Searching for cut-free sequent systems for SB and S5

Truls A. Pedersen

Abstract

In this paper we investigate the validity of the cut-elimination theorem for two Gentzen systems for the modal logics SB (KTB) and S5, including some variations of rule-set for the SB-system. The systems are formulated with the *impossibility* operator, $\div$, as their only modal connective. We show that the proposed reasoning systems are sound with respect to the semantics of $\mathcal{IC}$-algebras. We are not able to provide a full proof of the cut-elimination theorem for either of the systems, and we show a/the complicating factor which arises in the different approaches.

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Chapter 1

Introduction

1.1 History and Motivation

In 1918 C. I. Lewis published "*A Survey of Symbolic Logic*" [8] where he presented a logical system for *Strict Implication*. This system is now usually referred to as the Survey System. This was an attempt to rectify classical propositional logic's inability to adequately handle the notion of *implication*. Lewis' Survey System turned out not to be exactly what he had thought it would be; the axioms were found to be too permissive, and in 1932 [9] Lewis defines five systems, **S1**, **S2**, **S3**, **S4**, and **S5**. These systems and variants of these systems have since gained a lot of interest. Lewis belongs, according to [6], to the *syntactic tradition* of modal logic, defining no formal semantics for his systems. This, of course, makes the question of completeness impossible to answer for his logics. Lewis' syntactical logical systems were merged not long after they were published with the *algebraic tradition*, starting with Boole, in the form of a formal semantics being classes of modal algebras - a modal algebra is now defined as a Boolean algebra extended with modal operators.

The reasoning systems provided by Gentzen are powerful reasoning systems over a simple alphabet. The syntactic objects in his calculi are the *sequents*

$$\gamma_1, \gamma_2, \ldots, \gamma_n \vdash \delta_1, \delta_2, \ldots, \delta_m$$

where $\Gamma = \gamma_1, \gamma_2, \ldots, \gamma_n$ and $\Delta = \delta_1, \delta_2, \ldots, \delta_m$ are sequences of formulae. The sequent system is a very small extension of the manipulated language adding to the logical formulae only two symbols, comma(,) and turnstile($\vdash$)[1], thus the syntactic objects over which these sequent caluli operate are structurally simple. In the case of Gentzen's original calculus, LK, for classical logic, all the rules in the system, save the *cut*-rule, have the *subformula property*, superficially this is to say that the complexity of the premisses of a sequent is strictly less than the complexity of the conclusion. The consequence of this property is that in a search for a proof of a sequent (or formula), the premisses we are required to prove in a given step are always "simpler" than the conclusion, and thus we avoid the possibility of infinite proof-search and circularity. Gentzen was able to prove that the *cut*-rule is eliminable in the system; whatever could be proven in the system using the *cut*-rule, could also be proven without using it. The Gentzen sequent system has inspired the creation of several sequent systems for a variety of logics, including systems for modal logics.

Several reasoning systems for the modal logics SB and S5 have been proposed, but no pure cut-free Gentzen-style sequent systems have been supplied. Attempts to provide a pure Gentzen-style sequent system for S5 dates back to at least 1957 when, in [10], a pure Gentzen-style system for S5 was proposed. The provided system turned out not to fulfill cut elimination. And since then several versions of Gentzen systems and similar systems have been provided.

Some cut-free reasoning systems for S5 has been provided, however these systems always include some factor foreign to pure Gentzen-type systems; they either rely on structures (like hypersequents) more involved than

[1] Gentzen placed an arrow between the antecendent(Γ) and the succedent(Δ). The $\vdash$ symbol was originally used by G. Frege.

regular sequents, use deep inference, or even include semantic information in the inference rules. In this paper we investigate the cut-elimination theorem for some pure Gentzen-style sequent systems with algebraic semantic.

A hypersequent is a sequence of regular sequents, e.g.

$$\Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \ \ldots \ \mid \Gamma_n \vdash \Delta_n$$

In addition to the regular sequent rules, a hypersequent reasoning system will include more structural rules; these are seperated into *external* rules and *internal* rules. The internal rules are rules similar to those found in a standard Gentzen-style system. The examples given in [2] are those of the contraction rules. The external contraction rule is:

$$\frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta \mid H}{G \mid \Gamma \vdash \Delta \mid H}$$

where G and H represent possibly empty hypersequents, and the internal contraction rules are

$$\frac{G \mid A, A, \Gamma \vdash \Delta \mid H}{G \mid A, \Gamma \vdash \Delta \mid H} \quad \text{and} \quad \frac{G \mid \Gamma \vdash \Delta, A, A \mid H}{G \mid \Gamma \vdash \Delta, A \mid H}$$

Hypersequent calculi are extensions of the sequent calculi and, as such, all standard sequent calculi can be regarded as hyper sequent calculi. The hypersequent calculi bring with them a slightly more complex structure and the additional (external) rules required to manipulate the sequents, or components as they are also called. In [2] a cut-free hypersequent calculus, *GS5*, is given for S5 by extending a reasoning system for S4 by a single rule of *modalized splitting*:

$$\frac{G \mid \Box\Gamma_1, \Gamma_2 \vdash \Box\Delta_1, \Delta_2 \mid H}{G \mid \Box\Gamma_1 \vdash \Box\Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid H} \ (MS)$$

More recently [3] introduced the hypersequent calculus, $CSS5_S$ over a limited set of connectives, $\{\wedge, \neg\}$, consisting of the (modal) rules:

$$\frac{G \mid \alpha, \Box\alpha, M \vdash N}{G \mid \Box\alpha, M \vdash N} \ \Box A_1 \qquad \frac{G \mid M \vdash N \mid \ \vdash \alpha}{G \mid M \vdash N, \Box\alpha} \ \Box K$$

$$\frac{G \mid \Box\alpha, M \vdash N \mid \alpha, P \vdash Q}{G \mid \Box\alpha, M \vdash N \mid P \vdash Q} \ \Box A_2$$

where α is a formula, G a possibly empty hypersequent and M, N, P, Q are finite multisets of formulae. Considering the rules in a bottom-up way they seem informally to have a quite simple meaning, in terms of its relational semantic.

The syntactic objects for the hypersequent calculi are not much more complicated than the standard Gentzen-type systems, extending it by only one symbol ($|$), and the required rule-sets are quite simple and appealing and enjoy the subformula property.

The system provided in [4] is a Gentzen calculus over normal sequents where all the rules have the subformula property and which admits cut, however the modal rules have non-local requirements. These requirements are ensured by appropriate renaming of labels to keep track of *connections* percolating through a proof. Even though the syntactic objects of the calculus are exactly those of the standard Gentzen-type calculus, the formulae are those of monadic predicate logic, and no longer of modal predicate logic. This alteration permits formulae to be labled in a way which reflects the intended relational semantics. This is called an *impure* system because of this, e.g. [7]. The non-local requirements on the modal rules also represent a complicating factor; a derivation must be verified after it has been constructedby a given algorithm to constitute a proof.

Other systems are the Display Logic systems which adds unary *structural connectives* to the language manipulated by the reasoning system; $\circ$ representing a separator similar to comma (,) in standard Gentzen systems,

$*$ representing a shifting of the structure, and $\bullet$ representing that its argument is intentional. In [7] the basic rules are given as follows:

$$\begin{array}{cccc}
(1) & (2) & (3) & (4) \\[1em]
\dfrac{X \circ Y \vdash Z}{X \vdash Z \circ *Y} & \dfrac{X \vdash Y \circ Z}{X \circ *Z \vdash Y} & \dfrac{X \vdash Y}{*Y \vdash *X} & \dfrac{\bullet X \vdash Y}{X \vdash \bullet Y} \\[1em]
\dfrac{X \vdash Z \circ *Y}{Y \vdash *X \circ Z} & \dfrac{X \circ *Z \vdash Y}{*Y \circ X \vdash Z} & \dfrac{*Y \vdash *X}{X \vdash **Y} &
\end{array}$$

The structural inference, for example by (1), allows the shifting of a structure, Y, from the left hand side of the turnstile to the right hand side by applying a $*$ to it. The syntactic objects is a larger extension of the logical language than that of Gentzens system, and also, as is mentioned in [2], the structural connectives can be arbitrarily nested which usually violates the subformula property of the system.

In [9] the systems **S1** through **S5** were introduced with $\diamond$ as the single modal connective. Since then modal algebras have been, to our knowledge, represented as Boolean algebras extended with a unary modal connective representing possibility ($\diamond$ or m), neccessity ($\Box$ or l) or a combination of these ($\diamond$ and $\Box$ or otherwise m and l). The only modal connective in Lewis' earlier work [8] was the 'impossibility' operator. In said work, the unary connectives were negation ($-$) and impossibility ($\sim$) symbols which were chosen for "typographical convenience", stating only that $- \sim p$ should be interpreted as *possible* and $\sim -p$ as *neccessary.* Another early work, MacColl's logical system, was also equipped with an *impossibility constant*, η. Although it is not a crucial point, the fact that the early "ground-breakers" in modal logic found *impossibility* a primitive notion, also motivates an investigation of a modern system with impossibility. This fact alone is not the only motivation, of course, there is also a technical appeal in the modal algebra defined in terms of impossibility.

A modal algebra is an algebra $\langle \underline{A}, \wedge, \vee, -, 1, \Box \rangle$, where $\langle \underline{A}, \wedge, \vee, -, 1 \rangle$ is a Boolean algebra and $\Box$ is a unary operator which satisfies one or more axioms depending on the modal system studied. $\Diamond$ ("possibility") is commonly defined in terms of $\Box$ ("neccessity") as $\Diamond = -\Box-$, or $-\Box = \Diamond-$. Considering instead the modal operator $\div$ ("impossibility") the interdefinability of $\Box = \div-$ and $\Diamond = -\div$ is simply the associativity of $\div$ and $-$: $-(\div-) = (-\div)-$. An $\mathcal{IC}$-algebra ("intuitionistic-classical") introduced in [1] is an algebra $\langle \underline{A}, \wedge, \vee, -, \div, 1 \rangle$, where $\langle \underline{A}, \wedge, \vee, -, 1 \rangle$ is a boolean algebra and $\div$ is a unary operator satisfying one or more axioms depending on the modal algebra studied. Some common axioms are shown in Table 1.1 in terms of "impossibility" ($\div$) and in terms of the standard modal operators of "neccessity" ($\Box$) and "possibility" ($\Diamond$), where $\leq$ is defined in the usual way as $x \leq y$ *iff* $x \wedge y = x$ *iff* $x \vee y = y$.

In [1] the correspondence between topological algebras (Boolean algebras extended with the "*interior*"-operator, i) and $\mathcal{IC}$ algebras is shown. This established that $\mathcal{IC}$ algebras can be regarded as topological algebras where $\div$ is the *interior* of $-$, i.e. $\div x = i(-x)$. In topological algebras some elements are referred to as *open*, some as *closed*, and some as *clopen.* We do not delve deeply into the topic concerning topological algebras, but we will return to them somewhat in Chapter 4 where we describe a reasoning system for S5. A result from [1] is that $\mathcal{IC}$ algebras as semantics for S4 are topological algebras where the *open* elements forms a Heyting-algebra embedded in the algebra, over which $\div$ corresponds to intuitionistic negation.

The fact that $\div$ can be regarded as intuitionistic negation over a substructure and that the usual modalities ($\Box$ and $\diamond$) arise as interactions between boolean negation ($-$) and the impossibility operator/intuitionistic negation, ($\div$), motivates investigating a logic over $\div$.

And (without claiming philosophical merit) a dual connective, $- \div -$, could be described to express the the notion of 'falsifiability', another concept which has been considered interesting in philosophy, e.g., as the demarcation of the scope of empirical science.

1.2 Definitions and Conventions

In Table 1.1 are some common modal axioms, expressed in terms of "impossibility" ($\div$) and also in terms of "neccessity" ($\Box$) and "possibility" ($\diamond$).

K :	$\div(x \vee y) = \div x \wedge \div y$	$\Box(x \wedge y) = \Box x \wedge \Box y$
	$\div 0 = 1$	$\Box 1 = 1$
T :	$\div x \leq -x$	$\Box x \leq x$
B :	$x \leq \div \div x$	$x \leq \Box \Diamond x$
4 :	$\div x \leq \div - \div x$	$\Box x \leq \Box\Box x$
5 :	$- \div x \leq \div \div x$	$\Diamond x \leq \Box \Diamond x$

Table 1.1: Modal axioms

We name the modal logics as shown in Table 1.2, depending on the axioms satisfied by the $\div$ operator. S is often referred to by other names, e.g. T or M.

K	:	**K**
S	:	**K** + **T**
SB	:	**K** + **T** + **B**
S4	:	**K** + **T** + **4**
S5	:	**K** + **T** + **5**
	or	S4 + **B**

Table 1.2: Common modal logics

In attempting to provide a cut-free reasoning system for a modal logic, we extend the classical reasoning system, LK, as shown in Table 1.3, with rules for handling the new connective, $\div$, and then try to show that the (*cut*)-rule is admissible.

We consider the classical sequent calculus with sequents $\Gamma \vdash \Delta$, where Γ, Δ and are finite sets of formulae, and therefore we don't provide rules for contraction and permutation.

In an inference rule the sequent(s) above the line are called the *premiss/premises* of the rule, and the sequent below the line is called the *conclusion*. The formula(e) shown explicitly in the conclusion are called the *principal formula(e)*, and the formula(e) shown explicitly in the premises are called *side formula(e)*. The other formulae shown in a rule are called *parametric formulae* and we refer to these as the *context*. A rule has the *subformula property* if, and only if, each side formula is a subformula of a principal formula.

The following conventions are used in this text:
In formulae capital Latin letters, $A, B, C, \ldots$, refer to single, arbitrary formulae, and capital Greek letters, $\Gamma, \Delta, \Phi, \ldots$, refer to arbitrary finite sets of formulae. In algebraic expressions, capital Latin letters, $G, D, F, \ldots$, refer to arbitrary finite sets of algebraic expressions, and lower case Latin letters, $a, b, c, \ldots$, refer to an element in a model/algebra.

For a formula, A, we define the *complexity of the formula*, $|A|$, inductively:

- $|p| = 1$, for atomic p
- $|\neg A| = |\div A| = 1 + |A|$, and

$$\frac{}{p \vdash p}\ (ax)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg \vdash) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\vdash \neg)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge \vdash) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\vdash \wedge)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee \vdash) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vdash \vee)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to\vdash) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\vdash\to)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (W \vdash) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ (\vdash W)$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2 \vdash A, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (cut)$$

Table 1.3: The LK system and the *cut*-rule

- $|A \wedge B| = |A \vee B| = |A \to B| = 1 + |A| + |B|$.

The *derivation* of a sequent, $\Gamma_n \vdash \Delta_n$, is a finite sequence of sequents $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_{n-1} \vdash \Delta_{n-1}, \Gamma_n \vdash \Delta_n$ such that for each sequent $\Gamma_i \vdash \Delta_i$ there is a rule, $\dfrac{\Gamma_{j_1} \vdash \Delta_{j_1} \quad \ldots \quad \Gamma_{j_m} \vdash \Delta_{j_m}}{\Gamma_i \vdash \Delta_i}\ R$ with $j_l < i$ for each $1 \le l \le m$.

A derivation, δ, of a sequent $\Gamma_n \vdash \Delta_n$, where the last applied rule is R requiring m premises, is represented as

$$\delta = \left\{ \frac{\begin{array}{c} \vdots\ \delta_1 \\ \Gamma_{n-m} \vdash \Delta_{n-m} \end{array} \quad \ldots \quad \begin{array}{c} \vdots\ \delta_m \\ \Gamma_{n-1} \vdash \Delta_{n-1} \end{array}}{\Gamma_n \vdash \Delta_n}\ R \right.$$

Given a derivation, δ, the *length* of the derivation, $|\delta|$, is defined inductively. The length of a derivation of an axiom, i.e. applying a rule which takes no premises, is 1. The length of a derivation where the last rule is R taking m, $m \ge 1$, premises (as shown above) is $1 + \sum_{j=1}^{m} |\delta_j|$.

A rule is *admissible* in a system if whenever there are derivations for all the premises of the rule, then there is also a derivation of the conclusion of the rule.

In the (cut)-rule, the side formula, A, is called the *cut-formula*.

The standard approach of showing that the (cut)-rule is admissible in a system is by nested induction on the complexity of the cut-formula, then sub-induction on the length of the derivation of the left premiss, and finally on the length of the derivation of the right premiss. In the inductive step in such a proof, we generally split the case of a non-atomic cut-formulae occurring in premises which has derivations of length greater than 1, into cases where the cut-formula is principal in both cases, principal in one, but not the other, and not principal in either. Our attempted proofs are variations of such nested induction.

Given a model, $M = \langle \underline{M}, \wedge, \vee, -, 1, \div \rangle$, a valuation of a set of variables is a map $v : V \to \underline{M}$. Such a valuation induces a valuation of formulae, $\overline{v}$, defined inductively. $\overline{v}(x) = v(x)$ for a variable x, $\overline{v}(-A) = -\overline{v}(A)$ for a formula A, etc. Given a model, M, and a valuation v, the valuation of a formula, A, is denoted $[\![A]\!]_v^M$.

Chapter 2

Cut-free SB system

2.1 Introduction

To provide a cut-free reasoning system for the SB logic, several combinations of candidate rules have been considered. In none of the attempted systems have we been able to show admissibility of the (cut)-rule. In each of the following sections we provide a set of sound rules and show a/the difficulty in each case.

Other systems have been examined; in all of the systems we have attempted to show admissibility of the (cut)-rule we encounter a problem we have not been able to handle. The problem is always the lack of a ($\vdash \div$)-rule, allowing us to introduce a single $\div$ in the right hand side of a sequent.

A program, ABA - Chapter 3, has been developed for invalidating sequents/rules. However, as we shall see in Section 3.7, for any finite collection of finite SB models, there are formulae for which we can construct a counter model, but which does not have a counter model in the given collection. This means that, when dealing with SB, there will always be a possibility for *false negatives* when querying any finite collection of finite models for a counter model to a formula.

2.2 (K) (T) (B)

In trying to supply a cut-free Gentzen reasoning system for the system **SB**, we first try to complete a standard cut-elimination proof for straightforward rules motivated by the satisfied axioms. We try to supply sound rules with the subformula property. In any algebra satisfying **K.** we have the following: $x \leq y \Rightarrow \div x \geq \div y$. Also we have that $x \leq (y_0 \vee \ldots \vee y_n) \Rightarrow \div x \geq (\div y_0 \wedge \ldots \wedge \div y_n)$, motivating the straighforward rule $\dfrac{A \vdash \Gamma}{\div\Gamma \vdash \div A}$ (K). From **T.** it follows directly that $\div x \leq \neg x$, or $\div x \wedge \neg x = \div x$, motivating the rule $\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}$ (T). Similarly, from **B.** it follows that $\dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash \div \div A, \Delta}$ (B) is sound. All of these rules satisfy the subformula property.

We extend the LK system with these three rules

$$\frac{A \vdash \Gamma}{\div\Gamma \vdash \div A}\ (K) \qquad \frac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\ (T) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \div \div A, \Delta}\ (B)$$

Lemma 2.2.1. *The rules are sound, and (T) is also invertible.*

Proof.

- We show that $\dfrac{A \vdash \Gamma}{\div\Gamma \vdash \div A}$ (K) is sound by showing that such an inference perserves validity, i.e., assuming the sequent in the premiss is valid, so is the sequent in the conclusion. Given an arbitrary model, M, and an arbitrary valuation, v, assume

  $$[\![A]\!]_v^M \leq [\![\Gamma]\!]_v^M$$

  where $[\![\Gamma]\!]_v^M = [\![\gamma_1]\!]_v^M \vee [\![\gamma_2]\!]_v^M \vee \ldots \vee [\![\gamma_n]\!]_v^M$ and $\gamma_i \in \Gamma$ for $1 \leq i \leq n = |\Gamma|$.

  Let $a = [\![A]\!]_v^M$ and $\bigvee G = [\![\Gamma]\!]_v^M$, then

  $$\begin{array}{lll} a \leq \bigvee G & \Rightarrow^K & \div \bigvee G \leq \div a \\ & \Leftrightarrow^K & \bigwedge \div G \leq \div a \end{array}$$

  Thus in model M, under valuation v, we have $[\![\div\gamma_1]\!]_v^M \wedge [\![\div\gamma_2]\!]_v^M \wedge \ldots \wedge [\![\div\gamma_n]\!]_v^M \leq [\![\div A]\!]_v^M$ for $\gamma_i \in \Gamma$, which is to say that the sequent $\div\Gamma \vdash \div A$ is valid.

- $\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}$ (T) is sound and invertible. We show that in an arbitrary model, M, and for an arbitrary valuation, v, the sequent in the premiss is valid *iff* the sequent in the conclusion is valid.

  Let M be an arbitrary model and v be an arbitrary valuation. Let $a = [\![A]\!]_v^M$, $\bigwedge G = [\![\Gamma]\!]_v^M$ and $\bigvee D = [\![\Delta]\!]_v^M$:

  $$\begin{array}{lll} \bigwedge G \wedge \div a \leq a \vee \bigvee D & \Leftrightarrow & \bigwedge G \wedge \div a \wedge \neg a \leq \bigvee D \\ & \Leftrightarrow^T & \bigwedge G \wedge \div a \leq \bigvee D \end{array}$$

- We show that $\dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash \div\div A, \Delta}$ (B) is sound. Given an arbitrary model, M, and an arbitrary valuation, v, if the sequent in the premiss is valid in this model, and under this valuation, then so is the sequent in conclusion.

  Let $a = [\![A]\!]_v^M$, $\bigwedge G = [\![\Gamma]\!]_v^M$ and $\bigvee D = [\![\Delta]\!]_v^M$, then

  $$\bigwedge G \leq a \vee \bigvee D \quad \Rightarrow^B \quad \bigwedge G \leq \div\div a \vee \bigvee D$$

$\square$

We exchange, for convenience, the rule (T) with (T') allowing application to a set of formulae, a property which is applicable in simplifying one case in the attempt of a straightforward proof of cut-elimination. The new rule, (T'), is equivalent with multiple applications of the former, (T), and it is clear that we do not change the provable sequents.

