Simple Games

- Games: a mathematical formalism for rational interaction

Simple Games

- Games: a mathematical formalism for rational interaction
- What is the best solution concept? (Nash Equilibrium)

Types of Uncertainty

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Alleatory

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Markov Decision Process

Types of Uncertainty

Alleatory

Epistemic (Static)

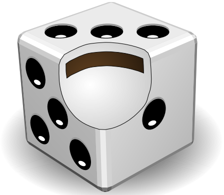
Markov Decision Process

Types of Uncertainty

Alleatory

Markov Decision Process

Epistemic (Static)



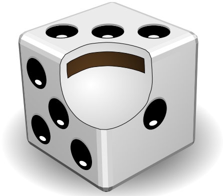
Reinforcement Learning

Types of Uncertainty

Alleatory

Markov Decision Process

Epistemic (Static)



Reinforcement Learning

Epistemic (Dynamic)

POMDP

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Reinforcement Learning

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Interaction

Types of Uncertainty

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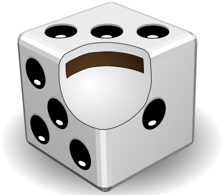
Epistemic (Static)		**Reinforcement Learning**
Epistemic (Dynamic)		**POMDP**
Interaction		**Game**

Normal Form Games

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- Alice and Bob are working on a homework assignment.

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Alice's Payoffs

B

A

	S	**W**
S	4	2
W	3	1

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Called a **Normal Form**, **Simple**, or **Bimatrix** Game

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Called a **Normal Form**, **Simple**, or **Bimatrix** Game

Question for today: What **solution concept** should we use for games?

Dominant Strategies

Dominant Strategies

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Dominant Strategies

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Definitions

- Action $a^i \in A^i$
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Action a^i is a deterministic best response to a^{-i} if

$$R^i(a^i, a^{-i}) \geq R^i(a^{i'}, a^{-i}) \quad \forall a^{i'}$$

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- **Dominant (Pure) Strategy**: Action a is a dominant strategy if it is a best response to every action taken by the other player.

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Is the dominant strategy equilibrium always the best outcome for the players?

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The Prisoner's Dilemma

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Player 2

Player 1

	S	**T**
S	-1, -1	-4, 0
T	0, -4	-3, -3

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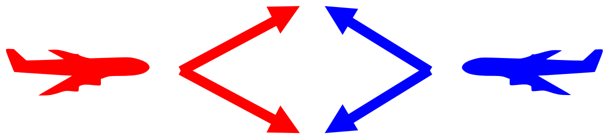
Do all games have a dominant strategy equilibrium?

Collision Avoidance Game

Collision Avoidance Game

Example: Airborne Collision Avoidance



Player 2

Player 1

	Up	Down
Up	Collision -5, -5	-1, 0
Down	0, -1	Collision -4, -4

Collision Avoidance Game

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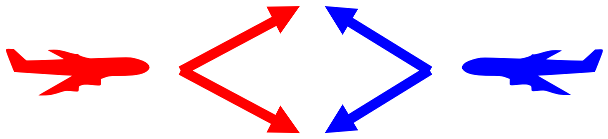

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Pure Nash Equilibrium: All players play a deterministic best response.

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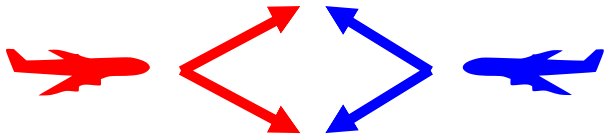

