Bayesian Network Learning

- Last time:
- Today:

Bayesian Network Learning

- Last time:
  - Conditional independence in Bayesian Networks
- Today:

Bayesian Network Learning

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  - Conditional independence in Bayesian Networks
  - Sampling from Bayesian Networks
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Bayesian Network Learning

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  - Sampling from Bayesian Networks
- **Today:**
  - Given a **Bayesian Network** and some **values**, how do we calculate the probability of **other values**?

Bayesian Network Learning

- **Last time:**
  - Conditional independence in Bayesian Networks
  - Sampling from Bayesian Networks
- **Today:**
  - Given a **Bayesian Network** and some **values**, how do we calculate the probability of **other values**? Inference
  - Given **data**, how do we **fit** a Bayesian network? Learning

Bayesian Network

Bayesian Network

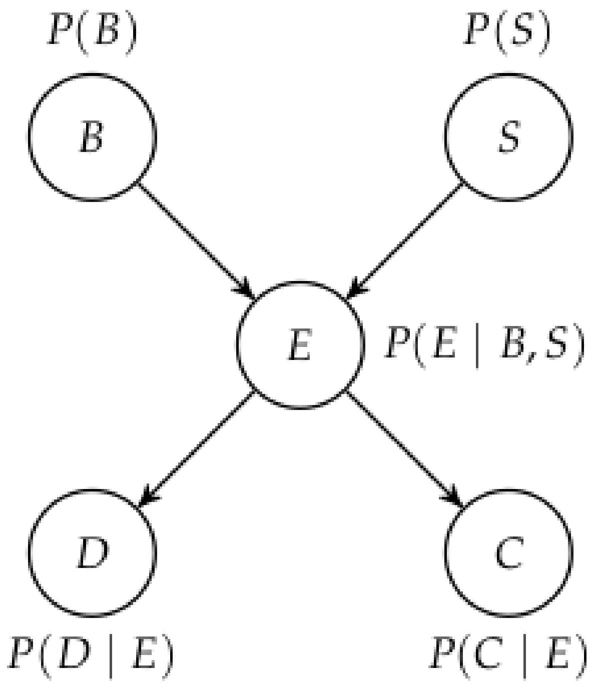
Structure

Bayesian Network

Structure

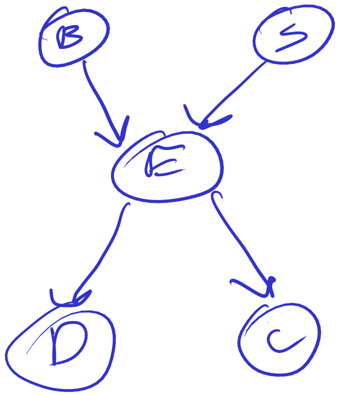
Parameters

Bayesian Network



B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

Structure G



Parameters θ

$P(B)$ $P(S)$

$P(E \mid B, S)$

$P(D \mid E)$ $P(C \mid E)$

Inference

Inputs

Outputs

Inference

Inputs

- Bayesian network structure

Outputs

Inference

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- Bayesian network parameters

Outputs

Inference

Inputs

- Bayesian network structure
- Bayesian network parameters
- Values of *evidence variables*

Outputs

Inference

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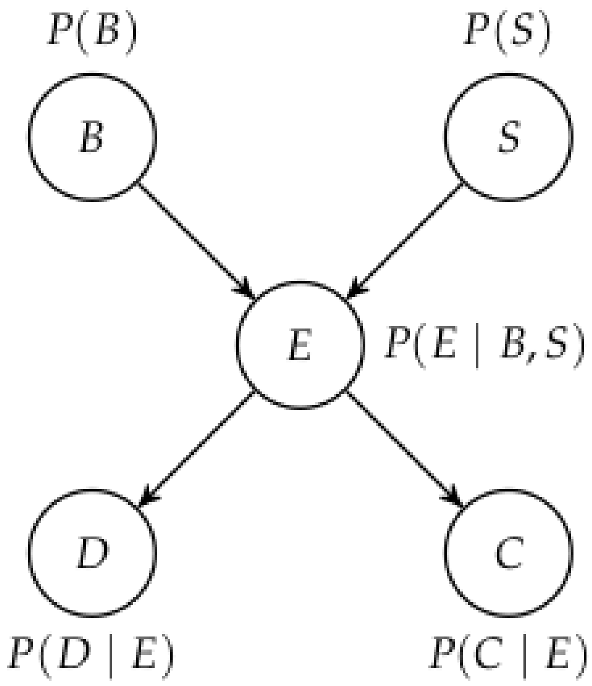
Outputs

- Posterior distribution of *query variables*

Inference



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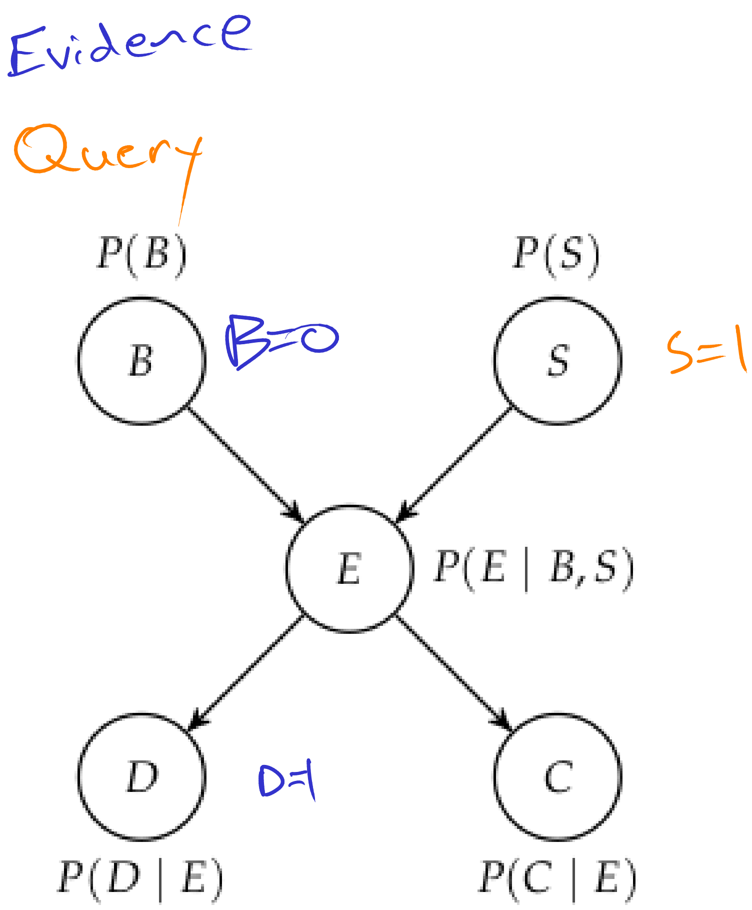

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Outputs

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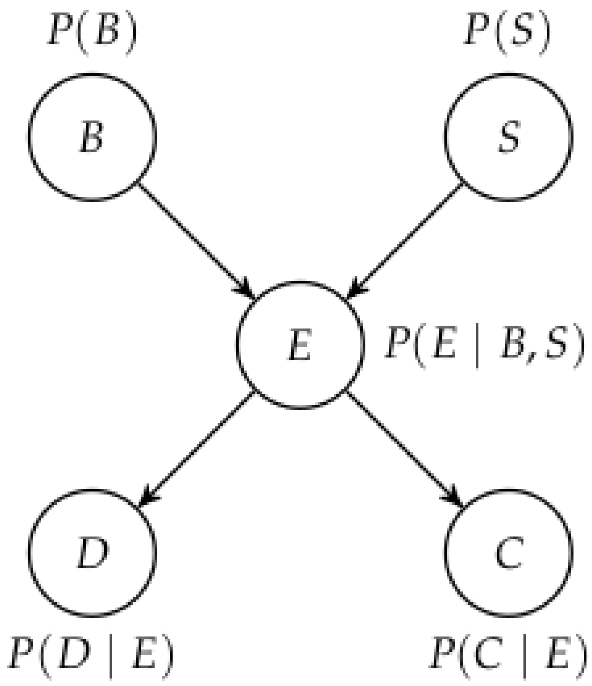
Given that you have detected a trajectory deviation, and the battery has not failed what is the probability of a solar panel failure?

Inference



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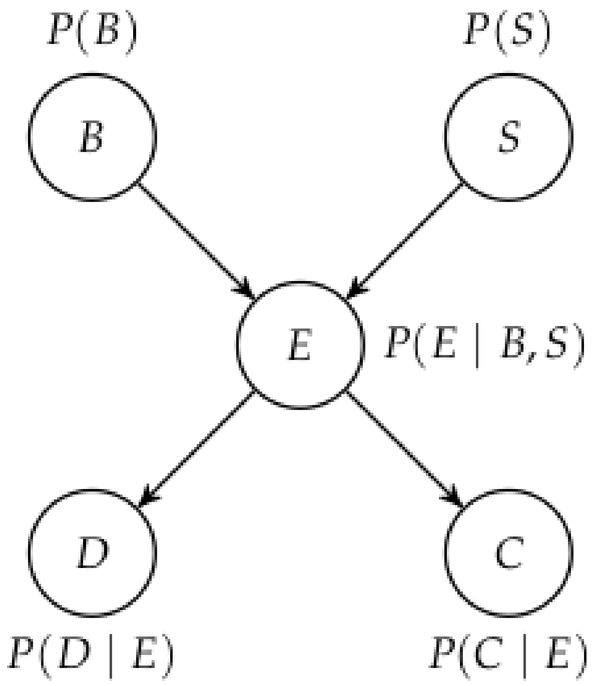
$$P(S = 1 \mid D = 1, B = 0)$$

Inference



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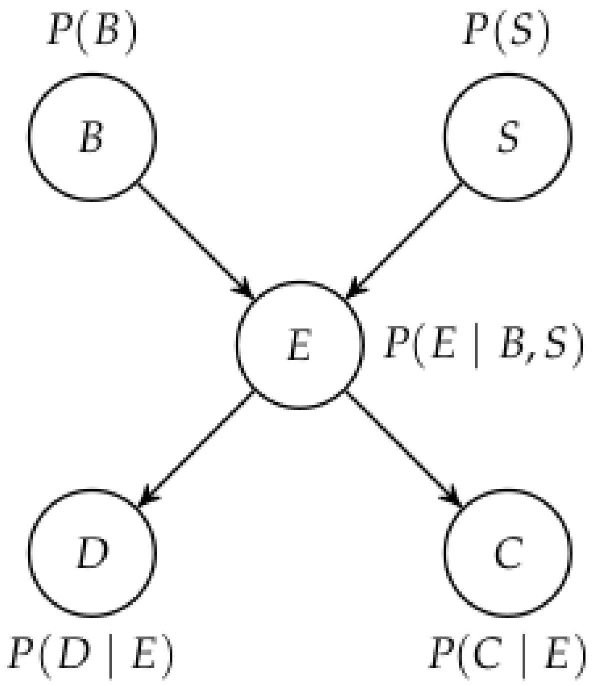
Exact

Inference



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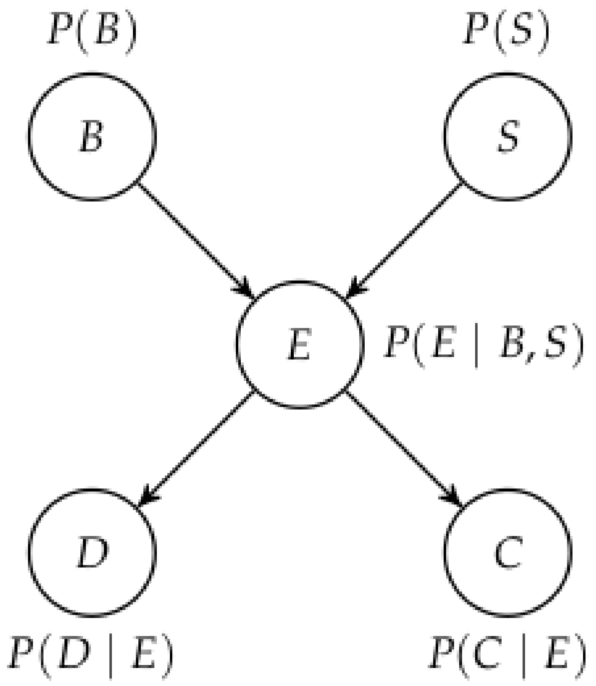
Exact NP-hard

