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Graph Theory Part -1

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Graph Theory Part-1

Content:

BASIC TERMINOLOGY

A graph G consists of two sets:

i. a non-empty set V whose elements are called vertices, nodes or points of G.

The group V(G) is called the vertex set of G.

ii. a set E of edges such that every edge $e \in E$ associated with ordered or unordered pairs of elements of V. The set E(G) is called the edge of G.

$$E(G) = \{(u, v) : u, v \in V(G)\},\ E(G) \subseteq V(G) \times V(G)$$

More formally, a graph G is an algebraic structure (V, E, Ψ) in which the set V is called the set of vertices, the set E is the set of edges and Ψ is a mapping from the set E to the set of unordered or ordered pairs of the elements of V.

Mostly, the graph G with vertices V and edges E is written as G = (V, E) or G(V, E), Committing the function Ψ.

If an edge e ∈ E is associated with an ordered pair (u, v) or an unordered pair (u, v), where u, v ∈ V, then e is said to connect u and v are called end points of e. An margin is said o be incident with the vertices it joins. Thus, the edge e that joins the vertices u and v is said to be incident on each of its end point u and v. any pair of vertices that is connected by an edge in a graph is called adjacent vertices.

In graph a vertex is not adjacent to another vertex is called an isolated vertex.

A graph G(V, E) is said to be finite if it has a finite number of vertices and finite number of Edges; otherwise, it is a infinite graph. If G is a finite, |V (G)| denotes the number of vertices in G and is called the order of G and |E (G)| denotes the number of edges in G and is called the size of G. We refer to a graph of order n and size m an (n, m) graph.

Although graphs are frequently stored in a computer as list of vertices and edges, they are pictured as diagrams in the plane in a naturals way. Vertex group of graph is described as a set of points in a plane and edge is represented by a line segment or an arc (not necessarily straight). The objects shown in figure 1.

It helps when discussing a graph to label each vertex, often with lower case letters as shown above. In figure1 (a), V = {a, b, c} and E = E{(a, b), (a, c), (b, c)} the member of vertices and edges are |v(G)| = 3 and |E(G)| = 3 in this graph, the vertices a and b, a and c and b and c are adjacent vertices.

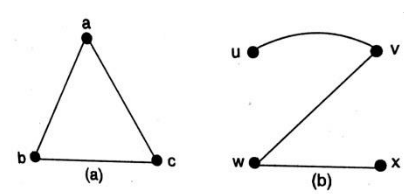


Figure 1

In figure 1(b), V = {u, v, w, x} and E = {(u, v), (v, w), (w, x)}

Here vertices u and v, v and w, w and x are adjacent, whereas u and w, u and x and v and x are non adjacent. The number of vertices and edges are |v(G)| = 4 and |E(G)| = 3.

The definition of a graph holds no reference to the length or the shape and the positioning of the edge or arc joining any pair of nodes, nor does it prescribe any ordering of positions of the nodes. Therefore, for a given graph there is no unique diagram that represents the graph, and it can happen that two diagrams that look entirely different from one another may represent the same graph. It is to be famed that, in drawing graph, it is immaterial whether the lines are drawn straight or curved, long or short, what is important is the incident between edges and vertices are the same in both cases.

UNDIRECTED AND DIRECTED GRAPH

An undirected graph G contains set V of vertices and a set E of edges such that each edge e ∈ E is associated with an unordered pair of vertices.

Figure 2(a) is an example of an undirected graph we can refer to an edge joining the vertex pair i and j as either (i, j) or (j, i).

A directed graph (or digraph) G contains a set V of vertices and a set E of edges such that e ∈ E is associated with an ordered pair of vertices. In other works, if each edge of the graph G e = (u, v) is represented by an arrow or directed curve from initial point u of e to the terminal point v. Figure 2(b) is an example of a directed graph.

