# Graph Theory Part-2

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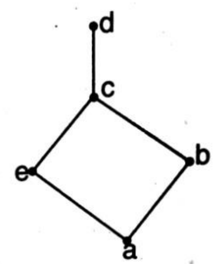
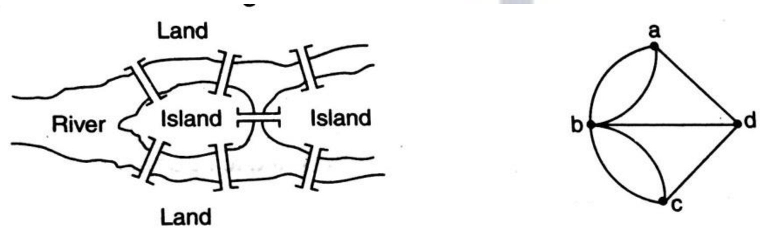
## EULERIAN AND HAMILTONIAN GRAPH




(a) (b)

One of the oldest problem involving graphs is the Konigsberg bridge problems. There were two islands linked to each other and to the banks of the Pregel River (earlier known as Konigsberg) by seven bridges shown in figure 14 (a). The problem was to begin at any of the four land areas to walk across each bridge once and to return the starting point. Euler drew a graph like figure 14(b) for the problem in which a and c represent the two river banks; b and d the two islands. The arcs joining them represent the seven bridges.

It is clear that the problem of walking each of the seven bridges exactly once and returning to the starting point is equivalent to finding a circuit in graph (b) that traverses each of the edge exactly once. Euler discovered a very simple criterion for determining whether such a circuit exists in a graph.

**EULERIAN GRAPH**

A circuit in a connected graph is an Euler circuit if it contains every edge of the graph exactly once. A connected graph with an Euler circuit is called an Euler graph or Eulerian graph.

If there is trial from vertex a to b in G and this trial traverses each edge in G exactly once, then the trial is called an Euler trail.

In figure 15, d c e a b c represents an Euler trial since it contains all the edges exactly once and vertex c is represented but start and end vertex are not the same. It is not an Euler circuit as starting and ending at the same vertex is not possible without repeating an edge cd.

Figure 15

The existence of Euler circuit and trial depends on the degree of vertices.

The next theorem provides necessary and sufficient condition for characterising Euler Graph.

**THEOREMS**

1. A nonempty connected graph G is Eulerian if and only if its vertices are all of even degree.
2. A connected graph contains an Euler trial, but not an Euler circuit, if and only if it has exactly tow vertices of odd degree.

**COROLLARY**

1. A directed multi-graph G has an Euler path if and only if it is unilaterally connected and the in degree of each vertex is equal to its out degree with the possible exception of two vertices, for which it may be that the in degree of one is larger than its out degree and the in degree of the other is one less then its out degree.
2. A directed multi graph G has n Euler circuit if and only if G is unilaterally connected and the in-degree of every vertex in G is equal to its out-degree.

Thus to determine whether a graph G has an Euler circuit, we note the follow points.

1. List the degree of all vertices in the graph.
2. If any value is zero, the graph is not connected and hence it can not have Euler path or Euler circuit.
3. If all the degrees are ever, then G has both Euler trial but no Euler circuit.
4. If exactly two vertices are old degree, then G has Euler trial but no Euler circuit.

**HAMILTONIAN GRAPH**

Hamiltonian graphs are named after Sir William Hamilton, an Irish mathematician who introduced the problems of finding a circuit in which all vertices of a graph appear exactly once.

A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as Hamiltonian cycle.

A graph G is called a Hamiltonian cycle if it contains a Hamiltonian cycle.

A Hamiltonian path is a simple path that contains all vertices of G where the end points may be distinct.

## THEOREMS

1. (**Dirac's Theorem)** A simple connected graph G with $n \geq 3$ vertices in Hamiltonian if deg (v) $\geq$ n/2 for every vertex v in G.

