Recap

- Alpha Vectors
- Best solver for discrete POMDPs:

SARSOP

POMDP Computational Complexity

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Sad facts ● 

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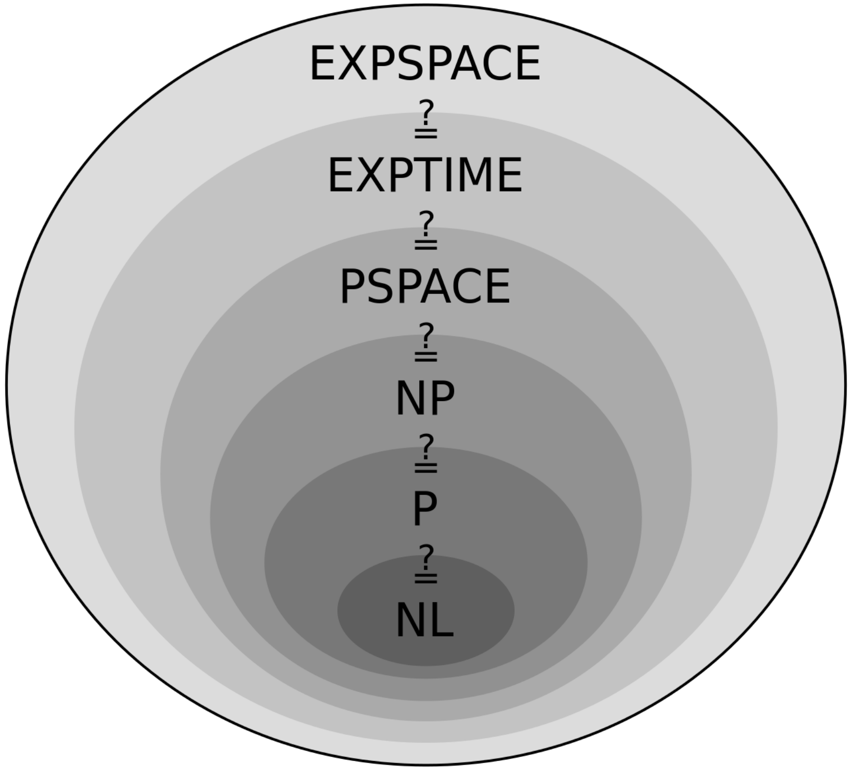

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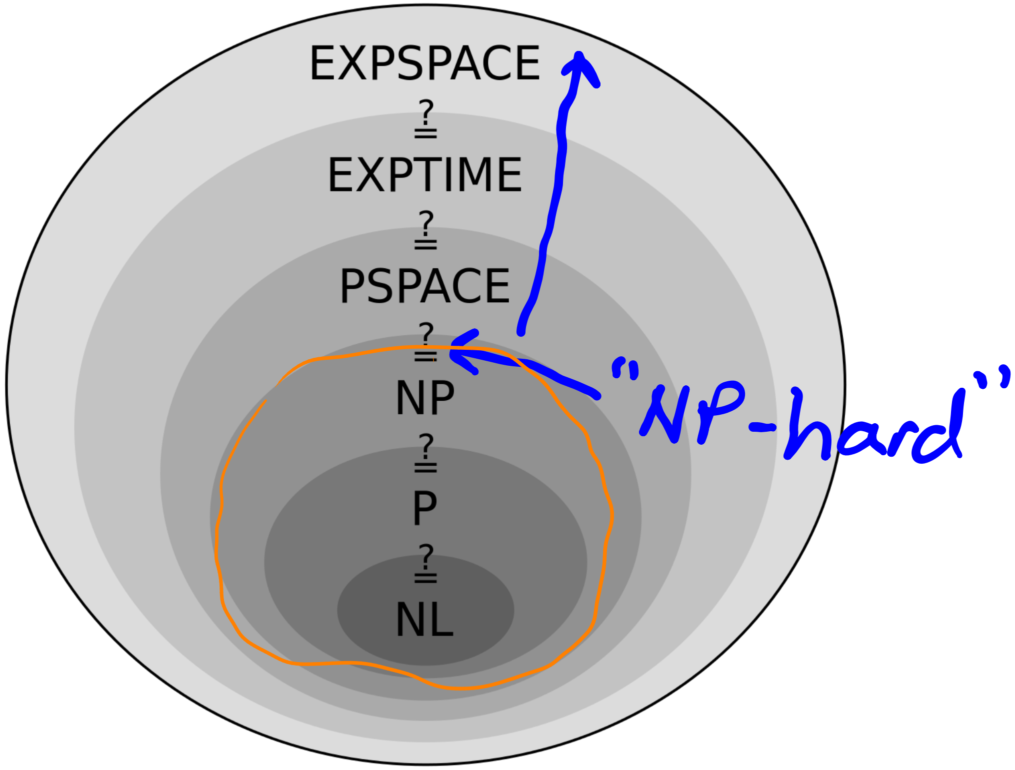

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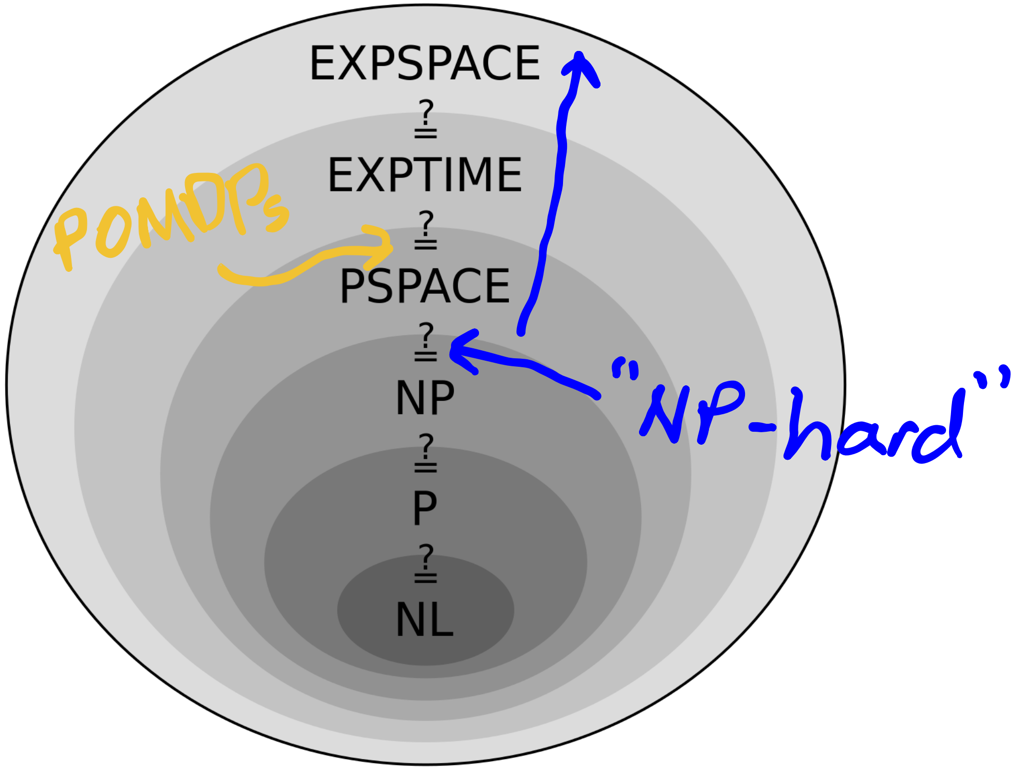