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Player 1	Up	Collision -5, -5	-1, 0
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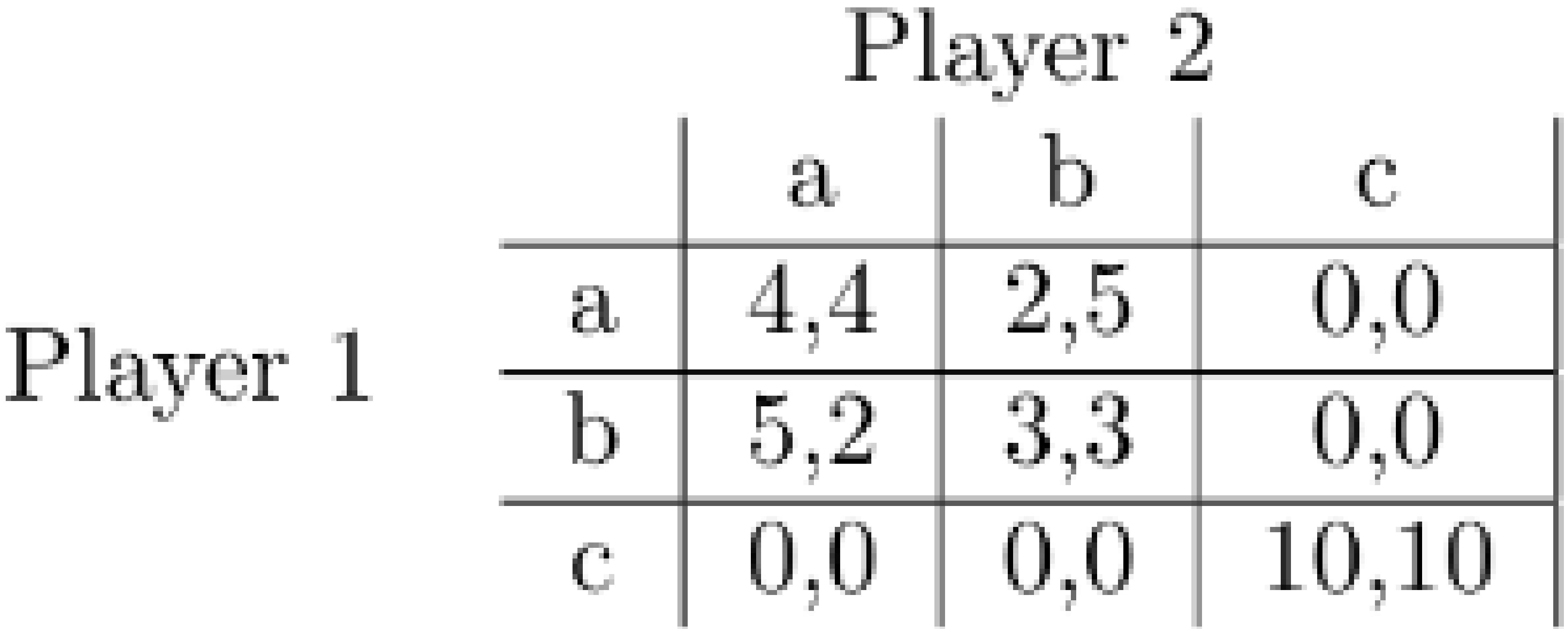
Pure Nash Equilibrium: All players play a deterministic best response.

Do all simple games have a pure Nash equilibrium?

Practice: Find Pure Nash Equilibria

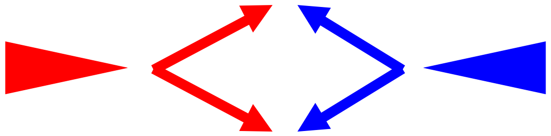


		Player 2		
		a	b	c
Player 1	a	4,4	2,5	0,0
	b	5,2	3,3	0,0
	c	0,0	0,0	10,10

Missile Defense Game

Missile Defense Game

Missile Defense (simplified)



		Defender	
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Attacker	Up	Collision -1, 1	1, -1
	Down	1, -1	Collision -1, 1

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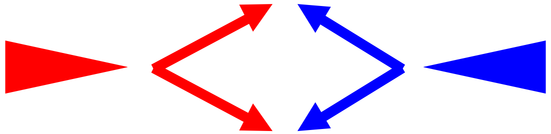

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No Pure Nash Equilibrium!

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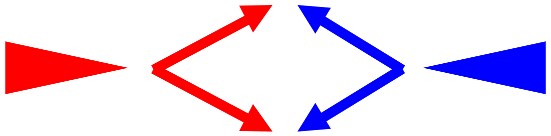

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Need a broader solution concept: Mixed Nash equilibrium.

