Guiding Question

- What does "Markov" mean in "Markov Decision Process"?

Stochastic Process

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- A stochastic process is a collection of R.V.s indexed by time.

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$v_t \sim \mathcal{U}(\{0, 1\})$ (i.i.d.) indep. ident. distributed

Bernoulli (0.5)

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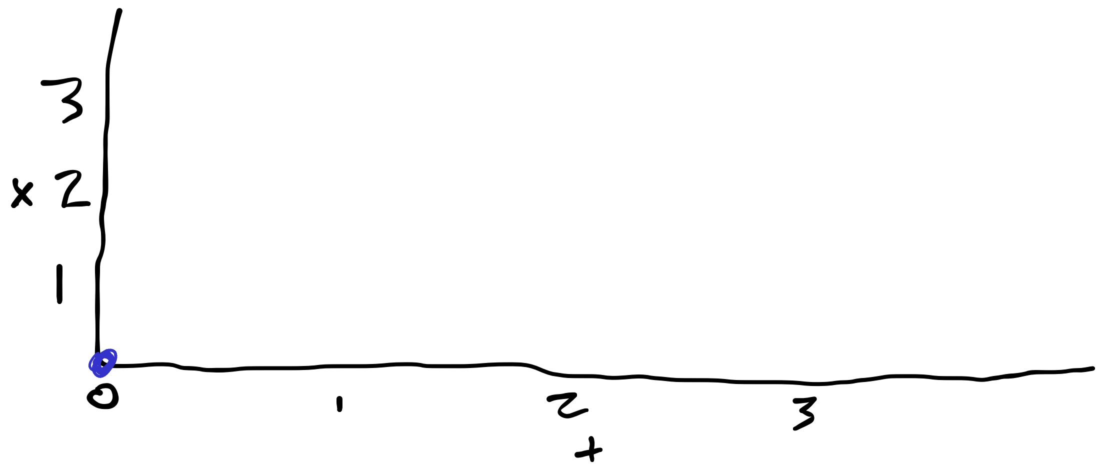

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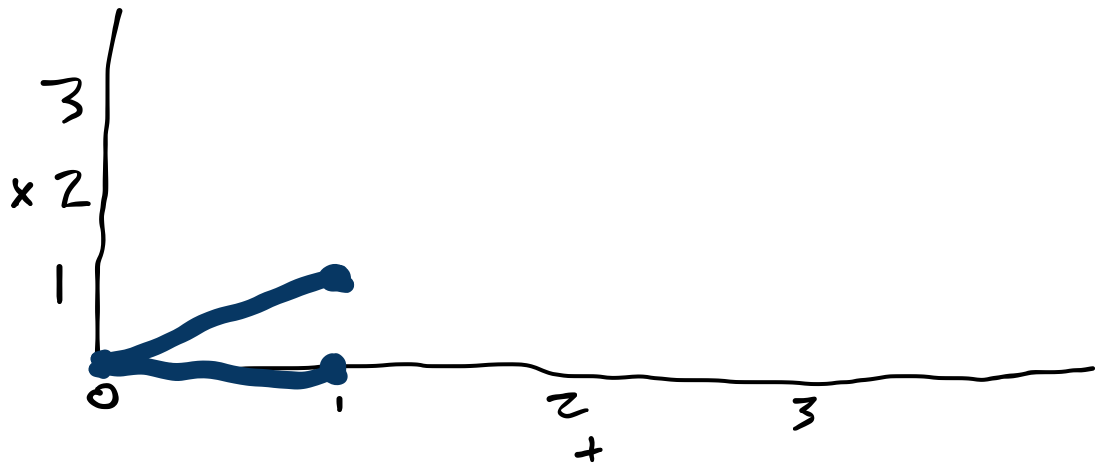

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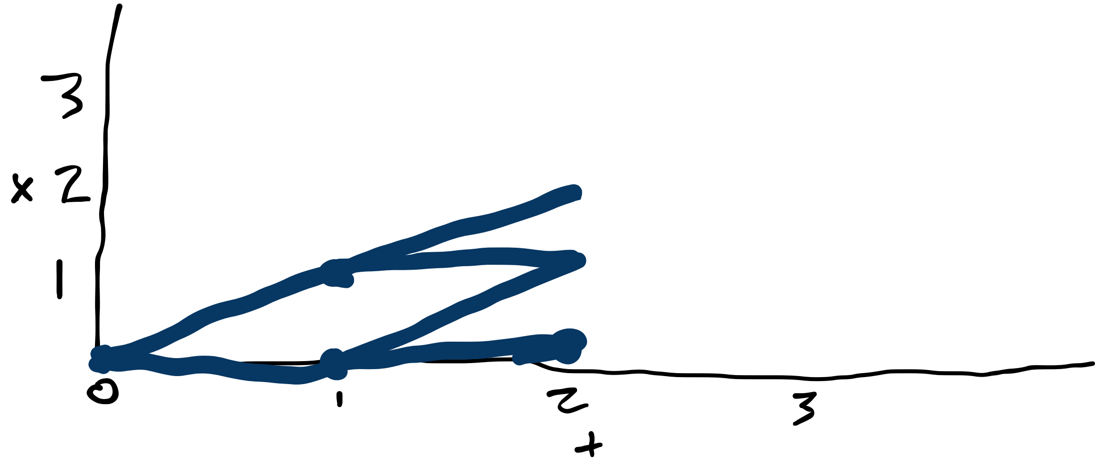

