Simple Games

Next Year: See notes within

- Last time:

- Today:

Simple Games

Next Year: See notes within

- **Last time:**
  - Inference in Bayesian networks
  - Learning Bayesian networks
- **Today:**

Simple Games

Next Year: See notes within

- **Last time:**
  - Inference in Bayesian networks
  - Learning Bayesian networks
- **Today:**
  - Games: a mathematical formalism for rational interaction
  - Nash and other equilibria

Types of Uncertainty

Types of Uncertainty

Alleatory

Types of Uncertainty

Alleatory

Markov Decision Process

Types of Uncertainty

Alleatory

Markov Decision Process

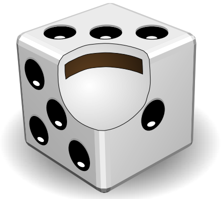
Epistemic (Static)

Types of Uncertainty

Alleatory

Markov Decision Process

Epistemic (Static)



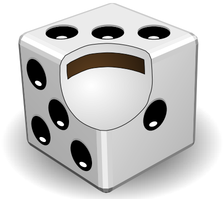
Reinforcement Learning

Types of Uncertainty

Alleatory

Markov Decision Process

Epistemic (Static)



Reinforcement Learning

Epistemic (Dynamic)

POMDP

Types of Uncertainty

Alleatory

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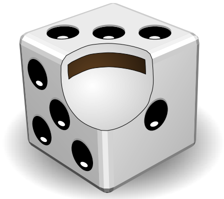

Reinforcement Learning

Epistemic (Dynamic)

POMDP

Interaction

Types of Uncertainty

Alleatory

Markov Decision Process

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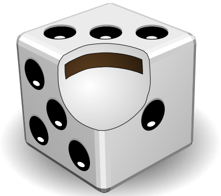

Reinforcement Learning

Epistemic (Dynamic)

POMDP

Interaction

Game

A win-win situation: International trade

Next Year: This example isn't great

A win-win situation: International trade

Next Year: This example isn't great

- Both Britain and Portugal need textiles and wine

A win-win situation: International trade

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- Both Britain and Portugal need textiles and wine
- Britain:
    - Producing wine: -3
    - Producing textiles: -1

A win-win situation: International trade

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A win-win situation: International trade

Next Year: This example isn't great

- Both Britain and Portugal need textiles and wine
- Britain:
  - Producing wine: -3
  - Producing textiles: -1
- Portugal:
  - Producing wine: -1
  - Producing textiles: -3
- No production capacity limits

A win-win situation: International trade

Next Year: This example isn't great

- Both Britain and Portugal need textiles and wine
- Britain:
  - Producing wine: -3
  - Producing textiles: -1
- Portugal:
  - Producing wine: -1
  - Producing textiles: -3
- No production capacity limits
- Each country can either
  - Produce their own goods
  - Trade at a price of 2

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		Portugal	
		T	P
Britain	T		
	P		