Approximate

Exact Inference

Exact Inference



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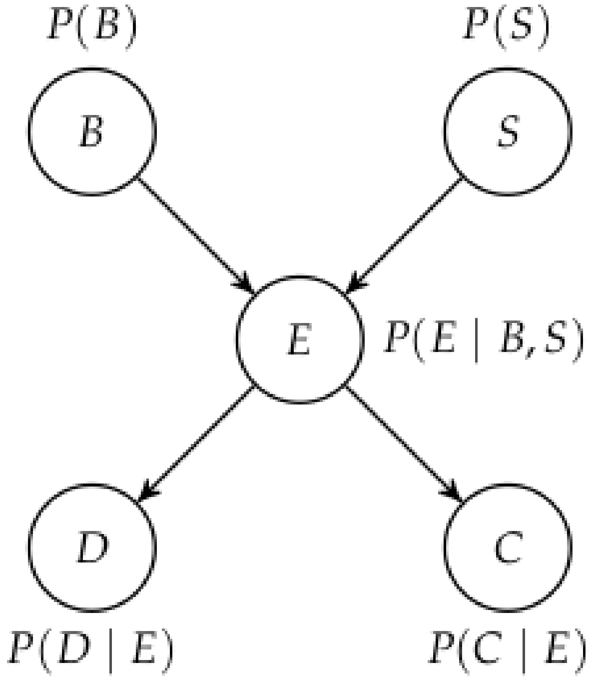

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$$P(S{=}1 \mid D{=}1, B{=}0)$$

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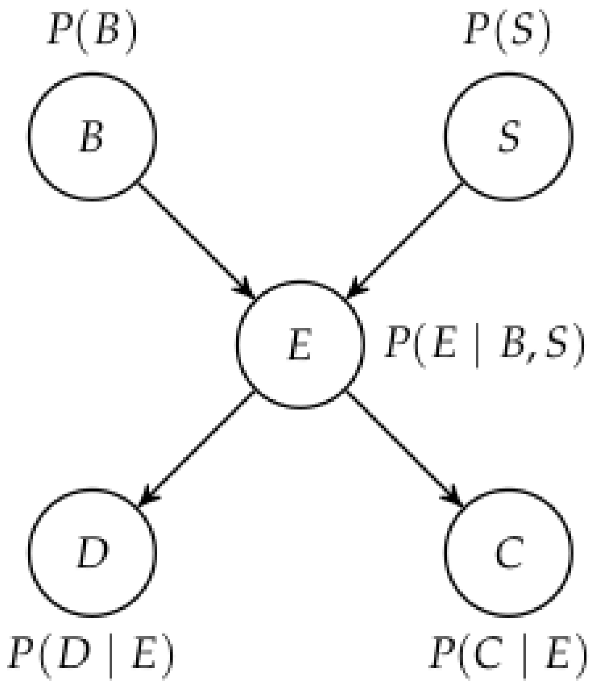

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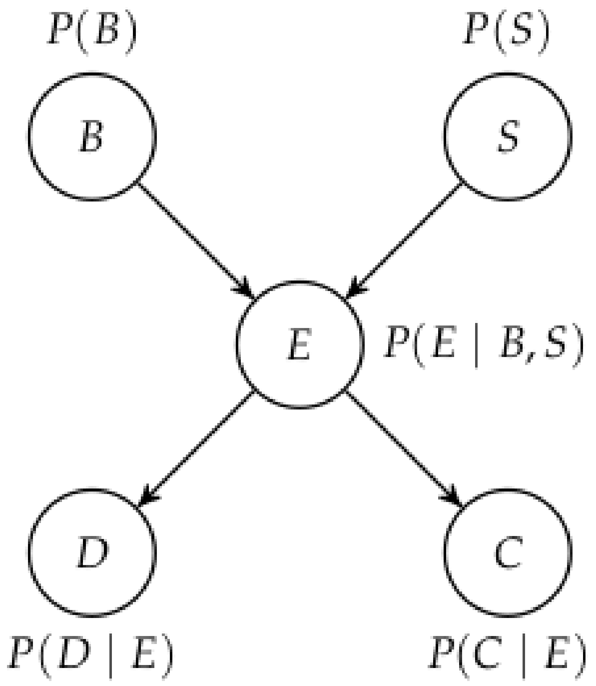
$$P(S{=}1 \mid D{=}1, B{=}0) \;=\; \frac{P(S{=}1, D{=}1, B{=}0)}{P(D{=}1, B{=}0)}$$

Exact Inference



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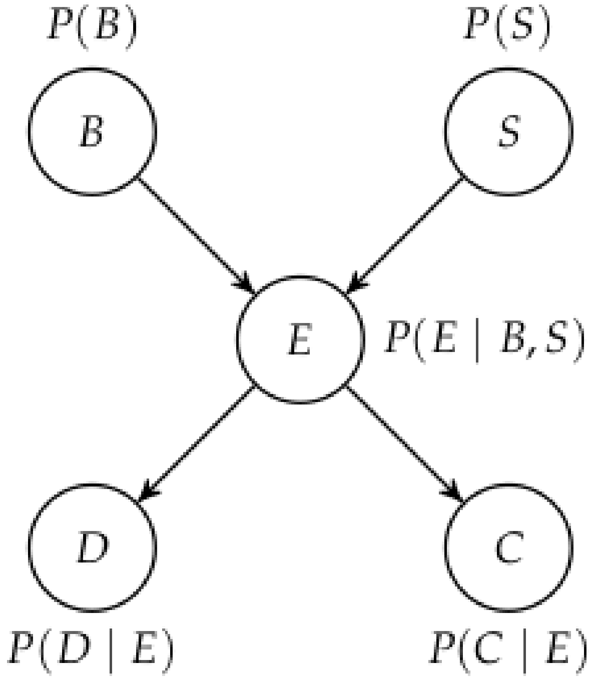
$$P(S{=}1, D{=}1, B{=}0) = \sum_{e,c} P(B{=}0, S{=}1, E{=}e, D{=}1, C{=}c)$$

Exact Inference



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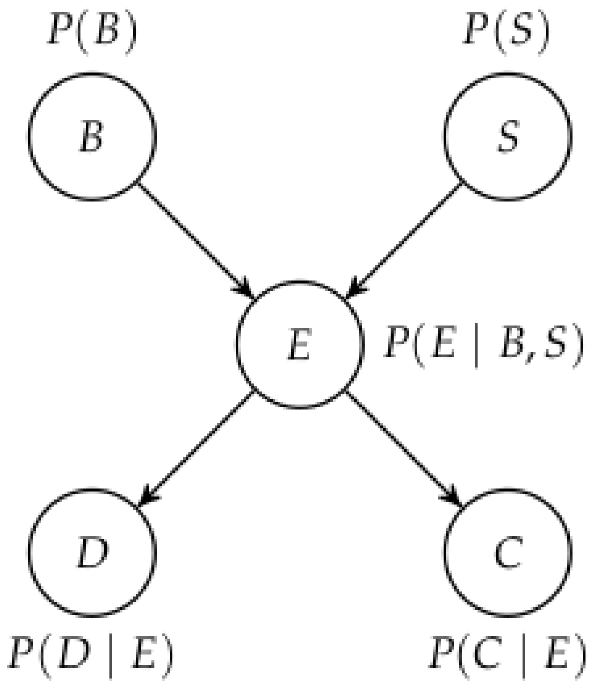

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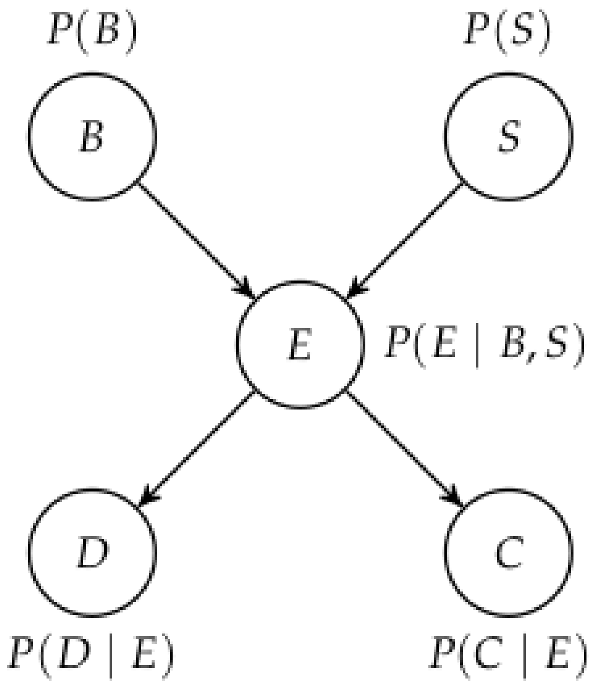
$$P(B{=}0, S{=}1, E, D{=}1, C)$$

$$= P(B{=}0)\, P(S{=}1)\, P(E \mid B{=}0, S{=}1)\, P(D{=}1 \mid E)\, P(C{=}1 \mid E)$$

Exact Inference



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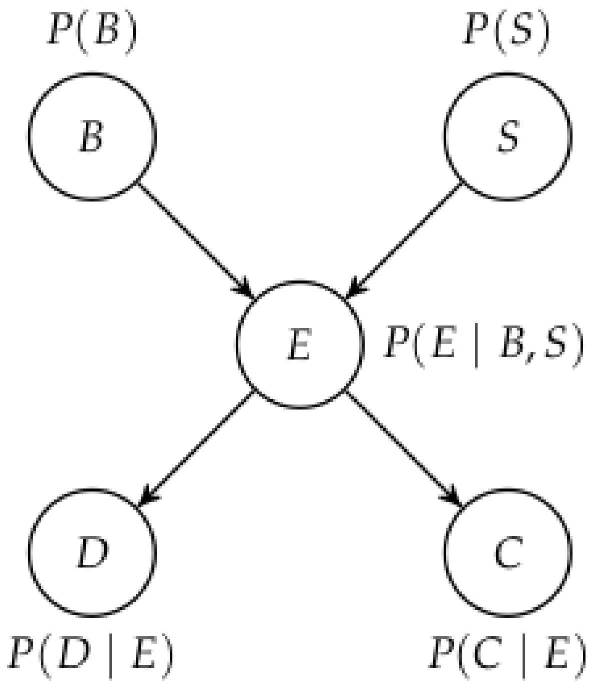
$2^5 = 32$ possible assignments,
but quickly gets too large

Exact Inference



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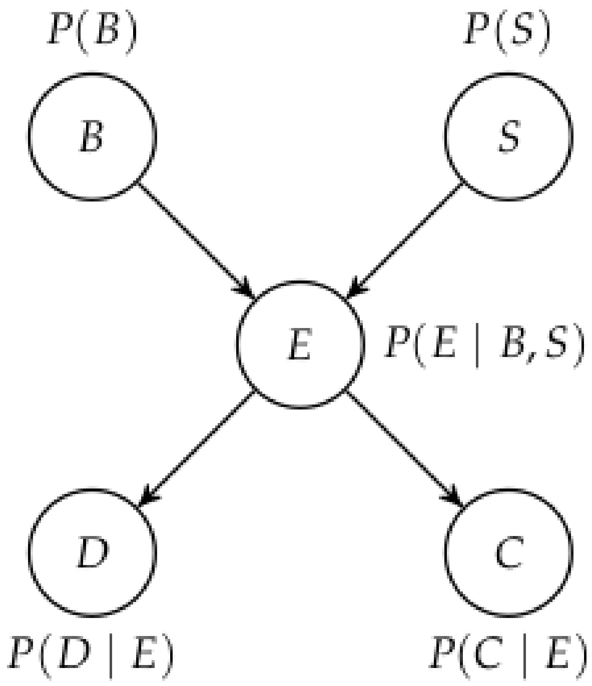

