Quantum Chaos and Measurements
(Working title)

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January 28, 2018

# 1 Introduction

Chaos in a classical system means that this system is completely determined by its initial conditions, but a small change in these initial conditions will lead to a major change in the behaviour of the system. This makes it practically impossible to predict the system's behaviour, since the initial conditions are not usually known with infinite precision.

In quantum mechanics, a system is only probabilistically determined by its initial conditions. .

FiXme Note: insert example from quantum top paper

# 2 Theory

## 2.1 The density operator

## 2.2 The kicked quantum top

An example of a quantum system displaying chaotic behaviour is the kicked top system [1], which we will model with the Hamiltonian

$$H = \hbar p F_y \sum_{n=0}^{\infty} f(t - n\tau) + \hbar \frac{\chi}{2F\tau} F_x^2. \tag{1}$$

The first term is the kick, described by a function with period $\tau$, which turns the top around the $y$-axis by an angle $p$. The second term rotates the system around the $x$-axis, where the $\chi$ parameter determines whether the dynamics are chaotic or regular. $F$ is the system spin, with projections

$$\mathbf{F} = (F_x, F_y, F_z). \tag{2}$$

In this project, we will study a system where $tau = 1$, $p = -0.99$, $\chi = 2$ and $F = 3$ in order to replicate and build upon the results in Ref. [1]. Although that reference has $p = 0.99$, it was found that the described dynamics were better replicated with a negative angle.

To illustrate the behaviour of the quantum spinning top we use a Bloch sphere with a Husimi quasi-probability distribution mapped upon it,

$$Q(\theta, \phi) = \frac{2F+1}{4\pi} \langle \theta, \phi | \rho | \theta, \phi \rangle, \tag{3}$$

where $|\theta, \phi\rangle$ is a spin coherent state, that is, a state which is maximally localized in the $(\theta, \phi)$ direction. To work in the spin basis we expand the spin coherent states in spin eigenstates [2]

$$|\theta, \phi\rangle = \sum_{m=-j}^{j} \sqrt{\binom{2j}{j+m}} \left(\cos\frac{\theta}{2}\right)^{j+m} \left(\sin\frac{\theta}{2}\right)^{j-m} e^{i(j-m)\phi} |jm\rangle. \tag{4}$$

Here we have used the usual designations of spin and projection, $j$ and $m$, but everywhere else the spin will be designated with $F$. Note that a spin vector $(F_x, F_y, F_z)$ can be translated directly to a spin coherent state $|\theta, \phi\rangle$, which will be maximally localized in the direction of the spin vector. To illustrate the Husimi distribution at different times we use the time propagation of the density operator [3]

$$\frac{dQ(\theta,\phi)}{dt} = \frac{2F+1}{4\pi} \langle\theta,\phi| \frac{d\rho}{dt} |\theta,\phi\rangle , \tag{5}$$

$$\frac{d\rho}{dt} = i[\rho, H(t)]. \tag{6}$$

Note that we are working in units where $\hbar = 1$. Eq. (6) constitutes a differential equation equivalent to the Schrödinger equation, and numerical integration of this allows us to propagate the system. The problem would be numerically solved faster if we propagated the states and constructed $\rho$ afterwards, but propagating $\rho$ allows us to introduce weak measurement as shown below.

### 2.3 Performing weak measurements

Usually, measurements in quantum mechanics are regarded as complete projections onto eigenstates, causing complete decoherence of the system. Quantum chaos is allowed by the indeterminacy of an uncollapsed system, but some degree of chaos may still be allowed if the measurements are weak (non-projective), only extracting partial information. We will implement such a weak measurement of the spin by using the stochastic master equation from Ref. [4],

$$d\rho_t = -i[H(t), \rho_t]dt + \sum_{k=x,y,z} \left(\kappa\mathcal{D}[\sigma_k]\rho_t dt + \sqrt{\eta\kappa}\mathcal{H}[\sigma_k]\rho_t dW_t\right). \tag{7}$$

The first term corresponds to Eq. (6), $\kappa$ is the probe field strength (measurement strength), $\sigma_k$ is the $k$'th Pauli spin matrix and $\eta$ is detector efficiency. For $\kappa = 0$, we perform no measurement and the system propagates according to the Schrödinger equation. For $\eta = 1$, we will remain in a pure state (on the surface of the Bloch sphere) throughout the propagation. By experimenting with the value of $\kappa$, we can study the effect of measurement strength on the evolution of our system. We are using the super-operators

$$\mathcal{D}[\sigma_k]\rho = \sigma_k\rho\sigma_k^\dagger - \frac{1}{2}\left\{\sigma_k^\dagger\sigma_k, \rho\right\} = \sigma_k\rho\sigma_k^\dagger - \rho, \tag{8}$$

$$\mathcal{H}[\sigma_k]\rho = \sigma_k\rho + \rho\sigma_k^\dagger - \mathrm{Tr}\left(\sigma_k\rho + \rho\sigma_k^\dagger\right)\rho, \tag{9}$$

since $\sigma_k$ is hermitian.

