Fuzzy Sets

Fuzzy Set

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A well defined collection of distinct objects is called a set. An object in a set is called an element or member of that set. A set A is well described by a function called characteristic function. If X be a universal set then the set A can be represented for all element $x \in X$ by its characteristic function.

$$\chi_c(x) = \begin{cases} \mathbf{1 \text{ if } x \in X} \\ \mathbf{0 \text{ otherwise}} \end{cases}$$

Thus in classical set theory , X_c (x) has only the value 0 (false) and 1(true). Such sets are called **crisp sets**

Consider the set S as

$S = \{ x \in R : x \geq 6 \}$

This representation implies that if $x \geq 6$, then x is a member of S; otherwise x is not a member of S. thus there is a clear distinction of the elements 'belonging' and not belonging to set ' or equivalently the transition from 'belonging' to 'not belong' to the set is abrupt.

Although classical sets are important tool into several areas of study in mathematic and computer science but they do not reflect the nature of human perceptions and thoughts which generally tend to be abstract and imprecise. For example, in the above representation if we consider the set of all students in a college whose height is more than or equal to 6ft. Then we would not classify a student whose height is 5.99ft as tall student. This distinction is intuitively unreasonable

Membership Function: In fuzzy set theory , the characteristics function is defined on a set F is generalized to a membership function that assign to $x \in X$, a value from the unit [0,1] instead of the two element set {0,1}. The membership function μ_F of a fuzzy set F is a function.

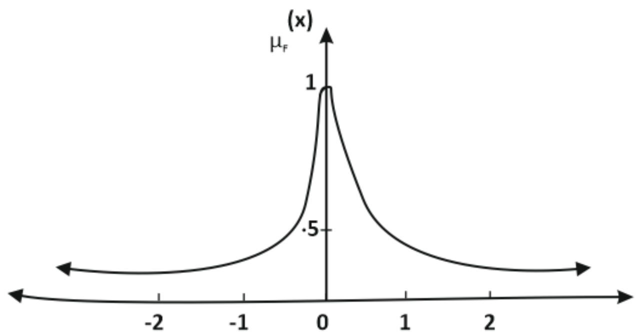
$\mu_F : X \rightarrow (0,1]$

SO, every element x from X has a membership degree $\mu_F(x) \in [0,1]$. F is completely determined by
the set of tuples

F={x, $\mu_F(x) | x \in X$}

Thus a fuzzy set is any set that allows its members to have different degree of membership , called membership function in the interval [0,1].

The graph of the function is shown in figure below. We can determine the membership grade of each real number in this fuzzy set , which signifies the degree to which that number is close to 0. For instance, the number 3 is assigned a grade of 0.1 , the number 1 a grade of 0.9 , and the number of 0 a grade of 1.



If X = {x_1, x_2,……..x_n}

Then a fuzzy set A of X could be written as

A={(x_1, $\mu_A(x_1)$), (x_2, $\mu_A(x_2)$)…..,(x_1, $\mu_A(x_n)$)}

Which sometimes is written in following way:

$$A = \left\{ \frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, ... \frac{\mu_A(x_n)}{x_n} \right\}$$

For example , the real numbers 3,1 and 0 with membership grades 0.1 , 0.09 and I can be written as:

A = {(3,0.1), (1,0.9), (0,1)}

A = $\{\frac{0.1}{3}, \frac{0.9}{1}, \frac{1}{0}, \}$

The problem of constructing meaning membership function in various contexts is a difficult task. A lot of research work is in progress in this area.

Some useful definitions:

Containment

Let X be a set($\neq \emptyset$) and A , B are two fuzzy sets of X with membership function μ_A (x) and μ_B (x) respectively . We say that the fuzzy set A is contained in the fuzzy set B if and only if

μ_A (x) $\leq \mu_B$ (x) for all x $\in$ X

We may say in another terminology , that A is a fuzzy subset of B , denoted by A $\subseteq$B

Normal Fuzzy Set: A fuzzy set A of a set X is called a normal fuzzy set if and only if

Max μ_A (x) = 1

x$\in$X

i.e. μ_A (x) =1 for at least one x$\in$X

Otherwise A is subnormal

For example If X = {1,2,3} and A ={1,2,3,4} and A = $\{\frac{1}{1},\frac{1}{2},\frac{2}{3},\frac{5}{4}\}$

Then A is a normal fuzzy set

The **height** of a fuzzy set A on the universal set X is the largest membership grade attained by any element of A i.e.

height(A) = max { μ_A (x) } x $\in$ X

The core of a fuzzy set A on the universal set X is the crisp set that contains only those elements of X for which μ_A (x) =1 i.e.