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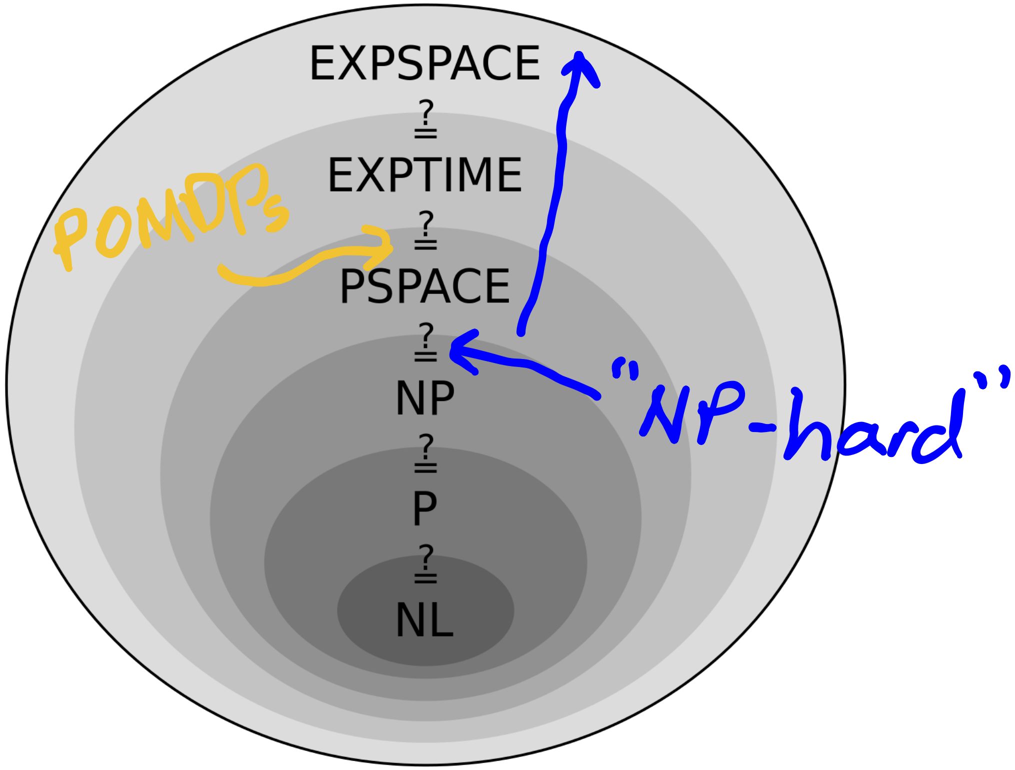

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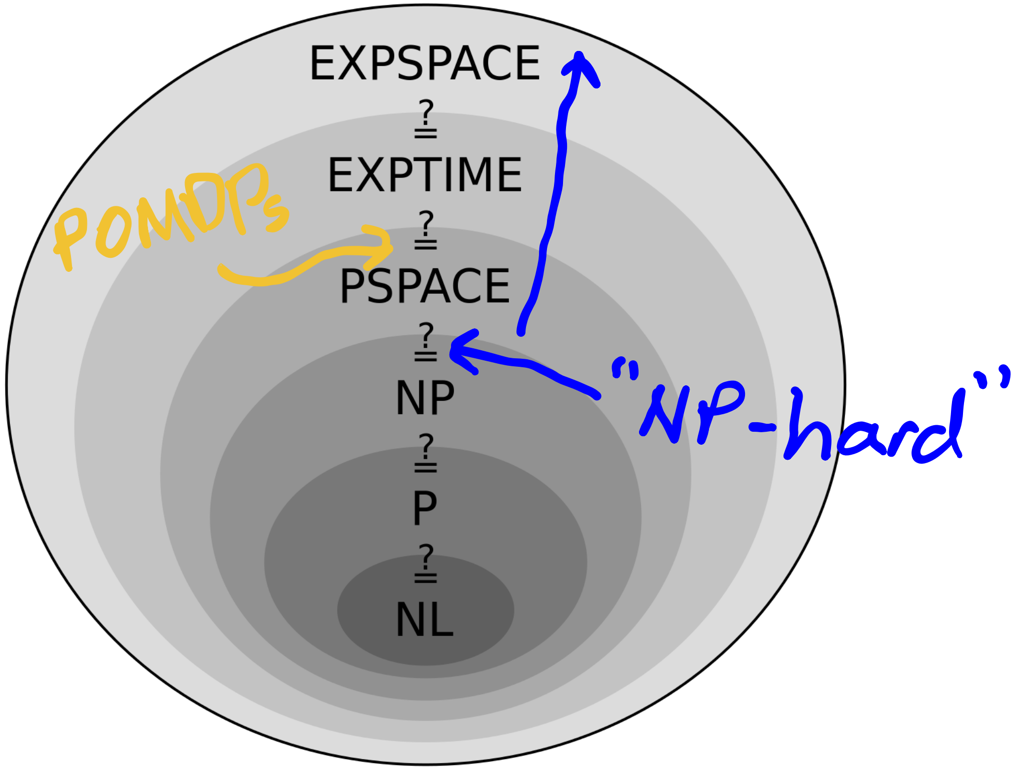
  (we think)



Approximate POMDP Solutions

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Numerical Approximations

(approximately solve original problem)

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Offline

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Last week

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Thursday

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Formulation Approximations

(solve a slightly different problem)

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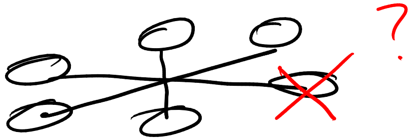
Formulation Approximations

(solve a slightly different problem)

Today!

Rotor Failure Example



$$s = (\underbrace{x, y, z, \dot{x}, \dot{y}, \dot{z}, \phi, \theta, \psi, p, q, r}_{\text{known}}, \underbrace{m_1, m_2 \cdots m_6}_{\text{unkown}})$$

$\cancel{m \in \{\text{nominal}, \text{failed}\}}$

Certainty - Equivalence

POMDP Objective

POMDP Objective

$$\pi^* = \underset{\pi : B \to A}{\operatorname{argmax}} \; \mathrm{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, \pi(b_t))\right]$$

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Certainty Equivalent

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$$\pi_{\mathrm{CE}}(b) = \pi_s(\underset{s\sim b}{\mathrm{E}}[s])$$

π^*_{MDP} $\pi_s(\hat{s})$ $\hat{s} = \underset{s\sim b}{\mathrm{E}}[s]$

$$b' = \tau(b, a, o)$$

Certainty Equivalent

Optimal for LQG ← Gaussian

Linear Quadratic

$S = \mathbb{R}^n$

$A = \mathbb{R}^m$

$O = \mathbb{R}^p$

$$T(\mathbf{s}' \mid \mathbf{s}, \mathbf{a}) = \mathcal{N}(\mathbf{s}' \mid \mathbf{T}_s\mathbf{s} + \mathbf{T}_a\mathbf{a}, \mathbf{\Sigma}_s)$$

$$O(\mathbf{o} \mid \mathbf{s}') = \mathcal{N}(\mathbf{o} \mid \mathbf{O}_s\mathbf{s}', \mathbf{\Sigma}_o)$$

$$R(s,a) = -s^\top R_s s - a^\top R_a a$$

Kalman Filter

$$b(\mathbf{s}) = \mathcal{N}(\mathbf{s} \mid \boldsymbol{\mu}_b, \mathbf{\Sigma}_b)$$

Prediction

$$\boldsymbol{\mu}_p \leftarrow \mathbf{T}_s\boldsymbol{\mu}_b + \mathbf{T}_a\mathbf{a}$$

$$\mathbf{\Sigma}_p \leftarrow \mathbf{T}_s\mathbf{\Sigma}_b\mathbf{T}_s^\top + \mathbf{\Sigma}_s$$

