Quantum Chaos and Measurements

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1 Introduction

Chaos in a classical system means that this system is completely determined by its initial conditions, but a small change in these initial conditions will lead to a major change in the behaviour of the system. This makes it practically impossible to predict the system's behaviour, since the initial conditions are not usually known with infinite precision.

In quantum mechanics, chaos does not exist in the same way. Two slightly different initial states will retain their overlap under unitary time propagation, and no exponential divergence will occur. Classically forbidden crossings in phase space between ordered trajectories are meaningless in quantum mechanics, since Heisenberg's uncertainty principle only allows approximate determination of any trajectory, and thus allows a classically chaotic trajectory to be partially ordered.

In this project we consider the quantum equivalent of the classically chaotic system of a kicked top. We will model it, attempt to quantify how chaotic its behaviour is, and finally consider the effect of weak, continuous measurement on the system.

We will work in units of $\hbar = 1$.

2 Theory

2.1 The kicked quantum top

An example of a quantum system displaying chaotic behaviour is the kicked top system [1], which we will model with the Hamiltonian

$$H = pF_y \sum_{n=0}^{\infty} f(t - n\tau) + \frac{\chi}{2F\tau} F_x^2. \tag{1}$$

The first term is the kick, described by a function with period τ, which turns the top around the y-axis by an angle p. The second term rotates the system around the x-axis, where the χ parameter determines whether the dynamics are chaotic or regular. F is the system spin, with projections

$$\mathbf{F} = (F_x, F_y, F_z). \tag{2}$$

In this project, we will study a system where $\tau = 1$, $p = -0.99$, $\chi = 2$ and $F = 3$ in order to replicate and build upon the results in Ref. [1]. Although that reference has $p = 0.99$, it was found that the described dynamics were better replicated with a negative angle.

To illustrate the behaviour of the quantum spinning top we use a Bloch sphere with a Husimi quasi-probability distribution mapped upon it,

$$Q(\theta, \phi) = \frac{2F+1}{4\pi} \langle \theta, \phi | \rho | \theta, \phi \rangle , \tag{3}$$

where $|\theta, \phi\rangle$ is a spin coherent state, that is, a state which is maximally localized in the (θ, ϕ) direction. All figures will go from blue (low probability) to red (high probability). To work in the spin basis we expand the spin coherent states in spin eigenstates [2]

$$|\theta, \phi\rangle = \sum_{m=-j}^{j} \sqrt{\binom{2j}{j+m}} \left(\cos\frac{\theta}{2}\right)^{j+m} \left(\sin\frac{\theta}{2}\right)^{j-m} e^{i(j-m)\phi} |jm\rangle . \tag{4}$$

Here we have used the usual designations of spin and projection, j and m, but everywhere else the spin will be designated with F. Note that a spin vector (F_x, F_y, F_z) can be translated directly to a spin coherent state $|\theta, \phi\rangle$, which will be maximally localized in the direction of the spin vector. To illustrate the Husimi distribution at different times we use the time propagation of the density operator [3]

$$\frac{dQ(\theta, \phi)}{dt} = \frac{2F+1}{4\pi} \langle \theta, \phi | \frac{d\rho}{dt} | \theta, \phi \rangle , \tag{5}$$

$$\frac{d\rho}{dt} = i[\rho, H(t)]. \tag{6}$$

Eq. (6) constitutes a differential equation equivalent to the Schrödinger equation, and numerical integration of this allows us to propagate the system. The problem would be numerically solved faster if we propagated the states and constructed ρ afterwards, but propagating ρ allows us to introduce weak measurement as shown below.

Fig. 1 shows the Husimi distribution averaged over 40 periods, with initial conditions in a classically non-chaotic part of the phase space. It can be seen that the dynamics are basically confined to this "order island". In Fig. 2 the initial conditions locate the spin in a smaller order island, with another order island close by, and we can see a classically forbidden tunnelling occurring. When we turn up the spin to $F = 10$, we can see that this tunnelling is suppressed (Fig. 3), in agreement with the correspondence principle. Finally, Fig. 4 shows the dynamics of a spin located outside the order islands, resulting in a Husimi distribution which spreads out across the Bloch sphere, but avoids the order islands (the "chaos sea").

