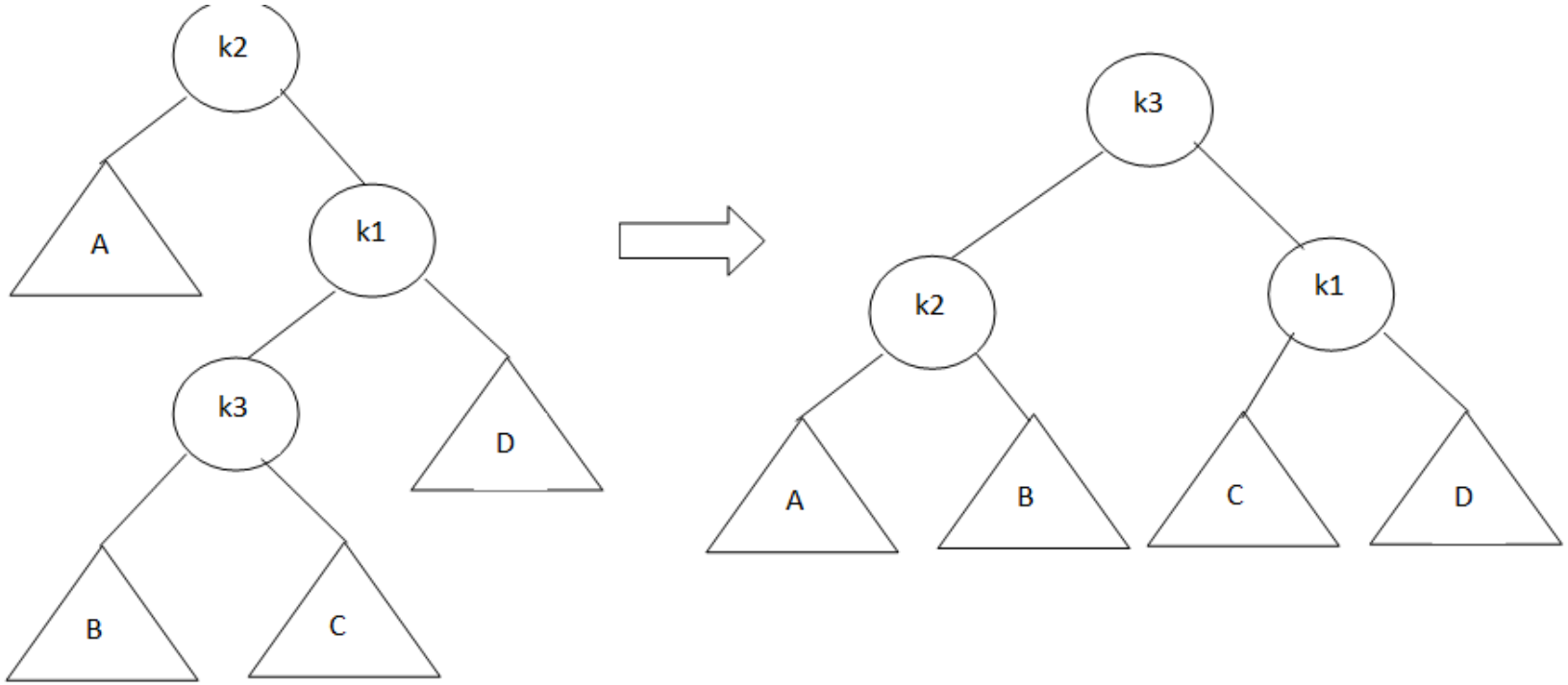
- After the rotation

AVL Trees - rotations - case 3

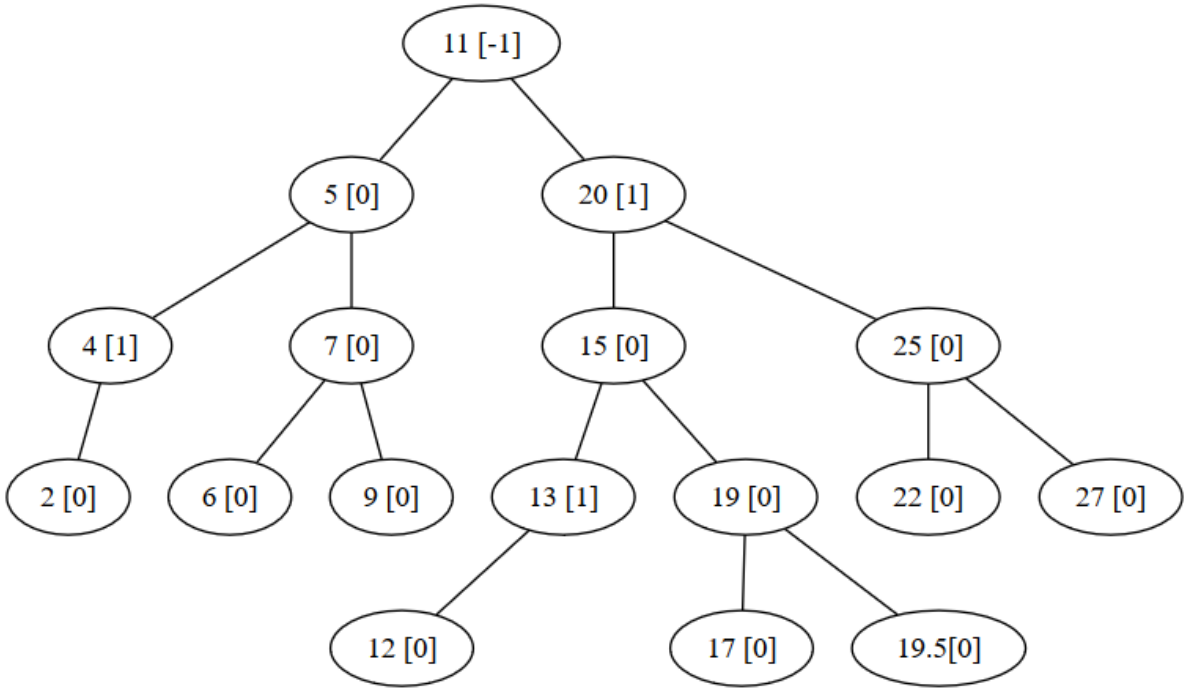


- Solution: **Double rotation to left**

AVL Trees - rotation - Double Rotation to Left

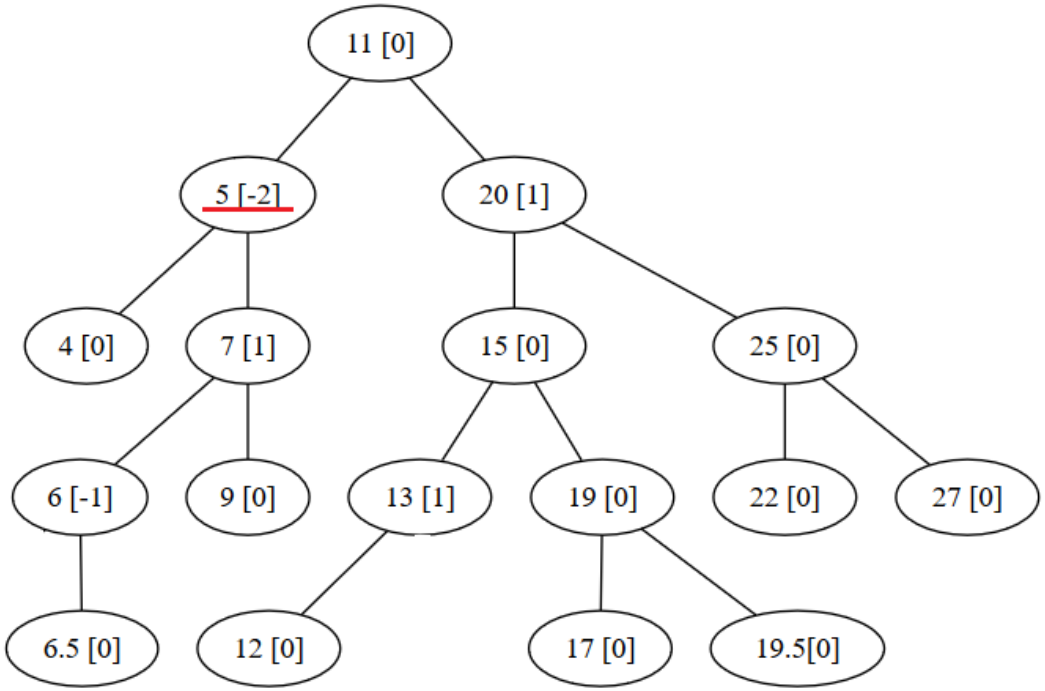


AVL Trees - rotations - case 3 example



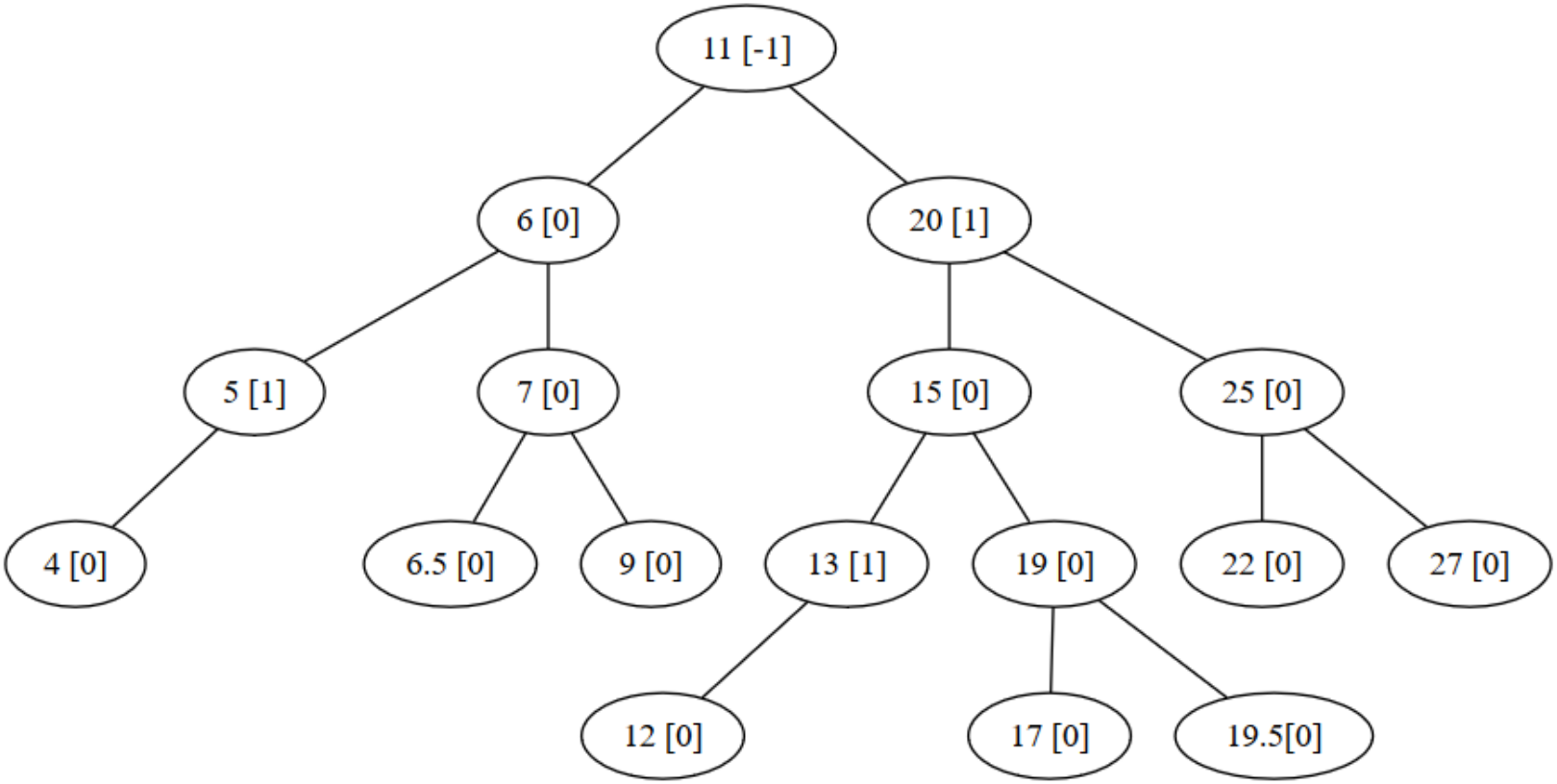
- Remove node with value 2 and insert value 6.5

AVL Trees - rotations - case 3 example



- Node 5 is imbalanced, because we inserted a new node (6.5) to the left subtree of the right child
- Solution: **double rotation to left**

AVL Trees - rotation - case 3 example



- After the rotation

AVL rotations example I

- Start with an empty AVL tree
- Insert 2

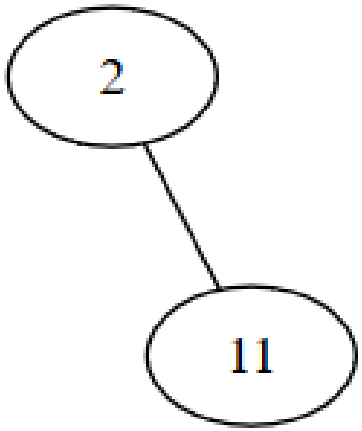
AVL rotations example II

- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example III

- No rotation is needed
- Insert 11

AVL rotations example IV

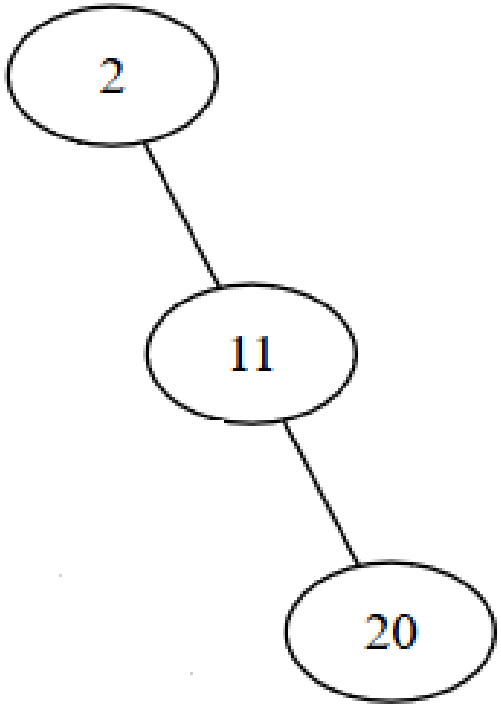


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example V

- No rotation is needed
- Insert 20

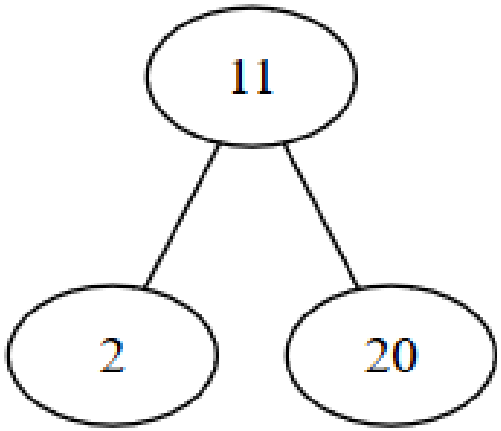
AVL rotations example VI



- Do we need a rotation?
- If yes, on which node and what type of rotation?