Update

$$\mathbf{K} \leftarrow \mathbf{\Sigma}_p\mathbf{O}_s^\top\left(\mathbf{O}_s\mathbf{\Sigma}_p\mathbf{O}_s^\top + \mathbf{\Sigma}_o\right)^{-1}$$

$$\boldsymbol{\mu}_b \leftarrow \boldsymbol{\mu}_p + \mathbf{K}\left(\mathbf{o} - \mathbf{O}_s\boldsymbol{\mu}_p\right)$$

$$\mathbf{\Sigma}_b \leftarrow (\mathbf{I} - \mathbf{K}\mathbf{O}_s)\mathbf{\Sigma}_p$$

$$\pi^*(b) = K_a \operatorname{mean}(b)$$

QMDP

POMDP Objective

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$$\underbrace{(S, A, T, R, O, Z, \gamma)}_{\text{solve}}$$

optimal Q-value for the fully observable MDP

$$\pi_{\text{QMDP}}(b) = \underset{a\in A}{\text{argmax}}\ \underset{s\sim b}{\mathrm{E}}\left[Q_{\text{MDP}}(s,a)\right]$$

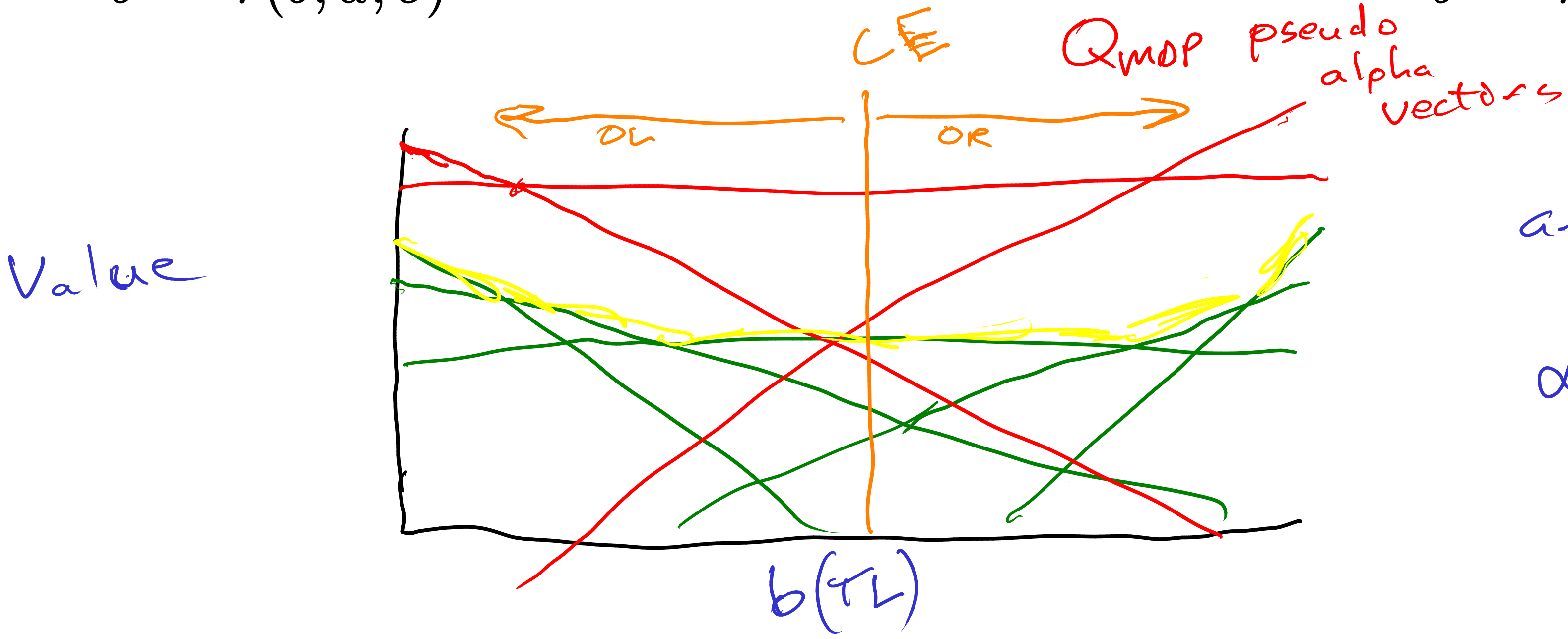
$$\underset{a\in A}{\text{argmax}}\ \pi_s\left(\underset{s\sim b}{\mathrm{E}}[s]\right)$$

CE

$$\underset{a\in A}{\text{argmax}}\ \alpha_a \cdot b$$

$$\alpha_a[s] = Q_{\text{MDP}}(s,a)$$

$$b' = \tau(b, a, o)$$



Example: Tiger POMDP with Waiting

$A = \{OL, OR, \text{Listen}, \text{Wait}\}$

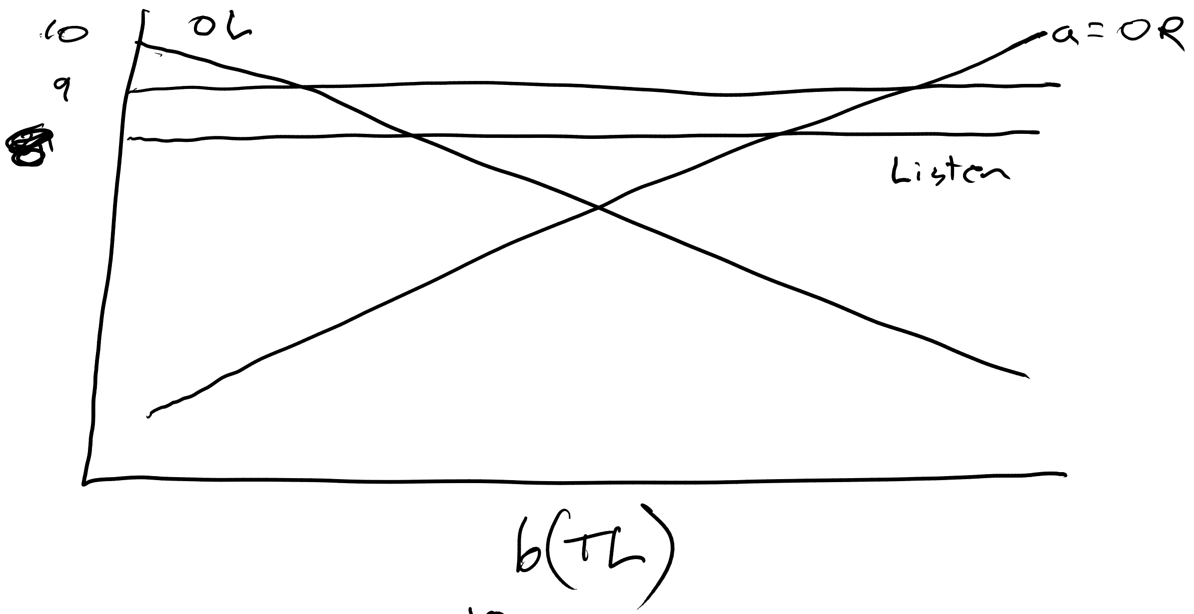
+10 if good

−100 if bad

−1 reward, 85% correct observation

0 reward, random observation

pseudo alpha vectors



$b(TL) = 50\%$

optimal = L

QMDP solution?

