Counting, Mathematical Induction and Discrete Probability Part-2

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Content:

1. **Mathematical Induction**

Mathematical Induction

The word induction means the method of inferring a general statement from the validity of particular cases. Mathematical induction is a technique by which one can prove mathematical statement involving positive integers.

Before describing the method of mathematical induction, let us try to understand its power. To do this, let us consider the statement:

$1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots\ldots. + n = n (n + 1)/2$

It easy to check that this statement is true for $n = 1$, $n = 2$and $n = 3$ etc.

From the above, one can not conclude that the statement is true for all positive n as one can never be sure that the statement does not fail for some untried value of n. But it is also impossible to substitute infinite number of possible values of *n*. Mathematical induction reduce the proof to a finite number of steps and guarantee that there is no positive n for which the statement fails to be determined.

A formal statement of **Principal of Mathematical Induction** can be stated as follows.

Let $P(n)$ be a statement that involves positive integer $n= 1,2,3,\ldots\ldots.$Then $P(n)$ is true for all positive integer n provided that

1. P(1) is true
2. $P(k+1)$ is true whenever $P(k)$ is true.

So, there are 3 steps of proof using the principle of mathematical induction.

Steps 1. (Inductive base) Verify that P(1) is true.

Steps 2. (Inductive hypothesis) Assume that P(k) is true for an arbitrary value of k.

Steps 3. (Inductive step) Verify that P(k + 1) is true on basis of the inductive hypothesis.

Note. (Change of inductive base) : The principle of mathematical induction defined above begins at $n = 1$ and proves that P(n) is true for $n > 1$. One can also begin with an integer different from 1, say, at $n = n_0$ and prove that for n = $k + 1$ assuming that the statement is true for $n = k$ ($k > n_0$).

Example. Show that

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2 = n\,(n+1)\,(2n+1)/\,6,\ n > 1$$

By mathematical induction.

Solution. Let P(n) be the given statement

1. Inductive base : For $n = 1$ we have

$1^2 = 1\,(1+1)\,(2+1)/6=1$

So , P(1) is true .

2. Inductive hypothesis : Assume that P(k) is true i.e.,

$$P(k) = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + k^2 = k\,(k+1)\,(2k+1)/6$$

3. Inductive Step : We wish to show the truth of P(k+1) i.e.,

$$P(k+1) = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + (k+1)^2 = [(k+1)\,(k+2)\,(2k+3)]/6$$

Which has been obtained by substituting k+1 for n is S (n).

$$\text{Now; } 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + (k+1)^2 = (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + k^2) + (k+1)^2$$

$$= k(k+1)\,(2k+1)/6 + (k+1)]$$

$$= (k+1)\,[(2k + 7k + 6)/\,6]$$

$$= (k+1)\,(k+2)\,(2k+3)/6$$

Which is P(k+1). That is P(k +1) is true whenever P(k) is true.

By the principal of mathematical induction P(n) is true for all positive integer n.

Example: Prove that the sum of cubes of n natural numbers is equal to ($n(n+1)^2)^2$ for all n natural numbers.

Solution:

In the given statement we are asked to prove:

$1^3+2^3+3^3+\cdots+n^3 = (n(n+1)^2)^2$

Step 1: Now with the help of the principle of induction in math let us check the validity of the given statement P(n) for n=1.

$P(1)=(1(1+1)^2)^2 = 1$ This is true.

Step 2: Now as the given statement is true for n=1 we shall move forward and try proving this for n=k, i.e.,

$1^3+2^3+3^3+\cdots+k^3= (k(k+1)^2)^2$.

Step 3: Let us now try to establish that P(k+1) is also true.