   The graph $K_5$ in figure 16 has Hamiltonian circuit. It has deg (v) = 4 for each v and also have V (G) = 5, so it satisfies the condition of the theorem.

   The graph in figure 16(b) has n = |V (G)| = 5 and has a vertex of degree 2. So it does not satisfy the condition of the theorem but nevertheless Hamiltonian. The next two theorems provides other sufficient conditions for graph to be Hamiltonian.

2. A simple connected graph with n vertices and m edges is Hamiltonian if $m \geq [(n - 1) (n - 2)]/2 + 2$.

   The Hamiltonian graph of figure 16(c) has n = 5, which given $[(n - 1) (n - 2)]/2 + 2 = 8$ and so it satisfies the condition and hence the conclusion of the theorem.

3. (**Ore's Theorem)** Let G be a single connected graph with $n \geq 3$ vertices. If $\deg(v) + \deg(w) \geq n$

   For each vertices v and w not connected by an edge i.e. for every pair of non-adjacent vertices of v and w, then G is Hamiltonian.

   For the graph in figure 16 (b), n = 5. There are these pairs of distinct vertices that are not connected by an edge.

   For <v, z> $\deg(v) + \deg(z) = 3 + 3 = 6 \geq 5$

   For <w, x> $\deg(w) + \deg(x) = 3 + 2 = 5 \geq 5$

   For <x, y> $\deg(x) + \deg(y) = 2 + 3 = 5 \geq 5$

   It may be noted that all the theorems started above provide only sufficient criterion. They are not necessary.

   A few helpful hints for trying to find a Hamilton cycle in a graph G = (V, E) is the given below:

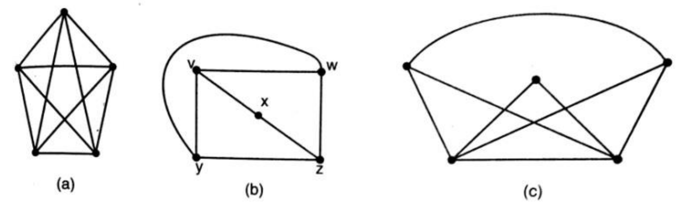



figure 16

1. If G has Hamilton cycle, then for all u ∈ V, deg(u) ≥ 2.
2. If a ∈ V and deg(a) = 2, then the two edges incident with vertex a must appear in every Hamilton circuit for G.
3. If a ∈ V and deg (a) > 2, then as we try to build a Hamilton cycle, once we pass through vertex a, any unused edges incident with a are deleted from further consideration.
4. A circular graph has both Hamiltonian path and cycle. The complete graph $K_n$ (n ≥ 3) is Hamiltonian. The complete bipartite graph $K_{m,n}$ is Hamiltonion if and only if m = n and n > 1. $Q_n$(n ≥ 2) and $W_n$(n ≥ 3) has a Hamiltonian cycle.

## GRAPH COLOURING

Graph colouring is an assignment of colours to elements of a graph subject to certain constraints. The starting point of graph colouring is the vertex colouring. The other colouring problems like edge colouring and region colouring can be transformed into vertex colouring. An edge colouring of a graph is just a vertex colouring of its line graph. The region colouring of a planar graph is the problem of colouring of vertex of its planar dual graph. In this chapter we present the basic results concerning vertex colouring and edge colouring of graphs and their applications.

## VERTEX COLOURING

The assign of colours to the vertices of G, one colour to each vertex, so that the adjacent vertices are assigned different colours is called the proper colouring of G or simply vertex colouring of G. A graph in which every vertex has been assigned a colour according to a proper colouring is called a properly coloured graph. The n - colouring of G is a proper colouring of G using n - colours. If G has n colouring, then G is said to be n - colourable.

**CHROMATIC NUMBER:** The chromatic number of a graph is the minimum number of colours needed for a proper colouring i.e. the minimum number of colours needed to assign colours to each vertex of G such that no two adjacent vertices are of same colour. It is denoted by x (G). Thus a graph G is K chromatic if x(G) = K.

