

Simple Games

- Last time:

- Today:

Simple Games

- **Last time:**
 - Inference in Bayesian networks
 - Learning Bayesian networks
- **Today:**

Simple Games

- **Last time:**

- Inference in Bayesian networks
- Learning Bayesian networks

- **Today:**

- Games: a mathematical formalism for rational interaction
- Nash and other equilibria

Types of Uncertainty

Types of Uncertainty

Alleatory

Types of Uncertainty

Alleatory



Markov Decision Process

Types of Uncertainty

Alleatory

Epistemic (Static)

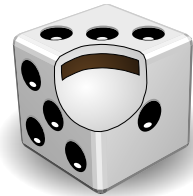


Markov Decision Process

Types of Uncertainty

Alleatory

Epistemic (Static)



Markov Decision Process

Reinforcement Learning

Types of Uncertainty

Alleatory



Markov Decision Process

Epistemic (Static)



Reinforcement Learning

Epistemic (Dynamic)



POMDP

Types of Uncertainty

Alleatory



Markov Decision Process

Epistemic (Static)



Reinforcement Learning

Epistemic (Dynamic)



POMDP

Interaction

Types of Uncertainty

Alleatory



Markov Decision Process

Epistemic (Static)



Reinforcement Learning

Epistemic (Dynamic)



POMDP

Interaction



Game

A win-win situation: International trade

A win-win situation: International trade

- Both Britain and Portugal need textiles and wine

A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: -1

A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: -1
- Portugal:
 - Producing wine: -1
 - Producing textiles: -3

A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: -1
- Portugal:
 - Producing wine: -1
 - Producing textiles: -3
- No production capacity limits

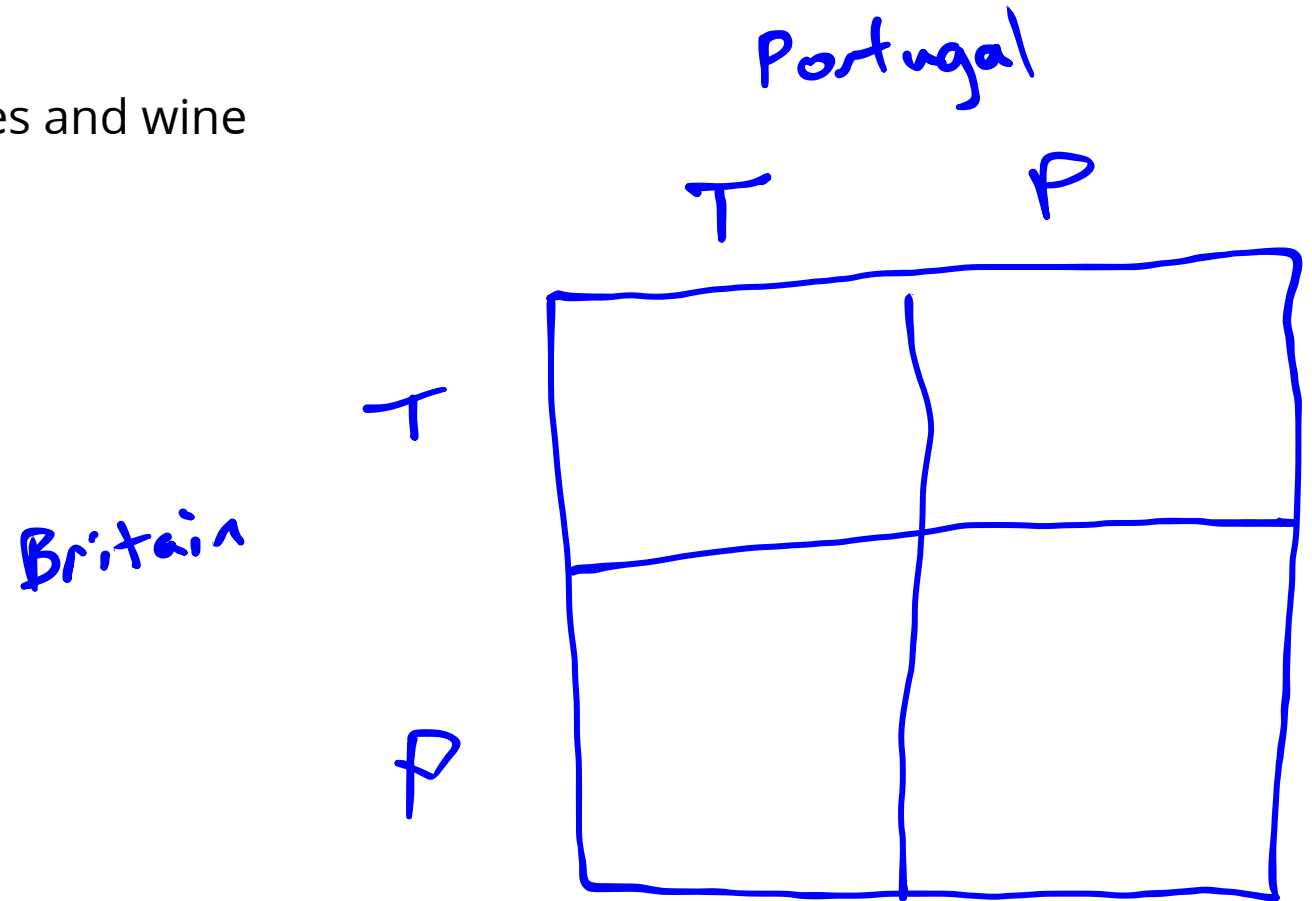
A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: -1
- Portugal:
 - Producing wine: -1
 - Producing textiles: -3
- No production capacity limits
- Each country can either
 - Produce their own goods
 - Trade at a price of 2

