DATA STRUCTURES AND ALGORITHMS

LECTURE 8

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In Lecture 7...

- Binary Heap
- Binomial Heap

Today

1. Written test
2. Hash tables

Written test - General Info

- Will be at the first half of seminar 5
- Will last 50 minutes
- **Everybody has to participate at the test with his/her own group!**

Written test - Subjects I

- Each subject will be of the form: Container(ADT) + Representation (a data structure) + Operation
- Containers:
  - Bag
  - Set
  - Map
  - MultiMap
  - List
  - sorted version of the above containers
  - (Sparse)Matrix

Written test - Subjects II

- Data structures
  - Singly Linked List with Dynamic Allocation
  - Doubly Linked List with Dynamic Allocation
  - Singly Linked List on an Array
  - Doubly Linked List on an Array

Written test - Subjects III

- Operation
  - Add
  - Remove
  - Search
  - For Sparse Matrix: Modify value (from 0 to non-zero or from non-zero to 0), search for element from position (i,j)

- Example of a subject: **ADT Set - represented on a doubly linked list on an array - operation: add**.

Written test - Grading

- **1p - Start**
- **0.5p - Specification of the operation** - header, preconditions, postconditions
- **0.5p - Short description of the container**
- **1.25p - representation**
  - 1p - structure(s) needed for the container
  - 0.25p - structure for the iterator for the container

Written test - Grading II

- **4.5p - Implementation of the operation in pseudocode**
  - If you have a data structure with dynamic allocation, you can use the *allocate* and *deallocate/free* operations. If you call any other function(s) in the implementation, you have to implement them.
  - If you have a data structure on an array, you do not need to write code for *resize*-ing the data structure, but you need to show where the resize part would be:

```
...
if s.firstEmpty = -1 then
    @resize
end-if
```

Written test - Grading III

- **1.25p - Complexity**
  - 0.25p - Best Case - with explanations
  - 0.5p - Worst Case - with computations
  - 0.5p - Average Case - with computations
- **1p - Style**
  - Is it general (uses TElem, TComp, a generic Relation)?
  - Is it efficient?
  - etc.

Example

Direct-address tables I

- Consider the following problem:
  - We have data where every element has a key (a natural number).
  - The universe of keys (the possible values for the keys) is relatively small, $U = \{0, 1, 2, \ldots, m-1\}$
  - No two elements have the same key
  - We have to support the basic dictionary operations: INSERT, DELETE and SEARCH

Direct-address tables II

- Example 1: Store data about different bus lines for a city's public transportation service
  - We can consider the bus number as a key, and the data to be stored as a value (satellite data)
  - The bus numbers belong to a relatively small interval - in Cluj-Napoca it is around 100
  - Bus numbers are unique
- Example 2: Store data about students based on their registration numbers (a number from the 1 - 9999 interval, but there may be unused numbers - students that left the university)

Direct-address tables III

- Solution:
  - Use an array T with m positions (remember, the keys belong to the $[0, m-1]$ interval)
  - Data about element with key k, will be stored in the *T[k]* slot
  - Slots not corresponding to existing elements will contain the value NIL (or some other special value to show that they are empty)

Operations for a direct-address table

function search(T, k) **is**:
//pre: T is an array (the direct-address table), k is a key
    search $\leftarrow$ T[k]
end-function

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//pre: T is an array (the direct-address table), x is an element
  T[key(x)] $\leftarrow$ x *//key(x) returns the key of an element*
end-subalgorithm

Operations for a direct-address table

function search(T, k) **is**:
//pre: T is an array (the direct-address table), k is a key
search ← T[k]
end-function

subalgorithm insert(T, x) **is**:
//pre: T is an array (the direct-address table), x is an element
T[key(x)] ← x *//key(x) returns the key of an element*
end-subalgorithm

subalgorithm delete(T, x) **is**:
//pre: T is an array (the direct-address table), x is an element
T[key(x)] ← NIL
end-subalgorithm

Direct-address table - Advantages and disadvantages

- Advantages of direct address-tables:
  - They are simple
  - They are efficient - all operations run in $\Theta(1)$ time.
- Disadvantages of direct address-tables - restrictions:
  - The keys have to be natural numbers
  - The keys have to come from a small universe (interval)
  - The number of actual keys can be a lot less than the cardinal of the universe (storage space is wasted)

Think about it

- Assume that we have a direct address T of length m. How can we find the maximum element of the direct-address table? What is the complexity of the operation?
- How does the operation for finding the maximum change if we have a hash table, instead of a direct-address table (consider collision resolution by separate chaining, coalesced chaining and open addressing)?

