2-D Geometrical Transforms and Viewing Part-1

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Content:

1. Transformations
2. Translation
3. Rotation
4. Scaling
5. Shearing

TRANSFORMATIONS

BASIC TRANSFORMATIONS

Consider the xy-coordinate system on a plane. An object (say Obj) in a place can be considered as a set of points. Each object point P has coordinates (x, y), so the object is the sum total of all its coordinate points (see in figure1). Let the object be moved to a new position. Many coordinate points P'(x', y') of a new object Obj' can be obtained from the original points P(x, y) by the application of a geometric transformation.

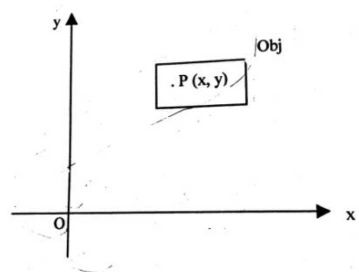


Figure 1

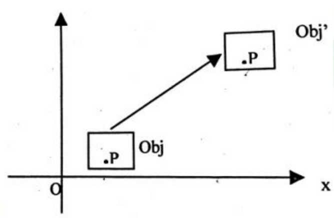
TRANSLATION

It is the process of changing the position of an object. Let an object point P(x, y) = xI + yJ be moved to P'(x', y') be the given translation vector V = t_xI + t_yJ, where t_x and t_y is the translation factor in x and y directions, such that

P' = P + V ....(1)

In component form, we have

$$Tv = \begin{cases} x' = x + t_x \, and \\ \quad y' = y + t_y \end{cases} \qquad ....(2)$$



$$(x',y',1)=(x,y,1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx & ty & 1 \end{bmatrix}$$

Example: Translate a square ABCD with the coordinates

A(0, 0), B(5, 0), C(5, 5), D(0, 5) by 2 units in x-direction and 3 units in y-direction.

Solution: we act as the given square, in matrix form, using homogeneous coordinate of vertices

as:

$$\begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{pmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \\ x4 & y4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 1 \\ 5 & 5 & 1 \\ 0 & 5 & 1 \end{pmatrix}$$

The translation factors are, tx = 2, ty = 3

The transformation matrix for translation:

:

$$T_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx & ty & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

New object point coordinates are:

$$[A'B'C'D'] = [ABCD].T_v$$

$$\begin{array}{c} A' \\ B' \\ C' \\ D' \end{array} \begin{bmatrix} x'_1 & y'_1 & 1 \\ x'_2 & y'_2 & 1 \\ x'_3 & y'_3 & 1 \\ x'_4 & y'_4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 5 & 0 & 1 \\ 5 & 5 & 1 \\ 0 & 5 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 7 & 3 & 1 \\ 7 & 8 & 1 \\ 2 & 8 & 1 \end{bmatrix}$$

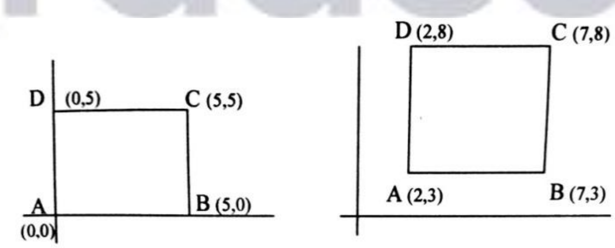
Thus, $A'(x'_1, y'_1) = (2, 3)$

$B'(x'_2, y'_2) = (7, 3)$

$C'(x'_3, y'_3) = (7, 8)$

$D'(x'_4, y'_4) = (2, 8)$

The graphical representation is given below:



a) Square before Translation   b) Square after Translation

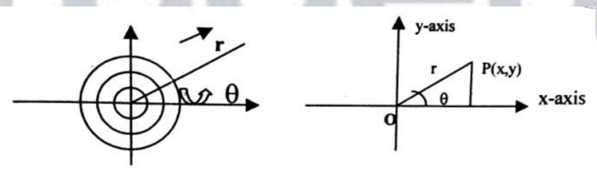
ROTATION

In 2-D rotation, an object is rotated by an angle θ with respect to the origin. This angle is supposed to be positive for anticlockwise rotation. There are two cases for 2-D rotation, case 1- rotation about the origin and case 2 rotation about an arbitrary point. If, the rotation is made about an arbitrary point, a set of basic transformation, i.e., composite transformation is required. For 3-D rotation involving 3-D objects, we need to specify both the angle of rotation and the axis of rotation, about which rotation ahs to be made. We will consider case 1 and in the next section we will consider case 2.