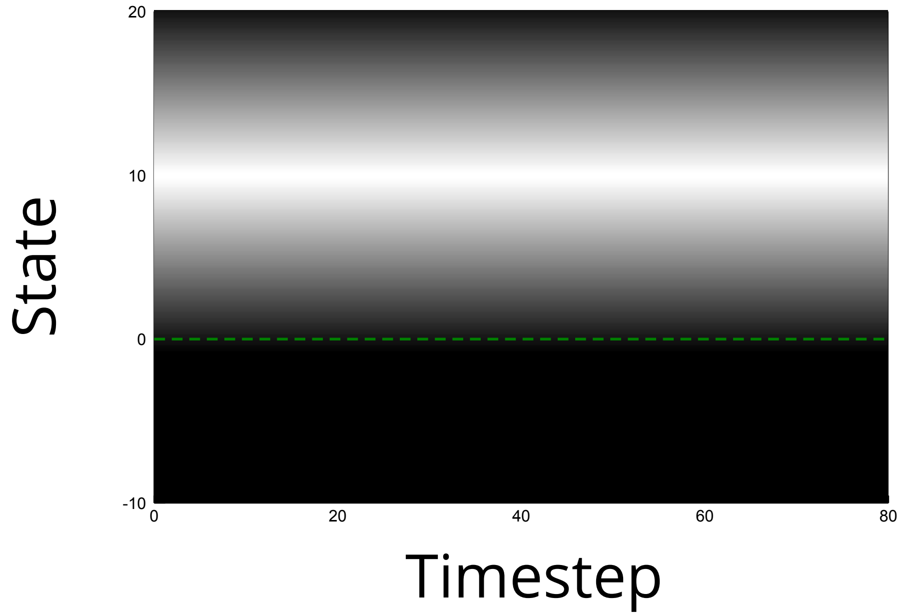
$Q_{MDP}(\text{any}, \text{open}) = 10$

$\gamma = 0.9$

+10

$-1 + \gamma \max_a Q_{MDP}(s,a)$

POMDP Example: Light-Dark



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$$s' = s + a \qquad o \sim \mathcal{N}(s, s - 10)$$

$$\mathcal{A} = \{-10, -1, 0, 1, 10\}$$

$$R(s,a) = \begin{cases} 100 & \text{if } a = 0, s = 0 \\ -100 & \text{if } a = 0, s \neq 0 \\ -1 & \text{otherwise} \end{cases}$$

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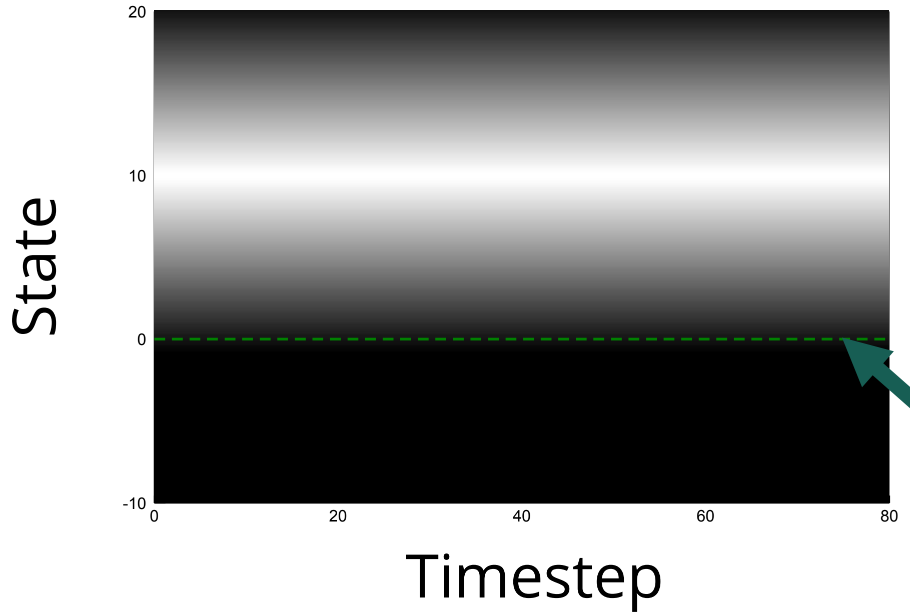

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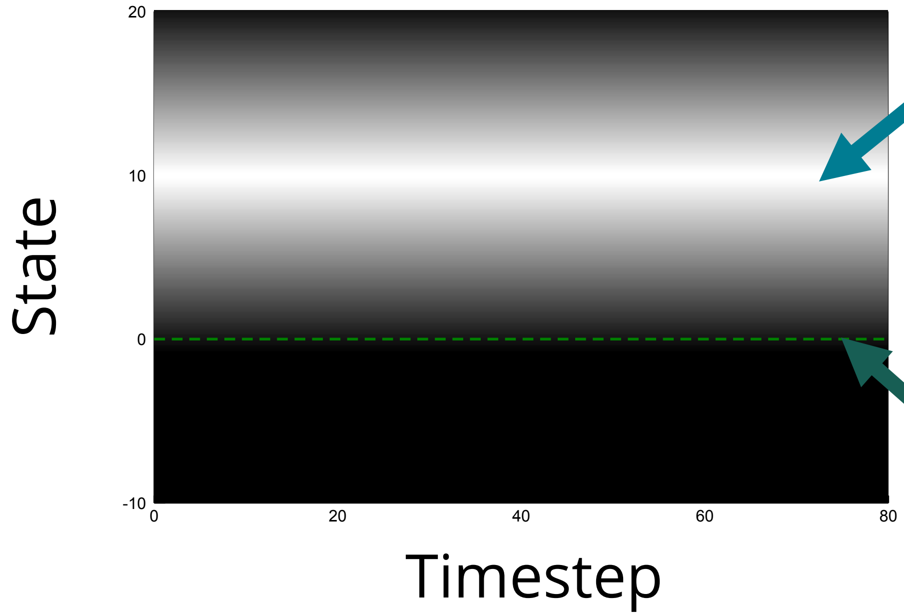

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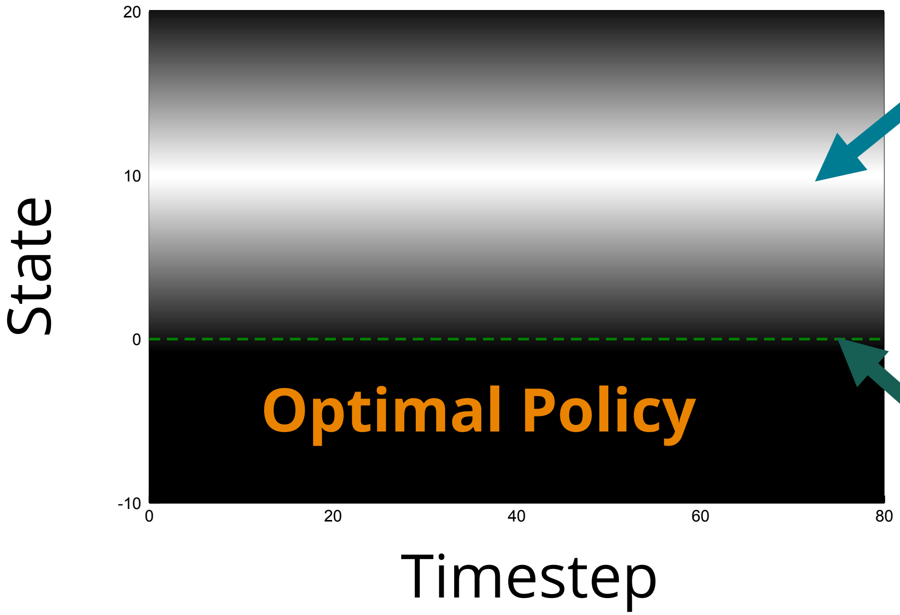