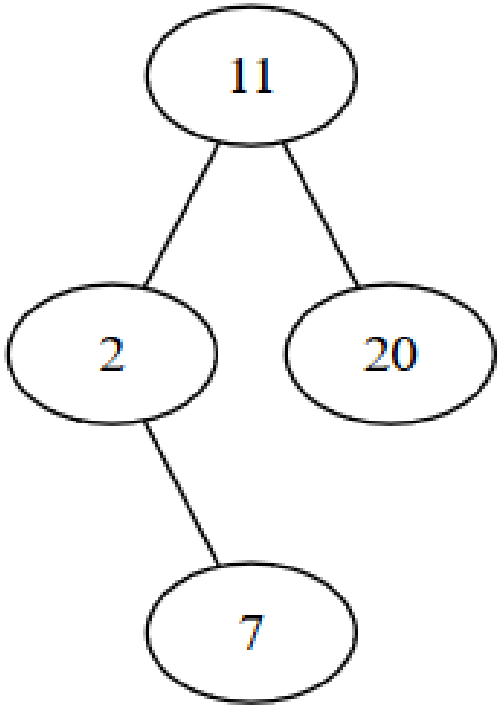
AVL rotations example VII

- Yes, we need a single left rotation on node 2
- After the rotation:



- Insert 7

AVL rotations example VIII

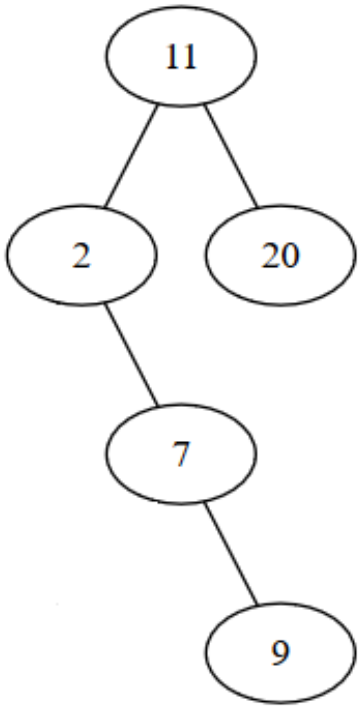


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example IX

- No rotation is needed
- Insert 9

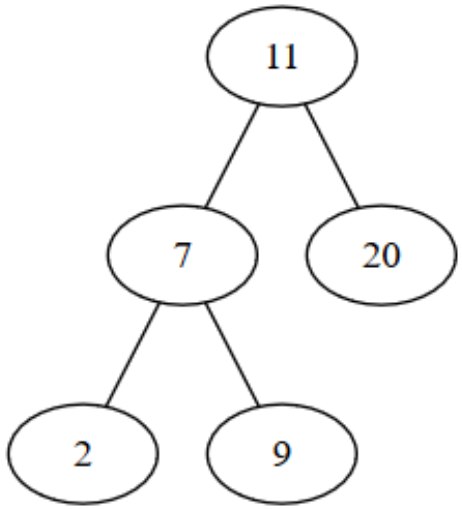
AVL rotations example X



- Do we need a rotation?
- If yes, on which node and what type of rotation?

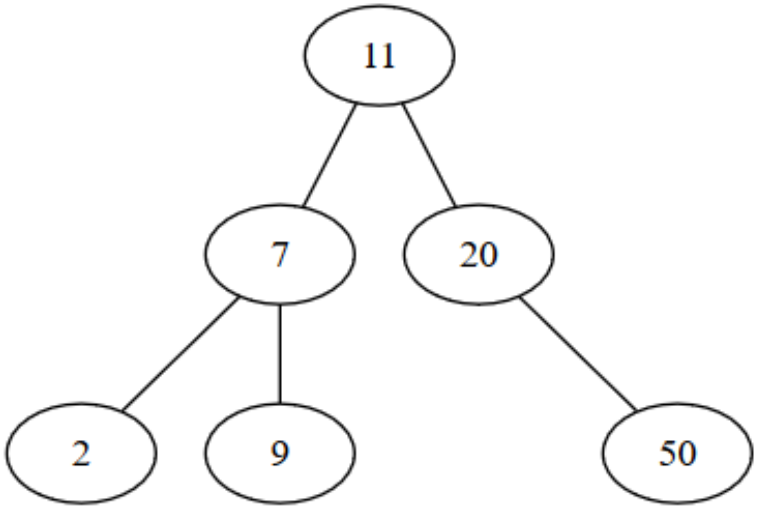
AVL rotations example XI

- Yes, we need a single left rotation on node 2
- After the rotation:



- Insert 50

AVL rotations example XII

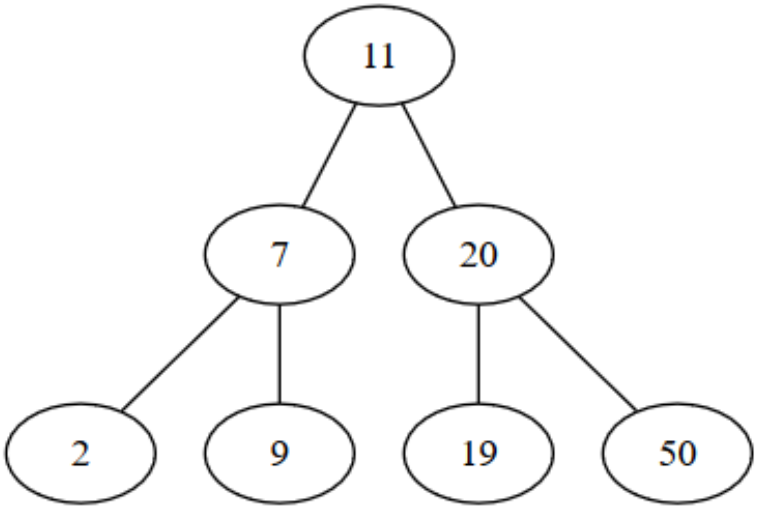


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XIII

- No rotation is needed
- Insert 19

AVL rotations example XIV

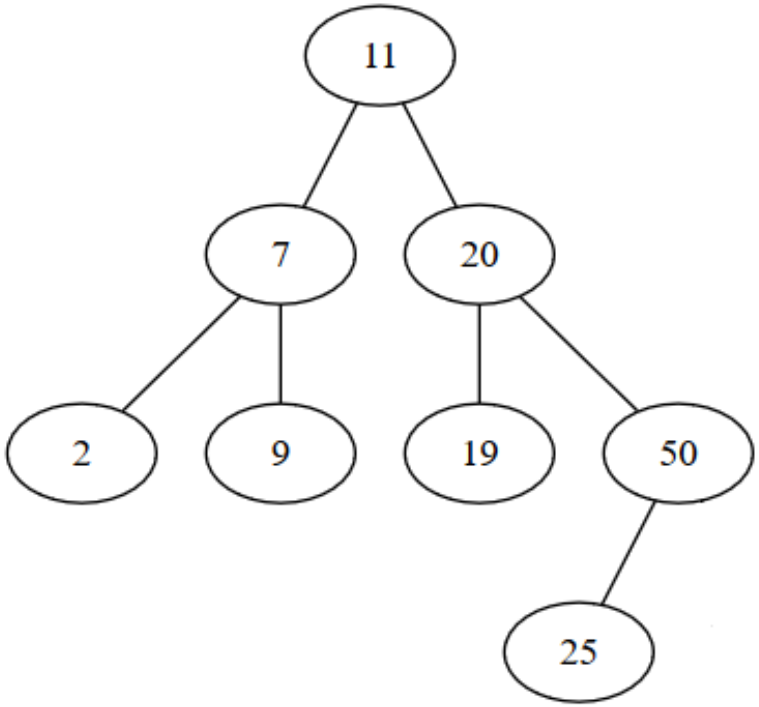


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XV

- No rotation is needed
- Insert 25

AVL rotations example XVI

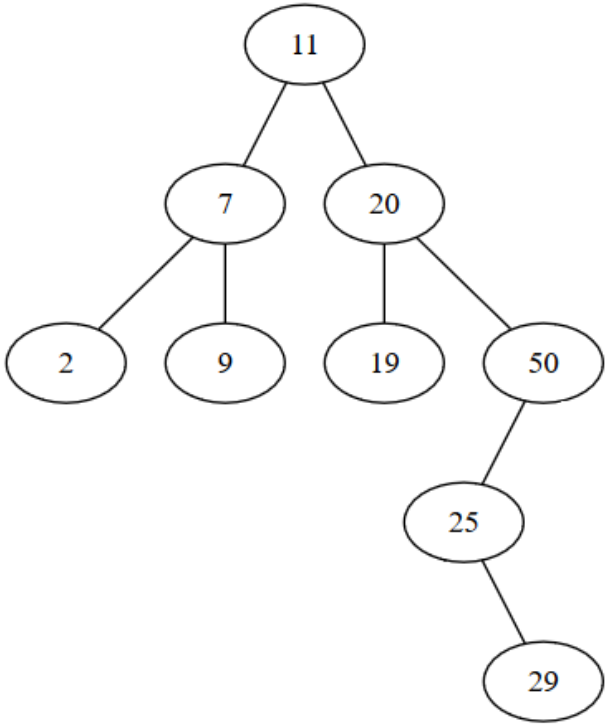


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XVII

- No rotation is needed
- Insert 29

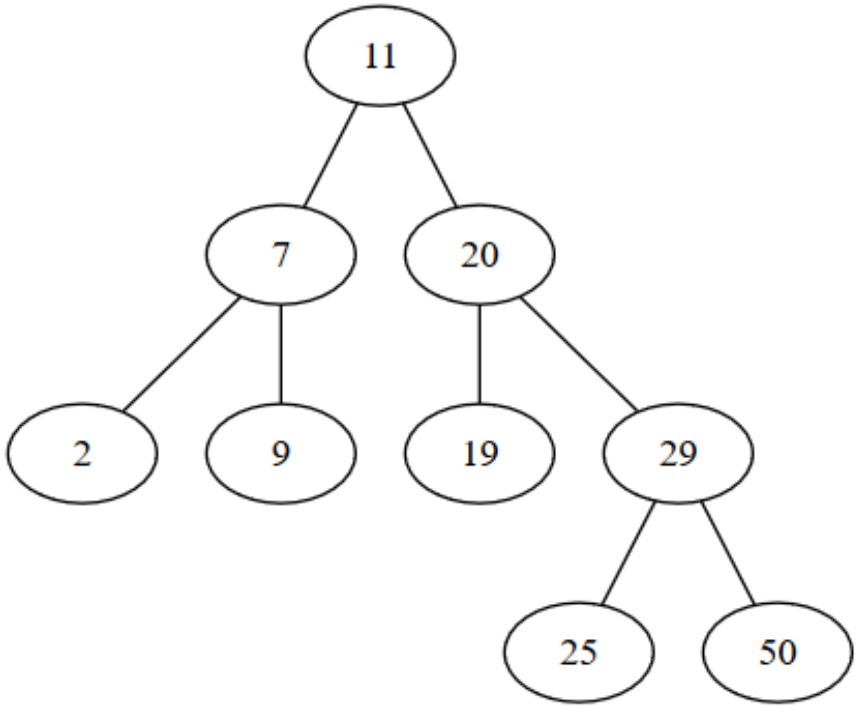
AVL rotations example XVIII



- Do we need a rotation?
- If yes, on which node and what type of rotation?

