Bayesian Networks

- Last time:
- Today:

Bayesian Networks

- Last time:
  - Online POMDP Methods
- Today:

Discrete: POMCP
DESPOT

Bayesian Networks

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Bayesian Networks

- Last time:
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  - How do we reason about independence in Bayesian Networks?

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  - Online POMDP Methods
- Today:
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  - How do we reason about independence in Bayesian Networks?
  - How do we sample from Bayesian Networks

Recall Quiz 1

Question 1. (30 pts) Let A, B, and C be three binary-valued random variables (binary-valued means the support is $\{0, 1\}$). Suppose we know the following pieces of information:

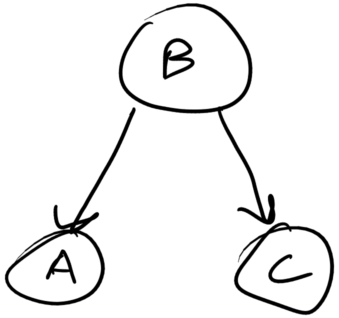
- $P(B = 1) = 0.25$
- $P(A = 1 \mid B = 1) = 0.7$
- $P(C = 1 \mid B = 1) = 0.8$
- $P(A = 1 \mid B = 0) = 0.4$
- $P(C = 1 \mid B = 0) = 0.4$
- $A \perp C \mid B$ (that is, A is conditionally independent of C given B)

a) What is $P(A = 1, B = 1)$?

b) Write the marginal distribution of C. $\leftarrow$ $P(C=1) = 0.5$

c) If $A = 1$, is $P(C = 1)$ less than, equal to, or greater than 50%?

$P(C=1 \mid A=1) = 0.55$



$\cancel{A \perp C}$

$A \perp C \mid B$

More Complex Example

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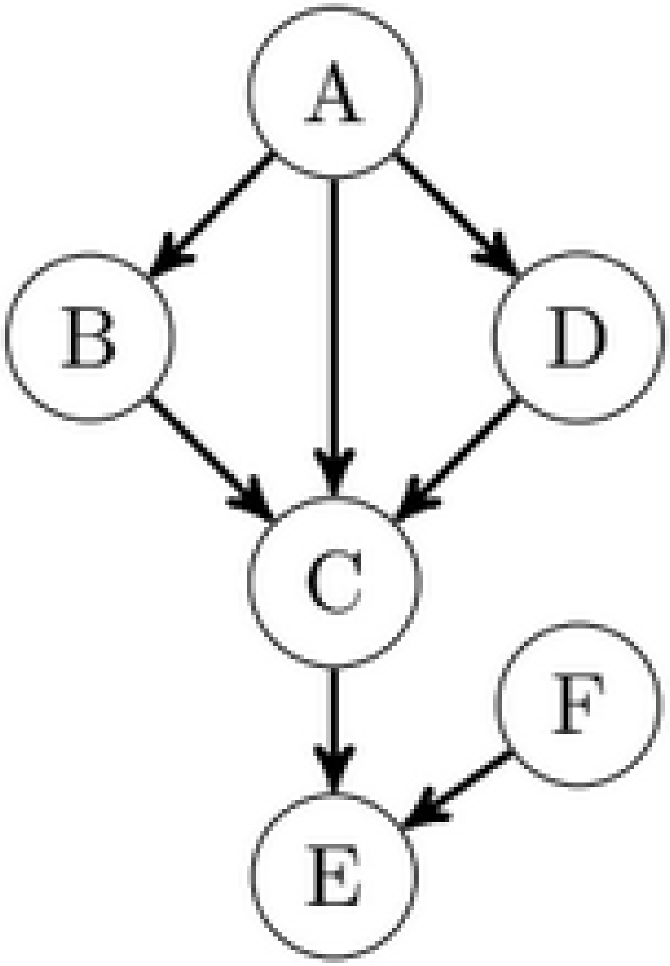
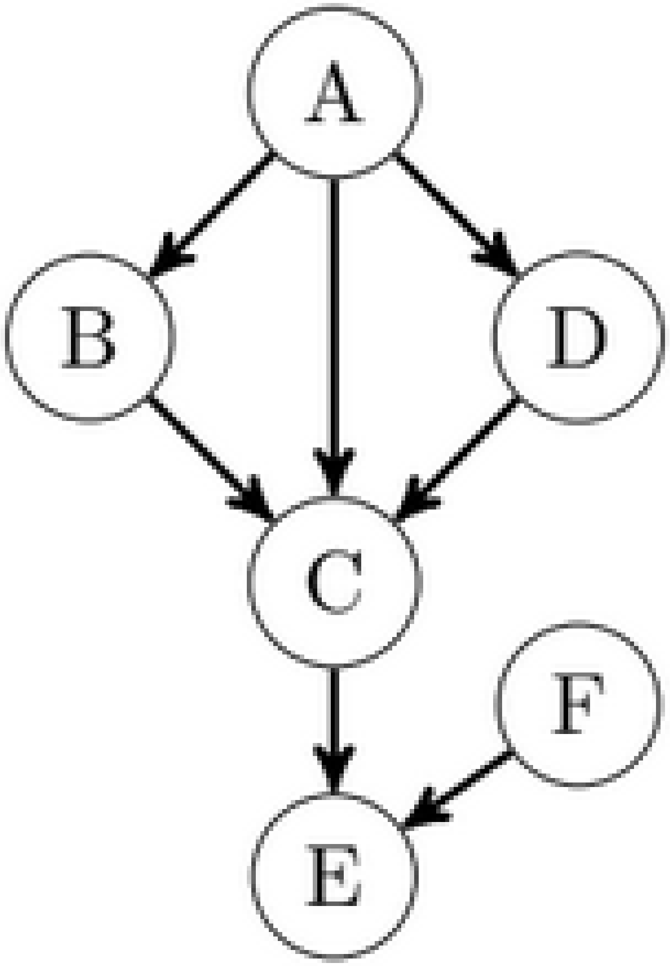


More Complex Example

$(B \perp D \mid A)$?



