Mathematical Logic Part-3

Mathematical Logic Part-3

Principal Conjunction Normal Forms (penf)

The dual of a minterm is called a maxterm. For a given number of variables the maxterm consists of disjunctions in which each variables or its negation, but not both, appears only once.

For example, for two variables *p* and *q*, there are maxterms given by

$p \vee q$, $p \vee \sim q$, $\sim p \vee q$ and $\sim p \vee \sim q$,

Maxterm for three variables *p* , *q* and *r* are:

$p \vee q \vee r \rightarrow$ $p \vee q \vee \sim r \rightarrow$ $p \vee \sim q \vee r \rightarrow$ $p \vee \sim q \vee \sim r \rightarrow$

$\sim p \vee q \vee r$ $\sim p \vee q \vee \sim r$ $\sim p \vee \sim q \vee r$ $\sim p \vee \sim q \vee \sim r$

each of the maxterms has the truth value *F* for exactly one combination of the truth values of the variables. Different maxterms have the truth value *F* for different combinations of the truth values of the variables.

Principal conjunctive normal form of a given formula can be defined as an equivalent formula consists of conjunctive of maxterm only. This is also called the **product of sums canonical form.** The process for obtaining principal conjunctive norm form is similar to the one followed for principal disjunctive normal form. For obtaining principal conjunctive norm of a, one can also construct the principal disjunctive normal of ~ a and apply negation.

The advantages of obtaining principal conjunctive normal form are:

(i) The principal conjunctive normal form is unique.

(ii) Every compound proposition, which is not a tautology, has an equivalent principal conjunctive normal form.

(iii) If the given compound proposition is a contradiction, then its principal conjunctive normal form will contain all possible maxterms of its components.

Example : Obtain the principal conjunctive normal form

(a) $P \wedge q$ using truth table.

(b) $(\sim p \rightarrow r) \wedge (q \leftrightarrow p)$ without using truth table.

(c) $A = (p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$ by constructing principal disjunctive normal form

Solution (a). Truth table of $p \wedge q$ is given below:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Form the columns $p \wedge q$ has truth value F for three combinations of the truth value of p and q. Now F in the second row of $p \wedge q$ correspond to maxterm $\sim p \vee q$. F in the third row p^ q correspond to maxterm $p \vee \sim q$ and F in the fourth row of $p \wedge q$ corresponds to maxterm p v ~ q. Hence the required principal conjunctive normal form of $p \wedge q$ is

$(\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee \sim q)$

(b) $(\sim p \rightarrow r) \wedge (q \leftrightarrow p) = (p \vee r) \wedge [(q \rightarrow p) \wedge (p \rightarrow q)]$

$= (p \vee r) \wedge [(\sim q \vee p) \wedge (\sim p \vee q)]$

$= [(p \vee r) \vee (\sim q \wedge \sim q)] \wedge [(\sim q \vee p) \vee (r \wedge \sim r)] \wedge [(\sim p \vee q) \vee (r \wedge r)]$

$= (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (\sim q \vee p \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

$= (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (\sim q \vee p \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee r)$

Which is the required principal conjunctive normal form

(c) Let $\sim A = \sim ((p \wedge q) \vee (\sim p \wedge q) \wedge (q \wedge r))$

$= \sim ((p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim(p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p)$

$= (\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim q \wedge \sim r \wedge \sim p)$

$V (\sim q \wedge \sim r \wedge p) = (\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge r)$

Now $\sim (\sim A) = (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

Which is the required principal conjunctive normal form.

Logic in Proof

A theorem is a proposition that can be proved to be true. An argument that establishes the truth of a theorem is called a proof. There are many different types of proof. In this section we shall look at some of the more common type.

Arguments

An argument is a process by which a conclusion is drawn from a set of propositions. The given set of propositions are called **premises or hypotheses.** The final proposition derived from the given proposition is called **conclusion.**

Sometimes an argument is written in the following form

$$\left.\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{array}\right\} \text{premises}$$
$$\overline{\therefore \quad c \longleftarrow \text{conclusion}}$$

An argument is said to be logically **valid argument** if and only if the conjunction of of the premises implies the conclusion i.e., if the premises are all true, the conclusion must also be true .The argument which yield a conclusion e from the premises P_1, P_2, P_3 ------- P_4 is valid if and only if the statement.

P_1, ^ P_2, ^ P_3 ^----------- ^ P_4 → c is a tautology.