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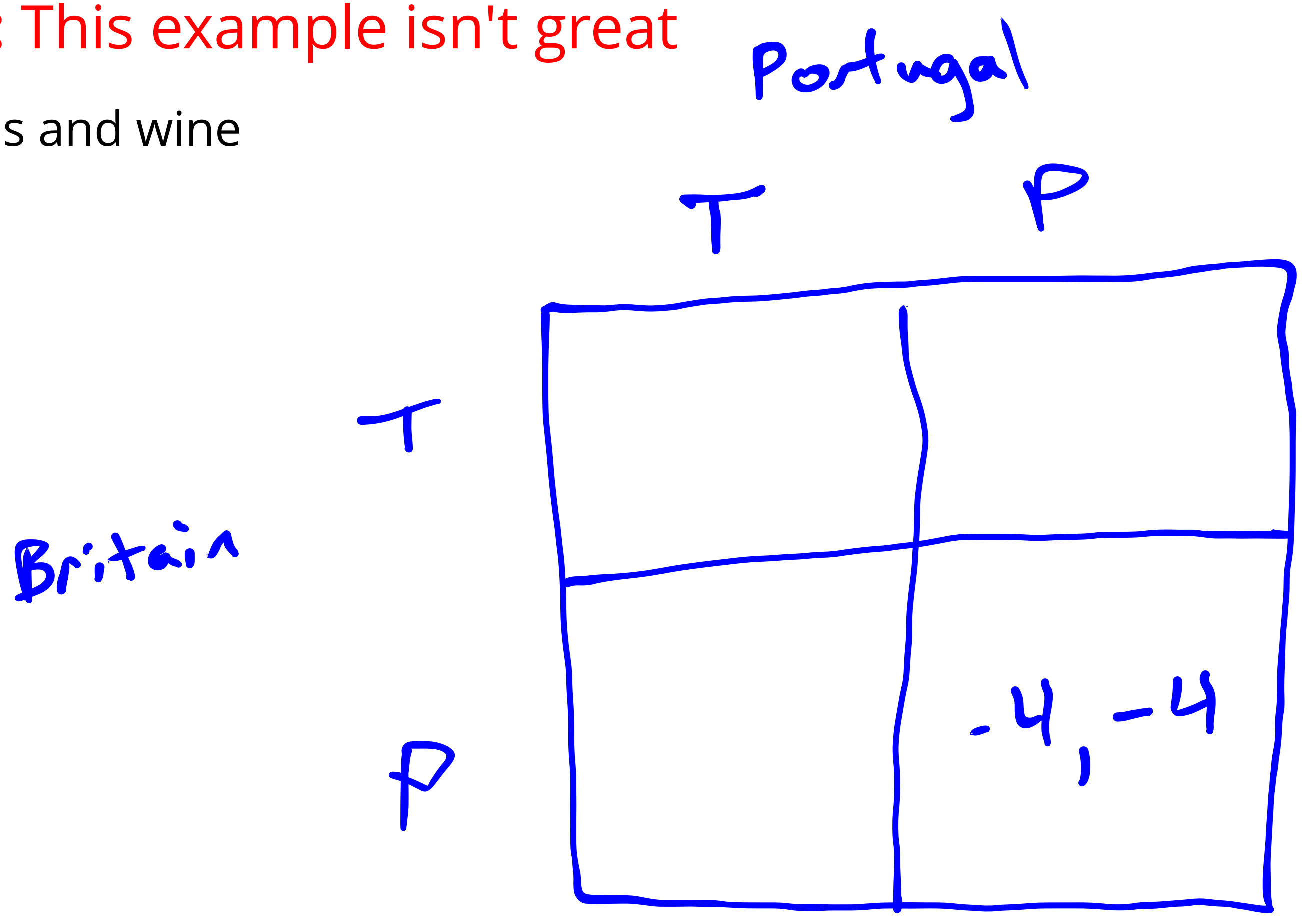

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Portugal

Britain	T	P
T		-3, -3
P	-3, -3	-4, -4

A win-win situation: International trade

Next Year: This example isn't great

- Both Britain and Portugal need textiles and wine
- Britain:
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Portugal

		T	P
Britain	T	-2, -2	-3, -3
	P	-3, -3	-4, -4

A win-win situation: International trade

Next Year: This example isn't great

- Both Britain and Portugal need textiles and wine
- Britain:
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Portugal

Britain

	T	P
T	-2, -2	-3, -3
P	-3, -3	-4, -4

-4, -2	-4, -3
-5, -3	-5, -4

A more surprising example:
The Prisoner's Dilemma

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The Prisoner's Dilemma

- 2 Alleged criminals are captured

A more surprising example: The Prisoner's Dilemma

- 2 Alleged criminals are captured
- Each can either keep silent or testify
  - other keeps silent -> minor conviction (1 year)
  - other testifies -> major conviction: 4 years
  - testify -> 1 year removed from sentence

Vocabulary and Notation

Vocabulary and Notation

Single Player	Joint

Vocabulary and Notation

	Single Player	Joint
• Action	$a^i \in A^i$	$a \in A$

Vocabulary and Notation

	Single Player	Joint
• Action	$a^i \in A^i$	$a \in A$
• Policy (strategy)	$\pi^i(a^{i}_{\cancel{i}})$	$\pi(a) = \prod_i \underline{\pi^i(a^{i}_{\cancel{i}})}$