Before starting case-1or case-2 you must know the relationship between **polar coordinate system** and **Cartesian system:**

Relation between **polar coordinate system** and **Cartesian system**

A frequently used non-cartesian system is Polar coordinate system. The following below figure shows a polar coordinate reference frame. In polar coordinate system a coordinate position is specified by r and θ, where r is a radial distance from the coordinate origin and θ is an angular displacements from the horizontal (see below figure). Positive angular displacements are counter clockwise. Angle θ is measured in degrees. One complete counter-clockwise revolution about the origin is treated as 360^0. A relation between Catesian and polar coordinate system is shown in below figure.



Consider a right angle triangle in above figure. Using the definition of trigonometric functions, we transform polar coordinates to Cartesian coordinates as:

$x = r.\cos\theta$

$y = r.\sin\theta$

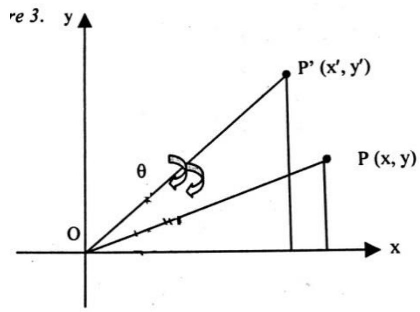
The inverse transformation from Cartesian to Polar coordinates is:

$r = \sqrt{(x^2+y^2)}$ and $\theta = \tan^{-1}(y/x)$

Case 1: Rotation about the origin

Given a 2-D point P(x, y), which we want to rotate, with respect to the origin O. the vector OP has a length 'r' and making a positive (anticlockwise) angle ϕ with respect to x-axis.

Let P'(x'y') be the result of rotation of point P by an angle ϕ about the origin, which is shown in below figure.



$P(x, y) = P(r.\cos\phi, r.\sin\phi)$

$P'(x', y') = P[r.\cos(\phi+\theta), r.\sin(\phi+\theta)]$

The coordinates of P' are:

$x' = r\sin(\theta+\phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi)$

$= x. \cos\theta - y.\sin\theta$ (where $x = r\cos\phi$ and $y = r\sin\phi$)

Similarly,

$y' = r\sin(\theta+\phi) = r(\sin\theta\cos\phi + \cos\theta.\sin\phi)$

$= x\sin\theta + y\cos\theta$

Thus,

$$R_\theta = \left\{ \begin{array}{l} x' = x.\cos\theta - y.\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{array} \right\} = R_\theta$$

Thus, we have obtained the new coordinate of point P after the rotation. In matrix from, the transformation relation between P' and P is given by:

$$(x'y')=(x,y)\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

This is P' = P.R_θ ....(5)

Where P' and P acts as object points in 2-D Euclidean system and R_θ is transformation matrix for **anti-clockwise** Rotation.

In terms of HCS, equation (5) becomes.

$$(x', y', 1) = (x, y, 1)\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \quad \text{---------(6)}$$

That is $P'_h = P_h.R_\theta$, ...(7)

Where P'_h and P_h acts as object points, after and before required transformation, in Homogeneous Coordinates and R_θ is called homogeneous transformation matrix for **anticlockwise** Rotation. P'_h, the new coordinates of a transformed object, can be found by multiplying previous object coordinate matrix, P_h, with the transformation matrix for Rotation R_θ.