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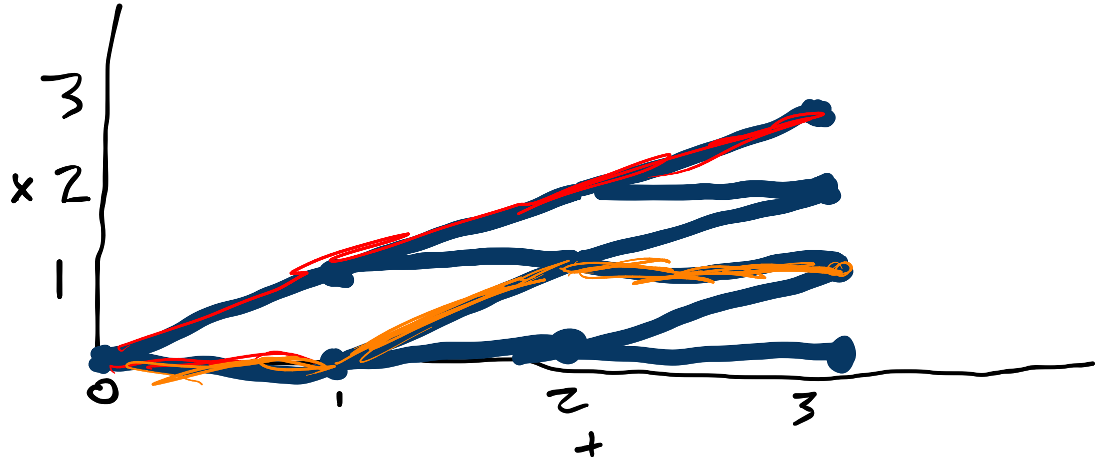

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Example:

In a *stationary* stochastic process (all in this class), this relationship does not change with time

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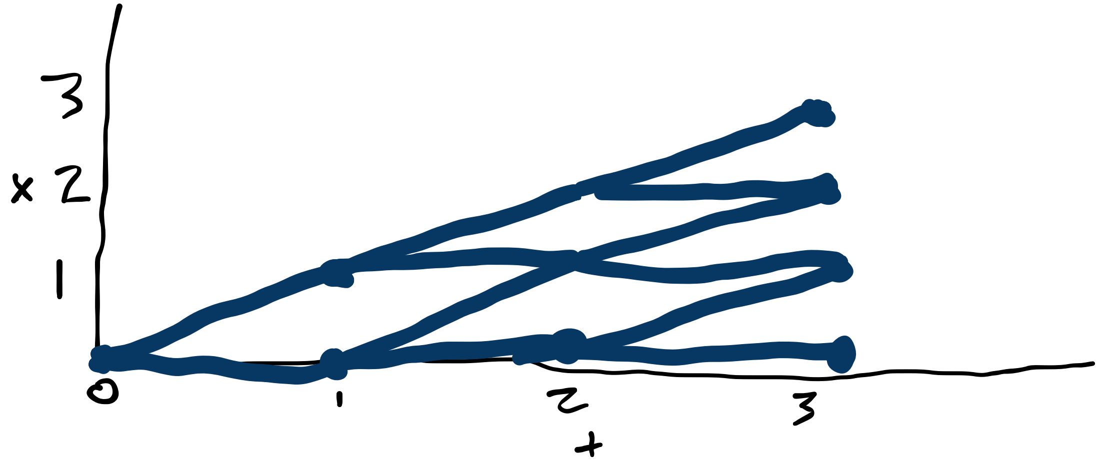

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$P(x_{t+1} \mid x_t = \bar{x})$

x_{t+1}	
$\bar{x}$	0.5
$\bar{x}+1$	0.5

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$$P(x_{t+1}) = \sum_{\bar{x}} P(x_{t+1}, x_t = \bar{x})$$

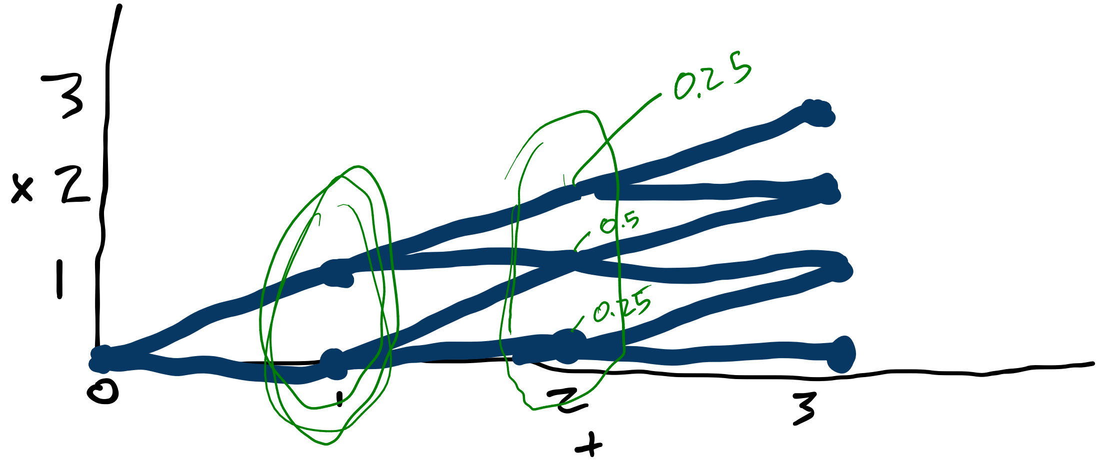
$$= \sum_{\bar{x}} P(x_{t+1} \mid x_t = \bar{x}) P(x_t = \bar{x})$$

x_1	$P(x_1)$
0	0.5
1	0.5

$$\underbrace{P(x_1 = 0 \mid x_0 = 0)}_{0.5} \underbrace{P(x_0 = 0)}_{1}$$

x_2	$P(x_2)$
0	0.25
1	0.5
2	0.25

$$\underbrace{P(x_2 = 0 \mid x_1 = 0)}_{0.5} \underbrace{P(x_1 = 0)}_{0.5} + \underbrace{P(x_2 = 0 \mid x_1 = 1)}_{0} \underbrace{P(x_1 = 1)}_{0.5}$$



