Set and Relation Part-2

Set theory Part-2
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Introduction
The word relation is used to indicate a relationship or association between two or more objects. When a relation indicates association between two elements when it is called a binary relation. In this chapter, we shall study binary relation, various types of relations and their properties, its matrix and graphical representation.

Relations
Let us consider two sets H and W where
H= $\{h_1, h_2, h_3\}$
W = $\{w_1, w_2, w_3\}$
then the cartesian products of H and W is
H x W = {(h1, w1), (h1, w2), (h1, w3), (h2, w1), (h2, w2), (h2, w3), (h3, w1), (h3, w2), (h3, w3)}
Consider any subset of H x W
Say R= $\{(h_1, w_2), (h_2, w_3), (h_3, w_1)\}$
Say every first element R is associated to second element by an association/relationship "husband of" then we can say that
h_1 is husband of w_2
h_2 is husband of w_3
h_3 is husband of w_1
It implies "h_1 is related to w_2" by a relation "husband of".
Let A and B are any two non- empty sets then a relation from set A to set B is defined as any subset of A x B. if an order pair (a, b) $\in$ R then we say that a R b and it is read as "a relates to b". similarly, if (a, b) $\notin$ R then we say that a- R b or a $\not R$ b and is read as "a does not relate to b".
If we consider $R_1 = \{(h_1, w_3), (h_2, w_2), (h_2, w_1)\}$
Clearly $R_1 \subset A \times B$. so, every possible subset of A x B represents a binary relation or simply a relation from set A to set B.
The statements
"R is a relation from A to B"
"R is a relation on A into B"
"R is a relation of A into B", have the same meaning.
Please do remember a R b can also be written as $a_R b$.
If {A} = m and {B} = n then numbers of elements in A x B = mn. Thus, the number of relations on A and B are 2^{mn}.

Domain and Range

Let R be a relation from a set A to set B. the domain of a relation R is defined as the set consisting of all the first elements of the ordered pairs belonging to R and the Range of the relation is the set of all the second elements of the ordered pairs of R.

$\therefore$Domain of R = $\{a \in : (a, b) \in R\}$

Range of R = $\{b \in B ; (a , b) \in R\}$

e.g. Let A = { 1, 2, 3}, B = {a, b, c}

Let R be relation from set A to B $\ni$

R = {(2, a), (2, c), (3, b), (3, c)}

then

Domain of R = {2, 3}

Range of R = { a, b, c}

Clearly Domain of R $\subseteq$ A and Range of R $\subseteq$ B

Moreover, domain of R = Range of R^{-1}

and range of R = Domain of R^{-1}

Types of Relations

Universal Relation

A relation R from set A to set B is said to be universal if

R = A x B

e.g. A = {a, b }, B = {c, d }

then R = {(a, c), (a, d), (b, c), (b, d)} is a universal relation from set A to set B.

Void, Null or Empty Relation

Any relation R is called empty or void relation from set A to B

If R = $\oint$

Inverse Relation

Let R be a relation from A to B. Then the relation R^{-1} from B to A is called the inverse relation of R if

$R^{-1} = \{(b, a) : (a, b) \in R\}$

e.g. A = {a, b}, B = {1, 2, 3} and R is relation from A to B

R = {(a, 1), (b, 2), (b, 3), (a, 3)}

Then R^{-1}= { (1, a), (3, a), (2, b), (3, b)}

Reflexive Relation

A Relation R defined on a set A is said to be reflexive if

a R a $\forall\ a \in A$

i.e., (a, a) $\in$ R $\forall$ a $\in$ A

Let A = { 1, 2, 3 } let R be a relation defined on A. If R is Reflexive then it must contain ordered pairs (1,1), (2, 2) and (3, 3).

Example: let A = {1, 2, 3, 4} let R be a relation on set A define as R = {(1, 1), (1, 2), (2, 2), (3,2)

(3,2), (3, 3), (4,1), (4,4)}

Clearly R is reflexive because it contains every ordered (1,1), (2,2), (3,3) and (4,4)

i.e. a R a $\forall$ a $\in$ A

consider, R_1 defined on A

R_1= {(1,1), (1,2), (3,2), (4,4), (3,3)}

Clearly R_1 is not reflexive because $2 \in A, (2,2) \notin R_1$.