Vocabulary and Notation for Mixed Strategies

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Single Player	Joint

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• Utility	$U^i(\pi) = \sum_a R^i(a)\pi(a)$	$U(\pi) = \sum_a R(a)\pi(a)$

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Utility	$U^i(\pi) = \sum_a R^i(a)\pi(a)$	$U(\pi) = \sum_a R(a)\pi(a)$

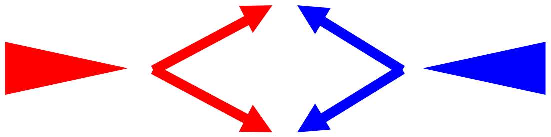
Best Response: Given a joint policy of all other agents, π^{-i}, a best response is a policy π^i that satisfies

$$U^i\left(\pi^i, \pi^{-i}\right) \geq U^i\left(\pi^{i\prime}, \pi^{-i}\right)$$

for all other $\pi^{i\prime}$.

Missile Defense Game

Missile Defense (simplified)



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Missile Defense Game

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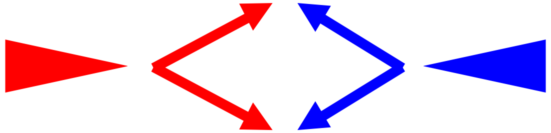

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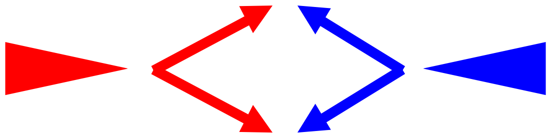

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Do all simple games have at least one Nash equilibrium?

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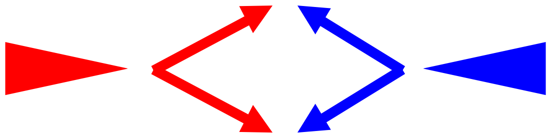

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Do all simple games have at least one Nash equilibrium?

Yes!! (might be mixed)

Every finite game has a Nash Equilibrium

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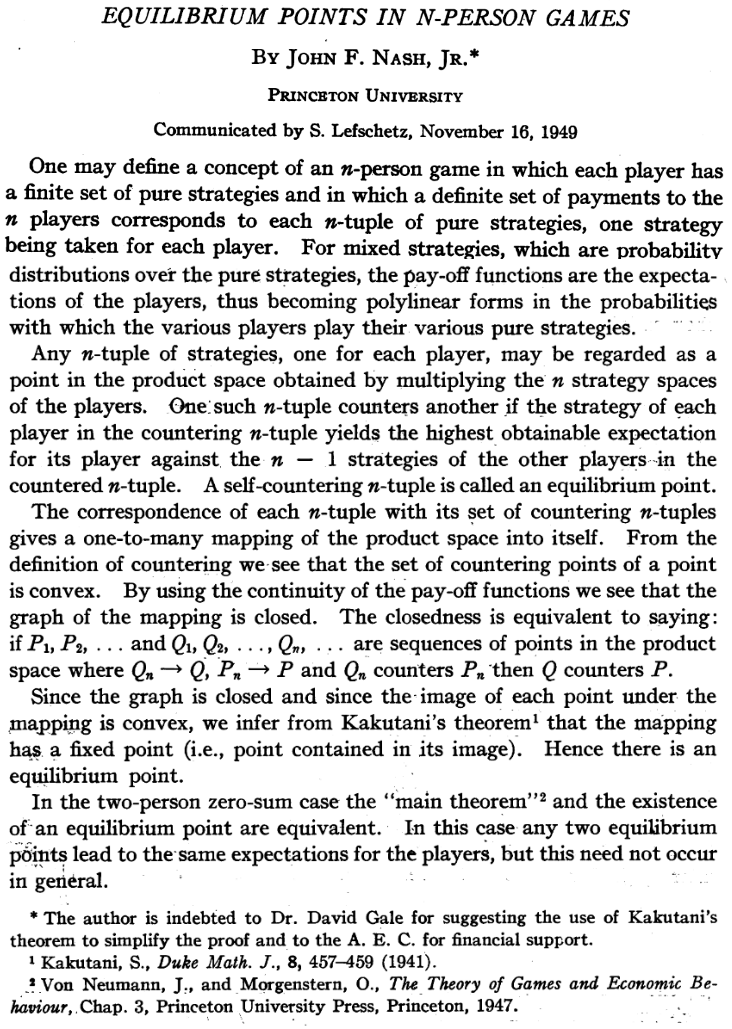

EQUILIBRIUM POINTS IN N-PERSON GAMES

BY JOHN F. NASH, JR.[*]

