Mathematical Logic Part-2

Mathematical Logic Part-2

Derived connectives

1. **NAND:** It means negation of conjunction of two statements. Assume p and q be two propositions. N and of p and q is a proposition which is false when both p and q are true otherwise true. It is denoted by $p \uparrow q$ and $p \uparrow q \equiv \sim(p \wedge q)$

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

2. **NOR**: it means negation of disjunction of two statements. Assume p and q be two propositions. Nor of p and q is a proposition which is true when both p and q are false, otherwise false. It is denoted by $p \downarrow q$ and $p \downarrow q \equiv \sim(p \vee q)$

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Note that

(i) $\sim p \equiv p \downarrow p$

(ii) $p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$

(iii) $p \vee q \equiv (p \downarrow q) \downarrow (p \downarrow q)$

3. **XOR (exclusive OR):** assume p and q be two proposition. The exclusive or (XOR) of p and q, denoted by $p \oplus q$ is the proposition that is true when exactly one of p and q is true but not both and is false otherwise

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Properties of exclusive OR

(i) $p \oplus q \equiv q \oplus p$ (Commutative)

(ii) $(p \oplus q) \oplus r \equiv p \oplus (p \oplus r)$ (Associative)

(iii) $p \wedge (q \oplus r) \equiv (p \wedge q) \oplus (p \wedge r)$ (Distributive)

Tautologies and contradictions

A compound proposition that is always true for all possible truth values of its variables or, in other words contain only T in the last column of its truth table is called tautology. A compound proposition that is always false for all possible values of its variables or, in other words contain only F in the last column of its truth table is called a contradiction. Finally a proposition that is neither a tautology nor a contradiction is called a contingency.

Example: prove that the following propositions are tautology
P v ~ p
Solution: the truth table of the given proposition is shown below. Since the truth value is TRUE for all possible values of the propositional variables which can be seen in the last column of the table, the given proposition is tautology.

P	~P	P v ~P
T	F	T
F	T	T

Some important implications which can be proved by truth tables are given below:

1. $p \Rightarrow p \vee q$
2. $\sim p \Rightarrow p \rightarrow q$
3. $q \Rightarrow p \rightarrow q$
4. $\sim(p \rightarrow q) \Rightarrow p$
5. $\sim(p \rightarrow q) \Rightarrow \sim q$
6. $p \wedge (p \rightarrow q) \Rightarrow q$
7. $\sim q \wedge (p \rightarrow q) \Rightarrow \sim p$
8. $\sim p \wedge (p \vee q) \Rightarrow q$
9. $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$
10. $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$

Well formed formula

A statement is an expression which is a string consisting of variables, parentheses and connective symbols. A grammatically correct expression is called a well formed formula, which is abbreviated wff and can be pronounced as, “woof”. A well formed formula can be generated by the following rules:

1. All variables and constants are well-formed formula.
2. if P is a well-formed formula then P is also a well- formed formula.
3. A statement formula consists of variables, parentheses and connectives is recursively well formed formula if it can be obtained by finitely applying the above rules.

Functionally complete set of connectives

Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives from the set is called a functionally complete set of connectives. It does not contain a connective that can be expressed in term of the other connectives of the set is called a minimal functionality complete set of connectives.

We know the following logical equivalence:

$p \rightarrow q \equiv \sim p \vee q$

there are two ways to express the bi-conditional:

$p \rightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

$p \rightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

thus all the conditional and bi-conditional can be replaced by three connectives $\wedge$, $\vee$, $\sim$.

De morgan's law we have

$P \wedge q \equiv \sim(\sim p \vee \sim q)$

$P \vee q \equiv \sim(\sim p \wedge \sim q)$

The first equivalence means that it is also possible to obtain a formula which is equivalent to given formula in which conjunction is eliminated. A similar procedure is possible for the elimination thus, in any formula, we can replace first all the bi-conditionals, then the conditionals, finally all the conjunctions or all the disjunctions to obtain an equivalent formula. This formula will contain either the negation and disjunction or the negation and conjunction. Thus the set of connectives $(\sim, \wedge)$, $(\sim, \vee)$are minimal functionality complete set.