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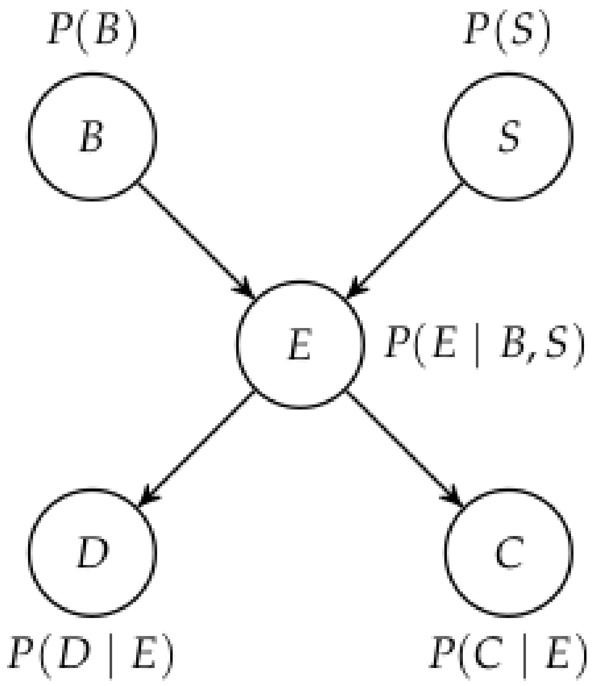
Product

X	Y	$\phi_1(X, Y)$
0	0	0.3
0	1	0.4
1	0	0.2
1	1	0.1

Y	Z	$\phi_2(Y, Z)$
0	0	0.2
0	1	0.0
1	0	0.3
1	1	0.5

X	Y	Z	$\phi_3(X, Y, Z)$
0	0	0	0.06
0	0	1	0.00
0	1	0	0.12
0	1	1	0.20
1	0	0	0.04
1	0	1	0.00
1	1	0	0.03
1	1	1	0.05

Exact Inference



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Condition

X	Y	Z	$\phi(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

$Y = 1$

X	Z	$\phi(X,Z)$
0	0	0.09
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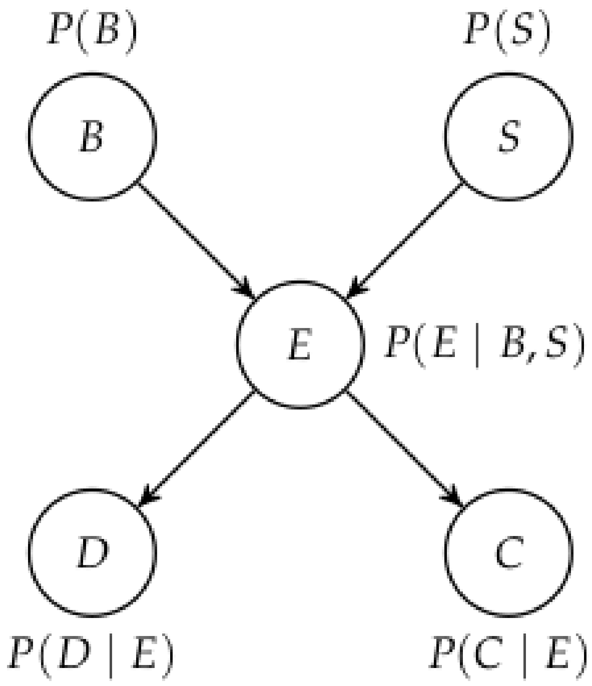
Exact Inference



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Marginalize

X	Y	Z	$\phi(X, Y, Z)$
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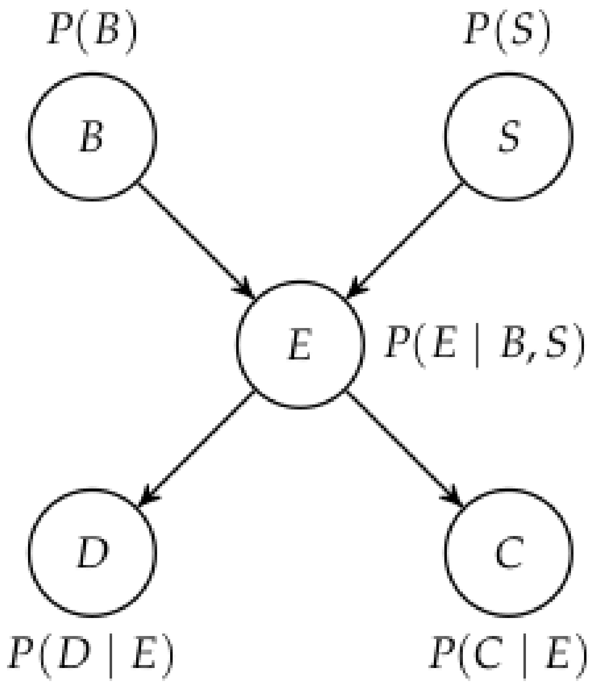
X	Z	$\phi(X, Z)$
0	0	0.17
0	1	0.68
1	0	0.03
1	1	0.12

Exact Inference



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```
struct ExactInference end

