Laboration in Automatic Control

Control of an inverted pendulum

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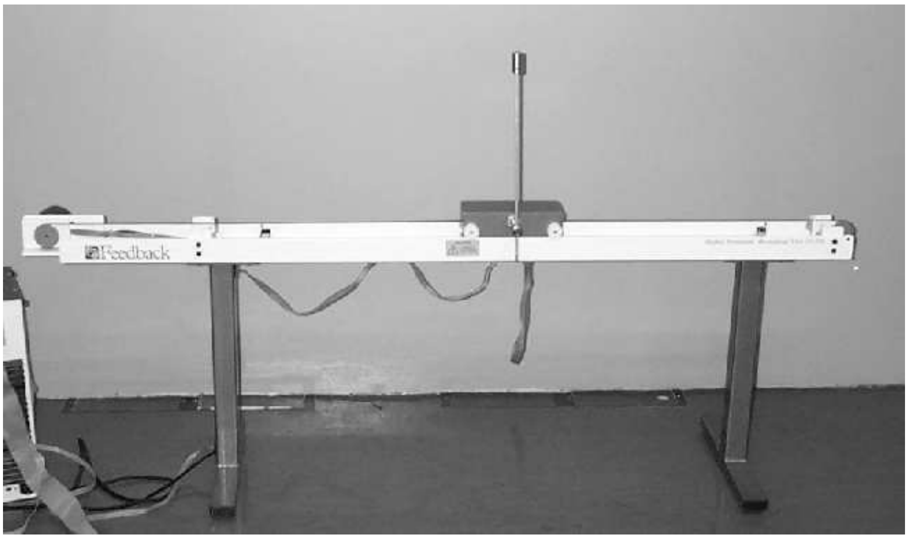

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Contents

1 Introduction

The aim with this laboratory exercise is to show that state-space models and state feedback are useful, and sometimes even necessary, to control some systems.

As an example of a control problem that is very difficult without state-space models, this lab session will use a simulation model of a so-called inverted pendulum. An example of an inverted pendulum is shown in Figure 1. This problem is indeed motivated from real life. The systems shown in Figure 2 can both be seen as examples of the problem with inverted pendulums. If the output is the angle of the pendulum relative to the vertical axis (in upright position), we realize that the system is unstable, since the pendulum will fall down if we release it with a small angle. To stabilize the system, i.e., to keep the pendulum in upright position, a feedback control system must be used.

The goal with this lab session is to design a control system that keeps the pendulum in upright position. In the first part of the lab session we will assume that all the states can be measured, while in the second part, only the cart position is measurable. In the last part it is assumed that both cart position and pendulum angle can be measured.