Vocabulary and Notation

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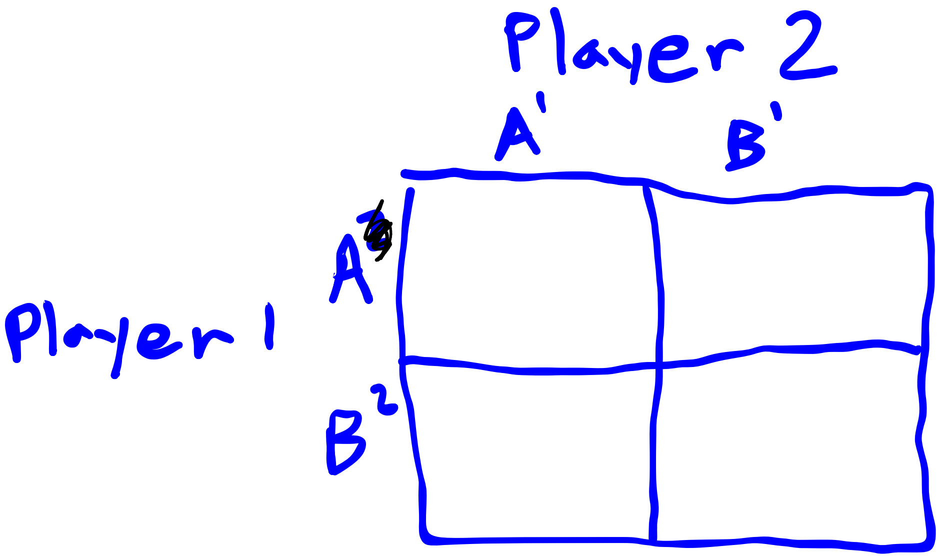

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Player 2

Player 1		A^1	B^1
	A^2	$(R^1(\cdot), R^2(\cdot))$	
	B^2		

Vocabulary and Notation

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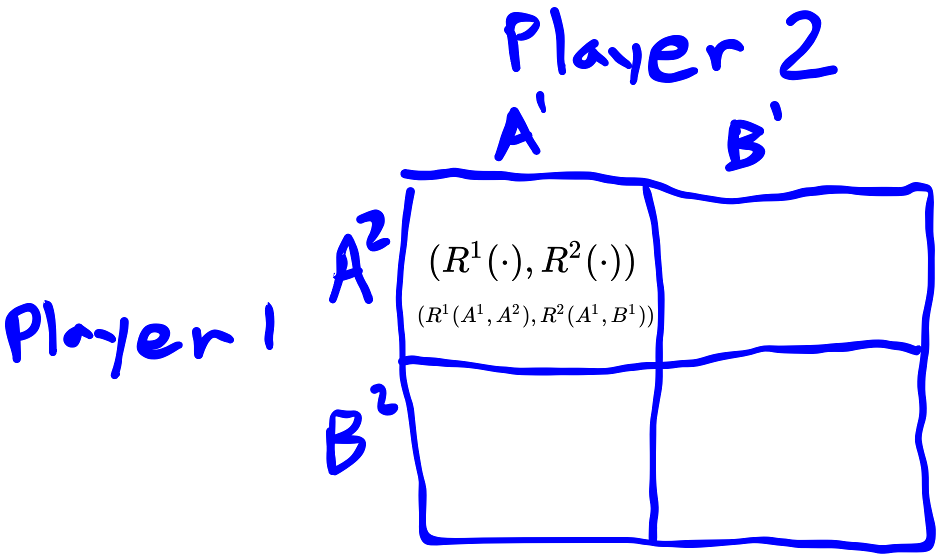

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Player 2

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A^2	$(R^1(\cdot), R^2(\cdot))$ $(R^1(A^1, A^2), R^2(A^1, B^1))$	$(R^1(\cdot), R^2(\cdot))$
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Robber 2

Robber 1

Vocabulary and Notation

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Robber 2

Robber 1	S	T
S		
T		

Vocabulary and Notation

	Single Player	Joint
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(R^1, R^2)

1, 2	

Player 2

Player 1	A^2	B^2
A^1	$(R^1(\cdot), R^2(\cdot))$ $(R^1(A^1, A^2), R^2(A^1, A^2))$	$(R^1(\cdot), R^2(\cdot))$
B^1	$(R^1(\cdot), R^2(\cdot))$	$(R^1(\cdot), R^2(\cdot))$

Robber 2

Robber 1	S	T
S	-1, -1	-4, 0
T	0, -4	-3, -3

Best Response

Best Response

Let π^{-i} be the joint policy of all *other* players.

