Conditional Gaussian

$$\Delta = -\frac{1}{2}(\vec{x} - \vec{\mu})^T \Sigma^{-1} (\vec{x} - \vec{\mu})$$

$$\vec{x} = \begin{pmatrix} x_a \\ x_b \end{pmatrix}; \quad \vec{\mu} = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix},$$

$$\begin{aligned}
\Rightarrow \Delta = & -\frac{1}{2}(x_a - \mu_a)^T \Lambda_{aa} (x_a - \mu_a) && \text{(1) quadratic term} \\
& \left.\begin{aligned} & -\frac{1}{2}(x_b - \mu_b)^T \Lambda_{ba} (x_a - \mu_a) && (2) \\ & -\frac{1}{2}(x_a - \mu_a)^T \Lambda_{ab} (x_b - \mu_b) && (3) \end{aligned}\right\} && \text{linear term.} \\
& - \ldots\ldots && \text{(only dependent on } x_b, \mu_b)
\end{aligned}$$

$$\Rightarrow \boxed{\Lambda_{aa} = \Sigma_{a|b}^{-1}} \quad \text{or} \quad \boxed{\Sigma_{a|b} = \Lambda_{aa}^{-1}}$$

(2)+(3): $\Rightarrow$ linear in x_a

$$+\frac{1}{2} x_a^T (\Lambda_{aa} \mu_a) + \frac{1}{2} (\mu_a^T \Lambda_{aa} x_a)$$

$$-\frac{1}{2} (x_b - \mu_b)^T \Lambda_{ba} x_a - \frac{1}{2} x_a^T \Lambda_{ab} (x_b - \mu_b)$$

$$= x_a^T \underbrace{\left[\Lambda_{aa} \mu_a - \Lambda_{ab} (x_b - \mu_b)\right]}_{\Sigma_{a|b}^{-1} \mu_{a|b}} \qquad (A)$$

why?

$$\frac{1}{2}(x_a - \mu_{a|b})^T \Sigma_{a|b}^{-1} (x_a - \mu_{a|b})$$

$$= \underbrace{x_a^T \Sigma_{a|b}^{-1} x_a}_{\text{quadr.}} - \underbrace{x_a^T \Sigma_{a|b}^{-1} \mu_{a|b}}_{\text{linear}} + \ldots$$

$$\Rightarrow \quad \Sigma_{a|b}^{-1} \mu_{a|b} = \Lambda_{aa} \mu_a - \Lambda_{ab}(x_b - \mu_b) \quad (B)$$

$$(A,B) \Rightarrow \boxed{\mu_{a|b} = \Sigma_{a|b} \left[\Lambda_{aa} \mu_a - \Lambda_{ab}(x_b - \mu_b)\right]}$$