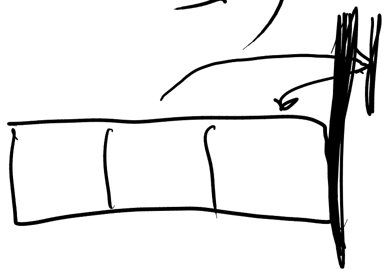
$$\pi^*(s) = \operatorname{argmax}\left(R(s,a) + \gamma E[V^*(s')]\right)$$



Last Time

a^1

$$R(s,a^1) + \gamma E[V^\pi(s') \mid a^1] = R(s,a^2) + \gamma E[V^\pi(s') \mid a^2]$$

1) Math

< Breakout

2) Julia

Last Time

$V(s)$

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- How do we reason about the **future consequences** of actions in an MDP?

Last Time

- How do we reason about the **future consequences** of actions in an MDP?
- What are the basic **algorithms for solving MDPs**?

Policy Iteration
evaluate
update
Value Iteration
bellman's

Guiding Questions

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- Does value iteration always converge?
- Is the value function unique?

First, Does Policy Iteration Converge?

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Theorem: Policy iteration converges to an optimal policy for a finite MDP in finite time.

π^1, π^2, π^3

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4. By (1), (2), and (3), the policy will improve until it finds the optimal policy, and it will always find the optimal policy.

Value Iteration: The Bellman Operator

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Algorithm: Value Iteration

while $\|V - V'\|_\infty < \epsilon$

$V \leftarrow V'$

$V' \leftarrow B[V]$

return V'

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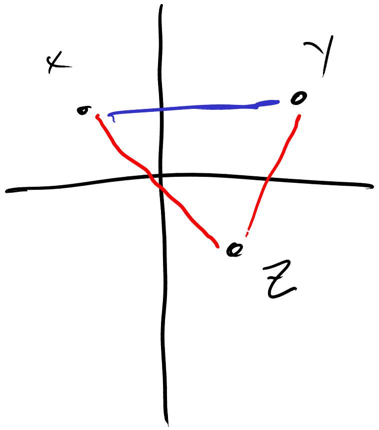
triangle inequality

$$d(x,y) = \sqrt{\sum_i (x_i - y_i)^2}$$



Contraction Mappings

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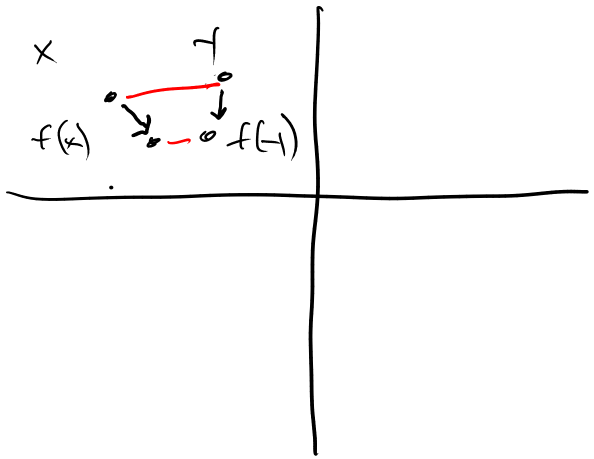
Definition: A *contraction mapping* on metric space (M, d) is a function $f : M \to M$ satisfying

$$d(f(x), f(y)) \leq \alpha\, d(x, y)$$

for some α, $0 \leq \alpha \leq 1$ and all x and y in M.