Produce our own
-4
Trade
-1-2

A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: -1
- Portugal:
 - Producing wine: -1
 - Producing textiles: -3
- No production capacity limits
- Each country can either
 - Produce their own goods
 - Trade at a price of 2



A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: -1
- Portugal:
 - Producing wine: -1
 - Producing textiles: -3
- No production capacity limits
- Each country can either
 - Produce their own goods
 - Trade at a price of 2

Portugal

T P

Britain

T P

	T	P
T		
P		-4, -4

A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: -1
- Portugal:
 - Producing wine: -1
 - Producing textiles: -3
- No production capacity limits
- Each country can either
 - Produce their own goods
 - Trade at a price of 2

Portugal

T P

Britain

T P

	T	P
T		-3, -3
P	-3, -3	-4, -4

A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: -1
- Portugal:
 - Producing wine: -1
 - Producing textiles: -3
- No production capacity limits
- Each country can either
 - Produce their own goods
 - Trade at a price of 2

Portugal

T P

Britain

T P

T	-2, -2	-3, -3
P	-3, -3	-4, -4

A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: ~~-1~~ **-2**
- Portugal:
 - Producing wine: -1
 - Producing textiles: -3
- No production capacity limits
- Each country can either
 - Produce their own goods
 - Trade at a price of 2

Britain

Portugal

	T	P
T	-2, -2	-3, -3
P	-3, -3	-4, -4

A win-win situation: International trade

- Both Britain and Portugal need textiles and wine
- Britain:
 - Producing wine: -3
 - Producing textiles: ~~-1~~ **-2**
- Portugal:
 - Producing wine: -1
 - Producing textiles: -3
- No production capacity limits
- Each country can either
 - Produce their own goods
 - Trade at a price of 2

Portugal

T P

T	-2, -2	-3, -3
P	-3, -3	-4, -4

Britain

-4, -2	-4, -3
-5, -3	-5, -4

P.

A more surprising example: The Prisoner's Dilemma

A more surprising example: The Prisoner's Dilemma

- 2 Alleged criminals are captured

A more surprising example:

The Prisoner's Dilemma

- 2 Alleged criminals are captured
 - Each can either keep silent or testify
- ■ other keeps silent -> minor conviction (1 year)
- ■ other testifies -> major conviction: 4 years
- testify -> 1 year removed from sentence

Ariana

	S	T
Jesse S	-1, -1	-4, 0
T	0, -4	-3, -3

Vocabulary

- Action, $a^i \in A^i$
- Policy, π^i (strategy)
- Joint action, a
- Joint policy, π (strategy profile)
- Reward, $R^i(a)$ ✓
- Joint reward, $R(a)$
- Joint utility, $U(\pi) = \sum_a R(a)\pi(a)$

Best Response: Given a joint policy of all other agents, π^{-i} , a best response is a policy π^i that satisfies

$$U^i(\pi^i, \pi^{-i}) \geq U^i(\pi^{i'}, \pi^{-i})$$

for all other $\pi^{i'}$.