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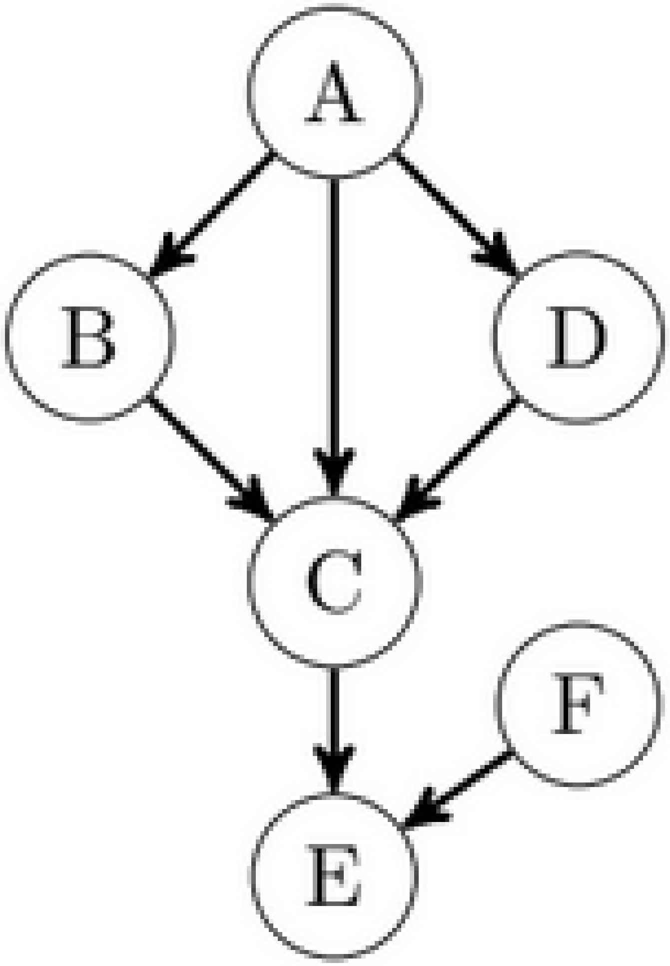

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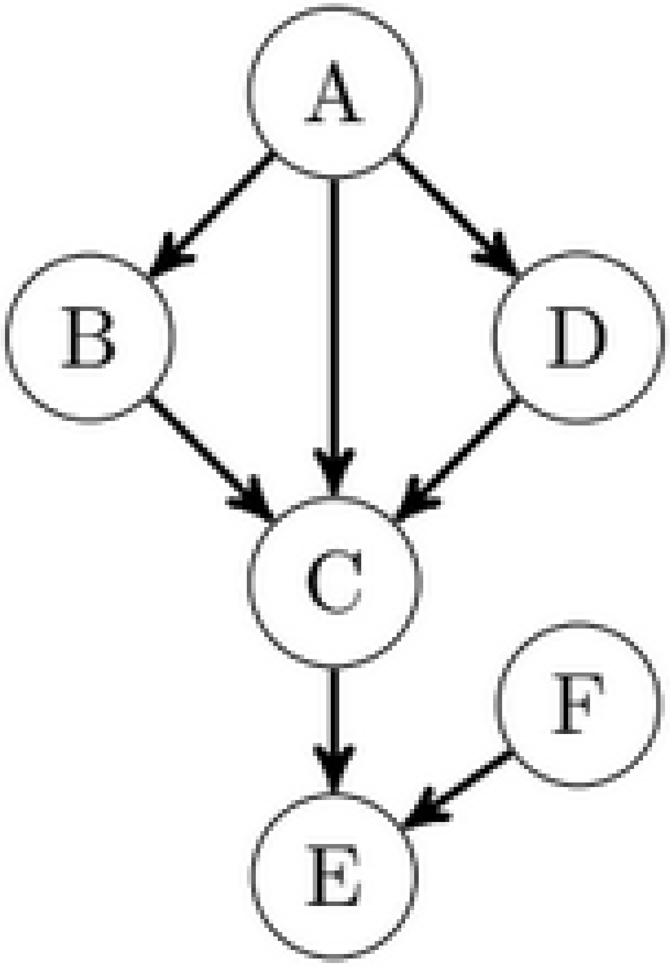

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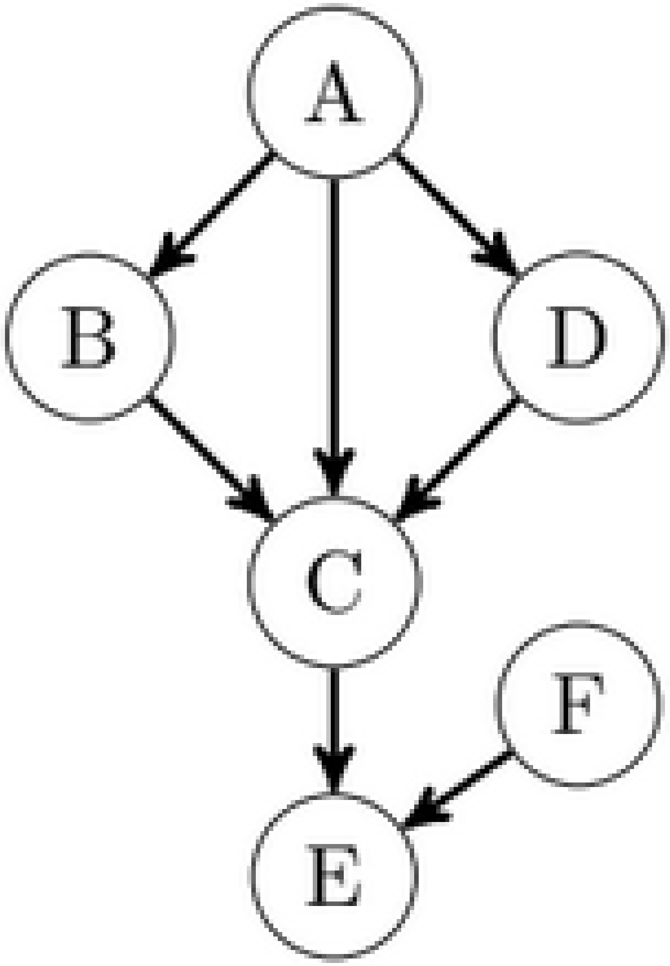


