Simple Games

- Last time:

- Today:

Simple Games

- Last time:
  - Inference in Bayesian networks
  - Learning Bayesian networks
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Simple Games

- **Last time:**
  - Inference in Bayesian networks
  - Learning Bayesian networks
- **Today:**
  - Games: a mathematical formalism for rational interaction
  - Nash and other equilibria

Types of Uncertainty

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Alleatory

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Markov Decision Process

Types of Uncertainty

Alleatory

Epistemic (Static)

Markov Decision Process

Types of Uncertainty

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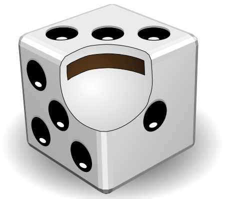

Reinforcement Learning

Types of Uncertainty

Alleatory

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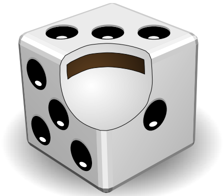

Reinforcement Learning

Epistemic (Dynamic)

POMDP

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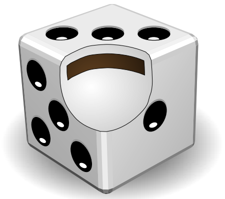

Reinforcement Learning

Epistemic (Dynamic)

POMDP

Interaction

Types of Uncertainty

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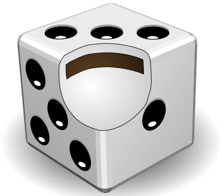
Epistemic (Static)		Reinforcement Learning
Epistemic (Dynamic)		POMDP
Interaction		Game

A win-win situation: International trade

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  - Producing wine: -3
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- Each country can either
  - Produce their own goods
  - Trade at a price of 2

Produce our own
-4
Trade
-1 -2

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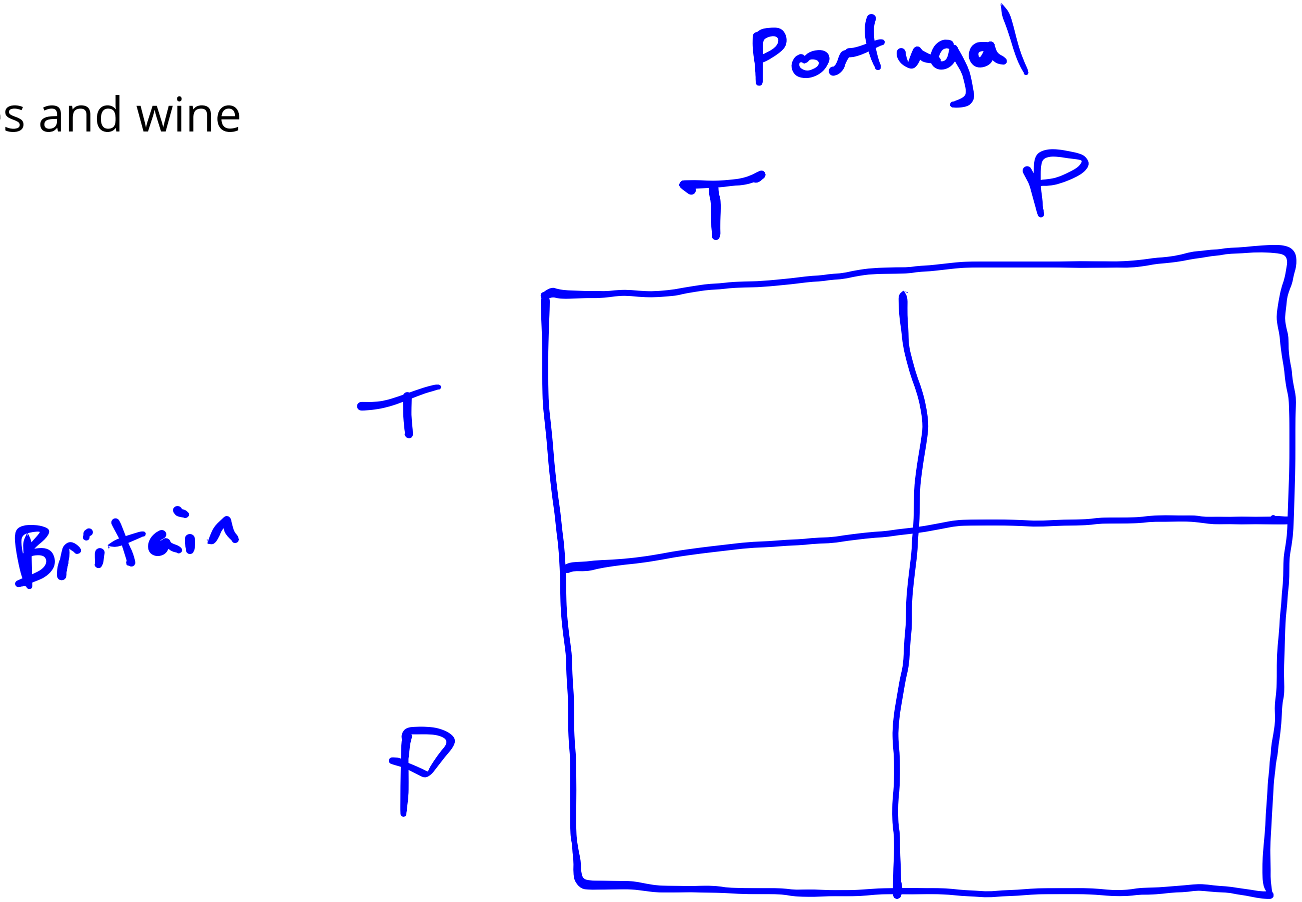