We consider the system which extends LK with the following three rules.

$$\frac{A \vdash \Gamma}{\div\Gamma \vdash \div A}\ (K) \qquad \frac{\Gamma, \div\Phi \vdash \Phi, \Delta}{\Gamma, \div\Phi \vdash \Delta}\ (T') \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \div\div A, \Delta}\ (B)$$

An attempt to show admissibility of (*cut*) in this system can be a proof by induction on the complexity of the cut-formula, then the length of the derivation of the left premiss, and the length of the derivation of the right premiss. In the cases where the last applied rule in the right premiss is (B), we have a problematic case. If the cut-formula is principal in both premisses, the last rule applied in the left premiss must be $(W\vdash)$, (K) or (T'). When the last rule applied in the left premiss is $(W\vdash)$ we can weaken with A instead of $\div\div A$. For the (K) and (T') cases we get very similar problems. In both of these cases, establishing that the sound rule $\dfrac{\Gamma, \div\div A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (B')$ is admissible, we would solve these cases. No proof has been found for the derivability of this rule. In the case when the last applied rule in the derivation of the left premiss is (K) and the last rule applied in the derivation of the right premiss is (B) we have the following situation. (We see that we could have handled this case if the (B') rule was admissible.)

$$\dfrac{\dfrac{B \vdash \div A, \Gamma}{\div\Gamma, \div\div A \vdash \div B}\ (K) \quad \dfrac{\Gamma' \vdash A, \Delta'}{\Gamma' \vdash \div\div A, \Delta'}\ (B)}{\div\Gamma, \Gamma' \vdash \div B, \Delta'}\ (cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{B \vdash \div A, \Gamma}{\div\Gamma, \div\div A \vdash \div B}\ (K)}{\div\Gamma, A \vdash \div B}\ (B') \quad \Gamma' \vdash A, \Delta'}{\div\Gamma, \Gamma' \vdash \div B, \Delta'}\ (cut)$$

However, as we can not show admissibility of this rule, we need to handle this case differently. We look at the possible derivations of the left premiss, and at the last rule applied with $\div A$ principal. Unless $\div A$ was introduced by weakening, it was introduced by either (K) or (B). If it was introduced by (B), then $A = \div A'$

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{A \vdash \Gamma_2}{\div\Gamma_2 \vdash \div A}\ (K)\\ \vdots \\ B \vdash \div A, \Gamma\end{array}}{\div\Gamma, \div\div A \vdash \div B}\ (K) \quad \dfrac{\Gamma' \vdash A, \Delta'}{\Gamma' \vdash \div\div A, \Delta'}\ (B)}{\div\Gamma, \Gamma' \vdash \div B, \Delta'}\ (cut) \quad \text{or} \quad \dfrac{\dfrac{\begin{array}{c}\dfrac{\Gamma_2 \vdash A', \Delta_2}{\Gamma_2 \vdash \div\div A', \Delta_2}\ (B)\\ \vdots \\ B \vdash \div\div A', \Gamma\end{array}}{\div\Gamma, \div\div\div A' \vdash \div B}\ (K) \quad \dfrac{\Gamma' \vdash \div A', \Delta'}{\Gamma' \vdash \div\div\div A', \Delta'}\ (B)}{\div\Gamma, \Gamma' \vdash \div B, \Delta'}\ (cut)$$

If $\div A$ (i.e. $\div\div A'$) was introduced by the (B)-rule, we can drop both applications of (B) and apply *cut* on the formula, $\div A'$, with lower complexity. Otherwise, if $\div A$ was introduced by the (K)-rule we have the following situation

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{A \vdash \Gamma_2}{\div\Gamma_2 \vdash \div A}\ (K)\\ \vdots\ \delta \\ B \vdash \div A, \Gamma\end{array}}{\div\Gamma, \div\div A \vdash \div B}\ (K) \quad \dfrac{\Gamma' \vdash A, \Delta'}{\Gamma' \vdash \div\div A, \Delta'}\ (B)}{\div\Gamma, \Gamma' \vdash \div B, \Delta'}\ (cut)$$

An attempt to lift the application of (*cut*) above either the first or second application of (K), yields the following derivations (provided that δ can be repeated in the new context):

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{A \vdash \Gamma_2 \quad \Gamma' \vdash A, \Delta'}{\Gamma' \vdash \Gamma_2, \Delta'}\ (cut)}{\Gamma', \div\Gamma_2 \vdash \Gamma_2, \Delta'}\ (W\vdash)}{\Gamma', \div\Gamma_2 \vdash \Delta'}\ (T')\\ \vdots\ \delta \\ \Gamma', B \vdash \Gamma, \Delta'\end{array} \quad \text{or} \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{A \vdash \Gamma_2}{\div\Gamma_2, A \vdash \Gamma_2}\ (W\vdash)}{\div\Gamma_2, A \vdash}\ (T')\\ \vdots \\ A, B \vdash \Gamma\end{array} \qquad \Gamma' \vdash A, \Delta'}{\Gamma', B \vdash \Gamma, \Delta'}\ (cut)$$

From either of these we can infer $\div\Gamma, \Gamma', B \vdash \Delta'$ by $(W\vdash)$ and (T'), but inferring $\div\Gamma, \Gamma' \vdash \div B, \Delta'$, as required, would be unsound. No sound way of handling this case has been found.

Similarly, when the last rule in the derivation of the left premiss of the (cut)-rule was (T'), we would be able to apply (cut) to a formula of lower complexity if we could show admissibility of (B'):

$$\dfrac{\dfrac{\Gamma, \div\div A, \div\Phi \vdash \div A, \Phi, \Delta}{\Gamma, \div\div A, \div\Phi \vdash \Delta}\,(T) \qquad \dfrac{\Gamma' \vdash A, \Delta'}{\Gamma' \vdash \div\div A, \Delta'}\,(B)}{\Gamma, \div\Gamma', \div\Phi \vdash \Delta}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, \div\div A, \div\Phi \vdash \div A, \Phi, \Delta}{\Gamma, \div\div A, \div\Phi \vdash \Delta}\,(T)}{\Gamma, A, \div\Phi \vdash \Delta}\,(B') \qquad \Gamma' \vdash A, \Delta'}{\Gamma, \div\Gamma', \div\Phi \vdash \Delta}\,(cut)$$

Since we fail to establish admissibility of the (B')-rule, we must either find another way of dealing with the shown cases, or try to formulate new rules which express more closely the characteristics of the system.

2.3 (K) (T″) (B) (B′)

We extend the LK system with the following four rules:

$$
\frac{A \vdash \Gamma}{\div\Gamma \vdash \div A}\ (K) \qquad \frac{\Gamma \vdash \Psi, \Delta}{\Gamma, \div\Psi \vdash \Delta}\ (T'') \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \div\div A, \Delta}\ (B) \qquad \frac{\Gamma, \div\div A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (B')
$$

The motivation for trying to complete a proof of admissibility of the (cut)-rule in this system is that, as we saw earlier, in the system (K) (T') (B), we can handle the shown problematic cases in the proof for admissibility of the (cut)-rule if we had admissibility of the now introduced (B')-rule. Soundness of the (B')-rule follows directly from the **B.** axiom. We replace the (T')-rule with an alternative rule, (T''), which is sound, but not invertible. This new rule does not affect the problematic case we have shown, nor the case we will encounter in this system. The new rule, (B'), does not have the subformula property.

Lemma 2.3.1. *The rule* $\dfrac{B \vdash \div A, \Gamma}{\div\Gamma, A \vdash \div B}$ (kb) *is admissible*

Proof. We apply first (K) to the premiss, then (B'):

$$
\cfrac{\cfrac{B \vdash \div A, \Gamma}{\div\Gamma, \div\div A \vdash \div B}\ (K)}{\div\Gamma, A \vdash \div B}\ (B')
$$

□

In a proof by induction on the complexity of the cut formula, subinduction on the length of the derivation of the left premiss and then on the length of the derivation of the right premiss, we look at all the cases where the cut formula is principal in both the left and the right premiss. In all cases where we increase the number of $\div$ in the cut formula, we manage to apply cut on a formula with a lower complexity. When we look at other cases, e.g. the case where the cut formula is principal in both premisses, the left hand side premiss' last applied rule is (B') and the right hand side premiss' last applied rule is (K), we get a more difficult situation. We look at how the principal formula in the premiss of (B') was introduced and apply the (kb)-rule to handle the case:

$$
\begin{array}{c}
\dfrac{C \vdash A, \Gamma_4}{\div\Gamma_4, \div A \vdash \div C}\ (K) \\
\vdots \\
\dfrac{\div A \vdash \Gamma_3}{\div\Gamma_3 \vdash \div\div A}\ (K) \\
\vdots \\
\dfrac{B \vdash \div\div A, \Gamma_2}{\div\Gamma_2, \div\div\div A \vdash \div B}\ (K) \\
\vdots \\
\cfrac{\cfrac{\Gamma, \div\div\div A, \vdash \Delta}{\Gamma, \div A \vdash \Delta}\ (B') \qquad \cfrac{A \vdash \Gamma'}{\div\Gamma' \vdash \div A}\ (K)}{\Gamma, \div\Gamma' \vdash \Delta}\ (cut)
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
\dfrac{C \vdash A, \Gamma_4}{\div\Gamma_4, \div A \vdash \div C}\ (K) \\
\vdots \\
\dfrac{\div A \vdash \Gamma_3}{\div\Gamma_3 \vdash \div\div A}\ (K) \\
\vdots \\
\dfrac{B \vdash \div\div A, \Gamma_2}{\div\Gamma_2, \div A \vdash \div B}\ (kb) \\
\vdots \\
\cfrac{\Gamma, \div A, \vdash \Delta \qquad \cfrac{A \vdash \Gamma'}{\div\Gamma' \vdash \div A}\ (K)}{\Gamma, \div\Gamma' \vdash \Delta}\ (cut)
\end{array}
$$

The complexity of the cut-formula is equal in the left hand side and the right hand side, and we can apply the induction hypothesis if we have a shorter derivation of the left hand premiss (or same length of derivation of the left premiss, but shorter derivation of the right premiss). However, treating the (kb) rule as the two applications ((K), then (B')) in the proof of its admissibility, the lengths of the derivations are identical (not shorter).

In SB $\div A$ is equivalent to $\div\div\div A$, so we could attempt to show that when the principal formula in (B') has the form $\div A'$, then the length of the derivation of the conclusion is no longer than the length of the derivation of the premiss, i.e. that the rule $\dfrac{\Gamma, \div\div\div A \vdash \Delta}{\Gamma, \div A \vdash \Delta}$ is admissible without lengthening the derivation. This would handle the above case. In attempting to prove this claim we again encounter the problem of repeated applications of (K) (simillar to the above example).

Alternatively, trying to lift the cut above either of the applications of (K) we get the following situations:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash \Gamma' \quad C \vdash A, \Gamma_4}{C \vdash \Gamma', \Gamma_4}\,(cut)}{\div\Gamma', \div\Gamma_4 \vdash \div C}\,(K)}{\vdots}}{\begin{array}{c}\dfrac{\div\Gamma' \vdash \Gamma_3}{\div\Gamma', \div\Gamma_3 \vdash}\,(T'') \\ \vdots \\ \div\Gamma', B \vdash \Gamma_2\end{array}} \qquad \text{or} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{C \vdash A, \Gamma_4}{\div\Gamma_4, \div A \vdash \div C}\,(K) \\ \vdots \\ \div A \vdash \Gamma_3\end{array} \quad \dfrac{A \vdash \Gamma'}{\div\Gamma' \vdash \div A}\,(K)}{\div\Gamma' \vdash \Gamma_3}\,(cut) \\ \dfrac{}{\div\Gamma', \div\Gamma_3 \vdash}\,(T'') \\ \vdots \\ \div\Gamma', B \vdash \Gamma_2\end{array}$$

Inferring now, for example, either $\div\Gamma', \div\Gamma_2 \vdash \div B$, $\div\Gamma_2 \vdash \Gamma', \div B$, $\div\Gamma' \vdash \Gamma_2, \div B$ or $\vdash \Gamma', \Gamma_2, \div B$ which is, or from which we could obtain, the required conclusion, would be unsound.

2.4 (KB) (T′′) (B)

As a final attempt we try to include a version of the (kb)-rule. We extend the LK system with the following three rules:

$$\dfrac{A \vdash \Phi, \div\Psi}{\div\Phi, \Psi \vdash \div A}\,(KB) \qquad \dfrac{\Gamma \vdash \Psi, \Delta}{\Gamma, \div\Psi \vdash \Delta}\,(T'') \qquad \dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash \div\div A, \Delta}\,(B)$$

The new rule, (KB), is motivated by the problems we encountered in the previous systems. This system is different from the earlier system, (K) (T'') (B), only in that that (KB) has replaced (K), where we had that if $\dfrac{B \vdash \div A, \Gamma}{\div\Gamma, A \vdash \div B}$ is admissible, then also the (cut)-rule is admissible. This necessary rule is now just an application of the (KB)-rule.

Lemma 2.4.1. *The new rule, (KB), is sound.*

Proof. Given an arbitrary model, M, and an arbitrary valuation, v, let $a = [\![A]\!]_v^M$, $\bigvee F = [\![\Phi]\!]_v^M$ and $\bigvee \div P = [\![\div\Psi]\!]_v^M$. If the sequent in the premiss is valid, then so is the sequent in the conclusion:

$$\begin{array}{lll} a \leq \bigvee F \vee \bigvee \div P & \Rightarrow^K & \div \bigvee F \wedge \div \bigvee \div P \leq \div a \\ & \Leftrightarrow^K & \bigwedge \div F \wedge \bigwedge \div\div P \leq \div a \\ & \Rightarrow^B & \bigwedge \div F \wedge \bigwedge P \leq \div a. \end{array}$$

□

It is clear that (KB) does not have the subformula property. In an attempt to show admissibility of (cut) we procede by induction on the complexity of the cut formula, then by the length of the derivation of the left premiss and finally the length of the derivation of the right premiss. When the cut formula is principal in both premisses and the last rule applied is (KB) in both the derivation of the left premiss and in the derivation of the right premiss we have two situations, depending on whether the (KB)-application in the left premiss increased or decreased the number of $\div$ in the cut formula. We have one of the following:

$$\dfrac{\dfrac{D \vdash A, \Phi, \div\Psi}{\div\Phi, \Psi, \div A \vdash \div D}\,(KB) \quad \dfrac{A \vdash \Phi', \div\Psi'}{\div\Phi', \Psi' \vdash \div A}\,(KB)}{\div\Phi, \div\Phi', \Psi, \Psi' \vdash \div D}\,(cut) \qquad \text{or} \qquad \dfrac{\dfrac{D \vdash \Phi, \div\div A, \div\Psi}{\div\Phi, \div A, \Psi \vdash \div D}\,(KB) \quad \dfrac{A \vdash \Phi', \div\Psi'}{\div\Phi', \Psi' \vdash \div A}\,(KB)}{\div\Phi, \div\Phi', \Psi, \Psi' \vdash \div D}\,(cut)$$

In the first case, where (KB) *increases* the complexity of the cut formula we cut with A before applying (KB). In the other case, we get a problem similar to what we have seen in the previous system. We could look at the possible derivations of $\div\div A$ and try to construct a premiss which differs only in $\div\div A$ being replaced by A as:

$$\begin{array}{c} \delta \\ \vdots \\ D \vdash \Phi, \div\div A, \div\Psi \end{array} \quad \leadsto^{?} \quad \begin{array}{c} \delta' \\ \vdots \\ D \vdash \Phi, A, \div\Psi \end{array}$$

However, the right hand side sequent is stronger than the left hand side sequent (unless A has the form $\div A'$). An attempt to lift the application of (cut) above the problematic application of (KB) could be attempted; we look at the possible derivations of $\div\div A$ and the only problematic case is when $\div\div A$ was last principal in an application of (KB). Again looking at the possible derivations we have the four possible cases of introducing $\div A$, one of which is problematic:

$$\dfrac{\dfrac{\begin{array}{c} \dfrac{\begin{array}{c}\dfrac{E \vdash \Phi'', \div\div A, \div\Psi''}{\div\Phi'', \div A, \Psi'' \vdash \div E}\,(KB) \\ \vdots \\ \div A \vdash \Phi', \div\Psi' \end{array}}{\div\Phi', \Psi' \vdash \div\div A}\,(KB) \\ \vdots \\ D \vdash \div\div A, \Phi, \div\Psi \end{array}}{\div\Phi, \div A, \Psi \vdash \div D}\,(KB) \qquad \dfrac{A \vdash \Gamma}{\div\Gamma \vdash \div A}\,(KB)}{\div\Gamma, \div\Phi, \Psi \vdash \div D}\,(cut)$$

We can see that we need to apply the (KB)-rule to obtain $\div E$ on the right hand side of the sequent, and we could apply the (KB)-rule and then the (cut)-rule with the our right hand side premiss, this, however leads nowhere, since we, by applying (cut) at this stage, obtain

$$\begin{array}{c} \dfrac{\dfrac{E \vdash \Phi'', \div\div A, \div\Psi''}{\div\Phi'', \div A, \Psi'' \vdash \div E}\,(KB) \qquad \dfrac{A \vdash \Gamma}{\div\Gamma \vdash \div A}\,(KB)}{\div\Gamma, \div\Phi'', \Psi'' \vdash \div E}\,(cut) \\ \vdots \\ \dfrac{\div\Gamma \vdash \Phi', \div\Psi'}{\div\Gamma, \div\Phi', \Psi' \vdash}\,(T'') \\ \vdots \\ \div\Gamma, D \vdash \Phi, \div\Psi \end{array}$$

Inferring now $\div D$ in the right hand side of a sequent would be an unsound step. No sound way of handling this case has been found.

2.5 Conclusion

Other systems have been tried, all without success. For example we have examined systems where the (K) rule is replaced with $\dfrac{A \vdash \div\Delta}{\Delta \vdash \div A}\ (K')$ together with the (B)-rule and variations of the (T)-rule, where we get a problem similar to what we have already seen. We therefore have to conclude that the problem with designing a cut-free sequent system for KTB is not overcome with our approach.

Chapter 3

(In)validating rules

3.1 Introduction

Attempts described in the previous chapter were complicated by the difficulties to determine unsoundness of various proposed rules. This led to the implementation of the ABA program which proved very useful for this purpose. Using it, a conjecture was posed that ABA is sufficient, i.e. every invalid formula could be invalidated by searching for a counter example to it in a collection of SB models. The conjecture turned out to be wrong. Proof of this fact, in Section 3.7, is of independent interest, giving additional insight into SB algebras.

3.2 The ABA program

ABA, Augumented Boolean Algebras, is a program written in Java^{TM} which main assignment is to invalidate a sequent/rule in a boolean algebra augumented with a single unary operation. The user will define the operation, $\sim$, by specifying a set of axioms the algebra must satisfy with respect to this operation. The program has a "shell-like" behaviour allowing direct interaction with the collection of found algebras (hereafter "models"). The program has two shells corresponding to semantical specification and syntactic testing/scanning. A meaningful session could consist of the three steps:

1. Specify axioms defining/constraining $\sim$,
2. search for models satisfying the specified axioms,
3. search for counter examples within these models invalidating a sequent/rule.

3.3 Representation of augumented boolean algebras

The program was developed to investigate collections of impossibility algebras, however since the program allows the user to specify arbitrary non-logic axioms for the unary operator studied, i.e. the operator need not reflect "impossibility", the name *augumented* Boolean algebras was adopted.

An augumented Boolean algebra $A = \langle \underline{A}, \wedge, \vee, -, 1, \sim \rangle$, is a finite algebra where $\langle \underline{A}, \wedge, \vee, -, 1 \rangle$ is a Boolean algebra, and $\sim$ satisfies a given set of axioms. A finite Boolean algebra has carrier set, $\underline{A}$, isomorphic to $\wp(B)$ for some finite set B. When specifying a Boolean algebra with $\wp(B)$ as carrier set we can define $\wedge = \cap$, $\vee = \cup$ and $-$ as $B \setminus$. To represent $\wp(B)$ on a computer, a convenient approach is to let the elements of $\wp(B)$ be represented by all binary strings of length $|B|$, where we impose an ordering on the elements of B and let the i-th bit be 1 *iff* that element is an element in the corresponding subset of B. The ordering of the elements of

B is irrelevant as we are interested in all sets isomorphic to $\wp(B)$. The set operations can now be realized by the standard bit-wise operators on binary strings: $\cap =$*bitwise AND*, $\cup =$ *bitwise OR*, and $- =$ *bitwise NEG*.

In ABA, the values are represented by the primitive `int`-type for efficiency reasons. This violates the straightforwardness of our operation definitions slightly. The *bitwise NEG* operation on an `int` negates all bits used to represent an `int` number (32, 64, ... bits), and thus our algebra would not be closed under $-$. After negating all bits we preform a *bitwise AND* with the "size mask" of our representation ($2^{n+1} - 1$).

Given a positive number n, we readily generate such a powerset-Boolean algebra by specifying the top-most element (which corresponds to the "size mask" we need in the realization of the $-$ operation). To augment this algebra we need to specify the $\sim$-operator. In one specific model, $\sim$ is a total function, $\wp(B) \to \wp(B)$, and the elements of $\wp(B)$ are represented by all numbers, $0 \leq m < 2^{n+1}$. To represent this, ABA stores an `int` array, `op_ar`, of length 2^{n+1}, such that for any value, m, $\sim m =$`op_ar[m]`.

The values in a model are represented by the collection of all natural numbers $0 \ldots 2^n - 1$, for some n, and are printed to the screen in binary representation. The collection will contain all *strings* over $\{0, 1\}$ of length n. In this program the length is $\geqslant 2$. If we let V denote this collection of values, the operation can be considered as a subset $\sim \ \subseteq V \times V$ where we require that each value appears exactly once as the first element in a pair(i.e. $\forall v \in V : \exists! u \in V : \langle v, u \rangle \in \sim$).

A model is written to the screen using only one line of text representing the definition of the $\sim$ operation in this model. One (not very interesting) model might be { `<00,00>, <01,00>, <10,00>, <11,00>` }. This model can be visualized as in Figure 3.1.

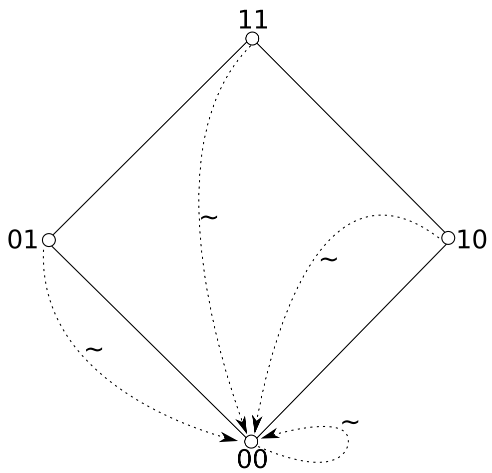

Figure 3.1: Visualization of example model

In Figure 3.1, the dotted arrows represent the result of applying the $\sim$ operation, i.e. the result of applying $\sim$ to the value at the source of an arrow is the value which is the arrow's target. Each value will always have one, and only one, outgoing arrow. When a counter model is being displayed, the model will be described in this way and then the list of variable assignments which constitutes the counter example will follow.

3.4 Interaction: Syntactic testing/scanning

When the program is executed it enters the shell for syntactic testing/scanning. The prompt has the form:
`<axioms>#<models>>`
where *`<axioms>`* is a string of the (short) names of the axioms currently taken into consideration, and *`<models>`* is the number of discovered models satisfying these axioms. The first time the program is run, the prompt will

be `#0>` meaning that no axioms are considered and no models have been found.

3.4.1 Commands

Entering `?` will give a list of available commands. The available commands are shown in table 3.1.

`axioms`	starts the "axiom management" part of the program.
`scan`	scans for models satisfying the currently considered axioms.
`save`	saves the discovered models.
`list axioms`	lists the currently considered axioms.
`list models`	lists the discovered models. (See Representation of models)
`sequent`	attempt to find a counter example to a sequent.
`rule`	attempt to find a counter example to a rule.
`exit`	exits the program.

Table 3.1: ABA commands in search-mode

3.4.2 Discovering models

To extend the list of models, type `scan` at this level. For a given set of axioms the first time this command is issued the program will search through all extensions of 4-element boolean algebras. The next time, all extensions of 8-element boolean algebras, and so on. To complete the scanning process for all 2^n-element algebras, the program tests $(2^n)^{(2^n)}$ models. Such a scanning obviously becomes very time-consuming very quickly. Two or three times (i.e. up to extensions of 16-element boolean algebras) should be possible on a personal computer, at least if one or more of the currently considered axioms allow for simplifying the search (see Semantic Specification).

3.4.3 Searching for counter models

To search for counter model(s) of a sequent the user will enter `sequent` and then enter the sequent which (s)he would like to refute. Refuting a sequent, $\Gamma \vdash \Delta$, is to find some model and some valuation of the variables in the sequent, v, such that $[\![\Gamma]\!]_v \nleqslant [\![\Delta]\!]_v$. If some such valuation of the variables exists in any of the discovered models, a counter example will be printed to the screen and the user has three choices: 1) enter "`,`" to get the next counter example within the same model (if it exists), 2) enter "`;`" to get the next counter example in the next model (if it exists), or 3) enter anything else and return to the program prompt.

