Set and Relation Part-1

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Set Theory Part-1

Content:

Set theory

Introduction

Set is the most basic term in mathematics and computer science. Hardly any discussion in either subject can proceed without set. We shall discuss the basic concepts of sets, its types, applications and various properties associated to it.

Basic definition

A set is a well-defined collection of objects, called elements or members of the set. A set can be listing its elements between braces {}. We denote sets by capital letters A, B, C,… and elements by lower case letters a, b, c,…

The following are some examples of sets:

1. The collection of first four natural numbers is a set containing elements 1, 2, 3, 4
2. The collection of nine planets in our universe is a set.
3. The collection of capitals of all the countries all over the world is a set.

The symbol ‘ ∈’ is used to express that an element belongs to (is contained in) A.

For example,

A={1, 2, 3, 4, 5}

Here 5 is in A, so we write 5 ∈ A(read as 5 belongs to A)

But 6 is not in A, so we write 6∉ A(read as 6 does not belong to A).

Some important sets are listed below:

N= the set of natural numbers

= {1, 2, 3,…}

Z = the set of integers

= {0, ± 1, ±2, ± 3,…}

Q= the set of rational numbers

R=the set of real numbers

C= the set of complex numbers

Z^+= the set of positive numbers

={1, 2, 3}

R^+ = the set of all real positive numbers

Forms of set

1. **Roster form or tabular form** in which elements are separated from each other using comma and then enclosed them in braces {}.
Let set A contains elements 1, 2, 3 then we write A= {1, 2, 3}.

2. **Set builder form or rule form** in which a set is represented by describing its elements in terms of one or several characteristic properties which help us to decide whether a given element is an element of set or not. For example, the set of integers can be represented as
I= {x : x is an integer}
Similarly the set of natural numbers can be represented as
N={ x : x is a natural number}
Here x : x or x/x is read as x such that x.

Types of sets

1. **Empty set:** A set which does not contain any element is called an empty set. It is also known as void or null set and is denoted by ∅ .
Eg., (i) A={x : x $\in$ R and x^2 +1 = 0}
Here x $\in$ R and x^2 +1=0
i.e., x^2= -1
x = ± $\notin$ R
hence there does not exist real value of x such that x^2 + 1 =0
A=∅.
(ii) A = {x : x is a man having age more than 600 years}

I real life, we cannot find any person having age more than 600 years, so A= ∅

2. **Singleton set:** a set which contains exactly one elements is called singleton set.

   Each of the following set is a singleton set:

   {a}. {b}, {∅}.

3. **Finite set:** A set considering of a finite number of elements is called a finite set. In a finite set if we start counting the different members of the set, the counting process comes to an end. For example, if A is the set of the months, then A is finite set as it contains 12 elements.

   The number of elements contained in a finite set is referred as its **cardinality or the size** of the set. The size or the cardinality of any set A is denoted by |A| (read as size A) or n(A) (read as number of elements in A)

   For example: if A ={a, b, c, d}

   |A| =4

4. **Infinite set:** A set consisting of infinite number of element is called an infinite set.

   For example: N, R, Q are all infinite sets.
5. **Equivalent sets:** two sets A and B are said to be equivalent if

   |A| =|B|

   e.g., A={1, 2, 3, 4} , B={a, b, c,d}

   here |A| =4 and |B|=4

   so A and B are equivalent.

   Sub set: set A is called subset of set B, written as A ⊆ B, if every element of set A is also an element of set B. here set B is called **superset** which contains A so if A ⊆ B then

$x \in A \Rightarrow x \in B$

Remark: $A \subseteq B$ is read as A is subset of B or A is contained in B or B contains A.

A subset can be **proper or improper**. If every element of set A is in set B and $n(A) < n(B)$ then A is called proper sub set of B and is written as $A \subset B$. for example, $N \subset Z$.

Since a set is always contained in itself so $A \subseteq A$. so A is an improper subset of itself. A is an improper subset of B if every element of A is in B and $n(A)=n(B)$. This is possible when $A=B$.

Equality of two sets

Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$ i.e.., every element of set A is also an element of set B and every element of set B is an element of set A.

For example:

A={x : x is letter of word FLOW}

B=(x : x is a letter of word WOLF}

A={f, l, o, w}

B={w, o, l, f}

Clearly $A \subseteq B$ and $B \subseteq A$

$A=B$

Some important theorems

Theorem1: every set is subset of itself.i.e.$A \subseteq A$.

Theorem 2: $\emptyset$ is subset of every set.

Theorem 3: The total number of subsets of a finite set containing n element is 2^n.

7 .Power set

The set of all the subsets of a given set A is called the power set of A and it is denoted by P(A).