Player 1

Britain

Portugal

	T	P
T	<u>-2, -2</u>	-3, -3
P	-3, -3	-4, -4

Equilibria

$-2, -2$

$-4, -4$

Equilibria

Dominant Strategy Equilibrium

$-2, -2$

$-4, -4$

Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.

-2, -2

-4, -4

Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.
- A joint policy where all agents use a dominant strategy is called a *dominant strategy equilibrium*.

-2, -2

-4, -4

Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.
- A joint policy where all agents use a dominant strategy is called a *dominant strategy equilibrium*.

-1, -1	-4, 0
0, -4	-3, -3

Portugal

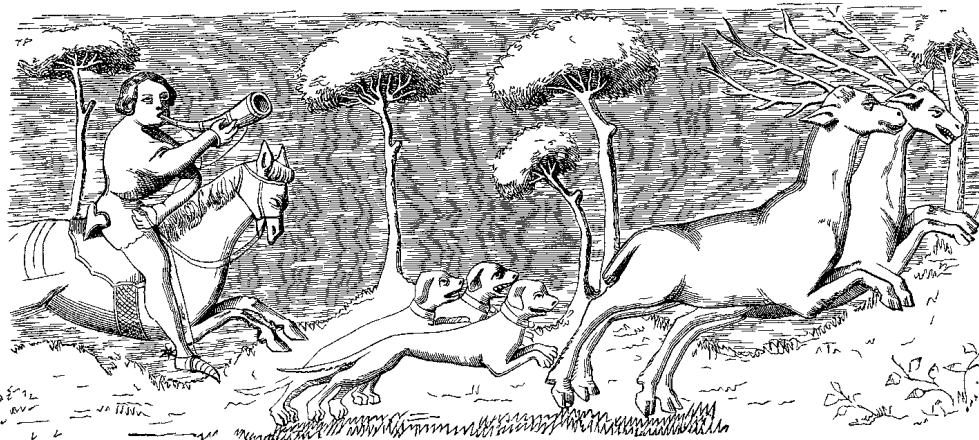
	T	P
T	-2, -2	-3, -3
P	-3, -3	-4, -4

Britain

Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.
- A joint policy where all agents use a dominant strategy is called a *dominant strategy equilibrium*.



Portugal

T P

Britain

T

-2, -2	-3, -3
-3, -3	-4, -4

P

Stag Hunt

Stag Hare

Stag

4, 4	1, 3
3, 1	2, 2

Hare

Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.
- A joint policy where all agents use a dominant strategy is called a *dominant strategy equilibrium*.

Nash Equilibrium

Portugal

	T	P
Britain		
T	-2, -2	-3, -3
P	-3, -3	-4, -4

Stag Hunt

	Stag	Hare
Stag	4, 4	1, 3
Hare	3, 1	2, 2

Equilibria

Dominant Strategy Equilibrium

- A *dominant strategy* is a policy that is a best response to all other possible agent policies.
- A joint policy where all agents use a dominant strategy is called a *dominant strategy equilibrium*.

Nash Equilibrium

- A *Nash equilibrium* is a joint policy in which all agents are following a best response

Player 2
Portugal

	T	P
Player 1 Britain	T	P
	-2, -2	-3, -3
	-3, -3	-4, -4

Handwritten notes: Arrows point from (-3, -3) to (-2, -2) and from (-4, -4) to (-3, -3), indicating that (T, T) is a dominant strategy equilibrium.

Stag Hunt

	Stag	Hare
Stag	4, 4	1, 3
Hare	3, 1	2, 2

Handwritten notes: Arrows point from (1, 3) to (4, 4) and from (3, 1) to (4, 4), indicating that (Stag, Stag) is a dominant strategy equilibrium.