Normal forms

By comparing truth tables, one determine whether two logical expressions P and Q are equivalent. But the process is very tedious when the number of variables increases. A better method is to transform the expressions P and Q to some standard forms of expression P' and Q' such that a simple comparison of P' and Q' shows whether P=Q, the standard forms are called normal forms or canonical forms. There are two types of normal form: disjunctive normal form and conjunctive normal form.

Disjunctive normal form

In an logical expression, a product of the variables and their negations is called an elementary product. For example, $p \wedge \sim q$, $\sim p \wedge \sim q$, $\sim p \wedge q$ are some elementary product in two variables. A sum of the variables and their negations is called an elementary sum. For example, $p \vee q$, $p \vee \sim q$, $\sim p \vee \sim q$ are elementary sums in two variables. A part of the elementary sum of product which is itself an one elementary sum of product is called a factor of the original sum of product. The elementary sums or products satisfy the following properties. We only state them without proof.

1. An elementary product is identically false if and only it contains at least one pair of factor in which one is negation of the other.
2. An elementary sum is identically true if and only if it contains at least one pair of factors in which one is the negation of the other.

A logical expression is said to be in disjunctive normal form if it is the sum of elementary products. For example, p v(q^ r) and p v (~q ^ r) are in disjunctive normal form, p ^ (q v r) is not in disjunctive normal form.
Procedure to obtain a disjunctive normal form of a given logical expression
Three steps are required to obtain a disjunctive normal form through algebraic manipulations.

1. Remove all→ and → by an equivalent expression containing the connectives ^, v, ~ only.
2. Eliminate ~ before sums and products by using the double negation or by using De-Morgan's laws.
3. Apply the distributive law until a sum of elementary product is obtained.

Example: obtain the disjunctive normal forms of the followings:

P v (~p → (q v(q → ~r)))

Solution:

$$
\begin{aligned}
(b)\ p \vee (\sim p \to (q \vee (q \to \sim r))) &\equiv p \vee (\sim p \to (q \vee (\sim q \vee \sim r))) \\
&\equiv p \vee (p \vee (q \vee (\sim q \vee \sim r))) \\
&\equiv p \vee p \vee q \vee \sim q \vee \sim r \\
&\equiv p \vee q \vee \sim q \vee \sim r
\end{aligned}
$$

Example: obtain a conjunctive normal form of the followings:
P ^ (p →q)

Solution:

$$p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$$

Principal disjunctive normal form

Let p and q be two statement variables, then p ^ q, p ^ ~ q, ~p ^ q and ~p ^ ~ q are called min terms of p and q. it may be noted that none of the min terms should contain both a variable and its negation. For given two variables, there are 2^2 min terms. The number of min terms in n variable is 2^2 for example, min terms for the three variables p, q and r are :

$$p \wedge q \wedge r, p \wedge q \wedge \sim r, p \wedge \sim q \wedge r, p \wedge \sim q \wedge \sim r,$$
$$\sim p \wedge q \wedge r, \sim p \wedge q \wedge \sim r, \sim p \wedge \sim q \wedge r, \sim p \wedge \sim q \wedge \sim r$$

The truth table of the min terms of p and q are given below.

p	q	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$\sim p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

Note that each min term has the truth value T for exactly one combination of the truth values of the variables p and q. also no two min terms are equivalent.

Principal disjunctives normal form of a given formula can be defined as an equivalent formula consisting of disjunctives of min terms only. This is also called the sum of products canonical form.

The process for obtaining principal disjunctive norm is discussed below.

By truth value:

1. Construct a truth table of the given compound propositions.
2. For every truth value T of the given proposition, select the min term, which also has the value T for the same combination of the truth value of the statement variables.
3. The disjunctive of the min terms selected in 2 is the required principal disjunctive normal form.

Without constructing truth table.

1. Obtain a disjunctive normal form.
2. Drop elementary products which are contradictions
3. If p_I and $\sim p_i$ are missing in an elementary products α, replace α by($\alpha \wedge p_i$) v ($\alpha \wedge \sim p_i$)
4. Repeat step 3 until all elementary products are reduced to sum of min terms. Identical min terms appearing in the disjunction are deleted.