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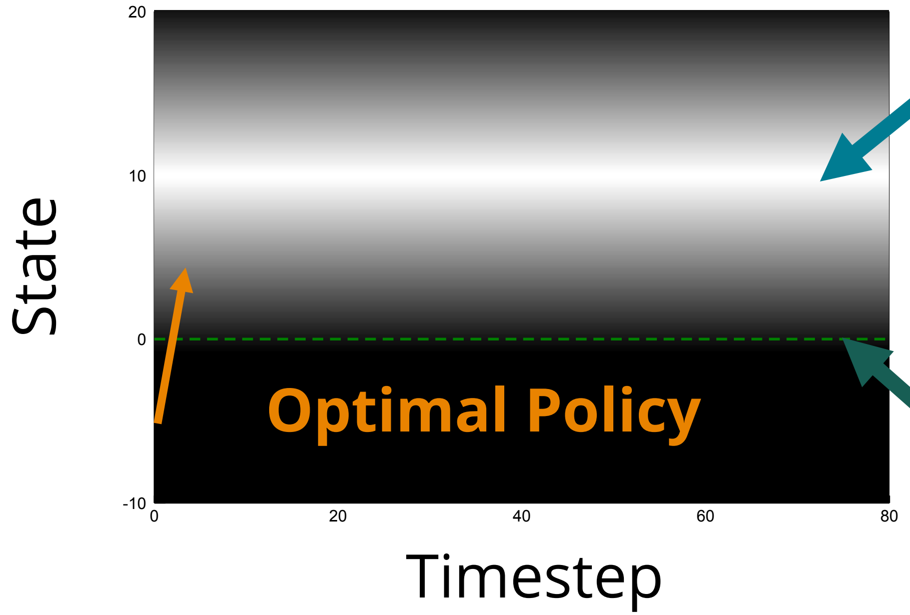

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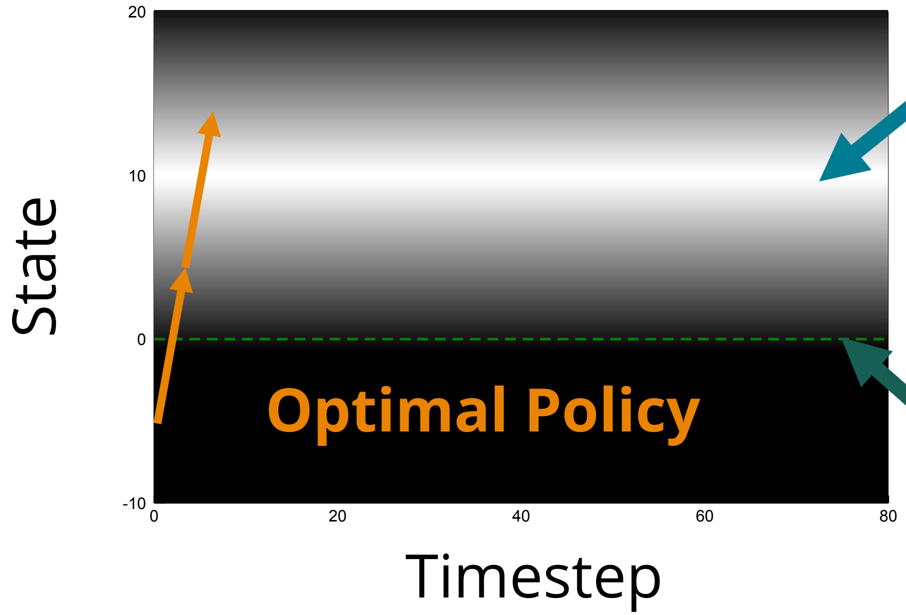

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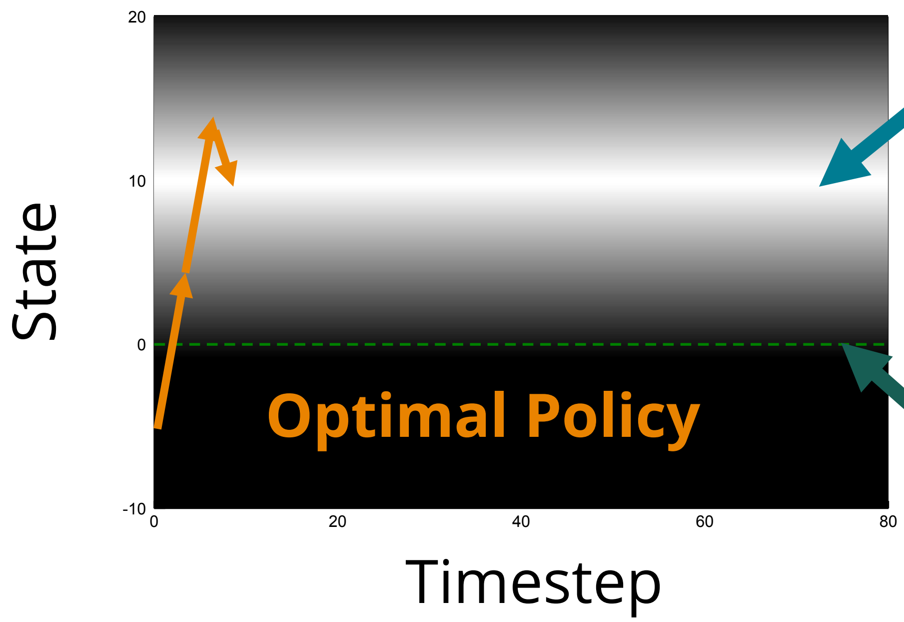

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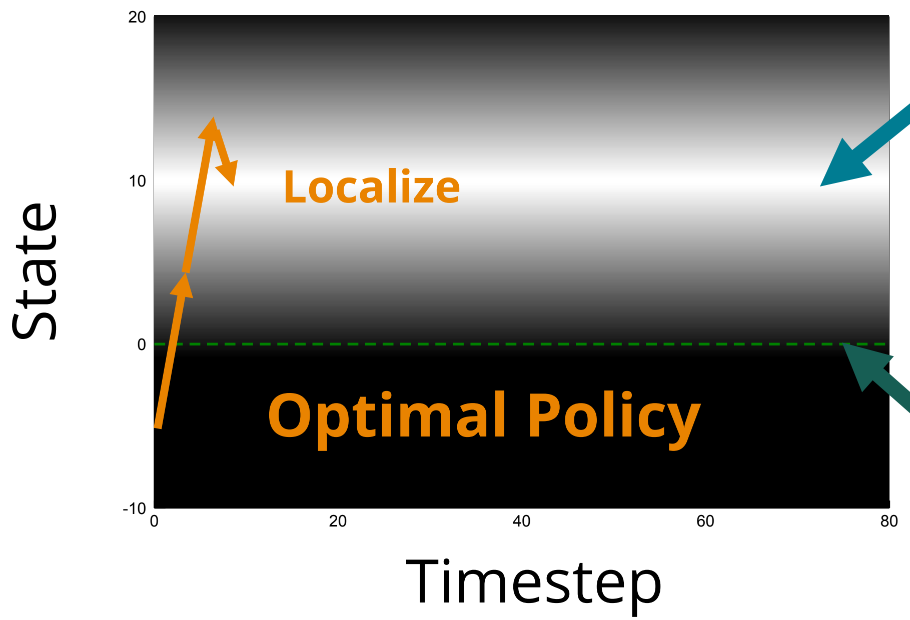