function infer(M::ExactInference, bn, query, evidence)
    ϕ = prod(bn.factors)
    ϕ = condition(ϕ, evidence)
    for name in setdiff(variablenames(ϕ), query)
        ϕ = marginalize(ϕ, name)
    end
    return normalize!(ϕ)
end
```

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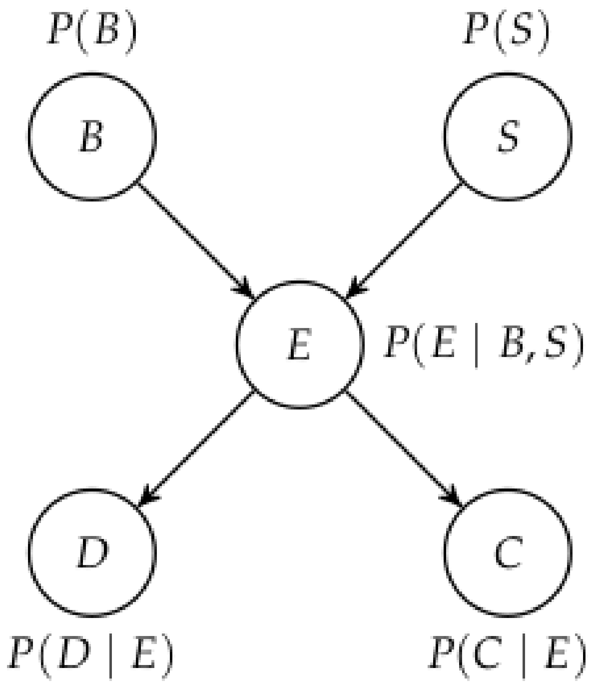

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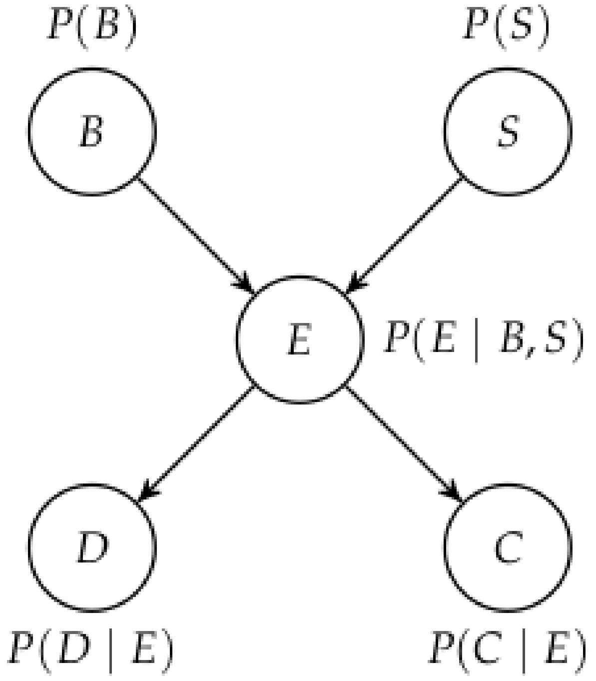
Exact Inference: Variable Elimination



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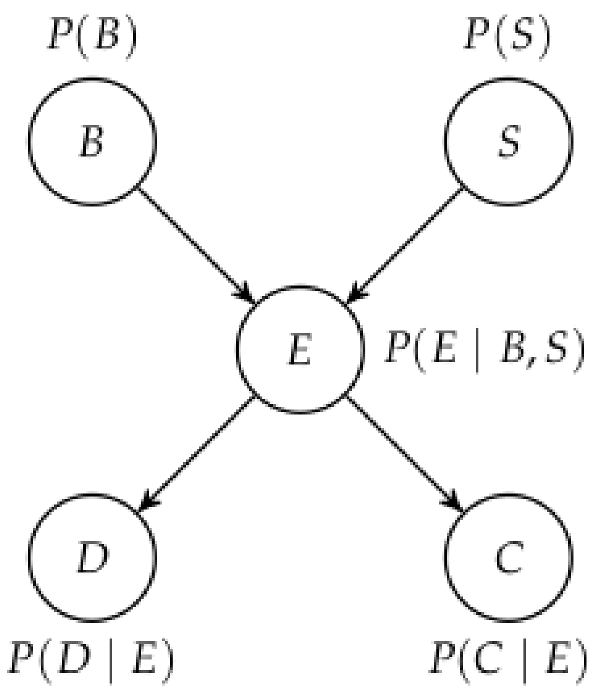

$P(B \mid d^1, c^1)$

D=1 C=1

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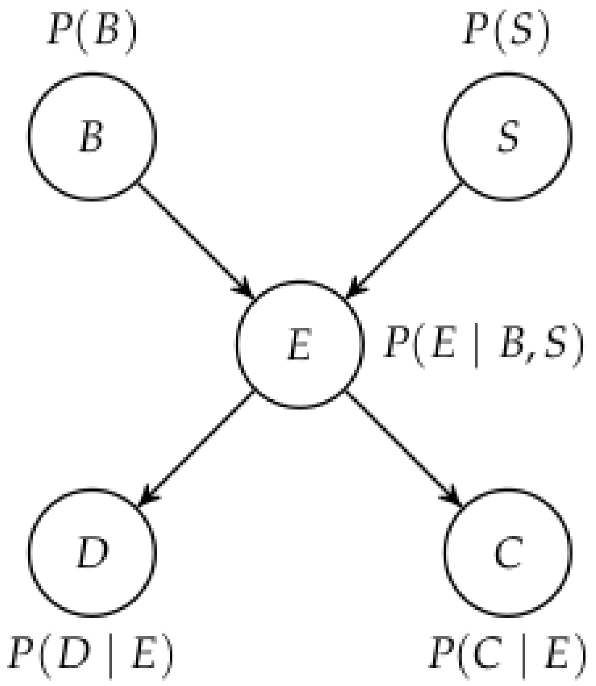
Exact Inference: Variable Elimination



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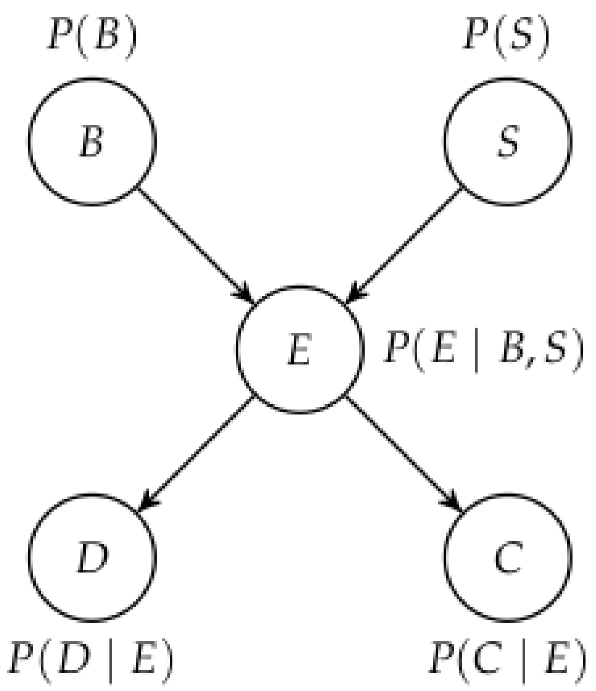
$P(B \mid d^1, c^1)$

Start with

$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Exact Inference: Variable Elimination



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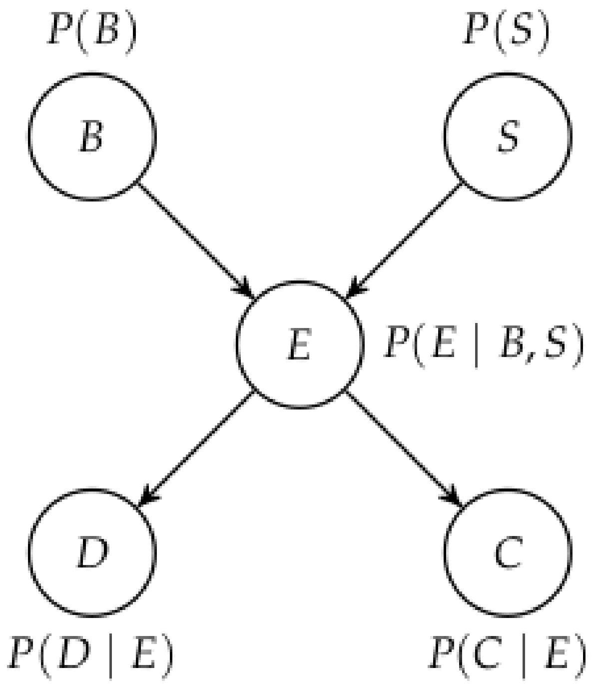
$P(B \mid d^1, c^1)$

Start with

$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

Exact Inference: Variable Elimination



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$P(B \mid d^1, c^1)$

Start with

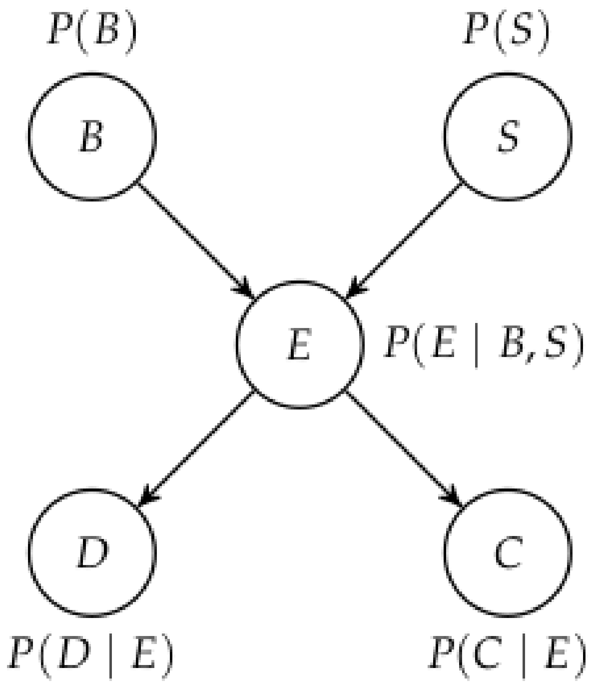
$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

Eliminate E

$$\phi_8(B, S) = \sum_e \phi_3(e, B, S)\phi_6(e)\phi_7(e)$$

Exact Inference: Variable Elimination



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$P(B \mid d^1, c^1)$

Start with

$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

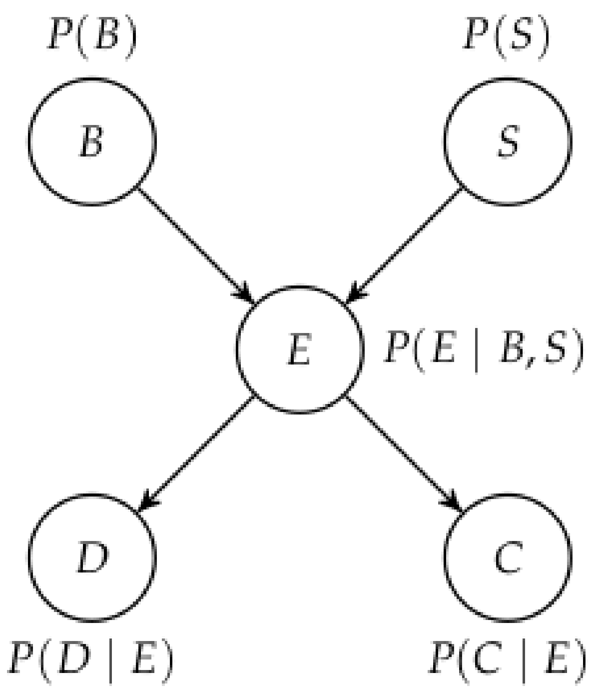
Eliminate E

$$\phi_8(B, S) = \sum_e \phi_3(e, B, S)\phi_6(e)\phi_7(e)$$

Eliminate S

$$\phi_9(B) = \sum_s \phi_2(s)\phi_8(B, s)$$

Exact Inference: Variable Elimination



B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

$P(B \mid d^1, c^1)$

Start with

$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

Eliminate E

$$\phi_8(B, S) = \sum_e \phi_3(e, B, S)\phi_6(e)\phi_7(e)$$

Eliminate S

$$\phi_9(B) = \sum_s \phi_2(s)\phi_8(B, s)$$

Variable Elimination

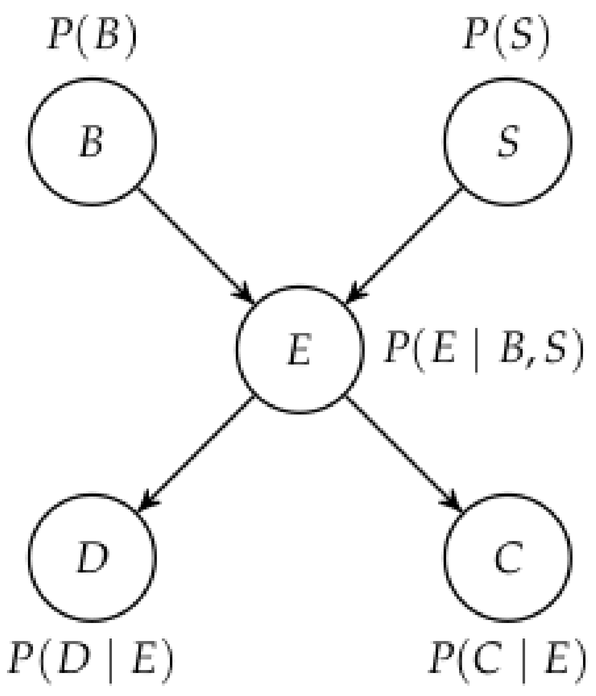
$$P(B \mid d^1, c^1) \propto \phi_1(B) \sum_s \left(\phi_2(s) \sum_e \left(\phi_3(e \mid B, s)\phi_4(d^1 \mid e)\phi_5(c^1 \mid e)\right)\right)$$

vs

Naive

$$P(B \mid d^1, c^1) \propto \sum_s \sum_e \phi_1(B)\phi_2(s)\phi_3(e \mid B, s)\phi_4(d^1 \mid e)\phi_5(c^1 \mid e)$$

Exact Inference: Variable Elimination



B battery failure
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$$P(B \mid d^1, c^1)$$

Start with