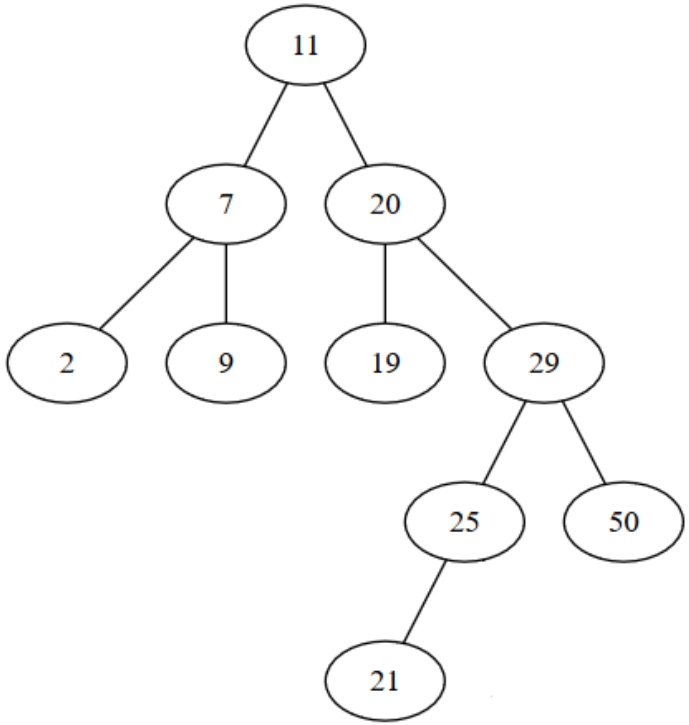
AVL rotations example XIX

- Yes, we need a double right rotation on node 50
- After the rotation



- Insert 21

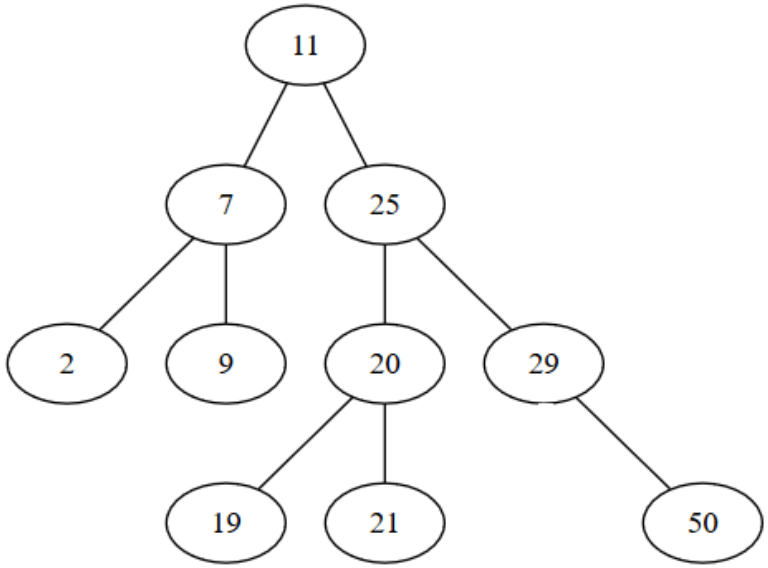
AVL rotations example XX



- Do we need a rotation?
- If yes, on which node and what type of rotation?

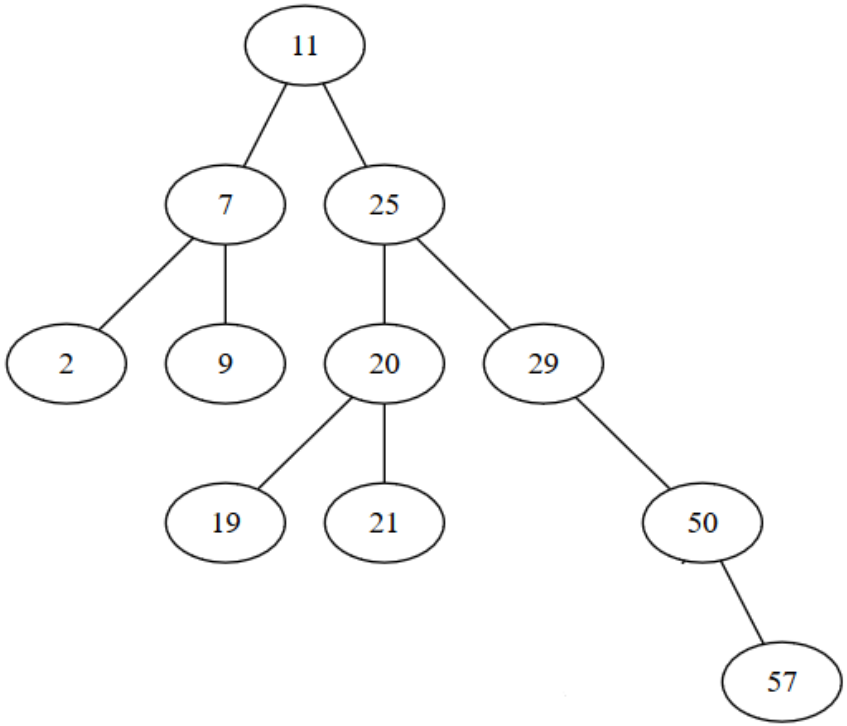
AVL rotations example XXI

- Yes, we need a double left rotation on node 20
- After the rotation



- Insert 57

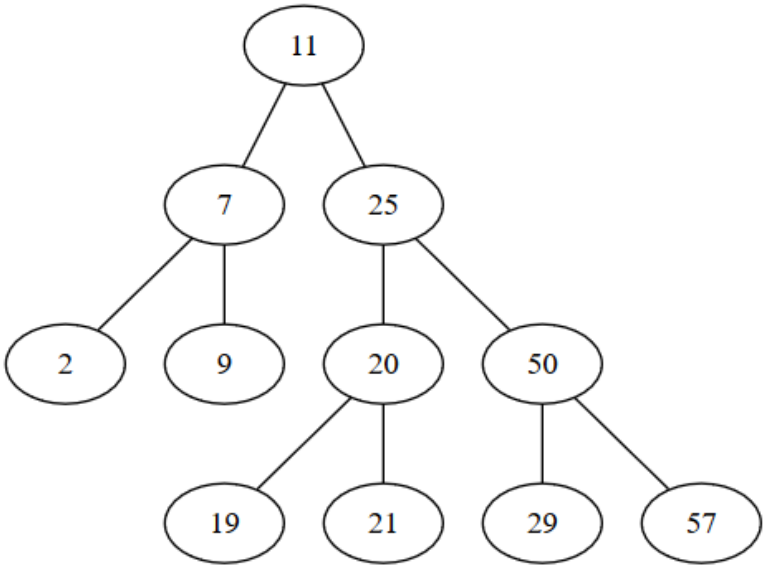
AVL rotations example XXII



- Do we need a rotation?
- If yes, on which node and what type of rotation?

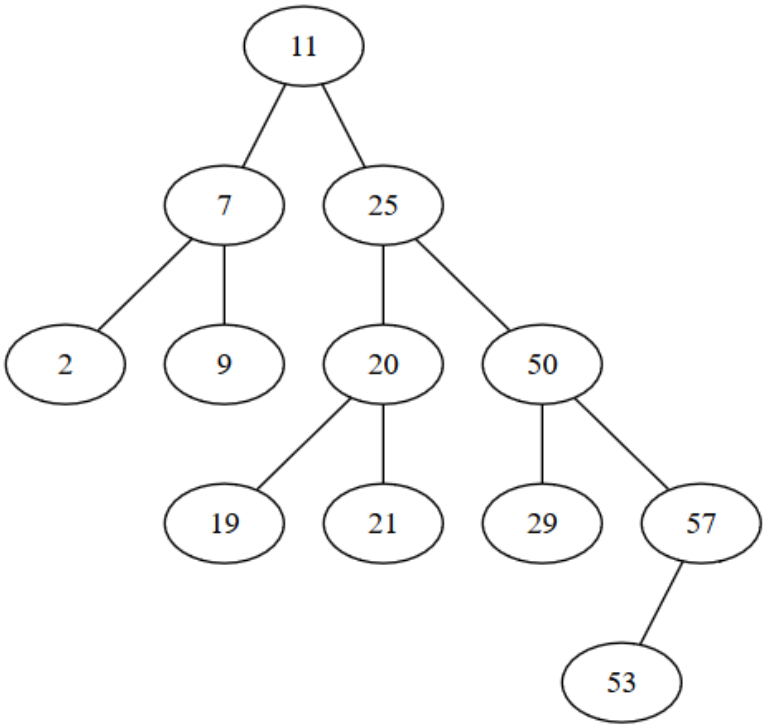
AVL rotations example XXIII

- Yes, we need a single left rotation on node 50
- After the rotation



- Insert 53

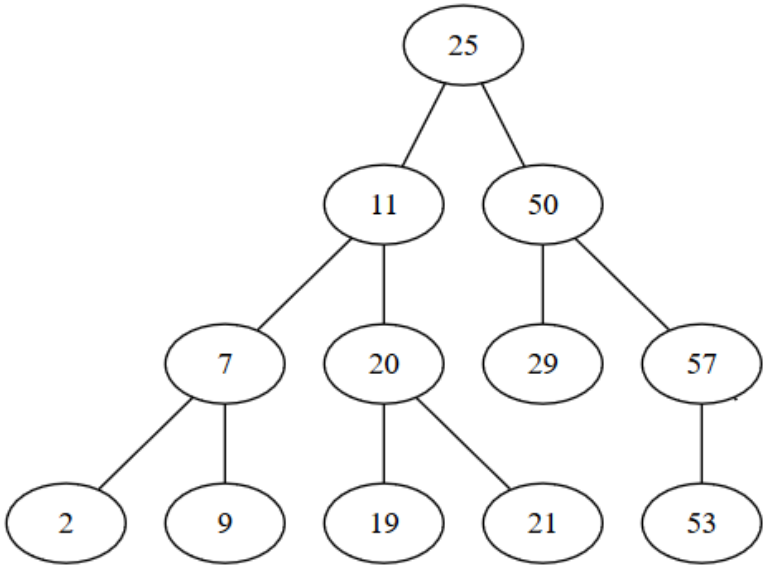
AVL rotations example XXIV



- Do we need a rotation?
- If yes, on which node and what type of rotation?

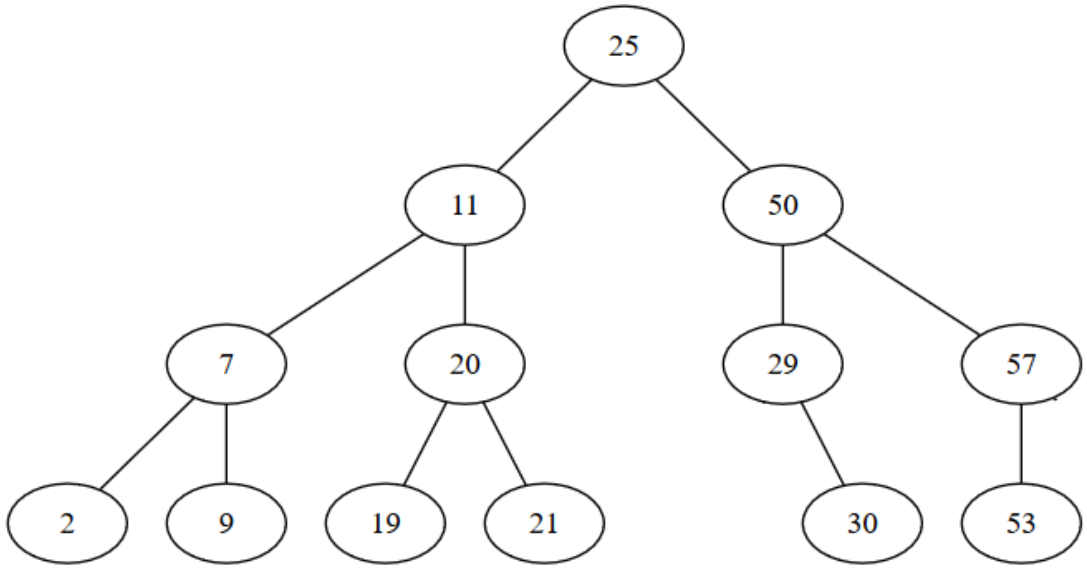
AVL rotations example XXV

- Yes, we need a single left rotation on node 11
- After the rotation



- Insert 30

AVL rotations example XXVI

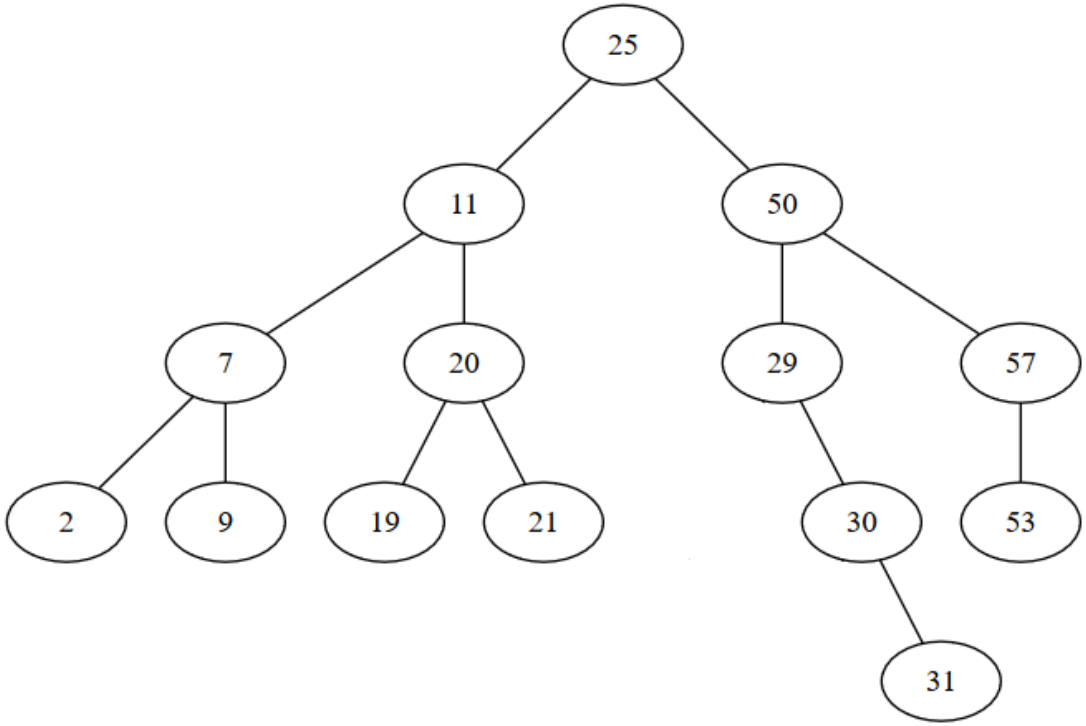


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XXVII

- No rotation is needed
- Insert 31

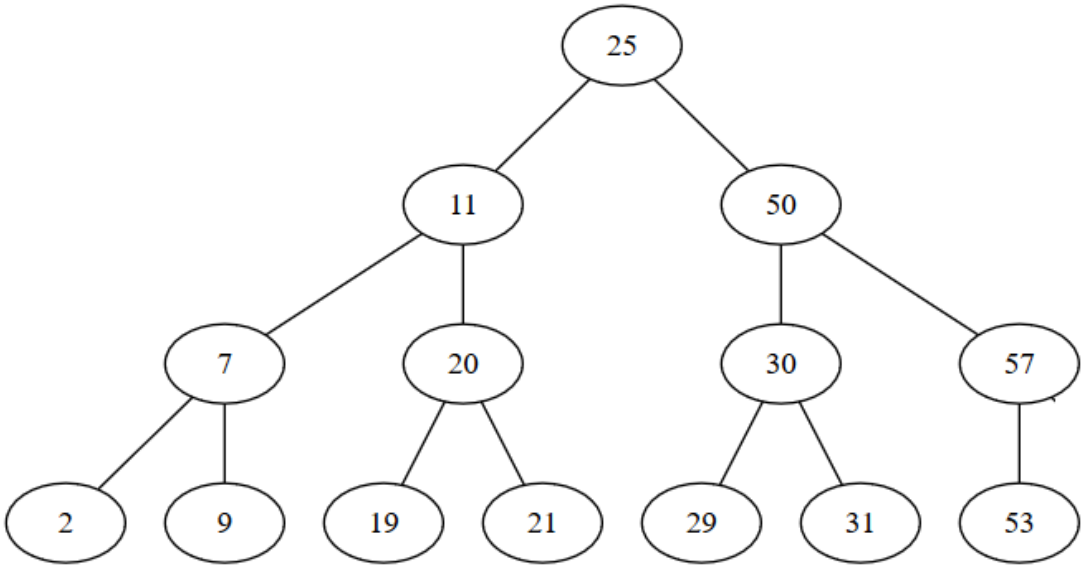
AVL rotations example XXVIII



- Do we need a rotation?
- If yes, on which node and what type of rotation?