Please do remember that both ∅ and A are elements of p(A).

If | A| = n then |P(A)| =2^n.

e.g., let A={1, 2}

then |P(A)= {∅, {1}, {2},A}

e.g.., A=(A, {b}, }

then P(A) ={∅, {A}, {{B}}, A}

8. Universal set

A set that contains all sets in a given context is called universal set.

For example: A={1, 4, 2, 3}

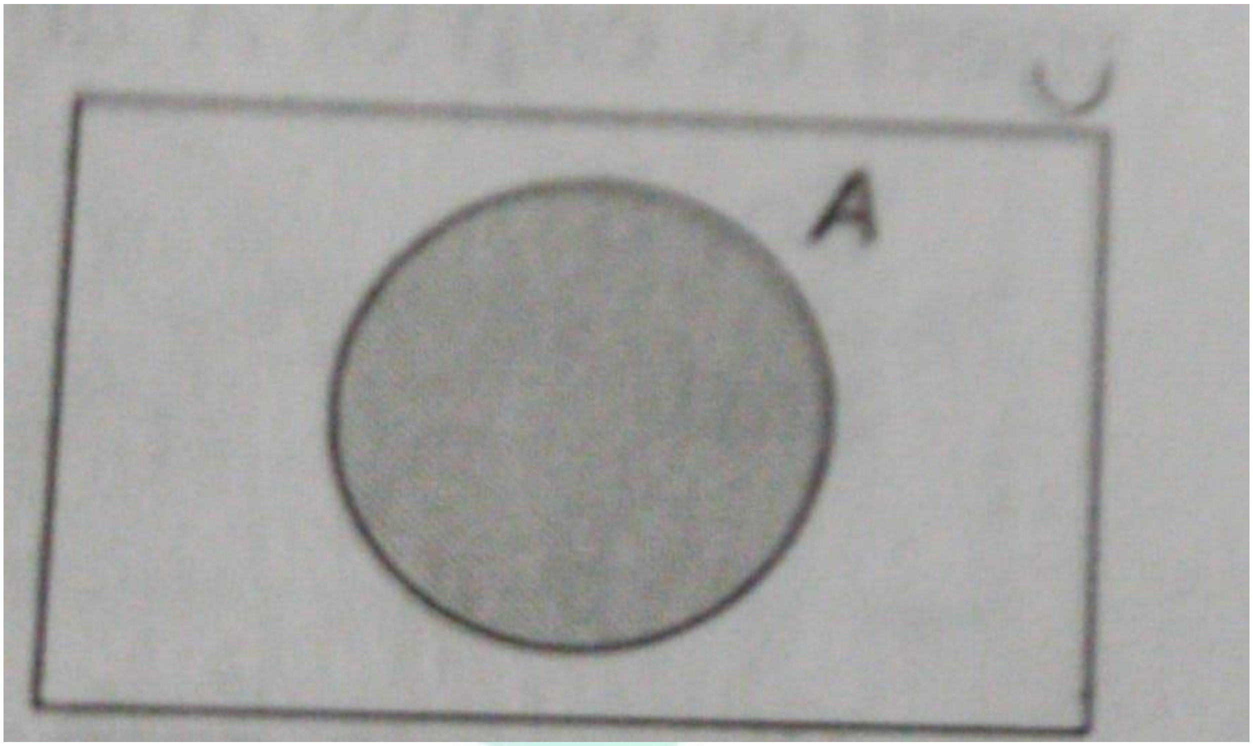
B= {3, 5, 2, 6, 7}

Then U= {1, 2, 3, 4, 5, 6, 7} can be taken as universal set.

Venn diagram

It named after mathematician Venn, are graphic representation of sets as enclosed areas in a plane. For instance the rectangle represents the universal set and the shaded region represents a set A.

Do remember here that A is contained in universal set U.