Note for **clockwise** rotation we have to put $\theta = -\theta$, thus the rotation matrix R_θ, in HCS, becomes.

$$R_{-\theta}=\begin{pmatrix}\cos(-\theta) & \sin(-\theta) & 0\\ -\sin(-\theta) & \cos(-\theta) & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

Example: Perform a 45^0 rotation of a triangle A(0, 0), B(1, 1), C(5, 2) about the origin.

Solution: We can act for the given triangle, in matrix form, using homogeneous coordinates of the vertices:

$$[ABC]=\begin{bmatrix} A & 0 & 0 & 1 \\ B & 1 & 1 & 1 \\ C & 5 & 2 & 1 \end{bmatrix}$$

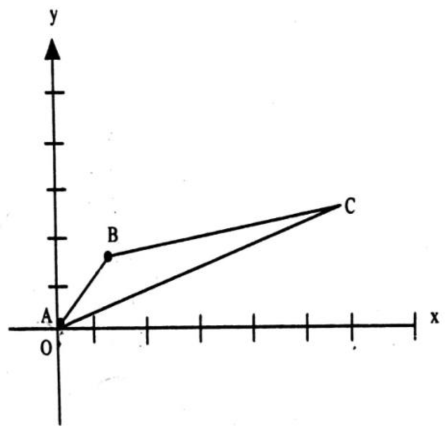
The matrix of rotation is: $R_\theta = R_{45^0} = \begin{bmatrix} \cos 45^0 & \sin 45^0 & 0 \\ -\sin 45^0 & \cos 45^0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, the new coordinates A, B, C, of the rotation triangle ABC can be found as:

$$[A'B'C']=[ABC].\,R_{45^\circ} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2/2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 1 \\ 3\sqrt{2}/2 & 7\sqrt{2}/2 & 1 \end{bmatrix}$$

Thus, A’ = (0, 0), B’ = (0, √2), C’ = (3√2/2, 7√2/2)

The following below figure shows the original, triangle [ABC] and figure shows triangle after the rotation.



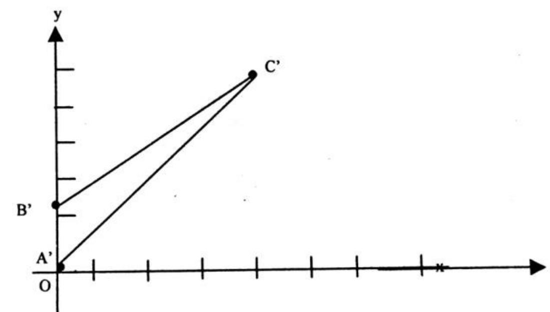


Figure (b)

SCALING

It is the process of expanding or compressing the dimensions (i.e., size) of an object. Important programing of scaling is in the development of viewing transformation, which is a mapping from a window used to clip the scene to a view port for displaying the clipped scene on the screen.

Let P(x, y) by any point of a given object and s_x and s_y by scalling factors in x and y directions respectively, then the coordinate of the scaled object can be obtained as:

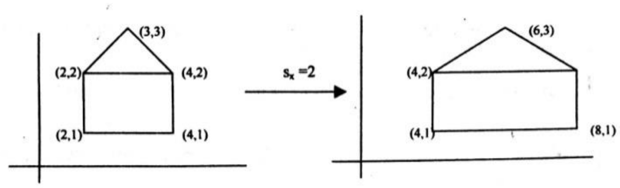
$x' = x.s_x$ ....(8)

$y' = y.\ s_y$

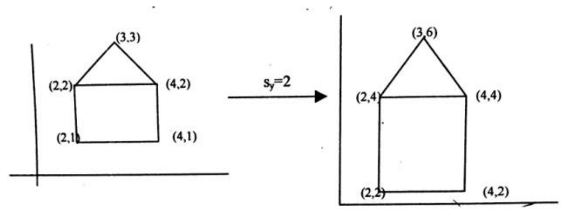
if the scale factor is $0 < s < 1$, then it reduces the size of an object and if it is more than 1, it magnifies the size of the object along an axis.

For example, assume $s_x > 1$.