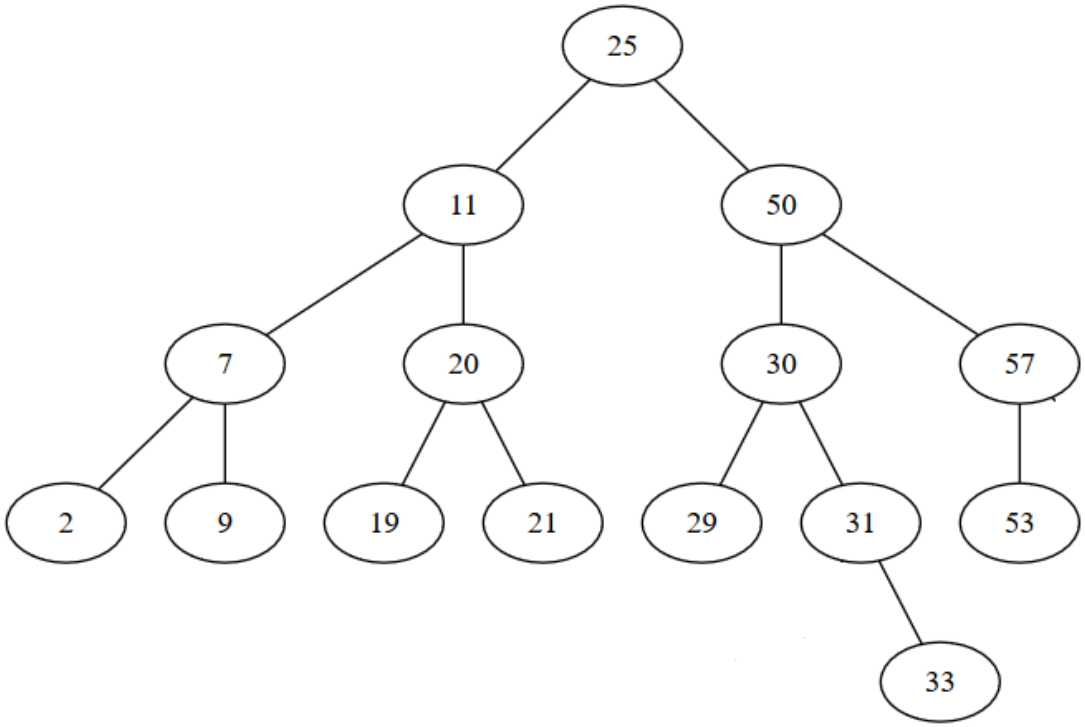
AVL rotations example XXIX

- Yes, we need a single left rotation on node 29
- After the rotation



- Insert 33

AVL rotations example XXX

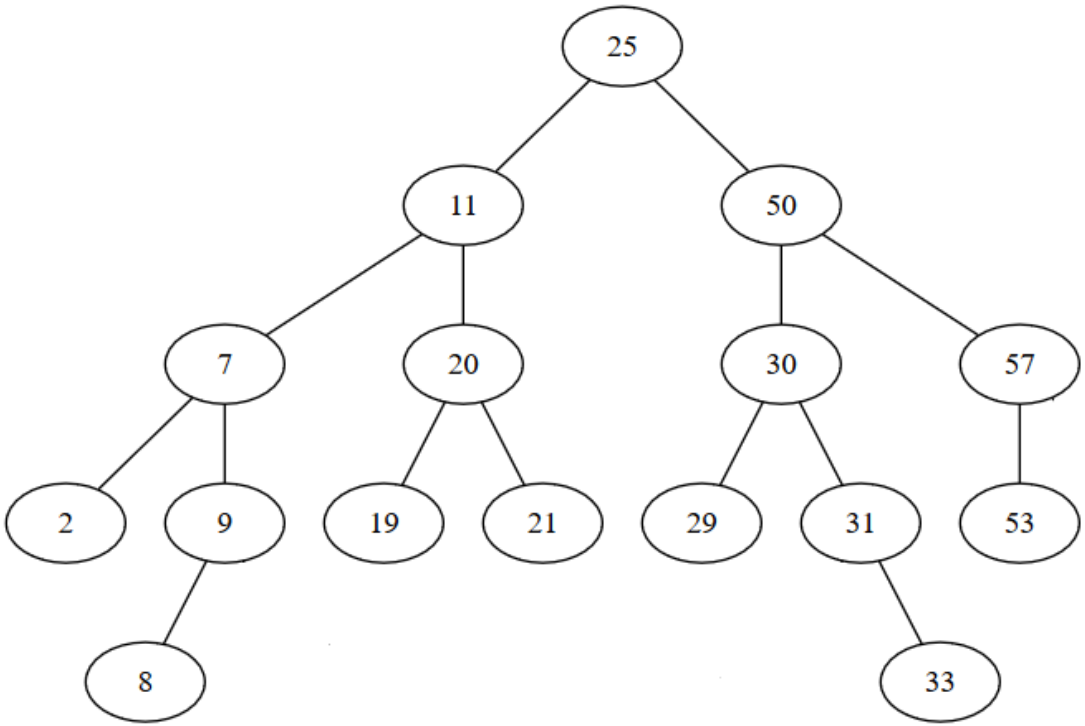


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XXXI

- No rotation is needed
- Insert 8

AVL rotations example XXXII

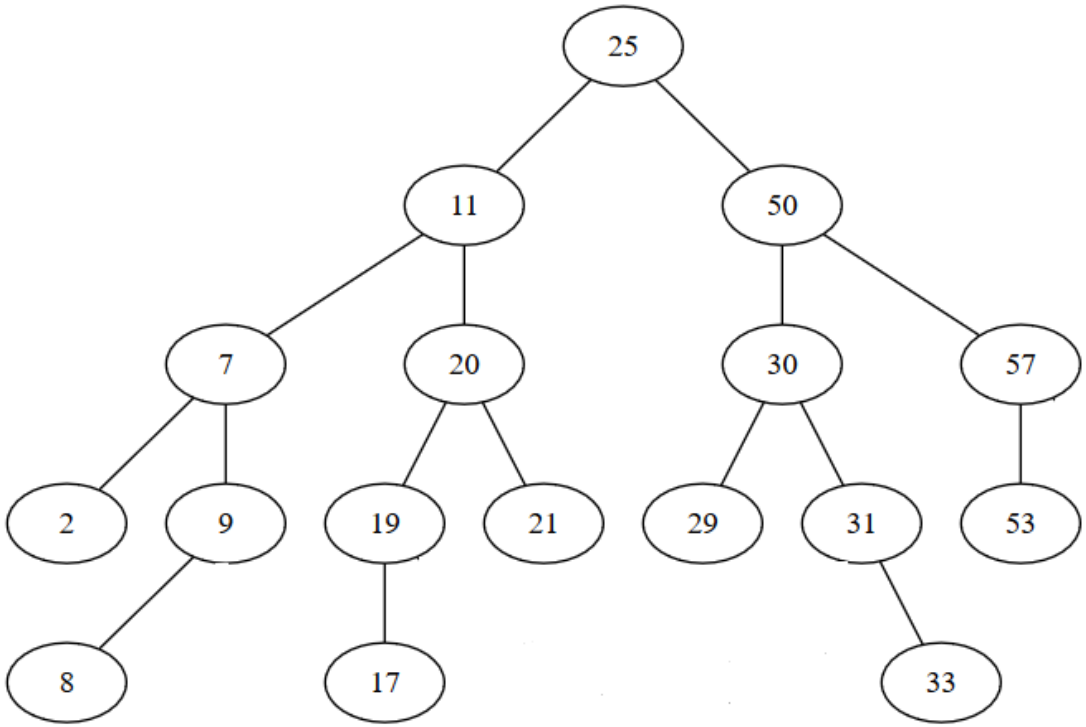


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XXXIII

- No rotation is needed
- Insert 17

AVL rotations example XXXIV

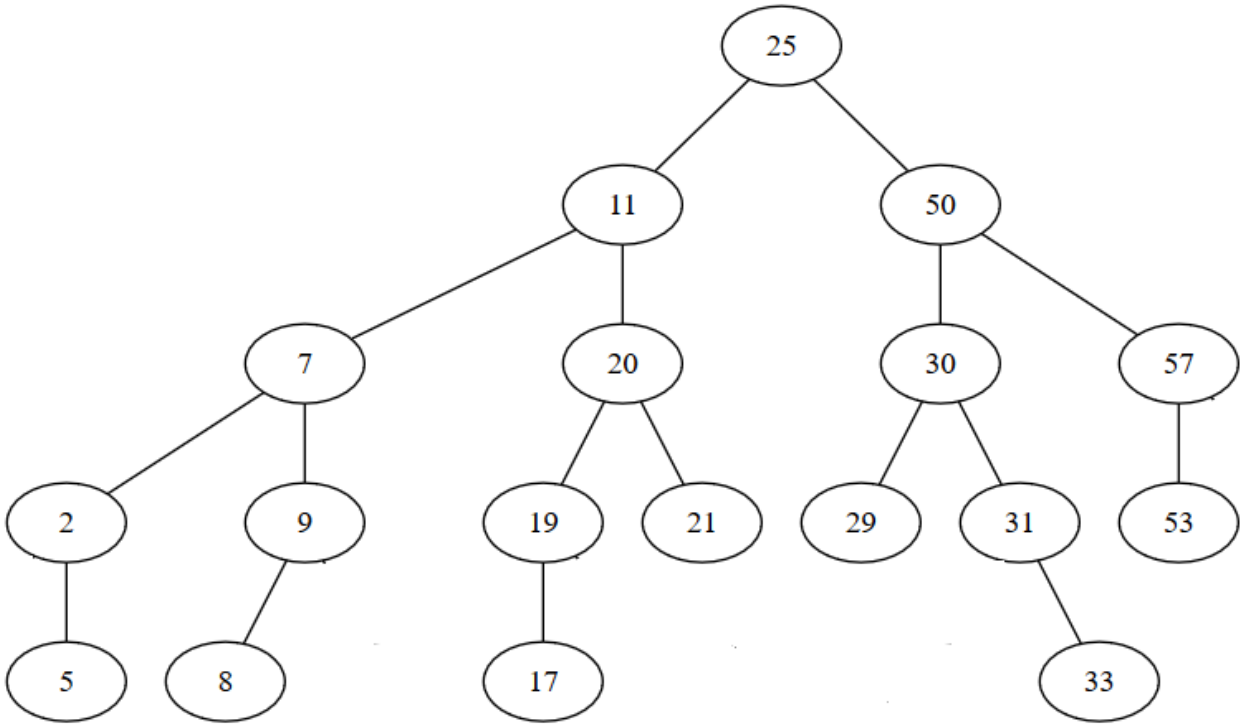


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XXXV

- No rotation is needed
- Insert 5

AVL rotations example XXXVI

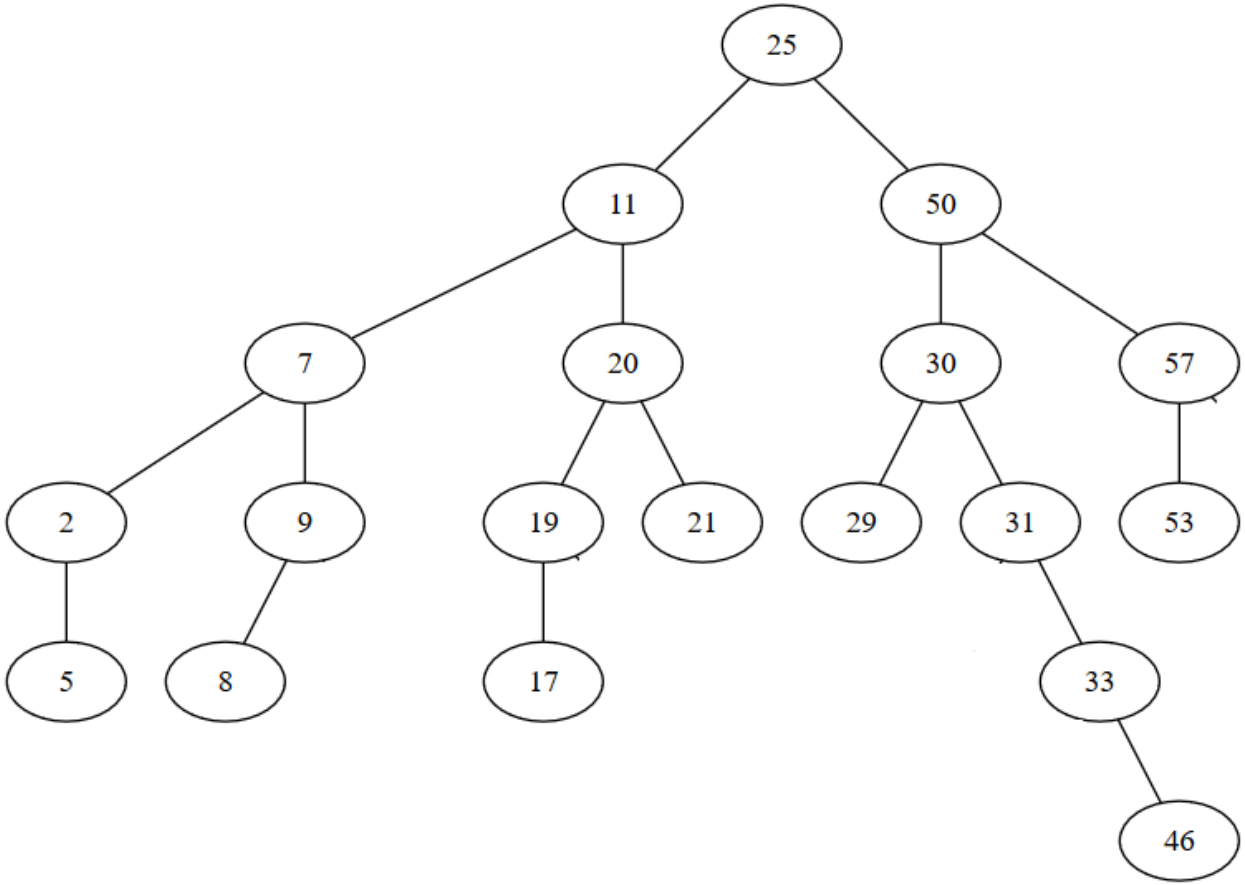