2.2 Performing weak measurements

Usually, measurements in quantum mechanics are regarded as complete projections onto eigenstates, causing complete decoherence of the system. The question is, how much can we retain the quantum chaotic/ordered dynamics

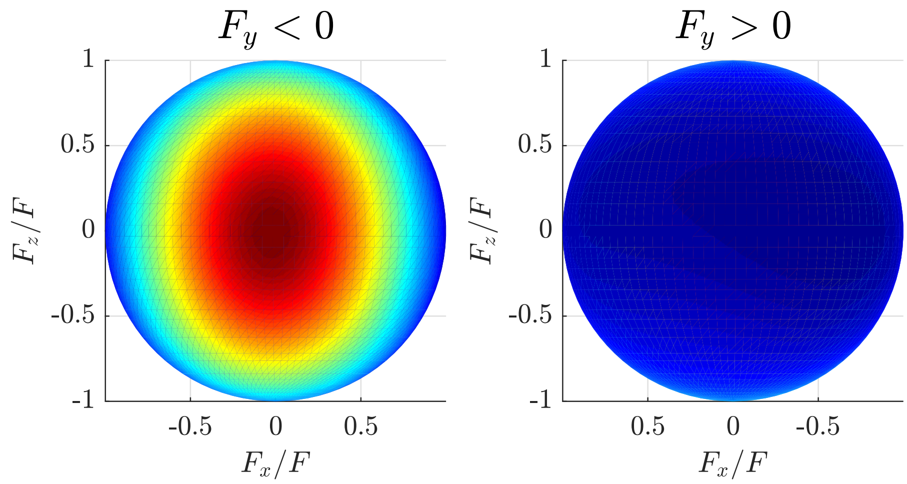

Figure 1: Major order island. Initial condition $(0, -0.99, -0.16)$.

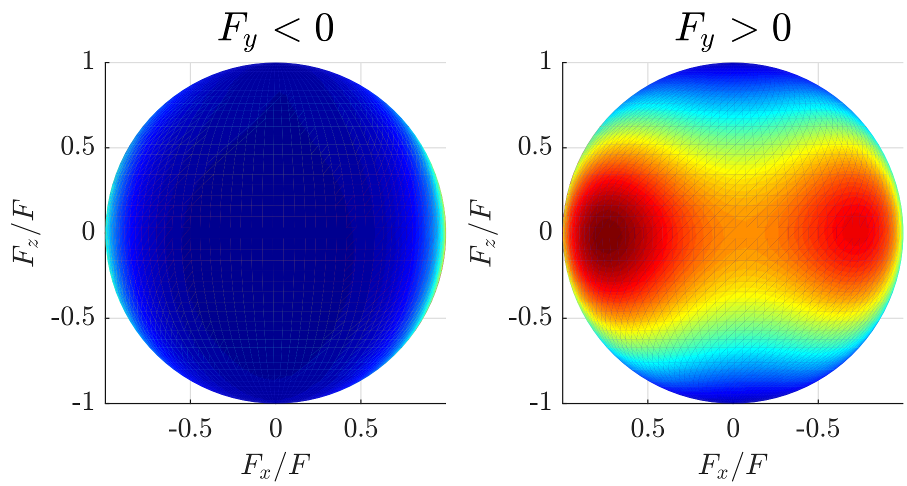

Figure 2: Tunneling between two order islands. Initial condition $(0.7, 0.7, -0.16)$.