Basic operations of sets

Union of sets

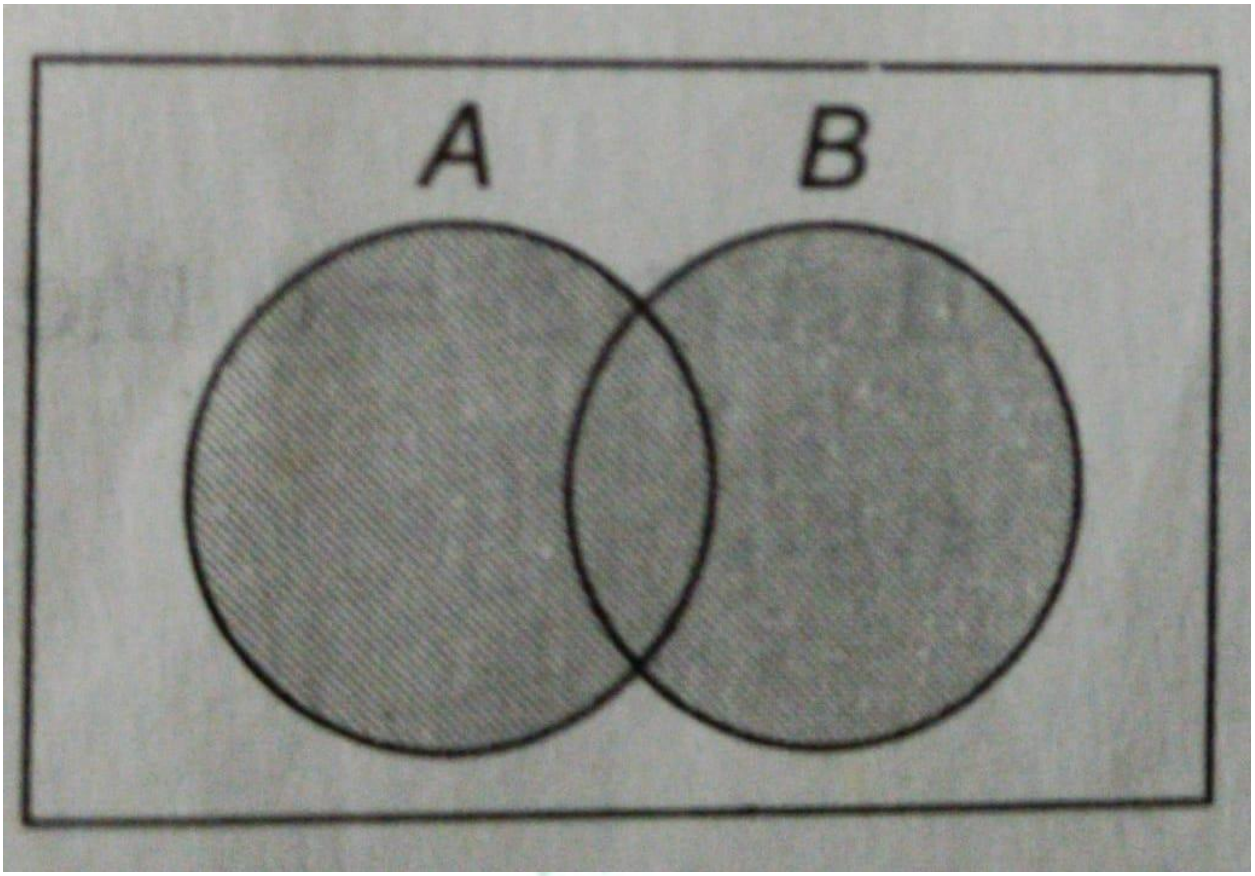
The union of two sets A and B, denoted by A ∪ B is any set C which contains every element of set A as well as of set B. it is also known as join or cup of sets A and B.

So $A \cup B = \{x : x \in A \text{ or } x \in B\}$

$A=\{1, 2, 3, 4\}$, $B=\{2,5,7\}$

$A \cup B=\{1, 2, 3, 4\} \cup \{2,5,7\}$

$= \{1, 2, 3, 4, 5, 6, 7\}$



Please do remember that

1. $A \cup B = B \cup A$
2. $A \cup (B \cup C) = (A \cup B) \cup C$
3. $A \cup \emptyset = A$
4. $A \cup U = U$

The union of finite number of sets $A_1, A_2 \ldots A_n$ is denoted by $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$