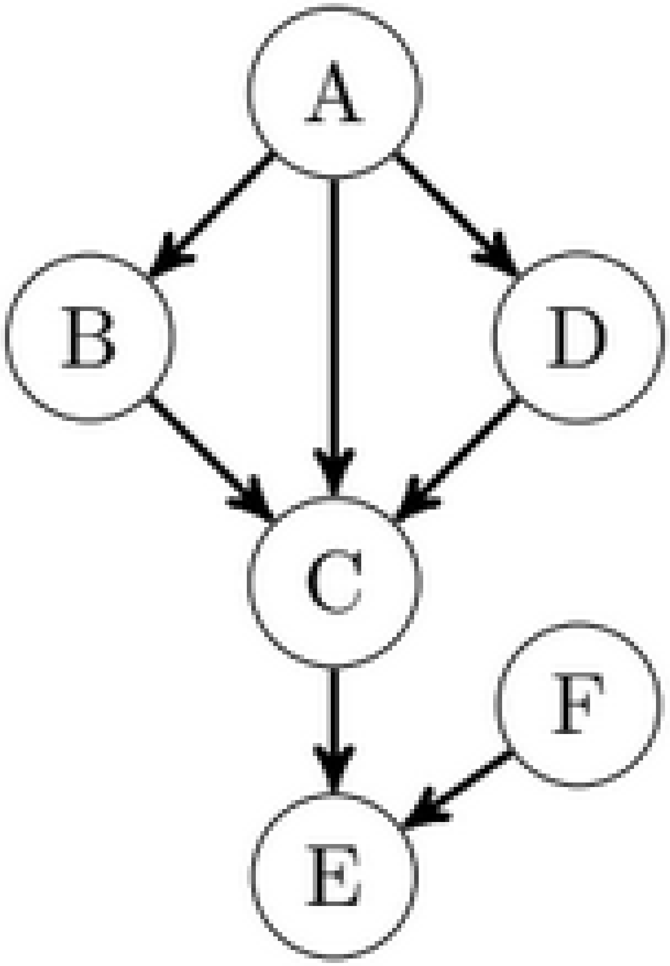
$(B \perp D \mid A)$?

Yes!

$(B \perp D \mid E)$?

No

More Complex Example



$(B \perp D \mid A)$?

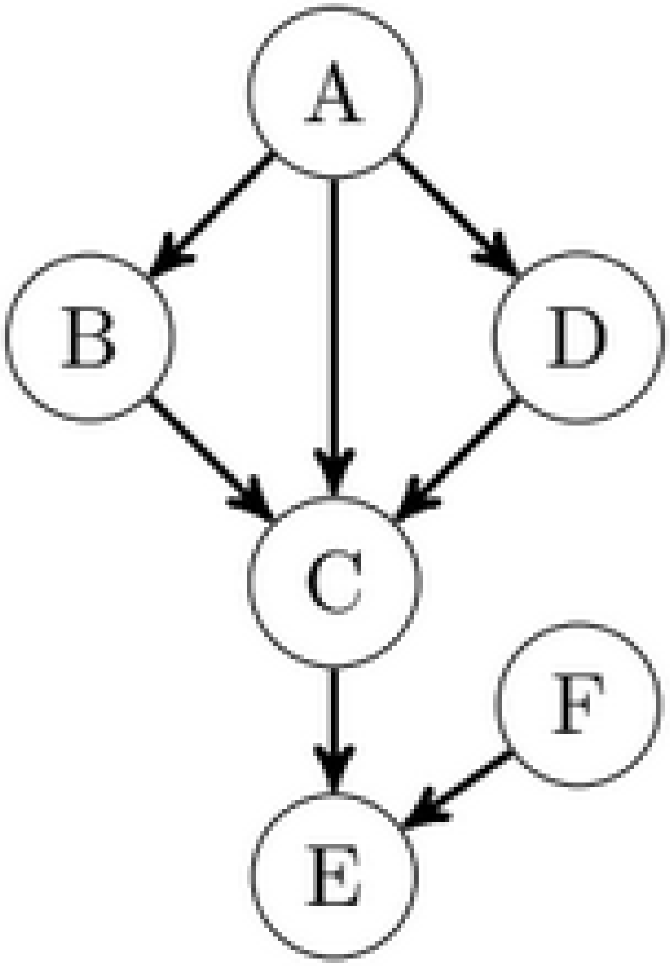
Yes!

$(B \perp D \mid E)$?

No

Why is this relevant?

More Complex Example



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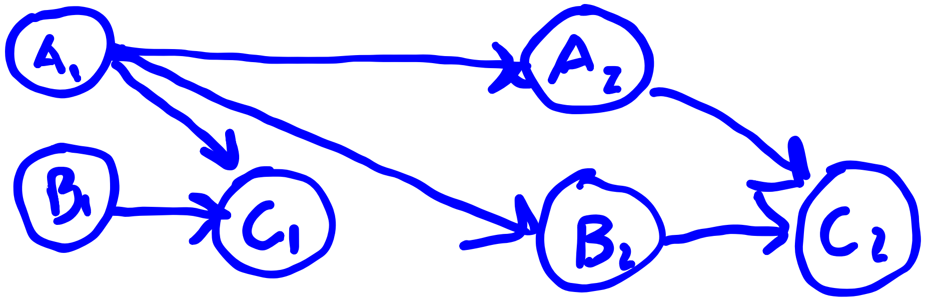