Hash tables

- Hash tables are generalizations of direct-address tables and they represent a *time-space trade-off*.
- Searching for an element still takes $\Theta(1)$ time, but as *average case complexity* (worst case complexity is higher)

Hash tables - main idea I

- We will still have a table T of size m (but now m is not the number of possible keys, $|U|$) - *hash table*
- Use a function h that will map a key k to a slot in the table T - *hash function*

$$h : U \to \{0, 1, ..., m-1\}$$

- Remarks:
  - In case of direct-address tables, an element with key k is stored in $T[k]$.
  - In case of hash tables, an element with key k is stored in $T[h(k)]$.

Hash tables - main idea II

- The point of the hash function is to reduce the range of array indexes that need to be handled $=>$ instead of $|U|$ values, we only need to handle m values.
- Consequence:
  - two keys may hash to the same slot $=>$ **a collision**
  - we need techniques for resolving the conflict created by collisions

A good hash function I

- A good hash function:
  - can minimize the number of collisions (but cannot eliminate all collisions)
  - is deterministic
  - can be computed in $\Theta(1)$ time

A good hash function II

- satisfies (approximately) the assumption of simple uniform hashing: **each key is equally likely to hash to any of the m slots, independently of where any other key has hashed to**

$$P(h(k) = j) = \frac{1}{m} \ \forall j = 0, ..., m-1 \ \ \forall k \in U$$

- Examples of bad hash functions:
  - $h(k) =$ constant number
  - $h(k) =$ random number
  - $h(k) = k \ mod \ 10$ - when $m > 10$
  - etc.

A good hash function III

- In practice we use heuristic techniques to create hash functions that perform well.
- Most hash functions assume that the keys are natural numbers. If this is not true, they have to be interpreted as natural number. In what follows, we assume that the keys are natural numbers.
- There are different methods of defining a hash function:
  - The division method
  - The multiplication method
  - Universal hashing

The division method

The division method

$h(k) = k \mod m$

For example:

$$m = 13$$
$$k = 63 => h(k) = 11$$
$$k = 52 => h(k) = 0$$
$$k = 131 => h(k) = 1$$

- Requires only a division so it is quite fast
- Experiments show that good values for m are primes not too close to exact powers of 2

The multiplication method I

The multiplication method

$h(k) = floor(m * frac(k * A))$ where

m - the hash table size

A - constant in the range $0 < A < 1$

$frac(k * A)$ - fractional part of $k * A$

For example

m = 13 A = 0.6180339887

k=63 => h(k) = floor(13 * frac(63 * A)) = floor(12.16984) = 12

k=52 => h(k) = floor(13 * frac(52 * A)) = floor(1.790976) = 1

k=129=> h(k)= floor(13 * frac(129 * A)) = floor(9.442999) = 9

The multiplication method II

- Advantage: the value of m is not critical, typically $m = 2^p$ for some integer p
- Some values for A work better than others. Knuth suggests $\frac{\sqrt{5}-1}{2} = 0.6180339887$

Universal hashing I

- If we know the exact hash function used by a hash table, we can always generate a set of keys that will hash to the same position (collision). This reduces the performance of the table.
- For example:

$m = 13$
$h(k) = k \mod m$
k = 11, 24, 37, 50, 63, 76, etc.

Universal hashing II

- Instead of having one hash function, we have a collection $\mathcal{H}$ of hash functions that map a given universe U of keys into the range $\{0, 1, \ldots, m-1\}$
- Such a collection is said to be **universal** if for each pair of distinct keys $x, y \in U$ the number of hash functions from $\mathcal{H}$ for which $h(x) = h(y)$ is precisely $\frac{|\mathcal{H}|}{m}$
- In other words, with a hash function randomly chosen from $\mathcal{H}$ the chance of collision between x and y, where $x \neq y$, is exactly $\frac{1}{m}$

Universal hashing III

Example 1

Fix a prime number $p >$ *the maximum possible value for a key from* U.
For every $a \in \{1, \ldots, p-1\}$ and $b \in \{0, \ldots, p-1\}$ we can define a hash function $h_{a,b}(k) = ((a * k + b) \ mod \ p) \ mod \ m$.

- For example:
  - $h_{3,7}(k) = ((3 * k + 7) \ mod \ p) \ mod \ m$
  - $h_{4,1}(k) = ((4 * k + 1) \ mod \ p) \ mod \ m$
  - $h_{8,0}(k) = ((8 * k) \ mod \ p) \ mod \ m$
- There are $p * (p-1)$ possible hash functions that can be chosen.