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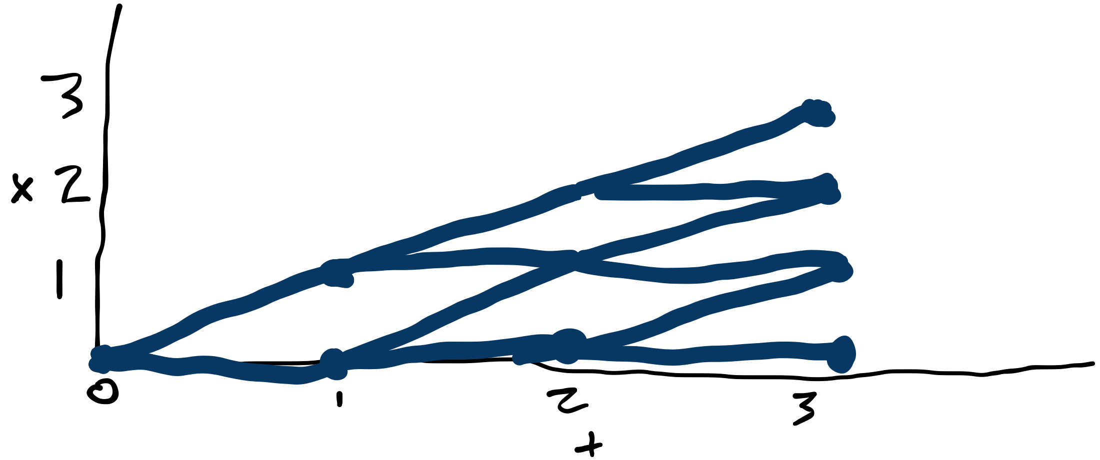
Joint



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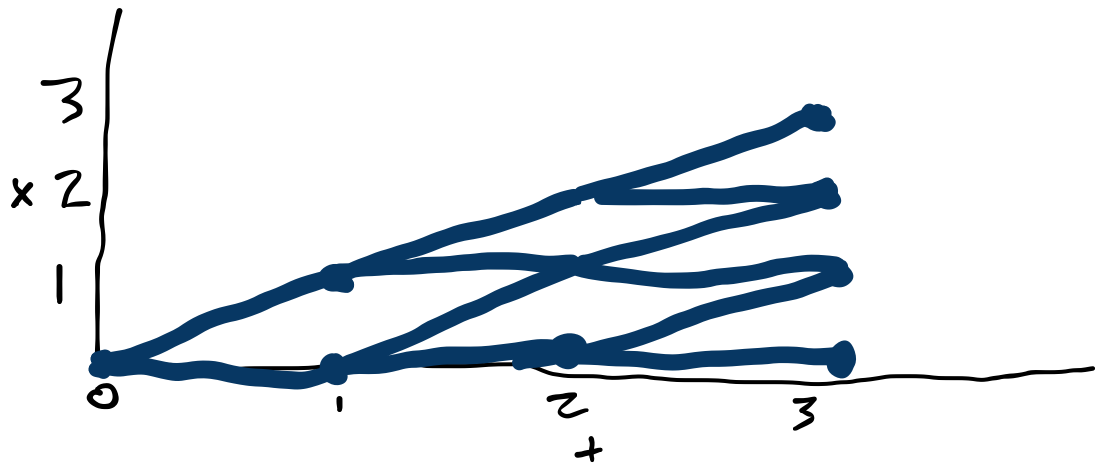