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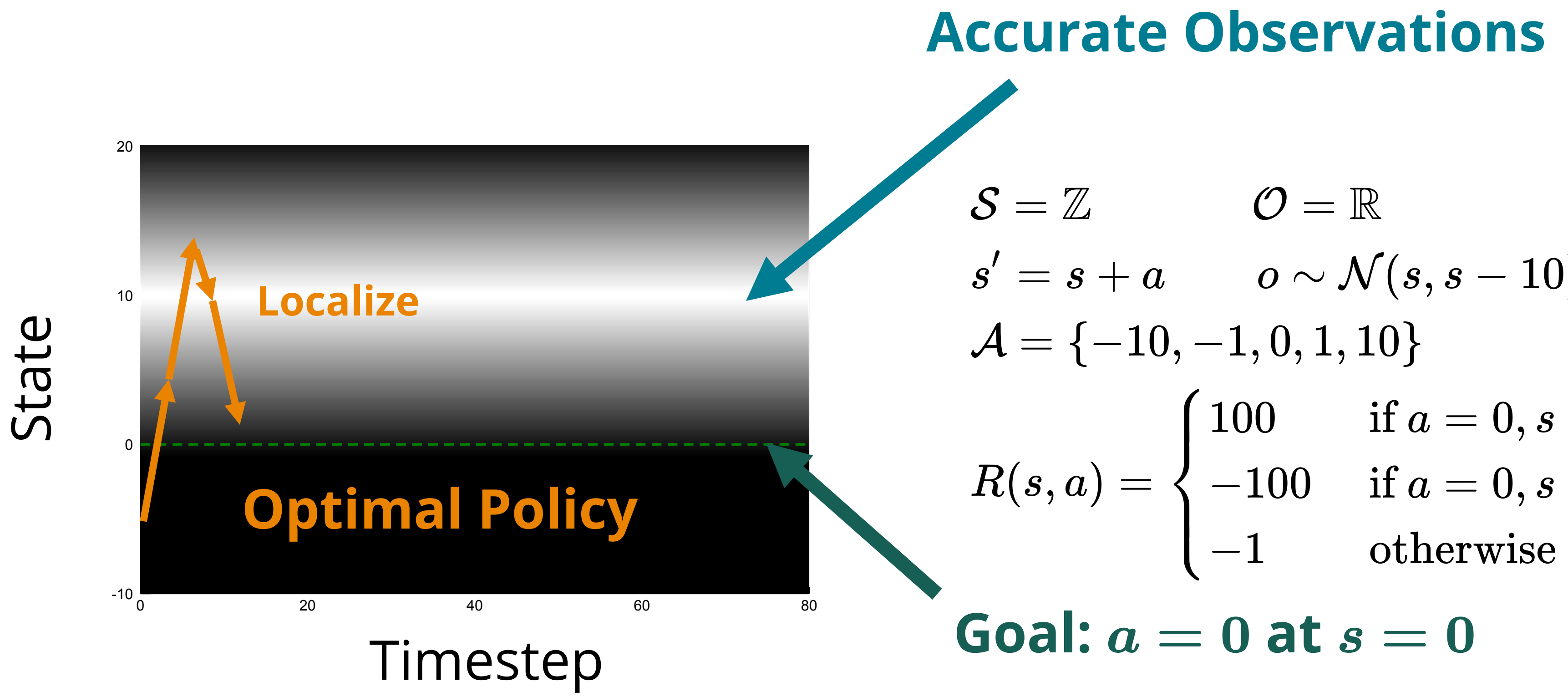

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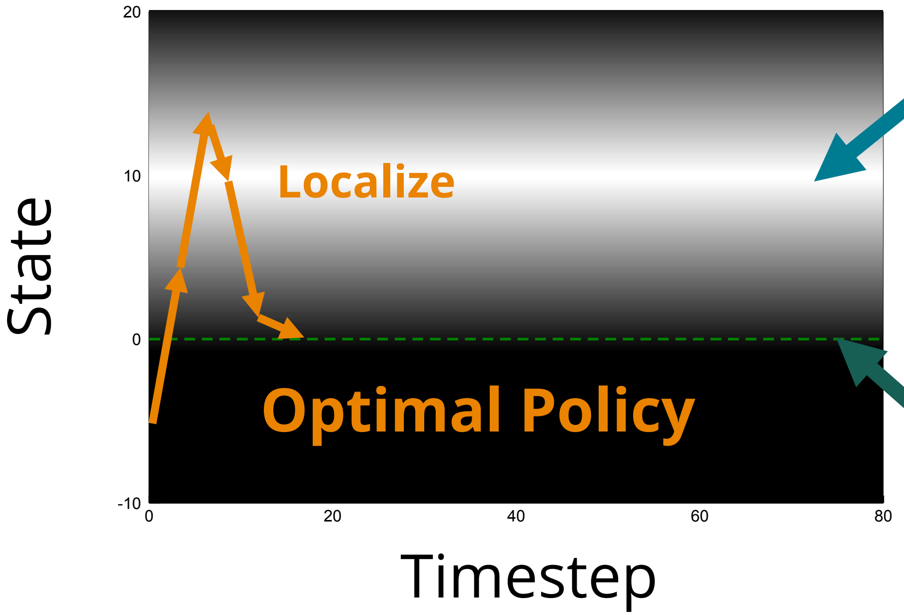

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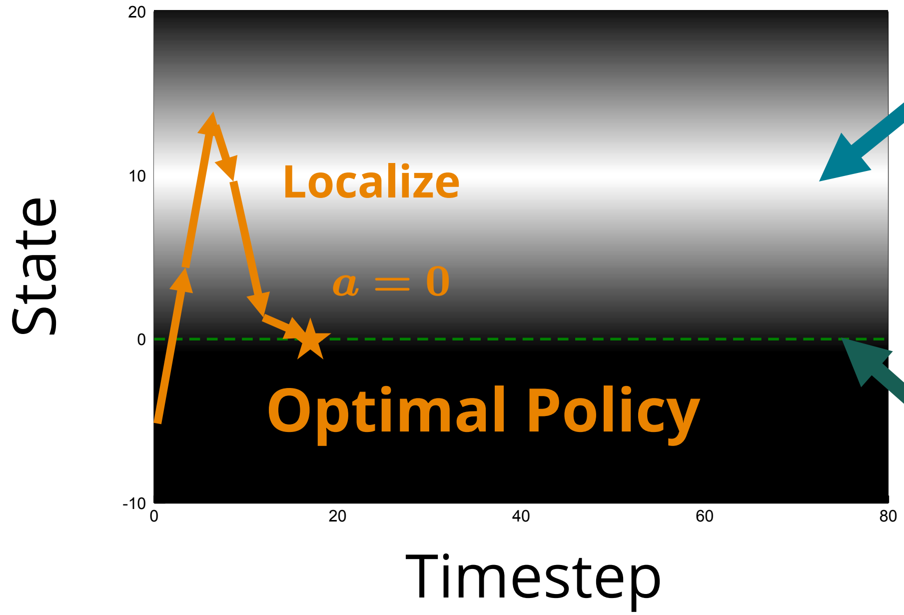

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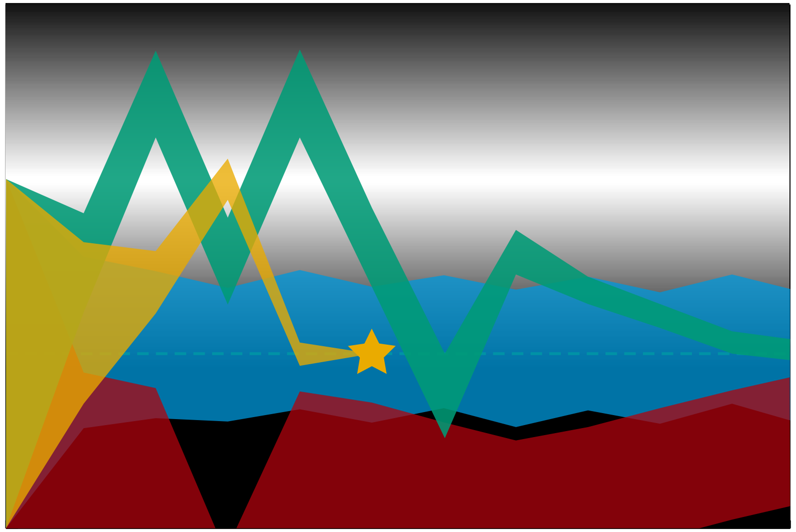
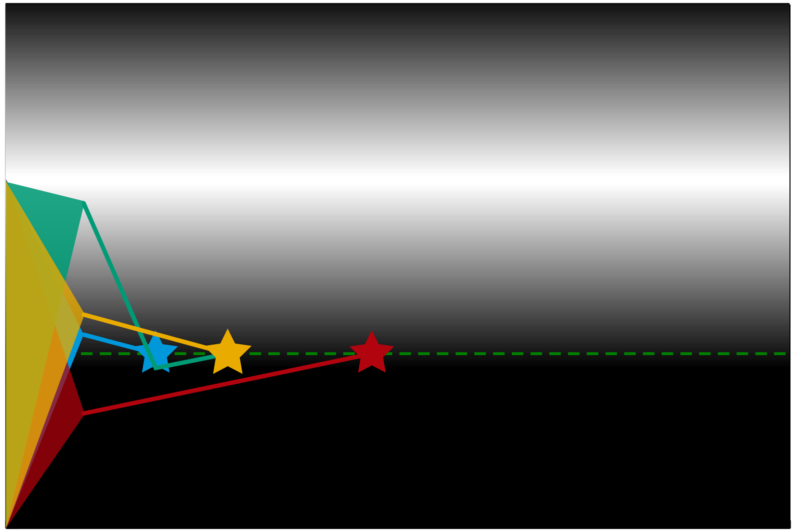
POMDP Solution



