gradeup

Context Free Language Part-4

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Content:-

1. Pumping Lemma For CFL'S

2. Questions on Pumping Lemma

THEOREMS

1. The family of context free languages is closed under union, concatenation and Kleene star-closure.
2. The family of context free language is not closed under intersection and complementation.
3. The intersection of a context-free language and a regular language is a context-free language.

THE PUMPING LEMMA FOR CFL'S

Let G be any CFG, then there is a constant, depending only on G, such that if W in in L(G) and $|W| \geq n$ then we may write W = $\mu vxyz$ such that

(a) $|vy| \geq 1$ then is either v or y is non-empty

(b) $|vxy| \leq n$ then for all $i \geq 0$

$\mu v^i xy^i z$ is in L(G).

Proof: Let our grammar be G = (V_n, V_t, P, S). Now in order to prove the theorem it is sufficient to show that there is a constant n such that any W ∈ L(G) of length greater than or equal to n has derivation of the form.

$$S \rightarrow \mu Xz$$

$$X \rightarrow vXy$$

$$X \rightarrow x$$

Now if we use above production as:

$$S \rightarrow \mu Xz \rightarrow \mu vXyz \overset{i-1}{\Longrightarrow} \mu v^i Xy^i z \rightarrow \mu v^i Xy^i z$$

Where μ, v, x, y, z ∈ Σ* and X, S ∈ V_n and $|vy| \geq 1$.

Then derivation X → vXy can be repeated any number of times to obtain $\mu v^i xy^i z$.

Example: Prove that language L = $\{a^n b^n c^n / n \geq 0\}$ is not context-free language.

Solution: We will pumping lemma for CFL's to prove that L is not CFL.

Here we are going to adopt the same approach as we follow in the pumping lemma for regular sets. That is first we assume that given language is regular then by applying pumping lemma on it we find an contradiction so automatically our assumption that given language is CFL will be wrong and in this way we will prove that language is not context-free.

So, let us assume that L = $\{a^n b^n c^n / n \geq 0\}$ is context-free and m be the constant obtained from the pumping lemma. Now let us pick the string

$$W = a^m b^m c^m, |W| > n$$

And according to pumping lemma let us break W in substrings as

$$W = \mu vxyz, |vy| \geq 1, |vxy| \leq m$$

Then lemma says that $\mu v^i xy^i z \in L$ for all $i \geq 0$.

Observations:

a. But this is not possible; because if v or y contains two symbols from {a, b, c} then $\mu v^2 xy^2 z$ contains a 'b' before an 'a' or a 'c' before 'b' then $\mu v^2 z$ will not be in L.
b. In other case if v and y each contains only a's, b's or only c's, then $\mu v^i xy^i z$ can not contain equal number of a's, b's and c's. So $\mu v^i xy^i z$ will not be L.
c. If both v and y contains all three symbols a, b and c then $\mu v^2 xy^2 z$ will not be in L by the same logic as in observation a.

So it is a contradiction to our statement (we assume that L is CFL).

So L is not context-free.

Example: Prove that language L = $\{WW / W \in \{a, b\}^*\}$ is not context-free.

Solution: Let us assume that language L is context-free and m must be constant obtained from the pumping lemma. Now

$$W = a^m b^m a^m b^m \in L$$

And according to pumping lemma we can write W in the following form

$$W = \mu vxyz, |vy| \geq 1, |vxy| \leq m$$

Then lemma says that $\mu v^i xy^i z \in L$ for all $i \geq 0$.

DECIDIABILITY

There ate however, some other fundamental questions about CFG that we can answer:

(a) Given a CFG, can G tell whether or not it generates any words at all? This is the question of emptiness.
(b) Given a CFG, we can tell whether or not the language it generate is finite or infinite? This is the question of finiteness.

(c) Given a CFG and a particular string of letter W, can be tell whether or not W can be generated by the CFG? This is the question of membership.

Example: show that the language L = $\{a^m\, b^m\, c^n / m \leq n \leq 2m\}$ is not context free.

Solution: Given language is:

L = $\{a^m\, b^m\, c^n / m \leq n \leq 2m\}$, we can prove that L is not context free by contradiction. Let us assume that L is context free, then it must follow the pumping Lemma for CFL's

Let w = $a^n\, b^n\, c^{2n} \in L$ where n is a constant obtained from pumping lemma.