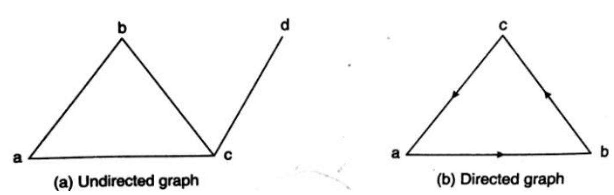


Figure 2

Suppose e = (u, v) is a directed edge in a diagram, then

i. u is called the initial vertex of e and v is the terminal vertex of e

ii. e is said to be incident from u and to be incident to v.

iii. u is adjacent to v, and v is adjacent from u.

Any edge of a digraph by its end-point, the edge is recognize to be directed from the first vertex towards the second. Graph, both directed and undirected, occur widely in all sorts of problems and before introducing more terminology we give examples of how graph arise in some familiar contexts.

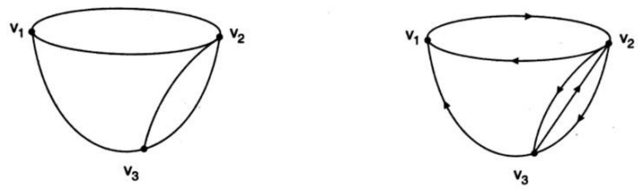
SIMPLE GRAPH, MULTIGRAPH AND PSUEDOGRAPH

An edge of a graph that joins a vertex to itself is called a loop or self loop i.e., a loop is an edge (v_1, v_2) where $v_i = v_j$.

In some directed as well as undirected graph, we may have contain pair of vertices joined by more than one edges, such edges are called multiple or parallel edges. Two edges (v_i, v_j) and (v_f, v_r) are parallel edges if $v_i = v_f$ and $v_f = v_r$. Note that in case of directed edges, the two possible edges between a pair of vertices which are opposite in direction are considered distinct. So more than one directed edge in a particular direction in the case of a directed graph is considered parallel.

A graph which has neither loops nor multiple edges i.e., where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called s simple graph. Figure 2(a) and (b) represents simple undirected and directed graph because the graph do not contain loops and the edges are all distinct.

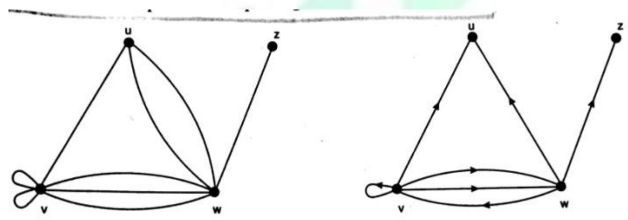
Graph which holds multiple edges is called a multigraph. In a multigraph, no loops are allowed.



(a) Undirected multigraph (b) Directed multigraph figure 3

In figure 3(a) there are two parallel edges joining nodes v_1 and v_2 and v_2 and v_3. In figure 3(b), there are two parallel edges associated with v_2 and v_3.

A graph with self-loops and multiple edges is called pseudograph.



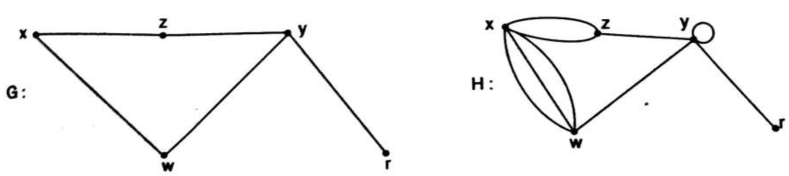
(a) Undirected pseudograph (b) Directed pseudograph

It famed that there is lack of standardisation of terminology in graph theory. Many words have almost obvious meaning, which are the same from book to book, but other terms are used differently by different authors.

DEGREE OF VERTEX

It is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v in a graph G may be denoted by $deg_G(v)$.