PRINCETON UNIVERSITY

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \to Q$, $P_n \to P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

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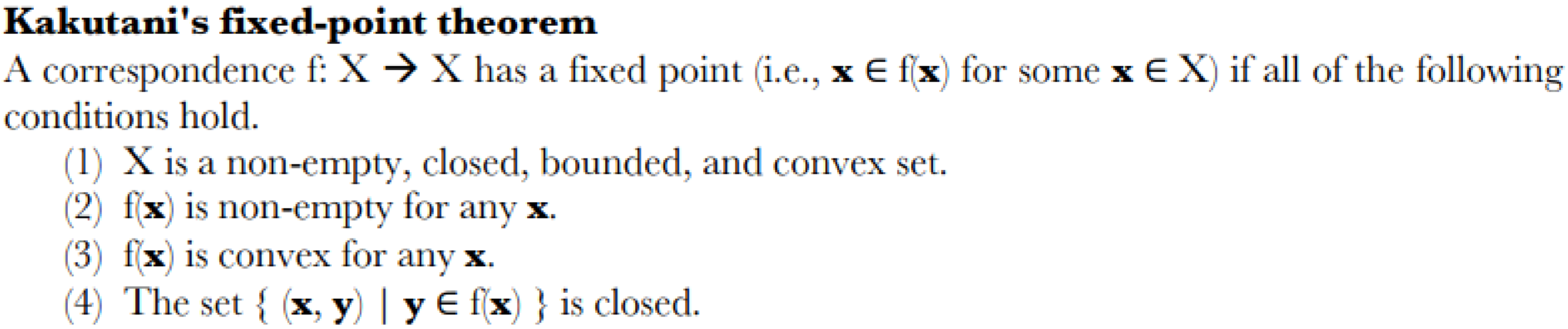
[*] The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the A. E. C. for financial support.

[1] Kakutani, S., *Duke Math. J.*, **8**, 457–459 (1941).

[2] Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behaviour*, Chap. 3, Princeton University Press, Princeton, 1947.

Every finite game has a Nash Equilibrium

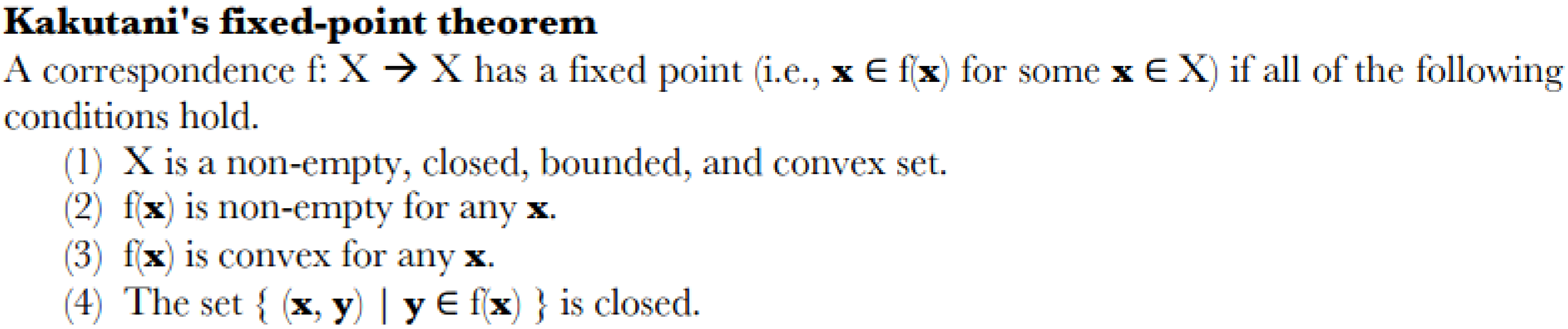


Kakutani's fixed-point theorem

A correspondence f: X → X has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

1. X is a non-empty, closed, bounded, and convex set.
2. $f(\mathbf{x})$ is non-empty for any $\mathbf{x}$.
3. $f(\mathbf{x})$ is convex for any $\mathbf{x}$.
4. The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

Every finite game has a Nash Equilibrium



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- Let x be a strategy profile, π.