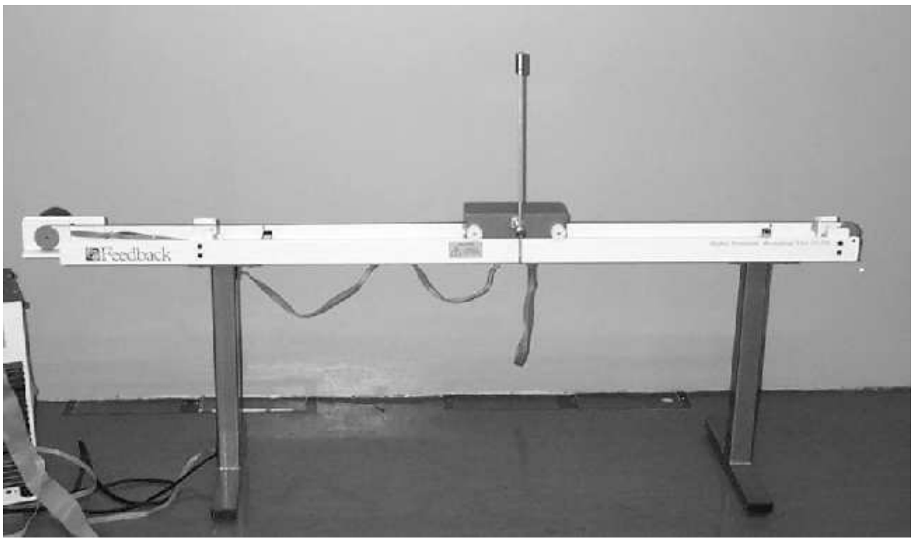
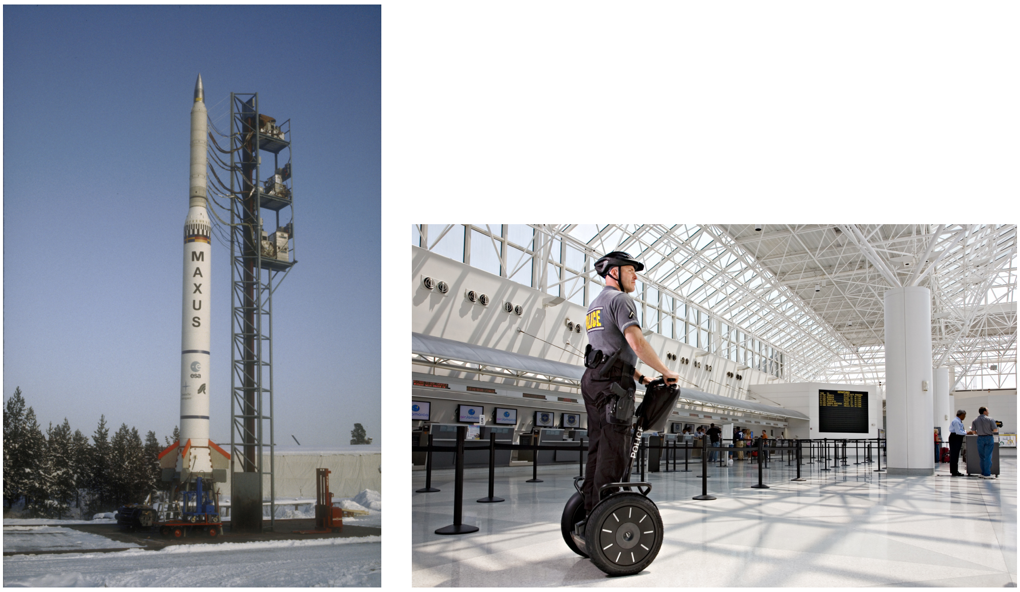


Figure 1: Inverted pendulum.



(a) MAXUS 1, Kiruna

(b) Segway® PT

Figure 2: Examples of real systems that behave as inverted pendulums. Figure 2(a): Rocket MAXUS 1 in starting position, Esrange Space Center, Kiruna, picture from the open image gallery of Rymdbolagetfoto, SSC [2011]. Figure 2(b): Segway® Personal Transporter (PT), a self-balancing vehicle, picture from the open image gallery of Segway Inc. [2011].

2 Description of the system

The inverted pendulum can be described with the figure below.

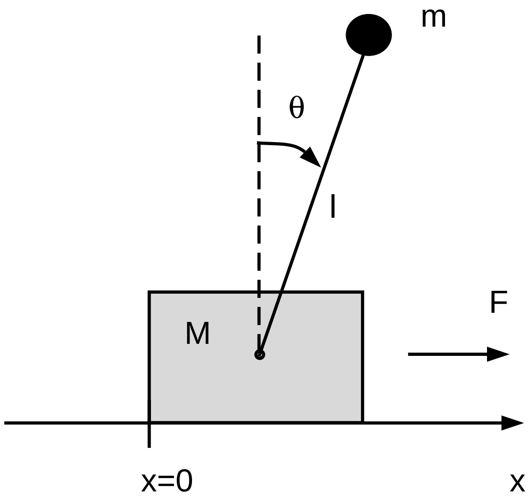


Figure 3: Description of the inverted pendulum.

With the notation x – cart position, θ – pendulum angle and F – applied force, the system can be described with the equations

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F \tag{1}$$

$$l\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta = -f_\theta\dot{\theta} \tag{2}$$

where M and m denotes the cart and pendulum mass, respectively, l the pendulum length, g the gravitational constant and f_θ the friction coefficient for the link where the pendulum is attached to the cart. A short derivation of the equations can be found in Appendix B.