In the graph G and H in figure 4 are given below:

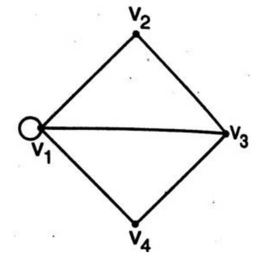


(a) (b)

In G as shown in figure 4(a)

$deg_G(x) = 2 = deg_G(z) = deg_G(w)$, $deg_G(y) = 3$, $deg_G(r) = 1$ and $deg_G(s) = 0$ and in H as shown in figure 4(b).

$deg_H(x) = 5$, $deg_H(z) = 3$, $deg_H(y) = 5$, $deg_H(w) = 4$ and $deg_H(r) = 1$.



A vertex of degree 0 is called isolated vertex. A vertex is pendant if and only if it has degree

1. Vertex s in the graph G is isolated and vertex r is pendant. A vertex of a graph is called odd-vertex or even vertex depending on whether its degree is odd or even.

Degree Sequence of Graph

In any graph g, we define

$\delta(G)$ = min {deg v: $\in$ V(G)} and

$\Delta(G)$ = max {deg v: $\in$ V(G)}

If $v_1, v_2, ..., v_n$ are the n vertices of G, and let $d_1, d_2, ..., d_n$ be their degrees.

If the sequence $(d_1, d_2, ... , d_n)$ is monotonically increasing i.e.

figure 5

$$\delta(G) = d_1 \leq d_2 \leq ..... \leq d_n = \Delta(G).$$

Then it is called the degree sequence of graph G.

For example, the degree sequence of the graph shown in figure 5.

is (2, 2, 3, 5) as deg (v_2) = deg (v_4) = 2 deg (v_3) = 3 and deg (v_1) = 5.

THEORENS

1. A simple graph with at least two vertices has at least two vertices of same dgree.
2. If G = (V, E) be an undirected graph with e edges. Then

$$\sum_{v \in V} deg_G(v) = 2\, e$$

i.e., the sum of degrees of the vertices in an undirected graph is even.

Corollary: In a non directed graph, the sum of odd degree vertices is even.

IN DEGREE AND OUT DEGREE

In a directed graph G, the out degree of a vertex v of G, denoted by $outdeg_G(v)$ or $deg_g^+(v)$, is the number of edges beginning from v and the indegree of v, denoted by $indeg_G(v)$ or $deg_G^-(v)$, is the number of edges ending at v. The sum of the in degree and out degree of a vertex is called total degree of the vertex. A vertex with zero in degree is called a source and a vertex with zero out degree is called a sink.

Theorem: If G = (V, E) be a directed graph with e edges, then

$$\sum_{v \in V} deg_G^+(v) = \sum_{v \in V} deg_G^-(v) = e$$

i.e., the sum of the outdegrees of the vertices of a diagraph G equals the sum of in degrees of the vertices which equal the number of edges in G.

TYPES OF GRAPHS

Some important types of graph are introduced here. These graph are often used as example and arise in many application.

NULL GRAPH

A graph which contains only isolated node is called a null graph i.e., the set of edges in a null graph is empty. Null graph is denoted on n vertices by N_n : N_4 is shown in figure 5. Note that every vertex of a null graph is isolated.

Diagram..

COMPLETE GRAPH

A simple graph G is defined to be complete if each vertex in G is connected with every other vertex i.e., if G contains exactly one edge between each pair of distinct vertices. A complete graph is usually denoted by K_n. It should famed that K_n has exactly $\frac{n(n-1)}{2}$ edges. The graph K_n for n = 1, 2, 3, 4, 5,6 are shown in figure 6.

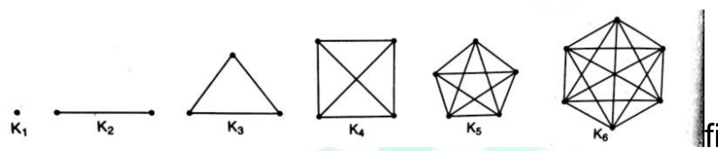


figure 6

REGULAR GRAPH

A graph in which all vertices are equal degree is called a regular graph. The degree of every vertex is r, then the graph is called a regular graph of degree r. Every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree n - 1. Also, note that if G has n vertices and is regular of degree r, then G has (1/2) r n edges.