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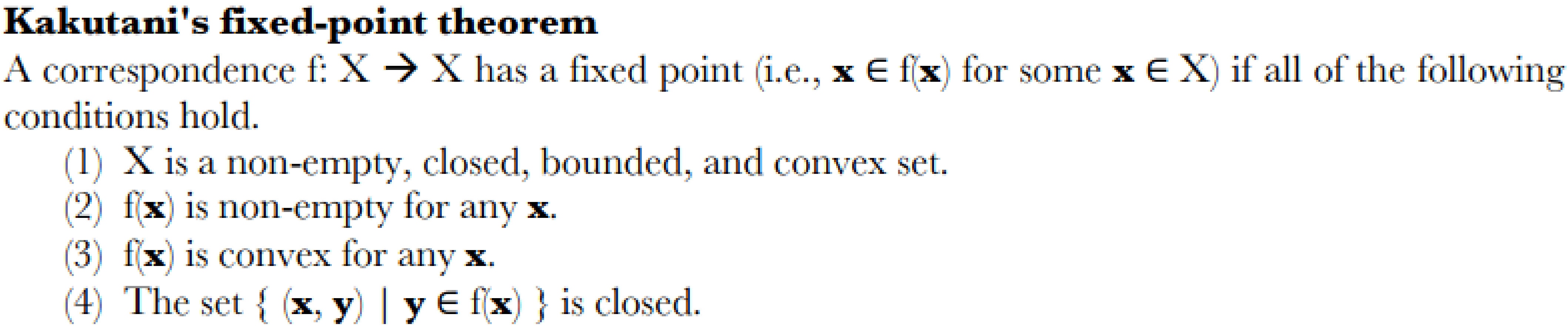

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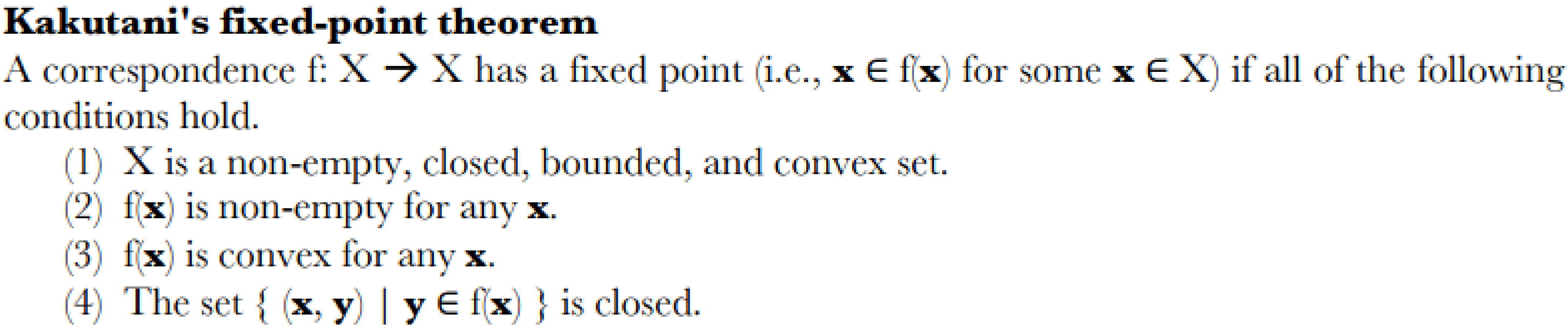
2. (2) f(**x**) is non-empty for any **x**.
3. (3) f(**x**) is convex for any **x**.
4. (4) The set { (**x**, **y**) | **y** ∈ f(**x**) } is closed.

- Let x be a strategy profile, π.
- Let f be BR, that is, the best response operator