To search for counter model(s) of a rule the user will enter `rule` and then enter the rule which (s)he would like to refute. Refuting a rule, $\dfrac{\Gamma_0 \vdash \Delta_0 \ldots \Gamma_n \vdash \Delta_n}{\Gamma_c \vdash \Delta_c}$, is to find some model in which all premises are valid and, for some valuation of the variables in the conclusion, the conclusion, $\Gamma_c \vdash \Delta_c$, is not valid. This is to validate the formula (3.1), by supplying a model, M, and a valuation, v, for which the conclusion is invalid.

$$\exists M[(\forall i : 0 \leqslant i \leqslant n \rightarrow \forall v' : [\![\Gamma_i]\!]_{v'} \leqslant [\![\Delta_i]\!]_{v'}) \wedge (\exists v : [\![\Gamma_c]\!]_v \nleqslant [\![\Delta_c]\!]_v)] \tag{3.1}$$

ABA searches for counter examples to a rule by searching through each discovered model. In assuming that the premises are valid we gather the valuations in which the premises are valid and test for a counter model to the conclusion under one of these valuations. This procedure correspond to iterating through the available models and testing each model by the following algorithm:

`let V :=` $\{$`v` $\mid [\![\Gamma_i]\!]_v \leq [\![\Delta_i]\!]_v\}$ for each premiss $\Gamma_i \vdash \Delta_i$
`if (for some valuation,` $v' \in$ `V :` $[\![\Gamma]\!]_{v'} \not\leq [\![\Delta]\!]_{v'}$`)` for conclusion $\Gamma \vdash \Delta$
`then report counter model,` v'

or, equivalently (for a rule with premises $\Gamma_i \vdash \Delta_i$ and conclusion $\Gamma \vdash \Delta$):

```
if for some valuation v :
        (for all premises Γi ⊢ Δi  :  [[Γi]]v ≤ [[Δi]]v) andalso [[Γ]]v ≰ [[Δ]]v
then report counter model, v
```

The input can span several lines and is ended with $**$, so if one would like to enter the rule

$$\frac{A \vdash B \quad C \vdash A}{C \vdash B}$$

one could type:

```
A |- B ; C |- A
___
C |- B
**
```

3.4.4 Syntax

Rules and sequents are entered into the program using the grammar shown in Table 3.2. In this language VAR is the regular expression ('a' ...'z'|'A' ...'Z')('0' ...'9'|'a' ...'z'|'A' ...'Z')* (i.e. a common *identifier* expression).
Whitespaces between tokens such as operators and variable names are ignored, e.g. `- A |- ~ A` and `-A|-~A` are treated identically.

$\rightarrow$	$\langle rule \rangle$	::=	$\langle sequent - list \rangle$ `___` $\langle sequent \rangle$	note: three consecutive ***underscore***-characters
	$\langle sequent - list \rangle$	::=	$\langle sequent \rangle$ **;** $\langle sequent - list \rangle$	
		|	$\langle sequent \rangle$	
$\rightarrow$	$\langle sequent \rangle$	::=	$[\langle formula - list \rangle]$ **|-** $[\langle formula - list \rangle]$	
	$\langle formula - list \rangle$	::=	$\langle cond \rangle$ **,** $\langle formula - list \rangle$	
		|	$\langle cond \rangle$	
	$\langle cond \rangle$	::=	$\langle disj \rangle$ [**->** $\langle cond \rangle$]	
	$\langle disj \rangle$	::=	$\langle conj \rangle$ [**|** $\langle disj \rangle$]	
	$\langle conj \rangle$	::=	$\langle uni \rangle$ [**&** $\langle conj \rangle$]	
	$\langle uni \rangle$	::=	**-** $\langle uni \rangle$	
		|	$\sim$ $\langle uni \rangle$	
		|	$\langle literal \rangle$	
	$\langle literal \rangle$	::=	**(** $\langle cond \rangle$ **)**	
		|	VAR	

Table 3.2: Syntax of ABA formulae

3.5 Interaction: Semantic specification

Enter `axioms` at the first prompt to enter the program's specification mode. In this mode the prompt will be `AX >`.

3.5.1 Commands

Entering `?` will give a list of available commands. The available commands are shown in figure 3.2.

`add`	specify a new axiom.
`list`	displays a list of available axioms.
`inc`	include some axiom.
`dis`	disregard some axiom.
`save`	saves the list of axioms.
`exit`	saves the list of axioms and exits this mode.

Figure 3.2: ABA commands for specifying axioms

3.5.2 Specify an axiom

To specify an axiom is to make this axiom available to the program. The specified axiom is added to the program's list of available axioms *and* is added to the list of considered axioms. To add a new axiom enter `add`. The user will be prompted for 1) (short) name of the axiom (not necessarily unique) and 2) the (in)equality expressing the axiom.

3.5.3 Listing available axioms

After some axiom(s) have been specified the user can review which axioms are available and which of these are currently being taken into consideration. The axioms are listed one entry per line. Axioms which are included are marked with a `+` at the begining of the line. Axioms which allow for simplifying the search for satisfying models are followed by `true` in the `Simplifies`-column. Simplifying expressions are expressions which have the form

$$\sim (\text{ VAR} \mid \text{ CONST }) (< \mid \ <= \mid \ = \mid \ >= \mid \ >) \text{ BOOL}$$

where VAR is a single variable, CONST is either `0` or `1` and BOOL is a boolean expression (an expression not containing the $\sim$-operation).

An example of such a list might be

```
  Name  Expr.               Simplifies
+ K     ~(x + y) = ~x * ~y
+ K     ~0 = 1              true
+ T     ~x <= -x            true
+ B     x <= ~~x
  4     ~x <= ~-~x
  5     -~x <= ~~x
```

In this case, axioms K, T and B are taken into consideration and axioms 4 and 5 are disregared when searching for satisfying models.

3.5.4 Changing the set of considered axioms

After axioms have been specified by the `add` command, the user may choose a subset of these to be the set of considered axioms. An axiom is included in the set of considered axioms by issuing the command `inc` *`name`* where *name* is the name of the axiom. An axiom can be removed from consideration by issuing the command `dis` *`name`*.

3.5.5 Syntax

(In)equalities are entered into the program using the grammar shown in Table 3.3. VAR is the regular expression starting with a letter (a...z, or A...Z) and followed by zero or more letters or digits.

$$
\begin{array}{llll}
\rightarrow & \langle expr \rangle & ::= & \langle join \rangle \ (\ \texttt{<} \mid \texttt{<=} \mid \texttt{=} \mid \texttt{>=} \mid \texttt{>}\ )\ \langle join \rangle \\
 & \langle join \rangle & ::= & \langle meet \rangle\ [\ +\ \langle join \rangle\ ] \\
 & \langle meet \rangle & ::= & \langle uni \rangle\ [\ *\ \langle meet \rangle\ ] \\
 & \langle uni \rangle & ::= & \texttt{-}\ \langle uni \rangle \\
 & & \mid & \sim \langle uni \rangle \\
 & & \mid & \langle atom \rangle \\
 & \langle atom \rangle & :\text{-} & (\ \langle join \rangle\ ) \\
 & & \mid & \mathrm{VAR} \\
 & & \mid & (\ \texttt{0} \mid \texttt{1}\ )
\end{array}
$$

Table 3.3: Syntax of ABA algebraic (in)equalities

3.6 Using the program

The expressions/sequents/rules entered into the program are parsed with a parser generated by ANTLR (`http://antlr.org/`) and therefore need the antlr library available when executed. The list of available axioms and lists of discovered models satisfying different axioms are all stored in a directory `aba-files` (created upon execution if it does not exist) in the *current directory*.

3.6.1 Demonstration

Try to find counter examples to the sequents $\sim X \vdash \neg X$ and $\neg X \vdash \sim X$ in KTB:
When the program is executed the required axioms already exist in the list of available axioms. The user selects the desired axioms, builds a collection of models satisfying these and looks for counter examples for the sequents. The first sequent gives no counter examples. The second sequent gives several counter examples and the user chooses to view seven counter examples in four different models. Then the user exits the program.

```
#0>axioms
AX >list
   Name  Expr.              Simplifies
   K     ~(x + y) = ~x * ~y
   K     ~0 = 1             true
   T     ~x <= -x           true
   B     x <= ~~x
   4     ~x <= ~-~x
   5     -~x <= ~~x
AX >inc K
AX >inc T
AX >inc B
AX >list
   Name  Expr.              Simplifies
 + K     ~(x + y) = ~x * ~y
 + K     ~0 = 1             true
 + T     ~x <= -x           true
 + B     x <= ~~x
   4     ~x <= ~-~x
   5     -~x <= ~~x
AX >exit
Saved "[...]/aba-files/axioms"
KTB#0>scan
~(x + y) = (~x * ~y)
~0 = 1
```

```
~x <= -x
x <= ~~x
Saved "[...]/aba-files/sys-KTB"
KTB#2>scan
~(x + y) = (~x * ~y)
~0 = 1
~x <= -x
x <= ~~x
Saved "[...]/aba-files/sys-KTB"
KTB#10>sequent
Enter sequent:
~X |- -X
KTB#10>sequent
Enter sequent:
-X |- ~X
Counter example in algebra with ~: { <00,11>, <01,00>, <10,00>, <11,00> }
  X 01
,
  X 10
,
Counter example in algebra with ~: { <000,111>, <001,000>, <010,000>, <011,000>, <100,000>,
[...]                                                  <101,000>, <110,000>, <111,000> }
  X 001
,
  X 010
,
  X 011
;
Counter example in algebra with ~: { <000,111>, <001,000>, <010,100>, <011,000>, <100,010>,
[...]                                                  <101,000>, <110,000>, <111,000> }
  X 001
;
Counter example in algebra with ~: { <000,111>, <001,010>, <010,001>, <011,000>, <100,000>,
[...]                                                  <101,000>, <110,000>, <111,000> }
  X 001

KTB#10>exit
```

3.7 Searching for counter models in a finite collection of models

Since the ABA program's brute force algorithm cannot gather a very large collection of models, a conjecture that some finite collection of models would suffice, i.e. that some finite collection of models would contain a counter model to any contingent formula. In this section we show that this is impossible. Given any finite collection of models, there is a contingent formula which does not have a counter model in this collection.

There are formulae which are indistinguishable for models upto a given size. In an SB algebra consisting of at most 2^{n+1} elements, the antecendent and consequent in the formula $(\div\neg)^n p \to (\div\neg)^{n+1} p$ are indistinguishable and the implication, thus, has no counter models with 2^{n+1} or fewer elements. We are also able to show that, for any n, we can construct a counter model for the formula $(\div\neg)^n p \to (\div\neg)^{n+1} p$.

Lemma 3.7.1. *For every SB-algebra $A = \langle \underline{A}, \wedge, \vee, \neg, \div, \mathbf{1}\rangle$ with $|\underline{A}| \leq 2^{n+1}$, and every formula ϕ : $A \models (\div\neg)^n\phi \to (\div\neg)^{n+1}\phi$*

Proof. Given a $v \in \underline{A}$, we form a sequence $v_0v_1 \dots v_l$ with $v_0 = v$ and $v_{i+1} = \div\neg v_i = (\div\neg)^{i+1}v_0$. For all i, $v_i \geq v_{i+1}$, by **T**. Let l be the least number such that $v_l = v_{l+1}$, then $l \leq n$ since

- if v is the top element, $v = \mathbf{1}$, then for the sequence is $v_0v_1 \dots v_l$, we have that $l = 0$, since $v_0 = \div\neg v_0$, by **K**, otherwise
- if $v < \mathbf{1}$, then for the sequence $v_0v_1 \dots v_l$, $l \leq n$ since the longest possible and strictly decreasing path in A not starting at the top element has length at most n. I.e. after at most n steps we obtain such a repetition $v_n = (\div\neg)^n v_0 = (\div\neg)^{n+1}v_0 = v_{n+1}$.

Since v was arbitrary, this shows that for an arbitrary valuation of ϕ, that $(\div\neg)^n\phi \to (\div\neg)^{n+1}\phi$ is satisfied in A. □

Conversely, an algebra with such a strictly decreasing sequence $v_0 > \div\neg v_0 > \dots > (\div\neg)^i v_0 > \dots > (\div\neg)^n v_0 > (\div\neg)^{n+1}v_0$, with $l \geq n+1$, provides a counter-model for the formula from Lemma 3.7.1 since $(\div\neg)^n v_0 > (\div\neg)^{n+1}v_0 \Rightarrow \neg(\div\neg)^n v_0 \vee (\div\neg)^{n+1} \neq \mathbf{1}$. Moreover, we have that an algebra, A, with such a strictly decreasing sequence with $l \geq n+1$, for any $m \leq n$, $A \not\models (\div\neg)^m p \to (\div\neg)^{m+1}p$. We show that such a counter-model exists for every n.

In Lemma 3.7.2 and its proof we use the fact that every element in a powerset of a finite set can be written as the join of atoms, i.e. for every $x \in \wp(B)$, $x = \bigvee x_i$ for atomic x_i (singleton sets).

Lemma 3.7.2. *For $n \geq 1$ and $N = \{1, 2, \dots, n\}$, $A_n = \langle \wp(N), \cup, \cap, -, \div, N\rangle$ with $\div$ defined as follows is an SB-algebra:*

1. *$\div i = N \setminus \{i-1, i, i+1\}$, for atomic i*
2. *$\div\emptyset = N$*
3. *$\div x = \bigwedge \div x_i$, for every other $x = \bigvee x_i$ where x is a non-empty, non-atomic element, and x_i are atomic*

In this definition, the value of $\div 1 = N \setminus \{0, 1, 2\} = N \setminus \{1, 2\}$, since $0 \notin N$. Similarly for $\div n = N \setminus \{n-1, n, n+1\} = N \setminus \{n-1, n\}$, since $n+1 \notin N$. This deviates from the definition of $\div$ for the other atomic values, since in all cases but the first and last (1 and n) *three* elements are removed from N. Schematically, $\div i$ for atomic i in A_n as described in **Lemma 3.7.2** for atomic i are obtained by removing the diagonal element i and its immediate neighbours from the i-th row in the $n \times n$ table:

	1	2	3	4	$\dots$	$n-3$	$n-2$	$n-1$	n
1	$-$	$-$	$+$	$+$		$+$	$+$	$+$	$+$
2	$-$	$-$	$-$	$+$		$+$	$+$	$+$	$+$
3	$+$	$-$	$-$	$-$		$+$	$+$	$+$	$+$
4	$+$	$+$	$-$	$-$		$+$	$+$	$+$	$+$
$\vdots$									
$n-3$	$+$	$+$	$+$	$+$		$-$	$-$	$+$	$+$
$n-2$	$+$	$+$	$+$	$+$		$-$	$-$	$-$	$+$
$n-1$	$+$	$+$	$+$	$+$		$+$	$-$	$-$	$-$
n	$+$	$+$	$+$	$+$		$+$	$+$	$-$	$-$

We now prove Lemma 3.7.2.

Proof.

K Let $x, y \in \wp(N)$. We consider the two cases of atomic values and non-empty, non-atomic values as one case in the proof.

1. if $x = \emptyset$ or $y = \emptyset$, say $x = \emptyset$ then
   $\div(x \vee y) = \div(\emptyset \vee y) = \div y = N \wedge \div y = \div x \wedge \div y$
2. otherwise $x = \bigvee_{i \in I} i$ and $y = \bigvee_{j \in J} j$ for $I \subseteq N$ and $J \subseteq N$:
   $\div(x \vee y) = \div(\bigvee_{i \in I} i \vee \bigvee_{j \in J} j) = \div \bigvee_{z \in I \cup J} z =^{(3)} \bigwedge_{z \in I \cup J} \div z = \bigwedge_{i \in I} \div i \wedge \bigwedge_{j \in J} \div j =^{(*)} \div \bigvee_{i \in I} i \wedge \div \bigvee_{j \in J} j = \div x \wedge \div y$
   The second last equation, $(*)$, holds since if x (resp. y) is atomic then $\bigwedge_{i \in I} i = \bigvee_{i \in I} i = i = x$ (resp. $\bigwedge_{j \in J} j = \bigvee_{j \in J} j = j = y$), and if x (resp. y) is not atomic, the equation follows directly from (3).

T Let $x \in \wp(N)$

1. if x is atomic then $\div x = N \setminus \{x-1, x, x+1\} \subseteq N \setminus \{x\} = -x$
2. if $x = \emptyset$, then $\div x = N = -x$
3. otherwise $x = \bigvee_{i \in I} i$ for $I \subseteq N$
   $\div x = \div \bigvee_{i \in I} i =^{(3)} \bigwedge_{i \in I} \div i \leq \bigwedge_{i \in I} -i = -\bigvee_{i \in I} i = -x$

B Let $x \in \wp(N)$

1. if x is atomic then $\div \div x = \div I$ where $I = N \subseteq \{x-1, x, x+1\}$ by (1) and $\forall j \in I : i \in \div j$, i.e. $i \in \bigwedge_{j \in I} \div j =^{(3)} \div \div \bigvee_{j \in I} j = \div \div x$ which shows that $x \leq \div \div x$
2. if $x = \emptyset$ then $\div \div x =^{(2)} \div N =^{(T)} \emptyset$
3. otherwise $x = \bigvee_{i \in I} i$ for $I \subseteq N$.
   $\div x = \div \bigvee_{i \in I} i =^{(3)} \bigwedge_{i \in I} \div i$, thus $j \in \div x \Leftrightarrow \forall i \in x : (i \neq j) \wedge (i \neq j-1) \wedge (i \neq j-1)$. Writing x as the ordered sequence $i_1 i_2 \ldots i_k$, with $i_l < i_{l+1}$, this means that $(*)$ $j \in \div x \Leftrightarrow \exists l : i_l + 1 < j < i_{l+1} - 1$ (with adjustments for $i_1 = 1$ and/or $i_k = n$). So $\div x = \bigvee_{j \in J} j$ where J is the collection of such js. Then $\div \div x = \div \bigvee_{j \in J} j =^{(3)} \bigwedge_{j \in J} \div j$. Writing now $\div \div x$ as such an ordered sequence $j_1 j_2 \ldots j_z$, by the above observation $(*)$, we have that $m \in \bigwedge_{j \in J} \div j \Leftrightarrow \exists p : j_p + 1 < m < j_{p+1} - 1$ (with adjustments for $j_1 = 1$ and/or $j_z = n$). For any $i \in x$, we have that $i-1, i, i+1 \notin J$, so that J has no members closer to i than $i-2, i+2$. By $(*)$ this means that $i \in \div \div x$ which, since $i \in x$ was arbitrary, establishes that $x \leq \div \div x$.

□

Constructing a model as described in Lemma 3.7.2 for $n = 1$, we get $N = \{1\}$, and all atomic values coincide with the top-element $\{1\} = \{n\} = N$. This simplest model can be visualized as the left-most model of the four models shown in Figure 3.3. The first four models constructed with this procedure can be visulalized as in Figure 3.3, these models, except the simplest model, A_1, was found using ABA.

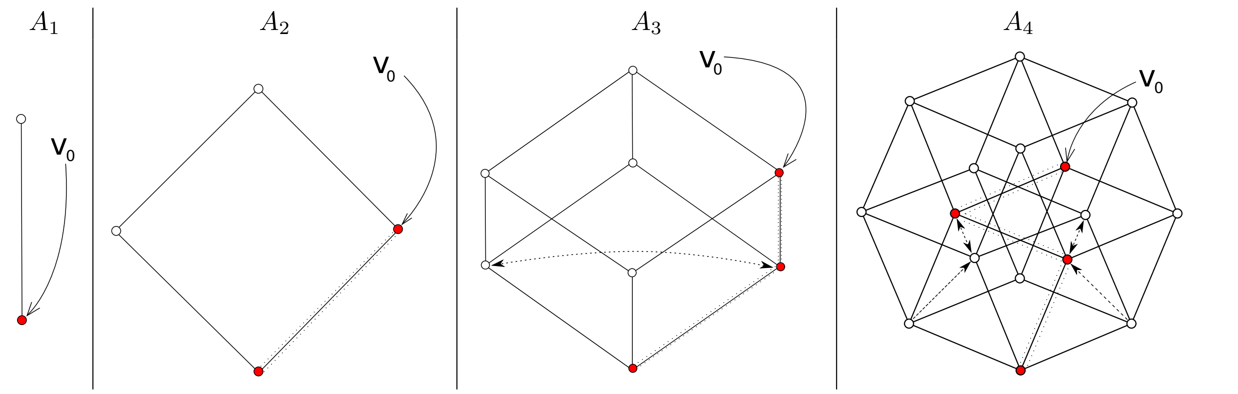

Figure 3.3: Counter models for $(\div\neg)^n \phi \rightarrow (\div\neg)^{n+1} \phi$

The v_0 value in the graphs in Figure 3.3 represents the greatest element in the strictly descending path we are seeking. In visualizing a model the dotted arrows represent applications of $\div$. For value, v, the dotted arrow representing $\div(v)$ is omitted if $v = \emptyset$, or $v = N$ or $\div(v) = \emptyset$. We permit this since $\div(\emptyset) = N$ and $\div(N) = \emptyset$ by K and T, and for any value v without an outgoing dotted arrow we have $\div(v) = \emptyset$.

Lemma 3.7.2 gives the following theorem according to which, for every $n \geq 1$, the formula $(\div\neg)^n p \to (\div\neg)^{n+1} p$ has a counter model, a model containing a sufficiently long strictly descending path.

Theorem 3.7.3. *For every n there exists an SB-algebra, A, such that for every $m < n : A \not\models (\div\neg)^m p \to (\div\neg)^{m+1} p$.*

Proof. For any n, an algebra A_{n+1} as defined in Lemma 3.7.2 (i.e. we let the set $N = \{1, 2, \ldots, n, n+1\}$), contains a descending sequence $v_0 v_1 \ldots v_n$ and $v_i > v_{i+1}$, with $l = n$ (where l is the least number such that $v_l = v_{l+1}$). This sequence can be obtained by letting $v_0 = \{1, \ldots, n\}$, then $\neg v_0 = \{n+1\}$ and $\div\neg v_0 = \div\{n+1\} = \{1, \ldots n-1\}$, i.e. $v_1 = \div\neg v_0 < v_0$. Generally, if $v_i = \{1, \ldots n-i\}$, then $\neg v_i = \{n-(i-1), \ldots, n\}$ and $\div\neg v_i = \bigwedge\{\div n-(i-1), \ldots, \div n\}$ by *(3)* in the definition, and thus $\div\neg v_i = \{1, 2, \ldots, n-(i+2)\}$. We see that we can obtain v_{i+1} by removing the largest number in v_i, this continues until v_i is empty. Schematically, the sequence is obtained by stepping downward in the following table

$\neg v_{i-1}$	1	2	3	$\ldots$	$n-4$	$n-3$	$n-2$	$n-1$	n	$n+1$	$v_i = (\div\neg)^i v_0$
	+	+	+		+	+	+	+	+	$-$	v_0
	+	+	+		+	+	+	+	$-$	$-$	v_1
n	+	+	+		+	+	+	$-$	$-$	$-$	v_2
$n-1, n$	+	+	+		+	+	$-$	$-$	$-$	$-$	v_3
$n-2, n-1, n$	+	+	+		+	$-$	$-$	$-$	$-$	$-$	v_4
$\vdots$											$\vdots$
$3, \ldots, n-2, n-1, n$	+	$-$	$-$		$-$	$-$	$-$	$-$	$-$	$-$	v_{n-1}
$2, 3, \ldots, n-2, n-1, n$	$-$	$-$	$-$		$-$	$-$	$-$	$-$	$-$	$-$	v_n
$1, 2, 3, \ldots, n-2, n-1, n$	$-$	$-$	$-$		$-$	$-$	$-$	$-$	$-$	$-$	$v_{n+1} = v_n$

We thus have a strictly descending sequence $v_0 > v_1 = \div\neg v_0 > \ldots > v_i = (\div\neg)^i v_0 > \ldots > v_n = (\div\neg)^n v_0 = \emptyset$, and for any $m < n : v_m > v_{m+1}$, i.e. the constructed algebra falsifies, as claimed, the formulae $(\div\neg)^m p \to (\div\neg)^{m+1} p$, under the assignment $\alpha(p) = v_0$. $\square$

We can now establish that given any finite collection of finite algebras, $\mathcal{C}$, there is a contingent formula which is valid in all $A \in \mathcal{C}$.

Given an arbitrary finite collection of finite algebras, $\mathcal{C}$, let n be a number such that for each $A \in \mathcal{C} : |A| \leq 2^{n+1}$. Then, for the formula $\phi = (\div\neg)^n p \to (\div\neg)^{n+1} p$, $A \models \phi$ for each $A \in \mathcal{C}$, by Lemma 3.7.1.
But $\not\models \phi$, by Theorem 3.7.3, i.e. there is an algebra A such that $A \not\models \phi$, namely A_{n+1}, as constructed in Lemma 3.7.2.