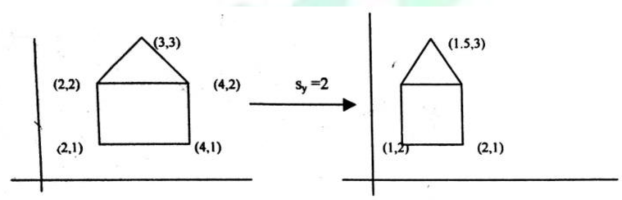
i. Consider $(x, y) \rightarrow (2.x, y)$ i.e., Magnification in x-direction with scale factor $s_x = 2$.



ii. Similarly, assume $s_y > 1$ and consider $(x, y) \rightarrow (x, 2.y)$, i.e. Magnification in y - direction with scale factor $s_y = 2$.



iii. Consider $(x, y) \rightarrow (x.s_x, y)$ where $0 < s_x = y_2 < 1$ i.e., Compression in x-direction with scale factor $s_x = ½$.



The general scaling is $(x, y) \rightarrow (x.s_x, y.s_y)$ i.e., magnifying or compression in both x and y directions depending on Scale factors s_x and s_y. We can act this in matrix form (2-D Euclidean system) as:

$$(x', y', 1) = (x, y, 1) \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{----(10)}$$

In terms of HCS, equation (9) becomes:

$$(x', y', 1) = (x, y, 1) \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{----(10)}$$

That is $P'_h = P_h.s_{sx,sy}$ ....(11)

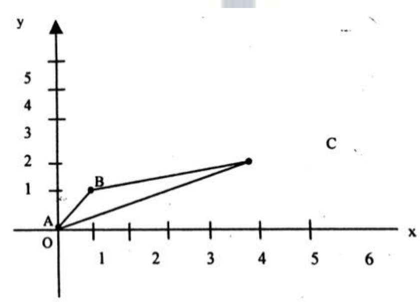
Example: Find the new coordinates of a triangle A(0, 0), B(1, 1), C(5, 2) after it has been (a) magnified to twice its size and (b) reduced to half its size.

Solution: Magnification and reduction can be attained by a uniform scaling of s units in both the x and y directions. If, s>1, the scaling produces magnification. If, s<1, the result is a reduction. The transformation can be written as: (x,y,1) → (s,x,s.y,1). In matrix form this becomes.

$$(x,y,1).\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} = (s.x,s.y,1)$$

We can represent the given triangle, shown in below figure, in matrix form, using homogeneous coordinates of the vertices as:

$$\begin{bmatrix} A & 0 & 0 & 1 \\ B & 1 & 1 & 1 \\ C & 5 & 2 & 1 \end{bmatrix}$$



(a) choosing s = 2

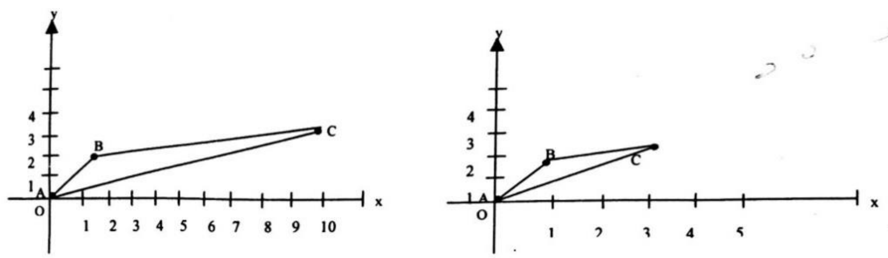
$$\text{The matrix of scaling is:} S_{s_x,s_y} = S_{2,2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the new coordinates A' B' C' of the scaled triangle ABC can be found as:

$$[A'B'C'] = [ABC].\ R_{2,2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \\ 10 & 4 & 1 \end{bmatrix}$$

Thus, A' = (0, 0), B' = (2, 2), C' = (10, 4)

(b) Similarly, here, s = ½ and the new coordinates are A''=(0, 0), B'' = (1/2, 1/2), C'' = (5/2,1). The following figure (b) shows the effect of scaling with $s_k = s_y = 2$ and (c) with $s_x = s_y = s = 1/2$.