Procedure for testing the validity of an Argument using Truth Table.

1. Identify the premises and conclusion of the argument.
2. Construct a truth table table showing the truth values of all premises and the conclusion.
3. Find the rows (called **critical rows**) in which all the premises are true.
4. In each critical row, determine whether the conclusion of the argument is also true.

(a) If in each critical row the conclusion is also true, then the argument form is valid.

(b) If there is at least one critical row in which the conclusion is false, the argument form is invalid.

The method to determine whether the conclusion logically follows from the given premises by constructing the relevant truth table is called truth table technique.

Rules of Inference.

The rules of inference are criteria for determining the validity of an argument. Any conclusion which is arrived by following these rules is called a valid conclusion, and the argument is called valid argument. The most familiar type of proof cases two fundamental rules of inference.

Fundamental Rule 1. If the statement in p is assumed as true and also the statement $p \rightarrow q$ is accepted as true, then, q must be true.

Symbolically it is written in the following pattern, where we use the familiar similar

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

In this presentation of an argument, the assertions $p \rightarrow q$ and p above the horizontal line are the **hypotheses** or **premises** and the assertion q below the line is the **conclusion.** The rule depicted is known as **modus ponens** or the **rule of detachment.** The term **modus** ponens is Latin word meaning "method of affirming" (since the conclusion is an affirmation). The validity of the argument can also be seen from the following truth table.

Premises			Conclusion	
p	q	$p \rightarrow q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F.

We are in Table 2.27 that there is only one case in which both premises re true, namely, the first case, and that is this case the conclusion is also true, hence the argument is valid.

Another way of starting that the above argument is valid is that $[(p \rightarrow q) \wedge p] \rightarrow q$ is a tautology.

Example : Represent the argument

If I study hard, then I get A' s

I study hard.

……………………………………………….

I get A' s

Symbolically and determine whether the argument is valid.

Solution. If we let

P : I study hard.

q : I get A' s

The argument may be written symbolically as

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

Hence, by modus ponens the argument is valid.

Example : Suppose that the implication " If the last digit of this number is a 5, then this number is divisible by 5" and its hypothesis. "The last digit of this number is a 5" are true. Then by modus ponens it follows that the conclusion of the implication. " This number is divisible by 5" is true.

Fundamental Rule 2. Whenever the two statement $p \rightarrow q$ and $q \rightarrow r$ are accepted s true then the statement $p \rightarrow r$ is accepted as true. Symbolically it can be represented as

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore \quad p \rightarrow r . \end{array}$$

This argument is known as a Hypothetical syllogism.

The truth table of the argument appears in Table 2.28

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Both premises are true as seen in the first, fifth, seventh, and eighth rows of the truth table.

Since in each case the conclusion is also true, the argument is valid. This rule is a valid rule of inference because

$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.

Example : Represent the argument

If it rains today, then we will not have a party today.

If we do not have party today, then we will have a party tomorrow.

....

Therefore, if it rains today, then we will have a party tomorrow.

Symbolically and determines whether the argument is valid.

Solution.. If we let

P : It s raining today.

q : we will not have a party today.

r : We will have a party tomorrow.

The argument is of the form

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore \rightarrow r \end{array}$$

Hence the argument is a hypothetical syllogism and thus the argument is valid.

Additional valid argument forms

There are other valid inferences which state that certain form of arguments are valid. Some of these rules of inference are nothing more than reinterpretation of the two fundamental rules in the light of the law of contraposition.

Modus tollens

The argument of the form

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

This argument is valid and is called modus tollens. Modus tollens is a latin word meaning " method of denying". It can be established by using a truth table.

Example : Represent the argument

If this number is divisible6, then it is divisible by 3.

This number is not divisible by 3.

This number is not divisible by 6.

Symbolically and determine whether the argument is valid.

Solution: if we let

P: the number is divisible by 6.

Q: it is divisible by 3.

The argument may be written as

$$\begin{array}{l} P \to q \\ \sim q \\ \hline \therefore \sim P \end{array}$$

Thus by modus tollens the argument is valid.

Addition

The following argument form is valid.

$$\frac{p}{\therefore\ p \vee q}$$

This argument form is used for making generalizations. If p is true, then, more generally, p or q is true for any other statement q.

Disjunctive syllogism

The following statement form is valid:

$$\begin{array}{l} p \vee q \\ \sim q \\ \hline \therefore\ p \end{array}$$

This argument states that when there are two possibilities and one can rule one out, the other must be the case.