The graph of figure 1 of n colouring for n = 2, 3, 4 are displayed, with positive integers in small circles designating the colours. Here x(G) = 2.

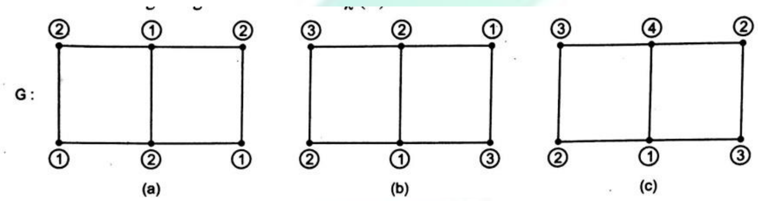



figure 1

In graph colouring, we consider the colouring of simple connected graphs only. This is so because the colouring of vertices of some component of a disconnected graph has no effect on the colouring of the other component. Self loops are not considered for colouring. Parallel edges between two vertices can be replaced by a single edge without affecting the adjacency of vertices.

**Note:**

i. A graph consisting of only isolated vertices is 1 - chromatic. The chromatic number of a null graph is also 1.
ii. X(G) $\leq$ n, where n is the number of vertices of G.
iii. A simple connected graph will one or more edges is at least 2 i.e. x(G) $\geq$ 2.
iv. Chromatic number of a graph having a triangle is at least 3 (because three colours are required to colours a triangle)

v. If $\deg(v) = d$ for a vertex $v$ in G, then at most $d$ colours are required for proper colouring of the vertices adjacent to $v$.

vi. Every k-chromatic graph has at least k vertices, say $v_1, v_2, \ldots, v_k$ such that $\deg(v_i) \geq k - 1$, $i = 1, 2, \ldots, k$.

vii. If H is a subgraph of G, then $x(H) \leq x(G)$.

**THEOREMS**

1. Every tree with two or more vertices is 2-chromatic.
2. **(Chromatic Number of a Bi-Partite Graph ($K_{m,n}$)** The chromatic number of a non-null graph is 2 if and only if the graph is bipartite.
3. **(Chromatic Number of a Complete Graph ($K_n$)** The chromatic number of a complete graph with n vertices ($K_n$) is n.
4. **(Chromatic Number of cycle ($C_n$)** The chromatic number of a cycle with n vertices ($C_n$) is

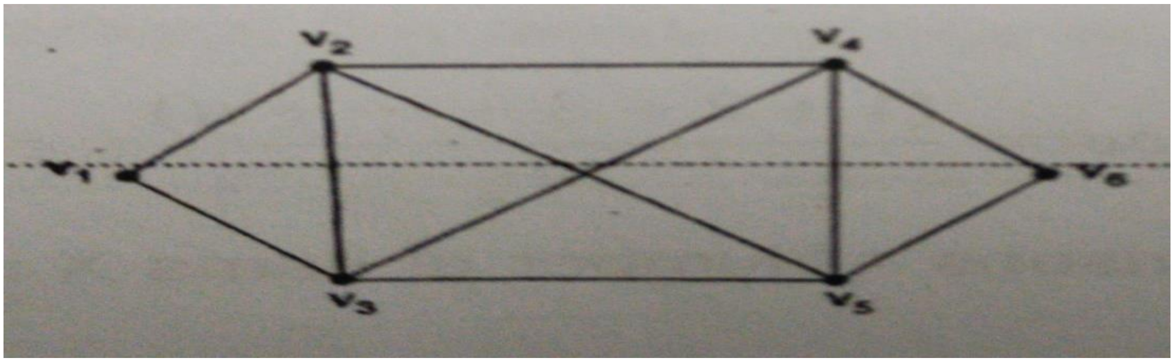
   i. 2 if n is even      ii. 3 if n is odd
5. A graph G with a least one edge is 2-chromatic if and only if one of the following conditions is satisfied.

a. G is a tree.