Core(A) = {x $\in$ X: μ_A (x) = 1}

A fuzzy set A on the universe X is said to be convex if for any three elements x,y,z $\in$ A and

Where x<y<z , we have

μ_A (y)$\geq$min{ μ_A (x) , μ_A (z) }

A fuzzy set A defined on real line R is said to be convex if for all x, y $\in$ A and $\lambda\in$ [0,1]

We have

μ_A [λ+ (1- λ)y] $\geq$ min{ μ_A (x) , μ_A (z) }

Support of a Fuzzy set

The support of a fuzzy set A of a set X is the crisp set that contains all the elements of X that have a non zero membership grade in A and denoted by sup(A) i.e.

Supp(A) = {x $\in$ X: μ_A (x) > 0}

The element x in X at which μ_A (x) = 0.5 is called the crossover point.

Example: Let X ={1,2,3,4,5,6,7,8,9,10} and fuzzy set A of X is given by

A ={(1,0), (2,0), (3,0.2), (4,0.5) ,(5,0.3),(6,0.4) ,(7,0) ,(8,0) (9,0) }

= {(3,0.2), (4,0.5) ,(5,0.3),(6,0.4)} where $\mu_{A(}$ (x) > 0

Thus supp(A) = {3,4,5,6} and x = 4 is the crossover point.

A special notation is often used for defining fuzzy sets with a finite support. Assume that x_i is an element of the support of fuzzy set A and that μ_i is its grade of membership in A. Then A is written as A = μ_i | x_1 + μ_2 | x_2 + μ_3 | x_3 .... μ_n | x_n

Where slash is used to link the elements of the support with their grades of membership in A and plus sign indicates , rather than any sort of algebraic addition that the listed pairs of elements and membership grades collectively from the definition of set A. for the case in which a fuzzy set A is defined on a universal set that is finite and countable, one may write

A = $\sum$ μ/x_i

α -cut or α - level set

The α - level set is the crisp set of elements that belong to the fuzzy set A at least to the degree α . In mathematical terms,

Aα = { x $\in$X: (x) $\geq$ α }

Where α is a real no such that 0 $\leq$ α $\leq$ 1

The strong α - level set or strong α cut is defined as

A^+_a = {x $\in$X: (x) > }

Example: suppose X = {1,2,3,4,5,6}. Consider the fuzzy set A of X given by A = {(1,0.2), (2,0.5),(3,0.7),(4,1),(5,0.8),(6,0.3)} .Then all possible α-level sets are:

$A_{0.2}$ = {1,2,3,4,5,6} $A_{0.3}$ = {2,3,4,5,6} $A_{0.5}$ = {2,3,4,5}

$A_{0.7}$ = {3,4,5} $A_{0.8}$ = {3,4} A_1 = {4}

Clearly A_0 = X and Aα = $\emptyset$ for all α > 1

Basic Operation On Fuzzy Sets:

Equal Sets:

Two fuzzy sets A and B are equal if

$\mu_A(x) = \mu_B(x)$ for all $x \in X$

and is written as A=B . If for atleast one $x \in X$, then A and B are said to be unequal and written $A \neq B$.

For example, if A = {(x_1,0.3), {(x_2,0.5) } B = {(x_1,0.2), {(x_2,0.5)} C = {{(x_1,0.3), {(x_2,0.5)}

Then $A \neq B$ since $\mu_A(x_1) \neq B\ \mu_B(x_1)$ through $\mu_A(x_2) = \mu_B(x_2)$

But A =C since $\mu_A(x_1) = \mu_B(x_1) = 0.3$ $\mu_A(x_2) = \mu_B(x_2) = 0.5$

Complement :

The complement of fuzzy set A is denoted by A^c (or A') and is defined by its membership as $\mu^c{}_A(x) = 1 - \mu_B(x)$ for all x.