$$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$$

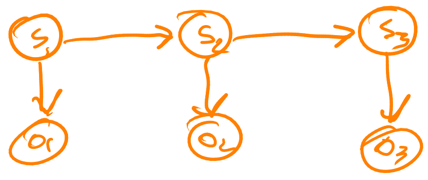
Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

Eliminate E

$$\phi_8(B, S) = \sum_e \phi_3(e, B, S)\phi_6(e)\phi_7(e)$$

Eliminate S

$$\phi_9(B) = \sum_s \phi_2(s)\phi_8(B, s)$$



$$P(B \mid d^1, c^1) \propto \phi_1(B) \sum_s \left(\phi_2(s) \sum_e \left(\phi_3(e \mid B, s)\phi_4(d^1 \mid e)\phi_5(c^1 \mid e) \right) \right)$$

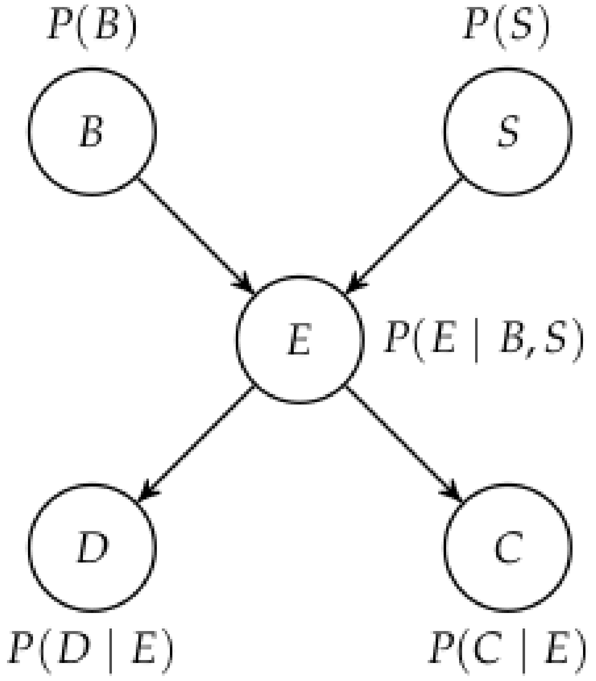
vs

$$P(B \mid d^1, c^1) \propto \sum_s \sum_e \phi_1(B)\phi_2(s)\phi_3(e \mid B, s)\phi_4(d^1 \mid e)\phi_5(c^1 \mid e)$$

Choosing optimal order is NP-hard

Approximate Inference

Approximate Inference: Direct Sampling



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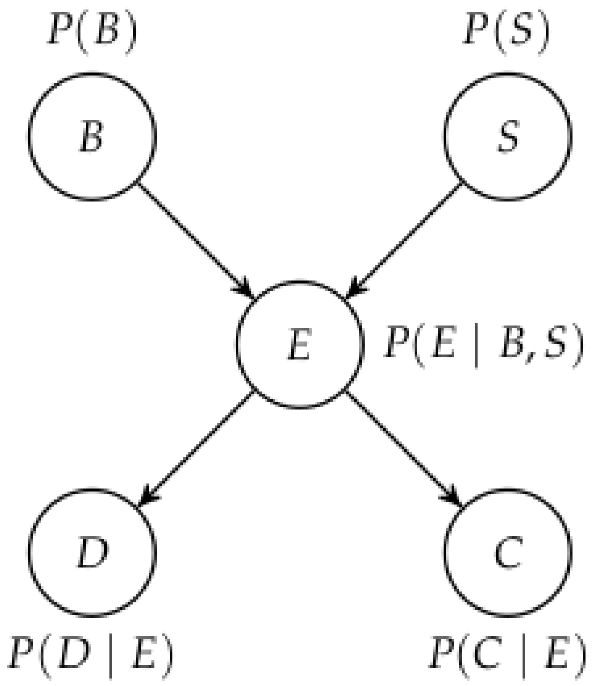
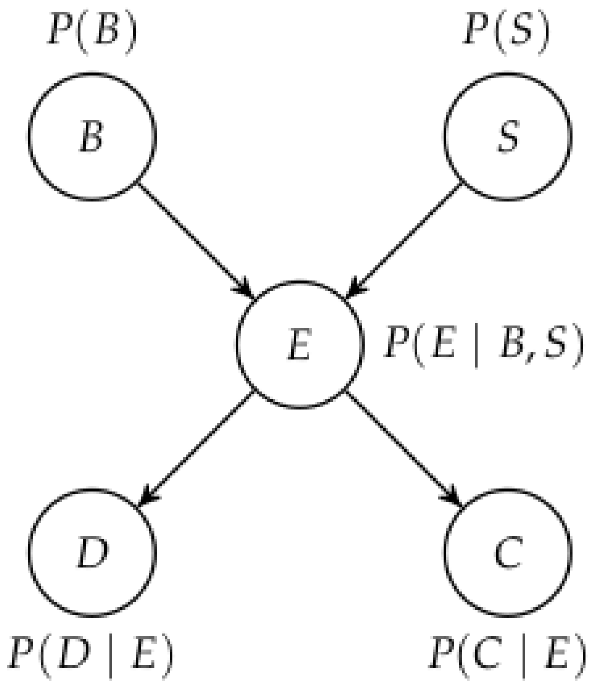
Approximate Inference: Direct Sampling



B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i (b^{(i)} = 1 \land d^{(i)} = 1 \land c^{(i)} = 1)}{\sum_i (\underline{d^{(i)} = 1} \land \underline{c^{(i)} = 1})}$$

Approximate Inference: Direct Sampling



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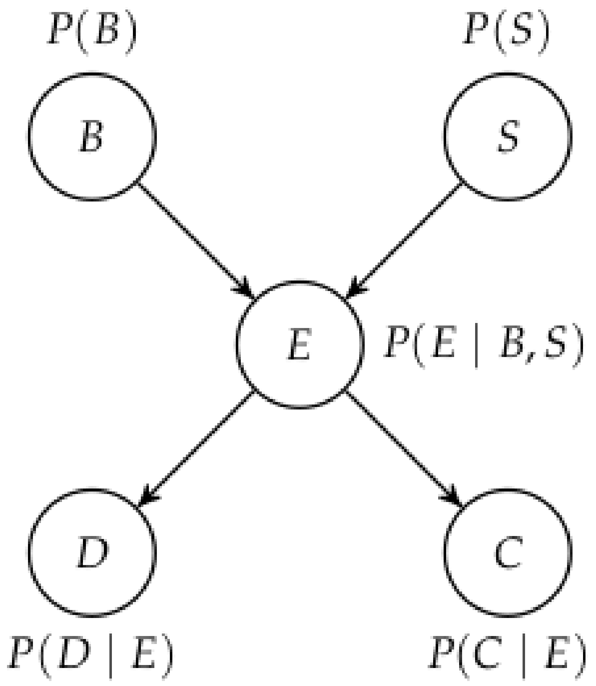
$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i (b^{(i)} = 1 \land d^{(i)} = 1 \land c^{(i)} = 1)}{\sum_i (d^{(i)} = 1 \land c^{(i)} = 1)}$$

$\frac{1}{2}$

Sample 1

B	S	E	D	C
0	0	1	1	0
0	0	0	0	0
1	0	1	0	0
1	0	1	1	1
0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	1	1	1	1
0	0	0	0	0
0	0	0	1	0

Approximate Inference: Direct Sampling



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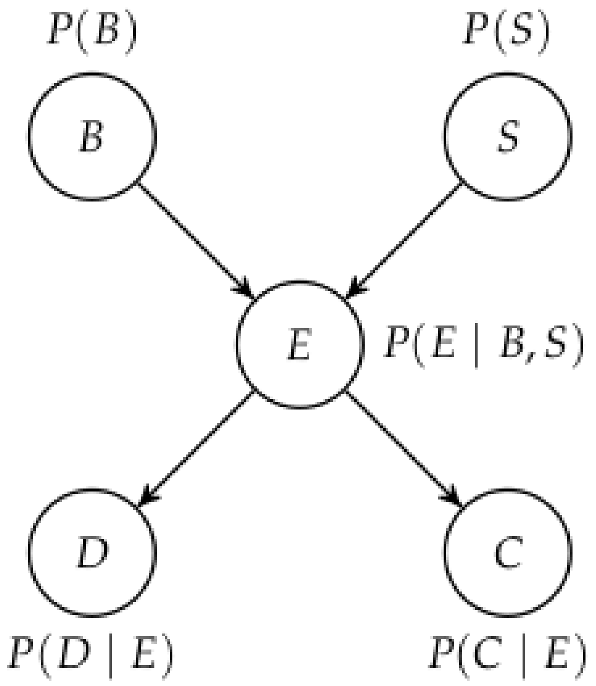
$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i (b^{(i)} = 1 \wedge d^{(i)} = 1 \wedge c^{(i)} = 1)}{\sum_i (d^{(i)} = 1 \wedge c^{(i)} = 1)}$$

B	S	E	D	C
0	0	1	1	0
0	0	0	0	0
1	0	1	0	0
1	0	1	1	1
0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	1	1	1	1
0	0	0	0	0
0	0	0	1	0

Analogous to

Approximate Inference: Direct Sampling



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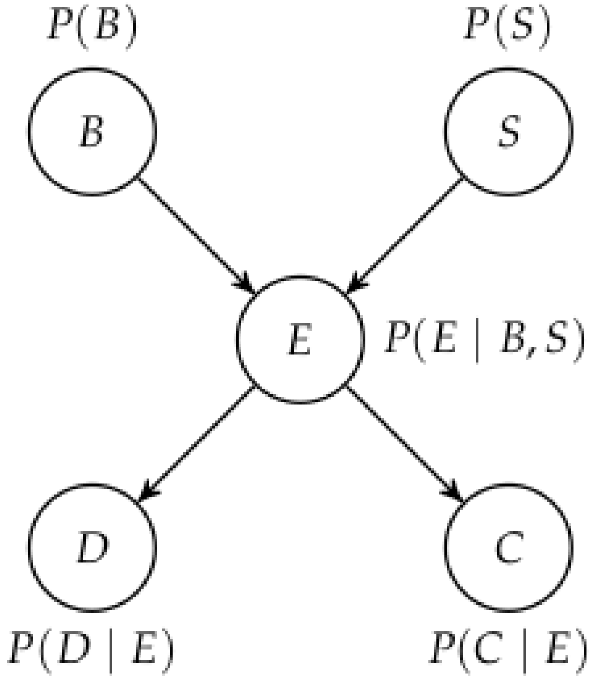
B	S	E	D	C
0	0	1	1	0
0	0	0	0	0
1	0	1	0	0
1	0	1	1	1
0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	1	1	1	1
0	0	0	0	0
0	0	0	1	0

Analogous to **unweighted particle filtering**

Approximate Inference: Weighted Sampling



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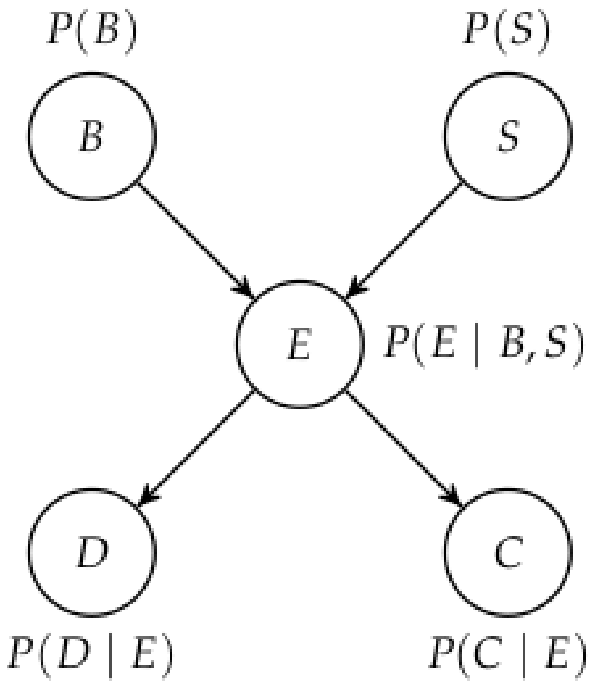
Approximate Inference: Weighted Sampling



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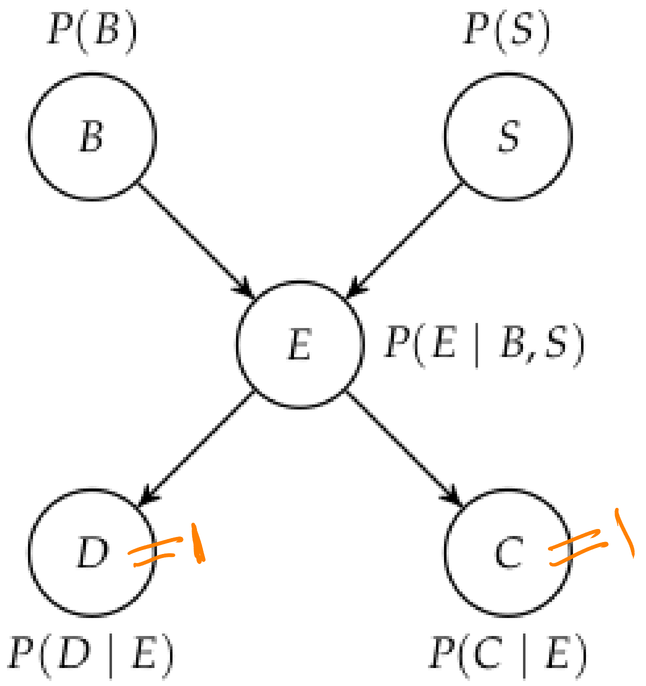
$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i w_i(b^{(i)} = 1 \wedge d^{(i)} = 1 \wedge c^{(i)} = 1)}{\sum_i w_i(d^{(i)} = 1 \wedge c^{(i)} = 1)}$$
