Counting, Mathematical Induction and Discrete Probability Part-3

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Content:

1. **Conditional Probability**
2. **Independent Variable**
3. **Bayes' Theorem**

Conditional Probability

The conditional probability of event B is the probability that the event take place given that you already have knowledge that event A has already taken place. The probability notation is given by P(B|A) which means the probability of B given A.

In this case where the two events A and B are independent where the event A does not affect the probability of event B then the conditional probability of the event event A is P(B).

However, if the two events A and B are not independent, the probability of intersection of A and B that is the probability of both the events occurring is denoted by:

P(A and B) = P(A)P(B|A).

This can help you to get the probability of P(B|A)which is obtained by

$P(B|A) = P(A \cap B)/P(A)$

Properties of Conditional Probability:

Property 1: If E and F are the events of the sample space say S , P(S|F) = P(F|F) =1

Property 2: If A and B are two events in a sample space S and F is an event of S such that

$P(F) \neq 0$, P((AUB)|F) = P(A|F) + P(B|F) - P((A∩B)|F).

Property3: P(A' |B) = 1- P(A|B)

Example: Given that E and Fare events such that

P (E) = 0.6, P (F) = 0.3 and P(E∩F) = 0.2

find P(E|F) and P (F|E).

Given: P (E)=0.6, P (F)=0.3, P (E ∩ F)=0.2

$P(E|F) = \frac{P(E \cap F)}{P(E)} = \frac{0.2}{0.3} = \frac{2}{3}$

$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{0.2}{0.6} = \frac{1}{3}$

Solution:

Example: If P (A)=0.8, P (B)=0.5 and P(B/A)=0.4, find

(i) P(A∩B)

(ii) P(A/B)

(iii)P(A∪B)

Solution:

(i) $P(B/A) = \frac{P(A \cap B)}{P(A)} \Rightarrow 0.4 = \frac{P(A \cap B)}{0.8}$

∴ P(A∩B) = 0.4 x 0.8 = 0.32

(ii) P(A/B) = $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.32}{0.5} = \frac{32}{50} = \frac{16}{25}$

(iii) P(A∪B) = P(A) + P(B) – P(A∩B)

= 0.8 + 0.5 – 0.32 = 1.30 – 0.32 = 0.98

Example: Evaluate P(A∪B) if 2P(A) = P(B) = 5/13 and P(A|B) = 2/5.

Solution:

Given:

$2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$.

$$\therefore \quad P(A) = \frac{5}{26}, P(B) = \frac{5}{13}$$

$$P(A \cap B) = P(A \mid B).P(B)$$

$$= \frac{2}{5} \times \frac{5}{13} = \frac{2}{13}$$

$$\text{Now } P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{26}$$

Example: Determine P(E/F):

A coin is tossed three times, where

(i) E: head on third toss F: heads on first two tosses.

(ii) E: at least two heads F : at most two heads

(ii) E: at most two tails F: at least one tail

Solution:

(i) E = Head occurs on third toss as {HHH, HTH, THH, TTH}

F : Heads on first two tosses = {HHH, HHT} E ∩ F = {HHH}

$$P(E \cap F) = \frac{1}{8}, \quad P(F) = \frac{1}{4}$$

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{8} \times 4 = \frac{1}{2}$$

(ii) E : At least two heads

= {HHT, HTH, THH, HHH}

F : At most two heads

= {TTT, HTT, THT, HTT, HHT, HTH, THH}

E ∩ F : {HHT, HTH, THH}

$$P(A \cap F) = \frac{3}{8}, P(F) = \frac{7}{8},$$

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{3}{8} \div \frac{7}{8} = \frac{3}{7}$$

(iii) E : At most two tails

= { HHT, THT, TTH, HHT, HTH, THH, HHH}

F : { THH, HTH, HHT, TTH, THT, HTT, TTT}

E ∩ F = { HTT, THT, TTH, THH, HTH, HHT}

$$P(E \cap F) = \frac{6}{8}, P(F)\frac{7}{8},$$

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{6}{8} \div \frac{7}{8} = \frac{6}{7}$$

Example: Black and a red die are rolled.

(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

(b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution:

(a) n(S) = 6 x 6 = 36

Let A represent obtaining a sum greater than 9 and B represents black die resulted in a 5.