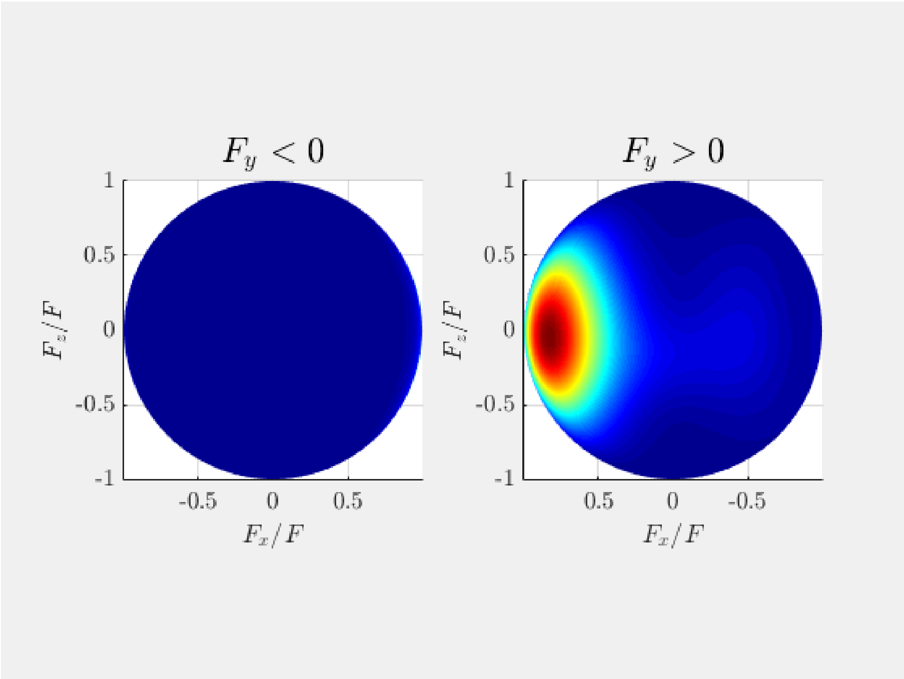

Figure 3: For larger spin $F = 10$, the tunneling between order islands gets suppressed. This agrees with the correspondence principle, since larger spin approaches the classical system. Initial condition $(0.7, 0.7, -0.16)$.

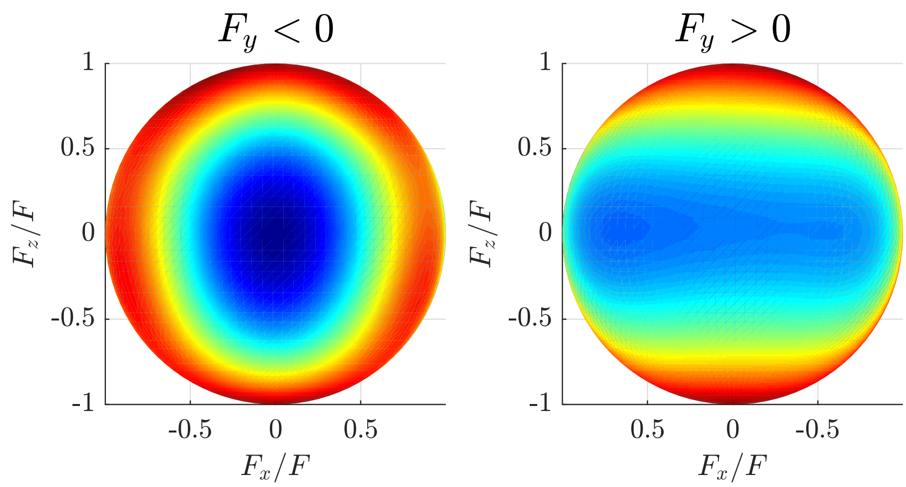

Figure 4: Chaos sea. Initial condition $(-0.94, -0.31, -0.16)$.

of the system if the measurements are weak (non-projective), only extracting partial information. We will implement such a weak measurement of the spin by using the stochastic master equation from Ref. [4],

$$d\rho_t = -i[H(t), \rho_t]dt + \sum_{k=x,y,z} \left(\kappa \mathcal{D}[F_k]\rho_t dt + \sqrt{\eta\kappa}\mathcal{H}[F_k]\rho_t dW_t\right). \tag{7}$$

The first term corresponds to Eq. (6), κ is the probe field strength (measurement strength) and η is detector efficiency. For $\kappa = 0$, we perform no measurement and the system propagates according to the Schrödinger equation. For $\eta = 1$, we will remain in a pure state (on the surface of the Bloch sphere) throughout the propagation. By experimenting with the value of κ, we can study the effect of measurement strength on the evolution of our system. We are using the super-operators

$$\mathcal{D}[\sigma_k]\rho = \sigma_k \rho \sigma_k^\dagger - \frac{1}{2}\left\{\sigma_k^\dagger \sigma_k, \rho\right\} \tag{8}$$

$$\mathcal{H}[\sigma_k]\rho = \sigma_k \rho + \rho \sigma_k^\dagger - \mathrm{Tr}\left(\sigma_k \rho + \rho \sigma_k^\dagger\right)\rho. \tag{9}$$

The dW_t factor is the Wiener increment [5], which at a time t is

$$dW_t = W_{t+dt} - W_t, \tag{10}$$

where W_t is an element from the Wiener process. This is a continuous stochastic process where each increment is taken from a normal distribution with mean 0 and variance dt, thus having each increment independent of previous increments.