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		Portugal	
		T	P
Britain	T		
	P		-4, -4

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Portugal

Britain	T	P
T	-2, -2	-3, -3
P	-3, -3	-4, -4

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  - Producing wine: -3
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Portugal

Britain	T	P
T	-2, -2	-3, -3
P	-3, -3	-4, -4

	-4, -2	-4, -3
	-5, -3	-5, -4

A more surprising example:
The Prisoner's Dilemma

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- 2 Alleged criminals are captured

A more surprising example: The Prisoner's Dilemma

- 2 Alleged criminals are captured
- Each can either keep silent or testify
  - other keeps silent -> minor conviction (1 year)
  - other testifies -> major conviction: 4 years
  - testify -> 1 year removed from sentence

		Ariana	
		S	T
Jesse	S	-1, -1	-4, 0
	T	0, -4	-3, -3

Vocabulary

- Action, $a^i \in A^i$
- Policy, π^i (strategy)
- Joint action, a
- Joint policy, π (strategy profile)
- Reward, $R^i(a)$
- Joint reward, $R(a)$
- Joint utility, $U(\pi) = \sum_a R(a)\pi(a)$

Best Response: Given a joint policy of all other agents, π^{-i}, a best response is a policy π^i that satisfies

$$U^i\left(\pi^i, \pi^{-i}\right) \geq U^i\left(\pi^{i'}, \pi^{-i}\right)$$

for all other $\pi^{i'}$.

Portugal

Player 1

Britain

	T	P
T	-2, -2	-3, -3
P	-3, -3	-4, -4

Equilibria

-2, -2

-4, -4

Equilibria

Dominant Strategy Equilibrium

-2, -2

-4, -4

Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.

-2, -2

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Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.
- A joint policy where all agents use a dominant strategy is called a *dominant strategy equilibrium*.

-2, -2

-4, -4

Equilibria

-1 -1	-4,0
0,-4	-3 -3

Dominant Strategy Equilibrium

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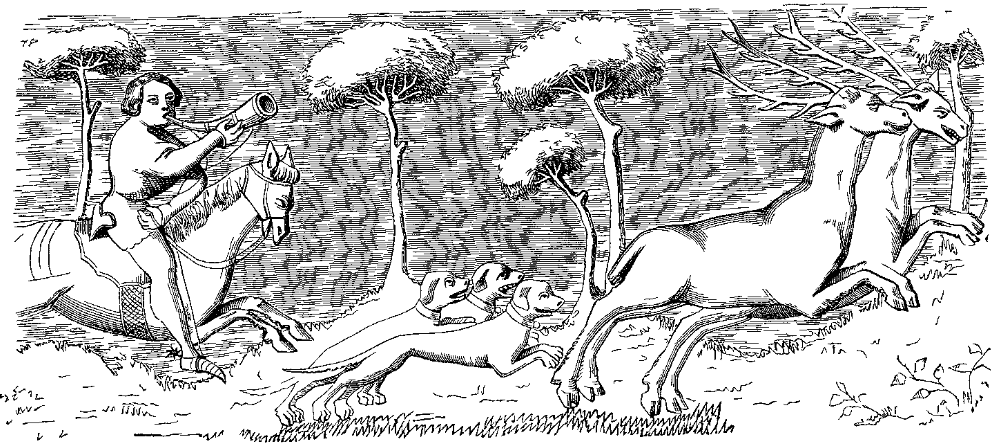

Portugal

Britain	T	P
T	-2,-2	-3,-3
P	-3,-3	-4,-4

Stag Hunt

	Stag	Hare
Stag	4,4	1,3
Hare	3,1	2,2

Equilibria

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Nash Equilibrium

Portugal

Britain	T	P
T	-2,-2	-3,-3
P	-3,-3	-4,-4

Stag Hunt

2.