With the following state variables

$$x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = \theta, \quad x_4 = \dot{\theta}$$

and some calculations, this yields the state space equations

$$\begin{aligned}
\dot{x}_1 &= \dot{x} = x_2 \\
\dot{x}_2 &= \ddot{x} = \frac{-mg\sin x_3\cos x_3 + mlx_4^2\sin x_3 + f_\theta m x_4\cos x_3 + F}{M + (1-\cos^2 x_3)m} \\
\dot{x}_3 &= \dot{\theta} = x_4 \\
\dot{x}_4 &= \frac{(M+m)(g\sin x_3 - f_\theta x_4) - (lmx_4^2\sin x_3 + F)\cos x_3}{l(M + (1-\cos^2 x_3)m)}
\end{aligned} \tag{3}$$

3 Simulation environment

3.1 Model

The system will be simulated in Simulink, and the simulation model to use in the first part of the lab session is shown in Figure 4.

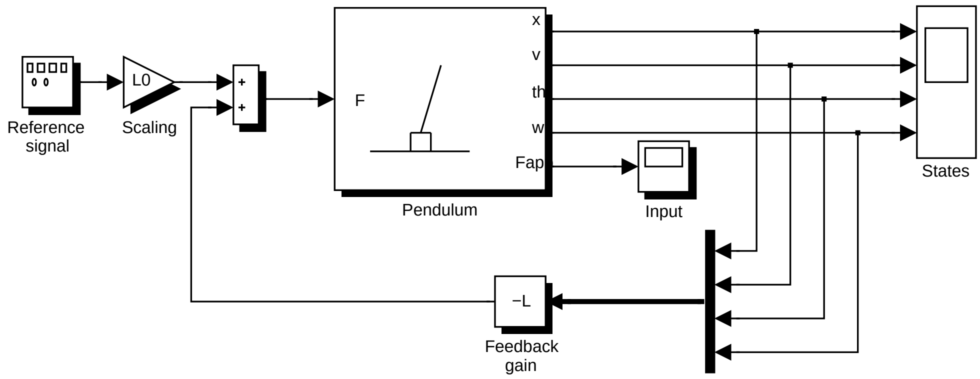


Figure 4: Simulink model.

3.2 Using the simulation model

- Start Matlab and type
  `initcourse TSRTXX` (with `XX=12, 15` or `19`)
  `syst1`
  in the command window in order to set the correct search path and open the Simulink model.

- Write `load penddata` to load the necessary variables.

- The simulation is started by selecting `Start` from the `Simulation` menu.

- As the simulation starts, an animation that shows the motion of the pendulum and the cart is started. See Figure 5.

- The simulation time can be modified using `Configuration Parameters` in the `Simulation` menu. The initial simulation time is 10 seconds.

- The initial state of the system is

$$x(0) = (0 \quad 0 \quad 0.2 \quad 0)^T$$

i.e., at $t = 0$ the cart is resting at $x = 0$ while the pendulum has a slope of 0.2 radians. The initial values of the cart position and pendulum position can be modified by changing the variables `x0` and `th0`, respectively.

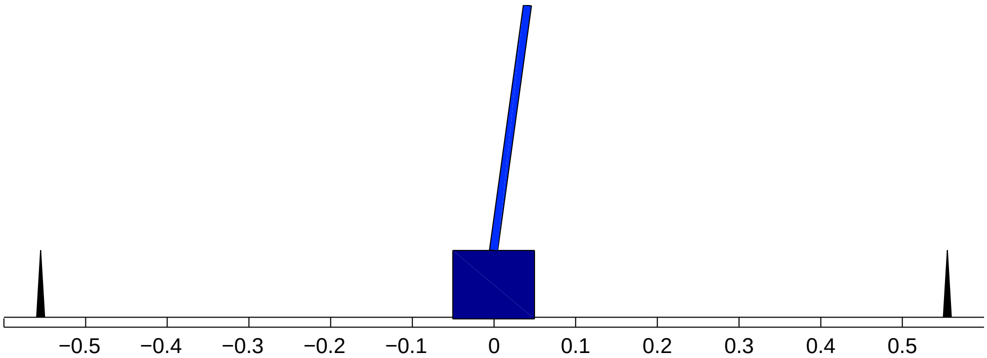


Figure 5: Animation of the movement of the cart and pendulum.

The simulation model uses the nonlinear differential equations, i.e., the simulated system is nonlinear. The feedback will however be calculated from the linearized model from Preparation Assignment 1 and 2. The matrices `A`, `B` and `C` are among the variables loaded above.