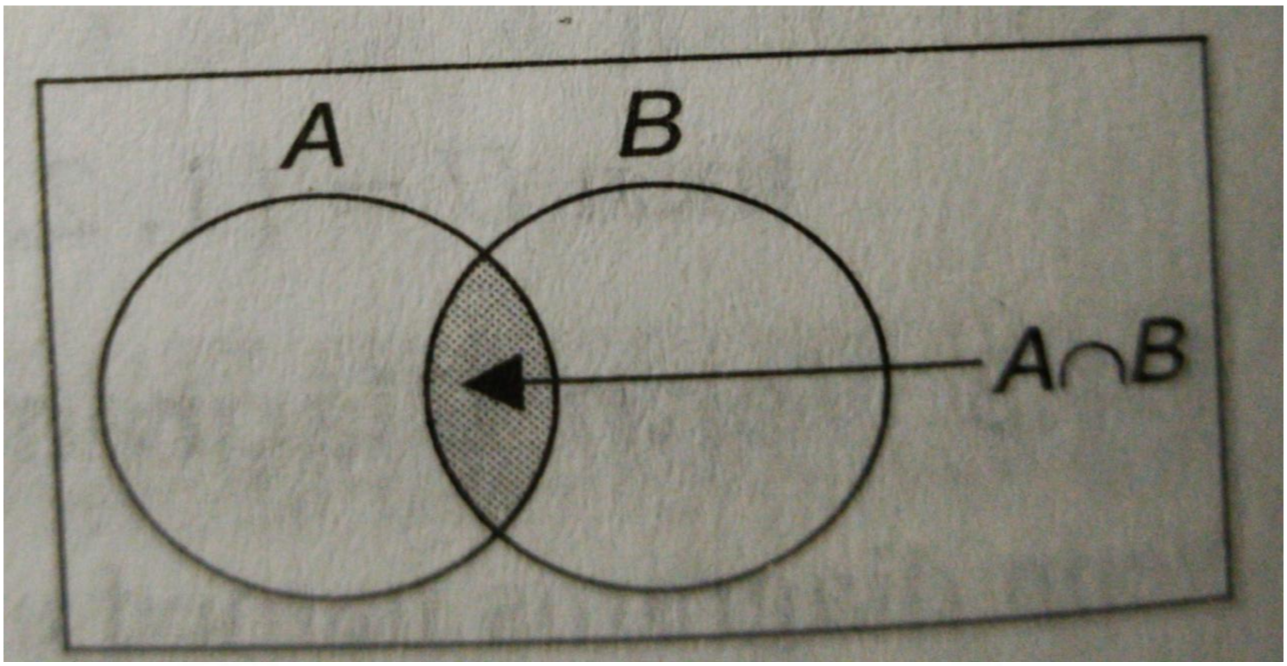
Intersection of sets

The intersection of two sets A and B, denoted by A ∩ B, is only set C which contains the common elements of set A and set B. it is also known as meet or cap of A and B.

So, if A= { 1, 2, 3, 4}

B={2, 4, 5, 6} then

A ∩ B = { 1, 2, 3, 4} ∩{2, 4, 5, 6} = {2,4}



Please do remember that

1. A ∩ B =B∩A
2. A ∩ (B∩C)=(A∩B) ∩ C
3. A ∩∅ = ∅
4. A ∩ U =A

The intersection of a finite number of sets A_1, A_2,... A_n is denoted by $A_1 \cap A_2 \cap ... \cap A_n$

If $A \cap B = \emptyset$ then A and B are called disjoint sets.

e.g., $A=\{x : x = 2n, n \in N\}$

$B= \{ x : x = 2n-1, n \in N\}$

$A \cap B =\{x$: x is an even natural number $\} \cap \{x$: x is an odd natural number$\}$

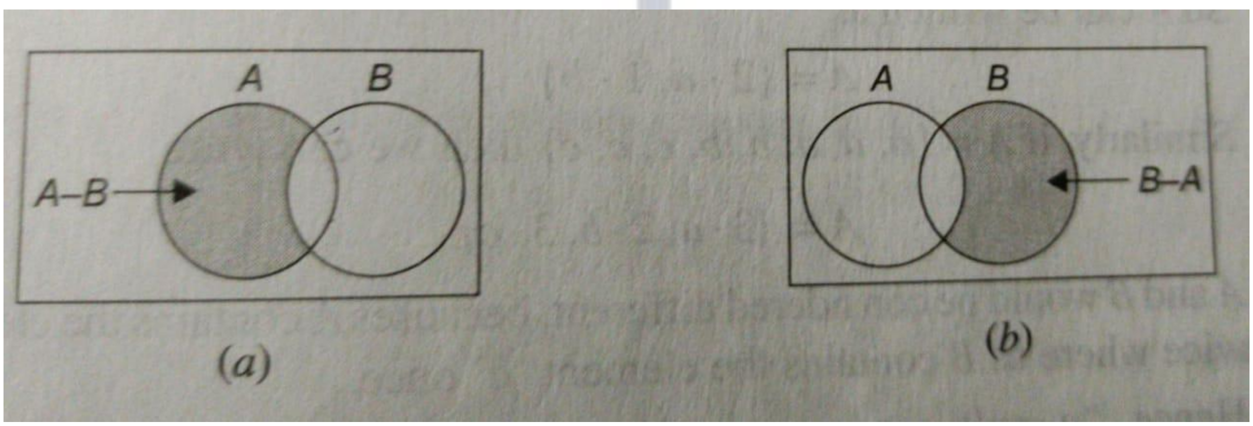
$A\cap B= \emptyset$

Difference of two sets

The difference of two sets A and B, denoted by A-B is a set which contains those elements of A which are not in B.

