2-D Geometrical Transforms and Viewing Part-2

2-D Geometrical Transforms and Viewing Part-2

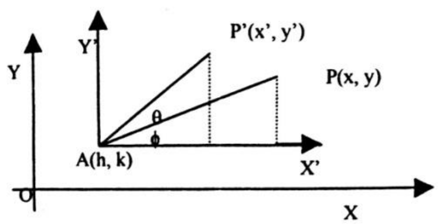
Content:

1. Rotation About A Point
2. Reflection About A Line

COMPOSITE TRANSFORMATIONS

ROTATION ABOUT A POINT

Given a 2-D point P(x, y), which we want to rotate, with respect to an arbitrary point A(h, k). Let P'(x' y') be the result of anticlockwise rotation of point P by angle θ about A, which is shown in below figure.



Since, the rotation matrix R_θ is defined only with respect to the origin, we need a set of basic transformations, which constitutes the composite transformation to compute the rotation about a given arbitrary point A, denoted by $R_{\theta,A}$. We can determine the transformation $R_{\theta,A}$ in three steps:

1. Translate the point A(h, k) to the origin O, so that the centre of rotation A is at the origin.
2. Perform the required rotation of θ degrees about the origin, and
3. Translate the origin back to the origin position A(h, k).

Using v=hI+kJ as the translation vector, we have the following sequence of three transformations:

$R_{\theta,A} = T_{-v}, R_\theta. T_v$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h & -k & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & k & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ (1-\cos\theta).h+k.\sin\theta & (1-\cos\theta).k-h.\sin\theta & 1 \end{pmatrix} \quad \text{-------(23)}$$

Example: Perform a 45^0 rotation of a triangle A(0, 0), B(1, 1), C(5, 2) about an arbitrary point P(-1, -1).

Solution: Given triangle ABC, as shown in below figure, can be represented in homogeneous coordinates of vertices as:

$$[ABC] = \begin{array}{c} A \\ B \\ C \end{array}\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{pmatrix}$$

From equation (23), a rotation matrix R_Q, A about a given arbitrary point A(h, k) is:

$$R_q, A = \begin{pmatrix} Cos\theta & Sin\theta & 0 \\ -Sin\theta & Cos\theta & 0 \\ (1-Cos\theta).h + k.Sin\theta & (1-Cos\theta).k - h.Sin\theta & 1 \end{pmatrix}$$

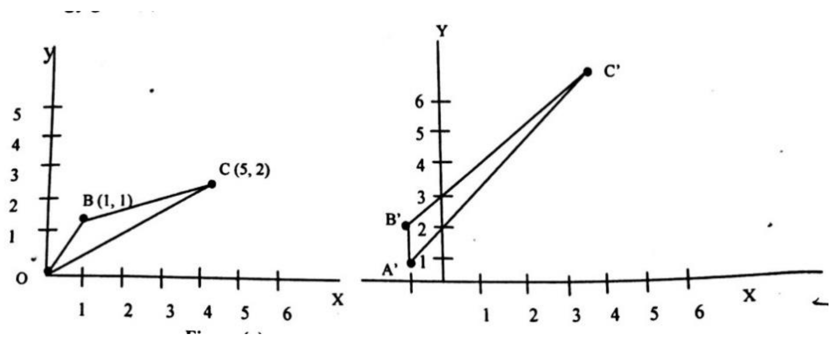
$$\text{Thus } R_{45^\circ}, A = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1 & (\sqrt{2}-1) & 1 \end{pmatrix}$$

So the new coordinates [A' B' C'] of the rotated triangle [ABC] can be found as:

$$[A'B'C'] = [ABC] . R_{45^\circ, A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1 & (\sqrt{2}-1) & 1 \end{pmatrix} =$$

$$\begin{array}{c} A' \\ B' \\ C' \end{array}\begin{bmatrix} -1 & (\sqrt{2}-1) & 1 \\ -1 & 2\sqrt{2}-1 & 1 \\ \left(\frac{3}{2}\sqrt{2}-1\right) & \left(\frac{9}{2}.\sqrt{2}-1\right) & 1 \end{bmatrix}$$