$$= \frac{\sum_i w_i(b^{(i)} = 1)}{\sum_i w_i}$$

Approximate Inference: Weighted Sampling



B battery failure
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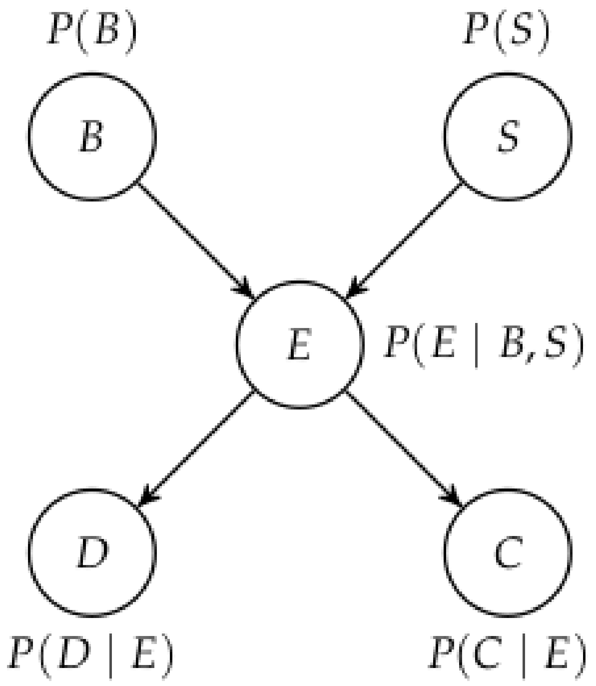
$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i w_i(b^{(i)} = 1 \wedge d^{(i)} = 1 \wedge c^{(i)} = 1)}{\sum_i w_i(d^{(i)} = 1 \wedge c^{(i)} = 1)}$$
$$= \frac{\sum_i w_i(b^{(i)} = 1)}{\sum_i w_i}$$

B	S	E	D	C	Weight
1	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	1	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$

Approximate Inference: Weighted Sampling



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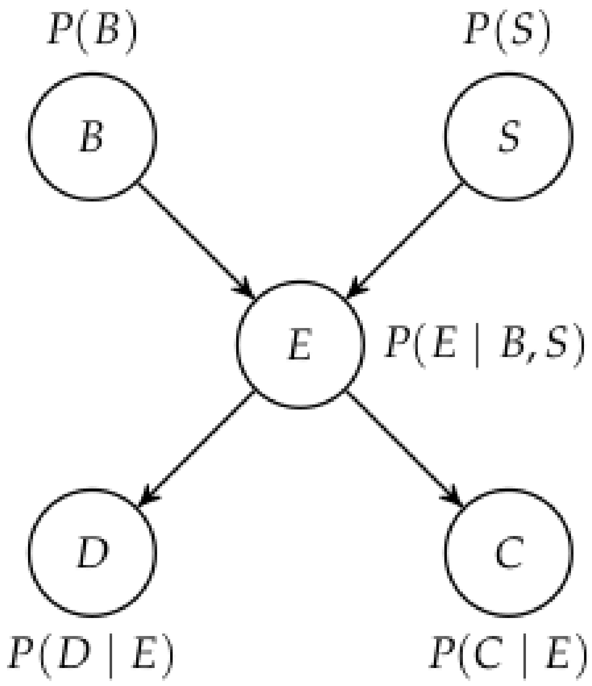
$$= \frac{\sum_i w_i(b^{(i)} = 1)}{\sum_i w_i}$$

B	S	E	D	C	Weight
1	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	1	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$

Analogous to

Approximate Inference: Weighted Sampling



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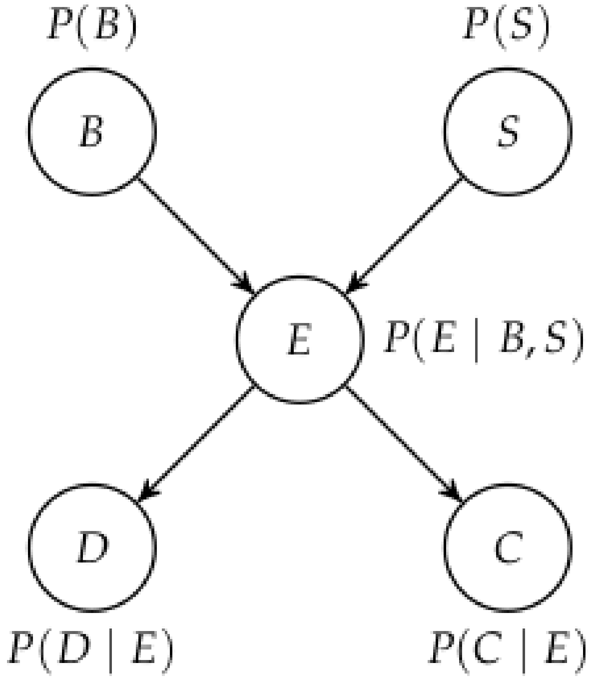
$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i w_i(b^{(i)} = 1 \wedge d^{(i)} = 1 \wedge c^{(i)} = 1)}{\sum_i w_i(d^{(i)} = 1 \wedge c^{(i)} = 1)} = \frac{\sum_i w_i(b^{(i)} = 1)}{\sum_i w_i}$$

B	S	E	D	C	Weight
1	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	1	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$

Analogous to **weighted particle filtering**

Approximate Inference: Gibbs Sampling



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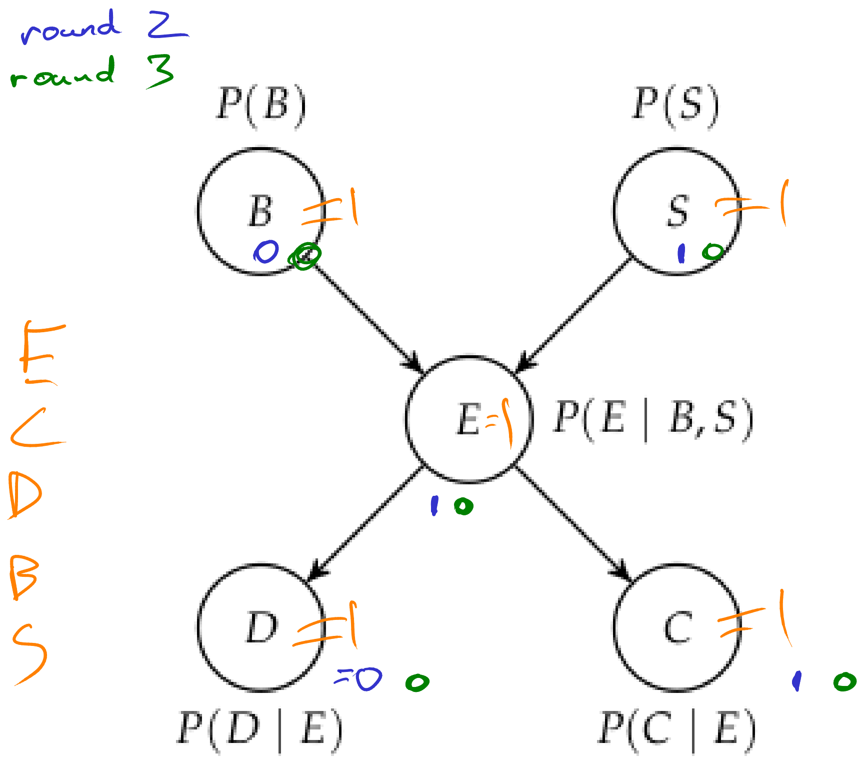
Approximate Inference: Gibbs Sampling



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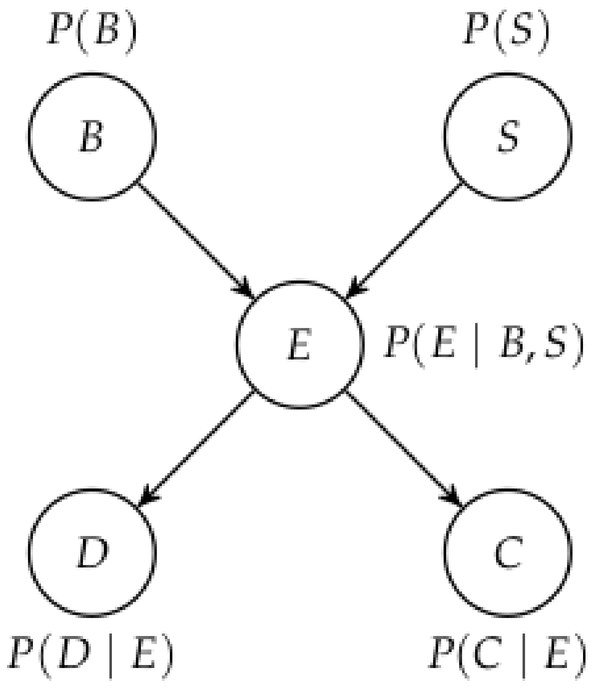
Markov Chain Monte Carlo (MCMC)

Approximate Inference: Gibbs Sampling



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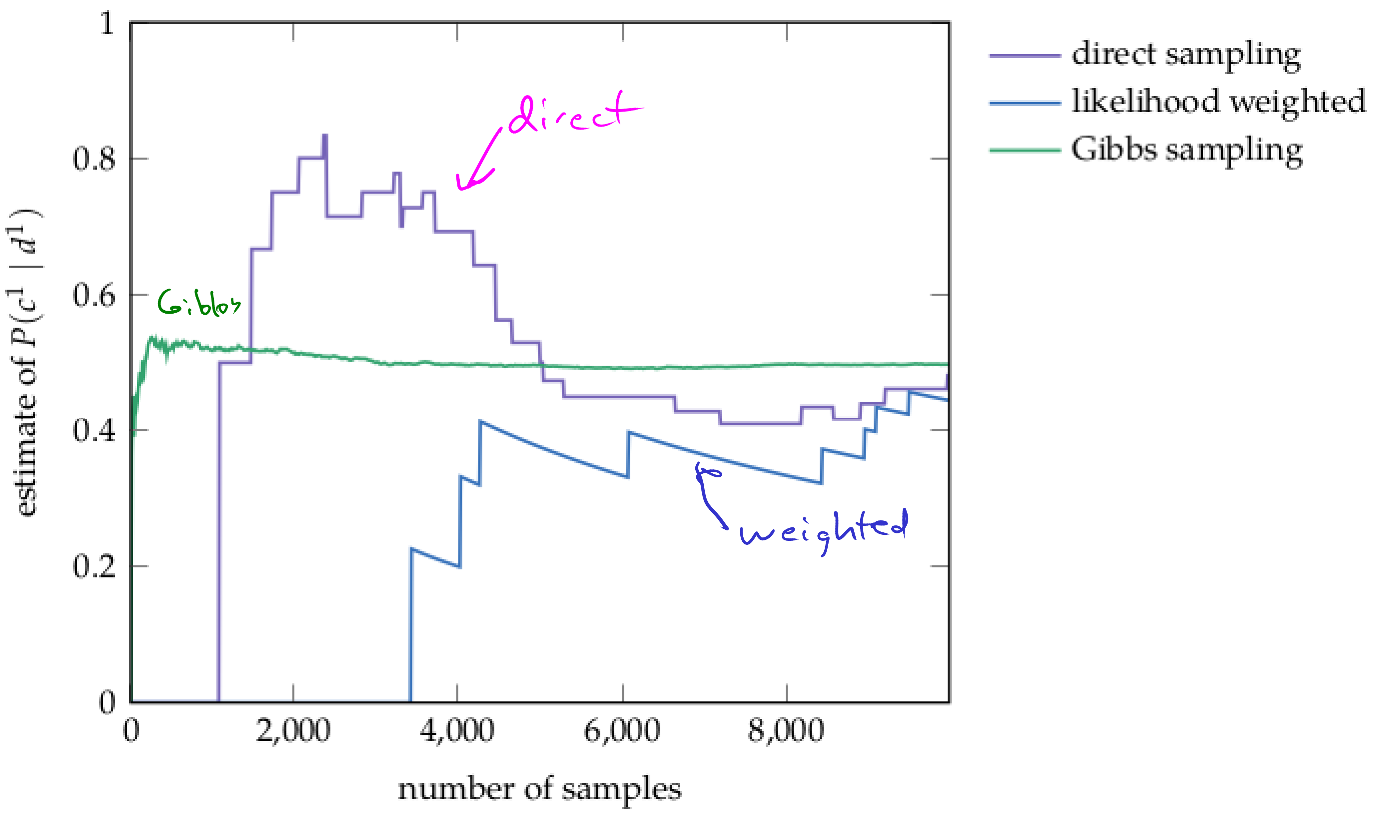
Markov Chain Monte Carlo (MCMC)



Learning

Bayesian Network Learning

Inputs

Outputs

Bayesian Network Learning

Inputs

- Data, D

Outputs

Bayesian Network Learning

Inputs

- Data, D
- Priors (?)

Outputs

Bayesian Network Learning

Inputs

- Data, D
- Priors (?)

Outputs

- Bayesian network structure, G

Bayesian Network Learning

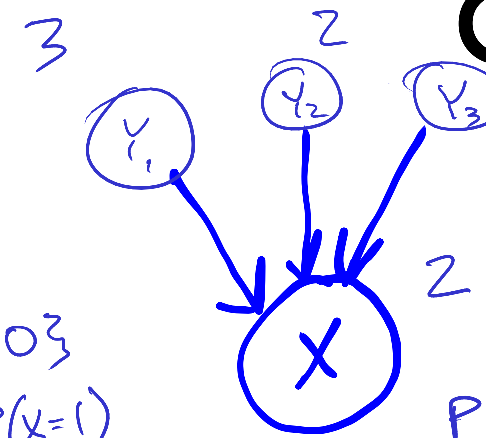
Inputs

- Data, D
- Priors (?)

Outputs

- Bayesian network structure, G
- Bayesian network parameters, θ

Counting Parameters



$X \in \{1, 0\}$

$\theta_1 = P(X=1)$

$P(X=0) = 1 - \theta_1$

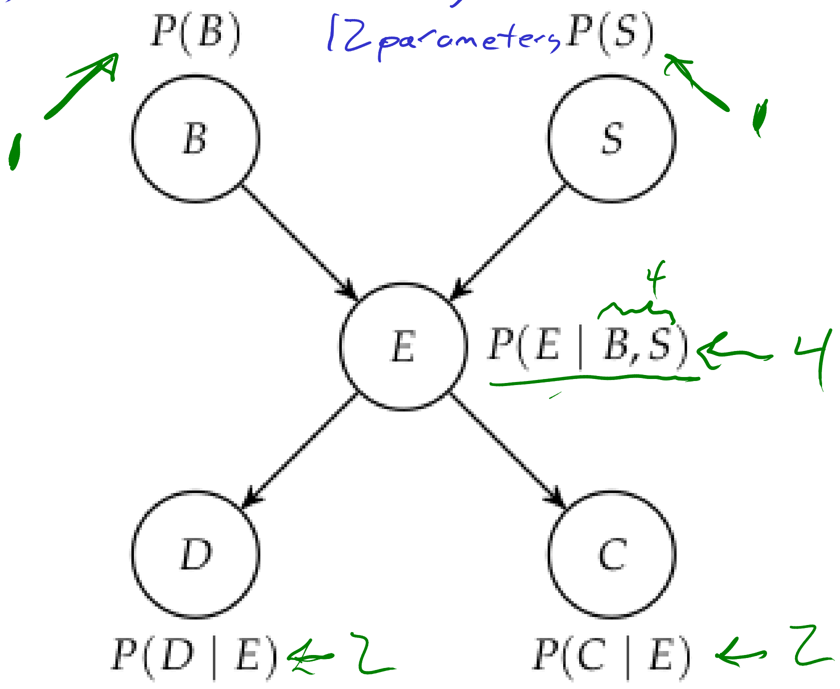
For discrete R.V.s:

$$\dim(\theta_X) = (|\text{support}(X)| - 1) \prod_{Y \in Pa(X)} |\text{support}(Y)|$$