Thus, A-B= { x: x $\in$ A but x $\notin$ B)

Similarly B-A = (x: x $\in$ B but x $\notin$ A)



e.g., if A={1, 2, 3,4} and B={3, 4, 5, 6}

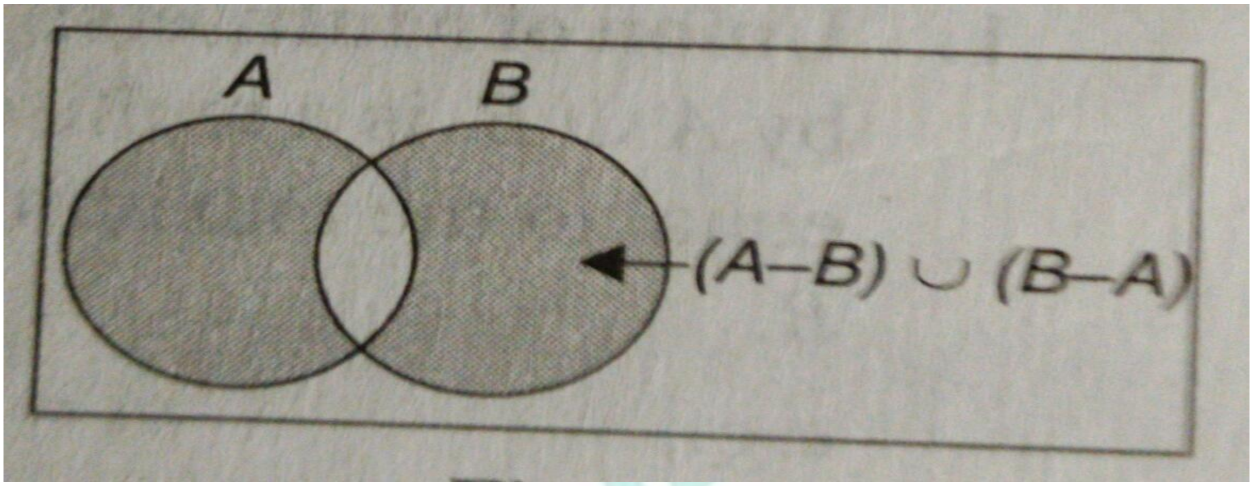
then A-B ={1, 2}

B - A ={5, 6}

Symmetric difference of two sets:

The symmetric difference of two sets A and B, denoted by A Δ B or A ⊕ B is defined by

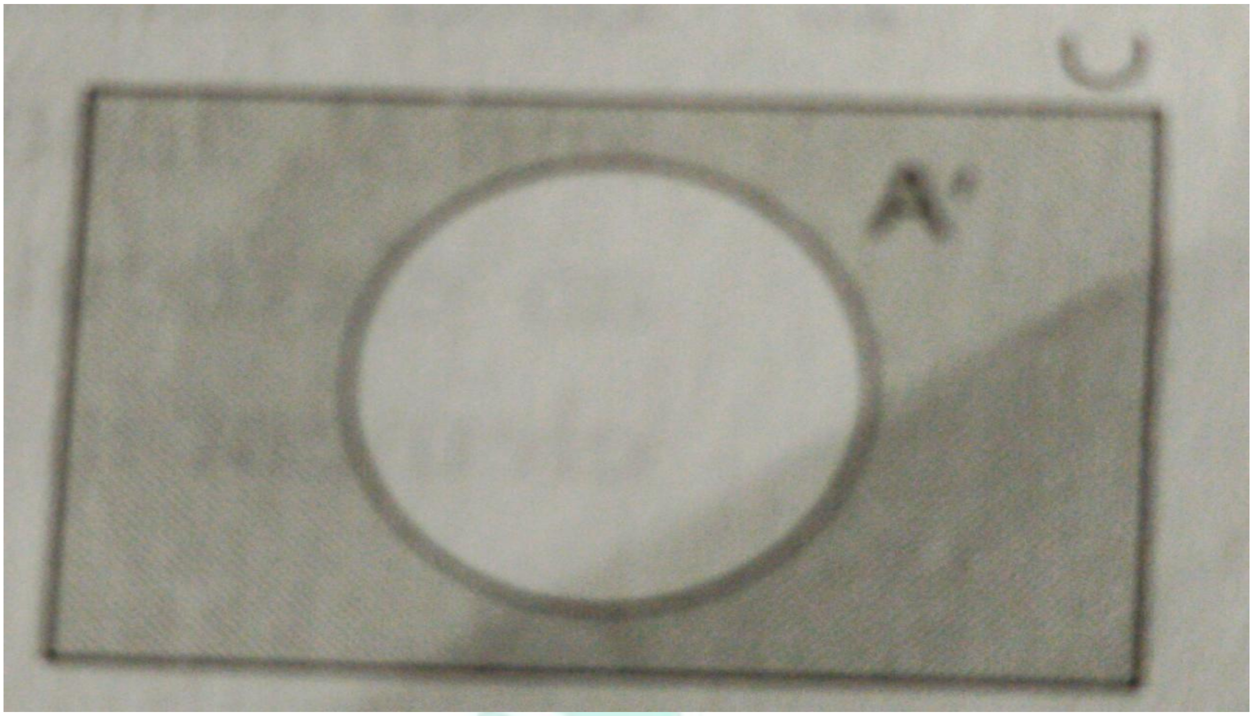
A ⊕ B = (A-B) U (B-A)