Chapter 4

Cut-free S5 system

4.1 Introduction

We have tried to complete a proof of the cut-elimination theorem for the system shown in Table 4.1. The system considers formulae only over a minimal set of classical contectives, $\{\neg, \wedge\}$. We show that the rules involving the $\div$ connective are sound with respect to $\mathcal{IC}$ algebras satisfying the **K**, **T** and **5** axioms. The proof is incomplete in that there is one case in which we seem to need an unproven lemma. In this section we introduce the rules and show soundness of those involving $\div$ and give the proof of admissibility for several rules, and lastly we show the partial proof of admissibility the mentioned lemma, $\dfrac{\Gamma \vdash \div A, \Delta}{\Gamma, A \vdash \Delta}$.

$$\frac{}{p \vdash p}\ (ax)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg \vdash) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\vdash \neg)$$

$$\frac{\Gamma A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge \vdash) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B \Delta}\ (\vdash \wedge)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (W \vdash) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ (\vdash W)$$

$$\frac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\ (\div \vdash) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \div A, \Delta}\ (\vdash \div) \quad \text{for } clopen(\Gamma, \Delta) \text{ } or\ clopen(A)$$

$$\frac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\ (\div\wedge \vdash) \qquad \frac{\Gamma \vdash \div A, \div B, \Delta}{\Gamma \vdash \div(A \wedge B), \Delta}\ (\vdash \div\wedge)$$

$$\frac{\Gamma \vdash \div\neg A \wedge \div\neg B, \Delta}{\Gamma \vdash \div\neg(A \wedge B), \Delta}\ (\vdash \div\neg\wedge)$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2 \vdash A, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (cut)$$

Table 4.1: The LK[S5] system and the *cut*-rule

In the ($\vdash\ \div$)-rules there are syntactic requirements that either the principal formula or all parametric formulae are *clopen*. We define *clopen* as follows:

Definition 4.1.1. *A formula, A, is clopen, clopen(A), iff*

1. $A = \div A'$, *or*
2. $A = \neg A'$ *and clopen(A'), or*
3. $A = B \wedge C$ *and clopen(B, C)*

4.2 Soundness and auxiliary results

An algebra satisfying **K**, **T** and **5**, also satisfies the **B** and **4** axioms and, furthermore, the **4** and **5** can be regarded as equalities. We show this in the following lemma.

Lemma 4.2.1. *Given an algebra, $A = \langle \underline{A}, \wedge, \vee, \neg, \div, \mathbf{1}\rangle$, where $\div$ satisfies **K**, **T** and **5***

1. *The **5** axiom can be regarded as an equality;* $\div \div x \leq \neg \div x$
2. $\div$ *satisfies the **B** axiom;* $x \leq \div \div x$
3. $\div$ *satisfies the **4** axioms;* $\div x \leq \div\neg \div x$
4. *The **4** axiom can be regarded as an equality;* $\div\neg \div x \leq \div x$

Proof.

1. $\neg \div x \leq \neg \div x \Rightarrow^{T} \div \div x \leq \neg \div x$
2. $\neg x \leq \neg x \Rightarrow^{T} \div x \leq \neg x \Rightarrow \neg\neg x \leq \neg \div x \Rightarrow x \leq \neg \div x \Rightarrow^{5} x \leq \div \div x$
3. By **5** we have $\neg \div x \leq \div \div x \Rightarrow^{K} \div \div \div x \leq \div\neg \div x \Rightarrow^{B} \div x \leq \div\neg \div x$
4. By **T** we have $\div x \leq \neg x \Rightarrow \neg\neg x \leq \neg \div x \Rightarrow x \leq \neg \div x \Rightarrow^{K} \div\neg \div x \leq \div x$

$\square$

In the proof of soundness of the proposed rules we will refer to the axioms and the result of Lemma 4.2.1 as follows:

	S5-algebra
K.	$\div(x \vee y) = \div x \wedge \div y$
	$\div 0 = 1$
T.	$\div x \leq \neg x$
B.	$x \leq \div \div x$
4.	$\div x = \div\neg \div x$
5.	$\neg \div x = \div \div x$

$\mathcal{IC}$ algebras are related to topological algebras, as shown in [1]. A topological algebra is an algebra $\langle T; \wedge, \vee, \neg, 1, c\rangle$ where $\langle T; \wedge, \vee, \neg, 1\rangle$ is a Boolean algebra and c, the *closure*-operator satisfies appropriate axioms. Alternatively, a topological algebra can be defined in terms of i, the *interior*-operator; the operators are mutually interdefinable; $c = \neg i \neg$ or $i = \neg c \neg$. The terms *open*, *closed* and *clopen* are inherited from this connection, an element, $t \in T$, is *open iff* $t = i(t)$, it is *closed* if $\neg t$ is *open* (or, equivalently, t is *closed iff* $t = c(t)$), and t is *clopen iff* it is both *open* and *closed*. In such an interpretation, $\div x = i(\neg x) = \neg c(x)$. Thus $\div x$ is an *open* element, but it is also *closed*; $\neg \div x =^{5} \div \div x$ (Lemma 4.2.1). We see that any element of the form $\div x$ is *clopen*. We extend this result slightly in Lemma 4.2.2.

Lemma 4.2.2. *For any formula, ϕ, if ϕ satsifies the syntactic definition of clopen give in Definition 4.1.1 then for any S5 $\mathcal{IC}$-algebra, $M = \langle A, \wedge, \vee, -, \div, 1\rangle$, and valuation v: $[\![\phi]\!]_v^M = \div a$ for some $a \in A$.*

Proof. By induction on the complexity of ϕ:

1. Base: $\phi = \div A$, given an arbitrary model, M, and valuation, v, $[\![\div A]\!]_v^M = \div[\![A]\!]$.
2. Inductive step: $\phi = \neg A$ where $clopen(A)$; by the induction hypothesis $[\![A]\!] = \div A'$ and given an arbitrary model, M, and valuation v, $[\![\neg A]\!]_v^M = \neg[\![A]\!]_v^M = \neg \div A' =^5 \div \div A'$
3. Otherwise: $\phi = B \wedge C$ where $clopen(B, C)$; by the induction hypothesis $[\![B]\!] = \div B'$ and $[\![C]\!] = \div C'$; For an arbitrary model, M, and valuation, v, $[\![B \wedge C]\!]_v^M = \left([\![B]\!]_v^M\right) \wedge \left([\![C]\!]_v^M\right) = (\div B') \wedge (\div C') =^K \div (B' \vee C')$.

□

Lemma 4.2.2 allows us to always refer to the valuation of a *clopen* formula as an expression starting with a $\div$. Lemma 4.2.2 can also be extended to include other connectives, we use the fact that it can also be extended to $\vee$ in the next Lemma. Over topological algebras, a general result is that if an *open* element (or *clopen* in the case of S5) is bound by an arbitrary element, then former element is bound by the interior of the latter; $i(x) \leq y \Rightarrow i(x) \leq i(y)$. We show that, for S5 $\mathcal{IC}$ algebras, four similar results hold; *clopen* elements bound by arbitrary elements are bound by both interior ($\div\neg$) and closure ($\neg\div$) of that element.

Lemma 4.2.3. *Let x be clopen*

1. $x \leq y \Rightarrow x \leq \neg \div y$
2. $x \leq y \Leftrightarrow x \leq \div\neg y$
3. $y \leq x \Leftrightarrow \neg \div y \leq x$
4. $y \leq x \Rightarrow \div\neg y \leq x$

Proof. Since x is *clopen*, we will refer to it as $\div x$ in all cases.

1. $\div x \leq y \Rightarrow \div y \leq \div \div x \Leftrightarrow^5 \div y \leq \neg \div x \Leftrightarrow \neg\neg \div x \leq \neg \div y \Leftrightarrow \div x \leq \neg \div y$
2. $\div x \leq y \Leftrightarrow \neg y \leq \neg \div x \Rightarrow^K \div\neg \div x \leq \div\neg y \Leftrightarrow^4 \div x \leq \div\neg y$, and
   $\div x \leq \div\neg y \Rightarrow^T \div x \leq \neg\neg y \Leftrightarrow \div x \leq y$
3. $y \leq \div x \Rightarrow \div \div x \leq \div y \Leftrightarrow^5 \neg \div x \leq \div y \Leftrightarrow \neg \div y \leq \neg\neg \div x \Leftrightarrow \neg \div y \leq \div x$, and
   $\neg \div y \leq \div x \Leftrightarrow \neg \div x \leq \neg\neg \div y \Leftrightarrow \neg \div x \leq \div y \Rightarrow^T \neg \div x \leq \neg y \Leftrightarrow \neg\neg y \leq \neg\neg \div x \Leftrightarrow y \leq \neg\neg \div x \Leftrightarrow y \leq \div x$
4. $y \leq \div x \Leftrightarrow \neg \div x \leq \neg y \Leftrightarrow \div\neg y \leq \div\neg \div x \Leftrightarrow^4 \div\neg y \leq \div x$

□

We now procede to show that the rules of the proposed system are indeed sound with respect to S5 algebras. We do not show the soundness of the rules for the classical connectives.

Lemma 4.2.4. *The new rules extending the LK system are sound*

1. $(\div \vdash)$ *is sound and invertible*
2. $(\div\wedge \vdash)$ *is sound and invertible*
3. $(\vdash \div)$ *with* $clopen(\Gamma, \Delta)$ *is sound and invertible*
4. $(\vdash \div)$ *with* $clopen(A)$ *is sound and invertible*
5. $(\vdash \div\wedge)$ *is sound*
6. $(\vdash \div\neg\wedge)$ *is sound and invertible*

Proof. In all cases we show that, given an arbitrary model, M, and an arbitrary valuation, v, satisfying the premiss/premisses of the topical rule also satisfies the conclusion.

1. Given an arbitrary model, M, and an arbitrary valuation, v, let $\bigwedge G = [\![\Gamma]\!]_v^M, \bigvee D = [\![\Delta]\!]_v^M$ and $a = [\![A]\!]_v^M$. Assume the premiss is valid, i.e. that $\bigwedge G \wedge \div a \leq a \vee \bigvee D$, it follows that, in this model, with this valuation, the conclusion is also valid:

$$\bigwedge G \wedge \div a \leq a \vee \bigvee D \Leftrightarrow \bigwedge G \wedge \div a \wedge \neg a \leq \bigvee D \Leftrightarrow^{T} \bigwedge G \wedge \div a \leq \bigvee D$$

2. It is shown in [5] in terms of the closure operator, C, in closure algebras that whenever b is *closed* then $C(x \wedge y) = C(x) \wedge y$ for any x in any closure algebra which is an S5 algebra. Interpreting C as $\neg\div$, we obtain that if $b = \neg \div b'$, i.e. that b is *closed*: $\neg \div (a \wedge b) = (\neg \div a) \wedge b$. Also that every *open* element is *closed*, letting $b = \div b'$ is a sufficient condition. Thus the result given is that $\neg \div (a \wedge \div b') = (\neg \div a) \wedge \div b'$, and it follows that $(*)\ \div(a \wedge \div b') = \div a \vee \neg \div b'$.

   We can show this[1] in terms of $\div$:
   1) $a \wedge \div b' \leq a \Leftrightarrow^{K} \div a \leq \div(a \wedge \div b') \Leftrightarrow \neg \div (a \wedge \div b') \leq \neg \div a$, and
   2) $a \wedge \div b' \leq \div b' \Leftrightarrow^{K} \div \div b' \leq \div(a \wedge \div b') \Leftrightarrow^{5} \neg \div b' \leq \div(a \wedge \div b') \Leftrightarrow \neg \div (a \wedge \div b') \leq \neg\neg \div b' \Leftrightarrow \neg \div (a \wedge \div b') \leq \div b'$.
   From 1) and 2) we get

$$\neg \div (a \wedge \div b') \leq \neg \div a \text{ and } \neg \div (a \wedge \div b') \leq \div b' \Leftrightarrow \neg \div (a \wedge \div b') \leq \neg \div a \wedge \div b'$$

   We also have that (from B) $a \wedge \div b' \leq \div \div (a \wedge \div b') \Leftrightarrow^{5} a \wedge \div b' \leq \neg \div (a \wedge \div b') \Rightarrow \neg \div a \wedge \div b' \leq \neg \div (a \wedge \div b')$ where the last implication holds since all elements, with a possible exception for a, are *clopen*(Lemma 4.2.3).

$$\begin{array}{lll} & \text{And thus } \neg \div (a \wedge \div b') \leq \neg \div a \wedge \div b' \text{ and } \neg \div (a \wedge \div b') \geq \neg \div a \wedge \div b' & \\ & \Leftrightarrow \quad \neg \div (a \wedge \div b') = \neg \div a \wedge \div b' & \\ & \Leftrightarrow \quad \neg \div (a \wedge \div b') = \neg \div a \wedge \neg\neg \div b' & \\ & \Leftrightarrow \quad \neg \div (a \wedge \div b') = \neg(\div a \vee \neg \div b') & \\ (*) & \Leftrightarrow \quad \div(a \wedge \div b') = \div a \vee \neg \div b' & \end{array}$$

   Given an arbitrary model, M, and an arbitrary valuation, v. Let $\bigwedge G = [\![\Gamma]\!]_v^M, \bigvee D = [\![\Delta]\!]_v^M, a = [\![A]\!]_v^M$ and $b = [\![B]\!]_v^M$. Validity of the premises yields validity of the conclusion:

$$\begin{array}{lll} \bigwedge G \wedge \div a \leq \bigvee D \text{ and } \bigwedge G \leq \div b \vee \bigvee D & \Leftrightarrow & \bigwedge G \wedge \div a \leq \bigvee D \text{ and } \bigwedge G \wedge \neg \div b \leq \bigvee D \\ & \Leftrightarrow & \bigwedge G \wedge (\div a \vee \neg \div b) \leq \bigvee D \\ & \Leftrightarrow^{(*)} & \bigwedge G \wedge \div(a \wedge \div b) \leq \bigvee D \end{array}$$

[1] We give an independent proof as the axioms specified for *closure algebras* in [5] are different than those used in our definition.

3. We can refer to clopen formulae as formulae starting with $\div$. Given an arbitrary model, M, and an arbitrary valuation, v, let $\bigwedge \div G = [\![\Gamma]\!]_v^M, \bigvee \div D = [\![\Delta]\!]_v^M, a = [\![A]\!]_v^M$. We can consider a bound by a clopen element and apply Lemma 4.2.3.

$$\bigwedge \div G \wedge a \leq \bigvee \div D \Leftrightarrow \bigwedge \div G \wedge \neg \div a \leq \bigvee \div D \Leftrightarrow \bigwedge \div G \leq \div a \vee \bigvee \div D$$

4. As A is a clopen formula, we denote its valuation as $\div a$. Let $\bigwedge G = [\![\Gamma]\!]_v^M, \div a = [\![A]\!]_v^M$ and $\bigvee D = [\![\Delta]\!]_v^M$.

$$\bigwedge G \wedge \div a \leq \bigvee D \Leftrightarrow \wedge G \leq \neg \div a \vee \bigvee D \Leftrightarrow^K \bigwedge G \leq \div \div a \vee \bigvee D$$

5. By B we have that $a \leq \div \div a$ and we also get $a \wedge b \leq (\div \div a) \wedge (\div \div b)$. By 5. we have that $\div a \vee \div b = \div(\div \div a \wedge \div \div b)$ and by K we have that if $x \leq y$ then $\div y \leq \div x$, so let $\bigwedge G = [\![\Gamma]\!]_v^M, a = [\![A]\!]_v^M, b = [\![B]\!]_v^M$ and $\bigvee D = [\![\Delta]\!]_v^M$, then

$$\begin{array}{lll} \bigwedge G \leq \div a \vee \div b \vee \bigvee D & \Leftrightarrow & \bigwedge G \leq \neg(\neg \div a \wedge \neg \div b) \vee \bigvee D \\ & \Leftrightarrow^5 & \bigwedge G \leq \neg(\div \div a \wedge \div \div b) \vee \bigvee D \\ & \Leftrightarrow^K & \bigwedge G \leq \neg \div (\div a \vee \div b) \vee \bigvee D \\ & \Leftrightarrow^5 & \bigwedge G \leq \div \div (\div a \vee \div b) \vee \bigvee D \\ & \Leftrightarrow^K & \bigwedge G \leq \div(\div \div a \wedge \div \div b) \vee \bigvee D \\ & \Rightarrow^{K+B} & \bigwedge G \leq \div(a \wedge b) \vee \bigvee D \qquad\qquad (a \wedge b \leq \div \div a \wedge \div \div b) \end{array}$$

Thus the conclusion is valid in M under valuation v.

6. Given an arbitrary model, M, and an arbitrary valuation, v, let $\bigwedge G = [\![\Gamma]\!]_v^M, a = [\![A]\!]_v^M, b = [\![B]\!]_v^M$ and $\bigvee D = [\![\Delta]\!]_v^M$. Assuming the premiss is valid, so is the conclusion:

$$\bigwedge G \leq \div\neg a \wedge \div\neg b \vee \bigvee D \Leftrightarrow^K \bigwedge G \leq \div(\neg a \vee \neg b) \vee \bigvee D \Leftrightarrow \bigwedge G \leq \div\neg(a \vee b) \vee \bigvee D$$

$\square$

4.3 Admissible rules

Lemma 4.3.1. *The following rules are admissible*

1. $\dfrac{\Gamma, \neg A \vdash \Delta}{\Gamma \vdash A, \Delta}$ *is admissible and the length of the derivation of the conclusion is not longer than the length of the derivation of the premiss.*

2. *The rule* $\dfrac{\Gamma, A \wedge B \vdash \Delta}{\Gamma, A, B \vdash \Delta}$ *is admissible.*

3. *The rules* $\dfrac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash A, \Delta}$ *and* $\dfrac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash B, \Delta}$ *are admissible and the length of the derivation of the conclusion is not longer than the length of the derivation of the premiss in either case.*

Proof.

1. The proofs of $\dfrac{\Gamma \vdash \neg A, \Delta}{\Gamma, A \vdash \Delta}$ and $\dfrac{\Gamma, \neg A \vdash \Delta}{\Gamma \vdash A, \Delta}$ are very similar and we show only the proof of the first rule. Assume we have a derivation, δ, for the premiss:

$$\begin{array}{c} \vdots\ \delta \\ \Gamma \vdash \neg A, \Delta \end{array}$$

Base case The shortest possible derivations of the premiss are (ax) followed by either $(\vdash W)$ or $(\vdash \neg)$: If the last rule applied is $(\vdash W)$ we weaken instead with A; the obtained derivation is not longer than the original derivation:

$$\dfrac{\dfrac{}{p \vdash p}\ (ax)}{p \vdash \neg A, p}\ (\vdash W) \quad \rightsquigarrow \quad \dfrac{\dfrac{}{p \vdash p}\ (ax)}{p, A \vdash p}\ (W \vdash)$$

Otherwise, if the last rule is $(\vdash \neg)$ we drop this application; the obtained derivation is shorter than the original derivation:

$$\dfrac{\dfrac{}{p \vdash p}\ (ax)}{\vdash p, \neg p}\ (\vdash \neg) \quad \rightsquigarrow \quad \dfrac{}{p \vdash p}\ (ax)$$

Induction step For a derivation , δ, in which some rule, say (R), is the last rule applied, longer than the base case, the derivation is of the form

$$\delta = \left\{ \dfrac{\begin{matrix} \vdots\ \delta_p \\ \Gamma, X \vdash Y, \Delta \end{matrix}}{\Gamma, X' \vdash Y', \neg A, \Delta}\ (R) \right.$$

If $\neg A$ is principal in (R) then (R) is either $(\vdash W)$ or $(\vdash \neg)$: If $\neg A$ is principal, the last rule applied in δ is either $(\vdash \neg)$ or $(\vdash W)$, in which case we have:

$$\dfrac{\begin{matrix} \vdots\ \delta_p \\ \Gamma, A \vdash \Delta \end{matrix}}{\Gamma \vdash \neg A, \Delta}\ (\vdash \neg) \quad \rightsquigarrow \quad \begin{matrix} \vdots\ \delta_p \\ \Gamma, A \vdash \Delta \end{matrix} \quad \text{or otherwise} \quad \dfrac{\begin{matrix} \vdots\ \delta_p \\ \Gamma \vdash \Delta \end{matrix}}{\Gamma \vdash \neg A, \Delta}\ (\vdash W) \quad \rightsquigarrow \quad \dfrac{\begin{matrix} \vdots\ \delta_p \\ \Gamma \vdash \Delta \end{matrix}}{\Gamma, A \vdash \Delta}\ (\vdash W)$$

In both cases we obtain the required sequent and the length of the derivation of the conclusion is no longer than that of original derivation.

Otherwise, if $\neg A$ is not principal, $\neg A$ is parametric in (R) and we can apply the induction hypothesis to obtain a new derivation for the required sequent. We show some cases, including the case of the $(\vdash \div)$ rule - the only rule with requirements on the context, and an example of a rule with more than one premiss:

- If the last rule applied in the derivation of the premiss is $(\vdash \div)$

$$\dfrac{\begin{matrix} \vdots\ \delta_p \\ \overline{\Gamma}, B \vdash \neg\overline{A}, \overline{\Delta} \end{matrix}}{\overline{\Gamma} \vdash \div B, \neg\overline{A}, \overline{\Delta}}\ (\vdash \div) \quad \rightsquigarrow^{IH} \quad \dfrac{\begin{matrix} \vdots\ \delta'_p \\ \overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta} \end{matrix}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\ (\vdash \div)$$

  In a derivation, δ, in which the $(\vdash \div)$ is applied with $\neg A$ parametric, then $\neg A$ is clopen - which implies that A is clopen.

- If the last rule applied is $(\vdash \div)_B$ with $\neg A$ as a parametric formula, we apply the induction hypothesis to the derivation of the premiss as follows:

$$\dfrac{\begin{matrix} \vdots\ \delta_p \\ \Gamma, \overline{B} \vdash \neg A, \Delta \end{matrix}}{\Gamma \vdash \div\overline{B}, \neg A, \Delta}\ (\vdash \div)_B \quad \rightsquigarrow^{IH} \quad \dfrac{\begin{matrix} \vdots\ \delta'_p \\ \Gamma, A, \overline{B} \vdash \Delta \end{matrix}}{\Gamma, A \vdash \div\overline{B}, \Delta}\ (\vdash \div)_B$$

- We show one example of a rule with more than one premiss, the $(\div\wedge\vdash)$-rule:

$$\dfrac{\dfrac{\vdots\ \delta_{p_1}}{\Gamma, \div B \vdash \neg A, \Delta} \quad \dfrac{\vdots\ \delta_{p_2}}{\Gamma \vdash \div C, \neg A, \Delta}}{\Gamma, \div(B \wedge \div C) \vdash \neg A, \Delta}\ (\div\wedge\vdash) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\vdots\ \delta'_{p_1}}{\Gamma, A, \div B \vdash \Delta} \quad \dfrac{\vdots\ \delta'_{p_2}}{\Gamma, A \vdash \div C, \Delta}}{\Gamma, A, \div(B \wedge \div C) \vdash \Delta}\ (\div\wedge\vdash)$$

  We apply the induction hypothesis to both premisses of the rule and obtain derivation of two sequents to which we can again apply the $(\div\wedge\vdash)$-rule and obtain the desired sequent; the length of the derivation is the same length as, i.e. no longer than, the length of the original derivation. We can apply the induction hypothesis to the derivations of both premisses since the length of the derivation of the conclusion is $|\delta| = 1 + |\delta_{p_1}| + |\delta_{p_2}|$, and thus the derivations of either premiss are shorter than that of the conclusion of the (R) rule; $|\delta_{p_1}| < |\delta|$ and $|\delta_{p_2}| < |\delta|$.