Every finite game has a Nash Equilibrium



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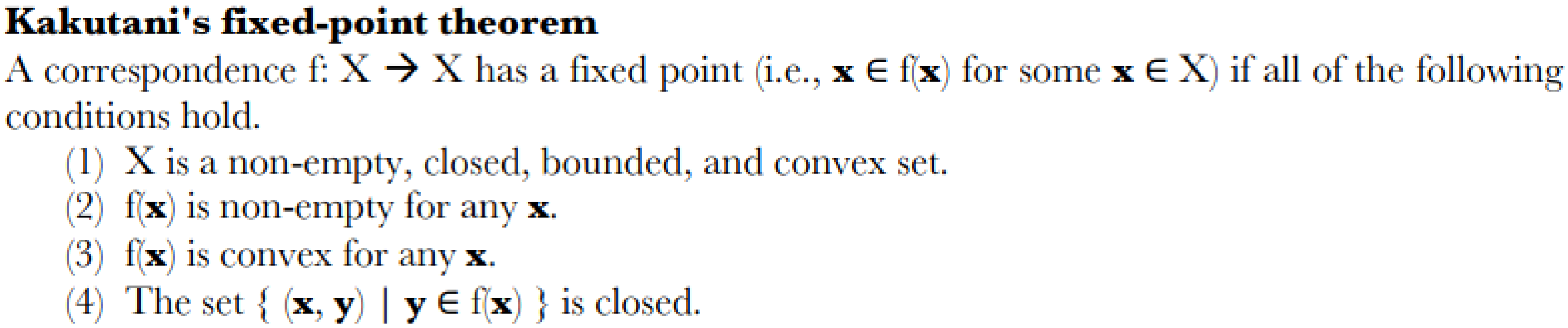
- A fixed point of BR is a Nash Equilibrium

Every finite game has a Nash Equilibrium



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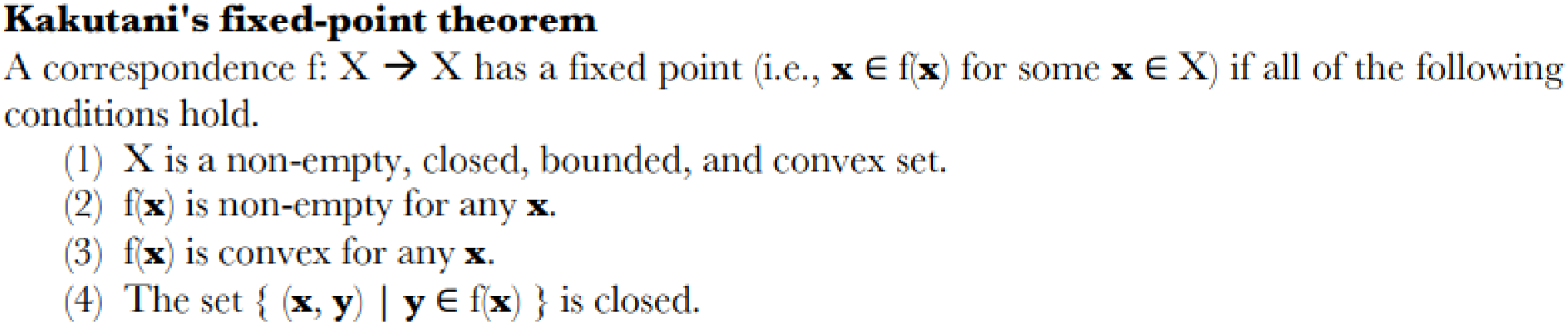
(2) $f(\mathbf{x})$ is non-empty for any $\mathbf{x}$.
(3) $f(\mathbf{x})$ is convex for any $\mathbf{x}$.
(4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

- Let x be a strategy profile, π.
- Let f be BR, that is, the best response operator
- A fixed point of BR is a Nash Equilibrium
- The BR operator and policy space for finite games meet the conditions above

Every finite game has a Nash Equilibrium



Kakutani's fixed-point theorem

A correspondence f: X → X has a fixed point (i.e., $\mathbf{x} \in \mathrm{f}(\mathbf{x})$ for some $\mathbf{x} \in \mathrm{X}$) if all of the following conditions hold.

1. X is a non-empty, closed, bounded, and convex set.
2. $\mathrm{f}(\mathbf{x})$ is non-empty for any $\mathbf{x}$.
3. $\mathrm{f}(\mathbf{x})$ is convex for any $\mathbf{x}$.
4. The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in \mathrm{f}(\mathbf{x}) \}$ is closed.