Irreflexive Relation

A relation R on a set A is said to be irreflexive if
a $\not R$ a $\forall$ a$\in A$
i.e., (a, a) $\notin R \; \forall \, \mathrm{a} \in \mathrm{A}$
Example. Let A = { 1, 2, 3}. Let R be a relation on set A
Solution R = {(1, 2), (1,3), (2,3)}
Clearly R is irreflexive because $\forall$ a $\in A \; a \not R \; a$.
i.e.for 1 $\in A, 1 \not R \, 1$
2 $\in A, 2 \not R \, 2$
3 $\in A, 3 \not R \, 3$
Non- Reflexive
A relation R on a set A is said to be non- reflexive if R is neither reflexive nor irreflexive
i.e., for some a $\in A, a \not R a$
and for some a $\in A, a \; R \; a$

Example A= {1, 2, 3} let R be a relation on A
R = {(1,1), (1,2), (2,2), (3,1), (3,2)}
Clearly R is non- reflexive because
For 1$\in A$ *and* $2 \in A$
(1,1) $\in R$ *and* $(2{,}2)$
But for 3 $\in A, (3{,}3) \notin R$.

Symmetric Relation
A relation R is defined on set A is said to be symmetric if
a R b → b R a, where a, b $\in A$
i.e. for any a, b ∈ A
(a,b) $\in R \rightarrow (b, a) \in R$
e.g. consider R = {(1,2),(2,1),(2,2),(3,1)(1,3)}
defined on
clearly R is symmetric
$\therefore$ (1,2) $\in R \rightarrow (2{,}1) \in R$
(2,2)$\in R \rightarrow (2{,}2) \in R$
(1,3) $\in R \rightarrow (3{,}1) \in R$
Whereas R_1= {(1,2), (1,3), (2,2), (3,1)}
Defined on A= {1, 2, 3} is not symmetric because
(1,2) $\in R$ -> (2,1)$\notin R$.

Asymmetric Relation
A relation R on set A is said to be asymmetric if
(a,b) $\in R \rightarrow (b, a) \notin R \; for \; a \; \neq b$

Anti-Symmetric
A relation R on a set A is called an anti- symmetric relation if
For a, b $\in A, a \; R \; b \; and \; b \; R \; a < -> a = b$
i.e. (a, b)$\in R \; and \; (b, a) \in R < -> \mathrm{a} = \mathrm{b}$
In other words
a$\neq b$ then either a $\not R$ b or b $\not R$ a or both.

Transitive
A relation R on a set A is called transitive If for a, b, c$\in A$
i.e. if (a, b) $\in Rand \; (b, c) \in R \; then \; (a, c) \in R$.

e.g. The relation "is parallel" to on the set of the lines in a plane is transitive because if a line l_1 is parallel to l_2 and if a l_2 is parallel to line l_3 then l_1 is parallel to l_3. A relation R on A is not transitive only when (a, b)$\in R, (b,c) \in R$ *but* $(a,c) \notin R$, *otherwise it is always transitive*

compatible Relation

A binary relation R on a set A is called compatible if it is reflexive and symmetric.

Less than Relation

A relation R from set A to B is said to be less than relation if

R= {(a, b)I a<b, a $\in A, b \in B$

e.g. let A = {1, 3}.B={2, 5}. Let R be a 'less than relation'
from set A and set B then.

R= {(1,2), (1,5),(3, 5)}

It clearly says that R contains all those ordered pairs of A x B whose domain elements are less than that of range elements.

Greater than Relation

A relation from set A to set B is said to be "greater than relation" if

R = {(a, b) {a>b, a$\in A, b \in B\}$

It clearly says that R contains all those ordered pairs of A x B whose domain elements are greater than that of range elements.