A= {46,64,55,36,63,45,54,65,56,66}

$$n\,(\mathrm{A}) = 10 \Rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{10}{216}$$

$$\mathrm{B} = \{51, 52, 53, 54, 55, 56\} \Rightarrow n\,(\mathrm{B}) = 6$$

$$\mathrm{P\,(B)} = \frac{6}{216},$$

$$A \cap B = \{55, 56\} \Rightarrow n(A \cap B) = 2$$

$$P(A \cap B) = \frac{2}{216},$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{216}}{\frac{6}{216}} = \frac{2}{6} = \frac{1}{3.}.$$

(b) Let A denotes the sum is 8

$\therefore$ A = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

B = Red die results in a number less than 4

either first or second die is red.

B = {(1, 1), (1, 2) (1, 3) (1, 4), (1, 5), (1, 6), (2, 1) (2, 2) (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2) (3, 3), (3, 4), (3, 5), (3, 6)}

$$A \cap B = \{(2, 6), (3, 5)\}$$

$$\therefore \quad P(A \cap B) = \frac{2}{36} = \frac{1}{18}, \; P(B) = \frac{18}{36} = \frac{1}{2}$$

$$\text{Hence } P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{1}{9}.$$

Independent Events :

Those events that when occurs does not affect any other event. Like if a coin is flipped in the air and the outcome is head . If you flip the coin again , the outcome is a tail. In both the cases, the occurrence of each event is independent of each other. If the probability of an outcome of an event say A is not affected by the probability of occurrence of another event B, it is said that A and B are two independent events.

In Interdependent event

P(A∩B) = P(A) X P(B)

Example: If, P(A) =3/5 and P(B) =1/5 find P (A ∩ B) if A and B are independent events.

Solution:

A and B are independent if P (A ∩ B)

$$= P(A) \times P(B) = \frac{3}{5} \times \frac{1}{5} = \frac{3}{25}$$

Example: Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution:

Number of exhaustive cases = 52

Number of black cards = 26

One black card may be drawn in 26 ways

∴ Probability of getting a black card,

$$P(A) = \frac{26}{52} = \frac{1}{2}$$

After drawing one card, number of cards left = 51

After drawing a black card number of black cards left = 25

∴ Probability of getting both the black cards,

$$P(A)\,P(B/A) = \frac{1}{2} \times \frac{25}{51} = \frac{25}{102}$$

Example: A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not

Solution:

When a coin is thrown, head or tail will occur

Probability of getting head P(A) = $\frac{1}{2}$

When a die is tossed 1,2,3,4, 5, 6 one of them will appear

∴ Probabillity of getting 3 = P(B) = $\frac{1}{6}$

When a die and coin is tosses, total number of cases are

H1,H2,H3,H4,H5,H6

T1,T2,T3,T4,T5,T6

Head and 3 will occur only in 1 way

∴ Probability of getting head and $3 = \frac{1}{12}$

i.e., $P(A \cap B) = \frac{1}{12}$, $P(A) \times P(B) = \frac{1}{12} \times \frac{1}{6} = \frac{1}{12}$

$\therefore \quad P(A \cap B) = P(A) \times P(B)$

$\Rightarrow$ Events A and B are independent.

Example: Given that the events A and B are such that P(A) = 1/2,P(A∪B) = 3/5 and P(B) = p. Find p if they are

(i) mutually exclusive

(ii) independent.

Sol. Let $P(A \cap B) = x$, Now $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$, $P(B) = P$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore \quad \frac{3}{5} = \frac{1}{2} + p - x$$

or $p - x = \frac{3}{5} - \frac{1}{2} = \frac{6-5}{10} = \frac{1}{10}$ ...(i)

(i) When events A and B are mutually exclusive

$x = 0, p = 1/10$

(ii) Whent events A and B are independent

$P(A \cap B) = P(A) \times P(B)$

$x = \frac{1}{2} \times p$ ...(ii)

Also $p - x = \frac{1}{10}$ from (i), subtracting value of

$x = \frac{p}{2}$ in $p - x = \frac{1}{10}$, we get

$$p - \frac{p}{2} = \frac{1}{10} \Rightarrow \frac{p}{2} = \frac{1}{10} \Rightarrow p = \frac{1}{5}$$

Example: Let A and B independent events P(A) = 0.3 and P(B) = 0.4. Find

(i) $P(A \cap B)$

(ii) $P(A \cup B)$

(iii) $P(A \mid B)$

(iv) $P(B \mid A)$

Solution:

P (A) = 0.3,
P (B) = 0.4
A and B are independent events
(i) ∴ $P(A \cap B) = P(A).\ P(B) = 0.3 \times 0.4 = 0.12.$
(ii) $P(A \cup B) = P(A) + P(B) - P(A).P(B)$
$= 0.3 + 0.4 - 0.3 \times 0.4 = 0.7 - 0.12 = 0.58.$

$$\text{(iii)}\ P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A).P(B)}{P(B)} = 0.3$$

$$\text{(iv)}\ P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A).P(B)}{P(A)} = 0.4$$

Bayes' Theorem:

Bayes, is a mathematical formula for determining conditional probability. Conditional probability is the likelihood of an outcome occurring, based on a previous outcome occurring. Bayes' theorem provides a way to revise existing predictions or theories (update probabilities) given new or additional evidence. In finance, Bayes' theorem can be used to rate the risk of lending money to potential borrowers.

Bayes' theorem is also called Bayes' Rule or Bayes' Law and is the foundation of the field of Bayesian statistics.

KEY TAKEAWAYS

- Bayes' theorem allows you to update predicted probabilities of an event by incorporating new information.

- It is often employed in finance in updating risk evaluation.

Bayes' theorem.. Let $A_1, A_2, \ldots, A_n$ be a set of mutually exclusive events that together form the sample space S. Let B be any event from the same sample space, such that P(B) > 0. Then,

$$P(A_k \mid B) = \frac{P(A_k \cap B)}{[P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)]}$$

Note: Invoking the fact that $P(A_k \cap B) = P(A_k)P(B \mid A_k)$, Baye's theorem can also be expressed as

$$P(A_k \mid B) = \frac{P(A_k)\,P(B \mid A_k)}{[P(A_1)\,P(B \mid A_1) + P(A_2)\,P(B \mid A_2) + \ldots + P(A_n)\,P(B \mid A_n)]}$$

Example: A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Solution:

Let A be the event that ball drawn is red and let E1 and E2 be the events that the ball drawn is from the first bag and second bag respectively. $P(E1) = \frac{1}{2}$, $P(E2) = \frac{1}{2}$.

P (A|E 1) = Probability of drawing a red ball from bag

$I = \frac{4}{8} = \frac{1}{2}$

$P(A|E_2)$ = Probability of drawing a red ball from bag

$II = \frac{2}{8} = \frac{1}{4}$

Therefore by Bayes' theorem

$P(E_1|A)$ = Probability that the red ball drawn is from bag I

$$= \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{4}} = \frac{2}{3}$$

Example: Of the students In a college, it is known that 60% reside In hostel and 40% are day scholars (not residing In hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student Is chosen at random from the college and he has an A- grade what Is the probability that the student is a hostlier?

Solution:

Let E1, E2 and A represents the following:
E1 = students residing in the hostel,
E2 day scholars (not residing in the hostel)
and A = students who attain grade A

Now $P(E_1) = \frac{60}{100}, P(E_2) = \frac{40}{100}$

$P(A|E_1) = \frac{30}{100}, P(A|E_2) = \frac{20}{100}$

Now by Bayes' theorem

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)}$$

$$= \frac{\frac{60}{100} \times \frac{30}{100}}{\frac{60}{100} \times \frac{30}{100} + \frac{40}{100} \times \frac{20}{100}} = \frac{9}{13}$$

Example: In answering a question on a multiple choice test, a student either knows the answer or 3 guesses. Let 3/4 be the probability that he knows the answer and 1/4 be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability 1/4 .What is the probability that the student knows the answer given that he answered it correctly?

Solution:

Let the event E1 = student knows the answer , E2 = He gusses the answer

P(E1) = $\frac{3}{4}$,P(E2) = $\frac{1}{4}$

Let A is the event that answer is correct, if the student knows the answer

$\Rightarrow$ Answer is correct $\therefore$ $P(A/E_1) = 1$

If he guesses the answer $\therefore$ $P(A/E_2) = \frac{1}{4}$

$\therefore$ Probability that a student knows the answer given that answer is correct is,

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{3}{4}\times 1}{\frac{3}{4}\times 1 + \frac{1}{4}\times\frac{1}{4}} = \frac{\frac{3}{4}}{\frac{13}{16}} = \frac{3}{4}\times\frac{16}{13} = \frac{12}{13}$$