3.3 Limitations

The simulation model has two additional nonlinearities, namely limitations on the cart movement and the applied force. For the cart position we have that

$$-x_{lim} \leq x_1(t) \leq x_{lim}$$

where x_{lim} is 0.5. In practice this means that the cart can only move between two stops as in Figure 5. The control signal to the system is the applied force, and in practice it will be generated by, e.g., an electric motor. From physical reasons the force that can be applied is limited,

$$-F_{sat} \leq F(t) \leq F_{sat}$$

where F_{sat} has the value 24. This means that the calculated and the applied control signal will be different when the calculated control signal is outside the interval above. The block `Input` in the simulation model shows the applied control signal.

4 Simulation without feedback

In this section the pendulum will be simulated without any feedback. The aim is to understand the physical meaning of the state variables. Initially all elements of the feedback vector `L` are zero. This means that no feedback is applied.

Assignment:

- Simulate the model using the given initial states and compare the animation and the graphs that are generated by the block `States`. Combine the four graphs and the variables cart position, cart velocity, pendulum angle and pendulum angular velocity.
- Modify the initial states and see how this affects the behavior of the system. The initial values of the states can be chosen by assigning values to the variables `x0, v0, th0` and `w0` respectively. Try, for example, to start the system in one of its equilibrium points.
- The matrix `A` has been obtained by linearization of the nonlinear model when the pendulum is in the upright position. Compute the eigenvalues of the matrix and compare the result to the behavior of the system. Similarly the matrix `A2` has been obtained by linearization when the pendulum is in the down position. Compare the eigenvalues and the behavior of the system.

Hints:

- Use Preparation Assignments 1-3.

Summary of results and observations:

5 State feedback

In this section it is assumed that all state variables can be measured exactly, i.e. without any measurement disturbances. The aim is to design a stabilizing feedback under these conditions.

Assignment:

- Design a stabilizing controller by directly placing the poles of the feedback system.
- Determine the limitations that affect the performance of the closed loop system.

Hints:

- The only variable that has to be modified is the state feedback vector `L`.
- A pole placement feedback can be computed with the Matlab command `acker`. Use `help acker` to see the syntax.
- Place, for example, all poles in $-\alpha$. Try different values of α in order to obtain different performance of the closed loop system.
- In this case it is enough to place all poles on the real axis.
- Check that the system has the original initial states.
- Use preparation assignment 4.

Summary of results and observations:

6 Feedback from cart position

In this part, the pendulum is to be stabilized when only the state x_1 can be measured, i.e., the cart position. Intuitively this means that the pendulum must be kept in upright position without "knowing" which angle the pendulum has.

The simulation model to use is shown in Figure 6 and it is opened by typing `syst2`. The model has been modified to include an observer, where the available measured signal x_1 and the applied control signal are used to estimate the states of the system. The measurement of the cart position also includes a measurement disturbance, which is created by the block denoted `Measurement disturbance`. The measurement disturbance consists of identically distributed random numbers in the interval $[-10^{-3}, 10^{-3}]$. A possible source for the disturbance would be a limited resolution in the sensor that measures the cart position. The block denoted `Sensor` contains a scale factor, representing the transformation from position to voltage. The scale factor is initially set to one.

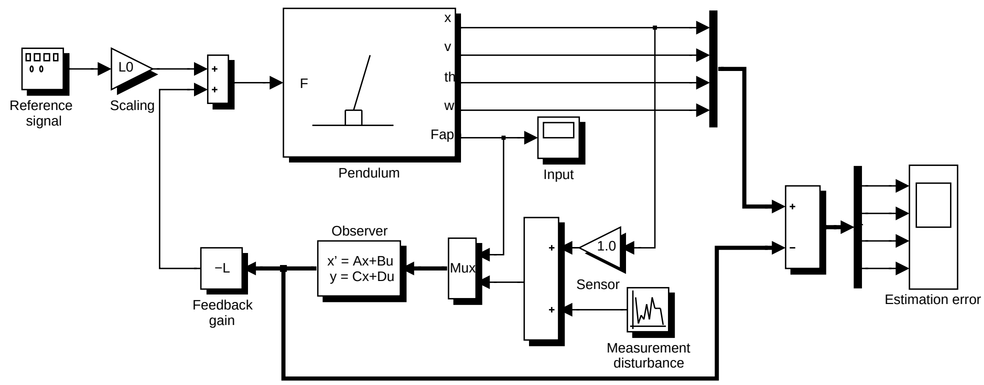


Figure 6: Simulink model with feedback from the cart position.