So, if R is greater than relation from set A = {1, 3} to set B {2, 5} then R = {(3, 2)}

Identity relation

A relation R defined from set A to set B is called identity relation if

R= {(a, b){a=b, a$\in A, b \in B\}$

It implies that all Domain set of R = Range set of R.

e.g., A= {1, 2, 3}, B={ 1, 3, 5}

Let R be an identity relation from set A to set B then

R = {(1,1), (5,5)

Circular Relation

A relation R is called circular if

(a, b) $\in R$ *and* $(b,c) \in R$

➔ (c, a) $\in R$

e.g. R= {(1, 3), (3, 2), (2, 1)}

on set A = {1, 2, 3} , R is circular

$\therefore$ $(1,3) \in R and\ (3,2) \in R$

→ $(2,1) \in$ R

Example: Let R be a relation on the set N of natural numbers defined by
R = {(a, b): a+3b=12 a,b ∈ N}
Find R , domain of R and Range of R

Solution: We have

$$a + 3b = 12 \Rightarrow a = 12 - 3b$$

Since $\quad b \in N$, so

Taking $\quad b = 1 \Rightarrow a = 9 \in \mathrm{N} \therefore (9, 1) \in R$

$\quad b = 2 \Rightarrow a = 6 \in \mathrm{N} \therefore (6, 2) \in R$

$\quad b = 3 \Rightarrow a = 3 \in \mathrm{N} \therefore (3, 3) \in R$

$\quad b = 4 \Rightarrow a = 0 \notin \mathrm{N} \therefore (0, 4) \notin R$

$\therefore \quad R = \{(9, 1), (6, 2), (3, 3)\}$

Domain of $\quad R$ = Set of first elements of R

$= \{9, 6, 3\}$

Range of $\quad R$ = Set of second element of R

$= \{1, 2, 3\}$

Example: Consider the following relation on
A = {1,2,3,4,5,6}
R = {(a,b) : |a-b| = 2}
Check whether R is reflexive , transitive , symmetric .

Solution: Here

$$R = \{(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3), (4, 6), (6, 4)\}$$

(*i*) Clearly R is not reflexive as $(a, a) \notin R \ \forall \ a \in A$

(*ii*) Clearly R is not transitive because $(1, 3) \in R$, and $(3, 1) \in R$ but $(1, 1) \notin R$

($i.e.$, $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R, a, b, c \in A$)

(*iii*) Clearly R is symmetric because

$$(a, b) \in R \Rightarrow (b, a) \in R, a, b \in A.$$

Theorems:
Theorem 1: Let R and S be two relations from A to B
a. If R⊆ S the S′⊆ R′
b. $\overline{(R \cap S)}$ = R′ U S′
c. $\overline{(R \cup S)}$ = R′∩ S′

Theorem 2: Suppose R and S be two relations from A to B
a. If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$
b. $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$
c. $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
Partial Order Relation:
A relation R on any set A is called A partially order relation if R reflexive , Anti-symmetric and Transitive
E.g. Let Z be the set of integers and R be relation "usual less than or equal to" on Z defined as $R = \{(a,b) : a \leq b, a,b \in Z \}$ then R is a partially ordered set as

(i) Reflexive:
$(a ,a) \in R \ \forall a \in A$

$a \leq a \ \forall a \in A$

(ii) **Anti- Symmetric:** Let a R b and b R a
Since a R b $\Rightarrow (a,b) \in R$
$\Rightarrow a \leq b$

and b R a $\Rightarrow (b,a) \in R$
➔ **B**$\leq \boldsymbol{a}$
(i) And (ii) → a=b

$\therefore R \text{ is anti} - \text{symmetric}$

(iii) **Transitive:** Let a R b and b R c
Since a R b → (a, b) $\in R \rightarrow a \leq b$
b R c → (b, c) $\in R \rightarrow b \leq c$
a$\leq c \rightarrow (a,b) \in R$

$\therefore R \text{ is transitive} \rightarrow \ a\, R\, c$

Equivalence Relation
Let A be a non-empty set. A relation R defined on set A is called equivalence relation if

(i) R is reflexive i.e., $\forall a \in A, (a,a) \in R$
(ii) R is symmetric i.e., (a, b) $\in R \rightarrow (b,a) \in R, a,b \in A$
(iii) R is transitive i.e., (a, b)$\in R \text{ and } (b,c) \in R \rightarrow (a,c) \in R, a,b,c \in A$
We find that symmetric and transitive properties of the relation R ensures the reflexive property of R. when R is symmetric.
(a, b) $\in R \rightarrow \ (b,a) \in R \text{ and}$
When R is transitive
(a, b), (b, a) $\in R \rightarrow (a,a) \in R$
Hence, (a, a)$\in R$
Thus, R is reflexive when R is symmetric and transitive.
But it is only true when every element a$\in A \text{ is related to some other element } b \ \in A$.
But if$\forall a \in A \nexists \text{ any } b \in A \ \ni (a,b) \in R \text{ then the symmetric and transitive relation } R \text{ may not be reflexive}$.
Equivalence classes
Let X be any non- empty set and R be an equivalent relation in X. Let ∊ for which a R x, is known as equivalence class of 'a' and is denoted as [a] or A.