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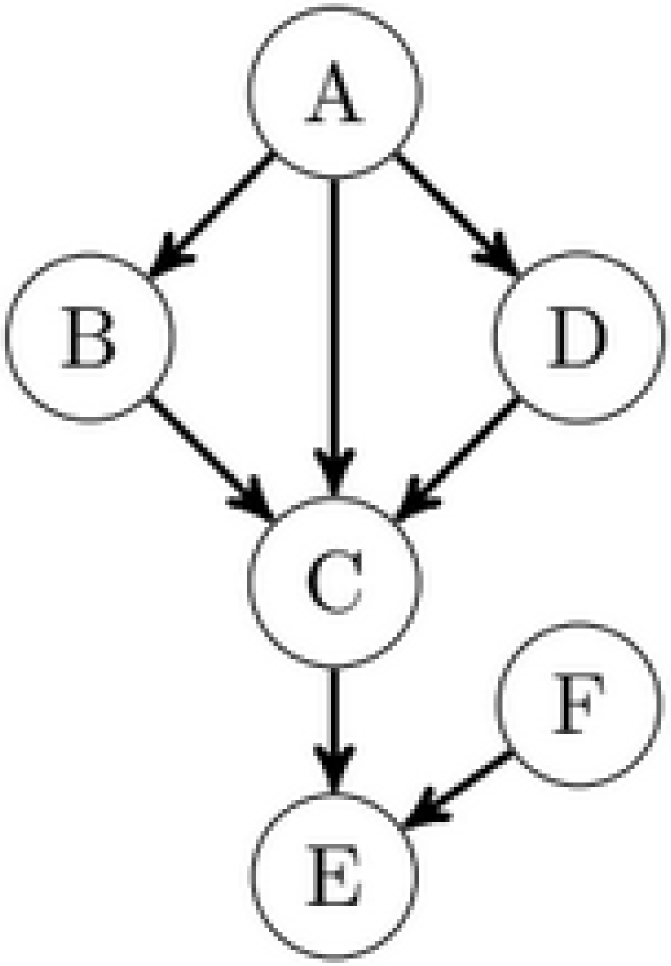


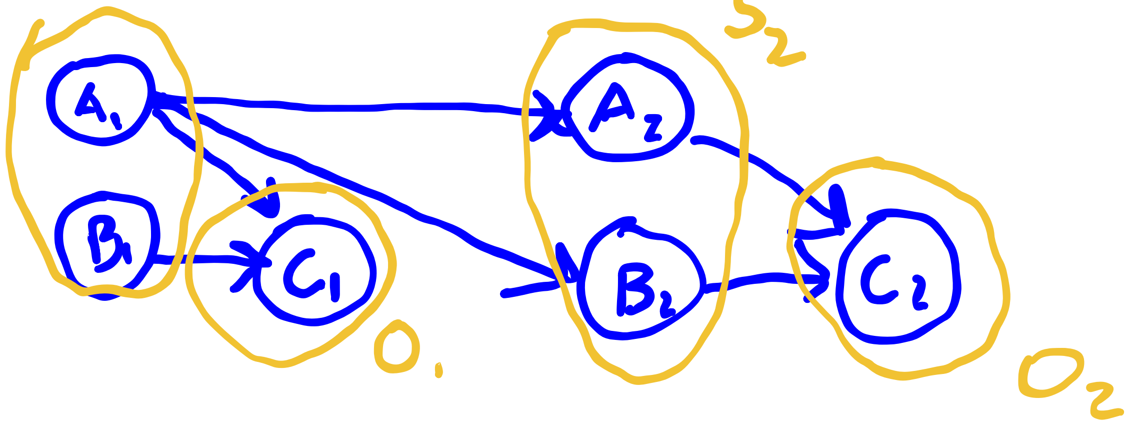
$(B \perp D \mid A)$?

Yes!

$(B \perp D \mid E)$?

No

Why is this relevant?



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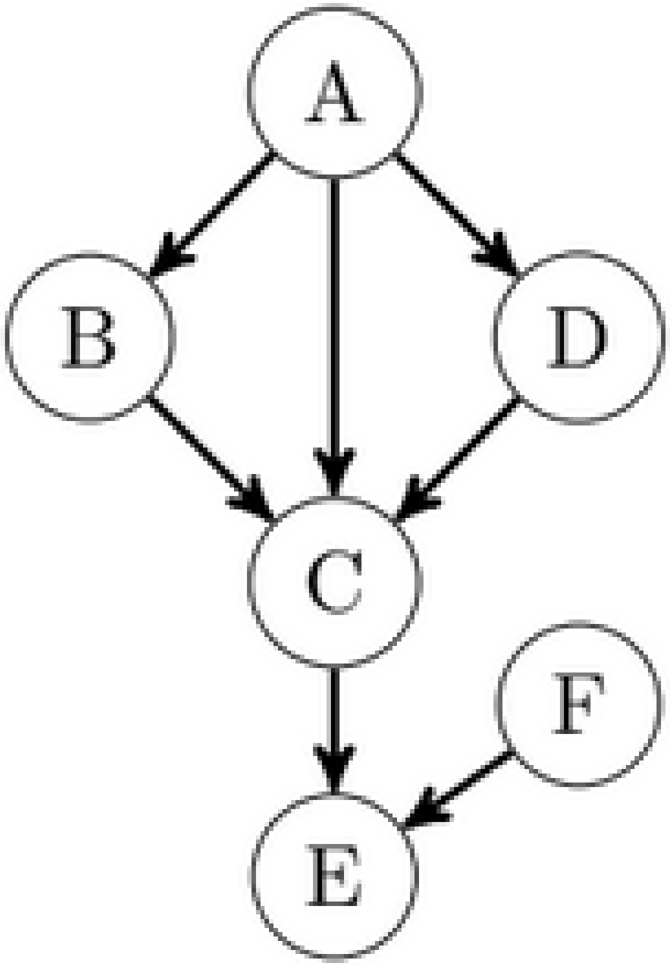