Thus if an element has a membership grade of 0.7 in a fuzzy set A, its membership grade in the complement if A will be 0.3

Example: If A = {(x_1,0) , (x_2,0.3), (x_3,0.5)}

Then A^c = {(x_1,1) , (x_2,0.7), (x_3,0.5)}

Since $\mu^c{}_A(x_1) = 1 - \mu_A(x) = 1-0 = 1$; $\mu^c{}_A(x_2) = 1 - \mu^c{}_A(x_2) = 1-0.3 = 0.7$

and $\mu^c{}_A(x_3) = 1 - \mu_A(x_3) = 1 - 0.5 = 0.5$

Union:

The union of two fuzzy sets A and B is a fuzzy set C given by

C = AUB

Where $\mu_A(x) = \max[\mu_A(x) , \mu_B(x)]$; $x \in X$

Example: If A = {(4,0.1),(6,0.5),(8,0.6),(10,0.7) }

B = {(4,0.2),(6,1),(8,0.4),(10,0.5)}

Then C = AUB = {(4,0.2),(6,1),(8,0.6),(10,0.7)}

Since $\mu_C(x_1) = \max[\mu_A(x_1) , \mu_B(x_1)] = \max[0.1,0.2] = 0.2$

$\mu_C(x_2) = \max[\mu_A(x_2) , \mu_B(x_2)] = \max[0.5,1] = 1$

$\mu_C(x_3) = \max[\mu_A(x_3) , \mu_B(x_3)] = \max[0.4,0.6] = 0.6$

$\mu_C(x_4) = \max[\mu_A(x_4) , \mu_B(x_4)] = \max[0.7 ,0.5] = 0.7$

Example: If X ={51,52,53,54}

A = 0.2/51 + 0.5/52 + 0.8/53 + 1/54

B = 1/51 + 0.8/52 + 0.5/53 + 0.2/54

And then what is the value of A U B ?

Solution: A U B = max[0.2,1]/51 + max[0.5,0.8]/52 + max[0.8,0.5]/53 + max [1,0.2]/54

= 1/51 + 0.8/52 + 0.8/53 + 1/54

= {(51,1),(52,0.8),(53,0.8),(54,1)}

Intersection: The intersection of two fuzzy sets A and B is a fuzzy set C given by :