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XXXVII

- No rotation is needed
- Insert 46

AVL rotations example XXXVIII



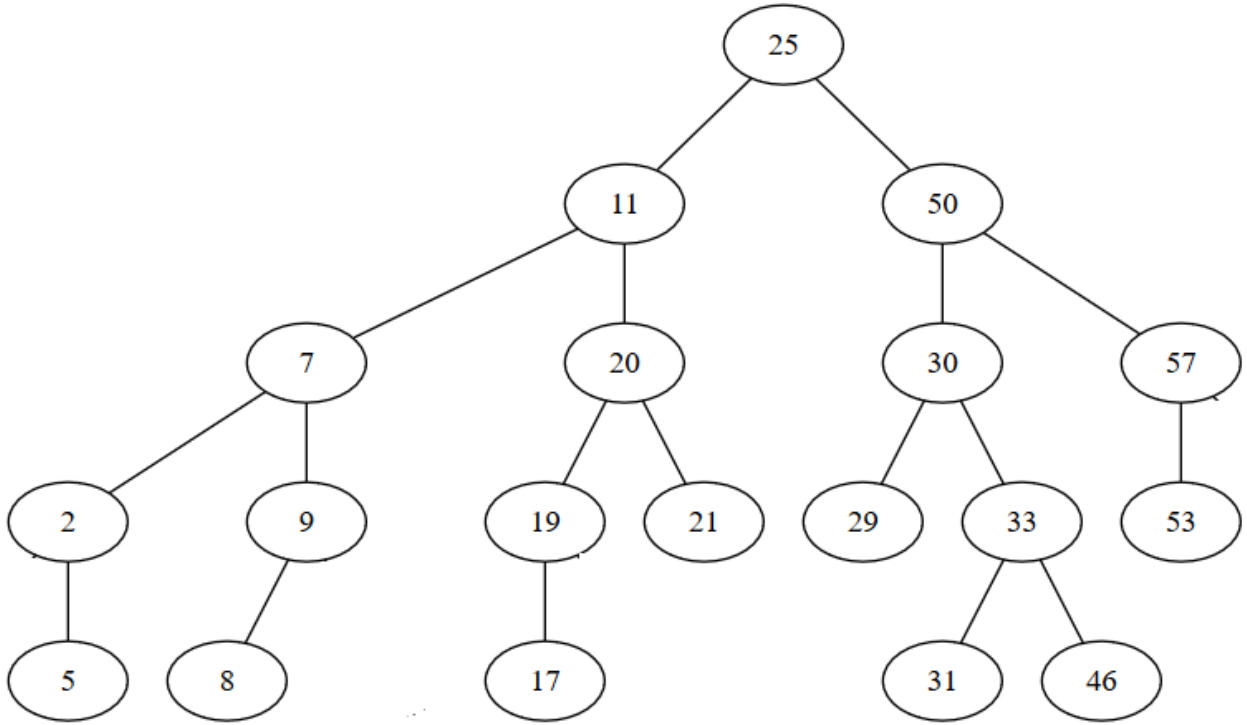
- Do we need a rotation?

AVL rotations example XXXIX

- If yes, on which node and what type of rotation?

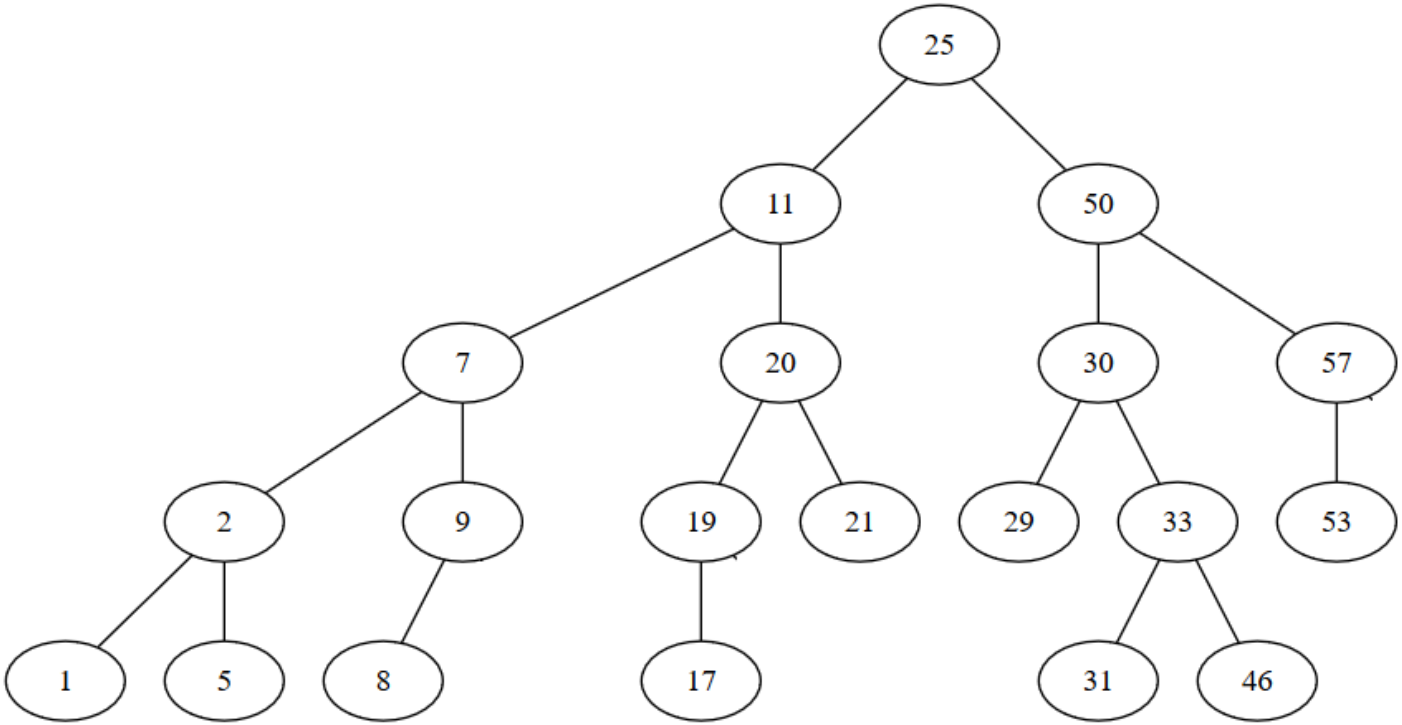
AVL rotations example XL

- Yes, we need a single left rotation on node 31
- After the rotation



- Insert 1

AVL rotations example XLI

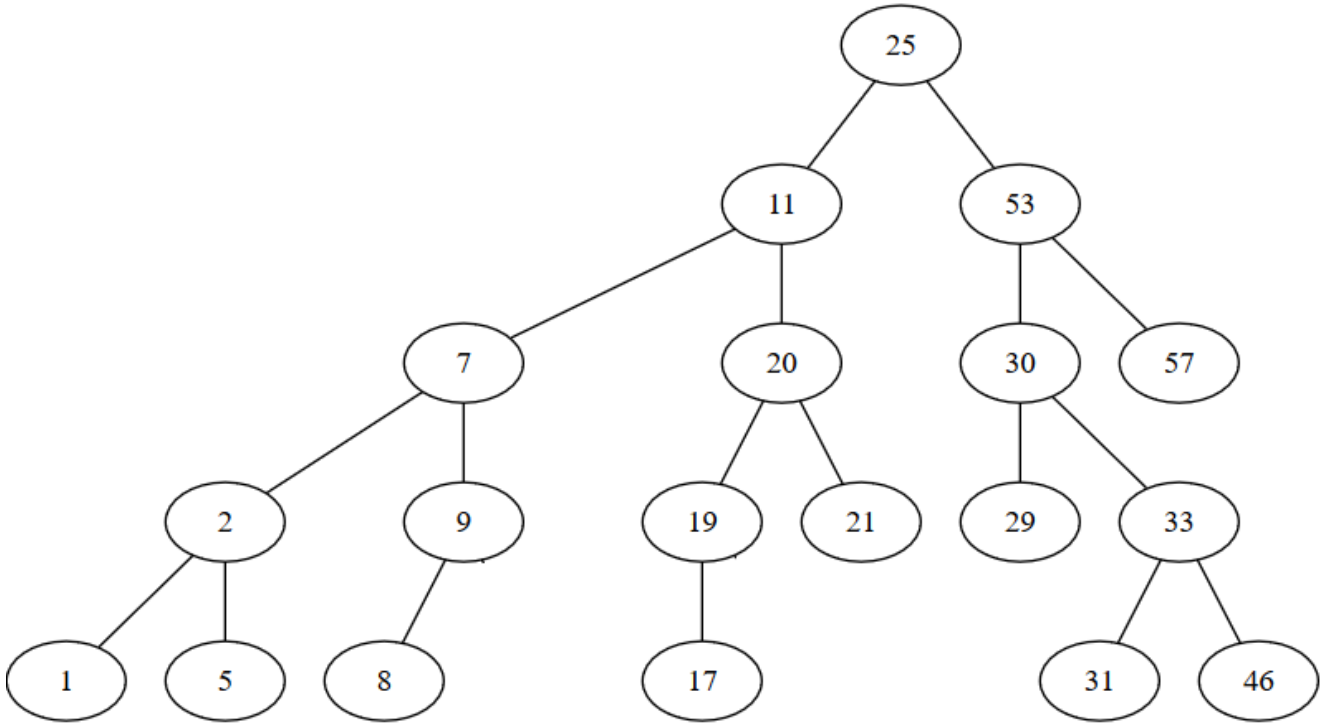


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XLII

- No rotation is needed
- Remove 50

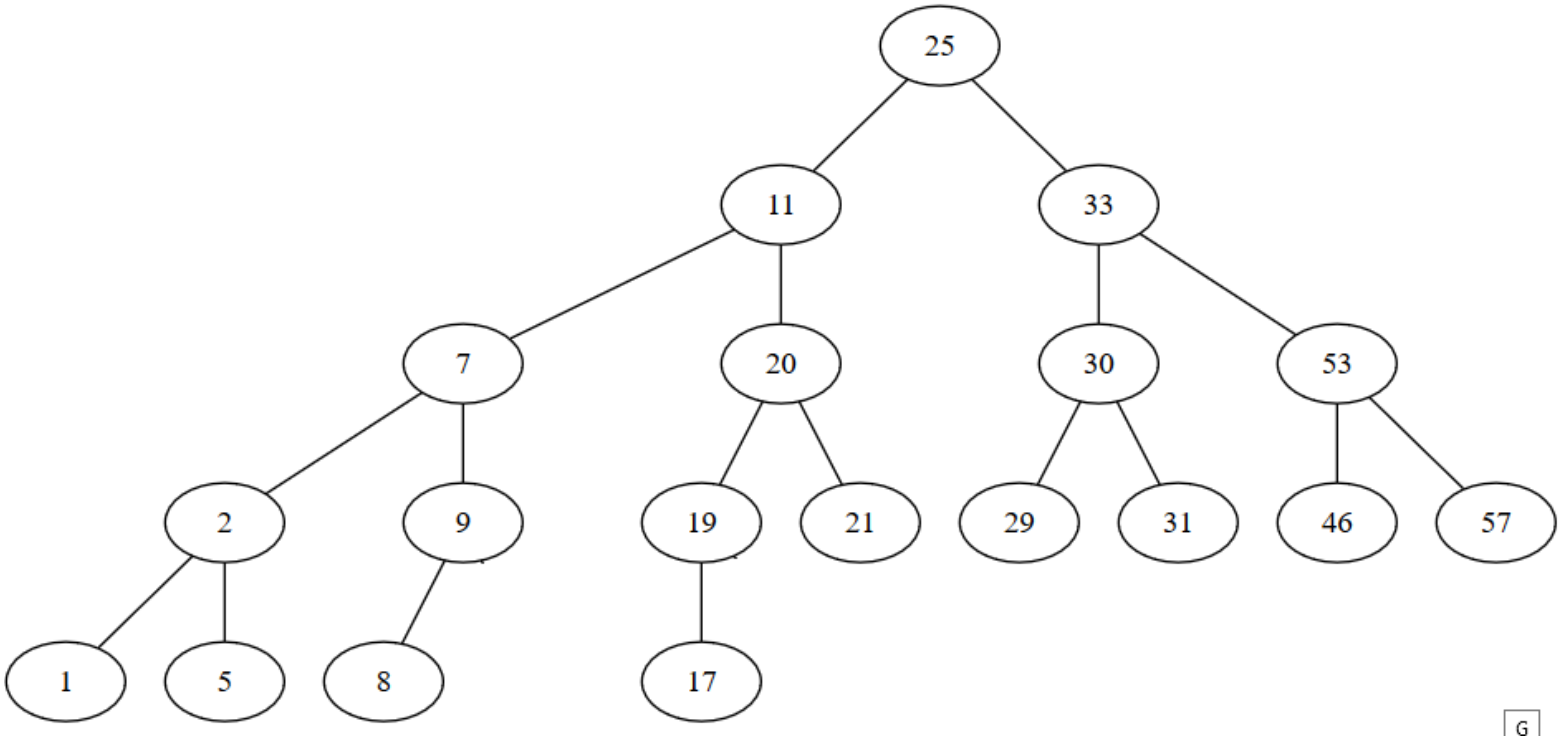
AVL rotations example XLIII



- Do we need a rotation?
- If yes, on which node and what type of rotation?