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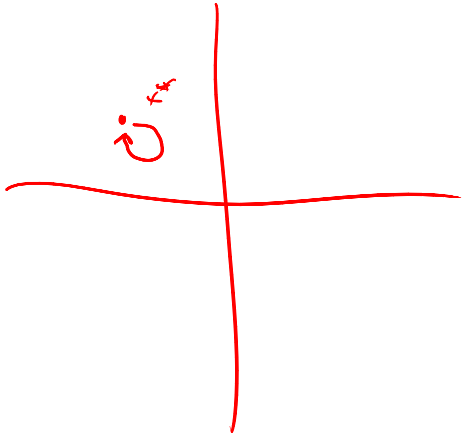

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Script: contraction_mapping.jl

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1. f has a single, unique fixed point x^*.
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Prove B is a contraction

first we need a metric

Max Norm

$V \in \mathbb{R}^{|S|}$

Max Norm

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2. If $\{x_k\}$ is a sequence defined by $x_{k+1} = f(x_k)$, then $\lim_{k\to\infty} x_k = x^*$.

Value Iteration Convergence

Theorem 1: Let $\{V_1, \ldots, V_\infty\}$ be a sequence of value functions for a discrete MDP generated by the recurrence $V_{k+1} = B[V_k]$. If $\gamma < 1$, then $\lim_{k\to\infty} V_k = V^*$.

Proof:

Lemma 2: B is a γ contraction mapping on $(\mathbb{R}^{|S|}, \|\cdot\|_\infty)$.

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Proof:

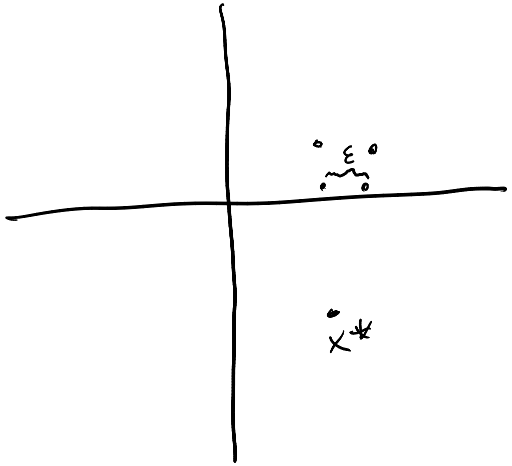
By Lemma 2 and Banach's theorem (part 2), repeated application of the Bellman operator always has a fixed point limit, $\hat{V}$.

By Banach's theorem (part 1), $\hat{V} = B[\hat{V}]$. Since $\hat{V}$ satisfies Bellman's equation, it is optimal and $\hat{V} = V^*$.

Guiding Questions

$\|V' - V\|_\infty < \varepsilon$

Guiding Questions



- Does value iteration always converge?
- Is the value function unique?

Yes

$S = \{0, 0.5, 1\}$

Breakout Rooms

Ecology MDP: Endangered Species

extinct

$S = \{0, 1\}$

$A = \{p, n\}$

$R(s,a) = R(s) + R(a)$

$R(1) = 1$
$R(0) = -1$
$R(p) = -1$
$R(n) = 1$

T

s, a, s'			$T(s'\|s,a)$
0	any	0	1.0
1	p	1	0.9
1	p	0	0.1
1	n	0	0.5
1	n	1	0.5

Find V^*
π^*

$V(s) = \max_a Q(s,a)$

$Q(s,a) = R(s,a) + \gamma E[V(s')]$

$Q^*(0,p) = R(0,p) + \gamma E[V^*(s')]$

$-2 + \gamma V^*(0)$

$Q^*(0,n) = 0 + \gamma V^*(0)$

$V^*(0) = 0 + \gamma V^*(0)$

$(1-\gamma) V^*(0) = 0$

$\boxed{V^*(0) = 0}$

$Q(1,p) = 0 + \gamma(0.9 Q(1,p) + \cancel{0.1 V^*(0)})$ $\to 0$

$Q(1,p) = 0.81 Q(1,p)$

$\boxed{Q(1,p) = 0}$

$Q(1,n) = 2 + \gamma(0.5 Q(1,n) + \cancel{0.5 V^*(0)})$ $\to 0$

$(1 - 0.45) Q(1,n) = 2$

$\boxed{Q(1,n) = 3.64}$

$V^*(1) = 3.64$