Geopolitics

US

Soviet Union

	Attack	Deter	Disarm
Attack	^{NE} -100, -100	-100, -101	+1, -100
Deter	-101, 100	^{NE} -1, -1	-1, 2
Disarm	-100, +1	-2, -1	0, 0

Rock-paper scissors

		agent 2		
		rock	paper	scissors
agent 1	rock	0,0	-1,1	1,-1
	paper	1,-1	0,0	-1,1
	scissors	-1,1	1,-1	0,0

A two-player game is **zero sum** if

$$\sum_i R^i(a) = 0 \quad \forall a$$

- Pure strategy: $\pi^i(a) \in \{0, 1\}$
- Mixed strategy: all other strategies

$$\pi^i(a^i) = \frac{1}{3}$$

$$0.5(1) + 0.25(-1) + 0.25(0)$$

General approach to find Nash Equilibria

General approach to find Nash Equilibria

0 if Nash Equilibrium

minimize $\sum_i (U^i - U^i(\pi))$

subject to $U^i \geq U^i(a^i, \pi^{-i})$ for all i, a^i

$\sum_{a^i} \pi^i(a^i) = 1$ for all i

$\pi^i(a^i) \geq 0$ for all i, a^i

policies of all other players

π is probability

General approach to find Nash Equilibria

$$\begin{aligned}
 &\underset{\pi, U}{\text{minimize}} && \sum_i (U^i - U^i(\pi)) \\
 &\text{subject to} && U^i \geq U^i(a^i, \pi^{-i}) \text{ for all } i, a^i \\
 &&& \sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i \\
 &&& \pi^i(a^i) \geq 0 \text{ for all } i, a^i
 \end{aligned}$$

4	1	6	5	4	3	2	1	6	5	4	3	2	1	
2	3	4	2	3	4	2	3	4	2	3	4	2	3	2 3
1	4	1	4	1	4	1	4	1	4	1	4	1	4	1 4
Heg. Stability	Samaritan _{su}	Samaritan _{su}	Clock _{su}	Clock _{su}	Endless	Called Bluff	Bully	Unfair	Skewed BoS	Asym BoS	Chicken			—
3	3	4	2	3	4	2	3	4	2	3	4	2	3	3 2
1	4	1	4	1	4	1	4	1	4	1	4	1	4	1 4
Samson	Asym Sd _{su}	Asym Sd _{su}	Cycle _{su}	Cycle _{su}	Inspector	Self-serving _{su}	Protector _{su}	Protector _{su}	Favorites _{su}	Battle of Sexes	Asym BoS			
4	3	4	2	3	4	2	3	4	2	3	4	2	3	3 1
2	1	4	1	4	1	4	1	4	1	4	1	4	1	2 4
Delliah	Asym Sd _{su}	Asym Sd _{su}	Pursuit	Pareto	Missile Crisis	Self-serving _{su}	Protector _{su}	Protector _{su}	Hero	Favorites _{su}	Skewed BoS			X
5	3	4	2	3	4	2	3	4	2	3	4	2	3	2 1
2	1	4	1	4	1	4	1	4	1	4	1	4	1	3 4
Hostage	Benevolence _{su}	Benevolence _{su}	2nd Best _{su}	2nd Best _{su}	Big Bully	Tragedy	Delight _{su}	Pure Delight	Protector _{su}	Protector _{su}	Unfair			
6	3	4	2	3	4	2	3	4	2	3	4	2	3	1 2
1	4	1	4	1	4	1	4	1	4	1	4	1	4	3 4
Blackmailer	Benevolence _{su}	Benevolence _{su}	2nd Best _{su}	2nd Best _{su}	Hamlet	Total Conflict	Mixed Delight	Delight _{su}	Protector _{su}	Protector _{su}	Bully			I
1	3	4	2	3	4	2	3	4	2	3	4	2	3	1 3
2	1	4	1	4	1	4	1	4	1	4	1	4	1	2 4
Id. Hegemony	Samaritan _{su}	Samaritan _{su}	Revelation	Alibi	Asym Pd	Asym Pd	Asym Pd	Asym Pd	Asym Pd	Asym Pd	Asym Pd			—
2	3	4	2	3	4	2	3	4	2	3	4	2	3	1 4
1	4	1	4	1	4	1	4	1	4	1	4	1	4	2 3
Win-win	C. Aligned _{su}	C. Aligned _{su}	C. Assurance _{su}	C. Assurance _{su}	Stag Hunt	Asym Pd	Hamlet	Big Bully	Missile Crisis	Inspector	Endless			—
3	3	4	2	3	4	2	3	4	2	3	4	2	3	1 4
1	4	1	4	1	4	1	4	1	4	1	4	1	4	3 2
R Assurance	Commons _{su}	Commons _{su}	Coordination _{su}	Coordination _{su}	R Assurance	Alibi	2nd Best _{su}	2nd Best _{su}	Pareto	Cycle _{su}	Clock _{su}			
4	3	4	2	3	4	2	3	4	2	3	4	2	3	2 4
1	4	1	4	1	4	1	4	1	4	1	4	1	4	3 1
R Assurance	Commons _{su}	Commons _{su}	Coordination _{su}	Coordination _{su}	R Assurance	Revelation	2nd Best _{su}	2nd Best _{su}	Pursuit	Cycle _{su}	Clock _{su}			X
5	3	4	2	3	4	2	3	4	2	3	4	2	3	3 4
2	1	4	1	4	1	4	1	4	1	4	1	4	1	2 1
Row Aligned	Harmony _{su}	Harmony-mix	Commons _{su}	Commons _{su}	Row Aligned _{su}	Samaritan _{su}	Benevolent _{su}	Benevolent _{su}	Asym Sd _{su}	Asym Sd _{su}	Samaritan _{su}			
6	3	4	2	3	4	2	3	4	2	3	4	2	3	3 4
1	4	1	4	1	4	1	4	1	4	1	4	1	4	1 2
Row Aligned	Harmony-pure	Harmony	Commons _{su}	Commons _{su}	Row Aligned _{su}	Samaritan _{su}	Benevolent _{su}	Benevolent _{su}	Asym Sd _{su}	Asym Sd _{su}	ActiveSam _{su}			I
2	3	4	2	3	4	2	3	4	2	3	4	2	3	2 4
1	4	1	4	1	4	1	4	1	4	1	4	1	4	1 3
No Conflict	C. Aligned _{su}	C. Aligned _{su}	C. Assurance _{su}	C. Assurance _{su}	Win-win	Id. Hegemony	Blackmailer	Hostage	Delliah	Samson	Heg. Stability			2
3	3	4	2	3	4	2	3	4	2	3	4	2	3	2
1	4	1	4	1	4	1	4	1	4	1	4	1	4	1