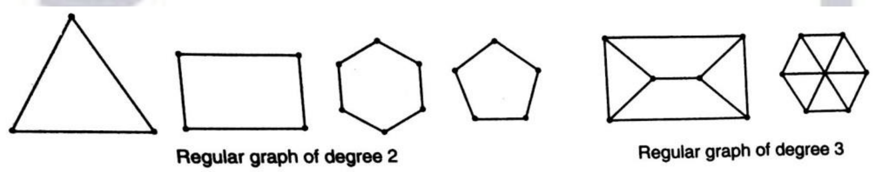
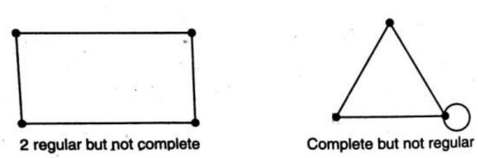


figure 7

Note that complete graph need not be regular and a regular graph need not be complete as the following examples illustrate.



PLATONIC GRAPH

Of special interest among the regular graphs are the so-called Platonic graph, the graphs formed by the vertices and edges of the five regular (Platonic) solids - The tetrahedron, icosahedrons. The graphs are shown in figure 7.

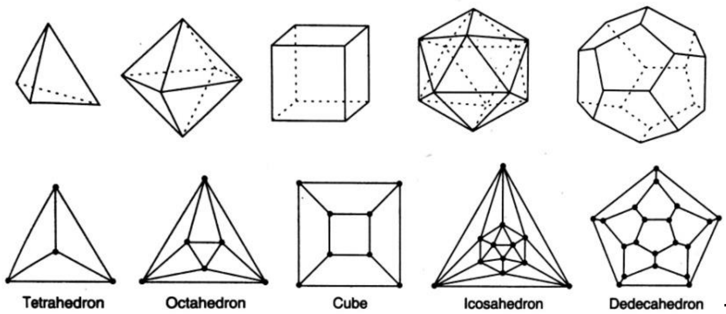


figure 7

CYCLE

The cycle graph C_n ($n \geq 3$), of length n is a connected graph which consists of n vertices v_1, v_2, ... v_n and n edges $\{v_1, v_2\}$. $\{v_2, v_3\}$, ... $\{v_{n-1}. V_n\}$. and $\{v_n, v_1\}$. The cycle C_3, C_4, C_5, and C_6 are shown in figure 8.

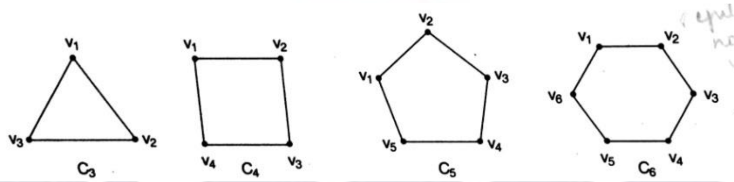


figure 8

C_n is a regular graph of degree 2.

WHEELS

The wheel graph W_n ($n > 3$) is obtained from C_{n-1} by adding a vertex v inside C_{n-1} and connecting it to every vertex in C_{n-1}. The wheels W_4, W_5, W_6 and W_7 are display in figure 9.

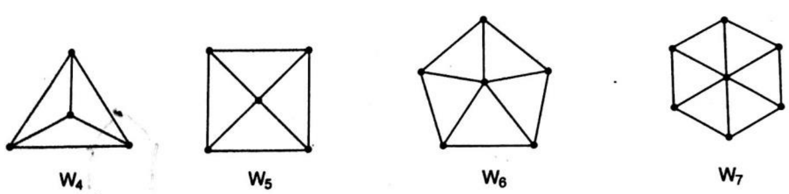


figure 9

W_n is a regular graph for n = 4. It has n vertices and 2n – 2 edges.