Joint

x0	x1	x2	P(x1, x2, x3)
0	0	0	0.25
0	0	1	0.25
0	1	1	0.25
0	1	2	0.25

Simulating a Stochastic Process

030-Stochastic-Processes.ipynb

Markov Process

Markov Process

- A stochastic process $\{s_t\}$ is *Markov* if

  $$P(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = P(s_{t+1} \mid s_t)$$

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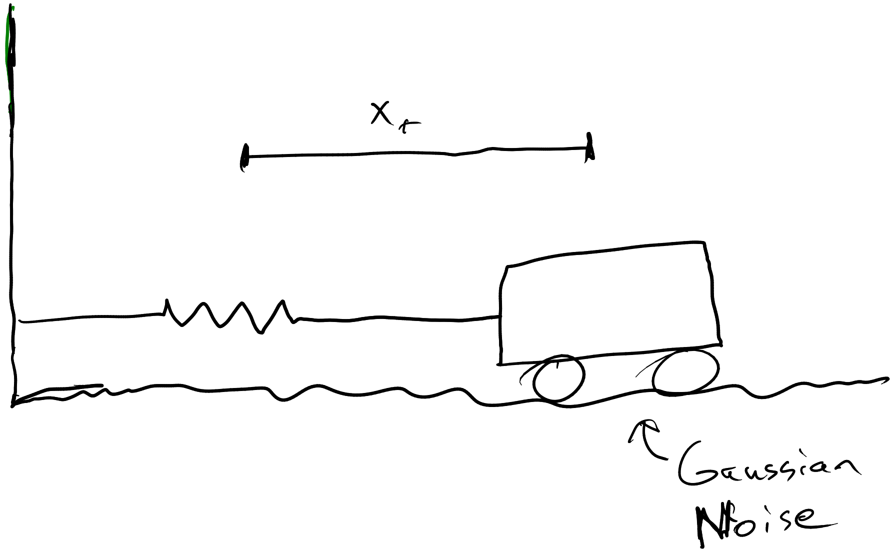
  $$P(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = P(s_{t+1} \mid s_t)$$
- s_t is called the "state" of the process



Is $\{x_t\}$ a Markov Process?

$v_t \sim N(\mu, \Sigma)$ (iid)

$$\underbrace{\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix}}_{y_{t+1}} = \begin{bmatrix} 1 & \Delta t \\ -\frac{k}{m}\Delta t & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}}_{y_t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_t$$

$$P(x_{t+1} \mid x_t, x_{t-1}, \ldots x_0) \overset{?}{=} P(x_{t+1} \mid x_t)$$

Is $\{y_t\}$ a Markov Process?

$$P(y_{t+1} \mid y_t \ldots y_0) \overset{?}{=} P(y_{t+1} \mid y_t)$$ Yes

Break

Break

- Suppose you want to create a Markov model that describes how many new COVID cases will be detected on a particular day. What information should be in the state of the model?

state: previous day's cases

n_t

k_1 : infectiousness

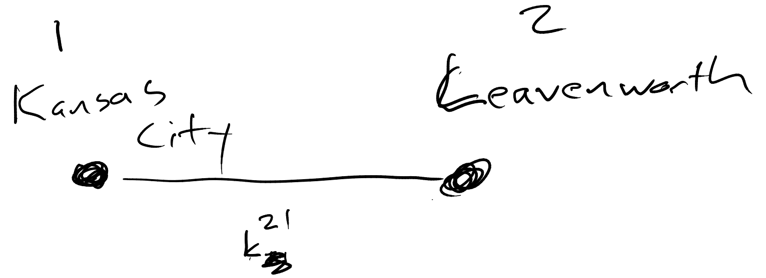
s_t : susceptible

$$n_{t+1} = s_t k_1 n_t + v_t$$

$$n_{t+1} = s_t \left(k_3 n_{t-3} + k_4 n_{t-4} \dots k_{14} n_{t-14} \right)$$

state : $(n_{t:t-14}, s_t)$

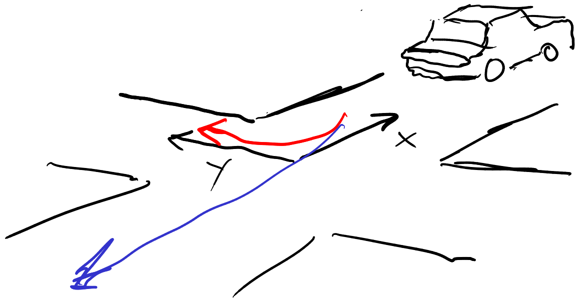
$k(d)$



$$n^i = s_t^i \left(k_3^{ii} n_3^i \dots + k^{ij} n^j \right.$$

Hidden Markov Model

(Often you can't measure the whole state)

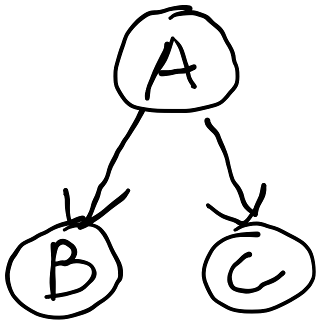


state : $(x, y, \dot{x}, \dot{y}, \text{intention})$

observation : $(x, y, \dot{x}, \dot{y})$

Bayesian Networks

Bayesian Networks



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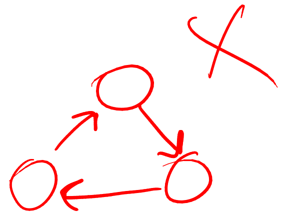
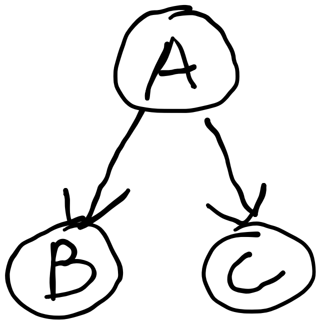


A *Bayesian Network* is a directed acyclic graph (DAG) that encodes probabilistic relationships between R.V.s



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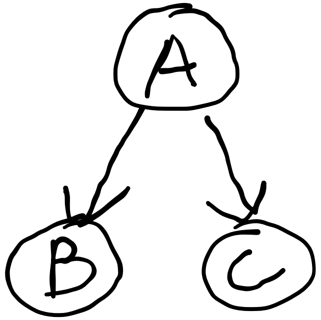

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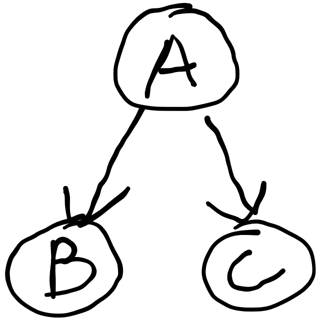