Every finite game has a Nash Equilibrium

Every finite game has a Nash Equilibrium

EQUILIBRIUM POINTS IN N -PERSON GAMES

BY JOHN F. NASH, JR.*

PRINCETON UNIVERSITY

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n -person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n -tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n -tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n -tuple counters another if the strategy of each player in the countering n -tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n -tuple. A self-countering n -tuple is called an equilibrium point.

The correspondence of each n -tuple with its set of countering n -tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if P_1, P_2, \dots and $Q_1, Q_2, \dots, Q_n, \dots$ are sequences of points in the product space where $Q_n \rightarrow Q$, $P_n \rightarrow P$ and Q_n counters P_n then Q counters P .

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem¹ that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"² and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

* The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the A. E. C. for financial support.

¹ Kakutani, S., *Duke Math. J.*, 8, 457-459 (1941).

² Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behaviour*, Chap. 3, Princeton University Press, Princeton, 1947.

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

A correspondence $f: X \rightarrow X$ has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

- (1) X is a non-empty, closed, bounded, and convex set.
- (2) $f(\mathbf{x})$ is non-empty for any \mathbf{x} .
- (3) $f(\mathbf{x})$ is convex for any \mathbf{x} .
- (4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

A correspondence $f: X \rightarrow X$ has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

- (1) X is a non-empty, closed, bounded, and convex set.
- (2) $f(\mathbf{x})$ is non-empty for any \mathbf{x} .
- (3) $f(\mathbf{x})$ is convex for any \mathbf{x} .
- (4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

- Let x be a strategy profile, π .