N-CUBE

The N-cube indicated by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bits strings that they represent differ in exactly one bit position. The graphs Q_1, Q_2, Q_3 are displayed in figure 10. Therefore Q_n has 2^n vertices and $n.2^{n-1}$ edges, and is regular of degree n.

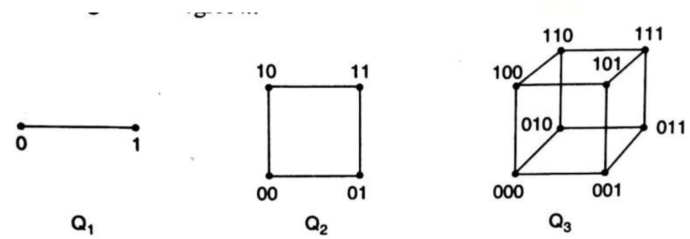
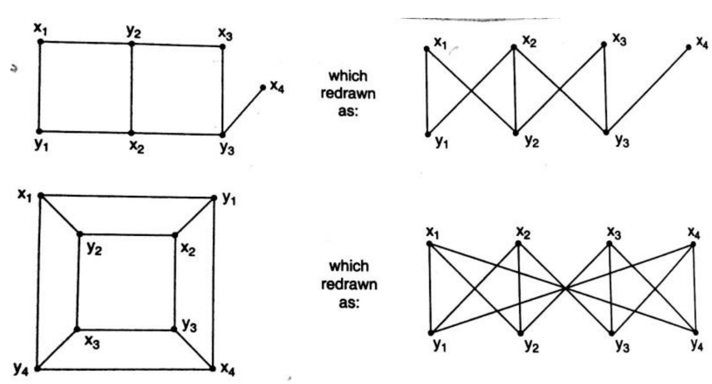


figure 10

BIPARTITE GRAPH

A graph G = (V, E) is bipa tite if the vertex set V can be partitioned into two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). (V_1, V_2) is called bipartition of G: Obviously, a bipartite graph can have no loop as loop connects the same vertex which is not permitted in bipartite graph.



COMPLETE BIPARTITE GRAPH

A bipartite graph G = (V, E) is said to be compete bipartite if each vertex of V_1 os connected to each vertex of V_2 where V_1 and V_z are the two distinct partition partitions of the vertex set V. Complete bipartite graph G is denoted by $K_{m,n}$, where m and n are the number of vertices m vertex sets V_1 and V_2. The compete bipartite graphs $K_{2,3}$, $K_{2,4}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$ are shown in figure 11. Note that $K_{m,n}$ has m + m vertices and m, n edges.

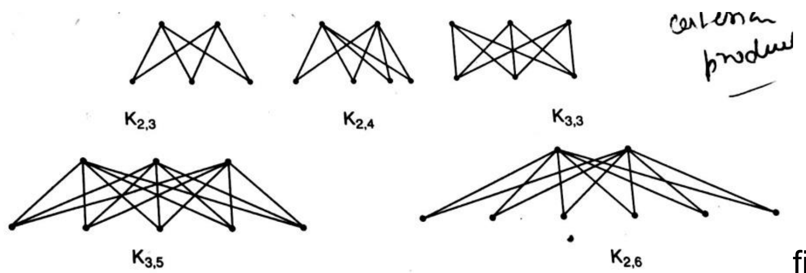


figure 11

NOTE: 1. Any graph $K_{1,n}$ is called a star graph.

2. A complete bipartite graph $K_{m,n}$ is not a regular if $m \neq n$.

3. K_5 and $K_{3,3}$ are called Kuratowski graphs.

A PROCEDURE TO CHECK IF GRAPH G IS BIPARTITE OR NOT

1. Arbitrarily select a vertex from G and include it into set V_1.
2. Consider the edges directly connected to that vertex and put the other and vertices of these edges into the set V_2.
3. Pick up one vertex from set V_2, and consider the edges directly connected to that vertex, and put the other end of these edges into the set V_1.
4. At each step, check if there is any edge among the vertices of set V_1 and set V_2.
   If so, the given graph is not bipartite graph, and then return.
   Elese continue 2 and 3 alternately until all the vertices are included in the union of sets V_1 and V_2.
5. If two computed sets are distinct, then the graph is bipartite.