SHEARING

They are used for modifying the shapes of 2-D or 3-D objects. The effect of a shear transformation looks like "pushing" a geometric object in a direction that is parallel to a coordinate plane (3D) or a coordinate axis (2D). How far a direction is pushed is determined by its shearing factor.

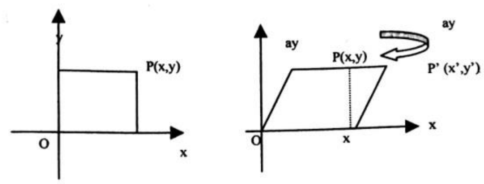
One familiar example of shear is that observed when the top of a book is moved relative to the bottom which is fixed on the table.

In case of 2-D shearing, we have two types namely x-shear and y-shear.

In x-shear, one can push in the x-direction, positive or negative, and keep the y-direction unchanged, while in y-shear, one can push in the y-direction and keep the x-direction fixed.

X-SHEAR ABOUT THE ORIGIN

Let an object point P(x, y) be moved to P'(x', y') in the x-direction, by the given scale parameter 'a', i.e., P'(x', y') be the result of x-shear of point P(x, y) by scale factor a about the origin, which is shown in below figure.



Thus, the point P(x, y) and P'(x', y') have the following relationship:

$$\left.\begin{aligned} x' &= x + ay \\ y' &= y \end{aligned}\right\} = Sh_x(a) \qquad \text{..........(11a)}$$

where 'a' is a constant (known as shear parameter) that measures the degree of shearing. If a is negative then the shearing is in the opposite direction.

Note that P(0, H) is taken into P'(aH, H). It follows that the shearing angle A (the angle through which the vertical adge was sheared) is given by:

Tan(A) = aH/H = a

So the parameter a is just the tan of the shearing angle. In matrix form (2-D Euclidean system), we have

$$(x',y')=(x,y)\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \qquad \text{-------(12)}$$

In terms of Homogeneous Coordinates, equation (12) becomes

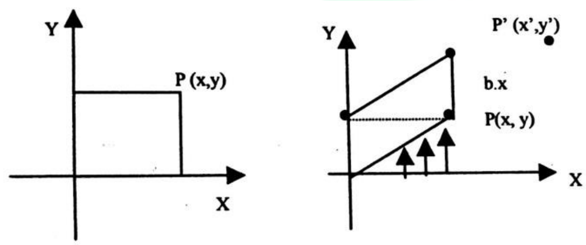
$$(x',y',1)=(x,y,1).\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{-------(13)}$$

That is, $P'_h = P_h SH_x(a)$ ....(14)

Where P_h and P'_h represents object points, before and after required transformation, in Homogeneous Coordinates and $Sh_x(a)$ is called homogeneous transformation matrix for x-shear with scale parameter 'a' in the x-direction.

Y-SHEAR ABOUT THE ORIGIN

Let an object P(x, y) be moved to P'(x', y') in the x-direction, by the given scale parameter 'b'. i.e., P'(x', y') be the result of y-shear of point P(x, y) by scale factor 'b' about the origin, which is shown in below figure.



Thus, the points P(x, y) and P'(x', y') have the following relationship:

$$\left.\begin{aligned} x' &= x \\ y' &= y + bx \end{aligned}\right\} = Sh_y(b) \quad \text{..........(15)}$$

where 'b' is a constant (known as shear parameter) that measures the degree of shearing. In matrix form, we have

$$(x',y')=(x,y)\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \qquad \text{----------(16)}$$

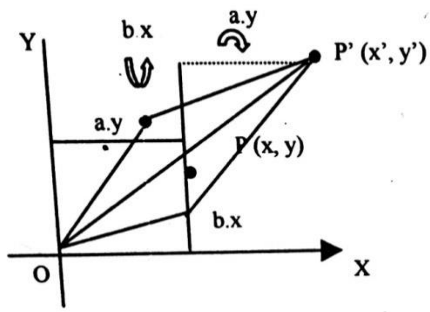
In terms of Homogeneous Coordinates, equation (16) becomes.

$$(x',y',1)=(x,y,1)\begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{---------(17)}$$

That is, $P'_h = P_h.Sh_y(b)$ ....(18)

XY-SHEARS ABOUT THE ORIGIN

Let an object point P(x, y) be moved to P'(x', y') as a result of shear transformation in both x and y directions with shearing factors a and b, respectively, as shown in below figure.