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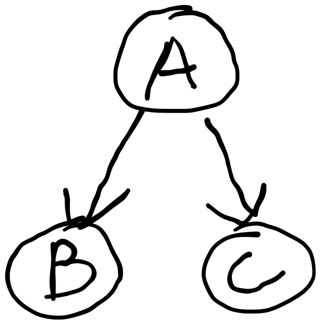

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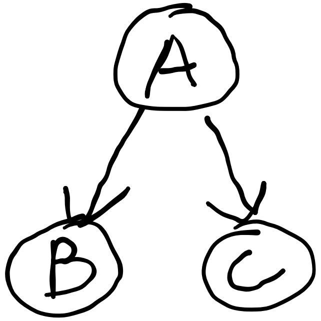
Concretely:

Bayesian Networks



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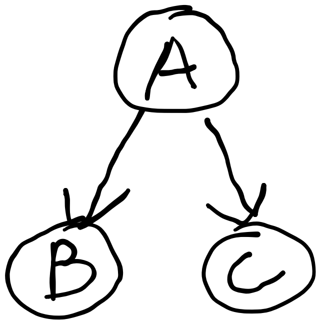
Concretely: $P(x_{1:n}) = \prod_i P(x_i \mid pa(x_i))$

Bayesian Networks



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Parents

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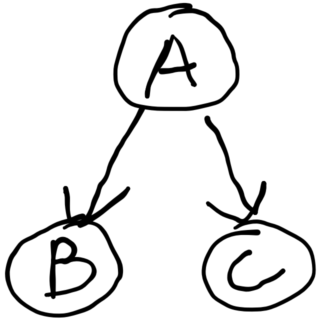

$P(A, B, C)$

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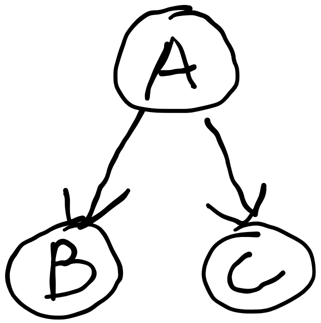

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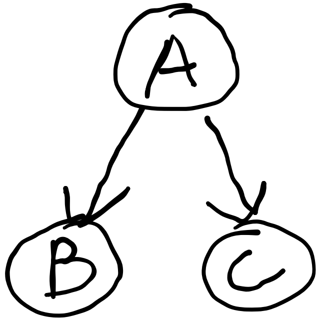

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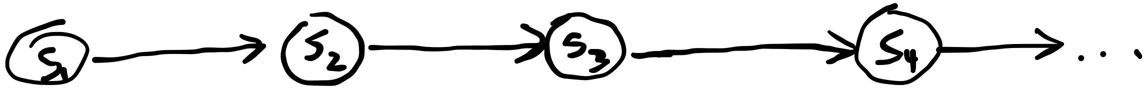

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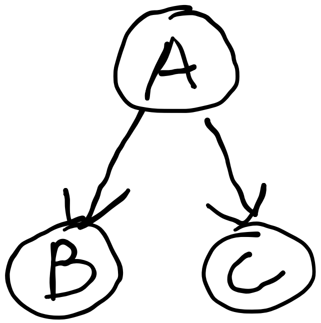

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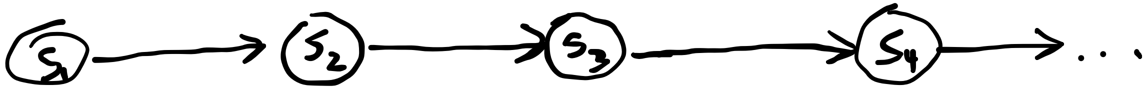

Hidden Markov Model

Bayesian Networks

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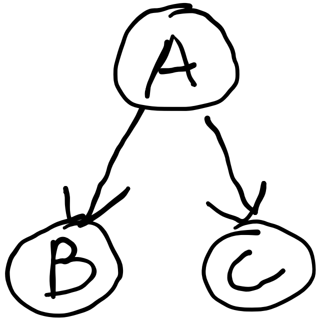

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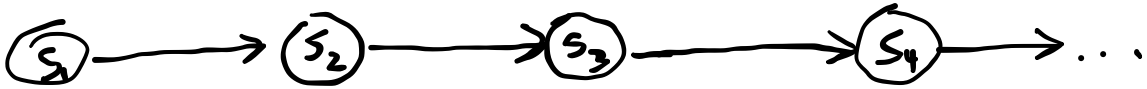
Markov Process



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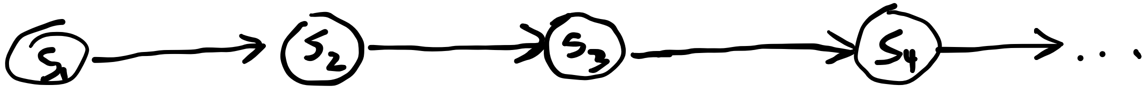

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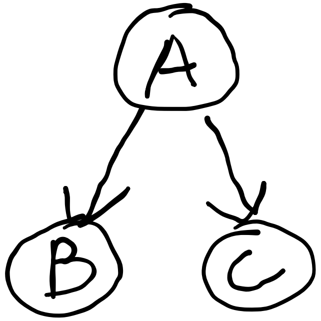