Complement of a set

The complement of a set A, denoted by A^C or A', is defined as

A’ = U -A = { x: x $\in$ U but x $\notin$ A }



Multi sets

Consider two sets

A={ a, a, b} and B={a, b}

having same elements namely a and b. however, in some applications it might be useful to allow repeated elements in a set. Such a set is called **multi set**. So, as multi sets A and B would be considered different, because A contains the elements ‘a’ twice whereas B contains the elements ‘a’ once.

So A can be written as

A= {2.a, 1.b}

Similarly, if A={a, a, a, b, b, c, c, c} then we can write

A={3.a, 2.b, 3.c}

A and B would be considered different, because A contains the element ‘a’ twice where as B contains the element ‘a’ once.

Hence, " a multi set is an unordered collection of elements in which an element can occur as a member more than once". The number of times an element repeat in a multi set is referred as its **multiplicity**. In the above example, the multiplicity of element 'a' is 3 where as the multiplicity of element b is 2.

Operations on multi sets

1. **Union of multi sets:** The union the two multi sets A and B, denoted by A ∪ B, is a multi sets in which the multiplicity of an element is equal to the maximum of the multiplicity of that element in A and B.

   e.g.,

   A = {3 .a, 2.b} = {a, a, a, b, b}

   B = { 2.a, 4.b} = { a, a, b, b, b, b}

   A ∪ B = { a, a, a, b, b, b, b}

   = { 3.a, 4.b}.

2. **Intersection of multi sets:** the intersection of two multi sets A and B, denoted by A ∩ B, is a multi set in which the multiplicity of an element is equal to the minimum of the multiplicity of that element in A and B.

   e.g., A = {3 .a, 2.b} = {a, a, a, b, b}

   B = { 2.a, 4.b} = { a, a, b, b, b, b}

   A ∩ B = {a, a, b, b}

   = {2.a, 2.b}

3. **Difference of multi sets:** the difference of multi sets A and B, denoted by A-B, is a multi set in which the multiplicity of an element in A-B is equal to the multiplicity in set A of that element minus the multiplicity in set B of that

element, if the difference is positive and if the difference is negative then the multiplicity is taken to be 0.

Thus, the multiplicity of an element 'a' in A-B is defined as

n_a (A) - n_a (B) , if n_a (A)> n_a (B)

0 , if n_a (A)< n_a (B)

Where n_a (A) means " number of time a repeats in A."

A={a, a, a, b, b} = { 3. a, 2. b}

B={a, a, b}= {2. a, 1. b}

C= {a, b, b, b }= {1. a, 3. b}

A-B = {1. a, 1. b} = {a, b}

A-C = {2.a} = {a,a}.

4. **Sum of Multi Sets:** The sum of two multi sets A and B , denoted by A + B, is the sum of multiplicities of elements sets A and B.

A = {a, a, b, c, c}= {2.a,1.b,2.c}

B = {a, a, a, b, b, c} = {3.a , 2.b , 1.c }

A+B = {5.a , 3.b,3.c}

={a, a, a, a, a, b, ,b, b ,c, c ,c}

Example: if A ={ 1, 2, 3, 4, 5} B={2, 3, 6, 9} then compute:

1. A ∪ B
2. A ∩ B
3. B - A

Solution:

1. $A \cup B = \{1, 2, 3, 4, 5, 6, 9\}$
2 . $A \cap B = \{2\}$
3. $B - A = \{3, 6, 9\}$

Example:

If $U= \{1, 2, 3, 4, 5, 6, 7\}$

$A= \{2, 4, 5\}$

$B= \{2, 3, 7\}$

Then compute

a) A’
b) B’
c) $A \cup B$
d) $A' \cap B'$

Solutions:

a) $A' = U-A = \{1, 3, 6, 7\}$

b) $B' = U - B = \{1, 4, 5, 6\}$

c) $A \cup B = (2, 3, 4, 5, 7\}$

d) $A' \cap B' = \{1,3,6,7\} \cap \{1,2,5,6\} = \{1,6\}$

Example:

If $A =\{n^7$: n is a positive integer}

$B= \{n^5$:n is a positive integer}

Solution :

$A=\{1,2^7, 3^7 , 4^7...\}$

$B= \{1, 2^5, 3^5 , 4^5...\}$

(32)7 here $\in A \rightarrow (2^5) \in A \rightarrow (2^7) \in A$

$(128)^5 \in A \rightarrow (128)^5 \in B$

So the elements of the form

$n^{35} = (n^5)^7$ of A will also be in B having form $(n^5)^7 = n^{35}$

So $A \cap B$ contains elements of the form $(n^5)^7$ or $(n^7)^5$

$A \cap B = \{n^{35} : n \in N\}$

Example:

prove that

$|P[P(P(\phi))]|=4$

Solution: : ϕ represents an empty set, so

$|\phi| = 0$

$|P(\phi)| = 2^0 = 1$

$|P(P(\phi))| = 2^1 = 2$

And $|P[P(P(\phi))]| = 2^2 = 4$

Overlapping sets:

Any two sets A and B are over lapped if they have at least one element in common i.e. , $A \cap B = \emptyset$.

Partition of a Set:

A partition of set A is a collection **δ** of non-overlapping(dis-joint) non-empty subsets of A whose union is the whole A. In other words a collection of subset $A_1, A_2, A_3, \ldots\ldots. A_n$ of set A is said to be a partition of set A if

1. $A_i \neq \emptyset$, where $A_i = 1$ to n i.e. each of $A_1, A_2, A_3, \ldots\ldots. A_n$ is non-empty
2. $A_i \cap A_j = \emptyset$ when $i \neq j$, i=1 to n i.e. A_i and A_j are pair wise disjoint
3. $A_1 \cup A_2 \cup A_3 \cup, \ldots\ldots. \cup A_n = A$

Algebra of sets

Properties of set(laws of sets)

- Commutative laws

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- Associative laws

$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- Identity laws

$$A \cup \phi = A$$
$$A \cup U = U$$

- **Complement laws**

$$A \cup \bar{A} = U$$
$$A \cap \bar{A} = \phi$$

- **Idempotent laws**

$$A \cup A = A$$

$$A \cap A = A$$

- **Absorption laws**

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

- Law of double complementation

$$\overline{(\overline{A})} = A$$

De- Morgan's laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

- 0/1laws

$$\overline{\phi} = U$$

$$\overline{U} = \phi$$

Example:

Two finite sets A and B have a and b number of elements respectively. The total number of subsets of the first set is 8 times the total number of subsets of the second set. Find the value of a-b.

Solution: The number of subsets A and B are 2^a and 2^b respectively.

Now number of subsets of A =8 (number of subsets of B)