$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = (k(k+1)^2)^2+(k+1)^3$$

$$\Rightarrow 1^3+2^3+3^3+\cdots+k^3+(k+1)^3=k^2(k+1)^4+(k+1)^3$$

$$= k^2(k+1)^2+4((k+1)^3)^4$$

$$=(k+1)^2(k^2+4(k+1))^4$$

$$=(k+1)^2(k^2+4k+4)^4$$

$$= (k+1)^2((k+2)^2)^4$$

$$=(k+1)^2(k+1+1)^2)^4$$

$$=(k+1)^2((k+1)+1)^2)^4$$

Example : **Show that 1 + 3 + 5 + … + (2n−1) = n^2**

Solution:

Step 1: Result is true for n = 1

That is $1 = (1)^2$ (True)

Step 2: Assume that result is true for n = k

$1 + 3 + 5 + \ldots + (2k-1) = k^2$

Step 3: Check for n = k + 1

i.e. $1 + 3 + 5 + \ldots + (2(k+1)-1) = (k+1)^2$

We can write the above equation as,

$1 + 3 + 5 + \ldots + (2k-1) + (2(k+1)-1) = (k+1)^2$

Using step 2 result, we get

$k^2 + (2(k+1)-1) = (k+1)^2$

$k^2 + 2k + 2 - 1 = (k+1)^2$

$k^2 + 2k + 1 = (k+1)^2$

$(k+1)^2 = (k+1)^2$

L.H.S. and R.H.S. are same.

So the result is true for n = k+1

By mathematical induction, the statement is true.

We see that the given statement is also true for n=k+1. Hence we can say that by the principle of mathematical induction this statement is valid for all natural numbers n.

Example: Show that 2^{2n}-1 is divisible by 3 using the principles of mathematical induction.

To prove: $2^{2n}-1$ is divisible by 3

Assume that the given statement be P(k)

Thus, the statement can be written as $P(k) = 2^{2n}-1$ is divisible by 3, for every natural number

Step 1: In step 1, assume n= 1, so that the given statement can be written as

$P(1) = 2^{2(1)}-1 = 4-1 = 3$. So 3 is divisible by 3. (i.e.3/3 = 1)

Step 2: Now, assume that P(n) is true for all the natural number, say k

Hence, the given statement can be written as

$P(k) = 2^{2k}-1$ is divisible by 3.

It means that $2^{2k}-1 = 3a$ (where a belongs to natural number)

Now, we need to prove the statement is true for n= k+1

Hence,

$P(k+1) = 2^{2(k+1)}-1$

$P(k+1)= 2^{2k+2}-1$

$P(k+1)= 2^{2k}.\ 2^{2} - 1$

$P(k+1)= (2^{2k}.\ 4)-1$

$P(k+1) =3.^{2k} +(2^{2k}-1)$

The above expression can be written as

$P(k+1) =3.2^{2k} + 3a$

Now, take 3 outside, we get

$P(k+1) = 3(2^{2k} + a)= 3b$, where “b” belongs to natural number

It is proved that p(k+1) holds true, whenever the statement P(k) is true.

Thus, $2^{2n}-1$ is divisible by 3 is proved using the principles of mathematical induction

Example: Prove that $2^n > n$ for all positive integers n.

Solution : Let P(n): $2^n > n$

When n =1, $2^1 > 1$. Hence P(1) is true.

Assume that P(k) is true for any positive integer k, i.e.,

$2^k > k$ ... (1)

We shall now prove that P(k +1) is true whenever P(k) is true.

Multiplying both sides of (1) by 2, we get

$2. 2^k > 2k$

i.e., $2^{k+1} > 2k = k + k > k + 1$

Therefore, P(k + 1) is true when P(k) is true. Hence, by principle of mathematical induction, P(n) is true for every positive integer n.

Example: For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

Solution: We can write

P(n) : $7^n - 3^n$ is divisible by 4.

We note that

P(1): $7^1 - 3^1 = 4$ which is divisible by 4.

Thus P(n) is true for n = 1

Let P(k) be true for some natural number k,

i.e., P(k) : $7^k - 3^k$ is divisible by 4.

We can write $7^k - 3^k = 4d$, where $d \in N$.

Now, we wish to prove that P(k + 1) is true whenever P(k) is true.

Now $7^{(k+1)} - 3^{(k+1)} = 7^{(k+1)} - 7.3^k + 7.3^k - 3^{(k+1)}$

$= 7(7^k - 3^k) + (7 - 3)3^k$

$= 7(4d) + (7 - 3)3^k$

$= 7(4d) + 4.3^k$

$= 4(7d + 3^k)$

From the last line, we see that $7^{(k + 1)} - 3^{(k + 1)}$ is divisible by 4. Thus, P(k + 1) is true

when P(k) is true. Therefore, by principle of mathematical induction the statement is

true for every positive integer n.