Then w = $\mu vxyz$ where $|vxy| \leq n$ and $|vy| \geq 1$ according to pumping lemma for CFL's $\mu v^{I}\, xy^{i}z$ = $\forall\, i \geq 0$ will belongs to the L.

Observation: here vy can not contain all the three symbols a, b and c. if vy contains only a's and b's the then we can choose i such that $\mu v^{I}\, xy^{i}z$ has more tha in occurrences of a (or b) and exactly 2n occurrences of c. This means $\mu v^{I}\, xy^{i}z \in L$, a contradiction by proper choice of i. thus given language L is not content free.

Example: Prove that the language L = $\{w/w \in \{a, b, c\}^* / n_a(w) < n_b(w)$ and $n_a(w) < n_w(c)\}$ is not context free.

Solution: Let us assume

$$w = a^n b^{n+1}\, c^{n+1}$$

$$= \mu vxyz,\ |vy| \geq 1,\ |vxy| \leq n$$

Then $\forall\, i \geq 0\ \mu v^{I}\, xy^{i}z \in L$

Observation: If v or y contains atleast one a, then vy contains c's. Therefore $\mu v^2\, xy^2z$ contains atleast as many a's and c's and can not be in L. if neither v nor y contains an a, then $\mu v^o\, xy^oz$ still contains n a's; since vy contains one of the other two symbols, $\mu v^o\, xy^oz$ contains fewer occurrences of that symbol then z does and therefore is not L. we have obtained contradiction and may conclude that L is not context free.

Example: Prove that L = $\{0^{2i} / i \geq 1\}$ is not CFL.

Solution: Let us assume that L is CFL. Then by pumping lemma, there are substrings u, v, x, y, z with w = $\mu vxyz = 0^{2r} \in L$, where $|vy| \geq 1$, $|vxy| \leq n$

$\rightarrow$ for $\forall\, i \geq 0\ \mu v^{I}\, xy^{i}z \in L$.

Since $|vy| \geq 1$, if follows that v = 0^P and y = 0^q where p + q ≥ 1. Since $i \geq 0\ \mu v^{I}\, xy^{i}z \in L$, we have $|\mu v^{I}xy^{i}z| = 2^r + (p + q)(i - 1)$ [that is $|uvu^{i-1}xyy^{i-1}z| \rightarrow |\ \mu vxyz.\ v^{i-1}y^{i-1}|$ is a power of 2, say 2j for some j.

$$(p + q)(i - 1) = 2^i - 2^r$$

$$(p + q)(i - 1) = 2^i - 2^r \quad \leftrightarrow \quad (p + q)(i - 1) + 2^r = 2^j$$

$$\rightarrow \quad 2^{r+1}(p + q) + 2^r = 2^j$$

$$\rightarrow \quad 2^r(2(p + q) + 1) = 2^j$$

The right hand side of the equality is power 2 but the left hand side of the equality in the above expression is a multiple of an odd integer, namely (2(p + q) + 1), which cannot be a power of 2. Thus it is not a CFL.

Example: Check that the language L = $\{a^m b^m c^n / m \le 2m)$ is not CFL.

Solution: Again we can prove L is not CFL by contradiction, that is let us assume that L is CFL, then it most follow pumping lemma as follows:

$$w = a^n b^n c^{2n} \qquad \text{n is constant from pumping lemma}$$

$$= \mu vxyz, \qquad |vy| \ge 1, \ |vxy| \le n$$

→ ∀ i ≥ 0, $uv^I xy^i z$ has more than 2n occurrence of a or b and exactly 2n occurrences of c. hence /
$uv^I xy^i z \in L$, which is a contradiction so L is not CFL.

FOLLOWING LANGUAGES ON ∑ = {a, b, c} ARE NOT CONTEXT-FREE

1. $L = \{a^n b^j c^k : k = j^n\}$
2. $L = \{ a^i b^j : i \ge (j - 1)^3\}$
3. $L = \{ a^i b^j c^k : k > i, k > j^n\}$
4. $L = \{ a^i b^i c^j / i \le j \le 2i\}$
5. $L = \{ww^R w / w \in \{a, b\}^*\}$

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