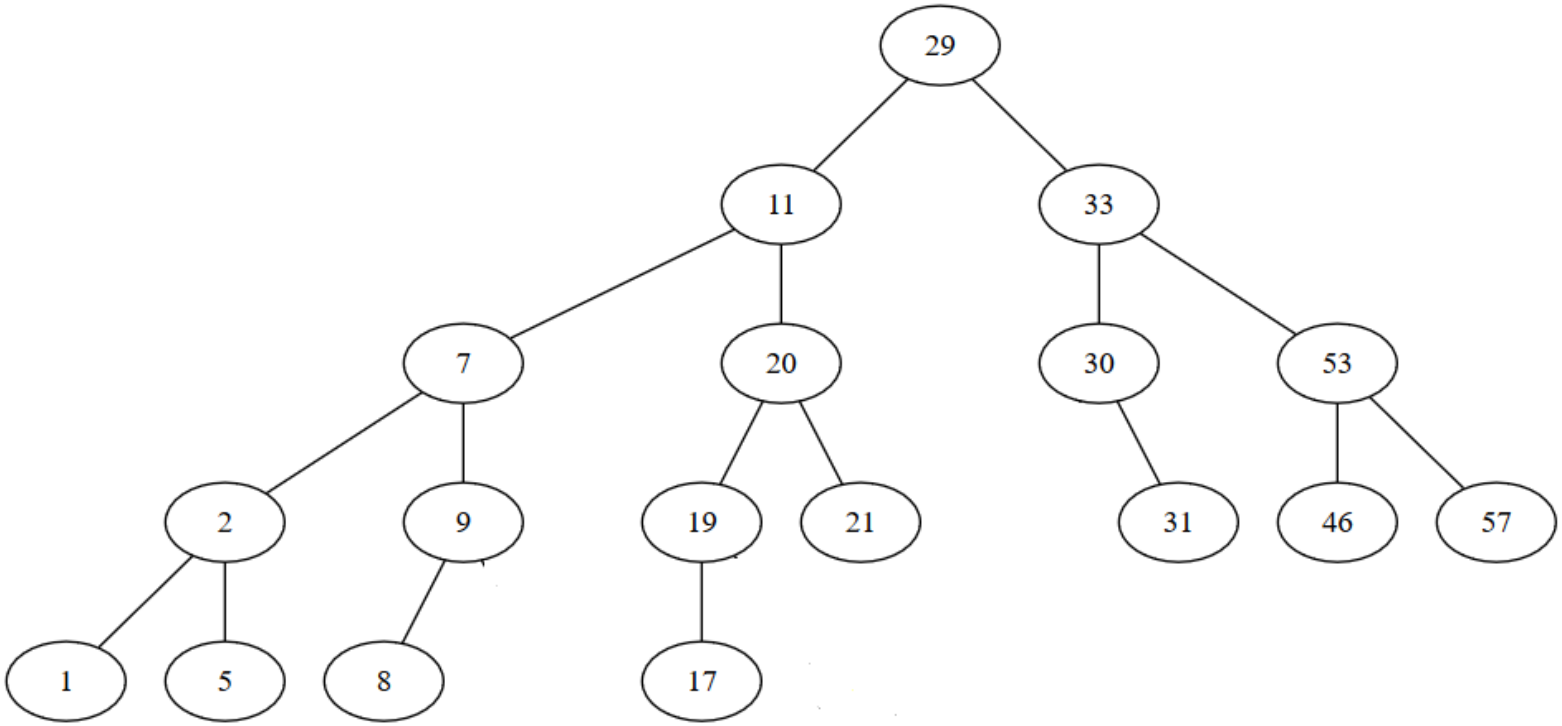
AVL rotations example XLIV

- Yes we need double right rotation on node 53
- After the rotation



- Remove 25

AVL rotations example XLV

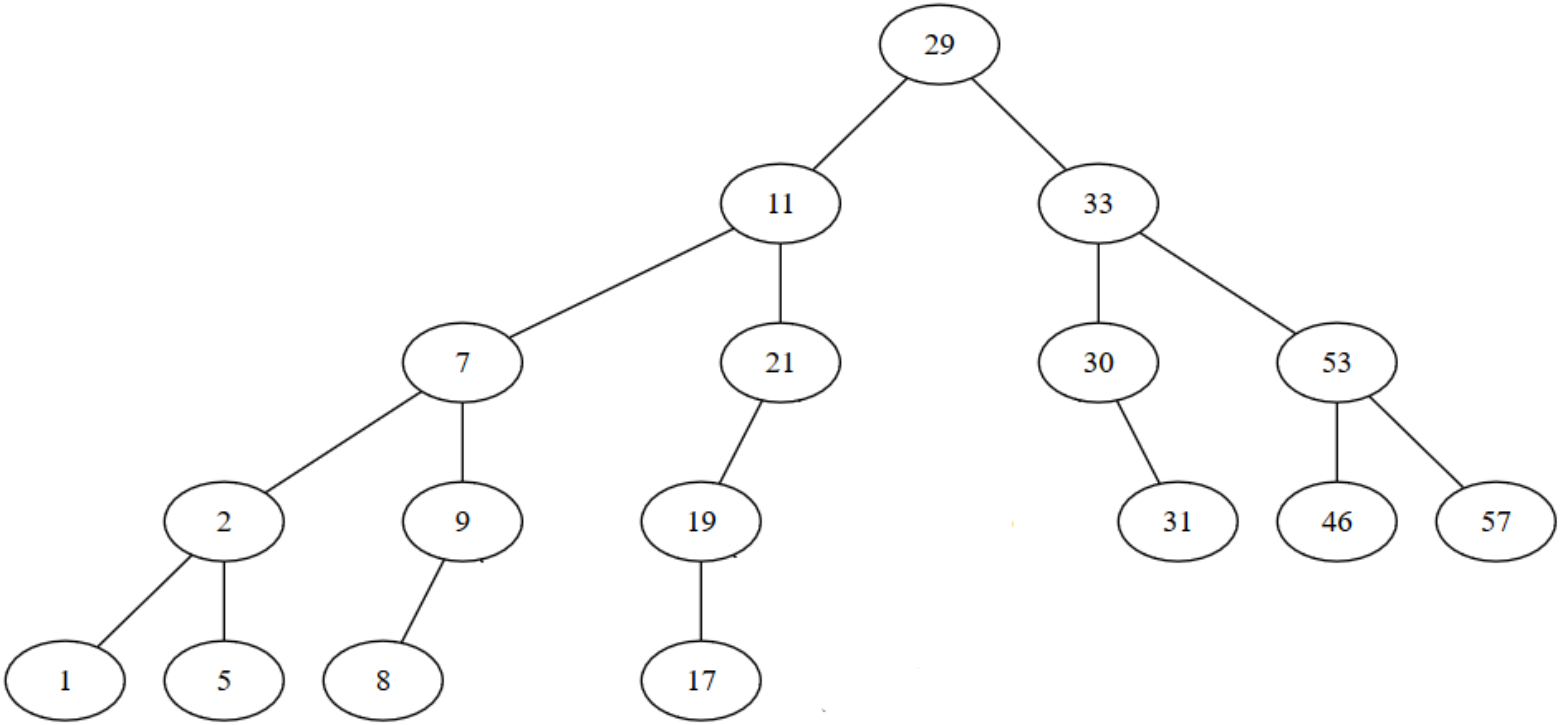


- Do we need a rotation?
- If yes, on which node and what type of rotation?

AVL rotations example XLVI

- No rotation is needed
- Remove 20

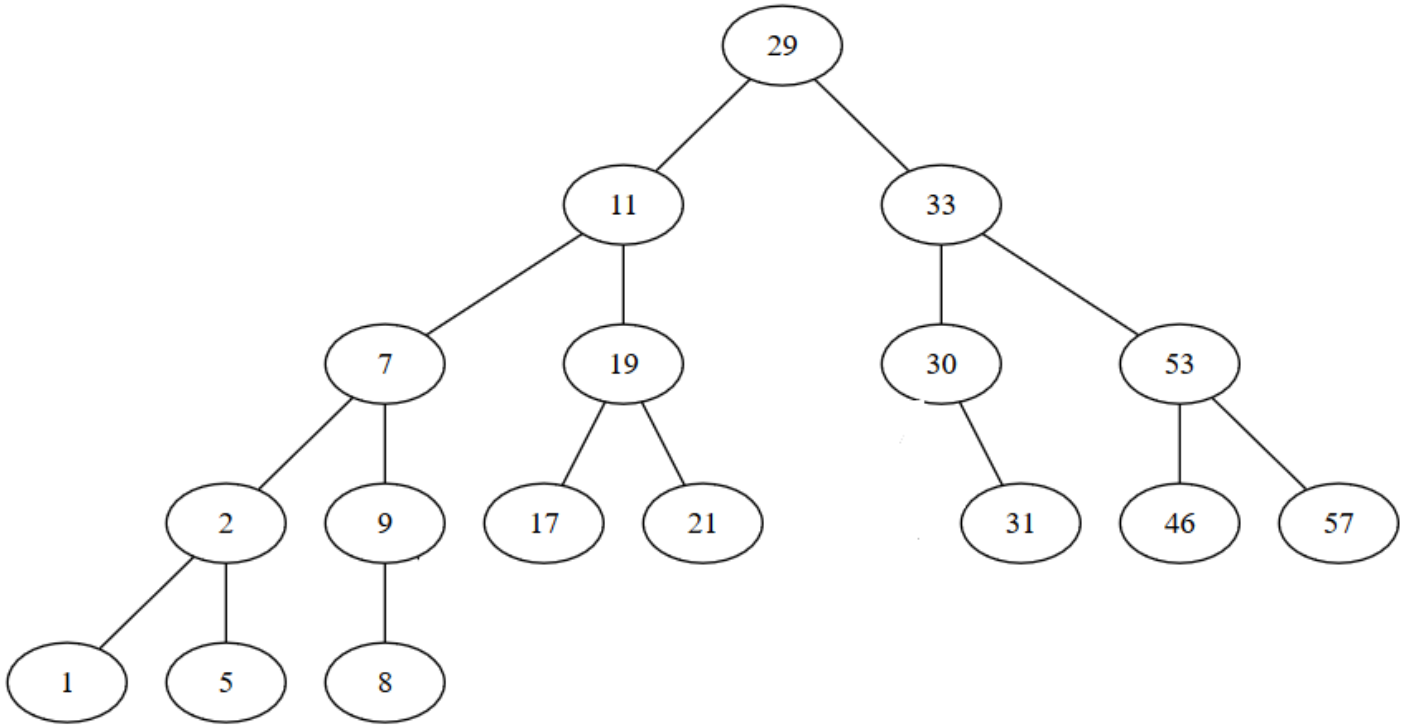
AVL rotations example XLVII



- Do we need a rotation?
- If yes, on which node and what type of rotation?