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

A correspondence $f: X \rightarrow X$ has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

- (1) X is a non-empty, closed, bounded, and convex set.
- (2) $f(\mathbf{x})$ is non-empty for any \mathbf{x} .
- (3) $f(\mathbf{x})$ is convex for any \mathbf{x} .
- (4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

- Let x be a strategy profile, π .
- Let f be BR , that is, the best response operator

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

A correspondence $f: X \rightarrow X$ has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

- (1) X is a non-empty, closed, bounded, and convex set.
- (2) $f(\mathbf{x})$ is non-empty for any \mathbf{x} .
- (3) $f(\mathbf{x})$ is convex for any \mathbf{x} .
- (4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

- Let x be a strategy profile, π .
- Let f be BR , that is, the best response operator
- A fixed point of BR is a Nash Equilibrium

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

A correspondence $f: X \rightarrow X$ has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

- (1) X is a non-empty, closed, bounded, and convex set.
- (2) $f(\mathbf{x})$ is non-empty for any \mathbf{x} .
- (3) $f(\mathbf{x})$ is convex for any \mathbf{x} .
- (4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

- Let x be a strategy profile, π .
- Let f be BR , that is, the best response operator
- A fixed point of BR is a Nash Equilibrium
- The BR operator and policy space for finite games meet the conditions above

Every finite game has a Nash Equilibrium

Kakutani's fixed-point theorem

A correspondence $f: X \rightarrow X$ has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

- (1) X is a non-empty, closed, bounded, and convex set.
- (2) $f(\mathbf{x})$ is non-empty for any \mathbf{x} .
- (3) $f(\mathbf{x})$ is convex for any \mathbf{x} .
- (4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

- Let x be a strategy profile, π .
- Let f be BR , that is, the best response operator
- A fixed point of BR is a Nash Equilibrium
- The BR operator and policy space for finite games meet the conditions above
- BR has a fixed point for every finite game, i.e. every finite game has a Nash Equilibrium

Battle of the Sexes

Battle of the Sexes

- Gabby and Max are going on a date

Battle of the Sexes

- Gabby and Max are going on a date
- Gabby wants to go to a football game

Battle of the Sexes

- Gabby and Max are going on a date
- Gabby wants to go to a football game
- Max wants to go to a movie (He is a rom-com superfan)

Battle of the Sexes

- Gabby and Max are going on a date
- Gabby wants to go to a football game
- Max wants to go to a movie (He is a rom-com superfan)

A hand-drawn blue payoff matrix for the Battle of the Sexes game. The matrix is a square divided into four quadrants by a horizontal and vertical line. The labels 'G' and 'M' are written in blue ink around the matrix. 'G' is written to the left of the matrix, and 'M' is written above the matrix. The quadrants are labeled with 'G' and 'M' as follows: the top-left quadrant is labeled 'G' above it and 'G' to its left; the top-right quadrant is labeled 'M' above it and 'G' to its left; the bottom-left quadrant is labeled 'G' above it and 'M' to its left; the bottom-right quadrant is labeled 'M' above it and 'M' to its left.

	G	M
G		
M		

Battle of the Sexes

- Gabby and Max are going on a date
- Gabby wants to go to a football game
- Max wants to go to a movie (He is a rom-com superfan)

		M	
		G	M
G	G	2, 1	0, 0
	M	0, 2	1, 2

Battle of the Sexes

- Gabby and Max are going on a date
- Gabby wants to go to a football game
- Max wants to go to a movie (He is a rom-com superfan)

Correlated Equilibrium

- A *correlated joint policy* is a single distribution over the joint actions of all agents.
- A *correlated equilibrium* is a correlated joint policy where no agent i can increase their expected utility by deviating from their current action to another.

A hand-drawn payoff matrix for the Battle of the Sexes game. The matrix is a 2x2 grid with 'G' and 'M' as row and column headers. The payoffs are written in the cells: (G, G) is 2,1; (G, M) is 0,0; (M, G) is 0,2; and (M, M) is 1,2.

	G	M
G	2,1	0,0
M	0,2	1,2

Recap

Recap

- Games provide a mathematical framework for analyzing interaction between rational agents

Recap

- Games provide a mathematical framework for analyzing interaction between rational agents
- Games may not have a single "optimal" solution; instead there are equilibria

Recap

- Games provide a mathematical framework for analyzing interaction between rational agents
- Games may not have a single "optimal" solution; instead there are equilibria
- If every player is playing a best response, that policy profile is a Nash Equilibrium

Recap

- Games provide a mathematical framework for analyzing interaction between rational agents
- Games may not have a single "optimal" solution; instead there are equilibria
- If every player is playing a best response, that policy profile is a Nash Equilibrium
- Every finite game has at least one Nash Equilibrium (pure or mixed)