$P(X \mid Pa(X)) = P(X \mid Y_1, Y_2, Y_3)$

$P(X \mid Y_1 = 1, Y_2 = 1, Y_3 = 1)$: 1 parameter

12 parameters

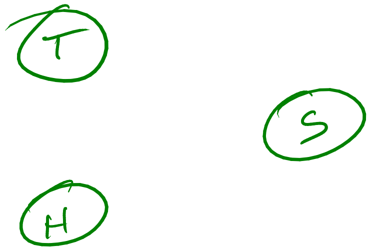


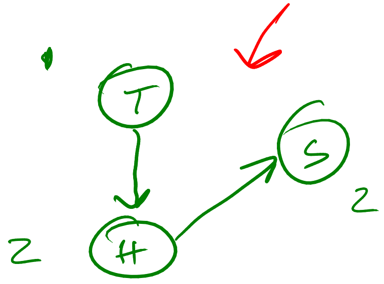
$1 + 1 + 4 + 2 + 2 = 10$ parameters

$C \sim N(\mu_C, \sigma_C)$

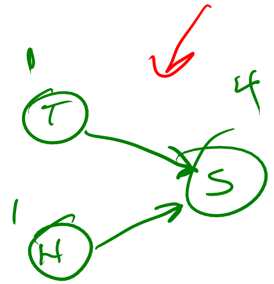
$C \sim N(aE + b, \sigma_C)$

Structure Learning Example





5 parameters



6 parameters

Parameter Learning

Parameter Learning

Maximum Likelihood

Parameter Learning

Maximum Likelihood

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_{i} P(o_i \mid \theta)$$

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_i P(o_i \mid \theta)$$

$$\hat{\theta} = \arg\max_{\theta} \sum_i \log P(o_i \mid \theta)$$

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_i P(o_i \mid \theta)$$

$$\hat{\theta} = \arg\max_{\theta} \sum_i \log P(o_i \mid \theta)$$

Multinomial:

$$P(B=1) = \frac{N\langle B=1\rangle}{N}$$

$$\hat{\theta}_i = \frac{n_i}{\sum_{j=1}^{k} n_j}$$

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_i P(o_i \mid \theta)$$

$$\hat{\theta} = \arg\max_{\theta} \sum_i \log P(o_i \mid \theta)$$

Multinomial:

$$\hat{\theta}_i = \frac{n_i}{\sum_{j=1}^{k} n_j}$$

Bayesian

$$\hat{\theta} = \mathbb{E}_{\theta \sim p(\cdot \mid D)}[\theta] = \int \theta p(\theta \mid D)\, \mathrm{d}\theta$$

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_i P(o_i \mid \theta)$$

$$\hat{\theta} = \arg\max_{\theta} \sum_i \log P(o_i \mid \theta)$$

Multinomial:

$$\hat{\theta}_i = \frac{n_i}{\sum_{j=1}^{k} n_j}$$

Bayesian

$$\hat{\theta} = \mathbb{E}_{\theta \sim p(\cdot \mid D)}[\theta] = \int \theta p(\theta \mid D) \, \mathrm{d}\theta$$

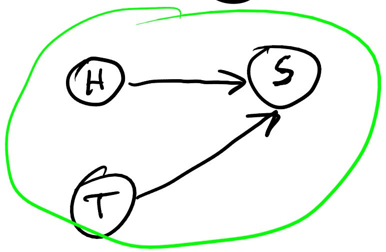
$p(\theta \mid D) \propto p(D \mid \theta) p(\theta)$

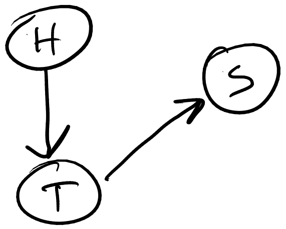
Multinomial:

$$p(\theta_{1:n} \mid \alpha_{1:n}, m_{1:n}) = \underline{\mathrm{Dir}(\theta_{1:n} \mid \alpha_1 + m_1, \ldots, \alpha_n + m_n)}$$

$$\frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j}$$

Structure Learning





Structure Learning

$P(G \mid D)$

Structure Learning

$$
\begin{aligned}
P(G \mid D) &\propto P(G)P(D \mid G) \\
&= P(G) \int P(D \mid \boldsymbol{\theta}, G)\, p(\boldsymbol{\theta} \mid G)\, \mathrm{d}\boldsymbol{\theta}
\end{aligned}
$$

Structure Learning

$$P(G \mid D) \propto P(G)P(D \mid G)$$
$$= P(G) \int P(D \mid \theta, G) p(\theta \mid G)\, \mathrm{d}\theta$$

Dirichlets

$$P(G \mid D) = P(G) \prod_{i=1}^{n} \prod_{j=1}^{q_i} \frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \prod_{k=1}^{r_i} \frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})}$$