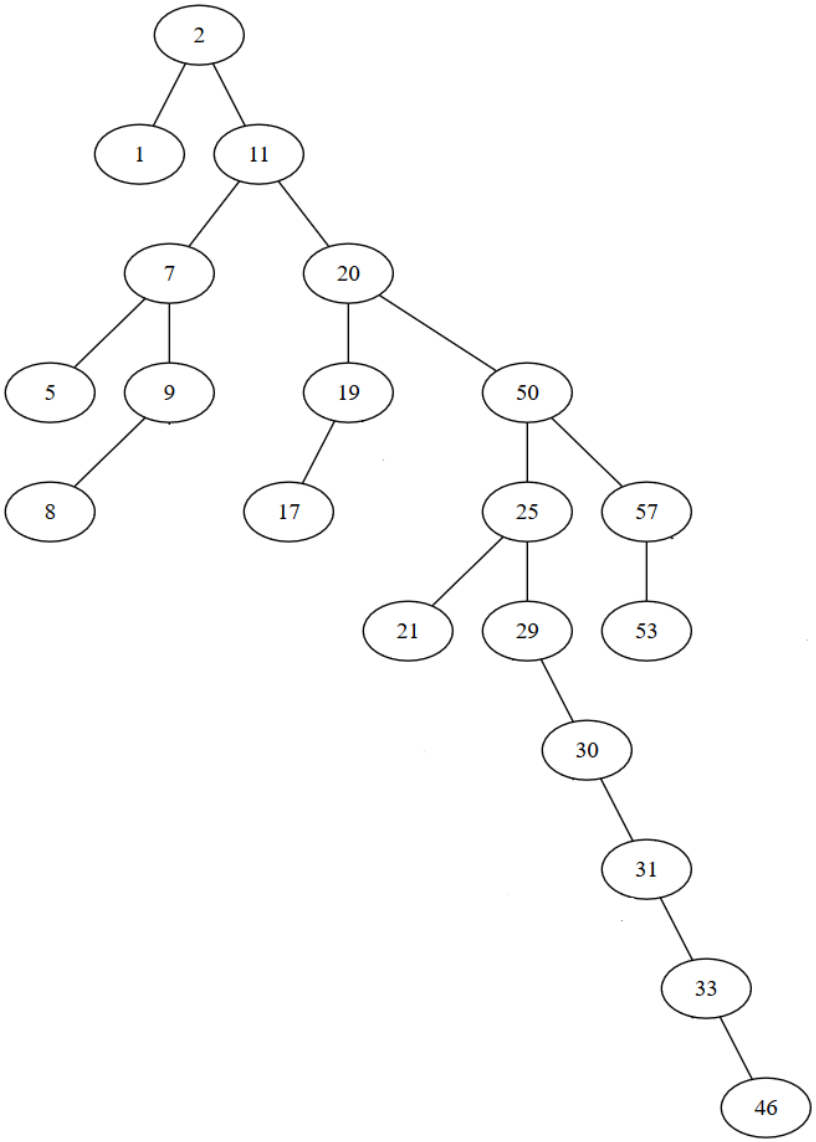
AVL rotations example XLVIII

- Yes, we need a single right rotation on node 21
- After the rotation



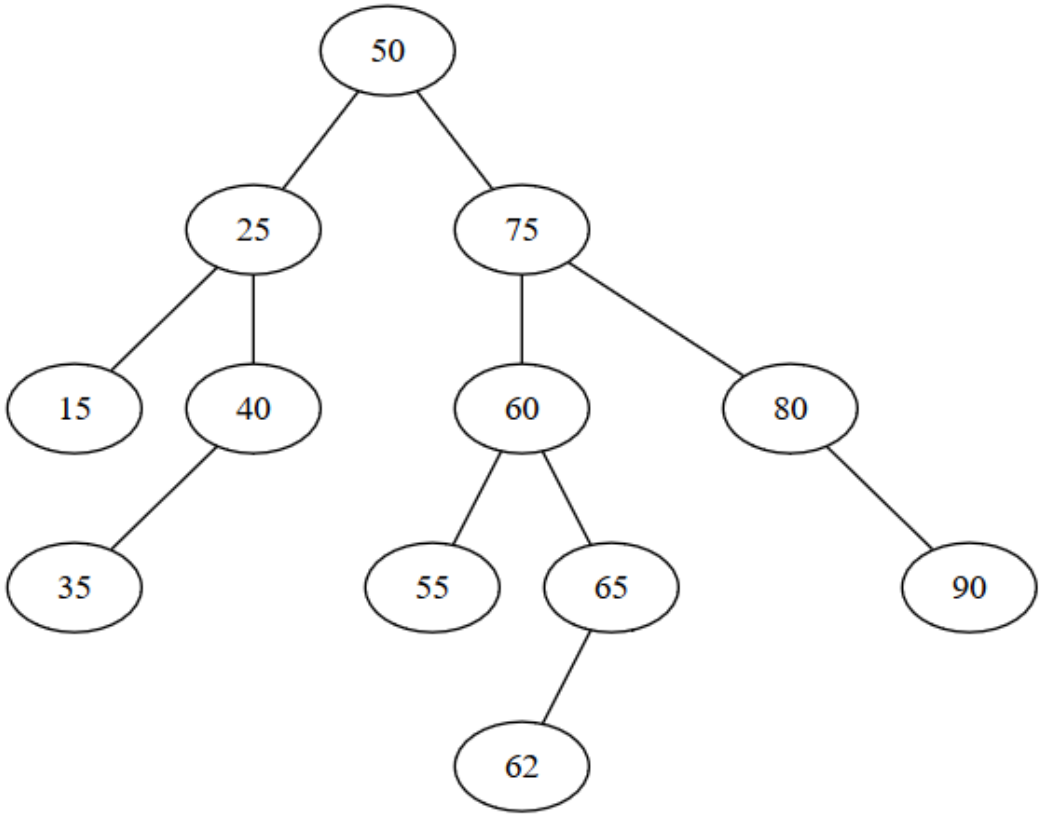
Comparison to BST

- If, instead of using an AVL tree, we used a binary search tree, after the insertions the tree would have been:


2
1
11
7
20
5
9
19
50
8
17
25
57
21
29
53
30
31
33
46

Rotations for remove

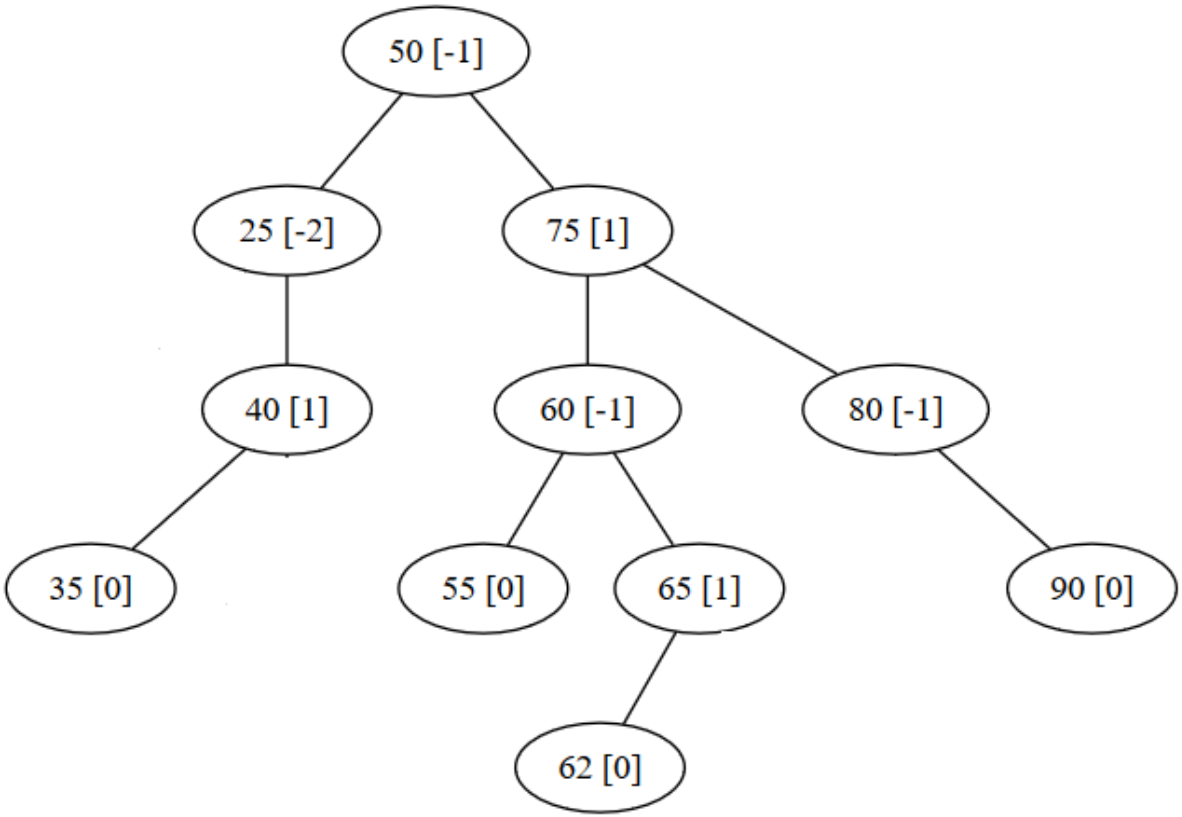
- When we remove a node, we might need more than 1 rotation:



- Remove value 15

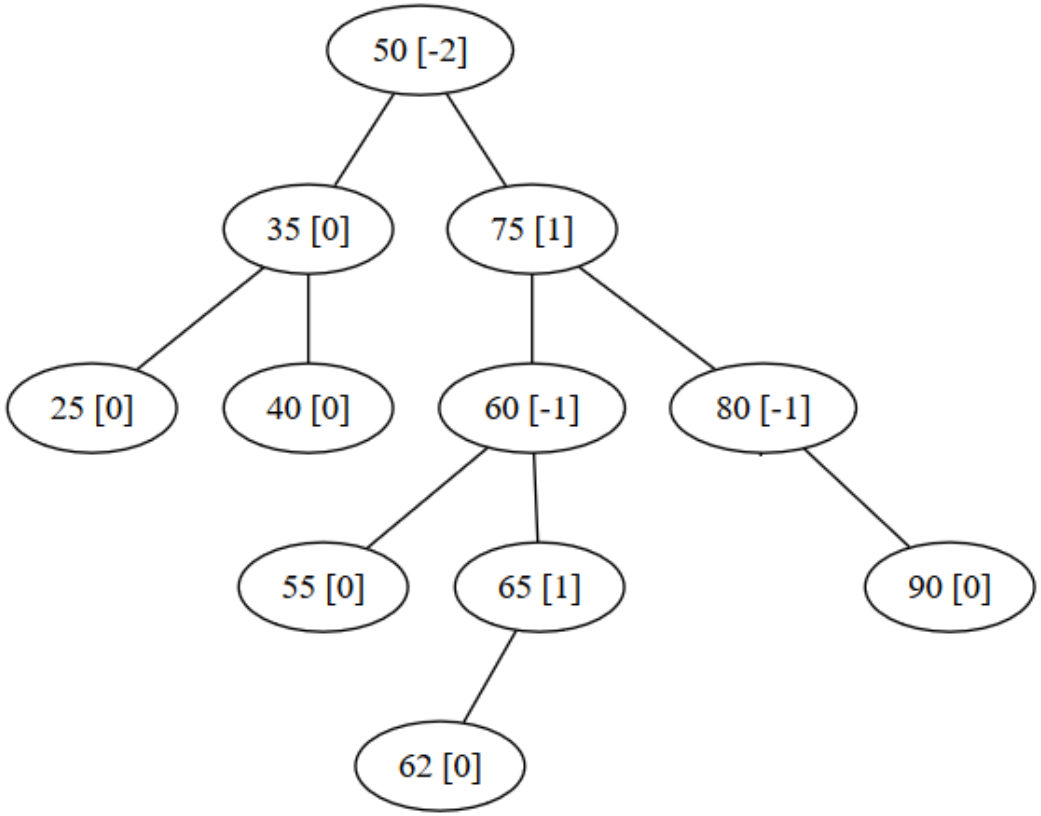
Rotations for remove

- After remove:



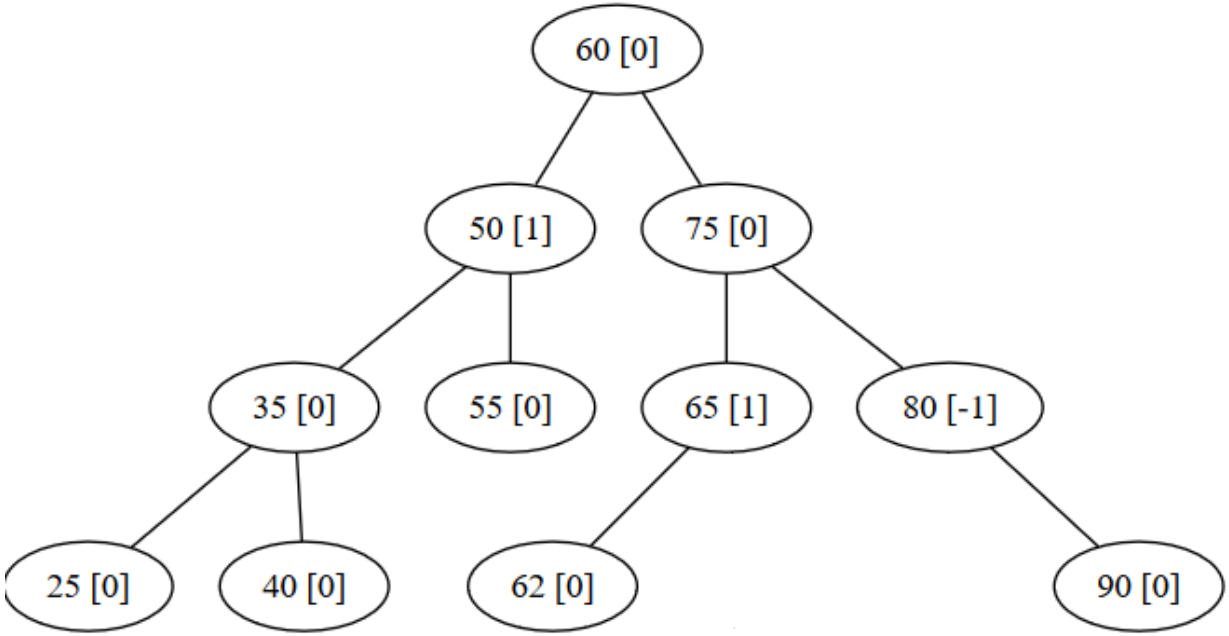
Rotations for remove

- After the rotation



Rotations for remove

- After the second rotation



AVL Trees - representation

- What structures do we need for an AVL Tree?

AVL Trees - representation

- What structures do we need for an AVL Tree?

AVLNode:
info: TComp *//information from the node*
left: ↑ AVLNode *//address of left child*
right: ↑ AVLNode *//address of right child*
h: Integer *//height of the node*

AVLTree:
root: ↑ AVLNode *//root of the tree*

AVL Tree - implementation

- We will implement the *insert* operation for the AVL Tree.
- We need to implement some operations to make the implementation of *insert* simpler:
  - A subalgorithm that (re)computes the height of a node
  - A subalgorithm that computes the balance factor of a node
  - Four subalgorithms for the four rotation types (we will implement only one)
- And we will assume that we have a function, *createNode* that creates and returns a node containing a given information (left and right are NIL, height is 0).

