Last Time

• What does "Markov" mean in "Markov Decision Process"?

• What is a **Markov decision process**?

- What is a **Markov decision process**?
- What is a **policy**?

- What is a **Markov decision process**?
- What is a **policy**?
- How do we **evaluate** policies?

Decision Network

Decision Network



Decision Network

Chance node

Decision Network

Chance node

Decision Network

Chance node

Decision node

Decision Network

Chance node

Decision node



Decision Network

Chance node

Decision node

Utility node

Decision Network

MDP Dynamic Decision Network

Chance node

Decision node

Utility node

Decision Network

MDP Dynamic Decision Network

Chance node

Decision node

Utility node



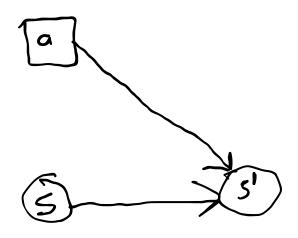
Decision Network



Decision node



MDP Dynamic Decision Network



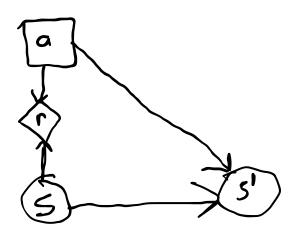
Decision Network



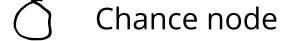
Decision node

Utility node

MDP Dynamic Decision Network



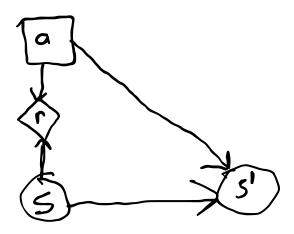
Decision Network



Decision node

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MDP Dynamic Decision Network



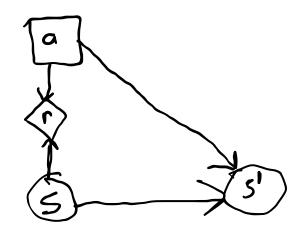
Decision Network



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MDP Dynamic Decision Network



$$ext{maximize} \quad \mathrm{E}\left[\sum_{t=1}^{\infty} r_t
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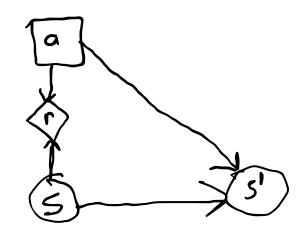
Decision Network







MDP Dynamic Decision Network



$$ext{maximize} \quad \mathrm{E}\left[\sum_{t=1}^{\infty} r_t
ight] \qquad \mathsf{Not well formulated!}$$

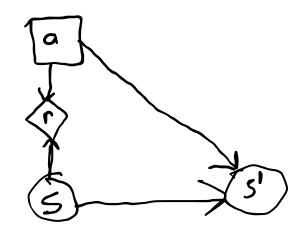
Decision Network



Decision node



MDP Dynamic Decision Network



1. Finite time

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$$\mathrm{E}\left[\sum_{t=0}^{T} r_{t}
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3. Discounting

$$\mathrm{E}\left[\sum_{t=0}^{\infty}\gamma^{t}r_{t}
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discount $\gamma \in [0,1)$

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4. Terminal States

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Infinite time, but a terminal state (no reward, no leaving) is always reached with probability 1.

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MDP "Tuple Definition"

 (S, A, T, R, γ)

 (S, A, T, R, γ) (and b in some contexts)

• S (state space) - set of all possible states

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 $\{1, 2, 3\}$

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 \mathbb{R}^2

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$$\{1,2,3\}$$
 \mathbb{R}^2 $\{0,1\} imes\mathbb{R}^4$

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ullet S (state space) - set of all possible states

$$\{1,2,3\} \qquad (x,y) \in \mathbb{R}^2 \quad \left\{0,1
ight\} imes \mathbb{R}^4$$

 (S, A, T, R, γ) (and b in some contexts)

ullet S (state space) - set of all possible states

$$\{1,2,3\} \hspace{0.1in} (x,y) \in \mathbb{R}^2 \hspace{0.1in} \{0,1\} imes \mathbb{R}^4$$
 {healthy, pre-cancer, cancer} $\hspace{0.1in} (s,i,r) \in \mathbb{N}^3$

 (S, A, T, R, γ) (and b in some contexts)

- ullet S (state space) set of all possible states
 - (healthy, pre-cancer, cancer) $(s,i,r)\in\mathbb{N}^3$
- *A* (action space) set of all possible actions

 $\{1,2,3\} \qquad (x,y) \in \mathbb{R}^2 \quad \{0,1\} imes \mathbb{R}^4$

 (S, A, T, R, γ) (and b in some contexts)

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- ullet S (state space) set of all possible states
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$$\{1,2,3\} \qquad (x,y) \in \mathbb{R}^2 \quad \left\{0,1
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$$\{1,2,3\}$$
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 (S, A, T, R, γ) (and b in some contexts)