Definition : Given a set X and an equivalence relation R ,and equivalence classis a subset of X of the form [a] = $\{x \in X: x\ R\ a\}$ where a is an element in X. [a] consists of those elements of X which are equivalent to a.
The set of all equivalence classes in X is given an equivalence relation R is denoted as X/Rand is called **quotient set** of X by R. (or read as x modulo R)
For example: if X is the set of all scooters and R is the equivalence relation of " having the same color" then one particular equivalence class consists of all red scooters so X/R means the set of all scooter colors.
Theorem 3 : If X is a non-empty set and R is a equivalence relation on A then the distinct equivalence classes of R form a partition of X.
Example: Let A = { 1,2,3,4} and R = {(1,1), (1,3),(2,2),(2,4), (3,1), (3,3), (4,2), (4,4)}. Show that R is an equivalence relation.
Solution: R = R = {(1,1), (1,3),(2,2),(2,4), (3,1), (3,3), (4,2), (4,4)}
Reflexive since (1,1),(2,2),(3,3) and (4,4) ∈ R,
i.e., (a, a) $\in R\ \forall\ a \in A$
Therefore, R is reflexive
Symmetric: Here R is symmetric because there does not exist any pair (a, b)$\in R$ for which (b, a) $\notin R$
Transitive: R is transitive because whether (a, b) $\in R$ and $(b, c) \in R$, then $(a, c) \in R$
(1,3) $\in R, (3,1) \in R \rightarrow (1,1) \in R$
(3,1) $\in R, (1,3) \in R \rightarrow (3,3) \in R$
(4,2) $\in R, (2,4) \in R \rightarrow (4,4) \in R$
(2,4) $\in R, (4,2) \in R \rightarrow (2,2) \in R$
Example : If R is an equivalence relation on set A then show that R^{-1} is also an equivalence relation.
Solution: Here R is given an equivalence relation on set A, so R is reflexive , symmetric and transitive.
We know that if (a, b) $\in A$ then $(b, a) \in R^{-1}$, a, b $\in A$
So we are to show that R^{-1} is an equivalence relation.
Reflexivity: Let a $\in A$
Since R is reflexive
so $(a, a) \in R\ \forall\ a \in A$
→ (a, a) $\in R^{-1}\ \forall\ a \in A$
∴ R^{-1} is reflexive
Symmetry: Let a, b$\in A$ $(a, b) \in R^{-1}$
∴ (b, a) ∈ R [∴ by definition of inverse relation]
→ (a, b) ∈ R [∴ R is symmetric]
→ (b, a) $\in R^{-1}$
Thus (a, b) $\in R^{-1}$
→ (b, a) $\in R^{-1}$
Thus, R^{-1} is symmetric
Transitive: Let a, b, c $\in A \ni (a, b) \in R^{-1}$ and (b, c) $\in R^{-1}$
To claim: (a, c) $\in R^{-1}$, a, c ∈ A
Now (a, b) $\in R^{-1}$ → (b, a) $\in R$
And (b, c) $\in R^{-1}$ → (c, b) ∈ R
Now
→ (c, b) ∈ R and (b, a) ∈ R
(c, a) ∈ R
(a, c) $\in R^{-1}$
Thus, R is transitive

Hence, R is an equivalence relation.

Combination Relations

Since relation from set A to set B are subsets of A x B; so two relations from set A to set B can be combined in the same way as two sets are combined i.e., we can find their union, intersection and even their difference.