1.	Stag	Hare
Stag	4,4	1,3
Hare	3,1	2,2

Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.
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Nash Equilibrium

- A *Nash equilibrium* is a joint policy in which all agents are following a best response

Player 2

Portugal

Player 1

Britain

	T	P
T	-2,-2	-3,-3
P	-3,-3	-4,-4

Stag Hunt

	Stag	Hare
Stag	4,4	1,3
Hare	3.1	2,2

Geopolitics

Soviet Union

US	Attack	Deter	Disarm
Attack	NE -100, -100	-100, -101	+1, -100
Deter	-101, -100	NE -1, -1	-1, -2
Disarm	-100, +1	-2, -1	0, 0

Rock-paper scissors

		agent 2 rock	agent 2 paper	agent 2 scissors
agent 1	rock	$0, 0$	$-1, 1$	$1, -1$
agent 1	paper	$1, -1$	$0, 0$	$-1, 1$
agent 1	scissors	$-1, 1$	$1, -1$	$0, 0$

A two-player game is **zero sum** if

$$\sum_i R^i(a) = 0 \quad \forall a$$

- Pure strategy: $\pi^i(a) \in \{0, 1\}$
- Mixed strategy: all other strategies

$\pi^i(a^i) = \frac{1}{3}$

$0.5(1) + 0.25(-1) + 0.25(0)$

General approach to find Nash Equilibria

General approach to find Nash Equilibria

0 if Nash Equilibrium

$$\underset{\pi, U}{\text{minimize}} \sum_i \left(U^i - U^i(\pi) \right)$$

policies of all other players

$$\text{subject to} \quad U^i \geq U^i(a^i, \pi^{-i}) \text{ for all } i, a^i$$

$$\sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i$$

π is probability

$$\pi^i(a^i) \geq 0 \text{ for all } i, a^i$$

General approach to find Nash Equilibria

$$
\begin{aligned}
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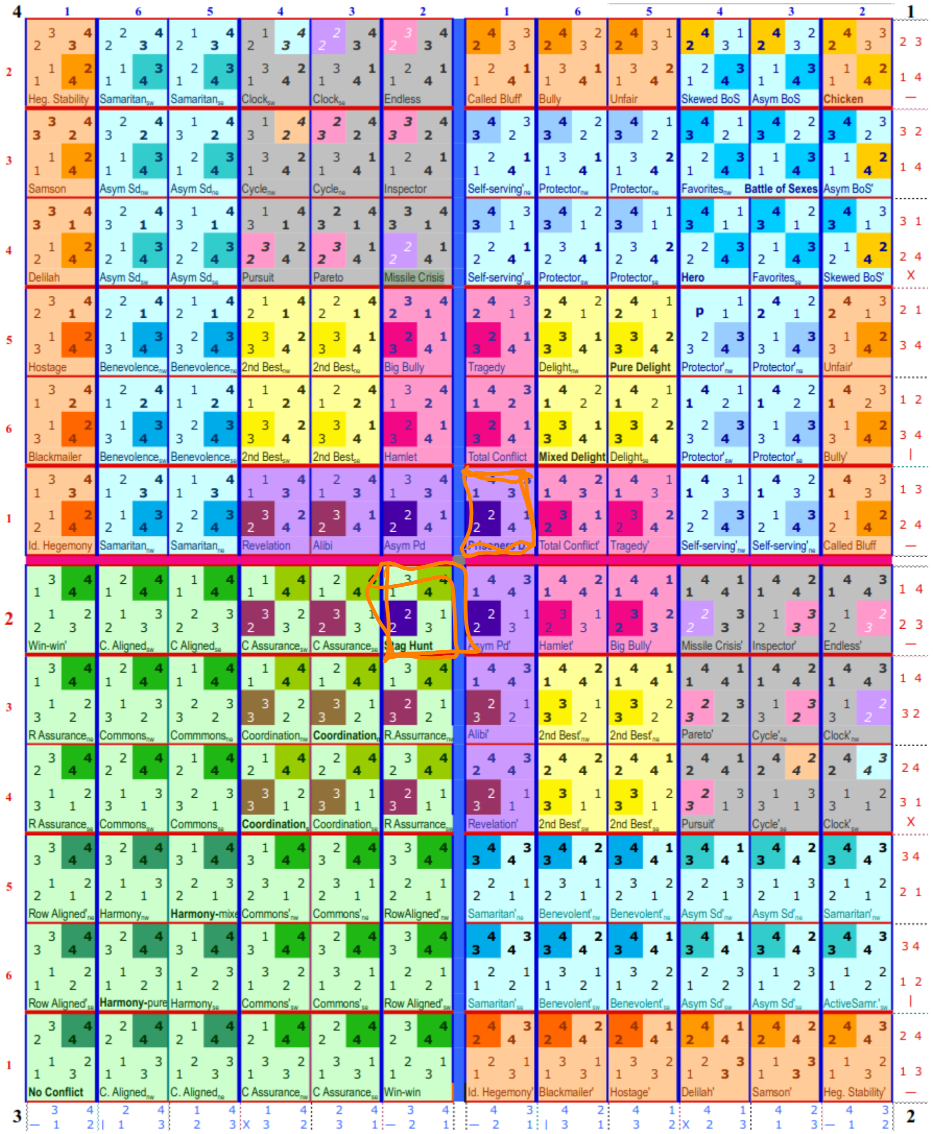
& \sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i \\
& \pi^i(a^i) \geq 0 \text{ for all } i, a^i
\end{aligned}
$$