π^i

$$\pi^{-i} = (\pi^1, \ldots, \pi^{i-1}, \pi^{i+1}, \ldots, \pi^n)$$

Best Response

Let π^{-i} be the joint policy of all *other* players.

$$\pi^{-i} = (\pi^1, \ldots, \pi^{i-1}, \pi^{i+1}, \ldots, \pi^n)$$

Best Response: Given a joint policy of all other agents, π^{-i}, a best response is a policy π^i that satisfies

$$U^i\left(\pi^i, \pi^{-i}\right) \geq U^i\left(\pi^{i'}, \pi^{-i}\right)$$

for all other $\pi^{i'}$.

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for all other $\pi^{i'}$.

$\pi^2(S)=1.0$ $\pi^2(T)=1.0$

Robber 2

		S	T
Robber 1	S	-1, -1	-4, 0
	T	0, -4	-3, -3

Best Response for Robber 1
Given Robber 2's strategy

Dominant Strategy Equilibria

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- A *dominant strategy* is a policy that is a best response to all other possible agent policies.

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Robber 2

Robber 1	S	T
S	-1, -1	-4, 0
T	0, -4	-3, -3

Testifying is a dominant strategy for Robber 1

Robber 2

Dominant Strategy Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.
- A joint policy where all agents use a dominant strategy is called a *dominant strategy equilibrium*.

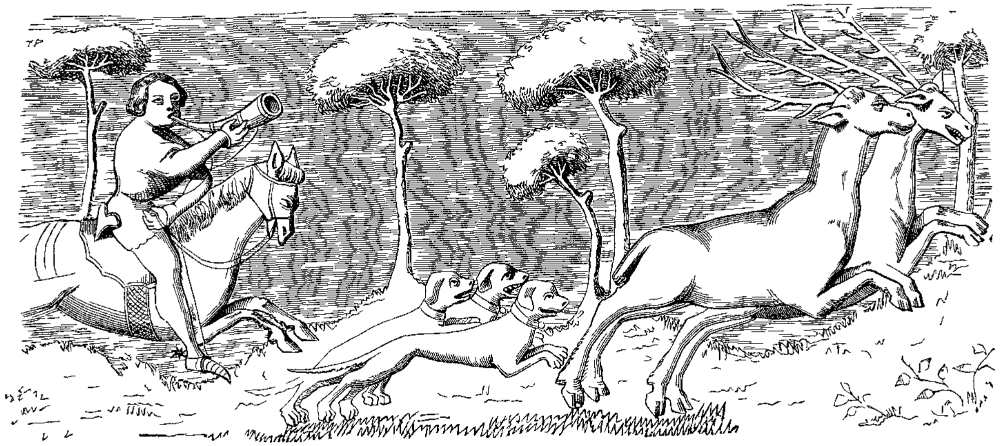
Robber 2

Robber1	S	T
S	-1, -1	-4, 0
T	0, -4	$\boxed{-3, -3}$

Both Robbers Testifying is a Dominant Strategy Equilibrium

Nash Equilibria

Nash Equilibria



	Stag	Hare
Stag	4, 4	1, 3
Hare	3, 1	2, 2

Nash Equilibria

Nash Equilibrium

	Stag	Hare
Stag	4, 4	1, 3
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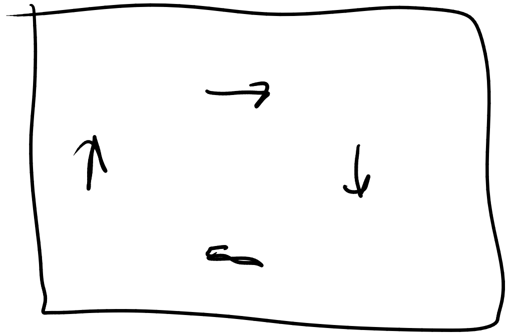
Nash Equilibria