b. Every cycle of G has even length

G is a bipartite graph

**Question 1:** Find the degree of each vertex of the following graph.




**Solution :** It is an undirected graph. Then

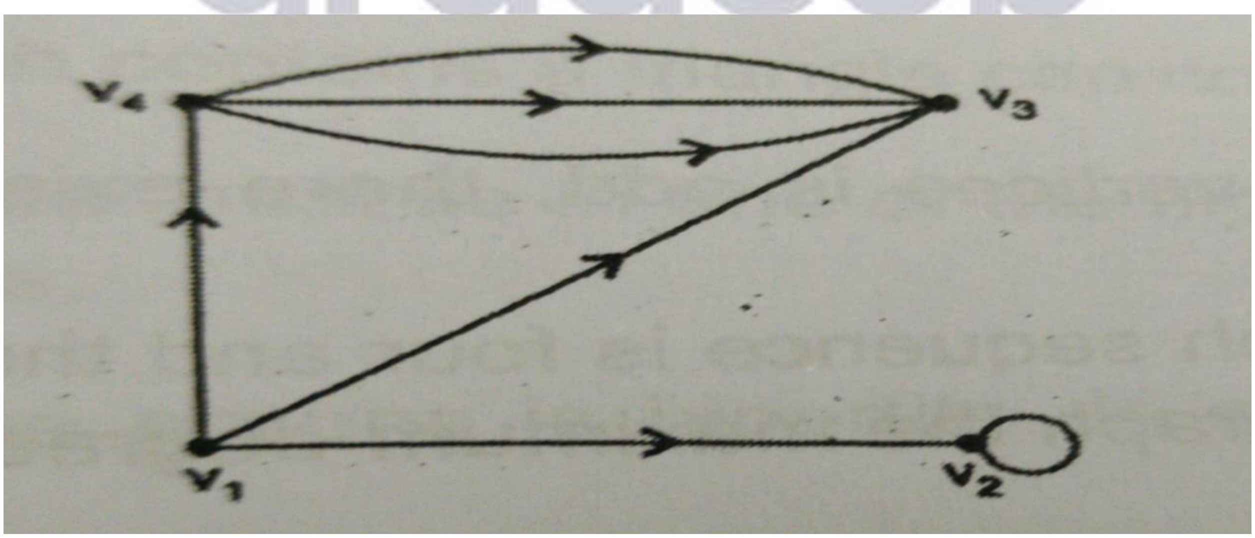
deg ($v_1$) = 2 deg ($v_2$) = 4

deg ($v_3$) = 4

deg ($v_4$) = 4 deg ($v_5$) = 4

deg ($v_5$) = 2

**Question 2:** Find the in degree out degree and of total degree of each vertex of the following graph.



**Solution :** It is a directed graph

in deg $(v_1) = 0$ out deg $(v_1) = 0$

total deg $(v_1) = 3$

in deg $(v_2) = 2$ out deg $(v_2) = 1$

total deg $(v_2) = 3$

in deg $(v_3) = 4$ out deg $(v_3) = 0$

total deg $(v_3) = 4$

in deg $(v_4) = 1$ out deg $(v_4) = 3$

total deg $(v_4) = 4$

**Question 3:** Show that the degree of a vertex of a simple graph G on n vertices can not exceed n - 1.

**Solution :** Let v be a vertex of G, since G is simple, no multiple edges or loop are allowed in G. Thus v can be adjacent to at most all the remaining n - 1 vertices of G. Hence v may have maximum degree n -1 in G. Then $0 < \deg_G (v)\ n - 1$ for all v E V(G).