Example: Represent the argument

Either ram is not guilty or shyam is telling the truth.

Shyam is not telling the truth.

Ram is not guilty

Symbolically and determine whether the argument is valid.

Solution: if we let

P: ram is not guilty.

q: shyam is telling the truth.

The argument can be written as

$$\begin{array}{l} p \vee q \\ \underline{\quad \sim q} \\ \therefore \; p \end{array}$$

Thus by disjunctive syllogism, the argument is valid. We list below some of the more important valid inferences along with the inferences discussed above in the following

Rule of Inference	Tautological Form	Name
a. $\dfrac{p}{\therefore p \vee q}$ b. $\dfrac{q}{\therefore p \vee q}$	a. $p \rightarrow (p \vee q)$ b. $q \rightarrow (p \vee q)$	Addition
a. $\dfrac{p \wedge q}{\therefore p}$ b. $\dfrac{p \wedge q}{q}$	a. $(p \wedge q) \rightarrow p$ b. $(p \wedge q) \rightarrow q$	Simplification
$\dfrac{p \quad q}{\therefore p \wedge q}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\dfrac{p \rightarrow q \quad p}{\therefore q}$	$[(p \rightarrow q) \wedge p] \rightarrow q$	Modus-Ponens
$\dfrac{p \rightarrow q \quad \sim q}{\therefore \sim p}$	$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$	Modus-tollens
$\dfrac{p \rightarrow q \quad q \rightarrow r}{\therefore p \rightarrow r}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical

table. Most of the rules follow from the two fundamental rules, De- Morgan's laws and the law of contraposition.

$p \vee q$ $\sim p$ $\therefore q$	$[(p \vee q) \wedge \sim p] \rightarrow q$	Disjunction Syllogism
$p \rightarrow q \wedge (r \rightarrow s)$ $p \vee r$ $\therefore q \vee s$	$(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r) \rightarrow (q \vee s)$	Constructive dilemma
$p \rightarrow q \wedge (r \rightarrow s)$ $\sim q \vee \sim s$	$(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\sim q \vee \sim s) \rightarrow (\sim p \vee \sim r)$	Destructive dilemma.

Example: show that t is a valid conclusion from the premises p →q, q →r, r →s, ~s and p v t.

Solution: the valid argument for deducting t from the given five premises is given as a sequence.

1.	$p \to q$	Premise (Given)
2.	$q \to r$	Premise (Given)
3.	$r \to s$	Premise (Given)
4.	$p \to r$	Hypothetical syllogism using 1 and 2
5.	$p \to s$	Hypothetical syllogism using 3 and 4
6.	$\sim s$	Premise (Given)
7.	$\sim p$	Modus tollens using 5 and 6
8.	$p \vee t$	Premise (Given)
9.	t	disjunctive syllogism using 7 and 8.

Thus we can conclude t from the given premises.

Example: s is a valid conclusion from the premises p→q, p→r, ~(q^r) and s v p.

Solution: The valid agreement for deducting s from the given premises is given as a sequence.

1.	$p \to q$	Premise (Given)
2.	$p \to r$	Premise (Given)
3.	$(p \to q) \wedge (p \to r)$	Using 1 and 2
4.	$\sim (q \wedge r)$	Premise (Given)
5.	$\sim q \vee \sim r$	Demorgan's law using 4.
6.	$\sim p \vee \sim p$	Destructive dilema using 3 and 5
7.	$\sim p$	Idempotent law using 6
8.	$s \vee p$	Premise (Given)
9.	s	Disjunctive syllogism using 7 and 8

Conditional proof (CP)

If a formula s can be derived from another r and a set of premises, then the statement (r→s) can be derived from the set premises alone.

Note: if the conclusion is of the form r→s, we will take r as and additional and derive s using the given premises and r.

Example: establish the validity of the following by conditional proof.

P ^ ((p →~q) v (r^s))→(q→r)

Solution: we assume q as an additional premise

1.	p	Premise (Given)
2.	$(p \to \sim q) \vee (r \wedge s)$	Premise (Given).
3.	q	Additional premise
4.	$p \wedge q$	Using 1, 3 and conjunction
5.	$\sim(\sim p \vee \sim q)$	Using 4 and De Morgen's law
6.	$\sim(p \to \sim q)$	Using 5 and $p \to q \equiv \sim p \vee q$
7.	$r \wedge s$	Using 2, 6 and disjunctive syllogism
8.	r	Using 7 and simplification
9.	$q \to r$	Using 3, 8 and CP

Methods of proof

In previous sections we discussed proofs in the setting and symbolism of the propositional calculus. The proofs used in everyday working mathematics are based on the same logical framework as the propositional calculus but their structure is not usually displayed in the format.