SUBGRAPHAS AND ISOMORPHIC GRAPHS

SUBGRAPH

Some graph applications are concerned with only a part of entire graph. Such a graph is called a subgraph of the original graph.

Consider a graph G = (V, E). A graph H = (V', E') is called a subgraph of G if the vertices and edges of H are contained in the vertices and edges o G, that is, if V' ___ V and E' ___ E.

So, if H is a subgraph of G, then

i. All the vertices of H are in G.
ii. All the edges of H are in G and
iii. Every edge of H has the same end points in H as in G.

Some sub graph of a graph G can be obtained by removing certain vertices and edges from G. It is appreciate that the removal of an edge leaves its points in place, whereas the removal of a vertex necessitates the removal of any edges with that vertex as an end point.

DIFFERENT TYPES OF SUBGRAPHS

i. If V' ___ V and E' ___ E, then H is called a proper subgraph of G.
ii. A subgraph H of G, is called a spanning subgraph of G if and only if V(H) = V(G).
   i.e., H contains all vertices G but not necessary all edges of G.
iii. A subgraph H(V' E') of G(V, E) is called the induced subgraph of G if V' ___ V and its edge set E' contains all those edges in G that joins the vertices of set V' in G.

iv. If a subset U of V and all the edges incident on the elements of U are deleted from a graph G(V, E), then the resulting subgraph is called a vertex deleted subgraph of G(V, E).

v. If a subset S of E from a graph G(V, E) is deleted, then the resulting subgraph is called an edge deleted subgraph of G.

For a given graph G, there can be many subgraphs. Let |V| = m and |E| = n. The total non-empty subsets of V is 2^m-1 and total subsets of E is 2^n. Thus, number of subgraph is equal to $(2^m - 1) \times 2^n$.

Then number of spanning subgraph is equal to 2^n because all vertices are to be included in a spanning subgraph. For example, in Q_3 we have |V| = 8, |E| = 12. Then total number of subgrpahs is

$$(2^8 - 1) \times 2^{12} = 127 \times 4096 = 520192.$$

And the total number of spanning subgraph is $2^{12} = 4096$.

ISOMORPHIC GRAPH

Two graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a functions $f : V_1 \to V_2$ such that

i. f is one-to-one onto i.e., f is bijectiv.

ii. {a, b} is an edge in E_1, if and only if {f(a), f(b)} is an edge in E_2 for any two elements a, b $\in V_1$.

The condition (ii) says that if vertices a and b are adjacent in G_1 then f(a) and f(b) are adjacent in G_2. In other words the function f preserves adjacency relationship and consequently the corresponding vertices in G_1 and G_2 will have the same degree. Any function f with the above properties is called an isomorphism between G_1 and G_2.

PATHS, CIRCUITS, CYCLES AND CONNECTIVITY

In this section we introduce some additional terminology associated with a graph.

WALK

A walk in a graph G is a finite alternating sequence.

$$V_0 - e_1 - v_1 - e_2 - v_2 - e_3 \text{ .... } - e_n - v_n$$

of vertices and edges of the graph such that each edge e_i in the sequence joins vertices v_{i-1} and v_i, $1 \leq i \leq n$. The end vertices v_0 and v_n are the end or terminal vertices of the walk. The vertices v_1, v_2, ... , v_{n-1} are called its internal vertices. The integer n, the number of edges in the walk is called the length of the walk. A walk is called open when the terminal vertices are distinct. For the same end terminal vertices, it is termed as closed. Note that a walk may repeat both vertices and edges.

SPECIAL TYPE OF WALK

A walk is called a trial if all its edges are distinct. A trial open or closed depends on whether its end vertices are distinct or not. A closed trial is called a circuit. A walk is called a path if all its vertices and edges are distinct. A path in which only repeated vertex is the first vertex is called a cycle to describe such a closed path.