Structure Learning

$$P(G \mid D) \propto P(G)P(D \mid G)$$
$$= P(G) \int P(D \mid \theta, G)p(\theta \mid G)\,\mathrm{d}\theta$$

$$P(G \mid D) = P(G) \prod_{i=1}^{n} \prod_{j=1}^{q_i} \frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \prod_{k=1}^{r_i} \frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})}$$

$$\log P(G \mid D)$$
$$= \log P(G) + \sum_{i=1}^{n} \sum_{j=1}^{q_i} \left(\log\left(\frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \right) + \sum_{k=1}^{r_i} \log\left(\frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})} \right) \right)$$

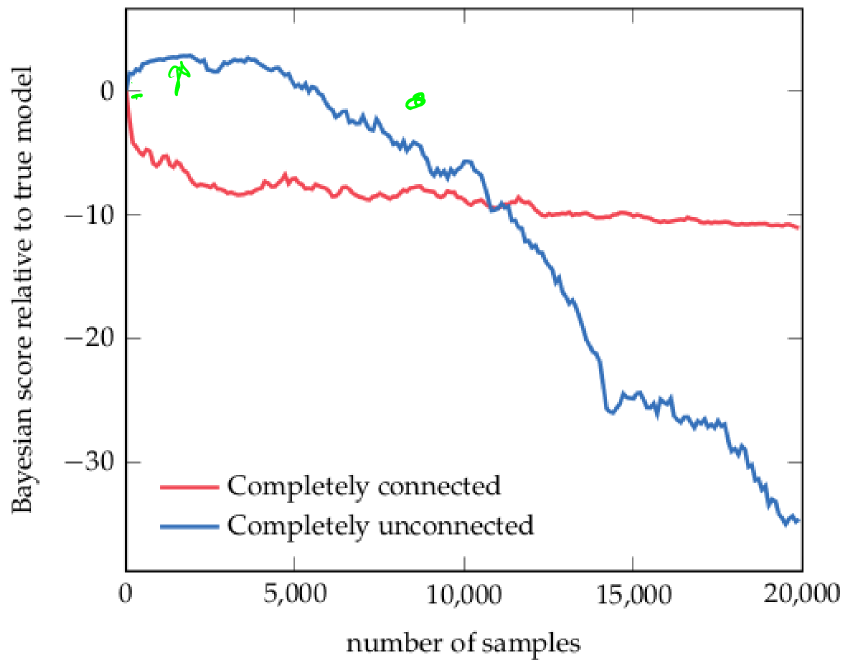
Structure Learning

$$P(G \mid D) \propto P(G)P(D \mid G)$$
$$= P(G)\int P(D \mid \theta, G)p(\theta \mid G)\,\mathrm{d}\theta$$

$$P(G \mid D) = P(G)\prod_{i=1}^{n}\prod_{j=1}^{q_i}\frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0}+m_{ij0})}\prod_{k=1}^{r_i}\frac{\Gamma(\alpha_{ijk}+m_{ijk})}{\Gamma(\alpha_{ijk})}$$

$$\log P(G \mid D)$$

$$= \log P(G) + \sum_{i=1}^{n}\sum_{j=1}^{q_i}\left(\log\left(\frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0}+m_{ij0})}\right) + \sum_{k=1}^{r_i}\log\left(\frac{\Gamma(\alpha_{ijk}+m_{ijk})}{\Gamma(\alpha_{ijk})}\right)\right)$$



Structure Learning

$$P(G \mid D) \propto P(G)P(D \mid G)$$
$$= P(G) \int P(D \mid \theta, G) p(\theta \mid G) \, d\theta$$

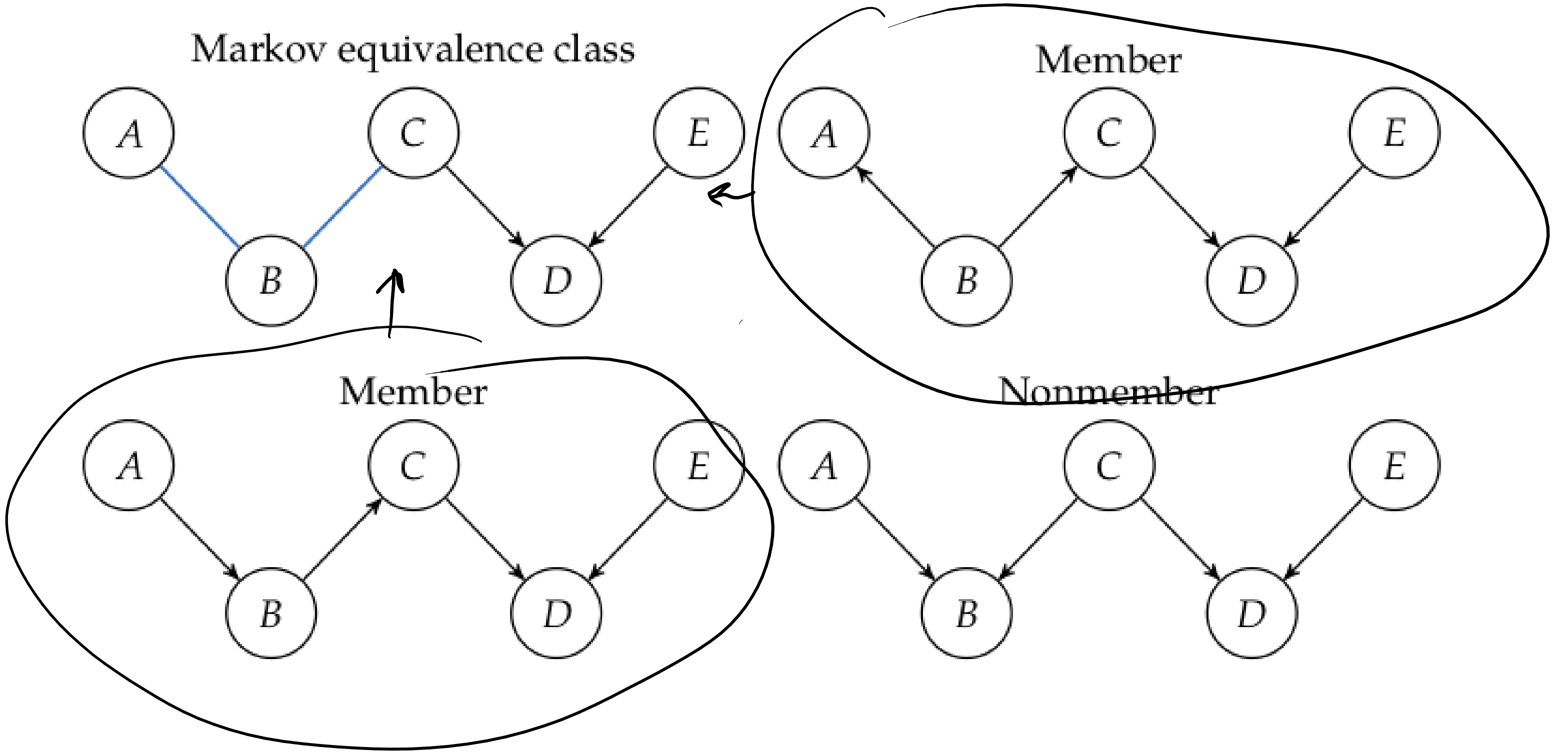
$$P(G \mid D) = P(G) \prod_{i=1}^{n} \prod_{j=1}^{q_i} \frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \prod_{k=1}^{r_i} \frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})}$$

$$\log P(G \mid D)$$
$$= \log P(G) + \sum_{i=1}^{n} \sum_{j=1}^{q_i} \left(\log \left(\frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \right) + \sum_{k=1}^{r_i} \log \left(\frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})} \right) \right)$$

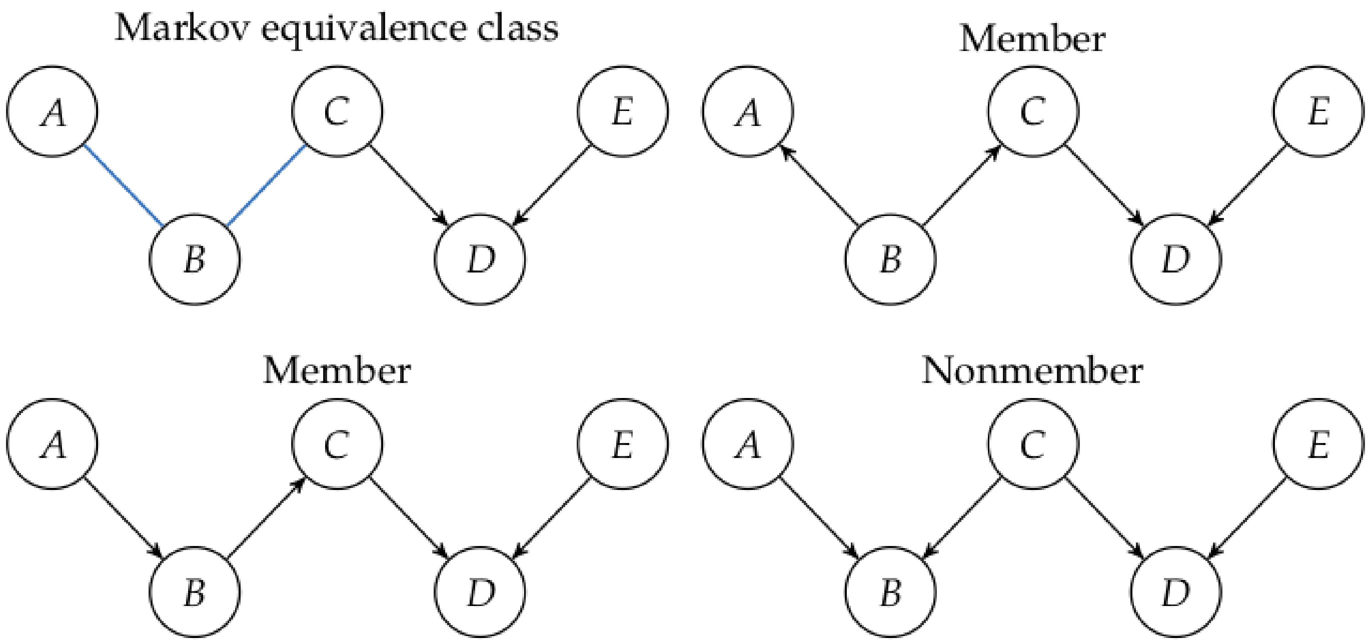
NP-Hard

Markov Equivalence Class

Markov Equivalence Class

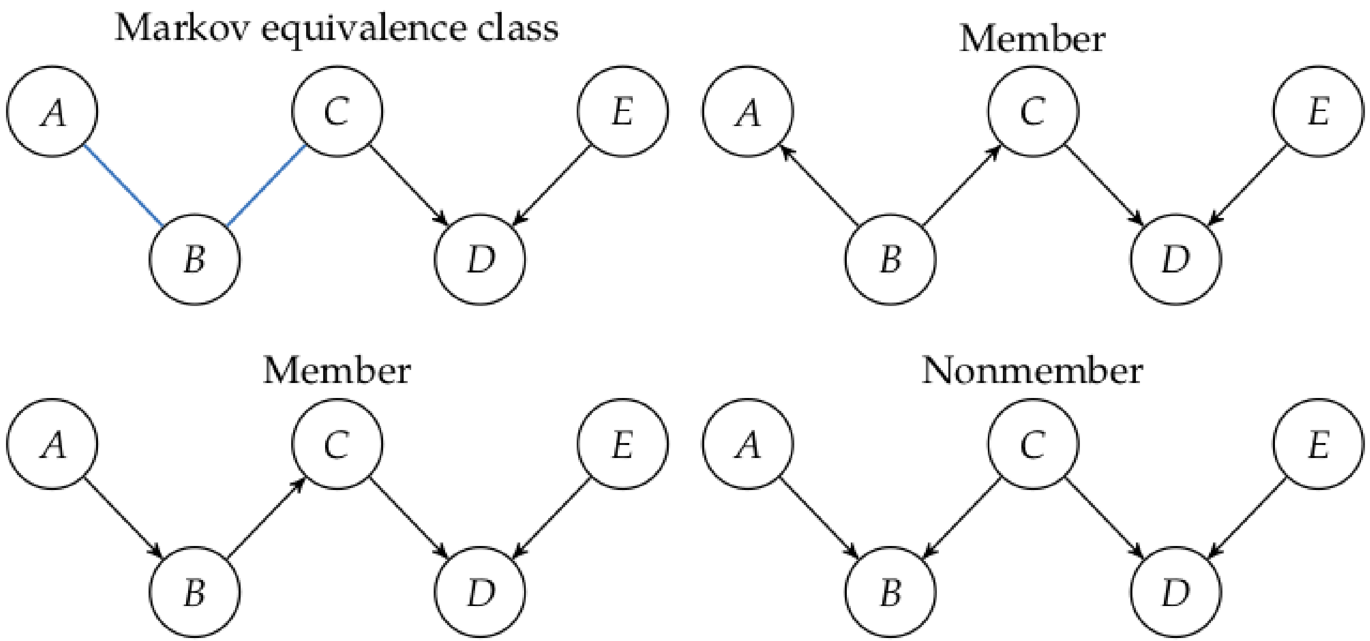


Markov Equivalence Class



Markov Equivalent iff

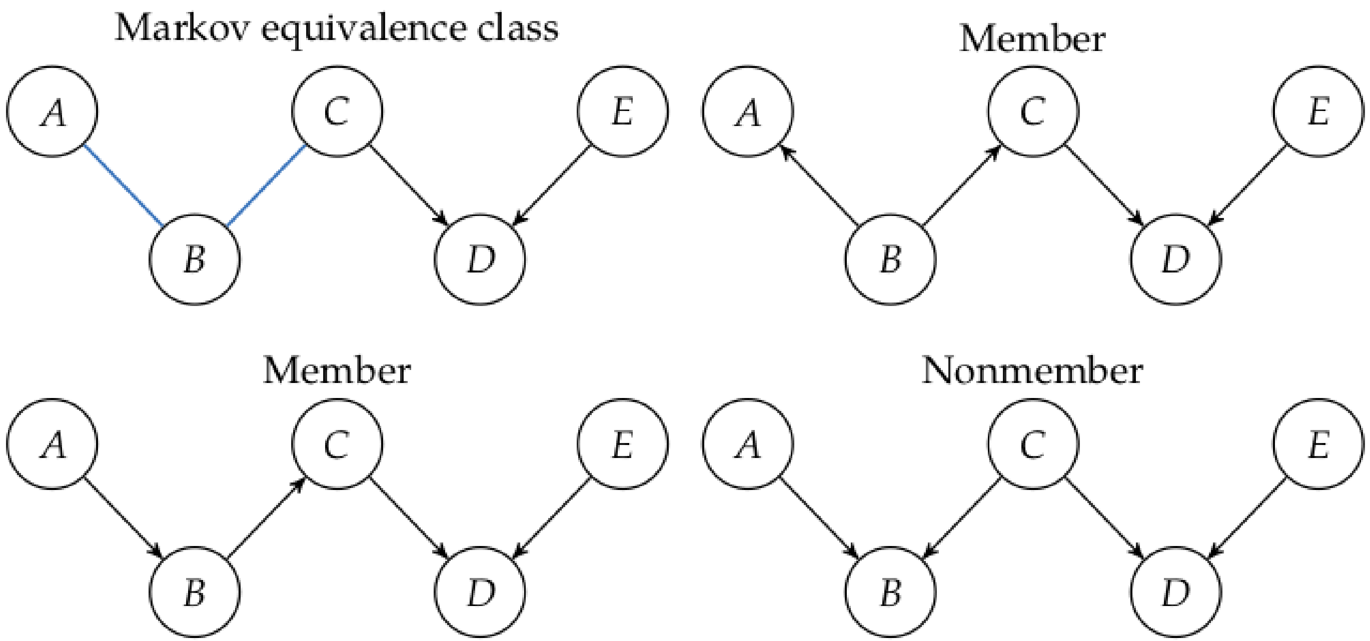
Markov Equivalence Class



Markov Equivalent iff

1. Same undirected edges

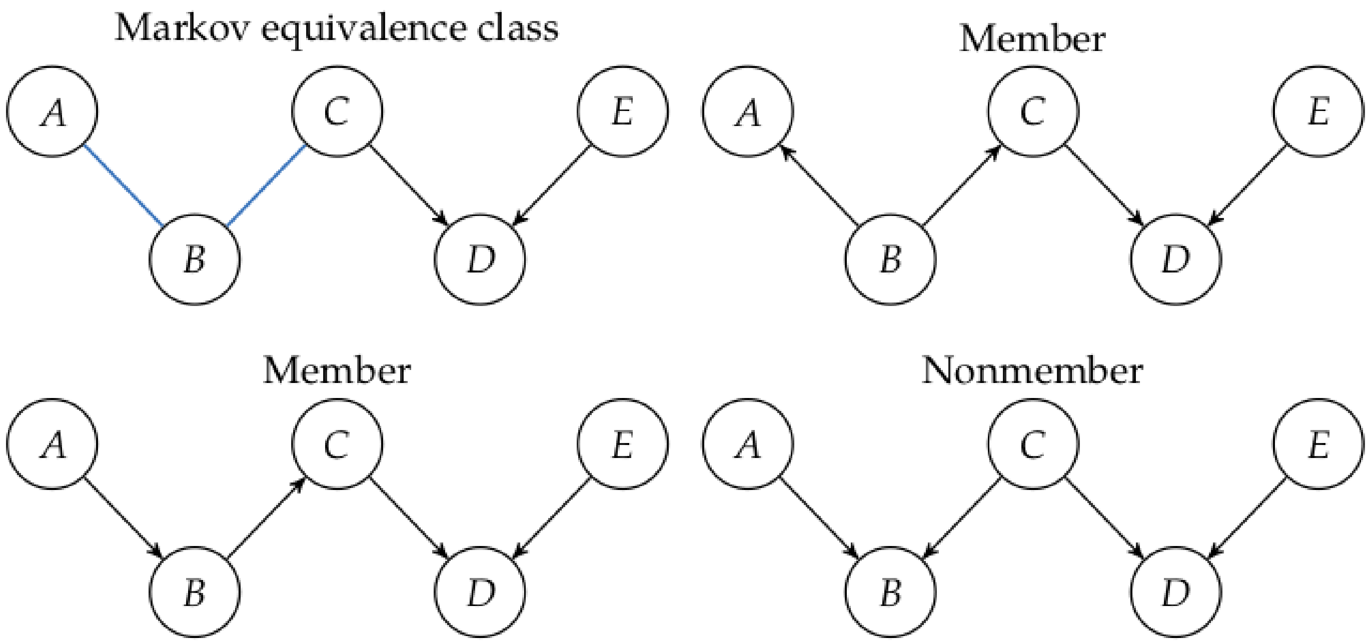
Markov Equivalence Class



Markov Equivalent iff

1. Same undirected edges
2. Same set of immoral v-structures

Markov Equivalence Class



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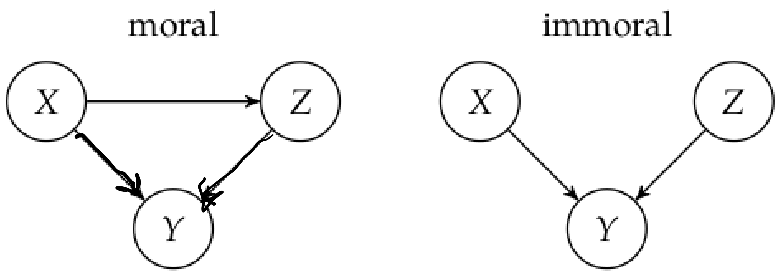

Recap

Inference

Inputs

$G, \Theta,$ evidence

Output

Probability of query variables given evidence

Learning

Inputs

Data, Prior

Output

G, Θ