**Question 4:**Show that the maximum number of edges in a simple graph with in vertices is

$$\frac{n(n-1)}{2}$$

**Solution :** By the handshaking theorem

$$\sum_{i=1}^{n} d\,(vi) = 2e$$

Where e is the number of edges with in the graph G

$\Rightarrow$ $d(v_1) + d(v_2) + \ldots.\ d(v_n) = 2e$

since the maximum degree of each vertex in the graph G can be (n -- 1). Therefore, equation (1) reduces to

$$(n-1) + (n-1) + \ldots \text{ to } n \text{ terms} = 2e$$

$$\Rightarrow \quad n(n-1) = 2e \Rightarrow e = \frac{n(n-1)}{2}$$

hence the maximum number of edges in any simple with n vertices is $\frac{n(n-1)}{2}$

**Question 5:** Is there a simple graph corresponding to the following degree sequence ?

(i) (1 , 1,2,3) (ii) (2,2,4,6) (iii) (1,1,1,1)

**Solution :**

i. Since the sun of degree of vertices is odd, there exist no graph corresponding to this degree sequence.

ii. Number of vertices in the graph sequence is four and the maximum degree of a vertex is 6, which is not possible as in a simple graph the maximum degree cannot exceed one less than the number of vertices.

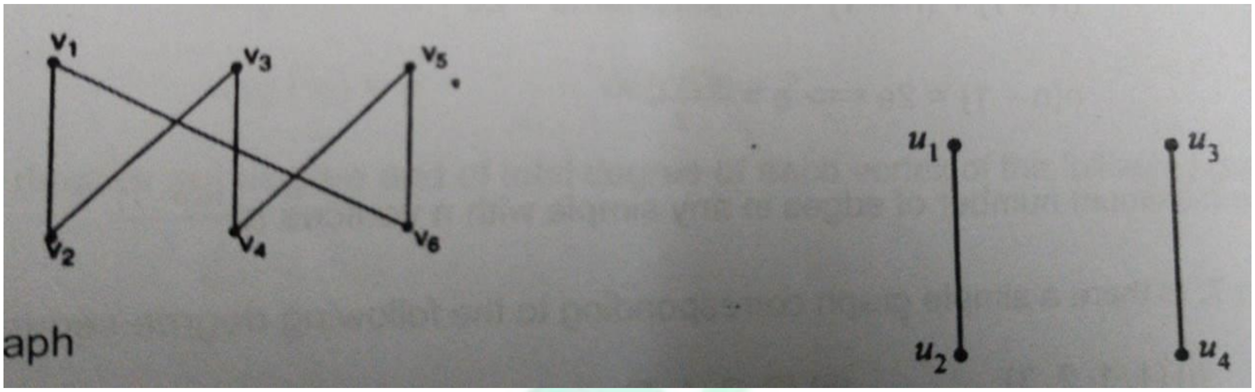
iii. The sun of the degrees of all vertices is 4, even . The number of odd vertices is 4,even. Hence a simple disconnected graph is possible which has 4 vertices of degree 1 each. The number of edges is 4/2=2.

**Question 6:** Does there exists a simple graph with seven vertices having degree sequence (1,2,2,4,5,6,6)

**Solution :** Here the sum of the degrees in 28, even . The number of odd vertices is 4, even. The maximum degree 6 does not exceed 7 -1 = 6. But two vertices have degree 6, each of these two vertices is adjacent with every other vertex . Hence the degree of each vertex is at least 2, so that the graph has no vertex of degree 1 which is a contradiction. Hence there does not exist a simple graph with the given degree sequence.

**Question 7:** Show that $C_6$ is a bipartite graph.

**Solution :** $C_6$ is a bipartite graph since its vertex set can be partitioned into the two sets $V_1$ = {$v_1$, $v_3$, $v_5$} and $V_2$ = {$v_2$, $v_4$, $v_6$}, and every edge of $C_6$ connects a vertex in $V_1$ and a vertex in $V_2$.