Direct proof

We are typically faced with a set of hypothesis H_1, $H_{2,\ldots\ldots}H_n$ from which we want to infer a conclusion C. one of the most natural sorts of proof is the direct proof in which we show

H_1 ^ H_2 ^ …….. H_n →C

We give a direct proof of the following example.

Indirect proof

Proofs that are not direct are called indirect. The two main types of indirect proof both use the negation of the conclusion, so they are often suitable when that negation is easy to state. The first type of proof is contrapositive proof. We give a proof by contrapositive of the statement in the following example.

Example: Let n be an integer. Prove that if n^2 is odd then n is also odd.

Solution: let p: n^2 is odd and q : n is odd. We have to prove p→q is true whenever both p and q are true. We prove the contrapositive i.e., ~q→~p. suppose n is not odd i.e., n is even. Let n=2k, k is any integer. Then $n^2 = (2k)^2 = 2(2k^2)$, so n^2 is even. This shows that if n is even then n^2 is also even, which is the contrapositive of the given statement. Hence, the given statement is true.

The second type of indirect proof is known as proof by contradiction. In this type of proof, we assume the opposite of what we are trying to prove and get a logical contradiction. Hence our assumption must have been false, and therefore what we originally required to prove must be true. Hence in order to infer a conclusion c from the premises H_1, H_2 .... H_n by this method, we include ~ c as an additional premise and derive a logical contradiction(F) using ~ c and given premises. We give a proof of the statement on the example by contradiction.

Example: prove that for every positive integer n, $n^3 + n$ is even.

Solution: case (i) suppose, n is even, then n=2k for some positive integer k.

Now $n^3 + n = 8k^3 + 2k = 2(4k^3 + k)$ which is even.

Case.(ii) suppose n is odd. Then n=2k +1 for some positive integer k.

Now $n^3 + n = (8k^3 + 12k^2 + 6k + 1) + (2k+1) = 2(4k^3 + 6k^2 + 4k + 1)$

Which is even.

Hence the sum $n^3 + n$ is even.

Fallacies

It is an error in reasoning that results in an invalid argument. Several common fallacies arise in incorrect arguments. They superficially resemble those that are valid by rules of inference, but are not in fact valid. In this section we discuss two types of fallacies.

1. The fallacy of affirming the consequent.
2. They fallacy of denying the hypothesis.

The fallacy of affirming the consequent has the following form

$$\begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore p \end{array}$$

Example: show that the following argument is invalid:

If Siddhartha solved this problem, then he obtained the answer 5.

Siddhartha obtained the answer 5.

Therefore, Siddhartha solved this problem correctly.

Solution: let p and q be the proposition as

P: Siddhartha solved this problem

Q: Siddhartha obtained the answer 5.

Then the argument is of the form: if p →q, then p. This argument is faulty because the conclusion can be false even though p →q and q are true. That is, [(p→q) ^ q]→p is not a tautology. It is possible, Siddhartha obtained the correct answer 5 by luck, guessing or prior knowledge but the arguments and intermediate steps are wrong. Hence the argument is invalid.

The fallacy under lying this invalid argument form is called the fallacy of affirming the consequent because the conclusion of the argument would follow from the premises p→q when replaced by its converse. Such a replacement is not allowed, because a conditional statement is not logically equivalent to its converse.

Example: test the validity of the following argument

If two sides of a triangle are equal, then the opposite angles are equal.

Two sides of a triangle are not equal.

Therefore, the opposite angles are not equal.

Solution: let p and q be the proposition as

P: two sides of a triangle are equal.

Q: the opposite angles are equal.

Hence ~ p: two sides of a triangle are not equal. Then this argument is of the form.

$$p \rightarrow q$$
$$\sim q$$
$$\overline{}$$
$$\therefore \sim p$$

The proposition $[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q$ is not a tautology, the argument is invalid. The fallacy under lying this invalid argument form is called the fallacy of denying the hypothesis assuming the inverse because the incorrect argument is of the form $p \rightarrow q$ and $\sim p$ imply $\sim q$ which is not allowed.