We can relate the stochastic master equation to an actual measurement of a spin component through the voltage

$$V_i(t) = \mathrm{Tr}(F_i \rho_t) + \frac{dW_t}{2\sqrt{\eta\kappa}dt} = \langle S_i \rangle + J \quad (i = x, y, z). \tag{11}$$

For η or κ small, the stochastic noise term J will dominate the measurement.

2.3 Chaos signature

The quantum spinning top has small areas on the Bloch sphere ("islands") of order, with the rest of the sphere (the "sea") exhibiting chaos. When propagating in the chaotic sea, the Husimi distribution will spread itself over large areas of the sphere, reflecting a large variance in possible spin vector states. Starting on an order island, however, the distribution will remain largely within the island, only allowing a small distribution of possible spin vector states on the island.

In other words, if we calculate the expectation value of each spin component at each time step, and then sum the variances for each component,

$$s^2 = \sum_{i=x,y,z} \sigma(F_i)^2 \tag{12}$$

we get a quantity which will be small for localized (non-chaotic) spin, and large for spread out (chaotic) spin. This can be time-averaged $\overline{s^2}$ to get an even

Table 1: Chaos signatures

Case	$\overline{s^2}$
Large order island	4.2
Tunneling order islands	6.1
Chaos sea	10.0

more focused quantification. Simulation without implemented measurement shows the behaviour listed in Table 1. The well-defined order island has a low variance, the chaos sea has a high variance, and the case of tunneling between two order islands has intermediary variance. These results suggest that it is a good measure.

Note that this measure depends on the size F of the spin, and can not be used to compare across different spins. For example, the $F = 10$ order island in Fig. 3 has $\overline{s^2} \simeq 32$.

2.4 Numerical implementation

In Ref [1], f is chosen to be a delta function which can be solved analytically. We have chosen a narrow Gaussian instead since this can be directly implemented numerically and is closer to the experimental reality.

The propagation of the system is done by solving the differential equation in Eq. (7) with Matlabs ODE45 solver. This can not be done directly since the differential equation is not ordinary but stochastic, and thus require a very different numerical solution strategy. The implementation of such a strategy is, however, beyond the scope of this project, and we will instead consider the ODE45 solution to be approximately correct. Since ODE45 has an adaptive step size, we must be careful with our implementation of the Wiener process. The numerical integration is done with finite time steps $dt \to \Delta t$, so we can construct an approximate Wiener process with a finite step

$$dW_t \to \Delta W_t = W_{t+(\Delta t)'} - W_t. \tag{13}$$

ODE45 has an adaptive step size, and must be allowed to treat the Wiener process as continuous, even though it is numerically generated as discrete. Thus we allow ODE45 to interpolate linearly between each discrete element. To conserve the property that each increment is independent of all previous increments, we must then require that no ODE45 time step is smaller than the resolution of our Wiener process,

$$(\Delta t)_{\min} > (\Delta t)'. \tag{14}$$

The minimum step size of Matlabs ODE45 solver can only be indirectly influenced by adjusting the tolerances of the solver. Thus, one must investigate the approximate smallest step size for a given solver tolerance, and make sure that $(\Delta t)'$ is smaller at all times.

The approximate nature of the ODE solver introduces negligible imaginary parts on the diagonal of ρ, which are ignored. The full MatLab code used