The $dW_t$ factor is the Wiener increment [5], which at a time $t$ is

$$dW_t = W_{t+dt} - W_t, \tag{10}$$

where $W_t$ is an element from the Wiener process. This is a stochastic process where each increment is taken from a normal distribution with mean 0 and variance $dt$, thus having each increment independent of previous increments.

### 2.4 Chaos signature

The quantum spinning top has small areas on the Bloch sphere ("islands") of order, with the rest of the sphere (the "sea") exhibiting chaos. When propagating in the chaotic sea, the Husimi distribution will spread itself over large areas of the sphere, reflecting a large variance in possible spin vector states. Starting on an order island, however, the distribution will remain largely

Table 1: Chaos signatures

| Case | $s^2$ |
|---|---|
| Large order island | 0.8 |
| Tunneling order islands | 1.6 |
| Chaos sea | 2.3 |

within the island, only allowing a small distribution of possible spin vector states on the island.

In other words, if we calculate the expectation value of each spin component at each time step, and then sum the variances (the squared standard deviation $\sigma$) for each component,

$$s^2 = \sum_{i=x,y,z} \sigma[\langle F_i \rangle(t)]^2 = \sum_{i=x,y,z} \sigma[\mathrm{Tr}(\rho(t)F_i)]^2 \tag{11}$$

we get a quantity which will be small for localized (non-chaotic) spin, and large for spread out (chaotic) spin. Simulation without implemented measurement shows the behaviour in Table 1. The well-defined order island has a low variance, the chaos sea has a high variance, and the case of tunneling between two order islands has intermediary variance. These results suggest that it is a good measure.

## 2.5 Numerical implementation

In Ref [1], $f$ is chosen to be a delta function which can be solved analytically. We have chosen a narrow Gaussian instead since this can be directly implemented numerically and is closer to the experimental reality. The propagation of the system is done by solving the differential equation in Eq. (7) with Matlabs ODE45 solver. This can not be done directly since the differential equation is not ordinary but stochastic, and thus require a very different numerical solution strategy. The implementation of such a strategy is, however, beyond the scope of this project, and we will instead consider the ODE45 solution to be approximately correct. Since ODE45 has an adaptive step size, we must be careful with our implementation of the Wiener process. Since the numerical integration is done with finite time steps $dt \to \Delta t$, we can construct an approximate Wiener process with a finite step

$$dW_t \to \Delta W_t = W_{t+(\Delta t)'} - W_t. \tag{12}$$

ODE45 has an adaptive step size, and must be allowed to treat the Wiener process as continuous, even though it is numerically generated as discrete. Thus we must allow ODE45 to interpolate linearly between each discrete element. To conserve the property that each increment is independent of all previous increments, we must then require that no ODE45 time step is smaller than the resolution of our Wiener process,

$$\Delta t > (\Delta t)'. \tag{13}$$

The minimum step size of Matlabs ODE45 solver can only be indirectly influenced by adjusting the tolerances of the solver. Thus, one must investigate the approximate smallest step size for a given tolerance, and make sure that $(\Delta t)'$ is smaller at all times.

The approximate nature of the ODE solver introduces negligible imaginary parts on the diagonal of $\rho$, which are ignored. The full MatLab code used for this project can be found in Ref. [**URL**].

## 3 Results

For spin half and $\kappa \neq 0$, one would expect constant $|\langle F \rangle| = 1/2$, but in the simulation the spin norm falls off slowly. When evolving according to Eq. (6), $|\langle F \rangle|$ is approximately constant in the spin half case (though with a strange dip around period 20, perhaps a numerical effect?), while with evolution according to Eq. (7) for $\kappa = 0.001$, we have the following progression in time (from a spin-coherent $\uparrow$ state)

$$\begin{aligned} \rho_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \rho_1 &= \begin{pmatrix} 0.0005 & (1+i) \cdot 10^{-4} \\ (1-i) \cdot 10^{-4} & 0.9995 \end{pmatrix} \\ \rightarrow \rho_2 &= \begin{pmatrix} 0.0010 & 0 \\ 0 & 0.9990 \end{pmatrix} \end{aligned} \tag{14}$$

This is before the system is pushed, and is thus completely because of the weak measurement. If we then consider $|F|$ for the last density matrix, the $F_x$ and $F_y$ terms will be zero, and

$$\begin{aligned} \mathrm{Tr}(\rho_2 F_z) &= \mathrm{Tr}\left( \begin{pmatrix} 0.5 & 0 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} 0.0010 & 0 \\ 0 & 0.9990 \end{pmatrix} \right) \\ &= \mathrm{Tr} \begin{pmatrix} 0.0005 & 0 \\ 0 & -0.4995 \end{pmatrix} = -0.4990. \end{aligned} \tag{15}$$

We can interpret this evolution as a mixing of the $|\uparrow\rangle$ and $|\downarrow\rangle$ states (in other words, the up or down $F_z$ polarization of the system).

## References


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5. Wikipedia. *Wiener process* `https://en.wikipedia.org/wiki/Wiener_process`.