QMDP

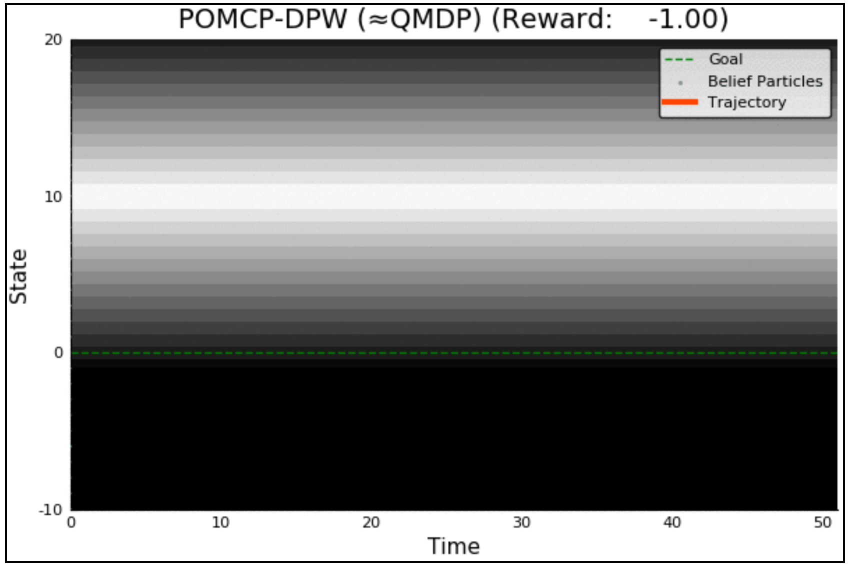


Same as **full observability** on the next step

Information Gathering

QMDP

Full POMDP



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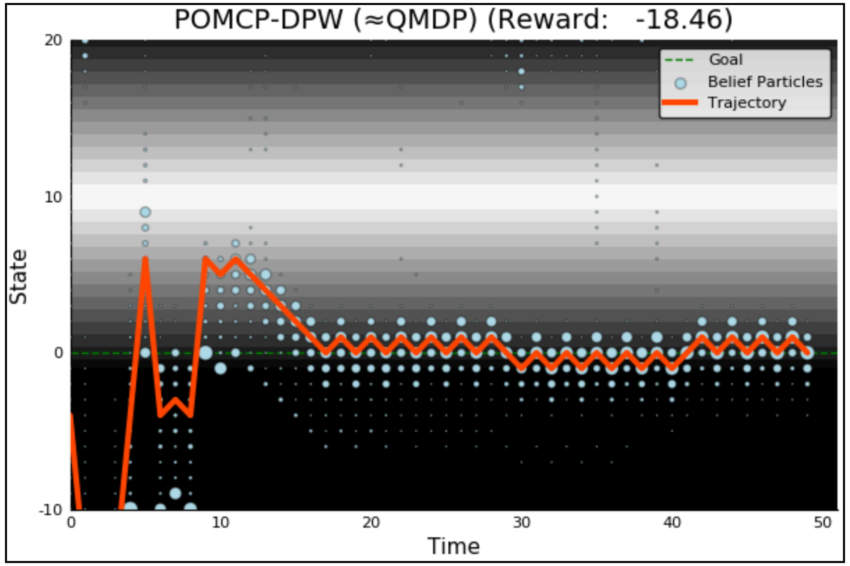
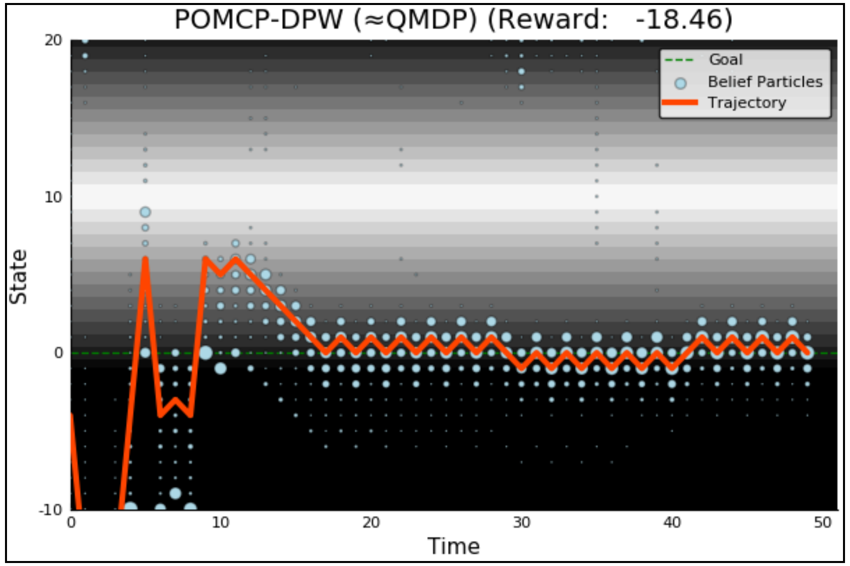