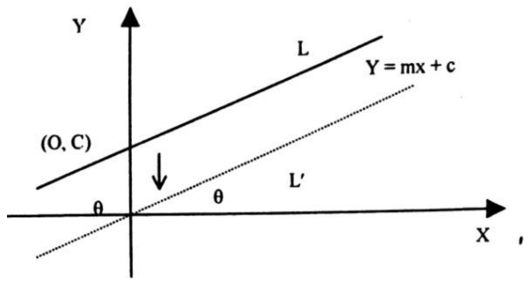
Thus, A' = (-1, $\sqrt{2}$ -1), B' = (-1 , 2$\sqrt{2}$ -1) and C' = $\left(\frac{3}{2}\sqrt{2}-1, \frac{9}{2}\sqrt{2}-1\right)$. The following below figure shows a given triangle, before and after the rotation.



REFLECTION ABOUT A LINE

Reflection is a transformation which generates the mirror image of an object. As discussed in the previous block, the mirror reflection help in achieving 8-way symmetry for the circle to simplify the scan conversion process. For reflection we need to know the reference axis or reference plane depending on whether the object is 2-D or 3-D.

Let the line L be represented by y=mx+c, where ‘m’ is the slope with respect to the x axis, and ‘c’ is the intercept on y-axis, as shown in below figure. Let P’ (x’, y’) by the mirror reflection about the line L of point P(x, y).



The transformation about mirror reflection about this line L consists of the following basic transformations:

1. Translate the intersection point A(0, c) to the origin, this shift the line L to L’.
2. Rotate the shifted line L’ by –θ degree so that the line L’ aligns with the x-axis.
3. Mirror reflection about x-axis
4. Rotate the x-axis back by θ degrees
5. Translate the origin back to the intercept point (0, c)

In transformation notation, we have

$$M_L = T_{-V}.R_{-\theta}.M_X.R_\theta.T_V \qquad \text{where } v=0I+cJ$$

$$M_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2.\cos\theta.\sin\theta & 0 \\ 2.\sin\theta.\cos\theta & \sin^2\theta - \cos^2\theta & 0 \\ -2.c.\sin\theta.\cos\theta & -c.(\sin^2\theta - \cos^2\theta)+c & 1 \end{pmatrix} \quad \text{----------(24)}$$

Let tanθ=m, the standard trigonometry yield $\sin\theta = m/\sqrt{(m^2+1)}$ and $\cos\theta = 1/\sqrt{(m^2+1)}$.

Substituting these values for sinθ and cosθ in the equation (24), we have:

$$M_L = \begin{pmatrix} (1-m^2)/(m^2+1) & 2m/(m^2+1) & 0 \\ 2m/(m^2+1) & (m^2-1)/(m^2+1) & 0 \\ -2cm/(m^2+1) & 2c/(m^2+1) & 1 \end{pmatrix} \quad \text{------------(25)}$$

SPECIAL CASES

1. If we put c = 0 and m = tanθ = 0 in the equation (25) than we have the reflection about the line y = 0 i.e., about x-axis. In matrix form:

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{----------(26)}$$

2. If c = 0 and m = tanθ = ∞ then we have the reflection about the line x = 0 i.e. about y-axis. In matrix form:

$$M_y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{---------(27)}$$

3. To get the mirror reflection about the line y = x, we have to put m = 1 and c = 0. In matrix form:

$$M_{y=x} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------(28)}$$

4. Similarly, to get the mirror reflection about the line y = -x, we have to put m = -1 and c = 0. In matrix form:

$$M_{y=-x} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------(29)}$$

5. The mirror reflection about the Origin (i.e., an axis perpendicular to the xy plane and passing through the origin).

$$M_{org} = \begin{matrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \text{----------(30)}$$

Example: Find the transformation matrix for the reflection about the line y = x.

Solution: The transformation for mirror reflection about the line y = x, consists of the following three basic transformations.

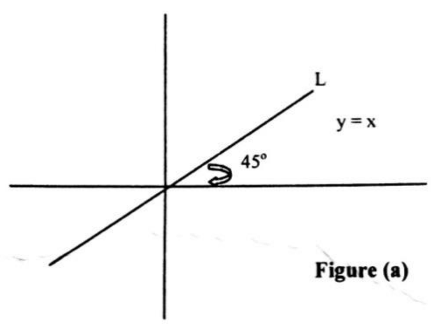


Figure (a)

1. Rotate the line L through 45^0 in clockwise rotation.
2. Perform the required Reflection about the x-axis.
3. Rotate back the line L by – 45^0

i.e.,

$M_L = R_{45^0} \cdot M_x \cdot R_{-45^0}$

$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos 45^\circ & +\sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ \sin 45^\circ & -\cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 90 & \sin 90 & 0 \\ \sin 90 & -\cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_y = x$$

Example: Reflect the diamond-shaped polygon whose vertices are A(-1, 0), B(0, -2), C(1, 0) and D(0, 2) about (a) the horizontal line y = 2, (b) the vertical line x = 2 and (c) the line y = x + 2.