Universal hashing IV

Example 2

If the key k is an array $< k_1, k_2, \ldots, k_r >$ such that $k_i < m$ (or it can be transformed into such an array).
Let $< x_1, x_2, \ldots, x_r >$ be a fixed sequence of random numbers, such that $x_i \in \{0, \ldots, m-1\}$.
$h(k) = \sum_{i=1}^{r} k_i * x_i \mod m$

Universal hashing V

Example 3

Suppose the keys are $u - bits$ long and $m = 2^b$.
Pick a random $b - by - u$ matrix (called h) with 0 and 1 values only.
Pick $h(k) = h * k$ where in the multiplication we do addition mod 2.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Using keys that are not natural numbers I

- The previously presented hash functions assume that keys are natural numbers.
- If this is not true there are two options:
  - Define special hash functions that work with your keys (for example, for real number from the [0,1) interval $h(k) = [k * m]$ can be used)
  - Use a function that transforms the key to a natural number (and use any of the above-mentioned hash functions) - *hashCode*

Using keys that are not natural numbers II

- If the key is a string s:
  - we can consider the ASCII codes for every letter
  - we can use 1 for *a*, 2 for *b*, etc.
- Possible implementations for *hashCode*
  - $s[0] + s[1] + ... + s[n-1]$
    - Anagrams have the same sum *SAUCE* and *CAUSE*
    - *DATES* has the same sum (D = C + 1, T = U - 1)
    - Assuming maximum length of 10 for a word (and the second letter representation), *hashCode* values range from 1 (the word *a*) to 260 (*zzzzzzzzzz*). Considering a dictionary of about 50,000 words, we would have on average 192 word for a *hashCode* value.

Using keys that are not natural numbers III

- $s[0] * 26^{n-1} + s[1] * 26^{n-2} + ... + s[n-1]$ where
  - n - the length of the string
  - Generates a much larger interval of *hashCode* values.
  - Instead of 26 (which was chosen since we have 26 letters) we can use a prime number as well (Java uses 31, for example).

Collisions

- When two keys, x and y, have the same value for the hash function $h(x) = h(y)$ we have a *collision*.
- A good hash function can reduce the number of collisions, but it cannot eliminate them at all:
  - Try fitting $m + 1$ keys into a table of size m
- There are different collision resolution methods:
  - Separate chaining
  - Coalesced chaining
  - Open addressing

The birthday paradox

- *How many randomly chosen people are needed in a room, to have a good probability - about 50% - of having two people with the same birthday?*
- It is obvious that if we have 367 people, there will be at least two with the same birthday (there are only 366 possibilities).

The birthday paradox

- *How many randomly chosen people are needed in a room, to have a good probability - about 50% - of having two people with the same birthday?*
- It is obvious that if we have 367 people, there will be at least two with the same birthday (there are only 366 possibilities).
- What might not be obvious, is that approximately 70 people are needed for a 99.9% probability
- 23 people are enough for a 50% probability

Separate chaining

- Collision resolution by chaining: each slot from the hash table T contains a linked list, with the elements that hash to that slot
- Dictionary operations become operations on the corresponding linked list:
  - $insert(T, x)$ - insert a new node to the beginning of the list $T[h(key[x])]$
  - $search(T, k)$ - search for an element with key k in the list $T[h(k)]$
  - $delete(T, x)$ - delete x from the list $T[h(key[x])]$

Hash table with separate chaining - representation

- A hash table with separate chaining would be represented in the following way (for simplicity, we will keep only the keys in the nodes).

Node:
key: TKey
next: ↑ Node

HashTable:
T: ↑Node[] *//an array of pointers to nodes*
m: Integer
h: TFunction *//the hash function*

Hash table with separate chaining - search

```
function search(ht, k) is:
//pre: ht is a HashTable, k is a TKey
//post: function returns True if k is in ht, False otherwise
    position ← ht.h(k)
    currentN ← ht.T[position]
    while currentN ≠ NIL and [currentN].key ≠ k execute
        currentN ← [currentN].next
    end-while
    if currentN ≠ NIL then
        search ← True
    else
        search ← False
    end-if
end-function
```

- Usually search returns the info associated with the key k

Analysis of hashing with chaining

- The average performance depends on how well the hash function h can distribute the keys to be stored among the m slots.
- **Simple Uniform Hashing** assumption: each element is equally likely to hash into any of the m slots, independently of where any other elements have hashed to.
- **load factor** α of the table T with m slots containing n elements
  - is n/m
  - represents the average number of elements stored in a chain
  - in case of separate chaining can be less than, equal to, or greater than 1.