2. We prove that $\dfrac{\Gamma, A \wedge B \vdash \Delta}{\Gamma, A, B \vdash \Delta}$ is admissible by induction on the length of the derivation of the premiss. Assume there is a derivation of the premiss

$$\dfrac{\vdots\ \delta}{\Gamma, A \wedge B \vdash \Delta}$$

Base case The shortest possible derivation of the premiss is by (ax) followed by $(W\vdash)$. We have:

$$\dfrac{\dfrac{}{p \vdash p}\ (ax)}{p, A \wedge B \vdash p}\ (W\vdash) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{p \vdash p}\ (ax)}{p, A \vdash p}\ (W\vdash)}{p, A, B \vdash p}\ (W\vdash)$$

Induction step Of the remaining cases we consider first such derivations where $A \wedge B$ is principal in the last rule applied in δ, this rule is either $(W\vdash)$ or $(\wedge\vdash)$:

- If the last rule applied is $(W\vdash)$ we weaken with A and B:

$$\dfrac{\dfrac{\vdots\ \delta}{\Gamma \vdash \Delta}}{\Gamma, A \wedge B \vdash \Delta}\ (W\vdash) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdots\ \delta}{\Gamma \vdash \Delta}}{\Gamma, A \vdash \Delta}\ (W\vdash)}{\Gamma, A, B \vdash \Delta}\ (W\vdash)$$

- If the last rule applied is $(\wedge\vdash)$, we drop this application:

$$\dfrac{\dfrac{\vdots\ \delta}{\Gamma, A, B \vdash \Delta}}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge\vdash) \quad \rightsquigarrow \quad \dfrac{\vdots\ \delta}{\Gamma, A, B \vdash \Delta}$$

If $A \wedge B$ is not principal in the last rule applied, we apply the induction hypothesis to the premiss of that rule. In all cases the rules can be applied as they were. We only two some examples of this, the example of when the context sensitive $(\vdash\div)$-rule is applied and an example of the case for the rule applied has two premisses:

- If last rule applied in the derivation of the premiss is the $(\vdash\div)$ rule requiring *clopen* context, then *clopen*$(A \wedge B)$ which imples that both *clopen*(A) and *clopen*(B):

$$\dfrac{\dfrac{\vdots\,\delta}{\overline{\Gamma}, \overline{A} \wedge \overline{B}, C \vdash \overline{\Delta}}}{\overline{\Gamma}, \overline{A} \wedge \overline{B} \vdash \div C, \overline{\Delta}}\,(\vdash \div) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\vdots\,\delta}{\overline{\Gamma}, \overline{A}, \overline{B}, C \vdash \overline{\Delta}}}{\overline{\Gamma}, \overline{A}, \overline{B} \vdash \div C, \overline{\Delta}}\,(\vdash \div)$$

- If the last rule applied is $(\vdash \wedge)$, we have

$$\dfrac{\overset{\vdots}{\Gamma, A \wedge B \vdash C, \Delta} \quad \overset{\vdots}{\Gamma, A \wedge B \vdash D, \Delta}}{\Gamma, A \wedge B \vdash C \wedge D, \Delta}\,(\vdash \wedge) \quad \rightsquigarrow^{IH} \quad \dfrac{\overset{\vdots}{\Gamma, A, B \vdash C, \Delta} \quad \overset{\vdots}{\Gamma, A, B \vdash D, \Delta}}{\Gamma, A \wedge B \vdash C \wedge D, \Delta}\,(\vdash \wedge)$$

3. The proofs of $\dfrac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash A, \Delta}$ and $\dfrac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash B, \Delta}$ are similar, we show some cases of the proof of the former. In both proofs we use the argument that if $clopen(A \wedge B)$ then also both $clopen(A)$ and $clopen(B)$ in the case when the formulae are parametric in an application of the $(\vdash \div)$-rule. Both proofs are proofs by induction on the length of the derivation of the premiss.
Assume there is a derivation

$$\dfrac{\vdots\,\delta}{\Gamma \vdash A \wedge B, \Delta}$$

Base case The shortest possible case is when (ax) is followed by an application of $(\vdash W)$:

$$\dfrac{\dfrac{}{p \vdash p}\,(ax)}{p \vdash A \wedge B, p}\,(\vdash W) \quad \rightsquigarrow \quad \dfrac{\dfrac{}{p \vdash p}\,(ax)}{p \vdash A, p}\,(\vdash W)$$

Induction step For longer derivations of the premiss, we first consider the cases where $A \wedge B$ is principal in the last rule applied in the derivation. If $A \wedge B$ is principal in the last rule applied in the sequent, then the last rule applied is either $(\vdash W)$ or $(\vdash \wedge)$, we have the following two cases:

$$\dfrac{\dfrac{\vdots\,\delta}{\Gamma \vdash \Delta}}{\Gamma \vdash A \wedge B, \Delta}\,(\vdash W) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdots\,\delta}{\Gamma \vdash \Delta}}{\Gamma \vdash A, \Delta}\,(\vdash W)$$

$$\dfrac{\overset{\vdots\,\delta_1}{\Gamma \vdash A, \Delta} \quad \overset{\vdots\,\delta_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}\,(\vdash \wedge) \quad \rightsquigarrow \quad \overset{\vdots\,\delta_1}{\Gamma \vdash A, \Delta}$$

Otherwise, if $A \wedge B$ is not principal in the last rule applied in the derivation, we apply the induction hypothesis to the derivation of its premiss and obtain derivations, δ_1' and δ_2', for two sequents to both of which we can apply the topical rule. We show only two examples, that of the $(\vdash \div)$ rule which requires that the context is clopen, and the $(\div\wedge \vdash)$-rule which has more than one premiss:

- If the last rule applied in the derivation is $(\vdash \div)$, then $A \wedge B$ is clopen, this implies that also both A and B are clopen:

$$\dfrac{\dfrac{\vdots\,\delta}{\overline{\Gamma}, C \vdash \overline{A \wedge B}, \overline{\Delta}}}{\overline{\Gamma} \vdash \div C, \overline{A \wedge B}, \overline{\Delta}}\,(\vdash \div) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\vdots\,\delta'}{\overline{\Gamma}, C \vdash \overline{A}, \overline{\Delta}}}{\overline{\Gamma} \vdash \div C, \overline{A}, \overline{\Delta}}\,(\vdash \div)$$

- If the last rule applied in the derivation is $(\div\wedge\vdash)$ we apply the induction hypothesis to both premisses, obtaining derivations for four sequents, we apply the $(\div\wedge\vdash)$ to both appropriate pairs:

$$\dfrac{\dfrac{\vdots\ \delta_1}{\Gamma, \div C \vdash A \wedge B, \Delta} \quad \dfrac{\vdots\ \delta_2}{\Gamma \vdash \div D, A \wedge B, \Delta}}{\Gamma, \div(C \wedge \div D) \vdash A \wedge B, \Delta}\ (\div\wedge\vdash) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\vdots\ \delta_1'}{\Gamma, \div C \vdash A, \Delta} \quad \dfrac{\vdots\ \delta_2'}{\Gamma \vdash \div D, A, \Delta}}{\Gamma, \div(C \wedge \div D) \vdash A, \Delta}\ (\div\wedge\vdash)$$

□

Lemma 4.3.2. *The following rules are admissible*

1. $\dfrac{\Gamma, \div\overline{A} \vdash \Delta}{\Gamma \vdash \overline{A}, \Delta}$ *with clopen*$(\overline{A})$
2. $\dfrac{\Gamma \vdash \div\overline{A}, \Delta}{\Gamma, \overline{A} \vdash \Delta}$ *with clopen*$(\overline{A})$
3. $\dfrac{\Gamma \vdash \div(A \wedge \div B), \Delta}{\Gamma, \div B \vdash \div A, \Delta}$
4. $\dfrac{\Gamma, \div\neg(A \wedge B) \vdash \Delta}{\Gamma, \div\neg A, \div\neg B \vdash \Delta}$
5. *The rules* $\dfrac{\Gamma, \div(A \wedge B) \vdash \Delta}{\Gamma, \div A \vdash \Delta}$ *and* $\dfrac{\Gamma, \div(A \wedge B) \vdash \Delta}{\Gamma, \div B \vdash \Delta}$ *are admissible*

Proof.

1. We prove that the rule $\dfrac{\Gamma, \div\overline{A} \vdash \Delta}{\Gamma \vdash \overline{A}, \Delta}$ is admissible by induction on the length of the derivation of the premiss. There are three rules in which $\div\overline{A}$ can be principal; $(W\vdash)$, $(\div\vdash)$ and $(\div\wedge\vdash)$. Assume there is a derivation, δ, with:

$$\dfrac{\vdots\ \delta}{\Gamma, \div\overline{A} \vdash \Delta}$$

Base case The shortest possible proof is an application of (ax) followed by an application of $(\vdash W)$:

$$\dfrac{\dfrac{}{p \vdash p}\ (ax)}{p, \div\overline{A} \vdash p}\ (\vdash W) \quad \rightsquigarrow \quad \dfrac{\dfrac{}{p \vdash p}\ (ax)}{p \vdash \overline{A}, p}\ (W\vdash)$$

Induction step If $\div\overline{A}$ is principal in the last rule applied in δ we have one of the three following cases:

- If the last rule applied is $(W\vdash)$, we weaken with $\overline{A}$ instead:

$$\dfrac{\dfrac{\vdots\ \delta}{\Gamma \vdash \Delta}}{\Gamma, \div\overline{A} \vdash \Delta}\ (W\vdash) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdots\ \delta}{\Gamma \vdash \Delta}}{\Gamma \vdash \overline{A}, \Delta}\ (\vdash W)$$

- If the last rule applied is $(\div\vdash)$, we have:

$$\dfrac{\dfrac{\vdots\ \delta}{\Gamma, \div\overline{A} \vdash \overline{A}, \Delta}}{\Gamma, \div\overline{A} \vdash \Delta}\ (\div\vdash) \quad \rightsquigarrow^{IH} \quad \dfrac{\vdots\ \delta}{\Gamma \vdash \overline{A}, \Delta}$$

where he active formula, $\div\overline{A}$, in the premiss of the $(\div\vdash)$-rule has shorter derivation than that of the conclusion of the rule, so we can apply the induction hypothesis.

- Finally, if the last rule applied is $(\div\wedge\vdash)$, the formula is of the form $A' \wedge \div B$; as it is clopen, we also know that A' is clopen: $\overline{A'} \wedge \div B$,

$$\dfrac{\Gamma, \div\overline{A'} \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(\overline{A'} \wedge \div B) \vdash \Delta} (\div\wedge\vdash) \quad \leadsto^{IH} \quad \dfrac{\Gamma \vdash \overline{A'}, \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma \vdash \overline{A'} \wedge \div B, \Delta} (\vdash\wedge)$$

If $\div\overline{A}$ is not principal in the last rule applied in the derivation, then we can apply the induction hypothesis to obtain a derivation with $\overline{A}$ in the right hand side of the sequent, then the topical rule can be applied as it were. $\overline{A}$ is *clopen*, so also in the case of the context sensitive rule, $(\vdash\div)$, this is possible. We show the examples of the context sensitive rule and a rule with two premisses:

- If the last rule applied in the derivation is $(\vdash\div)$:

$$\dfrac{\overline{\Gamma}, B \vdash \div\overline{A}, \overline{\Delta}}{\overline{\Gamma} \vdash \div\overline{A}, \div B, \overline{\Delta}} (\vdash\div) \quad \leadsto^{IH} \quad \dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}} (\vdash\div)$$

- If the last rule was, say $(\vdash\wedge)$, then we apply the induction hypothesis to both premisses and then apply $(\vdash\wedge)$ with the obtained sequents as premisses:

$$\dfrac{\Gamma \vdash \div\overline{A}, B, \Delta \quad \Gamma \vdash \div\overline{A}, C, \Delta}{\Gamma \vdash \div\overline{A}, B \wedge C, \Delta} (\vdash\wedge) \quad \leadsto^{IH} \quad \dfrac{\Gamma, \overline{A} \vdash B, \Delta \quad \Gamma, \overline{A} \vdash C, \Delta}{\Gamma, \overline{A} \vdash B \wedge C, \Delta} (\vdash\wedge)$$

2. We prove that the rule $\dfrac{\Gamma \vdash \div\overline{A}, \Delta}{\Gamma, \overline{A} \vdash \Delta}$ by nested induction on primarily the complexity of $\overline{A}$, $|\overline{A}|$, and secondarily on length of the derivation of the premiss. Assume there is a derivation of the premiss, δ:

$$\begin{array}{c} \vdots\ \delta \\ \Gamma \vdash \div\overline{A}, \Delta \end{array}$$

Since $\overline{A}$ is clopen, the lowest complexity A can have is 2, when $A = \div A'$, for atomic A'.

Base case The shortest possible proof of the premiss with $|\overline{A}| = 2$, say $\overline{A} = \div p$, is show below. We weaken by $\div p$ instead:

$$\dfrac{\dfrac{}{q \vdash q} (ax)}{q \vdash \div\div p, q} (\vdash W) \quad \leadsto \quad \dfrac{\dfrac{}{q \vdash q} (ax)}{q, \div p \vdash q} (W\vdash)$$

For longer derivations of the premiss where $|\overline{A}| = 2$, there are three rules which might be the last rule applied in the derivation with $\overline{A}$ principal. These are $(\vdash W)$, $(\vdash\div)$, and $(\vdash\div)_{\overline{A}}$:

- If the last rule applied is $(\vdash W)$, we weaken by $\overline{A}$ instead of $\div\overline{A}$ obtaining:

$$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \div\overline{A}, \Delta} (\vdash W) \quad \leadsto \quad \dfrac{\Gamma \vdash \Delta}{\Gamma, \overline{A} \vdash \Delta} (W\vdash)$$

- If the last rule applied is either $(\vdash \div)$ requiring *clopen* context or $(\vdash \div)_{\overline{A}}$, we drop this application:

$$\dfrac{\begin{matrix}\vdots \\ \overline{\Gamma}, \overline{A} \vdash \overline{\Delta}\end{matrix}}{\overline{\Gamma} \vdash \div\overline{A}, \overline{\Delta}}\ (\vdash \div) \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ \overline{\Gamma}, \overline{A} \vdash \overline{\Delta}\end{matrix} \quad \text{and} \quad \dfrac{\begin{matrix}\vdots \\ \Gamma, \overline{A} \vdash \Delta\end{matrix}}{\Gamma \vdash \div\overline{A}, \Delta}\ (\vdash \div)_{\overline{A}} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ \Gamma, \overline{A} \vdash \Delta\end{matrix}$$

In all cases where $\overline{A}$ was not principal in the last rule applied in the derivation, δ, we apply the induction hypothesis to the premiss. We can apply the induction hypothesis since the complexity of $\overline{A}$ is the same in the conclusion as in the premiss, but the length of the derivation of the premiss is shorter than that of the conclusion. Also, if $\overline{A}$ is parametric in the context sensitive $(\vdash \div)$-rule, we can do exactly this, since $clopen(\overline{A})$.

- If the last rule applied in the derivation of the premiss is $(\vdash \div)$, we have

$$\dfrac{\begin{matrix}\vdots \\ \overline{\Gamma}, B \vdash \div\overline{A}, \overline{\Delta}\end{matrix}}{\overline{\Gamma} \vdash \div\overline{A}, \div B, \overline{\Delta}}\ (\vdash \div) \quad \rightsquigarrow^{IH} \quad \dfrac{\begin{matrix}\vdots \\ \overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}\end{matrix}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\ (\vdash \div)$$

The above case and the remaining cases are identical to the corresponding cases in the induction step (for arbitrary $|\overline{A}|$), and are omitted here.

Induction step For arbitrary formulae $\overline{A}$ the main connective of $\overline{A}$ needn't be $\div$, there are now two more rules in which $\overline{A}$ can be principal. We consider first the cases where $\div\overline{A}$ is principal in the last rule applied in the derivation. The candidate rules are $(\vdash W)$, $(\vdash \div)$, $(\vdash \div)_{\overline{A}}$, $(\vdash \div\wedge)$ and $(\vdash \div\neg\wedge)$:

- If $\div\overline{A}$ was introduced with $(\vdash W)$, we weaken by $\overline{A}$ with $(W \vdash)$ instead:

$$\dfrac{\begin{matrix}\vdots\ \delta \\ \Gamma \vdash \Delta\end{matrix}}{\Gamma \vdash \div\overline{A}, \Delta}\ (\vdash W) \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\vdots\ \delta \\ \Gamma \vdash \Delta\end{matrix}}{\Gamma, \overline{A} \vdash \Delta}\ (W \vdash)$$

- If the last rule applied is $(\vdash \div)$ with clopen context, $clopen(\Gamma, \Delta)$ or $(\vdash \div)_{\overline{A}}$ we drop this application:

$$\dfrac{\begin{matrix}\vdots\ \delta \\ \overline{\Gamma}, \overline{A} \vdash \overline{\Delta}\end{matrix}}{\overline{\Gamma} \vdash \div\overline{A}, \overline{\Delta}}\ (\vdash \div) \quad \rightsquigarrow \quad \begin{matrix}\vdots\ \delta \\ \overline{\Gamma}, \overline{A} \vdash \overline{\Delta}\end{matrix} \quad \text{and} \quad \dfrac{\begin{matrix}\vdots\ \delta \\ \Gamma, \overline{A} \vdash \Delta\end{matrix}}{\Gamma \vdash \div\overline{A}, \Delta}\ (\vdash \div)_{\overline{A}} \quad \rightsquigarrow \quad \begin{matrix}\vdots\ \delta \\ \Gamma, \overline{A} \vdash \Delta\end{matrix}$$

- If the last rule applied was $(\vdash \div\wedge)$, let $A = A_1 \wedge A_2$. As $clopen(A)$, we also have $clopen(A_1)$ and $clopen(A_2)$.

$$\dfrac{\begin{matrix}\vdots\ \delta \\ \Gamma \vdash \div\overline{A_1}, \div\overline{A_2}, \Delta\end{matrix}}{\Gamma \vdash \div(\overline{A_1} \wedge \overline{A_2}), \Delta}\ (\vdash \div\wedge) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\vdots\ \delta \\ \Gamma \vdash \div\overline{A_1}, \div\overline{A_2}, \Delta\end{matrix}}{\Gamma, \overline{A_1} \vdash \div\overline{A_2}, \Delta}\ IH}{\Gamma, \overline{A_1}, \overline{A_2} \vdash \Delta}\ IH$$

Where the first application of the induction hypothesis is applied to $\overline{A_1}$ which has lower complexity than $\overline{A}$ and a shorter derivation than the conclusion, and the second application of the induction hypothesis is applied to $\overline{A_2}$ which has lower complexity than $\overline{A}$.

- Finally if the last rule applied was $(\vdash \div\neg\wedge)$, say $\overline{A} = \neg(\overline{A_1} \wedge \overline{A_2})$. We apply the admissible rules $\dfrac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash A, \Delta}$ and $\dfrac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash B, \Delta}$ (Lemma 4.3.1) obtaining derivations δ_1 and δ_2:

$$\dfrac{\begin{matrix}\vdots\ \delta\\ \Gamma \vdash \div\neg\overline{A_1} \wedge \div\neg\overline{A_2}, \Delta\end{matrix}}{\Gamma \vdash \div\neg(\overline{A_1} \wedge \overline{A_2}), \Delta}\ (\vdash \div\neg\wedge) \quad \rightsquigarrow^{L.4.3.1} \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \delta_1\\ \Gamma \vdash \div\neg\overline{A_1} \vdash \Delta\end{matrix}}{\Gamma, \neg\overline{A_1} \vdash \Delta}\ IH}{\Gamma \vdash \overline{A_1}, \Delta}\ (L.4.3.1) \quad \dfrac{\dfrac{\begin{matrix}\vdots\ \delta_2\\ \Gamma \vdash \div\neg\overline{A_2} \vdash \Delta\end{matrix}}{\Gamma, \neg\overline{A_2} \vdash \Delta}\ IH}{\Gamma \vdash \overline{A_2}, \Delta}\ (L.4.3.1)}{\Gamma \vdash \overline{A_1} \wedge \overline{A_2}, \Delta}\ (\vdash \wedge)}{\Gamma, \neg(\overline{A_1} \wedge \overline{A_2}) \vdash \Delta}\ (\neg \vdash)$$

If $\div\overline{A}$ was not principal in the last rule applied we apply the induction hypothesis to the premiss of this rule obtaining a sequent to which we can apply the same rule, obtaining in turn the required sequent. This is also the case for the context sensitive rule, $(\vdash \div)$, as $\overline{A}$ is *clopen*. In all the cases, if the last rule applied is (R), where $\div\overline{A}$ is parametric, we apply the induction hypothesis to the premiss/premisses and then apply (R). We show only the examples where (R) is the context sensitive $(\vdash \div)$-rule and one example where (R) is a rule with more than one premiss, here represented by the $(\div\wedge \vdash)$-rule:

- When (R) is the context sensitive $(\vdash \div)$-rule we have

$$\dfrac{\begin{matrix}\vdots\\ \overline{\Gamma}, B \vdash \div\overline{A}, \overline{\Delta}\end{matrix}}{\overline{\Gamma} \vdash \div B, \div\overline{A}, \overline{\Delta}}\ (\vdash \div) \quad \rightsquigarrow^{IH} \quad \dfrac{\begin{matrix}\vdots\\ \overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}\end{matrix}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\ (\vdash \div)$$

- When (R) is, e.g., $(\div\wedge \vdash)$, we have:

$$\dfrac{\begin{matrix}\vdots\\ \Gamma, \div B \vdash \div\overline{A}, \Delta\end{matrix} \quad \begin{matrix}\vdots\\ \Gamma \vdash \div C, \div\overline{A}, \Delta\end{matrix}}{\Gamma, \div(B \wedge \div C) \vdash \div\overline{A}, \Delta}\ (\div\wedge \vdash) \quad \rightsquigarrow^{IH} \quad \dfrac{\begin{matrix}\vdots\\ \Gamma, \overline{A}, \div B \vdash \Delta\end{matrix} \quad \begin{matrix}\vdots\\ \Gamma, \overline{A} \vdash \div C, \Delta\end{matrix}}{\Gamma, \overline{A}, \div(B \wedge \div C) \vdash \Delta}\ (\div\wedge \vdash)$$

3. We prove that $\dfrac{\Gamma \vdash \div(A \wedge \div B), \Delta}{\Gamma, \div B \vdash \div A, \Delta}$ is admissible by induction on the length of the derivation of the premiss, δ, where

$$\begin{matrix}\vdots\ \delta\\ \Gamma \vdash \div(A \wedge \div B), \Delta\end{matrix}$$

Base case The shortest possible proof is (ax) followed by $(\vdash W)$, in which case we have:

$$\dfrac{\dfrac{}{p \vdash p}\ (ax)}{p \vdash \div(A \wedge \div B), p}\ (\vdash W) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{p \vdash p}\ (ax)}{p, \div B \vdash p}\ (W \vdash)}{p, \div B \vdash \div A, p}\ (\vdash W)$$

Induction step We first consider the cases of when $\div(A \wedge \div B)$ was principal in the last rule applied in the derivation, δ, of the premiss. The possible rules are $(\vdash W)$, $(\vdash \div)$, $(\vdash \div)_{(A\wedge\div B)}$ and $(\vdash \div\wedge)$:

- If the last rule applied is $(\vdash W)$, we weaken by $\div B$ and $\div A$:

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma \vdash \Delta\end{array}}{\Gamma \vdash \div(A \wedge \div B), \Delta}\ (\vdash W) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma \vdash \Delta\end{array}}{\Gamma, \div B \vdash \Delta}\ (W \vdash)}{\Gamma, \div B \vdash \div A, \Delta}\ (\vdash W)$$

- If the last rule is $(\vdash \div)$ with *clopen* context, then we apply Lemma 4.3.1 and $(\vdash \div)$ obtaining the required sequent:

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ \overline{\Gamma}, A \wedge \div B \vdash \overline{\Delta}\end{array}}{\overline{\Gamma} \vdash \div(A \wedge \div B), \overline{\Delta}}\ (\vdash \div) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \delta\\ \overline{\Gamma}, A \wedge \div B \vdash \overline{\Delta}\end{array}}{\overline{\Gamma}, A, \div B \vdash \overline{\Delta}}\ (L.4.3.1)}{\overline{\Gamma}, \div B \vdash \div A, \overline{\Delta}}\ (\vdash \div)$$

- If the last rule applied in the derivation of the premiss is $(\vdash \div)_{A \wedge \div B}$, then since $A \wedge \div B$ is *clopen*, it follows that *clopen*(A):

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma, \overline{A} \wedge \div B \vdash \Delta\end{array}}{\Gamma \vdash \div(\overline{A} \wedge \div B), \Delta}\ (\vdash \div) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma, \overline{A} \wedge \div B \vdash \Delta\end{array}}{\Gamma, \overline{A}, \div B \vdash \Delta}\ (L.4.3.1)}{\Gamma, \div B \vdash \div \overline{A}, \Delta}\ (\vdash \div)$$

- Finally, if the last rule applied is $(\vdash \div\wedge)$, we apply the $\dfrac{\Gamma \vdash \div\overline{A}, \Delta}{\Gamma, \overline{A} \vdash \Delta}$-rule shown previously to be admissible:

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma \vdash \div A, \div\div B, \Delta\end{array}}{\Gamma \vdash \div(A \wedge \div B), \Delta}\ (\vdash \div\wedge) \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma \vdash \div A, \div\div B, \Delta\end{array}}{\Gamma, \div B \vdash \div A, \Delta}\ (L.4.3.2)$$

In all cases where the last rule applied in the derivation does not have $\div(A \wedge \div B)$ principal we apply the induction hypothesis to the premiss of the rule and then apply the given rule to the obtained sequent. This can also be done when the last rule applied is the context sensitive $(\vdash \div)$-rule:

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ \overline{\Gamma}, C \vdash \div(A \wedge \div B), \overline{\Delta}\end{array}}{\overline{\Gamma} \vdash \div(A \wedge \div B), \div C, \overline{\Delta}}\ (\vdash \div) \quad \rightsquigarrow^{IH} \quad \dfrac{\begin{array}{c}\vdots\ \delta\\ \overline{\Gamma}, \div B, C \vdash \div A, \overline{\Delta}\end{array}}{\overline{\Gamma}, \div B \vdash \div C, \div A, \overline{\Delta}}\ (\vdash \div)$$

4. We prove that $\dfrac{\Gamma, \div\neg(A \wedge B) \vdash \Delta}{\Gamma, \div\neg A, \div\neg B \vdash \Delta}$ is admissible by induction on the length of the derivation of the premiss: Assume there is a derivation, δ, for the premiss:

$$\begin{array}{c}\vdots\ \delta\\ \Gamma, \div\neg(A \wedge B) \vdash \Delta\end{array}$$

Base case The shortest possible proof is that of (ax) followed by $(W \vdash)$, in this case we weaken by $\div\neg A$ and $\div\neg B$ seperately:

$$\dfrac{\dfrac{}{p \vdash p}\,(ax)}{p, \div\neg(A \wedge B) \vdash p}\,(W\vdash) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{p \vdash p}\,(ax)}{p, \div\neg A \vdash p}\,(W\vdash)}{p, \div\neg A, \div\neg B \vdash p}\,(W\vdash)$$

Induction step There are two rules which can have the rule $\div\neg(A \wedge B)$ principal; $(W \vdash)$ and $(\div \vdash)$: If the formula is principal in the last rule applied in the derivation we have:

- We the last rule applied in the derivation is $(W \vdash)$, we have

$$\dfrac{\begin{matrix}\vdots\ \delta \\ \Gamma \vdash \Delta\end{matrix}}{\Gamma, \div\neg(A \wedge B) \vdash \Delta}\,(W\vdash) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\begin{matrix}\vdots\ \delta' \\ \Gamma \vdash \Delta\end{matrix}}{\Gamma, \div\neg A \vdash \Delta}\,(W\vdash)}{\Gamma, \div\neg A, \div\neg B \vdash \Delta}\,(W\vdash)$$

- Otherwise, the last rule applied in the derivation is $(\div \vdash)$ - we apply the induction hypothesis to the formula in the left hand side of the premiss:

$$\dfrac{\begin{matrix}\vdots\ \delta \\ \Gamma, \div\neg(A \wedge B) \vdash \neg(A \wedge B), \Delta\end{matrix}}{\Gamma, \div\neg(A \wedge B) \vdash \Delta}\,(\div\vdash) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \delta' \\ \Gamma, \div\neg A, \div\neg B \vdash \neg(A \wedge B), \Delta\end{matrix}}{\Gamma, \div\neg A, \div\neg B, A \wedge B \vdash \Delta}\,(L.4.3.1)}{\Gamma, \div\neg A, \div\neg B, A, B \vdash \Delta}\,(L.4.3.1)}{\Gamma, \div\neg A, \div\neg B, A \vdash \neg B, \Delta}\,(\vdash\neg)}{\Gamma, \div\neg A, \div\neg B \vdash \neg A, \neg B, \Delta}\,(\vdash\neg)}{\Gamma, \div\neg A, \div\neg B \vdash \neg B, \Delta}\,(\div\vdash)}{\Gamma, \div\neg A, \div\neg B \vdash \Delta}\,(\div\vdash)$$

If $\div\neg(A \wedge B)$ is not principal in an application of a rule in δ, we apply the induction hypothesis to the derivation of the premiss of the rule and then apply the same rule to the new sequent. We show some of the cases:

- If the last rule applied in δ is $(\vdash \div)$ with *clopen* context, we apply the induction hypothesis and then the $(\vdash \div)$-rule. As both $\div\neg(A \wedge B)$ and the obtained $\div\neg A$ and $\div\neg B$ are *clopen*, the context is *clopen* in the obtained derivation:

$$\dfrac{\begin{matrix}\vdots\ \delta \\ \overline{\Gamma}, \div\neg(A \wedge B), C \vdash \overline{\Delta}\end{matrix}}{\overline{\Gamma}, \div\neg(A \wedge B) \vdash \div C, \overline{\Delta}}\,(\vdash\div) \quad \rightsquigarrow^{IH} \quad \dfrac{\begin{matrix}\vdots\ \delta' \\ \overline{\Gamma}, \div\neg A, \div\neg B, C \vdash \overline{\Delta}\end{matrix}}{\overline{\Gamma}, \div\neg A, \div\neg B \vdash \div C, \overline{\Delta}}\,(\vdash\div)$$

- If the last rule applied is $(\vdash \wedge)$:

$$\dfrac{\begin{matrix}\vdots\ \delta_1 \\ \Gamma, \div\neg(A \wedge B) \vdash C, \Delta\end{matrix} \quad \begin{matrix}\vdots\ \delta_2 \\ \Gamma, \div\neg(A \wedge B) \vdash D, \Delta\end{matrix}}{\Gamma, \div\neg(A \wedge B) \vdash C \wedge D, \Delta}\,(\vdash\wedge)$$

$$\rightsquigarrow^{IH} \quad \dfrac{\begin{matrix}\vdots\ \delta'_1 \\ \Gamma, \div\neg A, \div\neg B \vdash C, \Delta\end{matrix} \quad \begin{matrix}\vdots\ \delta'_2 \\ \Gamma, \div\neg A, \div\neg B \vdash D, \Delta\end{matrix}}{\Gamma, \div\neg A, \div\neg B \vdash C \wedge D, \Delta}\,(\vdash\wedge)$$

5. We prove both the rules $\dfrac{\Gamma, \div(A \wedge B) \vdash \Delta}{\Gamma, \div A \vdash \Delta}$ and $\dfrac{\Gamma, \div(A \wedge B) \vdash \Delta}{\Gamma, \div B \vdash \Delta}$ are admissible by induction on the length of the derivation of the premiss:
We prove that the rule $\dfrac{\Gamma, \div(A \wedge B) \vdash \Delta}{\Gamma, \div A \vdash \Delta}$ is admissible, the proof of the other rule is similar.

Base case The shortest possible proof is that of (ax) followed by $(W \vdash)$:

$$\dfrac{\dfrac{}{p \vdash p}\,(ax)}{p, \div(A \wedge B) \vdash p}\,(W\vdash) \quad \rightsquigarrow \quad \dfrac{\dfrac{}{p \vdash p}\,(ax)}{p, \div A \vdash p}\,(W\vdash)$$

Induction step There are three rules which can have $\div(A \wedge B)$ as principal formula; $(W\vdash)$, $(\div\vdash)$ and $(\div\wedge\vdash)$. In these cases we obtain the required sequent as follows:

- If the last rule is $(W\vdash)$:

$$\dfrac{\stackrel{\vdots\ \delta}{\Gamma \vdash \Delta}}{\Gamma, \div(A \wedge B) \vdash \Delta}\,(W\vdash) \quad \rightsquigarrow \quad \dfrac{\stackrel{\vdots\ \delta}{\Gamma \vdash \Delta}}{\Gamma, \div A \vdash \Delta}\,(W\vdash)$$

- If the last rule is $(\div\vdash)$ we apply the induction hypothesis

$$\dfrac{\stackrel{\vdots\ \delta}{\Gamma, \div(A \wedge B) \vdash A \wedge B, \Delta}}{\Gamma, \div(A \wedge B) \vdash \Delta}\,(\div\vdash) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\stackrel{\vdots\ \delta_1'}{\Gamma, \div A \vdash A \wedge B, \Delta}}{\Gamma, \div A \vdash A, \Delta}\,(L.4.3.1)}{\Gamma, \div A \vdash \Delta}\,(\div\vdash)$$

- If the last rule applied is $(\div\wedge\vdash)$:

$$\dfrac{\stackrel{\vdots\ \delta_1}{\Gamma, \div A \vdash \Delta} \quad \stackrel{\vdots\ \delta_2}{\Gamma \vdash \div B, \Delta}}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge\vdash) \quad \rightsquigarrow \quad \stackrel{\vdots\ \delta_1}{\Gamma, \div A \vdash \Delta}$$

If $\div(A \wedge \div B)$ is not principal in the last rule applied in δ, we apply the induction hypothesis to its premiss and obtain derivations and obtain two sequents to which we can apply the rule and obtain the required sequents, eg. if the last rule applied is $(\vdash\div)$:

$$\dfrac{\stackrel{\vdots\ \delta}{\overline{\Gamma}, \div(A \wedge B), C \vdash \overline{\Delta}}}{\overline{\Gamma}, \div(A \wedge B) \vdash \div C, \overline{\Delta}}\,(\vdash\div) \quad \rightsquigarrow^{IH} \quad \dfrac{\stackrel{\vdots\ \delta_1'}{\overline{\Gamma}, \div A, C \vdash \overline{\Delta}}}{\overline{\Gamma}, \div A \vdash \div C \vdash \overline{\Delta}}\,(\vdash\div)$$

In the proof of admissibility of the other rule, $\dfrac{\Gamma, \div(A \wedge B) \vdash \Delta}{\Gamma, \div B \vdash \Delta}$, the case where the last rule applied is $(\div\wedge\vdash)$ differs somewhat. We have:

$$\dfrac{\stackrel{\vdots}{\Gamma, \div A \vdash \Delta} \quad \stackrel{\vdots}{\Gamma \vdash \div B, \Delta}}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge) \quad \rightsquigarrow \quad \dfrac{\dfrac{\stackrel{\vdots}{\Gamma \vdash \div B, \Delta}}{\Gamma, \div\div B \vdash \div B, \Delta}\,(W\vdash)}{\Gamma, \div\div B \vdash \Delta}\,(\div\vdash)$$

□

A conjecture and a partial proof

Conjecture 4.3.3.
The rule $\dfrac{\Gamma \vdash \div A, \Delta}{\Gamma, A \vdash \Delta}$ *is admissible.*

We show the partial proof of the admissibility of the rule $\dfrac{\Gamma \vdash \div A, \Delta}{\Gamma, A \vdash \Delta}$ and point out the unsolved case. The proof is a proof by induction with the complexity of A as primary induction parameter and the length of the derivation of the premiss as the secondary induction parameter, $\langle |A|, |\delta| \rangle$. If A is clopen, then the rule is admissible(Lemma 4.3.2). Throughout this proof we therefore assume that A is not clopen.

Base case We consider first a derivation of the premiss with atomic A. The shortest possible derivation is by $(\vdash W)$:

$$\dfrac{\dfrac{}{p \vdash p}\,ax.}{p \vdash \div A, p}\,(\vdash W) \quad\rightsquigarrow\quad \dfrac{\dfrac{}{p \vdash p}\,ax.}{p, A \vdash p}\,(W \vdash)$$

For a longer derivation, δ, we consider first the cases where A was principal in the last rule applied in the derivation; the last rule applied must be either $(\vdash W)$ or $(\vdash \div)$ with *clopen* context:

- If the last rule applied is $(\vdash W)$, we weaken by A instead:

$$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Gamma \vdash \Delta\end{array}}{\Gamma \vdash \div A, \Delta}\,(\vdash W) \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\vdots\,\delta\\ \Gamma \vdash \Delta\end{array}}{\Gamma, A \vdash \Delta}\,(W \vdash)$$

- If the last rule applied is $(\vdash \div)$, we drop this application:

$$\dfrac{\begin{array}{c}\vdots\,\delta\\ \overline{\Gamma}, A \vdash \overline{\Delta}\end{array}}{\overline{\Gamma} \vdash \div A, \overline{\Delta}}\,(\vdash \div) \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\delta\\ \overline{\Gamma}, A \vdash \overline{\Delta}\end{array}$$

If A is not principal in the last rule applied in the derivation, then we apply the induction hypothesis to the derivation of the premiss of this rule; we obtain a derivation for the sequent with A in the left-hand-side and show that we can still obtain the required conclusion. For all classical rules, the rules can be applied as they were:

- If the last rule applied is $(\neg \vdash)$:

$$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Gamma \vdash B, \div A, \Delta\end{array}}{\Gamma, \neg B \vdash \div A, \Delta}\,(\neg \vdash) \quad\rightsquigarrow^{IH}\quad \dfrac{\begin{array}{c}\vdots\,\delta'\\ \Gamma, A \vdash B, \Delta\end{array}}{\Gamma, A, \neg B \vdash \Delta}\,(\vdash \neg)$$

- If the last rule applied is $(\vdash \neg)$:

$$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Gamma, B \vdash \div A, \Delta\end{array}}{\Gamma \vdash \neg B, \div A, \Delta}\,(\vdash \neg) \quad\rightsquigarrow^{IH}\quad \dfrac{\begin{array}{c}\vdots\,\delta'\\ \Gamma, A, B \vdash \Delta\end{array}}{\Gamma, A \vdash \neg B, \Delta}\,(\vdash \neg)$$

- If the last rule applied is $(\wedge \vdash)$:

$$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Gamma, B, C \vdash \div A, \Delta\end{array}}{\Gamma, B \wedge C \vdash \div A, \Delta}\,(\wedge \vdash) \quad\rightsquigarrow^{IH}\quad \dfrac{\begin{array}{c}\vdots\,\delta'\\ \Gamma, A, B, C \vdash \Delta\end{array}}{\Gamma, A, B \wedge C \vdash \Delta}\,(\wedge \vdash)$$

- If the last rule applied is $(\vdash \wedge)$:

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ \Gamma \vdash B, \div A, \Delta\end{array} \quad \begin{array}{c}\vdots\,\delta_2\\ \Gamma \vdash C, \div A, \Delta\end{array}}{\Gamma \vdash B \wedge C, \div A \Delta}\,(\vdash \wedge) \quad\rightsquigarrow^{IH}\quad \dfrac{\begin{array}{c}\vdots\,\delta_1'\\ \Gamma, A \vdash B, \Delta\end{array} \quad \begin{array}{c}\vdots\,\delta_2'\\ \Gamma, A \vdash C, \Delta\end{array}}{\Gamma, A \vdash B \wedge C, \Delta}\,(\vdash \wedge)$$

If the last rule applied is any of remaining rules we have:

- If the last rule applied is $(\div \vdash)$:

$$\dfrac{\begin{array}{c}\vdots\,\delta\\ \Gamma, \div B \vdash B, \div A, \Delta\end{array}}{\Gamma, \div B \vdash \div A, \Delta}\,(\div \vdash) \quad\rightsquigarrow^{IH}\quad \dfrac{\begin{array}{c}\vdots\,\delta'\\ \Gamma, A, \div B \vdash B, \Delta\end{array}}{\Gamma, A, \div B \vdash \Delta}\,(\div \vdash)$$

- If the last rule applied is $(\div\wedge\vdash)$

$$\dfrac{\dfrac{\vdots\ \delta_1}{\Gamma, \div B \vdash \div A, \Delta} \quad \dfrac{\vdots\ \delta_2}{\Gamma \vdash \div C, \div A, \Delta}}{\Gamma, \div(B \wedge \div C) \vdash \div A, \Delta}\ (\div\wedge\vdash) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\vdots\ \delta_1'}{\Gamma, A, \div B \vdash \Delta} \quad \dfrac{\vdots\ \delta_2'}{\Gamma, A \vdash \div C, \Delta}}{\Gamma, A, \div(B \wedge \div C) \vdash \Delta}\ (\div\wedge\vdash)$$

- If the last rule applied is $(\vdash\div)_B$

$$\dfrac{\dfrac{\vdots\ \delta}{\Gamma, \overline{B} \vdash \div A, \Delta}}{\Gamma \vdash \div\overline{B}, \div A, \Delta}\ (\vdash\div)_B \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\vdots\ \delta'}{\Gamma, A, \overline{B} \vdash \Delta}}{\Gamma, A \vdash \div\overline{B}, \Delta}\ (\vdash\div)_B$$

- If the last rule applied is $(\vdash\div\wedge)$

$$\dfrac{\dfrac{\vdots\ \delta}{\Gamma \vdash \div B, \div C, \div A, \Delta}}{\Gamma \vdash \div(B \wedge C), \div A, \Delta}\ (\vdash\div\wedge) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\vdots\ \delta'}{\Gamma, A \vdash \div B, \div C, \Delta}}{\Gamma, A \vdash \div(B \wedge C), \Delta}\ (\vdash\div\wedge)$$

- If the last rule applied is $(\vdash\div\neg\wedge)$

$$\dfrac{\dfrac{\vdots\ \delta}{\Gamma \vdash \div\neg B \wedge \div\neg C, \div A, \Delta}}{\Gamma \vdash \div\neg(B \wedge C), \div A, \Delta}\ (\vdash\div\neg\wedge) \quad \rightsquigarrow^{IH} \quad \dfrac{\dfrac{\vdots\ \delta'}{\Gamma, A \vdash \div\neg B \wedge \div\neg C, \Delta}}{\Gamma, A \vdash \div\neg(B \wedge C), \Delta}\ (\vdash\div\neg\wedge)$$

Finally, in the case of the context sensitive $(\vdash\div)$-rule we have a derivation

$$\dfrac{\dfrac{\vdots\ \delta}{\overline{\Gamma}, B \vdash \div A, \overline{\Delta}}}{\overline{\Gamma} \vdash \div B, \div A, \overline{\Delta}}\ (\vdash\div)$$

If B is clopen, we apply the induction hypothesis and then the $(\vdash\div)_B$-rule:

$$\dfrac{\dfrac{\vdots\ \delta'}{\overline{\Gamma}, A, \overline{B} \vdash \overline{\Delta}}}{\overline{\Gamma}, A \vdash \div\overline{B}, \overline{\Delta}}\ (\vdash\div)_B$$

Otherwise, if B is not clopen then we can not apply the induction hypothesis directly to the premiss of the rule, as the context is then no longer clopen. The proof for this case is a special case of the proof presented in the induction step for the case corresponding to this. It is therefore omitted here.

Induction step Consider now the cases of an arbitrarily complex formula A. As in the base case we consider first the derivation in which A is principal in the last rule applied:

- If the last rule applied is $(\vdash W)$, we weaken by A with $(W\vdash)$ instead.

$$\dfrac{\dfrac{\vdots\ \delta}{\Gamma \vdash \Delta}}{\Gamma \vdash \div A, \Delta}\ (\vdash W) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdots\ \delta}{\Gamma \vdash \Delta}}{\Gamma, A \vdash \Delta}\ (W\vdash)$$

- If the last rule applied is $(\vdash\div)$, we drop this application

$$\dfrac{\dfrac{\vdots\ \delta}{\overline{\Gamma}, A \vdash \overline{\Delta}}}{\overline{\Gamma} \vdash \div A, \overline{\Delta}}\ (\vdash\div) \quad \rightsquigarrow \quad \dfrac{\vdots\ \delta}{\overline{\Gamma}, A \vdash \overline{\Delta}}$$

- If the last rule applied is $(\vdash \div)_A$, we drop this application

$$\dfrac{\begin{matrix}\vdots\ \delta\\ \Gamma, \overline{A} \vdash \Delta\end{matrix}}{\Gamma \vdash \div\overline{A}, \Delta}\ (\vdash \div)_A \quad \rightsquigarrow \quad \begin{matrix}\vdots\ \delta\\ \Gamma, \overline{A} \vdash \Delta\end{matrix}$$

- If the last rule applied is $(\vdash \div\wedge)$, let $A = (A_1 \wedge A_2)$. We have

$$\dfrac{\begin{matrix}\vdots\ \delta\\ \Gamma \vdash \div A_1, \div A_2, \Delta\end{matrix}}{\Gamma \vdash \div(A_1 \wedge A_2), \Delta}\ (\vdash \div\wedge) \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\vdots\ \delta\\ \Gamma, A_1, A_2 \vdash \Delta\end{matrix}}{\Gamma, (A_1 \wedge A_2) \vdash \Delta}\ (\wedge \vdash)$$

- If the last rule applied is $(\vdash \div\neg\wedge)$, let $A = \neg(A_1 \wedge A_2)$. We apply Lemma 4.3.1 to the premiss and then the induction hypothesis to both the obtain derivations. We have:

$$\dfrac{\begin{matrix}\vdots\ \delta\\ \Gamma \vdash \div\neg A_1 \wedge \div\neg A_2, \Delta\end{matrix}}{\Gamma \vdash \div\neg(A_1 \wedge A_2), \Delta}\ (\vdash \div\neg\wedge) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \delta'\\ \Gamma, \neg A_1 \vdash \Delta\end{matrix}}{\Gamma \vdash A_1, \Delta}\ (L.4.3.1) \quad \dfrac{\begin{matrix}\vdots\ \delta''\\ \Gamma, \neg A_2 \vdash \Delta\end{matrix}}{\Gamma \vdash A_2, \Delta}\ (L.4.3.1)}{\Gamma \vdash A_1 \wedge A_2, \Delta}\ (\vdash \wedge)}{\Gamma, \neg(A_1 \wedge A_2) \vdash \Delta}\ (\neg \vdash)$$

We consider the remaining cases in which A is not principal in the last rule applied in the derivation. As in the base case all the cases which do not depend on the context are trivial. In the non-trivial case of the context sensitive $(\vdash \div)$ rule, we have

$$\dfrac{\begin{matrix}\vdots\ \delta\\ \overline{\Gamma}, B \vdash \div A, \overline{\Delta}\end{matrix}}{\overline{\Gamma} \vdash \div B, \div A, \overline{\Delta}}\ (\vdash \div)$$

If B is clopen, we apply $(\vdash \div)_B$ instead. If B is not clopen then we consider the derivation of the premiss and the last rule applied with $\div A$ principal; either $(\vdash W)$, $(\vdash \div)$, $(\vdash \div\wedge)$ or $(\vdash \div\neg\wedge)$.