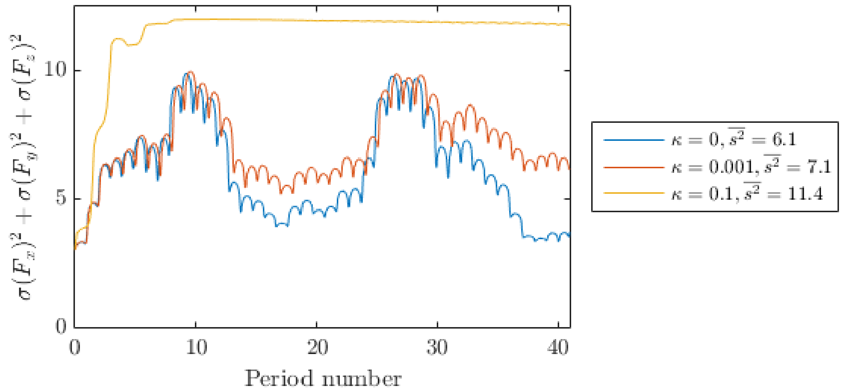

Figure 5: Evolution of the chaos measure for different values of κ. Independently of inital values, all systems evolve into very chaotic states with $\overline{s^2} \simeq 11$ for $\kappa = 0.1$.

for this project (including a tool for animating the evolution of the Husimi distribution) can be found in Ref. [6].

3 Results

All results are for $\eta = 1$. Changing this value gave very little difference except for slightly larger noise amplitude. Because of the stochastic nature of the results, it would have been natural to have included averages over many different propagations, but since the included results have very little dependence on the specifics of the noise, these have not been produced.

Fig. 5 shows the proposed measure Eq. (12) for different κ. Fig. 6 shows that the state purity $\mathrm{Tr}(\rho^2)$ quickly dips away from unity for $\kappa \neq 0$, showing that the results in the previous figure are most likely a numerical error. This is corroborated by Fig. 7, which shows that the large variance is not due to fluctuations of spin, but rather that it recedes into the Bloch sphere. These numerical errors most likely stem from the approximate implementation of the Wiener process, although Figs. 8 and 9 show that when considering the x-component voltage from Eq. (11), the noise behaves as predicted. The periodicity of the noise is caused by the ODE45 solver taking smaller steps near the Gaussian $f(t)$ pulses.

4 Discussion and conclusion

We have successfully simulated the quantum spinning top and quantized how chaotic its behaviour is through variance in spin component expectation value. However, the weak measurement has been implemented with significant numerical error, which is reflected in a drastic loss of state purity appearing for perfect detector efficiency. This is most likely caused by an erroneous approximation of the stochastic master equation as an ordinary differential equation, and subsequent attempt at implementation of the Wiener process into a numerical

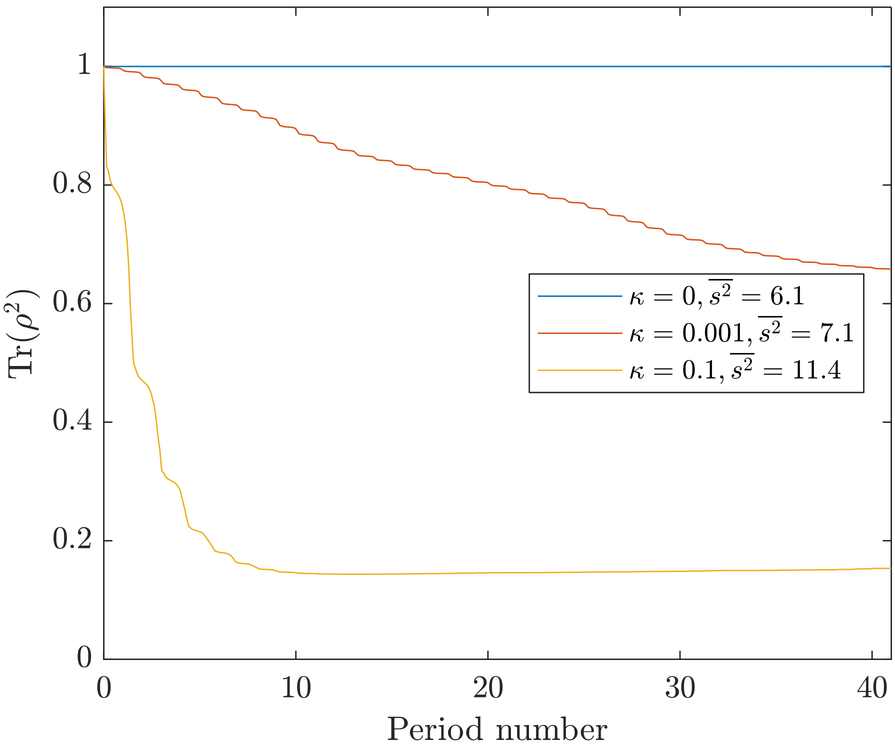

Figure 6: Evolution of the state purity $\mathrm{Tr}(\rho^2)$ for different values of κ. Since we have perfect detector efficiency, this should be constant $\mathrm{Tr}(\rho^2) = 1$. The significant loss of purity means there are significant numerical errors in the results.

