Last Time

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$\max_a R(s,a)$ ← Myopic

$\max \underbrace{R(s,a) + \gamma E[U(s')]}_{Q(s,a)}$

$\max_a Q^*(s,a)$

- How do we reason about the **future consequences** of actions in an MDP?

Last Time

- How do we reason about the **future consequences** of actions in an MDP?
- What are the basic **algorithms for solving MDPs**?

Policy Iteration
Evaluate policy
Improve policy

Value Iteration
Improve V estimate

Guiding Questions

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- Does value iteration always converge?
- Is the value function unique?

Value Iteration: The Bellman Operator

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Algorithm: Value Iteration

while $\|V - V'\|_\infty < \epsilon$

$V \leftarrow V'$

$V' \leftarrow B[V]$

return V'

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$$B[V](s) = \max_{a \in A} \left(R(s, a) + \gamma E\left[V\left(s'\right)\right]\right)$$

Value Iteration Convergence

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Theorem 1: Let $\{V_1, \ldots, V_\infty\}$ be a sequence of value functions for a discrete MDP generated by the recurrence $V_{k+1} = B[V_k]$. If $\gamma < 1$, then $\lim_{k\to\infty} V_k = V^*$.

Metrics

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Definition: Let M be a set. A *metric* on M is a function $d : M \times M \to [0, \infty)$ which satisfies the following three conditions for all $x, y, z \in M$:

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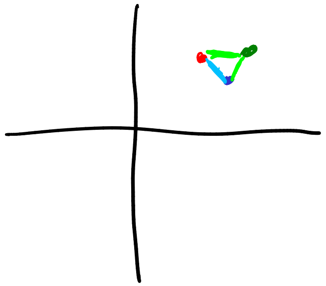
• x
• y
• z

$\|x - y\|_2$



Contraction Mappings

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<u>Definition</u>: A *contraction mapping* on metric space (M, d) is a function $f : M \to M$ satisfying

$$d(f(x), f(y)) \leq \alpha\, d(x, y)$$

for some α, $0 \leq \alpha \leq 1$ and all x and y in M.

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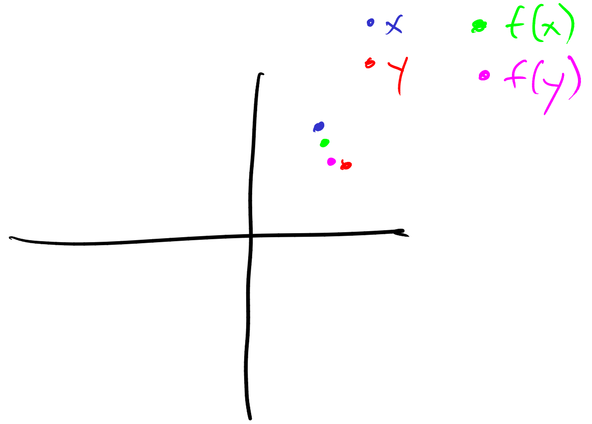
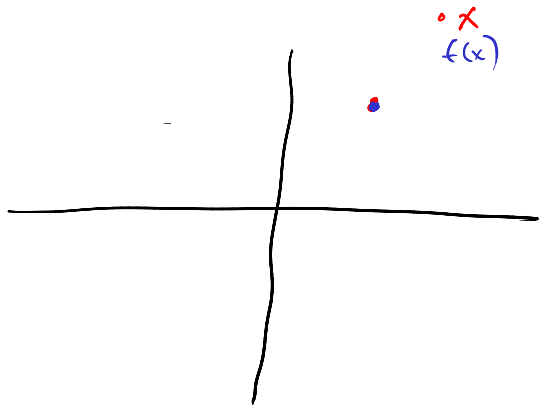


<u>Definition</u>: x^* is said to be a *fixed point* of f if $f(x^*) = x^*$.



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Script: contraction_mapping.jl

Banach's Theorem

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1. f has a single, unique fixed point x^*.
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Max Norm

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Proof:

$\|x\|_n = \left(\sum |x_i|^n\right)^{1/n}$

$\|x-y\|_1$

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3. $\max |x - z| = \max |x - y + y - z|$

Assumption: $|S| < \infty$

Max Norm

$S = \{1, 2, 3\}$

$\mathbb{R}^{|S|}$ = space of 3D vectors

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3. $\max|x-z| = \max|x-y+y-z|$
   $\leq \max(|x-y| + |y-z|)$
   $\leq \max|x-y| + \max|y-z|$

Bellman Operator Contraction

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$$\|B[V_1] - B[V_2]\|_\infty = \max_{s\in S} |B[V_1](s) - B[V_2](s)|$$

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$$\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

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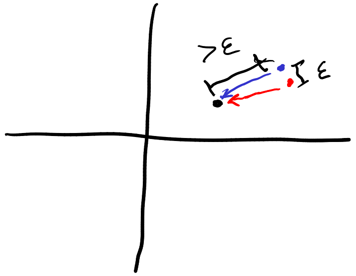

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By Lemma 2 and Banach's theorem (part 2), repeated application of the Bellman operator always has a fixed point limit, $\hat{V}$.

By Banach's theorem (part 1), $\hat{V} = B[\hat{V}]$. Since $\hat{V}$ satisfies Bellman's equation, it is optimal and $\hat{V} = V^*$.

Does Policy Iteration Converge?

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3. There are a finite number of possible policies
4. By (1), (2), and (3), the policy will improve until it finds the optimal policy, and it will always find the optimal policy.

Guiding Questions

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- Does value iteration always converge?
- Is the value function unique?

$S = \{1, 0.8 \ldots 0\}$

Break

Conservation MDP

$S = \{0, 1\}$ ← extinct (0)

$A = \{p, n\}$

$R(s,a) = R(s) + R(a)$

$R(1) = 1$

$R(0) = -1$

$R(p) = -1$

$R(n) = 1$

$$T^p = \begin{array}{c|cc} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0.1 & 0.9 \end{array}$$

$$T^n = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

$\gamma = 0.9$

Find V^*, π^*

$V^*(0) = \max_a \left(R(s,a) + \gamma E[V^*(s')]\right)$ ← Bellman's Optimality Equation

$\max_a \left(R(s,a) + 0.9\, V^*(0)\right)$ (a → n)

$V^*(0) = 0 + 0.9\, V^*(0)$

$(1 - 0.9) V^*(0) = 0$

$V^*(0) = 0$

$\pi^*(0) = n$

Two options: $\pi_p(1) = p$ $\quad \pi_n(1) = n$

$V^{\pi_p}(1) = R(1,p) + \gamma\left(0.1\, \cancel{V^{\pi_p}(0)} + 0.9\, V^{\pi_p}(1)\right)$ ← Policy Eval. Bellman's Exp. Eqn.

$0 + \cancel{\gamma}\left(0.9\, V^{\pi_p}(1)\right)$ ($\gamma \to 0.9$)

$(1 - 0.81) V^{\pi_p}(1) = 0$

$V^{\pi_p}(0) = 0$

$V^{\pi_n}(1) = R(1,n) + \gamma\left(\cancel{0.5\, V^{\pi_n}(0)} + 0.5\, V^{\pi_n}(1)\right)$ ($\to 0$)

$2 + \gamma\left(0.5\, V^{\pi_n}(1)\right)$

$(1 - 0.45) V^{\pi_n}(1) = 2$

$V^{\pi_n}(1) = 3.64 = V^*(1)$

$\pi^*(1) = n$