Assignment:

- Design a stabilizing controller where only the cart position is used in the feedback.
- Determine the limitations in the achievable performance of the observer and how these affect the closed loop system.

Hints:

- Change the initial pendulum angle `th0` to 0.1 rad.
- Use a state feedback vector `L` that corresponds to closed loop poles in the "middle" of the interval for λ that was determined in the previous section.
- Place, for example, the poles of the observer in $-\mu$. Try different values of μ in order to get different performance of the observer.
- The observer gain can be computed with the `acker` function, by writing `K=acker(A',C',p)'` where the row vector `p` contains the desired eigenvalues for the observer.
- The properties of the estimation error can be studied by opening the block `Estimation error` which plots the vector $\tilde{x}(t)$.
- Use preparation assignment 5.

Summary of results and observations:

7 Robusthetsanalys

In this section the robustness of the designed control system will be analyzed. This means that we will investigate how much the model and the true system can differ before the true closed loop system becomes unstable. The problem will be studied for the case when the properties of the sensor measuring the cart position are not known exactly. In the model it has been assumed that the constant that converts position to voltage is equal to one, and the goal here is to investigate how much the true constant can deviate from this value.

Mathematically, we have matrices `A, B` och `C`, and associated transfer function $G(s)$

$$Y(s) = G(s)U(s)$$

where U represent force on the cart and Y is position according to the sensor. An error in this sensor means that the true system is represented by

$$Y(s) = G^0(s)U(s) = (1+\alpha)G(s)U(s)$$

where α represent the deviation of the sensor constant from the value 1.

The feedback previously derived based on state-feedback is based on estimated states and can be described in state-space form as

$$U(s) = F_r(s)R(s) - F_y(s)Y(s)$$

where

$$F_y(s) = L(sI - A + KC + BL)^{-1}K$$

and

$$F_r(s) = 1 - L(sI - A + KC + BL)^{-1}B$$

and A, B och C are the matrices in the state-space model and L och K have been computed using pole-placement. The task now is to investigate robustness with respect to the error in the sensor constant.

7.1 Theoretical robustness analysis

Uppgifter:

- Use MATLAB to derive the transfer function $G(s)$ from `A, B` och `C`.
- Use MATLAB to derive the transfer function $F_y(s)$ based on `A, B, C, K` och `L`.
- Derive how large the error in the sensor constant can be, while still guaranteeing stability according to the robustness criteria.

Hints:

- Use Preparation Assignments 6 and 7.

Result:

7.2 Robustness-analysis through simulation

Uppgifter:

- Decide, by simulation, an upper bound on how large the error in the sensor constant can be.

Tips:

- The sensor constant in the model `syst2` can be modified by changing the constant in the block denoted `Sensor`.

Result:

8 Feedback from cart position and pendulum angle

In this section it is assumed that both cart position and pendulum angle can be measured, and that both variables are used in the feedback. A Simulink model for this situation is shown in Figure 7. The model is opened by typing `syst3`.