ODE45 solver. The results suggest that weak measurement pushes the system into a state with maximum spin component variance (a highly chaotic state), but this is most likely a result of the loss of state purity, since this pushes the state into the Bloch sphere. Better results will probably be achieved with a proper numerical implementation of the stochastic differential equation, something that might be easier to do with a more specific tool such as the QuTiP package for Python. This, however, is beyond the scope of this project.

References

1. Chaudhury, S., Smith, A., Anderson, B. E., Ghose, S. & Jessen, P. S. Quantum signatures of chaos in a kicked top. *Nature* **461,** 768–771 (Oct. 2009).
2. Gazeau, J.-P. in *Coherent states in quantum physics* 80–81 (Wiley, 2009).
3. Sakurai, J. J. & Napolitano, J. J. in *Modern Quantum Mechanics* 2nd New International Edition, 187 (Pearson, 2014).

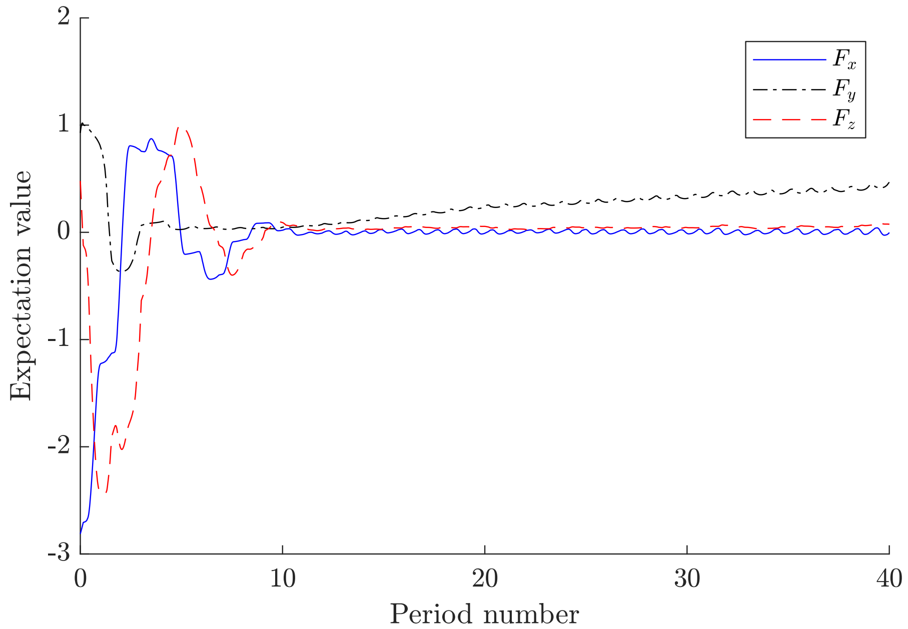

Figure 7: Evolution of expectation values of each spin component for the tunneling order islands, $\kappa = 0.1$. All fluctuation in spin value is significantly damped by the loss of purity, resulting in states more or less uniformly spread over the Bloch sphere.

4. Xu, P. *et al.* Measurement of the topological Chern number by continuous probing of a qubit subject to a slowly varying Hamiltonian. *Phys. Rev. A* **96,** 010101 (1 July 2017).
5. Wikipedia. *Wiener process* `https://en.wikipedia.org/wiki/Wiener_process`.
6. Michelsen, A. B. *quantum-chaos* `https://github.com/AndBM/quantum-chaos`.