AVL Tree - height of a node

subalgorithm recomputeHeight(node) **is:**
//pre: node is an ↑ AVLNode. All descendants of node have their height (h) set
//to the correct value
//post: if node ≠ NIL, h of node is set

AVL Tree - height of a node

```
subalgorithm recomputeHeight(node) is:
//pre: node is an ↑ AVLNode. All descendants of node have their height (h) set
//to the correct value
//post: if node ≠ NIL, h of node is set
    if node ≠ NIL then
        if [node].left = NIL and [node].right = NIL then
            [node].h ← 0
        else if [node].left = NIL then
            [node].h ← [[node].right].h + 1
        else if [node].right = NIL then
            [node].h ← [[node].left].h + 1
        else
            [node].h ← max ([[node].left].h, [[node].right].h) + 1
        end-if
    end-if
end-subalgorithm
```

- Complexity: $\Theta(1)$

AVL Tree - balance factor of a node

function balanceFactor(node) **is:**
//pre: node is an ↑ AVLNode. All descendants of node have their height (h) set
//to the correct value
//post: returns the balance factor of the node

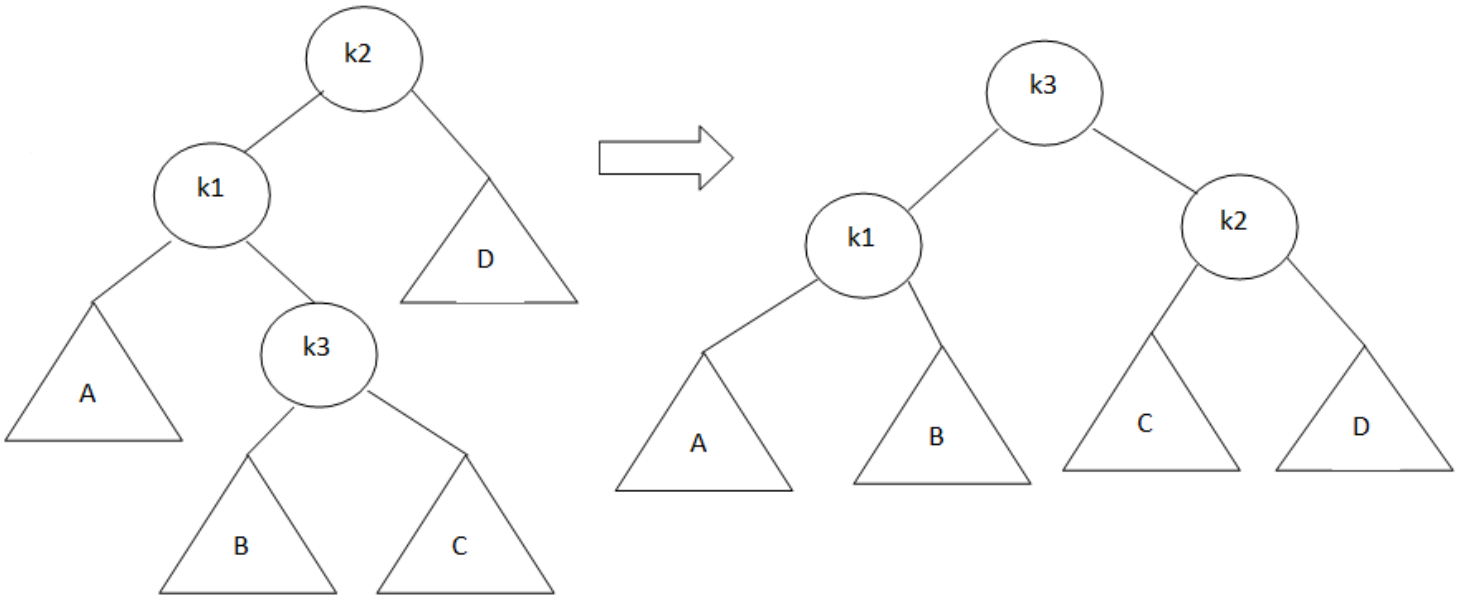
AVL Tree - balance factor of a node

```
function balanceFactor(node) is:
//pre: node is an ↑ AVLNode. All descendants of node have their height (h) set
//to the correct value
//post: returns the balance factor of the node
    if [node].left = NIL and [node].right = NIL then
        balanceFactor ← 0
    else if [node].left = NIL then
        balanceFactor ← -1 - [[node].right].h //height of empty tree is -1
    else if [node].right = NIL then
        balanceFactor ← [[node].left].h + 1
    else
        balanceFactor ← [[node].left].h - [[node].right].h
    end-if
end-subalgorithm
```

- Complexity: $\Theta(1)$

AVL Tree - rotations

- Out of the four rotations, we will only implement one, double right rotation (DRR).
- The other three rotations can be implemented similarly (RLR, SRR, SLR).



AVL Tree - DRR

function DRR(node) **is:** *//pre: node is an ↑ AVLNode on which we perform the double right rotation*
//post: DRR returns the new root after the rotation
    k2 ← node
    k1 ← [node].left
    k3 ← [k1].right
    k3left ← [k3].left
    k3right ← [k3].right

AVL Tree - DRR

function DRR(node) **is:** *//pre: node is an ↑ AVLNode on which we perform the double right rotation*
//post: DRR returns the new root after the rotation
    k2 ← node
    k1 ← [node].left
    k3 ← [k1].right
    k3left ← [k3].left
    k3right ← [k3].right
    //reset the links
    newRoot ← k3
    [newRoot].left ← k1
    [newRoot].right ← k2
    [k1].right ← k3left
    [k2].left ← k3right
//continued on the next slide

AVL Tree - DRR

```
//recompute the heights of the modified nodes
   recomputeHeight(k1)
   recomputeHeight(k2)
   recomputeHeight(newRoot)
   DRR ← newRoot
end-function
```

- Complexity: $\Theta(1)$

AVL Tree - insert

function insertRec(node, elem) **is**
//pre: node is a ↑ AVLNode, elem is the value we insert in the (sub)tree that
//has node as root
//post: insertRec returns the new root of the (sub)tree after the insertion
  if node = NIL **then**
    insertRec ← createNode(elem)
  else ifelem ≤ [node].info **then**
     [node].left ← insertRec([node].left, elem)
  else
    [node].right ← insertRec([node].right, elem)
  end-if
//continued on the next slide...

AVL Tree - insert

recomputeHeight(node)
balance $\leftarrow$ getBalanceFactor(node)
if balance $=$ -2 **then**

AVL Tree - insert

```
recomputeHeight(node)
balance ← getBalanceFactor(node)
if balance = -2 then
//right subtree has larger height, we will need a rotation to the LEFT
    rightBalance ← getBalanceFactor([node].right)
    if rightBalance < 0 then
```

AVL Tree - insert

```
recomputeHeight(node)
balance ← getBalanceFactor(node)
if balance = -2 then
//right subtree has larger height, we will need a rotation to the LEFT
    rightBalance ← getBalanceFactor([node].right)
    if rightBalance < 0 then
    //the right subtree of the right subtree has larger height, SRL
        node ← SRL(node)
    else
        node ← DRL(node)
    end-if
//continued on the next slide...
```

AVL Tree - insert

else if balance $= 2$ **then**
//left subtree has larger height, we will need a RIGHT rotation
leftBalance $\leftarrow$ getBalanceFactor([node].left)
if leftBalance > 0 **then**

AVL Tree - insert

```
    else if balance = 2 then
    //left subtree has larger height, we will need a RIGHT rotation
        leftBalance ← getBalanceFactor([node].left)
        if leftBalance > 0 then
        //the left subtree of the left subtree has larger height, SRR
            node ← SRR(node)
        else
            node ← DRR(node)
        end-if
    end-if
    insertRec ← node
end-function
```

AVL Tree - insert

- Complexity of the *insertRec* algorithm: $O(log_2 n)$
- Since *insertRec* receives as parameter a pointer to a node, we need a wrapper function to do the first call on the root

subalgorithm insert(tree, elem) **is**
//pre: tree is an AVL Tree, elem is the element to be inserted
//post: elem was inserted to tree
  tree.root $\leftarrow$ insertRec(tree.root, elem)
end-subalgorithm

- remove subalgorithm can be implemented similarly (start from the remove from BST and add the rotation part).

Evaluating an arithmetic expression

- We want to write an algorithm that can compute the result of an arithmetic expression:
- For example:
  - 2+3*4 = 14
  - ((2+4)*7)+3*(9-5) = 54
  - ((((3+1)*3)/((9-5)+2))-((3*(7-4)) + 6)) = -13

- An arithmetic expression is composed of *operators* (+, -, * or /), parentheses and *operands* (the numbers we are working with). For simplicity we are going to use single digits as operands and we suppose that the expression is correct.

Infix and postfix notations

- The arithmetic expressions presented on the previous slide are in the so-called *infix* notation. This means that the *operators* are between the two operands that they refer to. Humans usually use this notation, but for a computer algorithm it is complicated to compute the result of an expression in an infix notation.

- Computers can work a lot easier with the *postfix* notation, where the operator comes after the operands.

Infix and postfix notations

- Examples of expressions in infix notation and the corresponding postfix notations:

Infix notation	**Postfix notation**
1+2	12+
1+2-3	12+3-
4*3+6	43*6+
4*(3+6)	436+*
(5+6)*(4-1)	56+41-*
1+2*(3-4/(5+6))	123456+/-*+

- The order of the operands is the same for both the infix and the postfix notations, only the order of the operators changes
- The operators have to be ordered taking into consideration operator precedence and the parentheses

Infix and postfix notations

- So, evaluating an arithmetic expression is divided into two subproblems:
  - Transform the infix notation into a postfix notation
  - Evaluate the postfix notation
- Both subproblems are solved using stacks and queues.

Infix to postfix transformation - The main idea

- Use an auxiliary stack for the operators and parentheses and a queue for the result.
- Start parsing the expression.
- If an operand is found, push it to the queue
- If an open parenthesis is found, it is pushed to the stack.
- If a closed parenthesis is found, pop elements from the stack and push them to the queue until an open parenthesis is found (but do not push parentheses to the queue).

Infix to postfix transformation - The main idea

- If an operator (opCurrent) is found:
  - If the stack is empty, push the operator to the stack
  - While the top of the stack contains an operator with a higher or equal precedence than the current operator, pop and push to the queue the operator from the stack. Push opCurrent to the stack when the stack becomes empty, its top is a parenthesis or an operator with lower precedence.
  - If the top of the stack is open parenthesis or operator with lower precedence, push opCurrent to the stack.
- When the expression is completely parsed, pop everything from the stack and push to the queue.

Infix to postfix transformation - Example

- Let's follow the transformation of 1+2*(3-4/(5+6))+7

Infix to postfix transformation - Example

- Let's follow the transformation of 1+2*(3-4/(5+6))+7

Input	Operation	Stack	Queue
1	Push to Queue		1
+	Push to stack	+	1
2	Push to Queue	+	12
*	Check (no Pop) and Push	+*	12
(	Push to stack	+*(	12
3	Push to Queue	+*(	123
-	Check (no Pop) and Push	+*(-	123
4	Push to Queue	+*(-	1234
/	Check (no Pop) and Push	+*(-/	1234
(	Push to stack	+*(-/(	1234
5	Push to Queue	+*(-/(	12345
+	Check (no Pop) and Push	+*(-/(+	12345
6	Push to Queue	+*(-/(+	123456
)	Pop and push to Queue till (	+*(-/	123456+
)	Pop and push to Queue till (	+*	123456+/-
+	Check, Pop twice and Push	+	123456+/-*+
7	Push to Queue	+	123456+/-*+7
over	Pop everything and push to Queue		123456+/-*+7+

Infix to postfix transformation - Implementation

```
function infixToPostfix(expr) is:
    init(st)
    init(q)
    for elem in expr execute
        if @elem is an operand then
            push(q, elem)
        else if @ elem is open parenthesis then
            push(st, elem)
        else if @elem is a closed parenthesis then
            while @ top(st) is not an open parenthesis execute
                op ← pop(st)
                push(q, op)
            end-while
            pop(st) //get rid of open parenthesis
        else //we have operand
//continued on the next slide
```

Infix to postfix transformation - Implementation

```
        while not isEmpty(st) and @ top(st) not open parenthesis and @
top(st) has >= precedence than elem execute
            op ← pop(st)
            push(q, op)
        end-while
        push(st, elem)
    end-if
  end-for
  while not isEmpty(st) execute
    op ← pop(st)
    push(q, op)
  end-while
  infixtoPostfix ← q
end-function
```

- Complexity: $\Theta(n)$ - where n is the length of the sequence

Evaluation of expression in postfix notation

- Once we have the postfix notation we can compute the value of the expression using a stack
- The main idea of the algorithm:
  - Use an auxiliary stack
  - Start parsing the expression
  - If an operand if found, it is pushed to the stack
  - If an operator is found, two values are popped from the stack, the operation is performed and the result is pushed to the stack
  - When the expression is parsed, the stack contains the result

Evaluation of postfix notation - Example

- Let's follow the evaluation of 123456+/-*+7+

Evaluation of postfix notation - Example

- Let's follow the evaluation of 123456+/-*+7+

Pop from the queue	Operation	Stack
1	Push	1
2	Push	1 2
3	Push	1 2 3
4	Push	1 2 3 4
5	Push	1 2 3 4 5
6	Push	1 2 3 4 5 6
+	Pop, add, Push	1 2 3 4 11
/	Pop, divide, Push	1 2 3 0
-	Pop, subtract, Push	1 2 3
*	Pop, multiply, Push	1 6
+	Pop, add, Push	7
7	Push	7 7
+	Pop, add, Push	14

Evaluation of postfix notation - Implementation

```
function evaluatePostfix(q) is:
    init(st)
    while not isEmpty(q) execute
        elem ← pop(q)
        if @ elem is an operand then
            push(st, elem)
        else
            op1 ← pop(st)
            op2 ← pop(st)
            @ compute the result of op2 elem op1 in variable result
            push(st, result)
        end-if
    end-while
    result ← pop(st)
    evaluatePostfix ← result
end-function
```

- Complexity: $\Theta(n)$ - where n is the length of the expression

Evaluation of an arithmetic expression

- Combining the two functions we can compute the result of an arithmetic expression.
- How can we evaluate directly the expression in infix notation with one single function? *Hint: use two stacks.*
- How can we add exponentiation as a fifth operation?