- If $\div A$ was introduced by weakening, we weaken instead with A after the application of the $(\vdash \div)$:

$$\dfrac{\begin{matrix}\dfrac{\begin{matrix}\vdots\ \delta_1\\ \Gamma' \vdash \Delta'\end{matrix}}{\Gamma' \vdash \div A, \Delta'}\ (\vdash W)\\ \vdots\ \delta_2\\ \overline{\Gamma}, B \vdash \div A, \overline{\Delta}\end{matrix}}{\overline{\Gamma} \vdash \div B, \div A, \overline{\Delta}}\ (\vdash \div) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\vdots\ \delta_1\\ \Gamma' \vdash \Delta'\\ \vdots\ \delta_2\\ \overline{\Gamma}, B \vdash \overline{\Delta}\end{matrix}}{\overline{\Gamma} \vdash \div B, \overline{\Delta}}\ (\vdash \div)}{\overline{\Gamma}, A \vdash \div B, \overline{\Delta}}\ (W \vdash)$$

- If $\div A$ was last principal in an application of the $(\vdash \div\wedge)$-rule, say $A = A_1 \wedge A_2$, we postpone this application until after the application of $(\vdash \div)$ introducing $\div B$:

$$\dfrac{\begin{matrix}\dfrac{\begin{matrix}\vdots\ \delta_1\\ \Gamma' \vdash \div A_1, \div A_2, \Delta'\end{matrix}}{\Gamma' \vdash \div(A_1 \wedge A_2), \Delta'}\ (\vdash \div\wedge)\\ \vdots\ \delta_2\\ \overline{\Gamma}, B \vdash \div(A_1 \wedge A_2), \overline{\Delta}\end{matrix}}{\overline{\Gamma} \vdash \div B, \div(A_1 \wedge A_2), \overline{\Delta}}\ (\vdash \div) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \delta_1\\ \Gamma' \vdash \div A_1, \div A_2, \Delta'\\ \vdots\ \delta_2\\ \overline{\Gamma}, B \vdash \div A_1, \div A_2 \overline{\Delta}\end{matrix}}{\overline{\Gamma} \vdash \div B, \div A_1, \div A_2, \overline{\Delta}}\ (\vdash \div)}{\overline{\Gamma}, A_1 \vdash \div B, \div A_2, \overline{\Delta}}\ IH}{\overline{\Gamma}, A_1, A_2 \vdash \div B, \overline{\Delta}}\ IH}{\overline{\Gamma}, A_1 \wedge A_2 \vdash \div B, \overline{\Delta}}\ (\wedge \vdash)$$

- If $\div A$ was last principal in an application of the $(\vdash \div\neg\wedge)$-rule, say $A = \neg(A_1 \wedge A_2)$, we postpone this application and apply instead Lemma 4.3.1 to obtain two derivations for A_1 and A_2 seperately. We have the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_1\\ \Gamma' \vdash \div\neg A_1 \wedge \div\neg A_2, \Delta'\end{array}}{\begin{array}{c}\Gamma' \vdash \div\neg(A_1 \wedge A_2), \Delta'\\ \vdots\ \delta_2\\ \overline{\Gamma}, B \vdash \div\neg(A_1 \wedge A_2), \overline{\Delta}\end{array}}\,(\vdash \div\neg\wedge)}{\overline{\Gamma} \vdash \div B, \div\neg(A_1 \wedge A_2), \overline{\Delta}}\,(\vdash \div)$$

We apply Lemma 4.3.1 to the premiss of the $(\vdash \div\neg\wedge)$-rule obtaining derivations for the sequents δ_{1_1}: $\Gamma' \vdash \div\neg A_1, \Delta'$ and δ_{1_2}: $\Gamma' \vdash \div\neg A_2, \Delta'$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_{1_1}\\ \Gamma' \vdash \div\neg A_1, \Delta'\\ \vdots\ \delta_2\\ \overline{\Gamma}, B \vdash \div\neg A_1, \overline{\Delta}\end{array}}{\overline{\Gamma} \vdash \div B, \div\neg A_1, \overline{\Delta}}\,(\vdash \div)}{\overline{\Gamma}, \neg A_1 \vdash \div B, \overline{\Delta}}\,IH}{\overline{\Gamma} \vdash \div B, A_1, \overline{\Delta}}\,(L.4.3.1) \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_{1_2}\\ \Gamma' \vdash \div\neg A_2, \Delta'\\ \vdots\ \delta_2\\ \overline{\Gamma}, B \vdash \div\neg A_2, \overline{\Delta}\end{array}}{\overline{\Gamma} \vdash \div B, \div\neg A_2, \overline{\Delta}}\,(\vdash \div)}{\overline{\Gamma}, \neg A_2 \vdash \div B, \overline{\Delta}}\,IH}{\overline{\Gamma} \vdash \div B, A_2, \overline{\Delta}}\,(L.4.3.1)}{\overline{\Gamma} \vdash \div B, A_1 \wedge A_2, \overline{\Delta}}\,(\vdash \wedge)}{\overline{\Gamma}, \neg(A_1 \wedge A_2) \vdash \div B, \overline{\Delta}}\,(\neg \vdash)$$

Where both applications of the induction hypothesis are permitted since they have lower complexity than A.

- The last case is when $\div A$ was last principal in an application of the $(\vdash \div)$-rule. We have the following situation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_1\\ \overline{\Gamma'}, A \vdash \overline{\Delta'}\end{array}}{\begin{array}{c}\overline{\Gamma'} \vdash \div A, \overline{\Delta'}\\ \vdots\ \delta_2\\ \overline{\Gamma}, B \vdash \div A, \overline{\Delta}\end{array}}\,(\vdash \div)}{\overline{\Gamma} \vdash \div B, \div A, \overline{\Delta}}\,(\vdash \div)$$

The main idea of the argument for this case is that the non-*clopen* subformulae of B must have been introduced into the derivation of the conclusion either by weakening or by branching, in which cases we can either weaken approriately or ensure that the formulae which will contribute to B are already clopen when the branching occurs. To indicate this, consider the conclusion in the application which introduces $\div A$: all formulae are clopen. There are three rules which, in a derivation from a sequent of only clopen formulae, may introduce a non-clopen formula; $(W \vdash)$, $(\vdash W)$ and $(\vdash \wedge)$. In this partial proof, we ignore double negations, i.e. we assume that there is no formula $\neg\neg B' \in sub(B)$. We procede with induction on the complexity of B:

  - If B is atomic, $|B| = 1$, then the last rule in the derivation with B principal was $(W \vdash)$

$$
\begin{array}{c}
\vdots\ \delta_1 \\
\dfrac{\overline{\Gamma_1}, A \vdash \overline{\Delta_2}}{\overline{\Gamma_1} \vdash \div A, \overline{\Delta_2}}\ (\vdash \div) \\
\vdots\ \delta_2 \\
\dfrac{\Gamma_2 \vdash \div A, \Delta_2}{\Gamma_2, B \vdash \div A, \Delta_2}\ (W \vdash) \\
\vdots\ \delta_3 \\
\dfrac{\overline{\Gamma_3}, B \vdash \div A, \overline{\Delta_3}}{\overline{\Gamma_3} \vdash \div B, \div A, \overline{\Delta_3}}\ (\vdash \div)
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
\vdots\ \delta_1 \\
\overline{\Gamma_1}, A \vdash \overline{\Delta_2} \\
\vdots\ \delta_2 \\
\Gamma_2, A \vdash \Delta_2 \\
\vdots\ \delta_3 \\
\dfrac{\overline{\Gamma_3}, A \vdash \overline{\Delta_3}}{\overline{\Gamma_3}, A \vdash \div B, \overline{\Delta_3}}\ (\vdash W)
\end{array}
$$

– If $B = \neg B'$, then the last rule in which $\neg B'$ was principal is either $(W \vdash)$ or $(\neg \vdash)$. If it is $(W \vdash)$ then we weaken by $\div\neg B$ with $(\vdash W)$ instead (similar to what we did in the previous case), or, if it was $(\neg \vdash)$, we consider again the last rule in which B' was principal, i.e. the rule, (R) with B' principal, in the following:

$$
\begin{array}{c}
\vdots\ \delta_1 \\
\dfrac{\overline{\Gamma_1}, A \vdash \overline{\Delta_1}}{\overline{\Gamma_1} \vdash \div A, \overline{\Delta_1}}\ (\vdash \div) \\
\vdots\ \delta_2 \\
\dfrac{\Gamma_2, X \vdash Y, \div A, \Delta_2}{\Gamma_2, X' \vdash Y', B' \vdash \div A, \Delta_2}\ (R) \\
\vdots\ \delta_3 \\
\dfrac{\Gamma_3 \vdash B' \vdash \div A, \Delta_3}{\Gamma_3, \neg B' \vdash \div A, \Delta_3}\ (\neg \vdash) \\
\vdots\ \delta_4 \\
\dfrac{\overline{\Gamma_4}, \neg B' \vdash \div A, \overline{\Delta_4}}{\overline{\Gamma_4} \vdash \div\neg B', \div A, \overline{\Delta_4}}\ (\vdash \div)
\end{array}
$$

As we are assuming B' is not *clopen* there are only three candidates for (R); $(\vdash W)$, $(\vdash \neg)$ and $(\vdash \wedge)$. We ignore the case of $(\vdash \neg)$ as it would break the assumption of a subformula of B containing a double negation. In the case of $(\vdash W)$ we weaken by $\div\neg B'$ immediately after the conclusion instead:

$$
\begin{array}{c}
\vdots\ \delta_1 \\
\overline{\Gamma_1}, A \vdash \overline{\Delta_1} \\
\vdots\ \delta_2 \\
\Gamma_2, A \vdash \Delta_2 \\
\vdots\ \delta_3 \\
\Gamma_3, A \vdash \Delta_3 \\
\vdots\ \delta_4 \\
\dfrac{\overline{\Gamma_4}, A \vdash \overline{\Delta_4}}{\overline{\Gamma_4}, A \vdash \div\neg B', \overline{\Delta_4}}\ (\vdash W)
\end{array}
$$

If (R) is $(\vdash \wedge)$, say $B = B_1 \wedge B_2$:

$$
\begin{prooftree}
\AxiomC{$\vdots\ \delta_1$}
\noLine
\UnaryInfC{$\overline{\Gamma_1}, A \vdash \overline{\Delta_2}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_1} \vdash \div A, \Delta_2$}
\noLine
\UnaryInfC{$\vdots\ \delta_2$}
\noLine
\UnaryInfC{$\Gamma_2 \vdash B_1, \div A, \Delta_2$}
\AxiomC{$\vdots\ \delta_3$}
\noLine
\UnaryInfC{$\Gamma_2 \vdash B_2, \div A, \Delta_2$}
\RightLabel{$(\vdash \wedge)$}
\BinaryInfC{$\Gamma_2 \vdash B_1 \wedge B_2, \div A, \Delta_2$}
\noLine
\UnaryInfC{$\vdots\ \delta_4$}
\noLine
\UnaryInfC{$\Gamma_3 \vdash B_1 \wedge B_2, \div A, \Delta_3$}
\RightLabel{$(\neg \vdash)$}
\UnaryInfC{$\Gamma_3, \neg(B_1 \wedge B_2) \vdash \div A, \Delta_3$}
\noLine
\UnaryInfC{$\vdots\ \delta_5$}
\noLine
\UnaryInfC{$\overline{\Gamma_4}, \neg(B_1 \wedge B_2) \vdash \div A, \overline{\Delta_4}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_4} \vdash \div\neg(B_1 \wedge B_2), \div A, \overline{\Delta_4}$}
\end{prooftree}
$$

$$
\rightsquigarrow
\begin{prooftree}
\AxiomC{$\vdots\ \delta_1$}
\noLine
\UnaryInfC{$\overline{\Gamma_1}, A \vdash \overline{\Delta_1}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_1} \vdash \div A, \overline{\Delta_1}$}
\noLine
\UnaryInfC{$\vdots\ \delta_{2,4,5}$}
\noLine
\UnaryInfC{$\overline{\Gamma_4} \vdash B_1, \div A, \overline{\Delta_4}$}
\RightLabel{$(\neg \vdash)$}
\UnaryInfC{$\overline{\Gamma_4}, \neg B_1 \vdash \div A, \overline{\Delta_4}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_4} \vdash \div\neg B_1, \div A, \overline{\Delta_4}$}
\RightLabel{IH}
\UnaryInfC{$\overline{\Gamma_4}, A \vdash \div\neg B_1, \overline{\Delta_4}$}
\AxiomC{$\vdots\ \delta_3$}
\noLine
\UnaryInfC{$\Gamma_2 \vdash B_2, \div A, \Delta_2$}
\noLine
\UnaryInfC{$\vdots\ \delta_{4,5}$}
\noLine
\UnaryInfC{$\overline{\Gamma_4} \vdash B_2, \div A, \overline{\Delta_4}$}
\RightLabel{$(\neg \vdash)$}
\UnaryInfC{$\overline{\Gamma_4}, \neg B_2 \vdash \div A, \overline{\Delta_4}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_4} \vdash \div\neg B_2, \div A, \overline{\Delta_4}$}
\RightLabel{IH}
\UnaryInfC{$\overline{\Gamma_4}, A \vdash \div\neg B_2, \overline{\Delta_4}$}
\RightLabel{$(\vdash \wedge)$}
\BinaryInfC{$\overline{\Gamma_4}, A \vdash \div\neg B_1 \wedge \div\neg B_2, \overline{\Delta_4}$}
\RightLabel{$(\vdash \div\neg\wedge)$}
\UnaryInfC{$\overline{\Gamma_4}, A \vdash \div\neg(B_1 \wedge B_2), \overline{\Delta_4}$}
\end{prooftree}
$$

Where the applications of the (main) induction hypothesis is allowed since the lengths of the derivations are, in both cases, shorter than the original derivations.

- Otherwise, $B = B_1 \wedge B_2$. There are two possible rules in which B could have been last principal in; $(W \vdash)$ and $(\wedge \vdash)$. If it was $(W \vdash)$, we weaken by $\div B$ with $(\vdash W)$ instread. We consider the remaining case of $(\wedge \vdash)$. Since B is not *clopen*, at least one of B_1 and B_2 is also not *clopen*. We consider first the case where one is *clopen*, say B_1:

$$
\begin{prooftree}
\AxiomC{$\vdots\ \delta_1$}
\noLine
\UnaryInfC{$\overline{\Gamma_1}, A \vdash \overline{\Delta_1}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_1} \vdash \div A, \overline{\Delta_1}$}
\noLine
\UnaryInfC{$\vdots\ \delta_2$}
\noLine
\UnaryInfC{$\Gamma_2, \overline{B_1}, B_2 \vdash \div A, \Delta_2$}
\RightLabel{$(\wedge \vdash)$}
\UnaryInfC{$\Gamma_2, \overline{B_1} \wedge B_2 \vdash \div A, \Delta_2$}
\noLine
\UnaryInfC{$\vdots\ \delta_3$}
\noLine
\UnaryInfC{$\overline{\Gamma_3}, \overline{B_1} \wedge B_2 \vdash \div A, \overline{\Delta_3}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_3} \vdash \div(\overline{B_1} \wedge B_2), \div A, \overline{\Delta_3}$}
\end{prooftree}
\quad\rightsquigarrow\quad
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\overline{\Gamma_1}, A \vdash \overline{\Delta_1}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_1} \vdash \div A, \overline{\Delta_1}$}
\noLine
\UnaryInfC{$\vdots\ \delta_2$}
\noLine
\UnaryInfC{$\Gamma_2, \overline{B_1}, B_2 \vdash \div A, \Delta_2$}
\noLine
\UnaryInfC{$\vdots\ \delta_3$}
\noLine
\UnaryInfC{$\overline{\Gamma_3}, \overline{B_1}, B_2 \vdash \div A, \overline{\Delta}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_3}, \overline{B_1} \vdash \div B_2, \div A, \overline{\Delta}$}
\RightLabel{IH}
\UnaryInfC{$\overline{\Gamma_3}, A, \overline{B_1} \vdash \div B_2, \overline{\Delta}$}
\RightLabel{$(\vdash \div)_{B_1}$}
\UnaryInfC{$\overline{\Gamma_3}, A \vdash \div\overline{B_1}, \div B_2, \overline{\Delta}$}
\RightLabel{$(\vdash \div)$}
\UnaryInfC{$\overline{\Gamma_3}, A \vdash \div(\overline{B_1} \wedge B_2), \overline{\Delta}$}
\end{prooftree}
$$

The last remaining case is when neither B_1 nor B_2 are *clopen*. Both B_1 and B_2 must contain a non-*clopen* which was introduced by weakening or branching (through the $(\vdash \wedge)$-rule). We have not managed to complete this proof, nor construct any counterexamples of the claims validity. In fact, the constructed examples of derivations which satisfies the requirement of belonging to this case are readily derivable in a cut-free derivation in which no applications of the context sensitive ($\vdash \div$)-rule occur in the branch from the assumed application of ($\vdash \div$) (introducing A here) and to the conclusion.

The admissibility of the rule $\dfrac{\Gamma \vdash \div A, \Delta}{\Gamma, A \vdash \Delta}$ is the contigency of the cut-elimination proof shown in the following sections, in which there is one application of this rule. It appears that no further assumptions can be made about the formulae involved in the rule or about the derivation in which it occurs. The case in which it occurs is **II.a.1** of Section 4.4 where we treat the situation of having the cut-formula principal in only the left premiss (i.e. of the two rules applied last in the derivations of the left and right premiss, the cut-formula occurs as principal only in that of the derivation of the left premiss):

$$\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\div\vdash) \qquad \dfrac{\overline{\Gamma'}, B \vdash \overline{\Delta'}, \div A}{\overline{\Gamma'} \vdash \div A, \div B, \overline{\Delta'}}\,(\vdash\div)}{\Gamma, \overline{\Gamma'} \vdash \div B, \Delta, \overline{\Delta'}}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overline{\Gamma'}, B \vdash \div A, \overline{\Delta'}}{\overline{\Gamma'} \vdash \div A, \div B, \overline{\Delta'}}\,(\vdash\div)}{\overline{\Gamma'}, A \vdash \div B, \overline{\Delta'}}\,(C.4.3.3) \qquad \dfrac{\Gamma, \div A \vdash A, \Delta \qquad \dfrac{\overline{\Gamma'}, B \vdash \div A, \overline{\Delta'}}{\overline{\Gamma'} \vdash \div A, \div B, \overline{\Delta'}}\,(\vdash\div)}{\Gamma, \overline{\Gamma'} \vdash A, \div B, \Delta, \overline{\Delta'}}\,(cut)}{\Gamma, \overline{\Gamma'} \vdash \div B, \Delta, \overline{\Delta'}}\,(cut)$$

This case seems to require us to "remove" the $\div$ operator of the cut-formula, A, in the (originally right) premiss.

4.4 Admissibility of the *(cut)*-rule

We show that the *(cut)*-rule is admissible in the LK[S5] system by proof by mathematical induction on the complexity of the cut formula, the length of the derivation of the left premiss as the secondary parameter and the length of the derivation of the right derivation. We consider a proof in which an application of *(cut)* is applied and show that we can eliminate it. We consider all cases by seperating all possible applications of *(cut)* into the following cases based on where the cut-formula is principal: I) the cut-formula is principal in both premissess, II) only in the left-premiss, III) only in the right premiss, and IV) the cut-formula is not principal in either premiss. We first cover the cases I and II.

Cut-formula principal in both premisses or only in left premiss

If the cut-formula is principal in both premisses or only in the left premiss we seperate them into the following sub-cases:

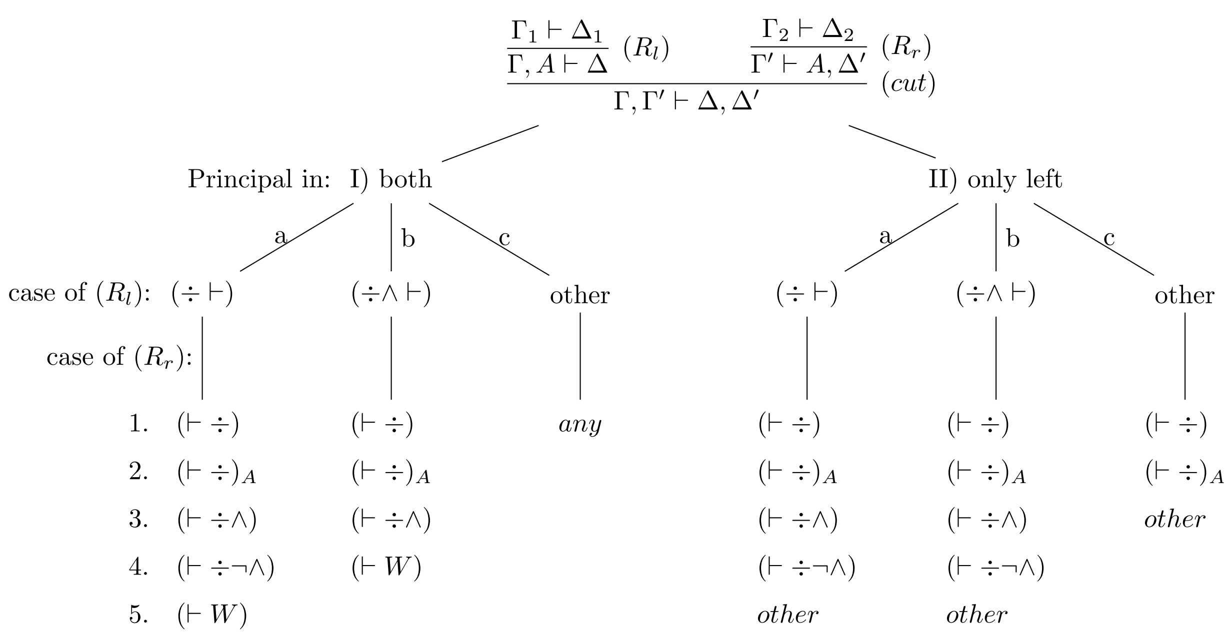

Figure 4.1: Proof map

After we have covered these cases, we discuss more briefly the remaining cases.

I. We look at the cases when the cut-formula is principal in both premissess.

I.a. When the last rule applied in the derivation of the left premiss is $(\div \vdash)$, we have four cases depending on which is the last rule applied in the derivation of the right premiss.