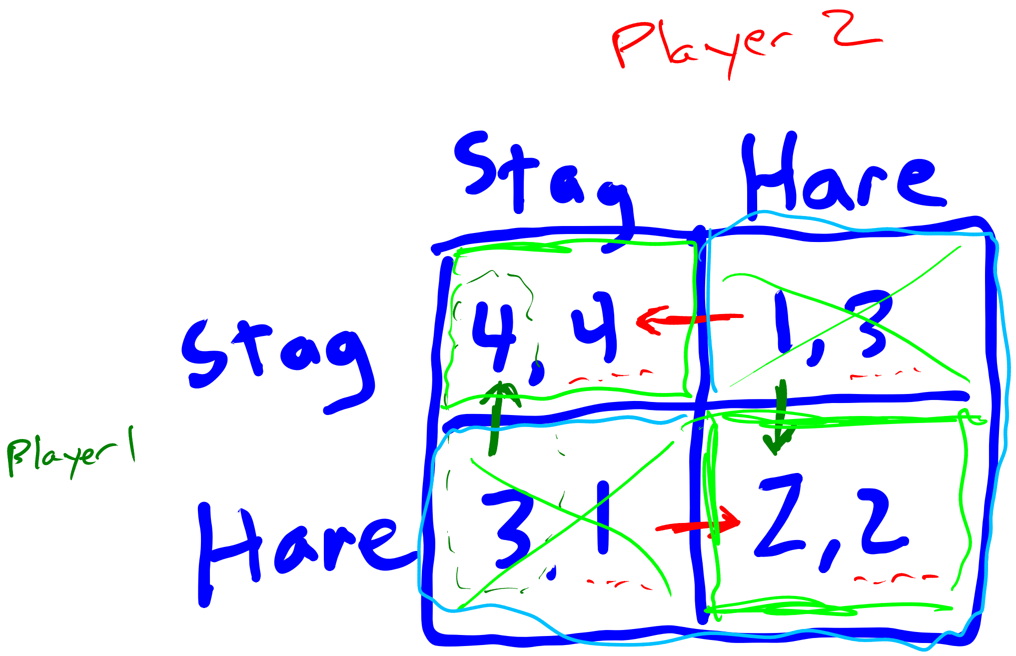
Nash Equilibrium

- A *Nash equilibrium* is a joint policy in which all agents are following a best response

Identifying Pure Nash Equilibria

1. Calculate Best responses of all players (draw arrows)

2. if a square doesn't have any out arrows it is a NE



Geopolitics

Rock-paper scissors

$\pi^2 = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$

is π^* a best response?

$U(\pi^{*1}, \pi^2) \overset{?}{\geq} U(\pi^{1\prime}, \pi^2)$

No

$\pi^{1\prime} = (0, 0, 1)$

		agent 2		
		rock	paper	scissors
agent 1	rock	$0, 0$	$-1, 1$	$1, -1$
	paper	$1, -1$	$0, 0$	$-1, 1$
	scissors	$-1, 1$	$1, -1$	$0, 0$

$\pi^{1\prime}(\text{rock}) \quad \pi^{1\prime}(\text{paper}) \quad \pi^{1\prime}(\text{scissors})$

$\pi^{1\prime} = (p_r, p_p, 1 - p_r - p_p)$

A two-player game is **zero sum** if

$$\sum_i R^i(a) = 0 \quad \forall a$$

- Pure strategy: $\pi^i(a^i) \in \{0, 1\}$
- Mixed strategy: all other strategies

since two-player

$\pi^1 = \pi^2$

Consider the strategy π^* where

$$\left(\pi^{*,i}(a^i) = \frac{1}{3} \quad \forall i, a^i\right)$$

Prove that this is a NE

Definition of Best Response

$$U^1(\pi^{*1}, \pi^{*2}) \geq U^1(\pi^{1\prime}, \pi^{*2}) \qquad \forall \pi^{1\prime}$$

$\pi^*(a) = \prod_i \pi^{*i}(a^i)$

$$U^1(\pi^{*1}, \pi^{*2}) = \sum_a R^1(a)\pi^*(a) = \frac{1}{3}\cdot\frac{1}{3}(0 + -1 + 1 + 1 + 0 + -1 + -1 + 1 + 0) = 0$$

$$U^1(\pi^{1\prime}, \pi^{*2}) = \sum_a R^1(a)\pi^{1\prime}(a^1)\pi^{*2}(a^2) = 0 \cdot p_r \cdot \frac{1}{3} + -1 \cdot p_r \cdot \frac{1}{3} + 1 \cdot p_r \cdot \frac{1}{3} + \dots + \dots$$

$$= \frac{1}{3} \cdot p_r \cdot (0 - 1 + 1) + \frac{1}{3} p_p (1 + 0 - 1) + \frac{1}{3}(1 - p_r - p_p)(0)$$

$$U^1(\pi^{1\prime}, \pi^{*2}) = 0$$

General approach to find Nash Equilibria

General approach to find Nash Equilibria

if 0 then π is NE

$$\begin{aligned}
\underset{\pi, U}{\text{minimize}} \quad & \sum_i \left(U^i - U^i(\pi) \right) \\
\text{subject to} \quad & U^i \geq U^i(a^i, \pi^{-i}) \text{ for all } i, a^i \\
& \sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i \\
& \pi^i(a^i) \geq 0 \text{ for all } i, a^i
\end{aligned}$$