So

$2^a = 8.2^b$

$2^a = 2^3.\ 2^b$

$2^a = 2^{3+b}$

a=3+b

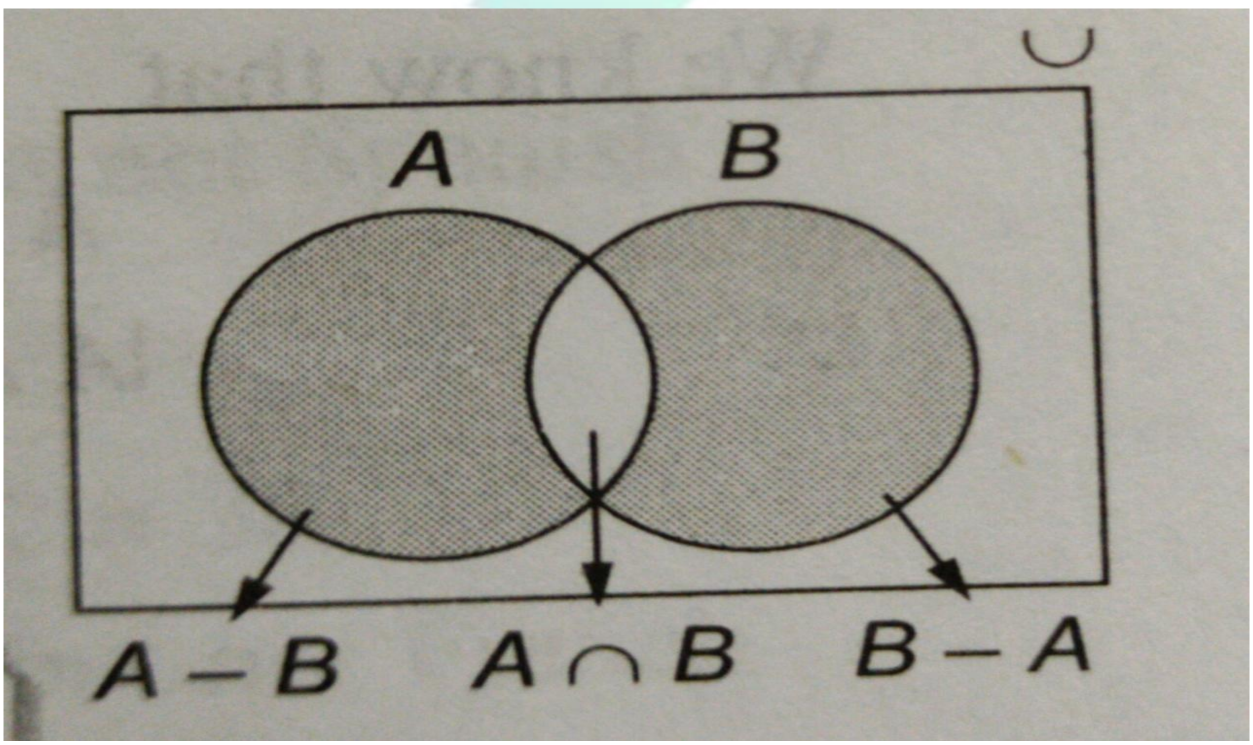
a-b =3

Principle of inclusion and exclusion

This principle is based on the cardinality of finite sets.

Let A and B be any two finite sets then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$



In other words to find $|A \cup B|$, we add $|A|$ and $|B|$ and then subtract $|A \cup B|$ i.e., **include** $|A|$ and $|B|$ and **exclude** $|A \cap B|$. that is why this principle is called " **Principle of inclusion and exclusion.**"

In the similar fashion

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

Remark: If A and B are disjoint sets i.e. $A \cap B = \emptyset$

$|A \cap B| = 0$

$|A \cup B| = |A| + |B| - 0$

$|A \cup B| = |A| + |B|$

Remark: If A,B,C are three pair wise disjoint sets i.e.

$A \cap B = \emptyset, B \cap C = \emptyset, C \cap A = \emptyset$ and $A \cap B \cap C = \emptyset$

$|A \cap B| = 0$, $|B \cap C| = 0$, $|C \cap A| = 0$ and $|A \cap B \cap C| = 0$

So $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

$|A \cup B \cup C| = |A| + |B| + |C|$

Example: For any two sets A and B

$n(A \cup B)=50$, $n(A)=30$, $n(B)=24$ then $n(A \cap B) = ?$

Solution :

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$50 = 30+24 - n(A \cap B)$
$n(A \cap B) = 54-50=4$

Example : A bag contains tickets bearing numbers 1 to 200. Find the number of tickets which bear a multiple of 3 or 4 or 5

Solution :

S={1,2,3,...200}

A={X : x ∈ S and divisible by 3}

B={X : x ∈ S and divisible by 4}

C={X : x ∈ S and divisible by 5}

|A|= 200/3=66

Where (x) represents greatest integer or floor function of x.

|B| = 200/4=50

|C| = 200/5=40

|A ∩ B| = [200/I.c.m(3,4)]= 16

|B ∩C| = [200/I.c.m(4,5)]= 200/20=10

|A ∩C| = [200/I.c.m(3,5)]= 200/15=13

|A ∩B ∩C|=[200/I.c.m(3,4,5)]= 200/60=3

|A ∪ B ∪ C|= |A|+|B|+|C|-|A ∩ B|-|C ∩A|-|B ∩C|+ |A ∩B ∩C|

=66+50+40-16-10-13+3=120

Example: Among 75 children who went to an amusement park where they could ride on merry-go-round, roller coaster and ferries wheel. It is known that 20 of them had taken all three rides and 55 had taken at least two of the three rides. Each ride costs rs.0.50 and total receipt of park is rs.70. Determine the number of children who did not try any of the rides.

Solution :

Total receipts of the park= rs70

Cost of a ride= rs.0.50

So total number of rides taken= 70/0.50=140

Here 20 of them had taken all the three rides

X=20

Now 55 of them had taken at least two rides i.e.2 or 3 rides.

|A ∩B∩C'|+ |A ∩C ∩B'|+ |B∩C ∩A'|+ |A ∩B ∩C|=55

y +z + w + x=55

y +z + w =35

35 students had taken exactly two rides

No. of students who took exactly one ride

=140 - (35*2+20*3)=10

No. of students who took ride

=20+35+10=65

So, no. of students who did not take any ride

=75-65=10