	Repeated Edge	Repeated Vertex	Starts and Ends at same points?
Walk (open)	Allowed	Allowed	No
Walk (closed)	Allowed	Allowed	Yes
Trail	No	Allowed	No
Circuit	No	Allowed	Yes
Path	No	No	No
Cycle	No	First and last only	Yea

For example, in the graph given in figure 12.

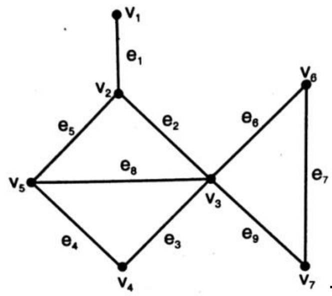


figure 12

i. The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_4 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2 - e_2 - v_3 - e_6 - v_6$ walk of length 8. It contains repeated vertices v_2, v_3 and v_5 and repeated edge e_5.

ii. The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_4 - e_3 - v_4 - e_4 - v_5$ is a trial. It contains repeated vertex v_5 but does not contain repeated edge.

iii. The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_4 - e_3 - v_4$ is a path. It does not contain repeated vertex and repeated edge.

iv. The sequence $v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2$ is a cycle. It does not contain repeated vertex and repeated edge except the first and last vertex.

THEOREMS

1. If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.
2. The minimum number of edges in a connected graph with n vertices is n - 1.
3. The minimum number of edges in a simple graph (not necessary connected) with n vertices is n - k, where k is the number of connected components of the graph.
4. A simple graph with n vertices and k components cannot have more than $\frac{(n-k)(n-k+1)}{2}$ edges

DISTANCE AND DIAMETER

In a connected graph G, the distance between the vertices u and v, denoted by d(u, v) is the length of the shortest path. In figure 13(a), d(a, f) = 2 and in figure 13(b), d(a, e) = 3.

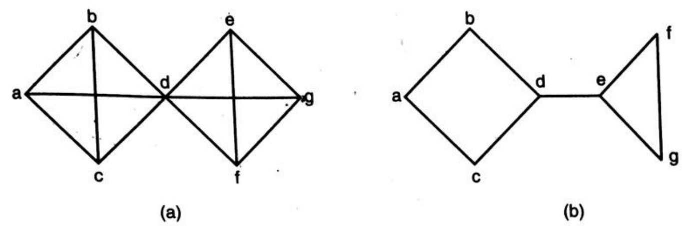


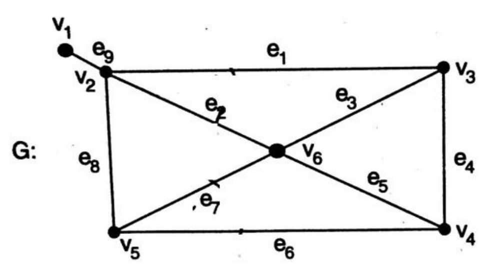
figure 13

The distance function as defined above the has the following properties. If u, v and w are any three vertices of a connected graph then

1. D(u, v) $\geq$ 0 and d(u, v) = 0 if u = v
2. D(u, v) = d(v, u) and
3. D(u, v) $\geq$ d(u, w) + d(w, v)

CUT SET AND CUT VERTICES

CUT SET: A cut set of a connected graph G is a set of edges whose removal (without removing the end vertices) from G leaves the G disconnected provided removal of no proper subsets of these edges disconnects G.



For example, in graph above

i. The edge $\{e_9\}$ is a cut set
ii. The set of edges $\{e_1, e_2, e_6, e_7\}$

iii. The set of edges $\{e_1, e_2, e_8, e_7\}$ is not a cut set because one of its proper subset $\{e_1, e_2, e_8\}$ is a cut set.