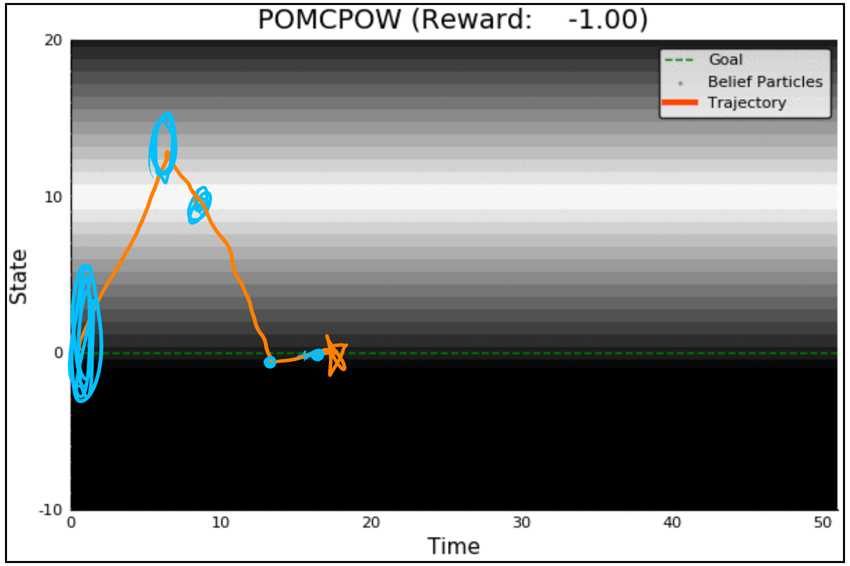
Information Gathering

QMDP

Full POMDP

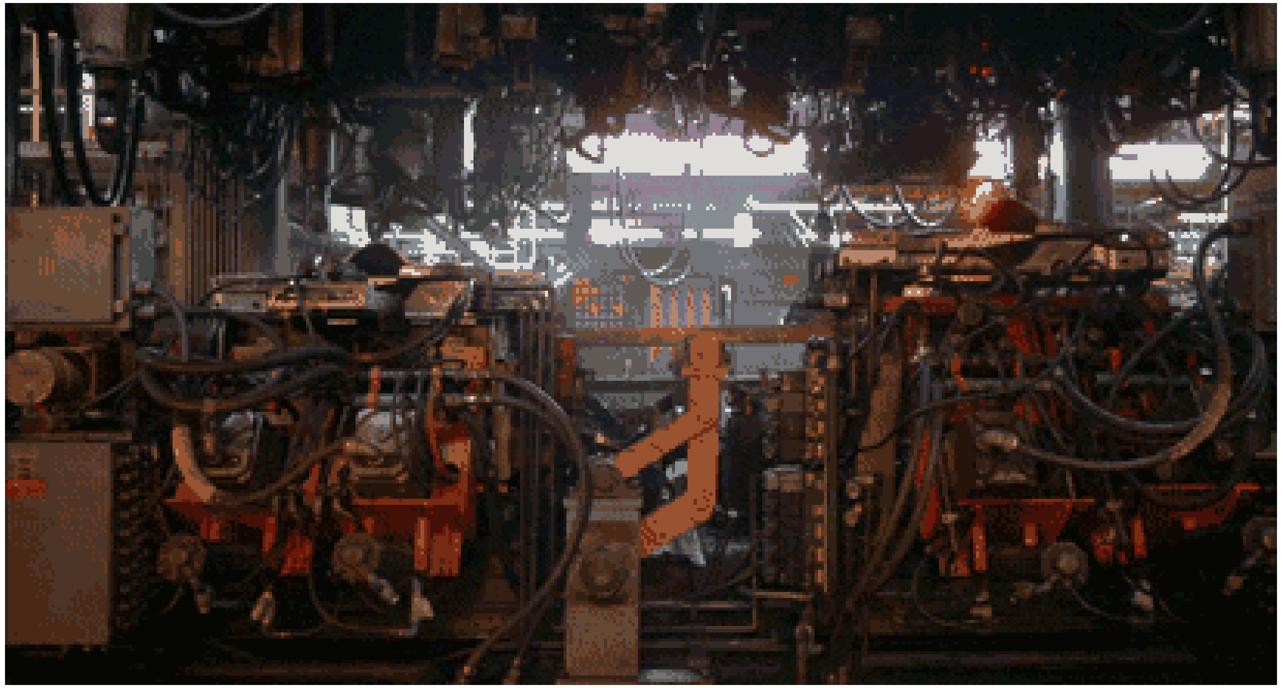
50





QMDP

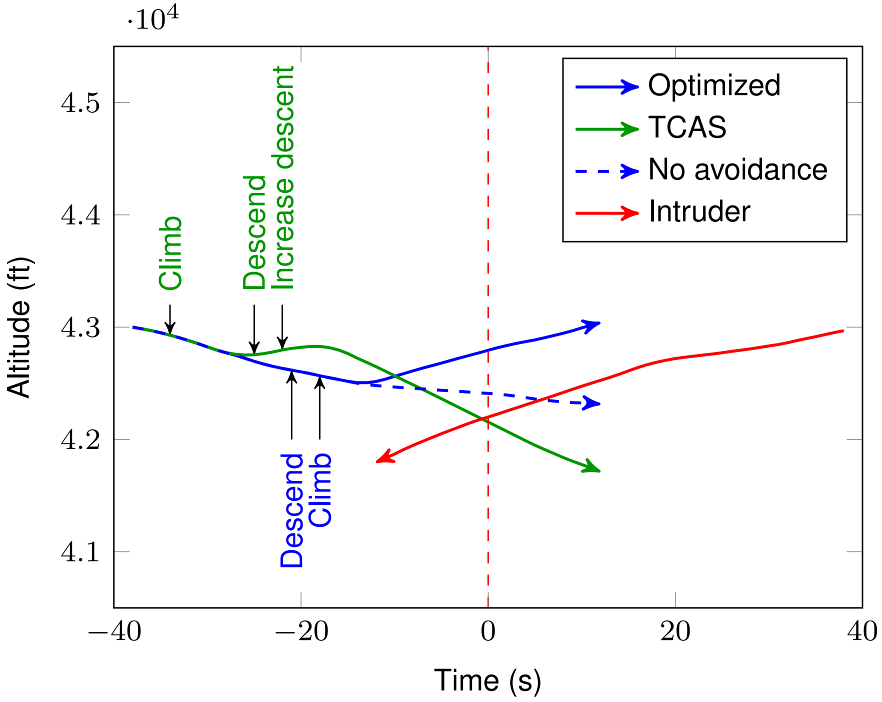
INDUSTRIAL GRADE



QMDP

ACAS X

[Kochenderfer, 2011]



Hindsight Optimization

POMDP Objective

$$\pi^* = \underset{\pi : B \to A}{\operatorname{argmax}} \, \mathrm{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, \pi(b_t))\right]$$

$$b' = \tau(b, a, o)$$

$$\pi_{HOP}(b) = \underset{a^i_{0:\infty}}{\operatorname{argmax}} \frac{1}{m} \sum_{i=1}^{m} \gamma^t R(s^i_t, a^i_t)$$

Subject to $\quad s^i_{t+1} = G(s^i_t, a^i_t, \phi^i_t)$

fixed outcome

$a^i_1 = a^j_1 \quad \forall i, j$

m scenarios

FIB

POMDP Objective

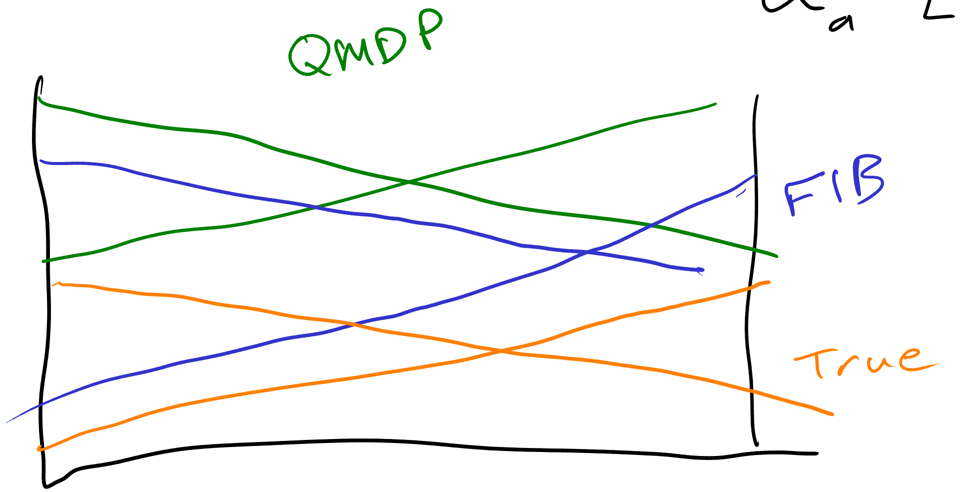
$$\pi^* = \underset{\pi:B\to A}{\operatorname{argmax}} \; \mathrm{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, \pi(b_t))\right]$$

$$b' = \tau(b, a, o)$$

$$\pi_{FIB}(b) = \underset{a \in A}{\operatorname{argmax}} \; \alpha_a^T b$$

iterate

$$\alpha_a^{(k+1)}[s] = R(s,a) + \gamma \sum_o \max_{a'} \sum_{s'} Z(o \mid a, s') \, T(s' \mid s,a) \, \alpha_{a'}^{(k)}[s']$$



k-Markov

POMDP Objective

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$$b' = \tau(b, a, o)$$

k previous observations

Open Loop

POMDP Objective

$$\pi^* = \underset{\pi : B \to A}{\operatorname{argmax}} \, \mathrm{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, \pi(b_t))\right]$$

$$\max_{a_1, a_2, \ldots a_T} E\left[\sum_{t=0}^{T} \gamma^t R(s_t, a_t)\right]$$

$$b' = \tau(b, a, o)$$