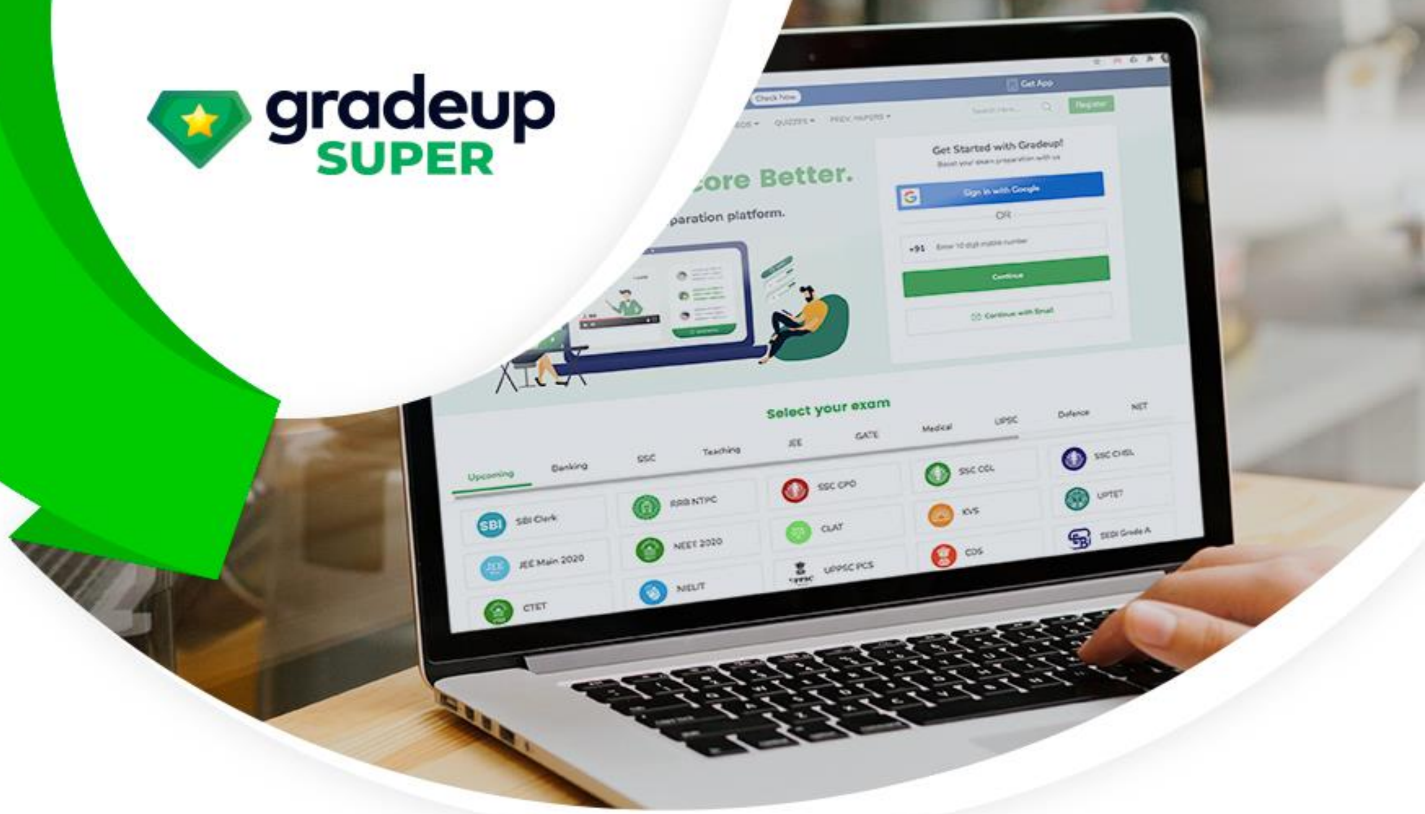
The advantages of obtaining principal disjunctive normal of a given formula is unique:

1. The principal disjunctive normal of a given formula is unique.
2. Two formulas are equivalent if and only if their principal disjunctive normal forms coincide.
3. If the given compound proposition is a tautology, then its principal disjunctive normal form will contain all possible min terms of its components.

Example: obtain the principal disjunctive normal form of P →q
Solution:
The truth table of p → q is
given below:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The column containing p → q has truth value T for three combination of the truth values of p and q. now T in the first row of p → q corresponds to min term $p \wedge q$, T in the third row of p → q corresponds to min term $\sim p \wedge q$ and T in the fourth row of p → q corresponds to min terms $\sim p \wedge \sim q$.


Select your exam