The points P(x, y) and P'(x', y') have the following relationship:

$$\left.\begin{aligned} x' &= x + ay \\ y' &= y + bx \end{aligned}\right\} = Sh_{xy}(a, b) \quad \text{..........(19)}$$

where 'ay' and 'bx' are shear factors in x and y directions, respectively. The xy-shear is also called simultaneous shearing for short.

In matrix form, we have,

$$(x', y') = (x, y)\begin{bmatrix} 1 & b \\ a & 1 \end{bmatrix} \quad \text{---------(20)}$$

In terms of Homogeneous Coordinates, we have

$$(x', y', 1) = (x, y, 1)\begin{bmatrix} 1 & b & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{---------(21)}$$

That is, $P'_h = P_h.Sh_{xy}(a, b)$ ....(22)

Example: A square ABCD is given with vertices A(0, 0), B(1, 0), C(1, 1), and D(0, 1). Illustrated the effect of a) x-shear b) y-shear and c) xy-shear on the given square, when a = 2 an b = 3

Solution: We can represent the given square ABCD, in matrix form, using homogeneous coordinates of vertices as:

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

a) The matrix of x-shear is:

$$Sh_x(a) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, the new coordinates A'B'C'D' of the x-shear object ABCD can be found as: [A'B'C'D'] = [ABCD]. $Sh_x(a)$

$$[A'B'C'D'] = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Thus, A' = (0, 0), B' = (1, 0), C' = (3, 1) and D' = (2, 1).

b) Similarly the effect shearing in the y direction can be found as: [A'B'C'D'] = [ABCD].$Sh_y(b)$

$$[A'B'C'D'] = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Thus, A' = (0, 0), B' = (1, 3), C' (1, 4) and D' = (0, 1)

c) Finally the effect of shearing in both directions can be found as: [A'B'C'D] = [ABCD].Sh_{xy}(a, b)

$$[A'B'C'D'] = \begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ 3 & 4 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

thus, A' = (0, 0), B' = (1, 3), C' = (3, 4) and D' = (2, 1).

Figure (a) shows the original square, figure (b)-(d) shows shearing in the x, y and both directions respectively.

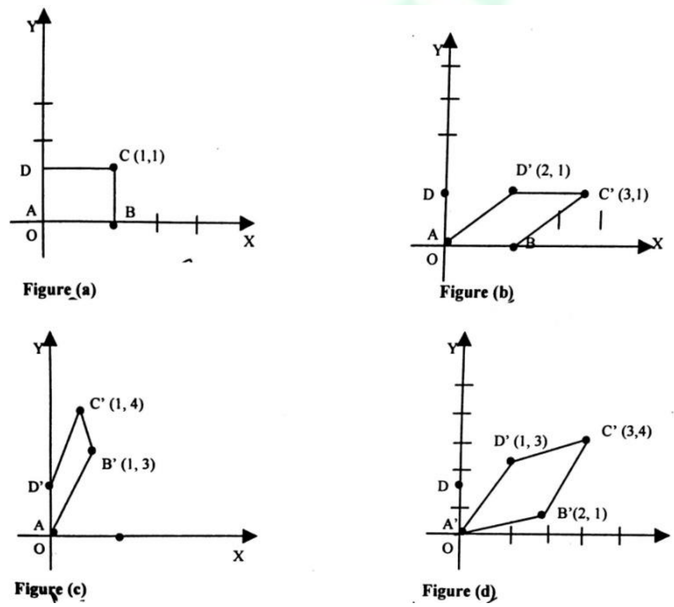


Figure (a) Figure (b) Figure (c) Figure (d)