Thus a cut set S of G satisfy the followings:

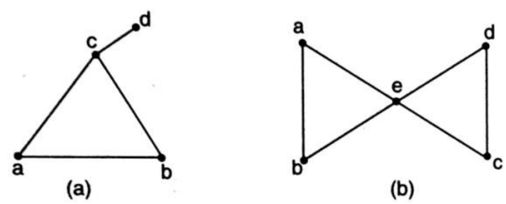
1. S is a subset of the edge set E of G.
2. Removal of edge/edges from a connected graph G disconnects the graph
3. No proper subset of G satisfy the condition.

Note: Every edge of a tree is a cut set.

CONNECTIVITY

To study the measure of connectedness of a graph G we consider the minimum number of vertices and edges to be removed from the graph in order to disconnect it.

EDGE CONNECTIVITY: Let G be a connected graph. Each cut set of G consists of a certain number of edges. The number of edges in the smallest cut set (i.e.. cut set containing fewest number of edges) is defined as the edge connectivity of G and denoted by $\lambda(G)$. The edge connectivity of the graph in (a) is one since the removal of edge between c and d results in a disconnected graph.



In (b), the edge connectivity is two since the removal of at least two edges viz (a, b) and (a, c) disconnected the graph.

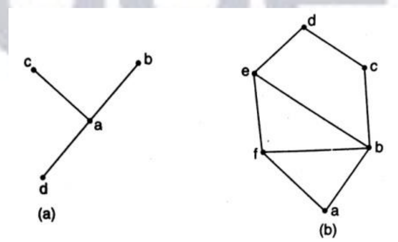
Note:

1. The edge connectivity of a tree is one because removal of an edge from the tree disconnects the graph.
2. The edge connectivity of the complete bipartite graph $K_{m,n}$ is a p where p = min (m, n)
3. The edge connectivity of a complete graph of n vertices is n - 1.

THEOREMS

1. The edge connectivity of a connected graph G cannot exceed the degree of the vertex having the smallest degree in G.
2. The edge connectivity is less than or equal [2e/n] where [2e/n] represents the integral part of 2e/n.

VERTEX CONNECTIVITY: The vertex connectivity of a connected graph G denoted by k(G) is defined as the minimum number of vertices whose removal (together with the edges incident on it) from G leaves the remaining graph disconnected. The vertex connectivity of the graph in figure (a) and (b) are one and two respectively.

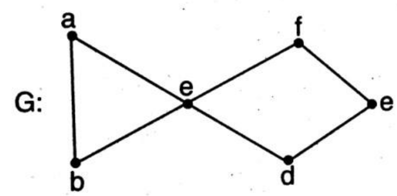


Note:

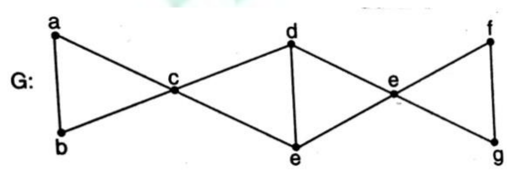
1. It G is a disconnected graph k(G) = 0
2. The complete graph K_n can not be disconnected by removing any number of vertices, but the removal of n - 1 vertices results in a trivial graph, hence $k(K_n)$ = n - 1.
3. The vertex connectivity of a path is one and then of cycle C_n ($n \geq 4$) is two.

SEPARABLE GRAPH: A connected graph G is said to be separable if its vertex connectivity is one.

All other graphs are known as non-separable. The graph shown below is separable.



CUT VERTEX: A cut vertex is a vertex in a separable graph whose removal disconnects the graph. The cut vertex is also called an articulation point. A given separable graph can have more than one cut vertex. The graph shown below is separable can have more than one cut vertex. The graph shown below is separable and both c and e are cut vertices.



THEOREM

1. A vertex v is a cut vertex of a connected graph if there exist two vertices x and y distinct from v such that every path between x and y passes through v.
2. The edge connectivity of a graph G cannot exceed the minimum degree of a vertex in G i.e. λ (G) $\leq$ δ (G).
3. The vertex connectivity of a graph G is always less than or equal to the edge connectivity of G i.e. k (G) $\leq$ λ (G)