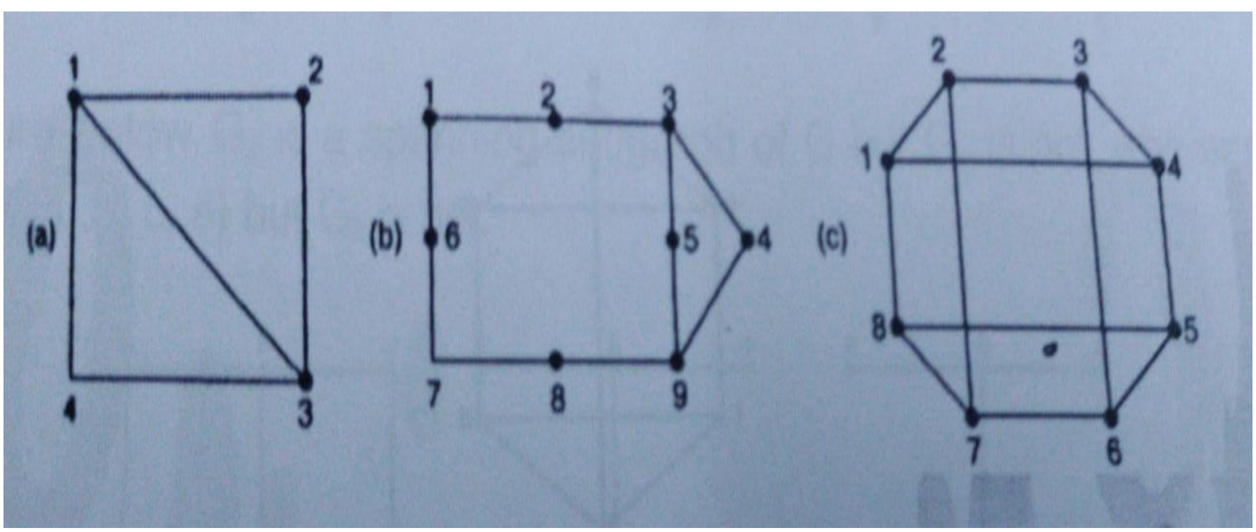



**Note:** $Q_n$ is a bipartic graph

**Question 8 :** (i) Prove that a graph which contains a triangle can not be bipartite.

**Solution :** At least two of the three vertices must lie in one of the bipartite sets since these two are joined by an edges, the graph can not be bipartite.

(ii) Determine whether or not each of the graphs is bipartite. In each case, give the bipartition sets or explain why the graph is not bipartite.