Example: consider A = {1,2,3}, B = {1,3,4}

Let R_1 and R_2 are two relations from set A to set B defined as

R_1 = {(1,1) (1,3) (2,3) (2,4)}

R_2 = {(1,1) (2,1) (3,1) (3,4)}

Then $R_1 \cup R_2$ = {(1,1) (1,3) (2,3) (2,4) (2,1) (3,4) (3,1)

$R_1 \cap R_2$ = { (1,1)}

$R_1 - R_2$ = {(1,3) (2,3) (2,4)}

R_2-R_1 = {(2,1) (3,1) (3,4)

$R_2 \oplus R_1$ = (R_1-R_2) $\cup$ ($R_2 - R_1$)

= {(1,3) (2,1) (2,3) (2,4) (3,1) (3,4)}

Complement Relation

A relation R_1 from a set A to set B is said to be complement of another relation R_2 from set A to set B if.

$R_1 \cup R_2$ = A x B and $R_1 \cap R_2 = \emptyset$

i.e., R_1 = A x B − R_2

or R_2= A x B − R_1

Here R_1 and R_2 are called complement of each other. The complement of a relation R is donated as R′ or R^c . So R $\cup$ R′ = A x B and R $\cap$ R′ = $\emptyset$

e.g., Let A= {a, b, c}, B= {1,2}

Let R be a relation from set A to set B

R= {(a, 1) (b,1) (b, 2)}

Then complement " R′ of R is given by"

R= A x B- R

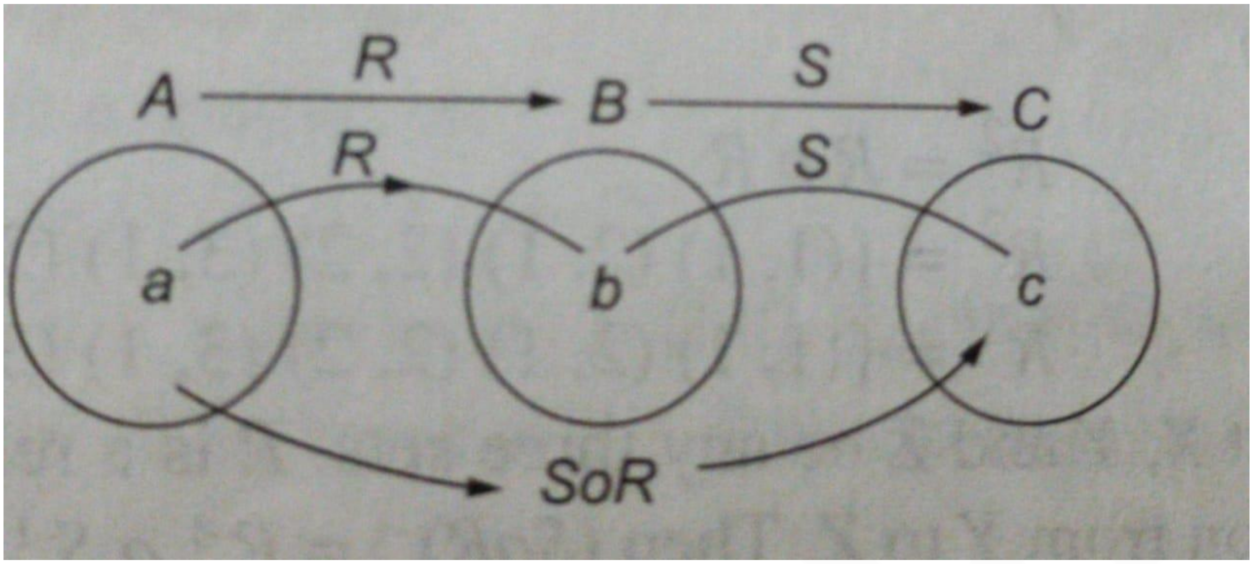
R′ = {(a, 2)(c,1), (c,2)}

Composition of into Relation

Let A, B, C be three sets and a relation R from set A to B be a R b where a $\in$ A

And b $\in B$and a relation S from set B to set C be b S c where b $\in$ B and c $\in$ C then the composite of R and S denoted by SoR is a relation from A to C consisting of ordered pairs (a, c) where a $\in A$ and c $\in C$

i.e., SoR = {(a, c) : $\exists$b $\in$ B is an element a R b and b S c }



Thus, a R b, bSc → a(SoR)c.
In other words , SoR ⊆ X x Z is defined by the rule that says (x, z) ∈ SoR if and only ∃ if an element y ∈ Y such that (x, y) ∈ R and (y, z) ∈ S.