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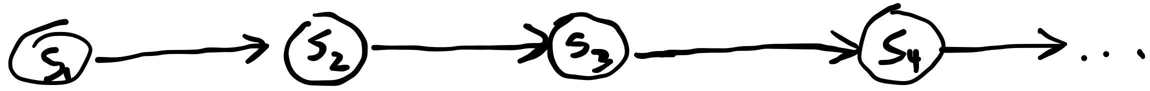

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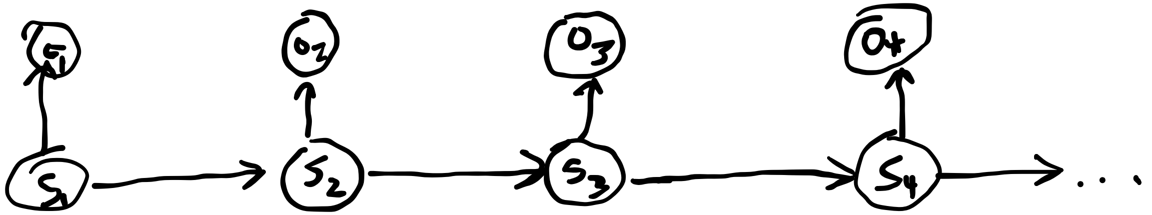

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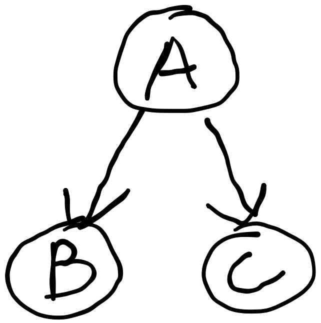

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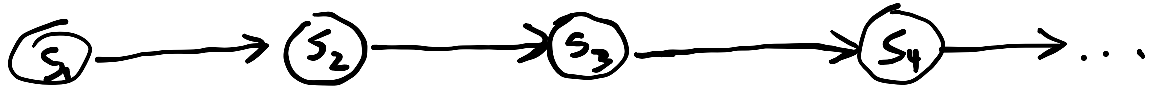
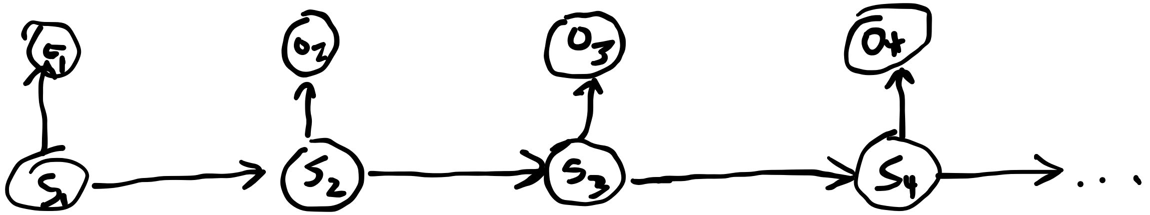


Hidden Markov Model



Dynamic Bayesian Network

Bayesian Networks

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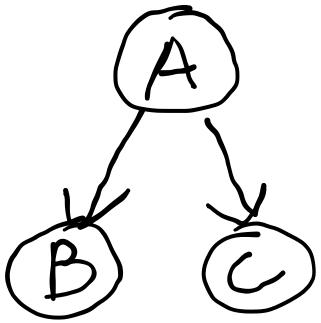

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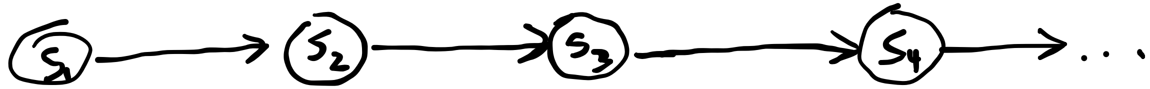
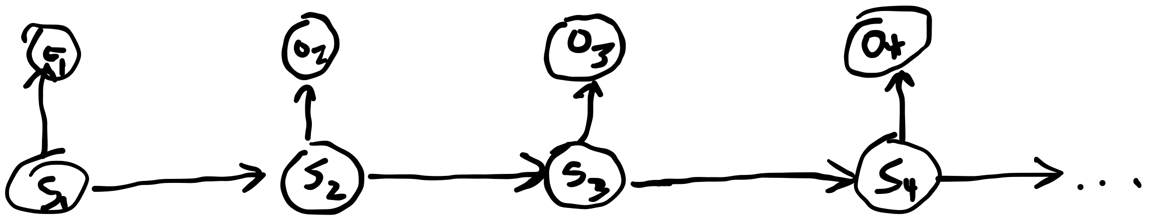
Parents

Markov Process



Hidden Markov Model



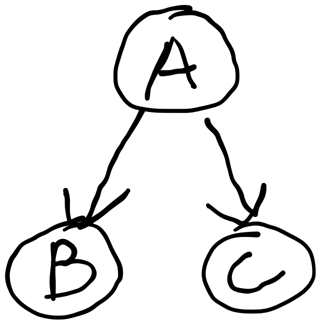
Dynamic Bayesian Network

(One step)

Bayesian Networks

A *Bayesian Network* is a directed acyclic graph (DAG) that encodes probabilistic relationships between R.V.s