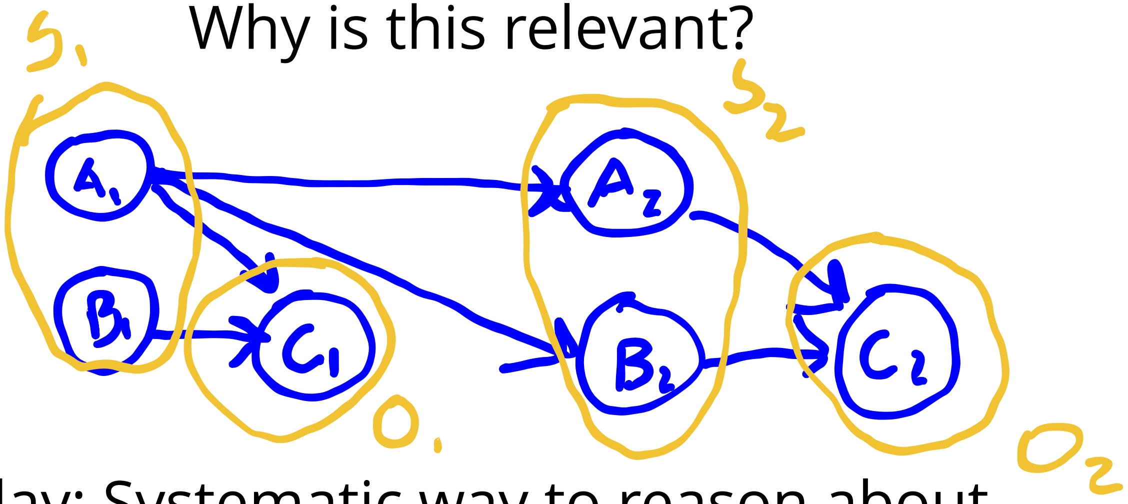
$(B \perp D \mid A)$?

Yes!

$(B \perp D \mid E)$?

No

Why is this relevant?



Today: Systematic way to reason about conditional independence

Review of Definitions

Review of Definitions

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**

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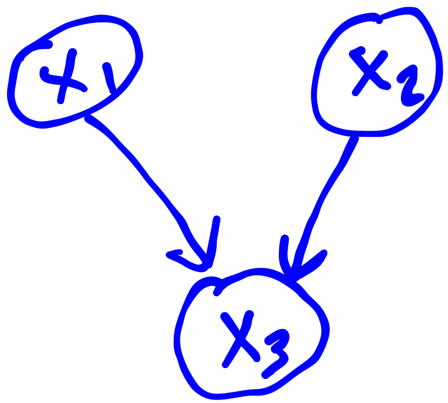
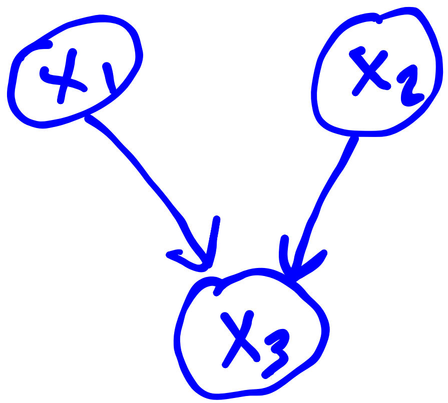
Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



Review of Definitions

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



- Node:

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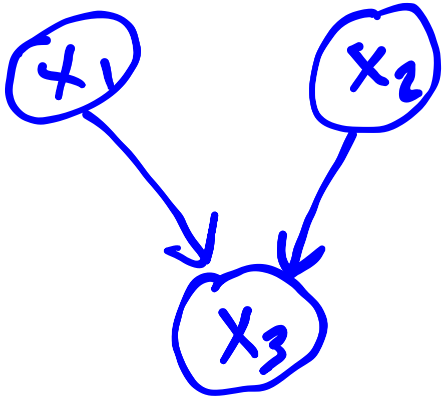
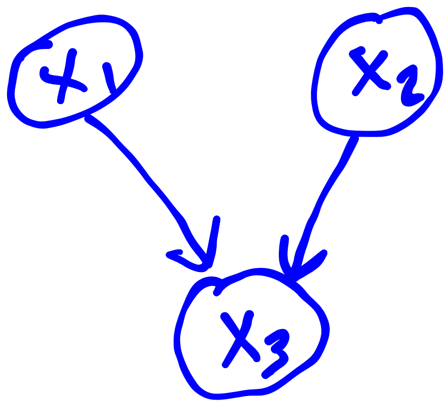


- Node: Random Variable

Review of Definitions

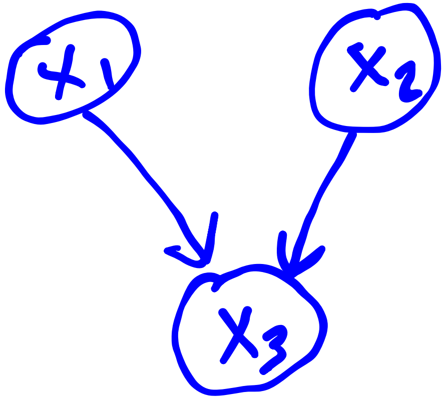
Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



- Node: Random Variable
- Edge: $P(X_i \mid X_1 \dots X_n)$

Review of Definitions

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**

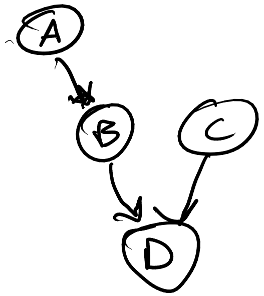


- Node: Random Variable
- Edge:

Parents

$$P(X_i \mid X_1 \dots X_n) = P(X_i \mid Pa(X_i))$$

$$P(D \mid A, B, C) = P(D \mid B, C)$$

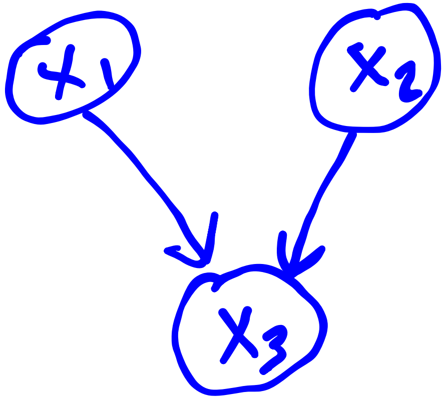


✗ $D \perp A$

✓ $D \perp A \mid B$

Review of Definitions

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



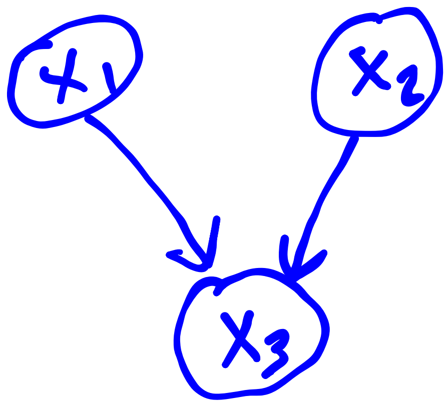
- Node: Random Variable
- Edge:

$$P(X_i \mid X_1 \ldots X_n) = P(X_i \mid Pa(X_i))$$

Independence

Review of Definitions

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



- Node: Random Variable
- Edge:

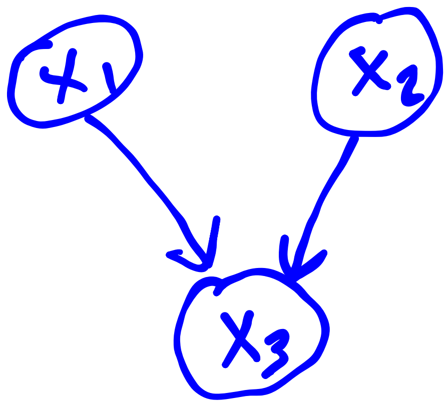
$$P(X_i \mid X_1 \dots X_n) = P(X_i \mid Pa(X_i))$$

Independence

$$P(X, Y) = P(X)\,P(Y)$$

Review of Definitions

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



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Conditional Independence

Review of Definitions

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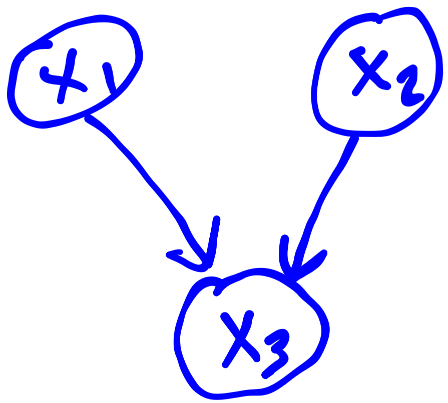

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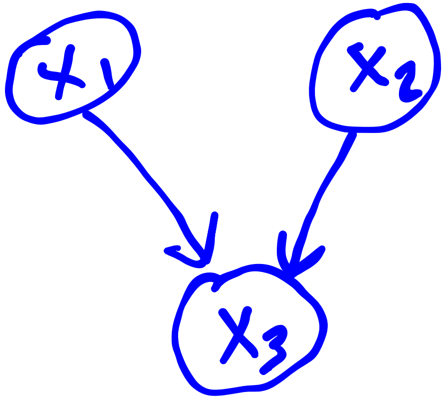
Conditional Independence

$$P(X,Y \mid Z) = P(X \mid Z)\,P(Y \mid Z)$$

Review of Definitions

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



- Node: Random Variable
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$$P(X_i \mid X_1 \dots X_n) = P(X_i \mid Pa(X_i))$$

Independence

$$P(X, Y) = P(X)\, P(Y)$$

$X \perp Y$

Conditional Independence

$$P(X, Y \mid Z) = P(X \mid Z)\, P(Y \mid Z)$$

$(X \perp Y \mid Z)$

Review of Definitions

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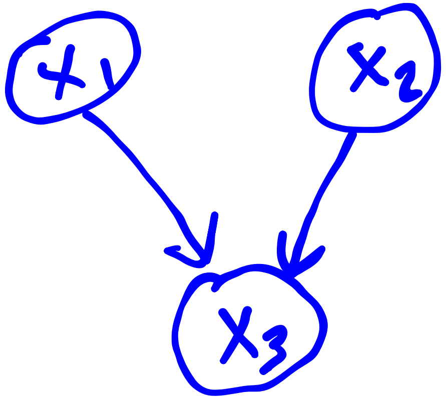

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Independence

$$P(X, Y) = P(X)\, P(Y)$$

$X \perp Y$

Conditional Independence

$$P(X, Y \mid Z) = P(X \mid Z)\, P(Y \mid Z)$$

$(X \perp Y \mid Z)$

$\Updownarrow$

$P(X \mid Z) = P(X \mid Y, Z)$

d-Separation

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Let $\mathcal{C}$ be a set of random variables.

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A *path* between A and B is *d-separated* by $\mathcal{C}$ if any of the following are true

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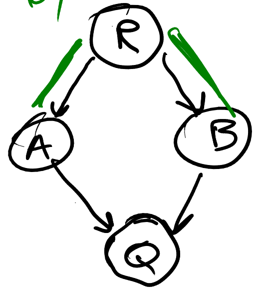
2. The path contains a *fork* $\underset{A}{X} \leftarrow \underset{R}{Y} \rightarrow \underset{B}{Z}$ such that $\underset{R}{Y} \in \mathcal{C}$
3. The path contains an *inverted fork* (v-structure) $\underset{A}{X} \rightarrow \underset{Q}{Y} \leftarrow \underset{B}{Z}$ such that $Y \notin \mathcal{C}$

$A \leftarrow R \rightarrow B$

✓ d-separated by R

$\mathcal{C} = \{R\}$



$A \rightarrow Q \leftarrow B$

$\mathcal{C} = \{Q\}$

not d-separated by $\{Q\}$

d-Separation

Let $\mathcal{C}$ be a set of random variables.

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We say that A and B are *d-separated* by $\mathcal{C}$ if all paths between A and B are d-separated by $\mathcal{C}$.