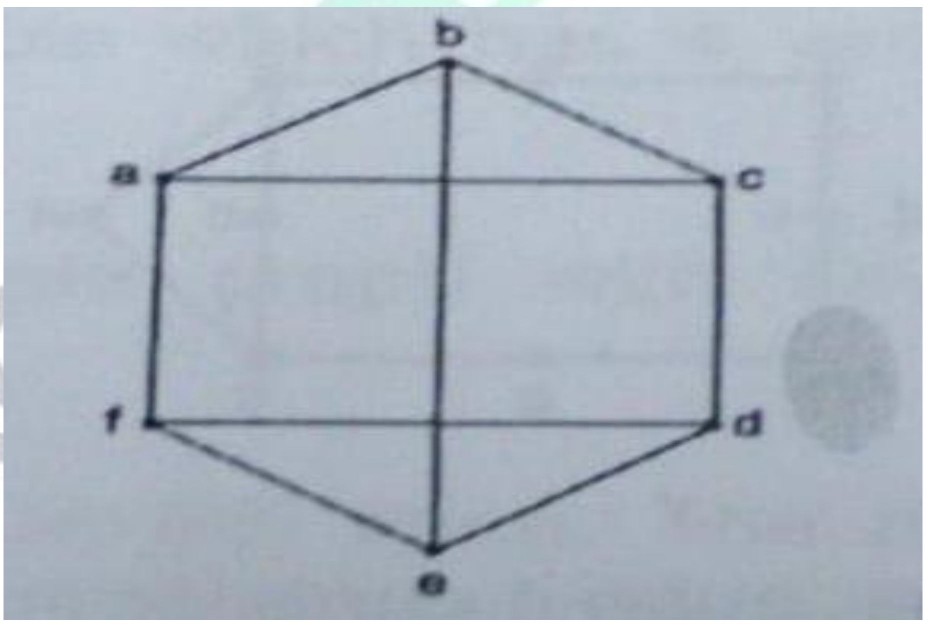


**Solution :** (a) The graph is not bipartite because it contains triangles (in face two triangle)

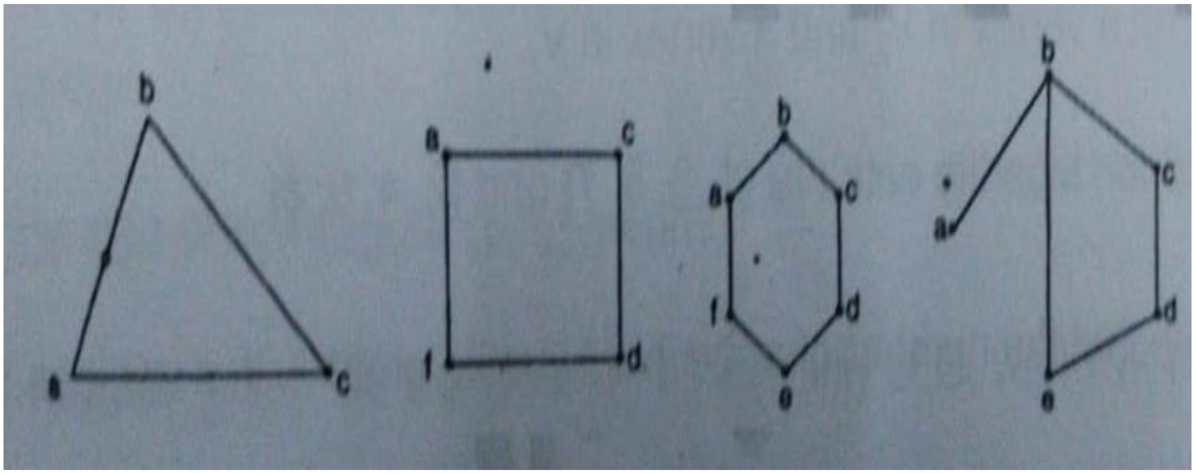
(b) This is a bipartite graph and the bipartite sets are $V_1$ = {1,3, , 7 , 9} and $V_2$ = { 2, 4,5,6,8} such that each edge is incident on a vertex in $V_1$ and a vertex in $V_2$ .