General approach to find Nash Equilibria

Any number of players, any number of actions

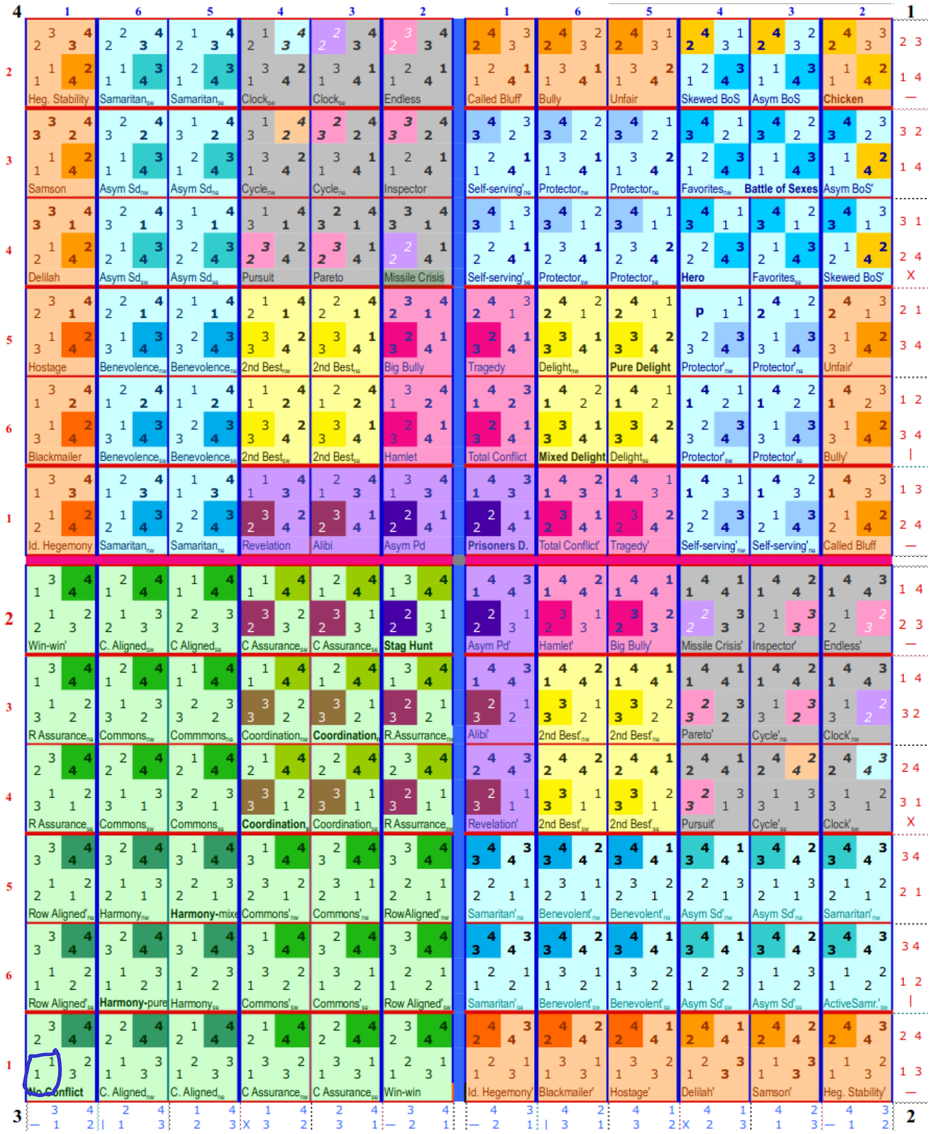
$$\underset{\pi, U}{\text{minimize}} \quad \sum_i \left(U^i - U^i(\pi) \right)$$

$$\text{subject to} \quad U^i \geq U^i(a^i, \pi^{-i}) \text{ for all } i, a^i$$

$$\sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i$$

$$\pi^i(a^i) \geq 0 \text{ for all } i, a^i$$

root-finding problem



Every finite game has a Nash Equilibrium

Every finite game has a Nash Equilibrium

EQUILIBRIUM POINTS IN N-PERSON GAMES

By John F. Nash, Jr.*

Princeton University

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \rightarrow Q$, $P_n \rightarrow P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

* The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the A. E. C. for financial support.