C = A ∩ B

Where $\mu_A(x) = \min [\mu_A(x) , \mu_B(x)]$; $x \in X$

Example: if A ={(3,0.1), (5,0.7), (7,0.7)}

B = {(3,0.4), (5,0.8), (7,0.3)}

Then C = A ∩ B = {(3,0.1), (5,0.7), (7,0.3)}

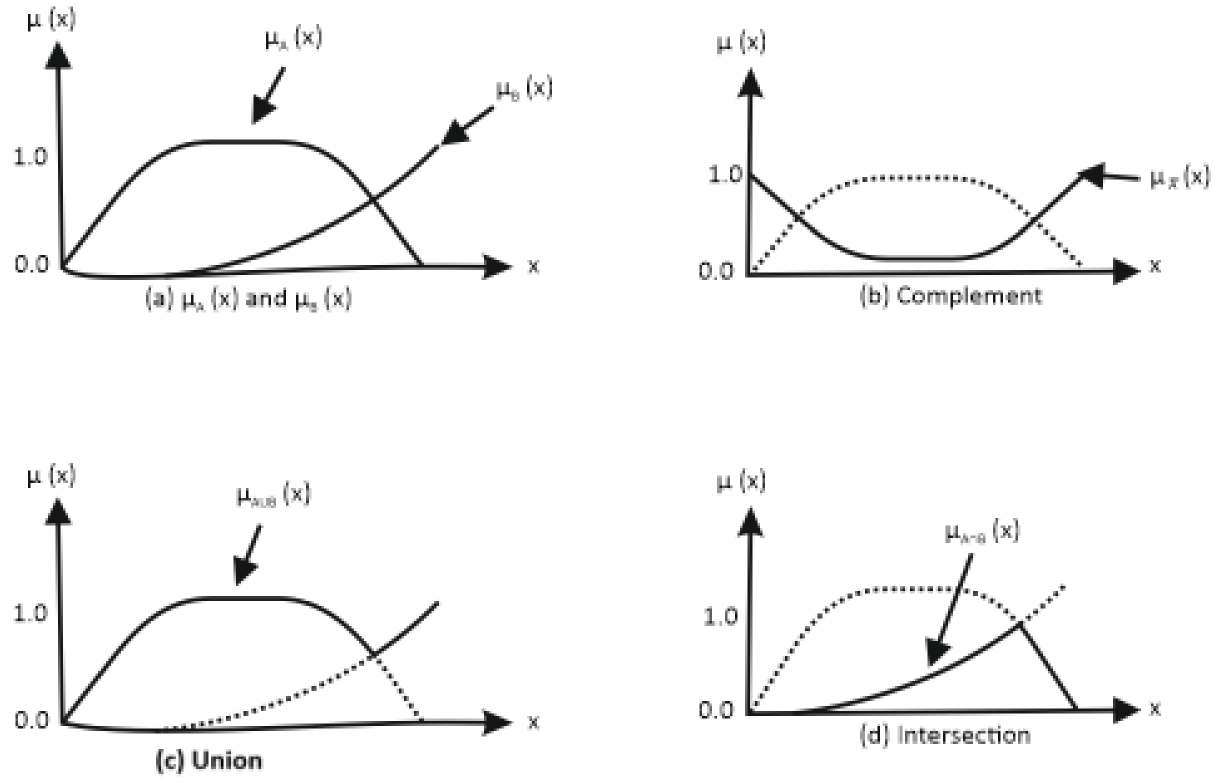
Since

$\mu_C(x_1) = \min [\mu_A(x_1) , \mu_B(x_1)] = \min[0.1 , 0.4] = 0.1$

$\mu_C(x_2) = \min [\mu_A(x_2) , \mu_B(x_2)] = \min[0.7 , 0.8] = 0.7$

$\mu_C(x_3) = \min [\mu_A(x_3) , \mu_B(x_3)] = \min[0.7 , 0.3] = 0.3$

These formulations of fuzzy complement , union and intersection perform identically to the corresponding crisp set operations when membership grades are restricted to the value 0 and 1. They are , therefore generation of the classical crisp set operations . Sometimes I is more convenient to give the graphs that represent the membership function as shown below.



Difference : The difference of two fuzzy sets A and B is defined by

A-B = A ∩ B^c

Example

If A = {(x_1,0.3) , (x_2,0.4), (x_3,0.5)}

B = {(x_1,0.2) , (x_2,0.6), (x_3,0.7)}

Then B^c = {(x_1,0.8) , (x_2,0.4), (x_3,0.3) }

Note that , except in particular cases A -B ≠ B-A

Properties of Fuzzy set operations:

The properties of fuzzy set operations that are common to crisp set operations are

Idempotent: AUA = A , A ∩A = A

Commutative: AUB = BUA , A ∩B = B ∩A

Associative: (AUB) U C = AU (BUC), (A∩B) ∩C = A∩(B∩C)

Distributive: A∩(BUC) = (A∩B) U (A∩C)

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Absorption: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

Identity: $A \cup \emptyset = A$, $A \cap U = A$

Double negation law(Involution):

If A is a fuzzy set then , $(A^c)^c = A$

De Morgan's Law:

If A and B are two fuzzy sets then

$(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$

Algebraic Sum: The algebraic sum of two fuzzy sets A and B is defined by the membership function as

$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\,\mu_B(x)$ for all $x \in X$

and written as A +B

Algebraic Product: The algebraic product of two fuzzy sets A and B is defined by two membership functions as

$\mu_{A.B}(x) = \mu_A(x) \,.\, \mu_B(x)$ for all $x \in X$

and written as A.B

in particular $\mu_{A.A}(x) = \mu_A{}^2(x) = [\mu_A(x)]^2$ for all $x \in X$

Example. If A = {(1,0.5), (2, 1), (3, 0.6)}

B = {(1.1) , (2, 0.6)}

Then A + B = {(1,1), (2,1), (3, 0.6)}

and A.B = {(1, 0.5),(2, 0.6), (3,0)}

Fuzzy Cartesian Product

Let $A_1, A_2, A_3, \ldots\ldots\ldots\ldots A_n$ be set in X_1 **,** X_2 **,** X_3 **……………,** X_n respectively. The Cartesian product of $A_1, A_2, A_3, \ldots\ldots\ldots\ldots A_n$ is a fuzzy set in the product space X_1 **x** X_2 **x** X_3 **……………x** X_n with the membership function as