- Nodes: R.V.s
- Edges: Direct probabilistic relationships



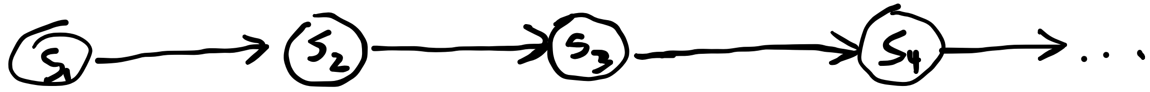
$P(A,B,C)$

$= P(A)P(B \mid A)P(C \mid A)$

Concretely: $P(x_{1:n}) = \prod_i P(x_i \mid pa(x_i))$

Parents

Markov Process



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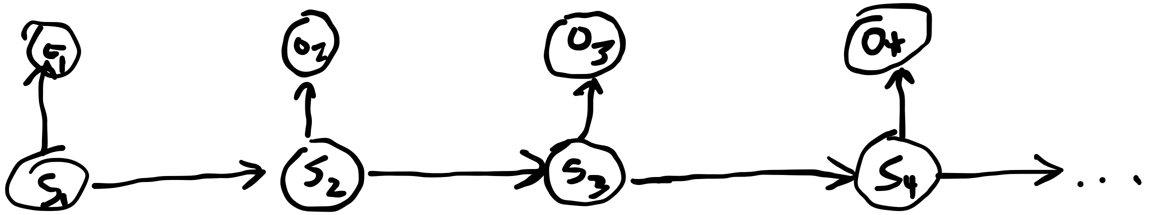
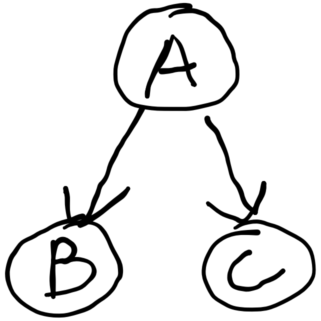


Bayesian Networks



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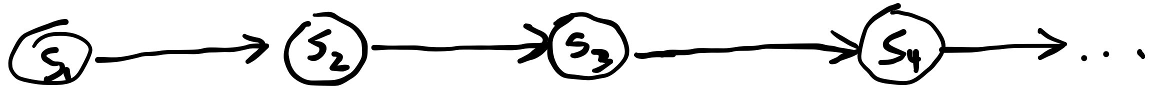

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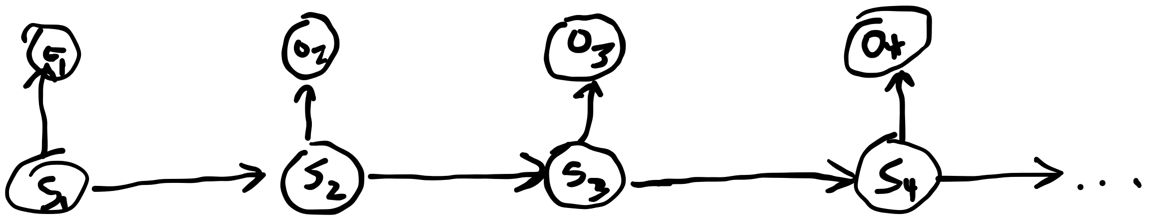

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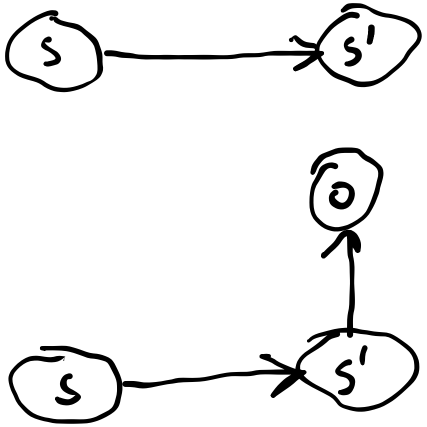

Decision Networks and MDPs

Decision Networks and MDPs

Decision Network

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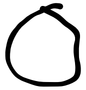

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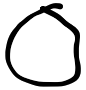 Chance node

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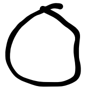
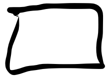
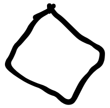
 Chance node

 Decision node



Decision Networks and MDPs

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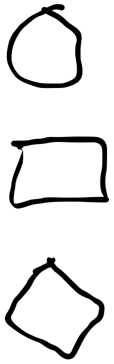

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MDP Dynamic Decision Network

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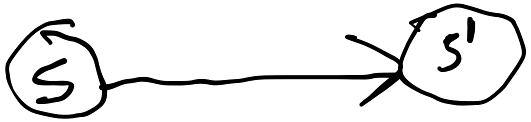

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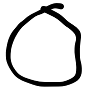
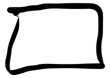
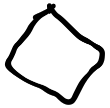
 Chance node

 Decision node

 Utility node

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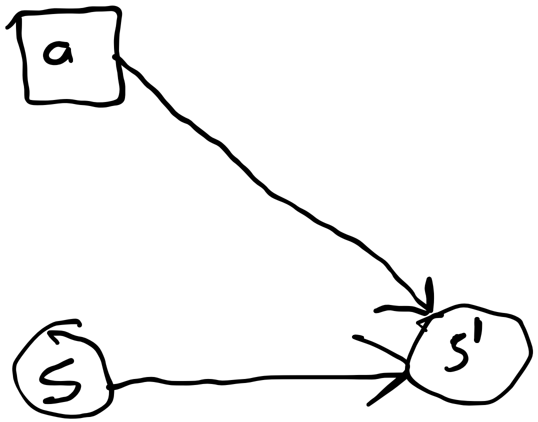