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{test, wait, treat}

T (transition distribution) - explicit or implicit ("generative")
 model of how the state changes

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$$s', r = G(s, a)$$

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$$R(s,a)$$
 or $R(s,a,s^\prime)$

$$s', r = G(s, a)$$

 (S, A, T, R, γ) (and b in some contexts)

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- $\{1, 2, 3\}$ • *A* (action space) - set of all possible actions \mathbb{R}^2 $\{0,1\} imes\mathbb{R}^2$ {test, wait, treat}
- T (transition distribution) explicit or implicit ("generative") $T(s' \mid s, a)$ model of how the state changes
- \bullet R (reward function) maps each state and action to a reward

R(s,a) or R(s, a, s')

• γ : discount factor

$$s', r = G(s, a)$$

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 $s^\prime, r = G(s,a)$

R(s, a, s')

• b: initial state distribution

MDP Example

Imagine it's a cold day and you're ready to go to work. You have to decide whether to bike or drive.

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• If you drive, you will have to pay \$15 for parking; biking is free.

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Imagine it's a cold day and you're ready to go to work. You have to decide whether to bike or drive.

- If you drive, you will have to pay \$15 for parking; biking is free.
- On 1% of cold days, the ground is covered in ice and you will crash if you bike, but you can't discover this until you start riding. After your crash, you limp home with pain equivalent to losing \$100.

Policies and Simulation

Policies and Simulation

• A *policy*, denoted with π , as in $a_t = \pi(s_t)$ is a function mapping every state to an action.

Break

Suggest a policy that you think is optimal for the icy day problem

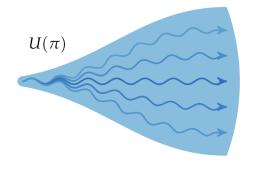
Utility

Policy Evaluation

Value Function-Based Policy Evaluation

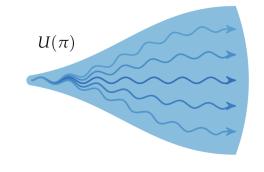
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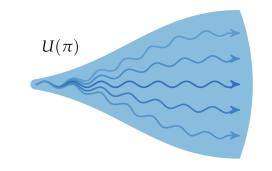
Let
$$au = (s_0, a_0, r_0, s_1, \ldots, s_T)$$
 be a *trajectory* of the MDP



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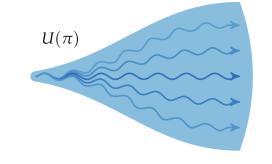
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$$U(\pi)pprox ar{u}_m = rac{1}{m}\sum_{i=1}^m \hat{u}^{(i)}$$

where $\hat{u}^{(i)}$ is generated by a rollout simulation

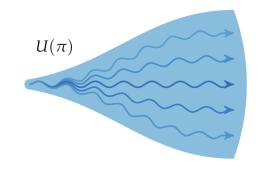
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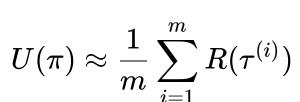
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How can we quantify the accuracy of \bar{u}_m ?

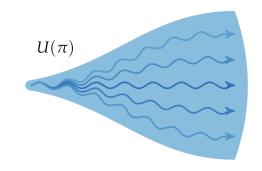
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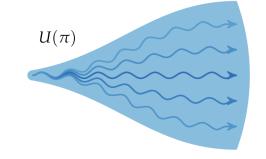
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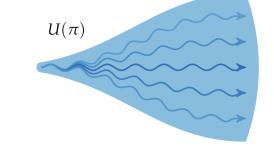
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C.L.T.

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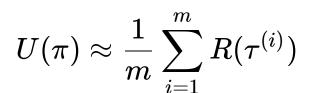
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C.L.T.
$$rac{ar{u}_m - U(\pi)}{\sigma_m/\sqrt{m}} \stackrel{d}{ o} \mathcal{N}(0,1)$$
 CLT not on exam

 Running a large number of simulations and averaging the accumulated reward is called *Monte Carlo Evaluation*

Let $\tau = (s_0, a_0, r_0, s_1, \dots, s_T)$ be a trajectory of the MDP



$$U(\pi)pproxar{u}_m=rac{1}{m}\sum_{i=1}^m\hat{u}^{(i)}$$

How can we quantify the accuracy of \bar{u}_m ?

 $U(\pi)$

C.L.T.
$$rac{ar{u}_m - U(\pi)}{\sigma_m/\sqrt{m}} \stackrel{d}{ o} \mathcal{N}(0,1)$$
 CLT not on exam

where
$$\hat{u}^{(i)}$$
 is generated by a rollout simulation

the Mean
$$ext{s.e.m.} = rac{ ext{std}(\hat{u})}{\sqrt{m}}$$

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