- Let x be a strategy profile, π.
- Let f be BR, that is, the best response operator
- A fixed point of BR is a Nash Equilibrium
- The BR operator and policy space for finite games meet the conditions above
- BR has a fixed point for every finite game, i.e. every finite game has a Nash Equilibrium

General approach to find Nash Equilibria

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$$
\begin{aligned}
\underset{\boldsymbol{\pi}, U}{\text{minimize}} \quad & \sum_i \Big(U^i - U^i(\boldsymbol{\pi}) \Big) \\
\text{subject to} \quad & U^i \geq U^i(a^i, \boldsymbol{\pi}^{-i}) \text{ for all } i, a^i \\
& \sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i \\
& \pi^i(a^i) \geq 0 \text{ for all } i, a^i
\end{aligned}
$$

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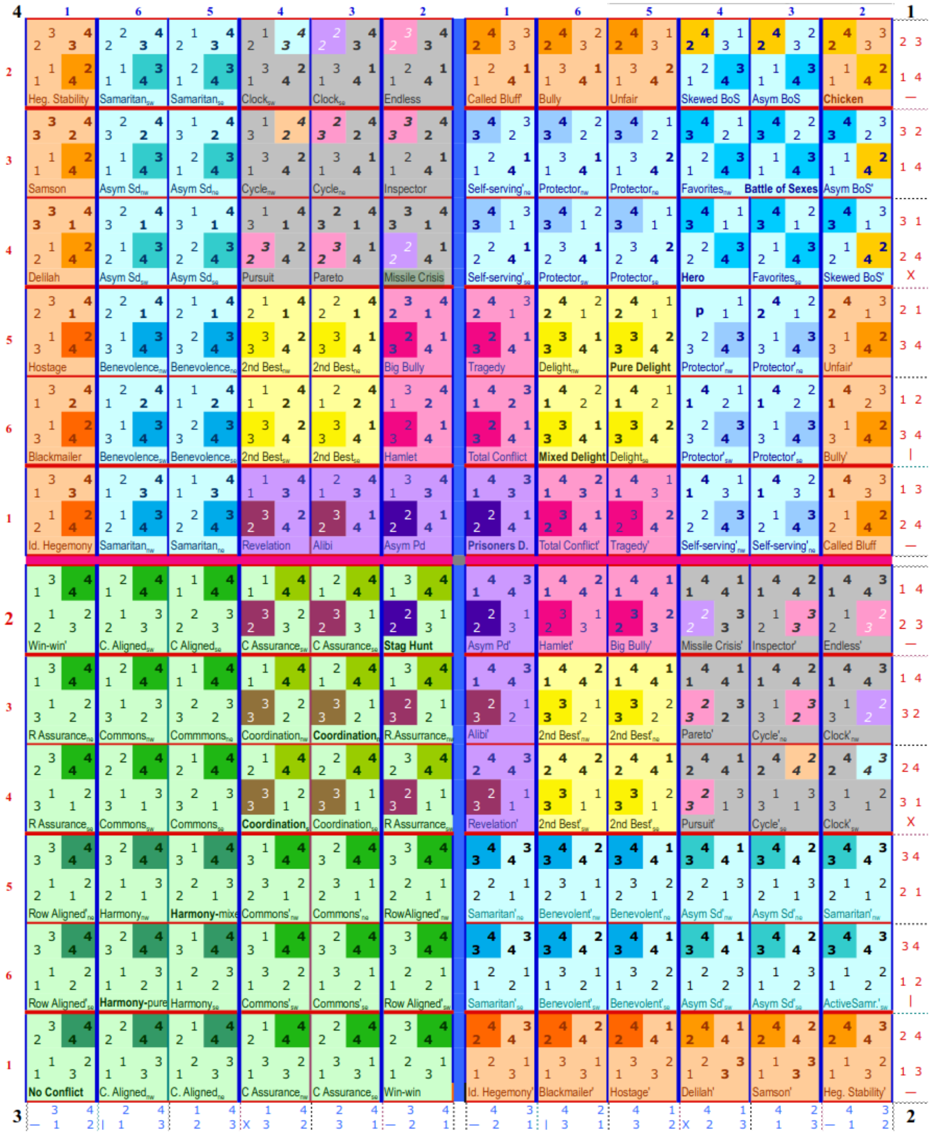

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- Games may not have a single "optimal" solution; instead there are equilibria
- If every player is playing a best response, that joint policy is a Nash Equilibrium
- Every finite game has at least one Nash Equilibrium (pure or mixed)

Battle of the Sexes

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		M	
		G	M
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Correlated Equilibrium

- A *correlated joint policy* is a single distribution over the joint actions of all agents.
- A *correlated equilibrium* is a correlated joint policy where no agent *i* can increase their expected utility by deviating from their current action to another.