[1] Kakutani, S., *Duke Math. J.*, 8, 457–459 (1941).

[2] Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behaviour*, Chap. 3, Princeton University Press, Princeton, 1947.

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

A correspondence f: X → X has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

(1) X is a non-empty, closed, bounded, and convex set.

(2) $f(\mathbf{x})$ is non-empty for any $\mathbf{x}$.

(3) $f(\mathbf{x})$ is convex for any $\mathbf{x}$.

(4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

A correspondence f: X → X has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

(1) X is a non-empty, closed, bounded, and convex set.

(2) $f(\mathbf{x})$ is non-empty for any $\mathbf{x}$.

(3) $f(\mathbf{x})$ is convex for any $\mathbf{x}$.

(4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

Joint policy

- Let x be a strategy profile, π.

Every finite game has a Nash Equilibrium

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- Let x be a strategy profile, π.
- Let f be BR, that is, the best response operator

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for VI
Banach's Theorem

- Let x be a strategy profile, π.
- Let f be BR, that is, the best response operator
- A fixed point of BR is a Nash Equilibrium
- The BR operator and policy space for finite games meet the conditions above

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

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- Let x be a strategy profile, π.
- Let f be BR, that is, the best response operator
- A fixed point of BR is a Nash Equilibrium
- The BR operator and policy space for finite games meet the conditions above
- BR has a fixed point for every finite game, i.e. every finite game has a Nash Equilibrium

Battle of the Sexes

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- Gabby and Max are going on a date

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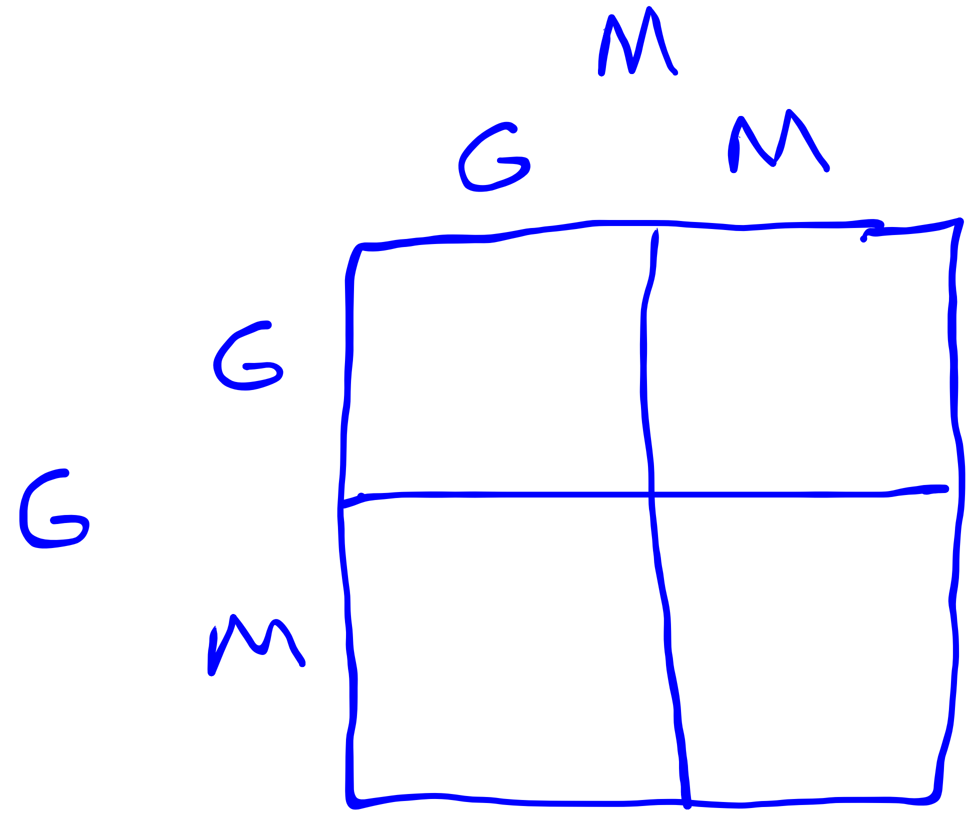