Analysis of hashing with chaining - Insert

- The slot where the element is to be added can be:
  - empty - create a new node and add it to the slot
  - occupied - create a new node and add it to the beginning of the list
- In either case worst-case time complexity is: $\Theta(1)$
- If we have to check whether the element already exists in the table, the complexity of searching is added as well.

Analysis of hashing with chaining - Search I

- There are two cases
  - unsuccessful search
  - successful search
- We assume that
  - the hash value can be computed in constant time ($\Theta(1)$)
  - the time required to search an element with key k depends linearly on the length of the list $T[h(k)]$

Analysis of hashing with chaining - Search II

- **Theorem:** In a hash table in which collisions are resolved by separate chaining, an unsuccessful search takes time $\Theta(1+\alpha)$, on the average, under the assumption of simple uniform hashing.
- **Theorem:** In a hash table in which collisions are resolved by chaining, a successful search takes time $\Theta(1+\alpha)$, on the average, under the assumption of simple uniform hashing.
- Proof idea: $\Theta(1)$ is needed to compute the value of the hash function and α is the average time needed to search one of the m lists

Analysis of hashing with chaining - Search III

- If $n = O(m)$ (the number of hash table slots is proportional to the number of elements in the table)
  - $\alpha = n/m = O(m)/m = O(1)$
  - searching takes constant time on average
- Worst-case time complexity is $\Theta(n)$
  - When all the nodes are in a single linked-list and we are searching this list
  - In practice hash tables are pretty fast

Analysis of hashing with chaining - Delete

- If the lists are doubly-linked and we know the address of the node: $\Theta(1)$
- If the lists are singly-linked: proportional to the length of the list

- **All dictionary operations can be supported in $\Theta(1)$ time on average.**
- In theory we can keep any number of elements in a hash table with separate chaining, but the complexity is proportional to α. If α is too large $\Rightarrow$ resize and rehash.

Example of separate chaining

- Consider a hash table of size $m = 11$ that uses separate chaining for collision resolution and a hash function with the division method
- Insert into the table the letters from *A SEARCHING EXAMPLE* (space is ignored)
- For each letter, the *hashCode* is the index of the letter in the alphabet.

Letter	A	S	E	R	C	H	I	N	G	X	M	P	L
HashCode	1	19	5	18	3	8	9	14	7	24	13	16	12

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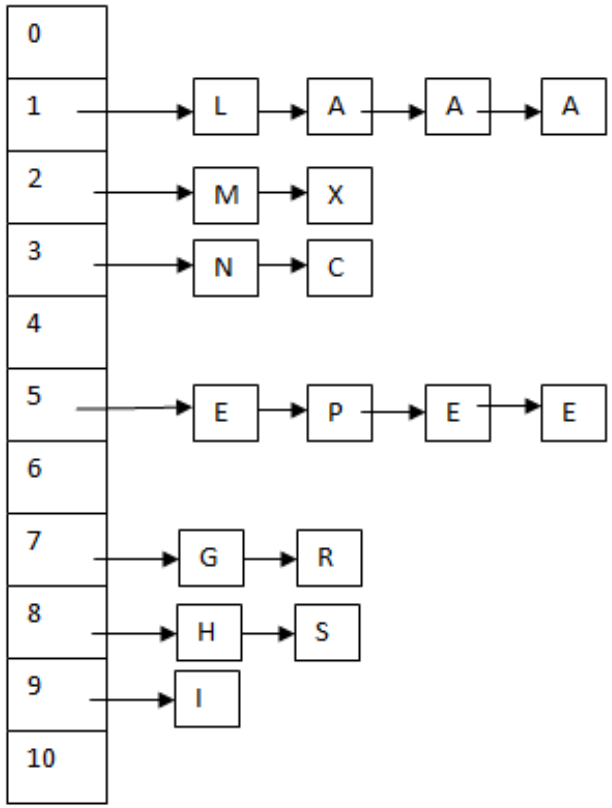
Letter	A	S	E	R	C	H	I	N	G	X	M	P	L
HashCode	1	19	5	18	3	8	9	14	7	24	13	16	12
h(Letter)	1	8	5	7	3	8	9	3	7	2	2	5	1

Example of separate chaining

- After the letters were inserted in an empty hash table:



- Load factor α: $17/11 = 1.54$ - the average length of a list