(c) This is bipartite and the bipartite sets are { 1,3,5,7} and {2,4,6,8}

**Question 9:** Consider the graph shown in below figure. Find the different subgraph of this graph.

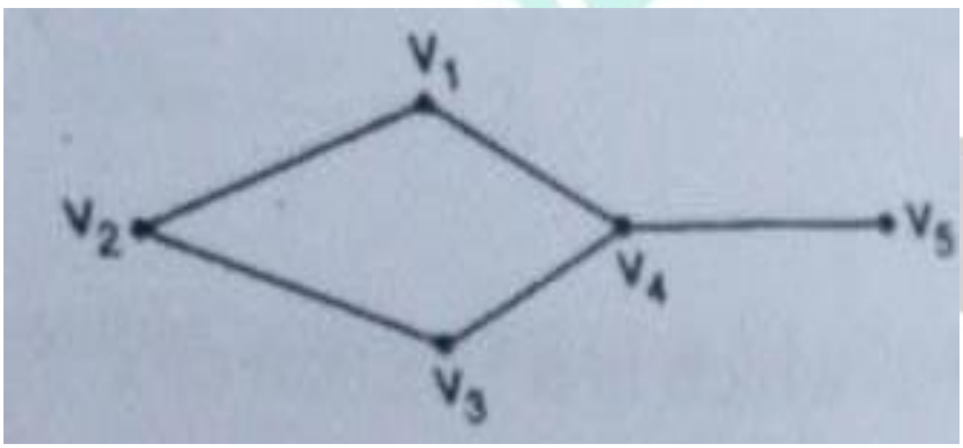



**Solution :** The following are sub graphs of the above graph.




**Note:** That the subgraphs do not have to be drawn the same way they appear in the presentation of G.

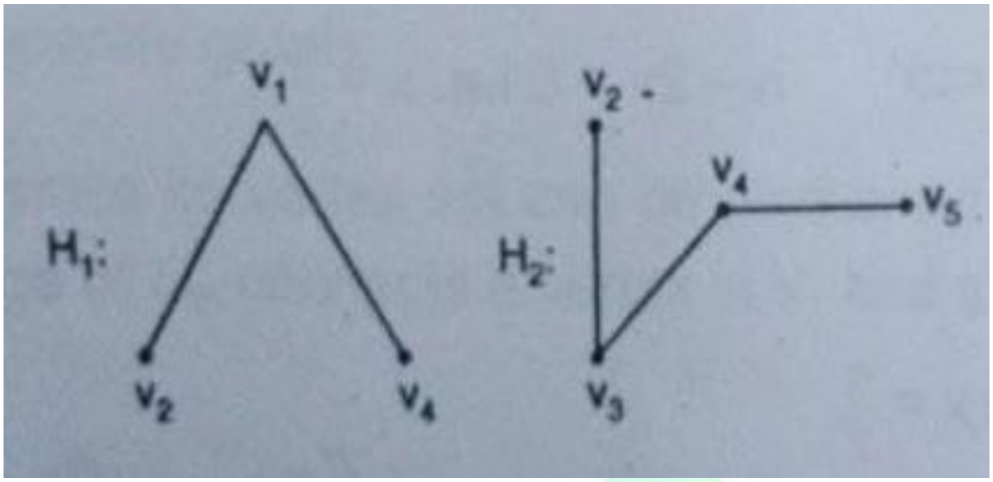
**Question 10:** Construct two edge deleted sub graph and two vertex deleted subgraphs

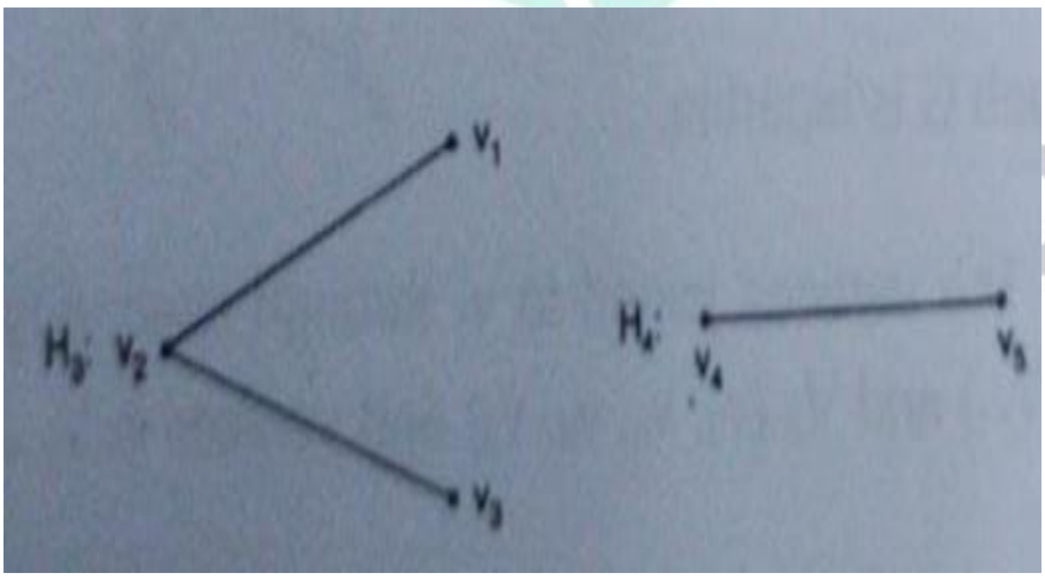



of a graph G shown below.

**Solution :** the graphs $H_1$ and $H_2$ are two edge deleted subgraphs of G.

The graphs $H_3$ and $H_4$ are vertex deleted subgraphs of G which are also edge deleted subgraphs of G.

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