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d-separation $\Rightarrow A \perp B \mid \mathcal{C}$

Proving Conditional Independence

1. The path contains a *chain* $X \rightarrow Y \rightarrow Z$ such that $Y \in \mathcal{C}$
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Proving Conditional Independence

1. Enumerate all paths between nodes in question

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Proving Conditional Independence

1. Enumerate all paths between nodes in question
2. Check all paths for d-separation

1. The path contains a *chain* $X \to Y \to Z$ such that $Y \in \mathcal{C}$
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Proving Conditional Independence

1. Enumerate all paths between nodes in question
2. Check all paths for d-separation
3. If all paths d-separated, then CE

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Proving Conditional Independence

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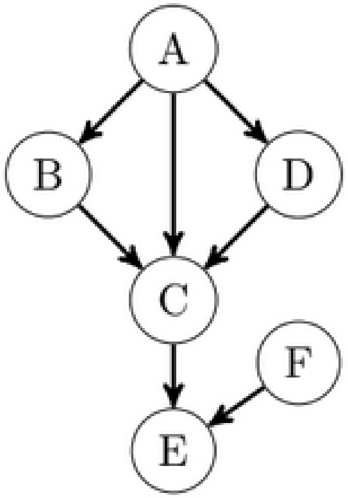

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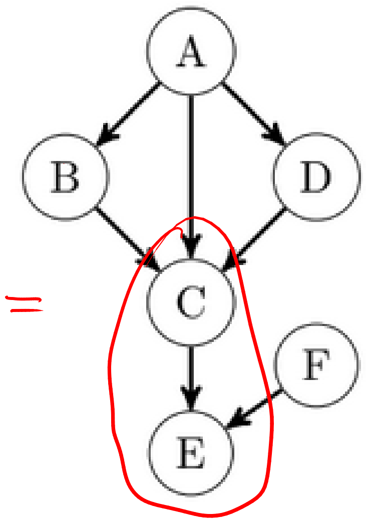
Proving Conditional Independence

1. Enumerate all paths between nodes in question
2. Check all paths for d-separation
3. If all paths d-separated, then CE

$\mathcal{C}$ =



Example: $(B \perp D \mid C, E)$?

$B \leftarrow A \rightarrow D$ not d-separated by $\mathcal{C}$

$B \rightarrow C \leftarrow D$ not d-separated by $\mathcal{C}$

$B \leftarrow A \rightarrow C \leftarrow D$ not d-separated

$B \rightarrow C \leftarrow A \rightarrow D$ not d-separated

not all paths are d-separated by $\mathcal{C}$

i.e. not ~~$B \perp D \mid C, E$~~

1. The path contains a *chain* $X \rightarrow Y \rightarrow Z$ such that $Y \in \mathcal{C}$
2. The path contains a *fork* $X \leftarrow Y \rightarrow Z$ such that $Y \in \mathcal{C}$
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Exercise

1. The path contains a *chain* $X \rightarrow Y \rightarrow Z$ such that $Y \in \mathcal{C}$
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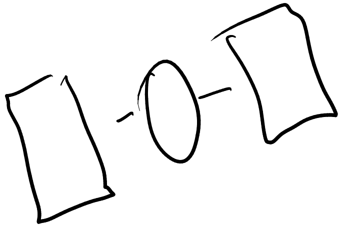
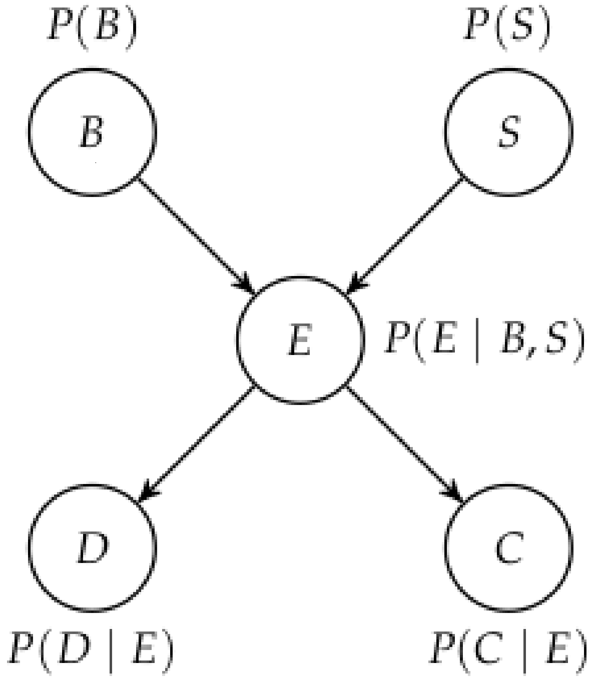
Exercise





B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

1. The path contains a *chain* $X \to Y \to Z$ such that $Y \in \mathcal{C}$
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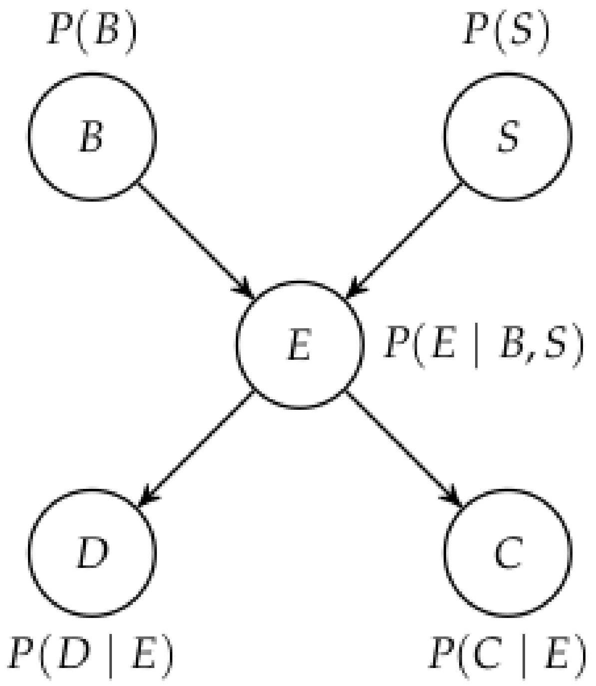

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Exercise

No

$D \perp C \mid B$?