Solution: We can represent the give polygon by the homogeneous coordinate matrix as

$$V=[ABCD] = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

a) The horizontal line y = 2 has an intercept (0, 2) on y axis and makes an angle of 0 degree with the x axis. So m = 0 and c = 2. Thus, the reflection matrix

$$M_L = T_{-v}.R_{-\theta}.M_x.R_\theta.T_{-v}, \quad \text{where } v=0I+2J$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

so the new coordinates A'B'C'D' of the reflected polygon ABCD can be found as:

[A'B'C'D'] = [ABCD]. M_L

$$= \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 1 \\ 0 & 6 & 1 \\ 1 & 4 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Thus, A' = (-1, 4), B'=(0, 6), C'=(1, 4) and D'=(0, 2).

b) The vertical line x = 2 has no intercept on y-axis and makes an angle of 90 degree with the x-axis. So m=tan90^0=∞ and c=0. Thus, the reflection matrix

$$M_L = T_{-v}.R_{-\theta}.M_y.R_\theta.T_{-v}, \quad \text{where } \mathbf{v}=2\mathbf{I}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

so the new coordinates A'B'C'D' of the reflected polygon ABCD can be found as:

[A'B'C'D'] = [ABCD]. M_L

$$= \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} . \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 1 \\ 4 & -2 & 1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{pmatrix}$$

thus, A'=(5, 0), B'=(4, -2), C'(3, 0) and D'=(4, 2)

c) The line y=+2 has an intercept (0, 2) on y-axis and makes an angle of 45^0 with the x-aixs. So m=tan45^0 =1 and c=2. Thus, the reflection matrix.

$$M_L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{pmatrix}$$

The required coordinates A', B', C', and D' can be found as:

[A'B'C'D']=[ABCD]. M_L

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{pmatrix}. = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 1 \\ -2 & 3 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Thus, A' = (-2, 1), B' = (-4, 2), C' = (-2, 3) and D' = (0, 2)

The effect of the reflected polygon, which is shown in below figure, about the line y = 2, x = 2, and y = x+2, and y=x+2 is shown in below figure, respectively.

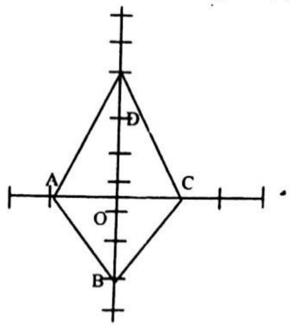


Figure (a)

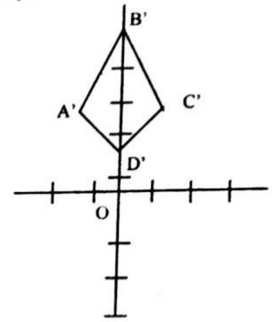


Figure (b)

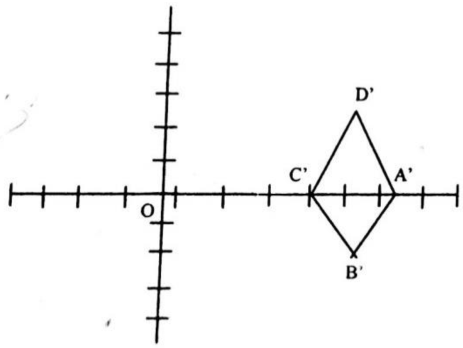


Figure (c)

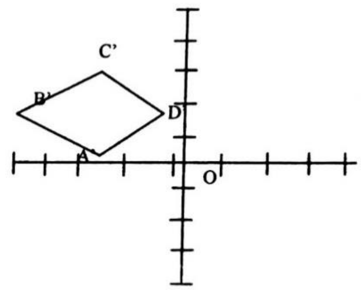


Figure (d)