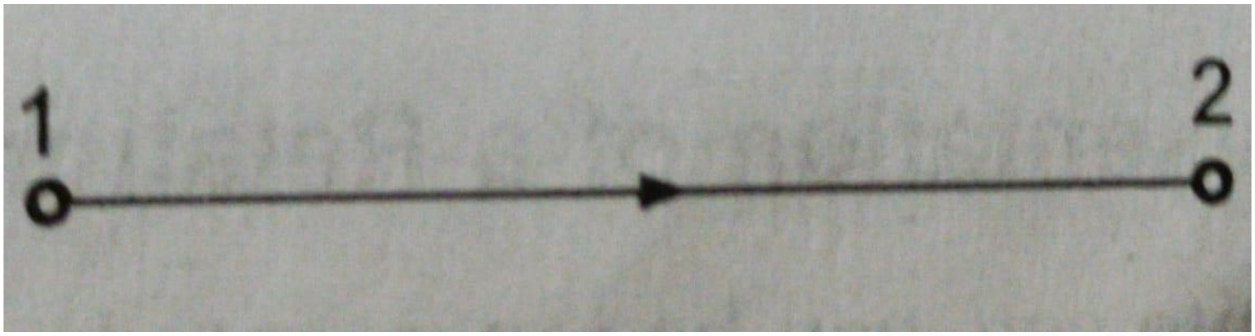
Law of Positive Integral Powers
Let R be a relation on set A then R^n, n ∈ N is defined as
$R^1 = R$ and $R^{n+1} = R^n$ o R
$R^3 = R^2$ o R = (R o R) o R and so on
Example Let R = {91,1) (1,2) (3,2) (4,2)} defined on A = {1,2,3,4} Find R^2 and R^2
Solution: We know that
R^2 = R o R
so R^2 = {(1,1) (1,2)}
and $R^3 = R^2$ o R = {(1,1)(1,2)}

Theorem 4: Let X, Y and Z be any three sets. R is a relation from X to Y and S is a relation from Y to Z. then $(SoR)^{-1} = R^{-1}$ o S^{-1}