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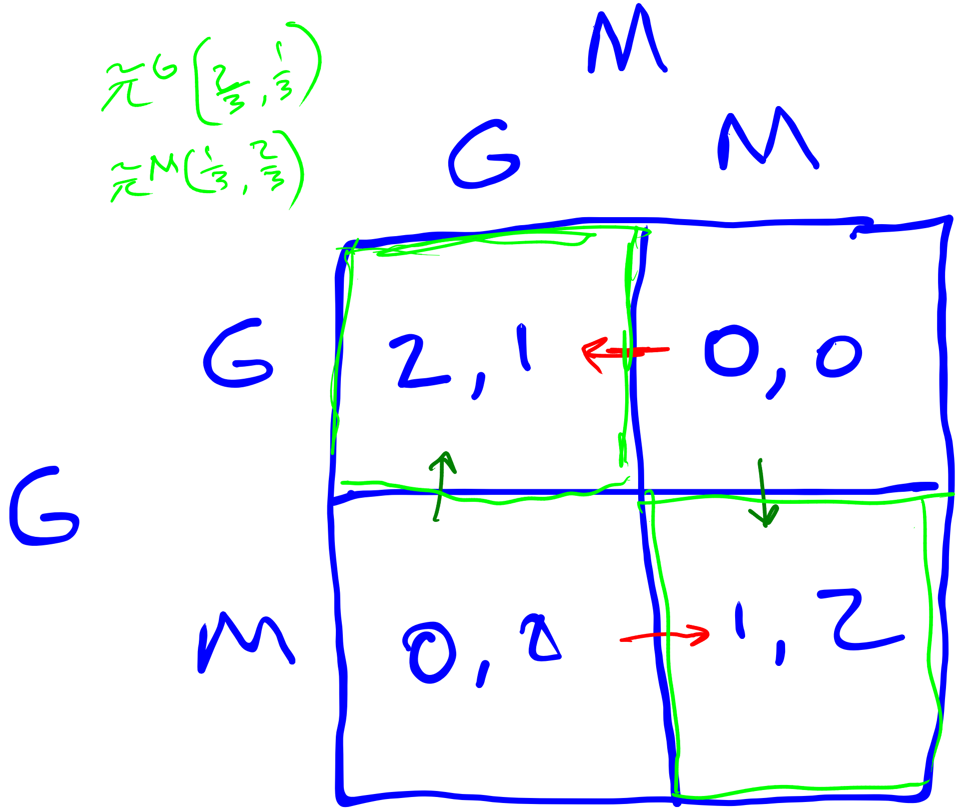

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Uncorrelated

$\pi(a) \neq \prod \pi^i(a^i)$

G \ M	G	M
G	2,1	0,0
M	0,0	1,2

Correlated Equilibrium

- A *correlated joint policy* is a single distribution over the joint actions of all agents.
- A *correlated equilibrium* is a correlated joint policy where no agent *i* can increase their expected utility by deviating from their current action to another.

$\pi(G,G) = \frac{1}{2}$
$\pi(M,M) = \frac{1}{2}$
$\pi(\cdot,\cdot) = 0$

$\pi(G,G) = a$
$\pi(M,M) = 1-a$
$\pi(\cdot,\cdot) = 0$

Recap

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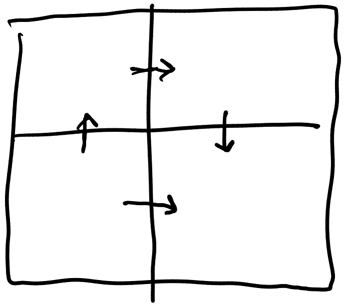
- Games may not have a single "optimal" solution; instead there are equilibria
- If every player is playing a best response, that joint policy is a Nash Equilibrium
- Every finite game has at least one Nash Equilibrium (pure or mixed)

Practice



		Player 2		
		a	b	c
Player 1	a	4,4	2,5	0,0
	b	5,2	3,3	0,0
	c	0,0	0,0	10,10