$\mu_{A_1 \times A_2 \times \ldots \times A_n}(x_1, x_2, \ldots\ldots\ldots x_n) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2) \ldots\ldots \mu_{A_n}(x_n))$

where $x_1 \in X_1$, $x_2 \in X_2$ .... $x_n \in X_n$

Fuzzy Relation:

Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets X_1 , X_2 , X_3 ...., X_n where the n-tuple (x_1 , x_2 , x_3 ...., x_n) may have varying degrees of membership within relation. The membership values indicate the strength of the relation between the tuples.

Binary relation:

Any relation between two sets X and Y is known as binary relation . The domain of a binary fuzzy relation R(X,Y) is the fuzzy set domR(X,Y) with the membership function.

$\mu_{\text{dom R}}(x) = \max_{y \in Y} \mu_R(x, y)$ for each $x \in X$

The range of a binary fuzzy relation R(X,Y) is the fuzzy set ran R(X,Y) with the membership function

$\mu_{\text{ran R}}(y) = \max_{x \in X} \mu_R(x, y)$ for each $y \in Y$

The height of a fuzzy relation R is a number h(R) defined by

$h(R) = \max_{y \in Y} \max_{x \in X} \mu_R(x, y)$

That is , the largest membership grade attained by any pair(x,y) in R . If h(R) = 1 , the fuzzy relation R is called normal , otherwise it is called subnormal.

The inverse of a fuzzy relation R is defined by

$\mu_R^{-1}(y, x) = \mu_R(x, y)$ for all $(x, y) \in X \times Y$

$$M_R = R(X,Y) = \begin{bmatrix} \mu_R(x_1, y_1) & \mu_R(x_1, y_2) \ldots\ldots\ldots \mu_R(x_1, y_m) \\ \mu_R(x_2 y_1) & \mu_R(x_2, y_2) \ldots\ldots\ldots \mu_R(x_2, y_m) \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \mu_R(x_n y_1) & \mu_R(x_n, y_2) \ldots\ldots\ldots \mu_R(x_n, y_m) \end{bmatrix}$$

A fuzzy binary relation can be represented as a two dimensional membership array whose entries correspond to ordered pairs in the universal set. These entries take values representing the membership grades of the corresponding order pairs.

Let X = {x_1 , x_2 , ……………….,x_n) and Y = {y_1, y_2,…………………,y_m}

The fuzzy relation can be expressed by an n x m matrix(known as membership matrix) as

$$M_R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 5 & 0 \\ 4 & 3 \end{bmatrix} \end{array}$$

Example: Let R(X,Y) be a fuzzy relation on X = {x_1 **,** x_2 **,** x_3} and Y = and Y = {y_1 **,** y_2 **,**} and a fuzzy binary relation is given by membership matrix as

$$M_{R^{-1}} = M_R^T = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 5 & 4 \\ 2 & 0 & 3 \end{bmatrix} \end{array}$$

Then the inverse of R(X,Y) is given by

Operations on fuzzy relations:

Let R and S be two fuzzy relations on X x Y

Union of R and S is given by

$\mu_{R\cup S}$ (x, y) = max (μ_R (x ,y) , μ_S (x, y))

Insertion of R and S is given by

$\mu_{R\cap S}$ (x ,y) = min (μ_R(x ,y) , μ_S (x ,y))

Complement of R is given by

μ_R^C (x ,y) =1 - μ_R (x ,y)

Algebraic Products is given by

μ_{RS} (x ,y) = μ_R (x ,y). μ_S (x ,y)

Algebraic Sum is given by

μ_{R+S} (x ,y) = μ_R (x ,y) + μ_S (x ,y) - μ_R (x, y). μ_S (x ,y)