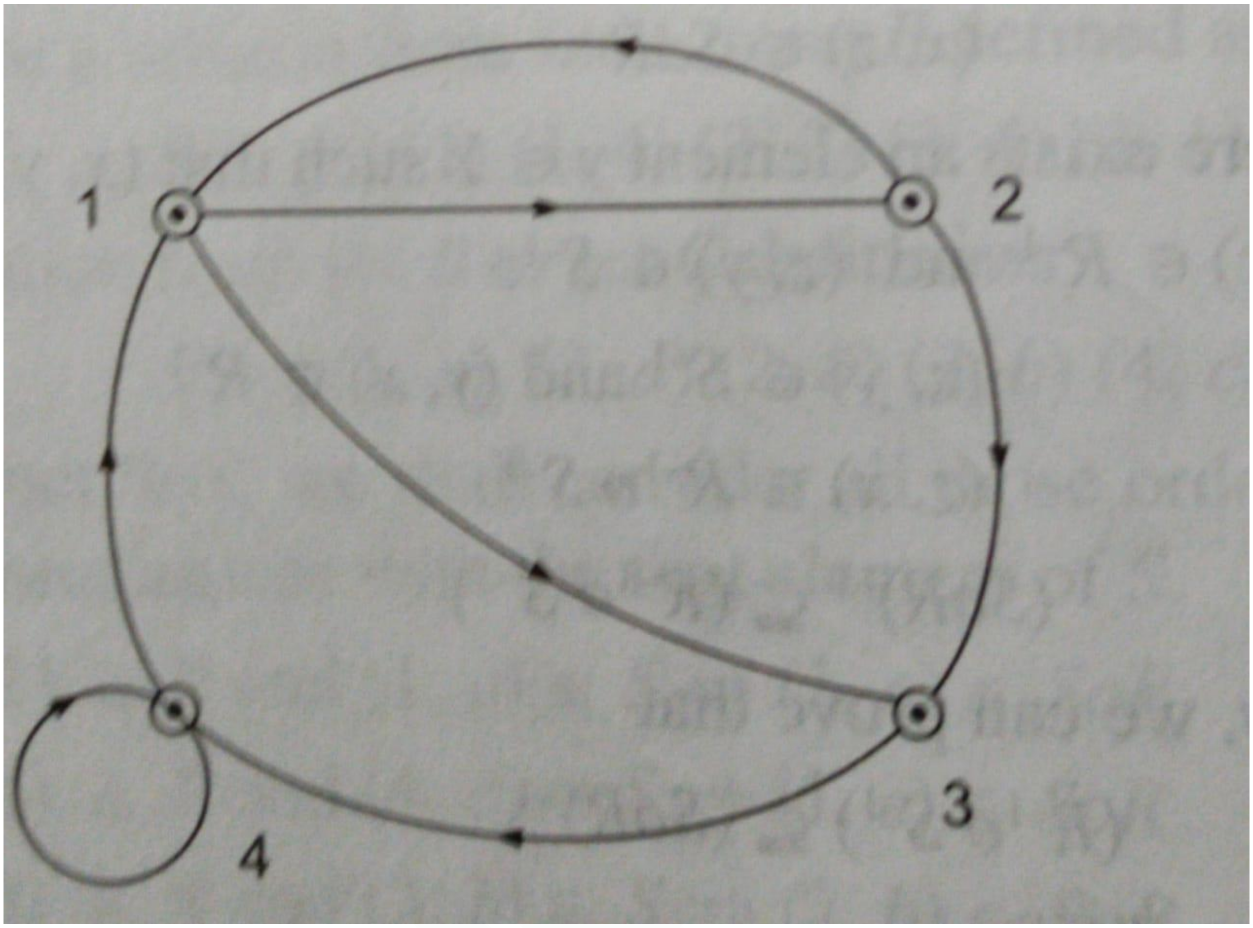
REPRESENTING A RELATION
We have learnt that a relation from set A to B is a subset of A x B which is collection of ordered pairs but a relation can more conveniently be represented using **digraph or matrix.**

Digraph or Directed Graph Representation
A convenient graphical representation of relation is using associated directed graph or digraph. Under this representation the various ordered pairs of given relation are represented using graph G(V, E), where Vis the set of vertices and E is the set of edges. If an ordered pair (a, b) ∈ R i.e., a Rb then there will be a directed edge (arc, or curved) from node (vertex or point) a to node b. Here 'a' is called Initial Vertex of edge e = (a, b) and b is called **Terminal Vertex** of edge e. But if a R b then there will be no edge between node a and node b. But if aRa then there will be an arc from vertex a' to itself. Such an arc is called a loop (self loop) .e.g.. Let R be a relation on
Set A = (1, 2, 3.4) 3
R = {(1, 2) (1, 3) (2, 1) (2, 3) (3, 4) (4, 1) (4, 4)

Since (1,2) ∈ R so there will a directed edge between node 1 and node 2
So we shall have



So the diagraph of R is given below:



directed graph representing a relation can be used to determine whether the relation is reflexive irreflexive, symmetric, transitive etc.

Reflexive Relation

The diagraph of a reflexive relation will have a self-loop at each vertex so that every ordered pair of the form (a, a) occurs in the relation. A digraph in which no vertex has a self loop is called an irreflexive but if self loops are there at some of the vertices but not at all the vertices then the relation is called non-reflexive.

Symmetric Relation

Digraphs in which for every edge (a, b) there is also an edge (b, a) are called symmetric digraphs. The digraph of symmetric relation is a symmetric digraph because for every directed edge from vertex a_j to a_i there is a directed edge from the vertex a_j to a_i Thus, in a symmetric digraph for every edge between distinct vertices there is an edge in the opposite directions but if there are not two edges in the opposite directions between distinct vertices then the digraph represents an anti-symmetric relation

Transitive Relation

In the digraph of a transitive relation when ever there is an edge from u vertex 'a' to a vertex 'b' and an edge from a vertex 'b' to a vertex 'c' there is an edge from vertex a to vertex c. It means if a R b and b Rc => a R c then the digraph will complete a triangle where each side is a directed edge with comet direction.