1. The path contains a *chain* $X \rightarrow Y \rightarrow Z$ such that $Y \in \mathcal{C}$
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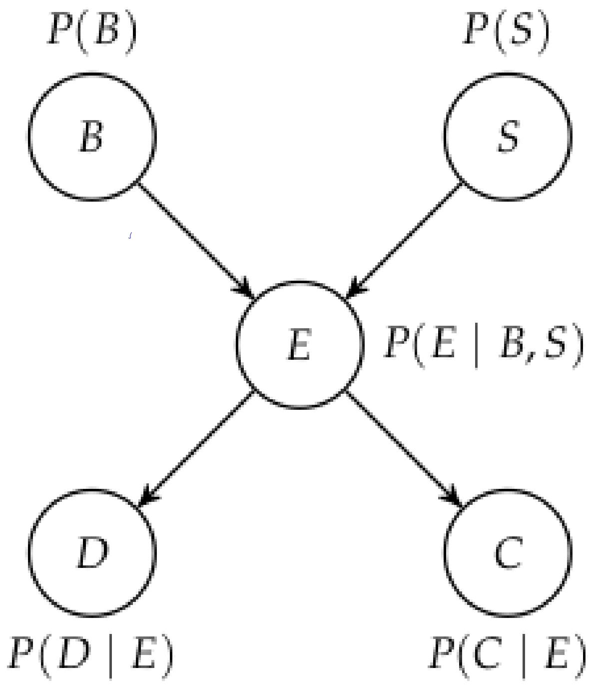

Exercise

enumerate paths
- check d-separation
- if all paths d-separated, then CE

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$D \perp C \mid B$? No

$D \leftarrow E \rightarrow C$

fork, but $E \notin \mathcal{C}$ $\Rightarrow$ not d-separated

$D \perp C \mid E$? Yes

$D \leftarrow E \rightarrow C$

fork, $E \in \mathcal{C}$ $\Rightarrow$ d-separated

d-separation

1. The path contains a *chain* $X \rightarrow Y \rightarrow Z$ such that $Y \in \mathcal{C}$
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Where do these rules come from?

https://kunalmenda.com/2019/02/21/causation-and-correlation/

Sampling from a Bayesian Network

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Given a Bayesian network, how do we sample from the joint distribution it defines?

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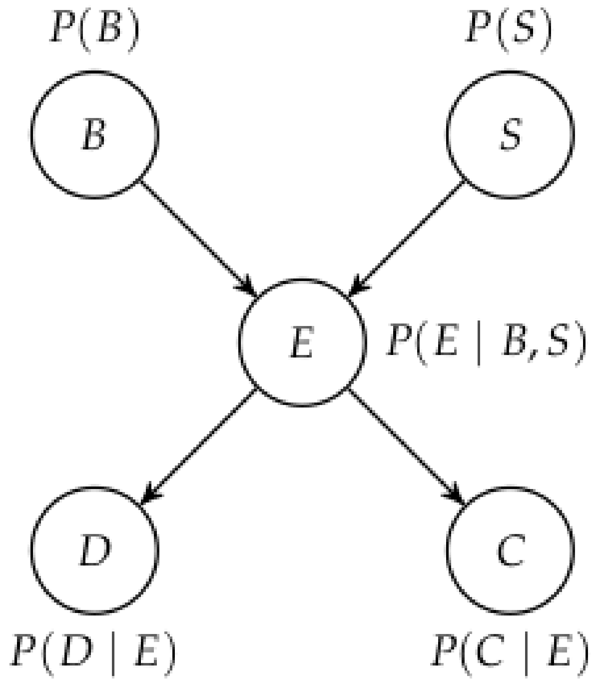

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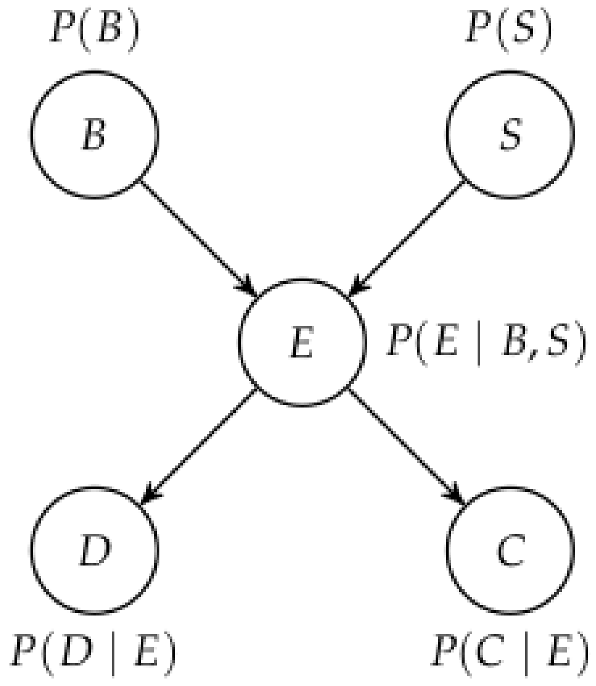
Sampling from a Bayesian Network

Given a Bayesian network, how do we sample from the joint distribution it defines?

1. Topoligical Sort (If there is an edge $A \to B$, then A comes before B)



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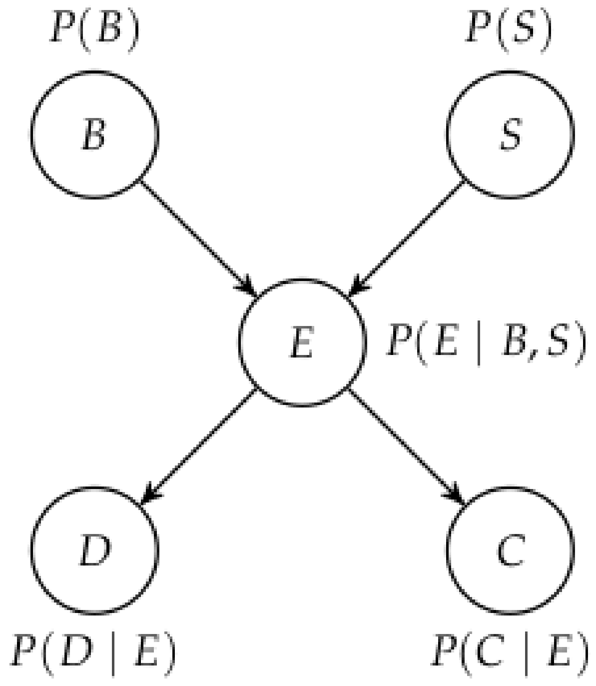
Sampling from a Bayesian Network

Given a Bayesian network, how do we sample from the joint distribution it defines?

1. Topoligical Sort (If there is an edge $A \to B$, then A comes before B)
2. Sample from conditional distributions in order of the topological sort



B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

Recap