Composition of Relations: Suppose R is a fuzzy relation defined on X x Y and S is a fuzzy relation defined on YxZ, then RoS is a fuzzy relation on X x Z . The fuzzy max-min composition is defined as

μ_{RoS} (x ,z) =max(min(μ_R (x ,y), μ_S (y,z)

y ∈ Y

Example: Let R and S be two fuzzy relations from X to Y given in the following matrix forms . Find

$$M_R = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 3 & 1 & 2 \\ 8 & 0 & 5 \end{bmatrix} \end{array} \qquad M_S = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 6 & 1 & 9 \\ 0 & 2 & 3 \end{bmatrix} \end{array}$$

a. R U S

b. R∩ S

c. R + S

d. R.S

e. R^c

Solution:

$$(a) M_{R\cup S} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 6 & 1 & 9 \\ 8 & 2 & 5 \end{bmatrix} \end{array}$$

$$(b) M_{R\cap S} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \end{array}$$

$$(c) M_{R+S} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 72 & 1 & 92 \\ 8 & 2 & 65 \end{bmatrix} \end{array}$$

$$(d)\, M_{R \cdot S} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 18 & 1 & 18 \\ 0 & 0 & 15 \end{bmatrix} \end{array}$$

$$(e)\, M_{R^c} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 7 & 0 & 8 \\ 2 & 0 & 5 \end{bmatrix} \end{array}$$

Example: let X = $\{x_1, x_2\}$, Y = $\{y_1, y_2\}$ and Z = $\{z_1, z_2, z_3, z_4\}$. Let R and S be fuzzy relation X x Y and Y x Z respectively , given below in the matrix form as

$$M_R = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 3 & 8 & 1 \\ 9 & 7 & 4 \end{bmatrix} \end{array} ;\; M_S \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{bmatrix} 7 & 6 & 4 & 1 \\ 4 & 1 & 7 & 2 \\ 5 & 9 & 6 & 8 \end{bmatrix} \end{array}$$

Find the composition of R and S in the matrix form.

Solution: $\mu_{ROS}(x_1, z_1)$ = max (min ($\mu_R(x_1, y_1)$, $\mu_s(y_1, z_1)$), min ($\mu_R(x_1, y_2)$, $\mu_S(y_2, z_1)$)

min ($\mu_R(x_1, y_3)$, $\mu_s(y_3, z_1)$))

= max (min (3, 7), min (8, 4) min (1, 5))

= max (3, 4, 5)=5

$\mu_{ROS}(x_1, z_2)$=max (min ($\mu_R(x_1, y_1)$, $\mu_s(y_1, z_2)$), min ($\mu_R(x_1, y_2)$, $\mu_2(y_2, z_2)$), min ($\mu_R(x_1, y_2)$,($\mu_s(y_3,z_2)$))

=max (min (3, 6), min (8, 1), min (1, 9)

=max (3, 1, 9)=9

$\mu_{ROS}(x_1, z_3)$=max (min ($\mu_R(x_1, y_1)$, $\mu_s(y_1, z_3)$), min ($\mu_R(x_1, z_2)$, $\mu_z(y_2, z_3)$), min ($\mu_R(x_1, y_3)$,$\mu_s(y_3,z_3)$))

=max (min (3, 4), min (8, 7), min (1, 6)

=max (3, 7, 6) = 7

$\mu_{ROS}(x_1, z_4)$=max (min ($\mu_R(x_1, y_1)$, $\mu_s(y_1, z_4)$), min ($\mu_R(x_1, y_2)$, $\mu_s(y_2, z_4)$), min ($\mu_R(x_1, y_3)$,$\mu_s(y_3,z_4)$))

=max (min (3, 1), min (8, 2), min (1, 8)

=max (1, 2, 8) = 8

Similarly, $\mu_{ROS}=(x_2, z_1)=7$, $\mu_{ROS}(x_2, z_2)=6$

$\mu_{ROS}=(x_2, z_3)=7$, $\mu_{ROS}(x_2, z_4)=4$

$$M_{ROS} = \begin{array}{c c} & \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 5 & 9 & 7 & 8 \\ 7 & 6 & 7 & 4 \end{bmatrix} \end{array}$$