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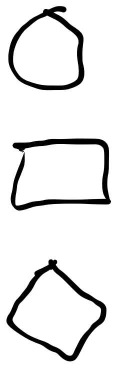

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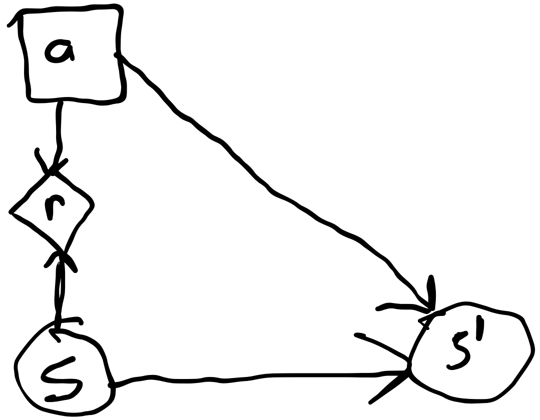

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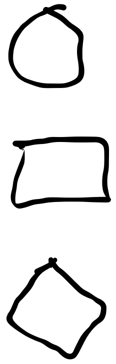

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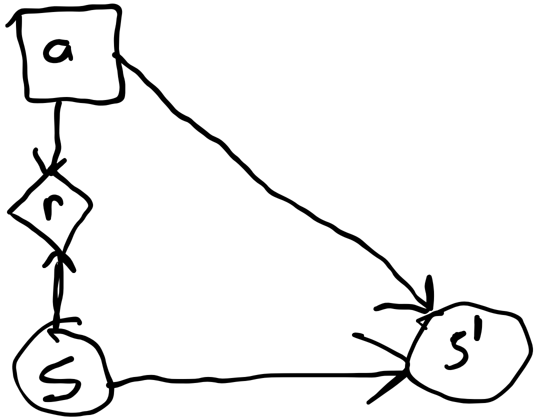

MDP Optimization problem

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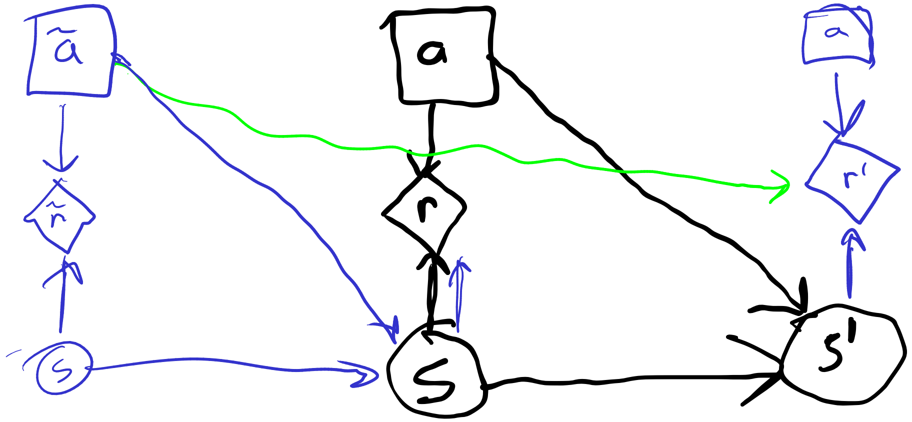

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$$\text{maximize} \quad \mathrm{E}\left[\sum_{t=1}^{\infty} r_t\right]$$

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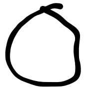
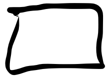
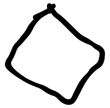
 Chance node

 Decision node

 Utility node

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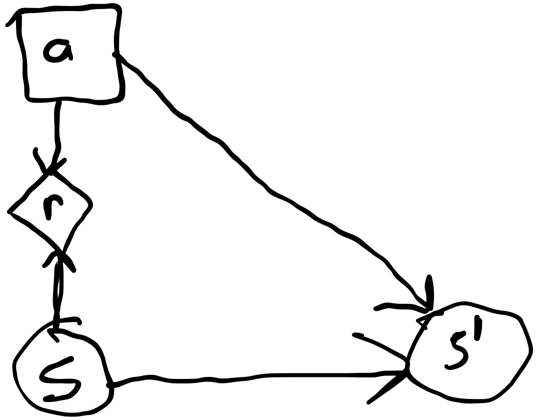

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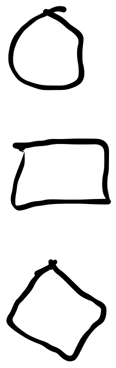

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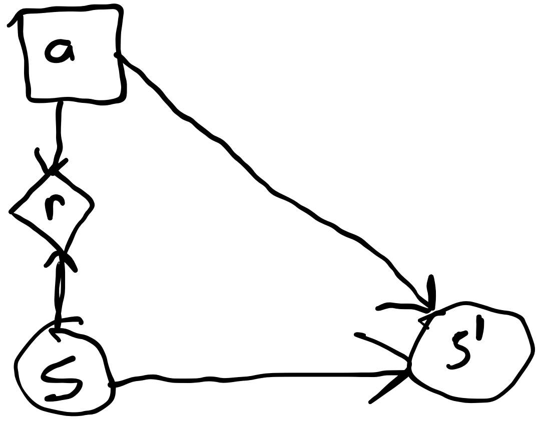

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4. Terminal States

Infinite time, but a terminal state (no reward, no leaving) is always reached with probability 1.

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?