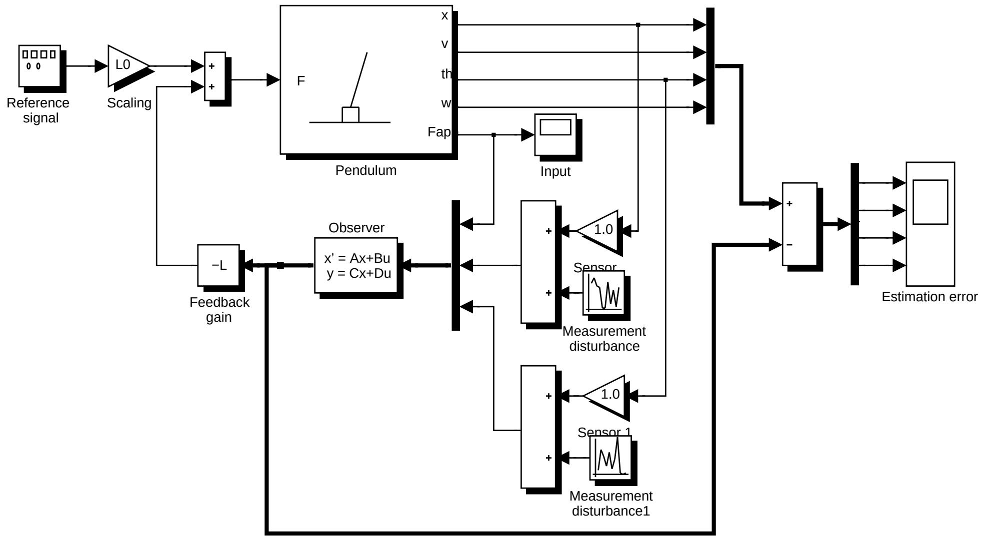


Figure 7: Simulink model with feedback from the cart position and pendulum angle.

Assignment:

- Determine a stabilizing feedback using cart position and pendulum angle.
- Study how the magnitude of the estimation error is affected by the use of an additional sensor.
- Study, by using simulation, how the robustness of the control system is affected by the use of the additional sensor.

Hints:

- The pendulum system will now have two outputs. The C-matrix must therefore be modified. Use preparation assignment 8.
- The gain in an observer using more than one measured signal can be computed using the function `place`. The function requires that the poles of the observer are different.

Summary of results and observations:

9 Non minimum phase system

This assignment is compulsory for TSRT12 and optional for TSRT15 and TSRT19.

Assume here that all state variables can be measured, i.e. use the Simulink model `syst1`.

Assignment:

Verify that the closed loop system, if the cart position is the output, is non-minimum phase.

Hints:

- What are the characteristics of a non-minimum phase system?

Summary of results and observations:

10 Extra assignment: LQ-optimization

This assignment is compulsory for TSRT12 and optional for TSRT15 and TSRT19.

Assume here that all state variables can be measured, i.e. use the Simulink model `syst1`.

Assignment:

- Determine a state feedback using linear quadratic optimization. Try different weight matrices in the quadratic criterion and observe how the choice affects the properties of the closed loop system.
- Investigate how the poles of the closed loop system are affected by the choice of weight matrices.

Hints:

- The state feedback vector obtained using LQ-optimization can be computed using the function `lqr`.
- Try, initially, to put weight on the cart position and input signal only.

11 Preparation Assignments

To complete the laboration, the following assigments must be completed. Also study the connection between these assignments and the assignments in the laboration using the "hints" that are given on the previous pages.

1. For a nonlinear system given by the state space model

$$\dot{x} = f(x, u)$$

all points (x_0, u_0) that satisfy

$$f(x_0, u_0) = 0$$

are denoted stationary points (equilibrium points). Verify that all points

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ n \cdot \pi \\ 0 \end{pmatrix} \qquad u_0 = F = 0$$

where n is an integer, are stationary points for the nonlinear system given by Equation (3).

2. The nonlinear state-space equations can be found in Equation (3). As an alternative to the normal linearization procedure, the linear model can in this case be obtained by introducing the approximations

$$\sin x_3 = x_3, \qquad \cos x_3 = 1, \qquad x_4^2 = 0, \qquad x_3 x_4 = 0$$

Verify that this results in a feedback model of the form

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -a & b \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & -d \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ e \\ 0 \\ -f \end{bmatrix}$$

3. Use the linearized model in the previous exercise. Insert the numerical values

$$M = 2.4, \qquad m = 0.23, \qquad l = 0.36, \qquad f_\theta = 0.1, \qquad g = 9.81$$

What is A and B?

4. For a controllable system, state feedback can place the poles of the feedback system arbitrarily. What will in practice prevent you from making the feedback system arbitrarily fast?

5. For an observable system, the poles of the observer can be placed arbitrarily. What will in practice prevent you from making the observer arbitrarily fast?

6. Suppose that a system is described by the model $G(s)$, while the true system is given by
$$G^0(s) = (1 + \alpha)G(s)$$
where α is a constant, with $\mid \alpha \mid < \alpha_{max}$. What relative model error does this correspond to, and what requirement according to the robustness criterion does this give for the absolute value of the complementary sensitivity function of the feedback system? How can this criterion be checked using Matlab?

7. (a) Assume that a state-space model is given by
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$
The transfer function $G(s)$ is then
$$G(s) = C(sI - A)^{-1}B$$
Write down how you derive the transfer function by the Matlab functions `ss` and `tf`.