Every finite game has a Nash Equilibrium

Every finite game has a Nash Equilibrium

EQUILIBRIUM POINTS IN N-PERSON GAMES

BY JOHN F. NASH, JR.*

PRINCETON UNIVERSITY

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \rightarrow Q$, $P_n \rightarrow P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

* The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the A. E. C. for financial support.

1 Kakutani, S., *Duke Math. J.*, 8, 457–459 (1941).

2 Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behaviour*, Chap. 3, Princeton University Press, Princeton, 1947.

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

A correspondence f: X → X has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

(1) X is a non-empty, closed, bounded, and convex set.

(2) $f(\mathbf{x})$ is non-empty for any $\mathbf{x}$.

(3) $f(\mathbf{x})$ is convex for any $\mathbf{x}$.

(4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

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- Let x be a strategy profile, π.

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- Let x be a strategy profile, π.
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- The BR operator and policy space for finite games meet the conditions above

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Battle of the Sexes

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- Gabby and Max are going on a date

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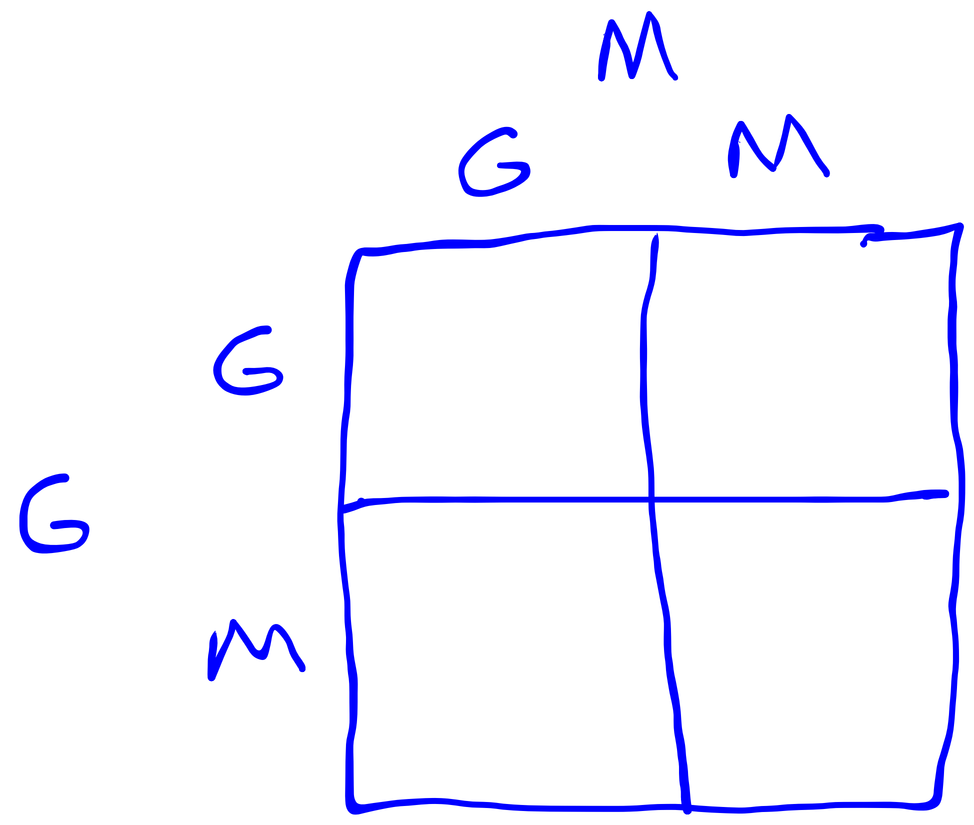

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M

		G	M
G	G	2,1	0,0
	M	0,2	1,2

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Correlated Equilibrium

- A *correlated joint policy* is a single distribution over the joint actions of all agents.
- A *correlated equilibrium* is a correlated joint policy where no agent *i* can increase their expected utility by deviating from their current action to another.

Recap

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- Games provide a mathematical framework for analyzing interaction between rational agents

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- Games provide a mathematical framework for analyzing interaction between rational agents
- Games may not have a single "optimal" solution; instead there are equilibria
- If every player is playing a best response, that policy profile is a Nash Equilibrium
- Every finite game has at least one Nash Equilibrium (pure or mixed)