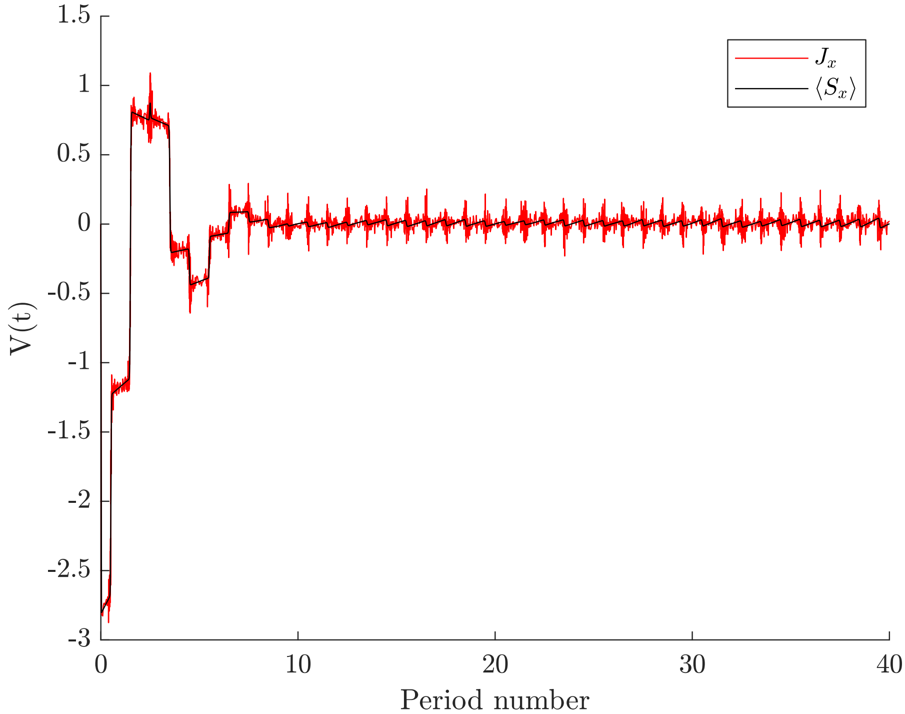

Figure 8: Output voltage for $\kappa = 0.1$. As expected, the noise does not dominate.

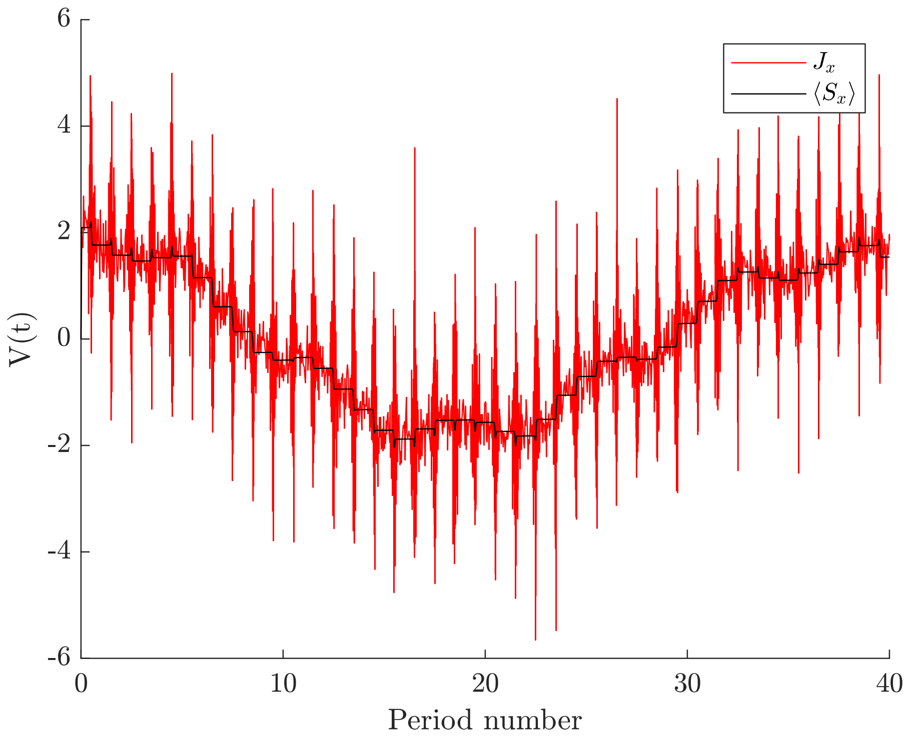

Figure 9: Output voltage for $\kappa = 0.001$. As expected, the noise dominates.