   (b) State-space control based on estimated states can, for the case $r(t) = 0$, be written
$$\begin{aligned} \dot{\hat{x}}(t) &= (A - KC - BL)\hat{x}(t) + Ky(t) \\ u(t) &= -L\hat{x}(t) \end{aligned}$$
and in transfer function form
$$U(s) = -F_y(s)Y(s)$$
where
$$F_y(s) = L(sI - A + KC + BL)^{-1}K$$
Write down how you derive the transfer function $F_y(s)$ by the Matlab functions `ss` `tf`. (Hint: compare to the form in (a). Can you see what the different matrices corresponds to here?)

8. Assume that both cart position and pendulum angle can be measured, i.e. the measurement signals are given by

$$y_1(t) = x_1(t)$$

and

$$y_2(t) = x_3(t)$$

respectively. Assume that the measured signal are collected in the vector

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

Determine the matrix C in the expression

$$y(t) = Cx(t)$$

9. What do the Matlab commands in Appendix A do?

A Useful Matlab commands

eig	Compute eigenvalues
acker	Compute a pole placement feedback vector
ss	Generate a state-space LTI object
tf	Generate a transfer function LTI object
bode	Draw a bode diagram
bodemag	Draw the amplitude curve of the bode diagram
nyquist	Draw a Nyquist diagram
pole	Compute poles
tzero	Compute zeros
place	Compute a pole placement feedback matrix

B Derivation of the state space model

The state space equations of the inverted pendulum can be derived from the Lagrange-equation

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}L\right] - \frac{\partial}{\partial q}L = \tau$$

where L denotes the Lagrange-function $L = K - V$ with K and V being the kinetic and potential energy, respectively. For the inverted pendulum, the kinetic energy is given by

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\Big(\big(\dot{x} + l\dot{\theta}\cos\theta\big)^2 + \big(l\dot{\theta}\sin\theta\big)^2\Big)$$

while the potential energy is given by

$$V = mgl\cos\theta$$

This gives the equations

$$\begin{aligned}(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta &= F\\ l\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta &= -f_\theta\dot{\theta}\end{aligned}$$

where the term $f_\theta\dot{\theta}$ describes the friction in the rotational link of the pendulum.

By selecting the state variables as

$$x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = \theta, \quad \text{and} \quad x_4 = \dot{\theta}$$

we get the equations

$$\begin{aligned}\dot{x}_1 &= \dot{x} = x_2\\ \dot{x}_2 &= \ddot{x} = \frac{ml}{M+m}\big(x_4^2\sin x_3 - \dot{x}_4\cos x_3\big) + \frac{F}{M+m}\\ \dot{x}_3 &= \dot{\theta} = x_4\\ \dot{x}_4 &= \ddot{\theta} = \frac{1}{l}\big(-\dot{x}_2\cos x_3 - f_\theta x_4 + g\sin x_3\big)\end{aligned} \tag{4}$$

These equations contain a so-called algebraic loop, since $\dot{x}_2$ depends on $\dot{x}_4$ and $\dot{x}_4$ in turn depends on $\dot{x}_2$. By eliminating this dependence, we finally get the nonlinear state equations

$$\begin{aligned}\dot{x}_1 &= \dot{x} = x_2\\ \dot{x}_2 &= \ddot{x} = \frac{-mg\sin x_3\cos x_3 + mlx_4^2\sin x_3 + f_\theta m x_4\cos x_3 + F}{M + (1-\cos^2 x_3)m}\\ \dot{x}_3 &= \dot{\theta} = x_4\\ \dot{x}_4 &= \frac{(M+m)(g\sin x_3 - f_\theta x_4) - (lmx_4^2\sin x_3 + F)\cos x_3}{l(M + (1-\cos^2 x_3)m)}\end{aligned} \tag{5}$$

References

Segway Inc. Rymdbolaget (Swedish Space Cooperation), bilddatabasen, 2011. URL: http://www.segway.com/about-segway/media-center/product-photos.php, nedladdad 2011-03-11.

SSC. Rymdbolaget (Swedish Space Cooperation), bilddatabasen, 2011. URL: http://www.ssc.se/images.aspx, nedladdad 2011-03-11.