I.a.1. If the last rule applied in the derivation of the right premiss is $(\vdash \div)$

$$\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}(\div \vdash) \quad \dfrac{\overline{\Gamma'}, A \vdash \overline{\Delta'}}{\overline{\Gamma'} \vdash \div A, \overline{\Delta'}}(\vdash \div)}{\Gamma, \overline{\Gamma'} \vdash \Delta, \overline{\Delta'}}(cut) \quad \rightsquigarrow \quad \dfrac{\overline{\Gamma'}, A \vdash \overline{\Delta'} \quad \dfrac{\Gamma, \div A \vdash A, \Delta \quad \dfrac{\overline{\Gamma'}, A \vdash \overline{\Delta'}}{\overline{\Gamma'} \vdash \div A, \overline{\Delta'}}(\vdash \div)}{\Gamma, \overline{\Gamma'} \vdash A, \Delta, \overline{\Delta'}}(cut)}{\Gamma, \overline{\Gamma'} \vdash \Delta, \overline{\Delta'}}(cut)$$

I.a.2. If the last rule applied in the derivation of the right premiss is $(\vdash \div)_A$

$$\dfrac{\dfrac{\Gamma, \div\overline{A} \vdash \overline{A}, \Delta}{\Gamma, \div\overline{A} \vdash \Delta}\,(\div\vdash) \quad \dfrac{\Gamma', \overline{A} \vdash \Delta'}{\Gamma' \vdash \div\overline{A}, \Delta'}\,(\vdash\div)_A}{\Gamma,\Gamma' \vdash \Delta,\Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\Gamma,\overline{A}\vdash\Delta' \quad \dfrac{\Gamma, \div\overline{A} \vdash \overline{A}, \Delta \quad \dfrac{\Gamma', \overline{A} \vdash \Delta'}{\Gamma' \vdash \div\overline{A}, \Delta'}\,(\vdash\div)_A}{\Gamma' \vdash \div\overline{A},\Delta'}\,(cut)}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(cut)$$

I.a.3. If the last rule applied in the derivation of the right premiss is $(\vdash\div\wedge)$

$$\dfrac{\dfrac{\begin{matrix}\vdots\;\delta_1\\ \Gamma, \div(A\wedge B) \vdash A\wedge B, \Delta\end{matrix}}{\Gamma, \div(A\wedge B) \vdash \Delta}\,(\div\vdash) \quad \dfrac{\begin{matrix}\vdots\;\delta_2\\ \Gamma' \vdash \div A, \div B, \Delta'\end{matrix}}{\Gamma' \vdash \div(A\wedge B), \Delta'}\,(\vdash\div\wedge)}{\Gamma,\Gamma' \vdash \Delta,\Delta'}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\vdots\;\delta_1\\ \Gamma, \div(A\wedge B) \vdash A\wedge B, \Delta\end{matrix} \quad \dfrac{\begin{matrix}\vdots\;\delta_2\\ \Gamma' \vdash \div A, \div B, \Delta'\end{matrix}}{\Gamma' \vdash \div(A\wedge B), \Delta'}\,(\vdash\div\wedge)}{\Gamma,\Gamma' \vdash A\wedge B,\Delta,\Delta'}\,(cut)}{\Gamma,\Gamma' \vdash A,\Delta,\Delta' \quad ; \quad \Gamma,\Gamma' \vdash B,\Delta,\Delta'}\,(L.4.3.1)$$

where the induction hypothesis allows the application of the (cut)-rule since the length of the derivation of the left premiss is shorter. We now have derivations for the sequents $\Gamma, \Gamma' \vdash A, \Delta, \Delta'$ and $\Gamma, \Gamma' \vdash B, \Delta, \Delta'$ in addition to the derivations for the originally given premisses:

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma,\Gamma \vdash B,\Delta,\Delta'}{\Gamma,\Gamma,\div B \vdash B,\Delta,\Delta}\,(W\vdash)}{\Gamma,\Gamma' \vdash \div B,\Delta,\Delta'}\,(\div\vdash) \quad \dfrac{\dfrac{\dfrac{\Gamma,\Gamma' \vdash A,\Delta,\Delta'}{\Gamma,\Gamma',\div A \vdash A,\Delta,\Delta'}\,(W\vdash)}{\Gamma,\Gamma',\div A \vdash \Delta,\Delta'}\,(\div\vdash) \quad \begin{matrix}\vdots\;\delta_2\\ \Gamma' \vdash \div A, \div B, \Delta'\end{matrix}}{\Gamma,\Gamma' \vdash \div B,\Delta,\Delta'}\,(cut)}{\Gamma,\Gamma' \vdash \Delta,\Delta'}\,(cut)$$

In both application of the (cut)-rule the complexity of the cut-formula is lower.

I.a.4. If the last rule applied in the derivation of the right premiss is $(\vdash\div\neg\wedge)$

$$\dfrac{\dfrac{\Gamma, \div\neg(A\wedge B) \vdash \neg(A\wedge B), \Delta}{\Gamma, \div\neg(A\wedge B) \vdash \Delta}\,(\div\vdash) \quad \dfrac{\Gamma' \vdash \div\neg A \wedge \div\neg B, \Delta'}{\Gamma' \vdash \div\neg(A\wedge B), \Delta'}\,(\vdash\div\wedge)}{\Gamma,\Gamma' \vdash \Delta,\Delta'}\,(cut)$$

We apply Lemma 4.3.1 to the premiss of the $(\vdash\div\neg\wedge)$ and obtain the sequents $\Gamma' \vdash \div\neg A, \Delta'$ and $\Gamma' \vdash \div\neg B, \Delta'$.

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \div\neg(A\wedge B) \vdash \neg(A\wedge B), \Delta}{\Gamma, \div\neg(A\wedge B) \vdash \Delta}\,(\div\vdash)}{\Gamma, \div\neg A, \div\neg B \vdash \Delta}\,(L.4.3.2) \quad \Gamma' \vdash \div\neg A, \Delta'}{\Gamma,\Gamma',\div\neg B \vdash \Delta,\Delta'}\,(cut) \quad \Gamma' \vdash \div\neg B, \Delta'}{\Gamma,\Gamma' \vdash \Delta,\Delta'}\,(cut)$$

I.a.5. If the last rule applied in the derivation of the right premiss is $(\vdash W)$, we have:

$$\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\div\vdash) \quad \dfrac{\Gamma' \vdash \Delta'}{\Gamma' \vdash \div A, \Delta'}\,(\vdash W)}{\Gamma,\Gamma' \vdash \Delta,\Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma' \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,(W\vdash) \quad \dfrac{\Gamma, \div A \vdash A, \Delta \quad \dfrac{\Gamma' \vdash \Delta'}{\Gamma' \vdash \div A, \Delta'}\,(\vdash W)}{\Gamma,\Gamma' \vdash A,\Delta,\Delta'}\,(cut)}{\Gamma,\Gamma' \vdash \Delta,\Delta'}\,(cut)$$

I.b. We consider the cases when the last rule applied in the derivation of the left premiss is $(\div\wedge\vdash)$

I.b.1. If the last rule applied in the derivation of the right premiss is $(\vdash\div)$

$$\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}(\div\wedge\vdash) \quad \dfrac{\overline{\Gamma'}, A \wedge \div B \vdash \overline{\Delta'}}{\overline{\Gamma'} \vdash \div(A\wedge\div B), \overline{\Delta'}}(\vdash\div)}{\Gamma, \overline{\Gamma'} \vdash \Delta, \overline{\Delta'}}(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Gamma,\div A\vdash\Delta \quad \dfrac{\dfrac{\overline{\Gamma'}, A\wedge\div B\vdash\overline{\Delta'}}{\overline{\Gamma'}, A, \div B \vdash \overline{\Delta'}}(L.4.3.1)}{\overline{\Gamma'}, \div B \vdash \div A, \overline{\Delta'}}(\vdash\div)}{\Gamma,\overline{\Gamma'},\div B\vdash\Delta,\overline{\Delta'}}(cut) \quad \Gamma\vdash\div B,\Delta}{\Gamma,\overline{\Gamma'}\vdash\Delta,\overline{\Delta'}}(cut)$$

I.b.2. $(\vdash\div)_{(A\wedge\div B)}$, with $clopen(A\wedge\div B)$, we get a case similar to the previous case. Since $clopen(A\wedge\div B)$, we also have $clopen(A)$.

$$\dfrac{\dfrac{\Gamma, \div \overline{A} \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(\overline{A} \wedge \div B) \vdash \Delta}(\div\wedge\vdash) \quad \dfrac{\Gamma', \overline{A} \wedge \div B \vdash \Delta'}{\Gamma' \vdash \div(\overline{A}\wedge\div B), \Delta'}(\vdash\div)_{(\overline{A}\wedge\div B)}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Gamma,\div \overline{A}\vdash\Delta \quad \dfrac{\dfrac{\Gamma', \overline{A}\wedge\div B\vdash\Delta'}{\Gamma', \overline{A}, \div B \vdash \Delta'}(L.4.3.1)}{\Gamma', \div B \vdash \div \overline{A}, \Delta'}(\vdash\div)_A}{\Gamma,\Gamma',\div B\vdash\Delta,\Delta'}(cut) \quad \Gamma\vdash\div B,\Delta}{\Gamma,\Gamma'\vdash\Delta,\Delta'}(cut)$$

I.b.3. If the last rule applied in the derivation of the right premiss is $(\vdash\div\wedge)$

$$\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}(\div\vdash) \quad \dfrac{\Gamma' \vdash \div A, \div\div B, \Delta'}{\Gamma' \vdash \div(A\wedge\div B), \Delta'}(\vdash\div\wedge)}{\Gamma, \Gamma' \vdash \Delta, \Delta'}(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Gamma,\div A\vdash\Delta \quad \Gamma'\vdash\div A,\div\div B,\Delta'}{\Gamma,\Gamma'\vdash\div\div B,\Delta,\Delta'}(cut) \quad \dfrac{\dfrac{\Gamma\vdash\div B,\Delta}{\Gamma,\div\div B\vdash\div B,\Delta}(W\vdash)}{\Gamma,\div\div B\vdash\Delta}(\div\vdash)}{\Gamma,\Gamma'\vdash\Delta,\Delta'}(cut)$$

Where both applications of the (cut)-rule are allowed since the cut formula, in both cases, are of lower complexity. $|\div\div B| = 1 + |\div B| < 2 + |A| + |\div B| = |\div(A\wedge\div B)|$.

I.b.4 If the last rule applied in the derivation of the right premiss is $(\vdash W)$ we have:

$$\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}(\div\wedge\vdash) \quad \dfrac{\Gamma' \vdash \Delta'}{\Gamma' \vdash \div(A\wedge\div B), \Delta'}(\vdash W)}{\Gamma, \Gamma' \vdash \Delta, \Delta'}(cut)$$

$$\rightsquigarrow \quad \dfrac{\Gamma,\div A\vdash\Delta \quad \dfrac{\dfrac{\dfrac{\Gamma'\vdash\Delta'}{\Gamma',\div B\vdash\Delta'}(W\vdash) \quad \Gamma\vdash\div B,\Delta}{\Gamma,\Gamma'\vdash\Delta,\Delta'}(cut)}{\Gamma,\Gamma'\vdash\div A,\Delta,\Delta'}(\vdash W)}{\Gamma,\Gamma'\vdash\Delta,\Delta'}(cut)$$

I.c.1 If the last rule applied in the derivation of the left premiss is neither $(\div\vdash)$ or $(\div\wedge\vdash)$, it is a "classical rule". Our cut-formula is not on the form $\div A$ and the last rule applied in the derivation of the right premiss is also a "classical rule". We give some examples:

- $$\dfrac{\dfrac{\Gamma\vdash A,\Delta}{\Gamma,\neg A\vdash\Delta}\,(\neg\vdash)\quad\dfrac{\Gamma',A\vdash\Delta'}{\Gamma'\vdash\neg A,\Delta'}\,(\vdash\neg)}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(cut)\quad\rightsquigarrow\quad\dfrac{\Gamma,A\vdash\Delta\quad\Gamma'\vdash A,\Delta'}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(cut)$$

- $$\dfrac{\dfrac{\Gamma\vdash A,\Delta}{\Gamma,\neg A\vdash\Delta}\,(\neg\vdash)\quad\dfrac{\Gamma'\vdash\Delta'}{\Gamma'\vdash\neg A,\Delta'}\,(W)}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(cut)\quad\rightsquigarrow\quad\dfrac{\dfrac{\Gamma'\vdash\Delta'}{\Gamma',A\vdash\Delta'}\,(W)\quad\Gamma\vdash A,\Delta}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(cut)$$

- $$\dfrac{\dfrac{\Gamma,A,B\vdash\Delta}{\Gamma,A\wedge B\vdash\Delta}\,(\wedge\vdash)\quad\dfrac{\Gamma'\vdash A,\Delta'\quad\Gamma'\vdash B,\Delta'}{\Gamma'\vdash A\wedge B,\Delta'}\,(\vdash\wedge)}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(cut)$$

$$\rightsquigarrow\quad\dfrac{\dfrac{\Gamma,A,B\vdash\Delta\quad\Gamma'\vdash A,\Delta'}{\Gamma,\Gamma',B\vdash\Delta,\Delta'}\,(cut)\quad\Gamma\vdash B,\Delta}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,(cut)$$

II. We look at the cases when the cut-formula is principal only in the left premiss.

II.a. When the last rule applied in the derivation of the left premiss is $(\div\vdash)$, we have four cases depending on which is the last rule applied in the derivation of the right premiss. In the shown cases, the cases where the last rule applied in the derivation of the right premiss, (R), is $(\vdash\div)$ are the most interesting since this is the only rule with requirements on the context or principal formula; in the cases of the remaining rules (R) can be applied after (cut), thus the cut-formula is the same, the length of the derivation of the left premiss remains the same, but the length of the derivation of the right premiss is shorter. We show examples of this for the cases when the last rule applied in the derivation of the right premiss is $(\vdash\div\wedge)$ or $(\vdash\div\neg\wedge)$.

II.a.1. If the last rule applied in the derivation of the right premiss is $(\vdash\div)$ we apply the rule C.4.3.3 which admissibility remains a conjecture. We have unfortunately not found another way of handling this case.

$$\dfrac{\dfrac{\Gamma,\div A\vdash A,\Delta}{\Gamma,\div A\vdash\Delta}\,(\div\vdash)\quad\dfrac{\overline{\Gamma'},B\vdash\overline{\Delta'},\div A}{\overline{\Gamma'}\vdash\div A,\div B,\overline{\Delta'}}\,(\vdash\div)}{\Gamma,\overline{\Gamma'}\vdash\div B,\Delta,\overline{\Delta'}}\,(cut)$$

$$\rightsquigarrow\quad\dfrac{\dfrac{\dfrac{\overline{\Gamma'},B\vdash\div A,\overline{\Delta'}}{\overline{\Gamma'}\vdash\div A,\div B,\overline{\Delta'}}\,(\vdash\div)}{\overline{\Gamma'},A\vdash\div B,\overline{\Delta'}}\,(C.4.3.3)\quad\dfrac{\Gamma,\div A\vdash A,\Delta\quad\dfrac{\overline{\Gamma'},B\vdash\div A,\overline{\Delta'}}{\overline{\Gamma'}\vdash\div A,\div B,\overline{\Delta'}}\,(\vdash\div)}{\Gamma,\overline{\Gamma'}\vdash A,\div B,\Delta,\overline{\Delta'}}\,(cut)}{\Gamma,\overline{\Gamma'}\vdash\div B,\Delta,\overline{\Delta'}}\,(cut)$$

II.a.2. when the last rule applied in the derivation of the right premiss is $(\vdash\div)_B$ we have

$$\dfrac{\dfrac{\Gamma,\div A\vdash A,\Delta}{\Gamma,\div A\vdash\Delta}\,(\div\vdash)\quad\dfrac{\Gamma',\overline{B}\vdash\div A,\Delta'}{\Gamma'\vdash\div A,\div\overline{B},\Delta'}\,(\vdash\div)_B}{\Gamma,\Gamma'\vdash\div\overline{B},\Delta,\Delta'}\,(cut)\quad\rightsquigarrow\quad\dfrac{\dfrac{\dfrac{\Gamma,\div A\vdash A,\Delta}{\Gamma,\div A\vdash\Delta}\,(\div\vdash)\quad\Gamma',\overline{B}\vdash\div A,\Delta'}{\Gamma,\Gamma',\overline{B}\vdash\Delta,\Delta'}\,(cut)}{\Gamma,\Gamma'\vdash\div\overline{B},\Delta,\Delta'}\,(\vdash\div)_B$$

II.a.3 when the last rule applied in the derivation of the right premiss is $(\vdash \div\wedge)$ we have

$$\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\div \vdash) \quad \dfrac{\Gamma' \vdash \div A, \div B, \div C, \Delta'}{\Gamma' \vdash \div A, \div(B \wedge C), \Delta'}\,(\vdash \div\wedge)}{\Gamma, \Gamma' \vdash \div(B \wedge C), \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\vdash \div) \quad \Gamma' \vdash \div A, \div B, \div C, \Delta'}{\Gamma, \Gamma' \vdash \div B, \div C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div(B \wedge C), \Delta, \Delta'}\,(\vdash \div\wedge)$$

Where the complexity of the cut-formula is the same, the length of the derivation of the left premiss is the same, but the length of the derivation of the right premiss is shorter.

II.a.4 when the last rule applied in the derivation of the right premiss is $(\vdash \div\neg\wedge)$ we have

$$\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\div \vdash) \quad \dfrac{\Gamma' \vdash \div A, \div\neg B \wedge \div\neg C, \Delta'}{\Gamma' \vdash \div A, \div\neg(B \wedge C), \Delta'}\,(\vdash \div\neg\wedge)}{\Gamma, \Gamma' \vdash \div\neg(B \wedge C), \Delta, \Delta'}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\vdash \div) \quad \Gamma' \vdash \div A, \div\neg B \wedge \div\neg C, \Delta'}{\Gamma, \Gamma' \vdash \div\neg B \wedge \div\neg C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div\neg(B \wedge C), \Delta, \Delta'}\,(\vdash \div\neg\wedge)$$

II.a.5 We give two examples from the remaining cases. We can apply the last rule from the derivation of the right premiss after cut.

- $$\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\div \vdash) \quad \dfrac{\Gamma', B \vdash \div A, \Delta'}{\Gamma' \vdash \div A, \neg B, \Delta'}\,(\vdash \neg)}{\Gamma, \Gamma' \vdash \neg B, \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\div \vdash) \quad \Gamma', B \vdash \div A, \Delta'}{\Gamma, \Gamma', B \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \neg B, \Delta, \Delta'}\,(\vdash \neg)$$

- $$\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\div \vdash) \quad \dfrac{\Gamma' \vdash \div A, B, \Delta' \quad \Gamma' \vdash \div A, C, \Delta'}{\Gamma' \vdash \div A, B \wedge C, \Delta'}\,(\vdash \wedge)}{\Gamma, \Gamma' \vdash B \wedge C, \Delta, \Delta'}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\div \vdash) \quad \Gamma' \vdash \div A, B, \Delta'}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(cut) \quad \dfrac{\dfrac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta}\,(\div \vdash) \quad \Gamma' \vdash \div A, C, \Delta'}{\Gamma, \Gamma' \vdash C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash B \wedge C, \Delta, \Delta'}\,(\vdash \wedge)$$

II.b. We now consider the cases where the last rule applied in the derivation of the left premiss is $(\div \vdash)$

II.b.1. When the last rule applied in the derivation of the right premiss is $(\vdash \div)$ we have

$$\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge \vdash) \quad \dfrac{\overline{\Gamma'}, C \vdash \div(A \wedge \div B), \overline{\Delta'}}{\overline{\Gamma'} \vdash \div C, \div(A \wedge \div B), \overline{\Delta'}}\,(\vdash \div)}{\Gamma, \overline{\Gamma'} \vdash \div C, \Delta, \overline{\Delta'}}\,(cut)$$

$$\rightsquigarrow \qquad \dfrac{\dfrac{\Gamma, \div A \vdash \Delta \qquad \dfrac{\dfrac{\overline{\Gamma'}, C \vdash \div(A \wedge \div B), \overline{\Delta'}}{\overline{\Gamma'} \vdash \div(A \wedge \div B), \div C, \overline{\Delta'}}\,(\vdash \div)}{\overline{\Gamma'}, \div B \vdash \div C, \div A, \overline{\Delta'}}\,(L.4.3.2)}{\Gamma, \overline{\Gamma'}, \div B \vdash \div C, \Delta, \overline{\Delta'}}\,(cut) \qquad \Gamma \vdash \div B, \Delta}{\Gamma, \overline{\Gamma'} \vdash \div C, \Delta, \overline{\Delta'}}\,(cut)$$

II.b.2 when the last rule applied in the derivation of the right premiss is $(\vdash \div)_C$ we have

$$\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge\vdash) \qquad \dfrac{\Gamma', \overline{C} \vdash \div(A \wedge \div B), \Delta'}{\Gamma' \vdash \div\overline{C}, \div(A \wedge \div B), \Delta'}\,(\vdash \div)_C}{\Gamma, \Gamma' \vdash \div\overline{C}, \Delta, \Delta'}\,(cut)$$

$$\rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge\vdash) \qquad \Gamma', \overline{C} \vdash \div(A \wedge \div B), \Delta'}{\Gamma, \Gamma', \overline{C} \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div\overline{C}, \Delta, \Delta'}\,(\vdash \div)_C$$

II.b.3 We consider the case when the last rule applied in the derivation of the right premiss is $(\vdash \div\wedge)$

$$\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge\vdash) \qquad \dfrac{\Gamma' \vdash \div(A \wedge \div B), \div C, \div D, \Delta'}{\Gamma' \vdash \div(A \wedge \div B), \div(C \wedge D), \Delta'}\,(\vdash \div\wedge)}{\Gamma, \Gamma' \vdash \div(C \wedge D), \Delta, \Delta'}\,(cut)$$

$$\rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge\vdash) \qquad \Gamma' \vdash \div(A \wedge B), \div C, \div D, \Delta'}{\Gamma, \Gamma' \vdash \div C, \div D, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div(C \wedge D), \Delta, \Delta'}\,(\vdash \div\wedge)$$

II.b.4 We consider the case when the last rule applied in the derivation of the right premiss is $(\vdash \div\neg\wedge)$

$$\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge\vdash) \qquad \dfrac{\Gamma' \vdash \div(A \wedge \div B), \div\neg C \wedge \div\neg D, \Delta'}{\Gamma' \vdash \div(A \wedge \div B), \div\neg(C \wedge D), \Delta'}\,(\vdash \div\neg\wedge)}{\Gamma, \Gamma' \vdash \div\neg(C \wedge D), \Delta, \Delta'}\,(cut)$$

$$\rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge\vdash) \qquad \Gamma' \vdash \div(A \wedge B), \div\neg C \wedge \div\neg D, \Delta'}{\Gamma, \Gamma' \vdash \div\neg C \wedge \div\neg D, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div\neg(C \wedge D), \Delta, \Delta'}\,(\vdash \div\neg\wedge)$$

II.b.5 for the remaining cases of the last rule applied in the derivation of the right premiss we can apply (cut) before applying this last rule as they do not depend on the context (as in **II.a.5** and in the previous three cases)

$$\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge\vdash) \qquad \dfrac{\Gamma', \div A, X \vdash Y, \Delta'}{\Gamma', \div A, X' \vdash Y', \Delta'}\,(R_r)}{\Gamma, \Gamma_2 \vdash \Delta, \Delta_2}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta}\,(\div\wedge \vdash) \qquad \Gamma', \div A, X \vdash Y, \Delta'}{\Gamma, \Gamma', X \vdash Y, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', X' \vdash Y', \Delta, \Delta'}\,(R_r)$$

II.c. We consider the cases where the last rule applied in the derivation of the left premiss is a "classical rule". We show the cases of the context-sensitive rule $(\vdash \div)$, $(\vdash \div)_A$, and also some examples of the remaining cases.

II.c.1. If the last rule applied in the derivation of the right premiss is $(\vdash \div)$

$$\dfrac{\dfrac{\Gamma \vdash \overline{A}, \Delta}{\Gamma, \neg\overline{A} \vdash \Delta}\,(\neg\vdash) \quad \dfrac{\overline{\Gamma'}, B \vdash \neg\overline{A}, \overline{\Delta'}}{\overline{\Gamma'} \vdash \neg\overline{A}, \div B, \overline{\Delta'}}\,(\vdash\div)}{\Gamma, \overline{\Gamma'} \vdash \div B, \Delta, \overline{\Delta'}}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overline{\Gamma'}, B \vdash \neg\overline{A}, \overline{\Delta'}}{\overline{\Gamma'} \vdash \neg\overline{A}, \div B, \overline{\Delta'}}\,(\vdash\div)}{\overline{\Gamma'}, \overline{A} \vdash \div B, \overline{\Delta'}}\,(L.4.3.1) \qquad \Gamma \vdash \overline{A}, \Delta}{\Gamma, \overline{\Gamma'} \vdash \div B, \Delta, \overline{\Delta'}}\,(cut)$$

If the last rule applied in the derivation of the left premiss is $(\wedge\vdash)$, we apply $(L.4.3.1)$ to the right premiss and obtain $\overline{\Gamma'}, C \vdash \overline{A}, \overline{\Delta'}$ and $\overline{\Gamma'}, C \vdash \overline{B}, \overline{\Delta'}$:

$$\dfrac{\dfrac{\Gamma, \overline{A}, \overline{B} \vdash \Delta}{\Gamma, \overline{A} \wedge \overline{B} \vdash \Delta}\,(\wedge\vdash) \quad \dfrac{\overline{\Gamma'}, C \vdash \overline{A} \wedge \overline{B}, \overline{\Delta'}}{\overline{\Gamma'} \vdash \overline{A} \wedge \overline{B}, \div C, \overline{\Delta'}}\,(\vdash\div)}{\Gamma, \overline{\Gamma'} \vdash \div C, \Delta, \overline{\Delta'}}\,(cut) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma, \overline{A}, \overline{B} \vdash \Delta \quad \dfrac{\overline{\Gamma'}, C \vdash \overline{B}, \overline{\Delta'}}{\overline{\Gamma'} \vdash \overline{B}, \div C, \overline{\Delta'}}\,(\vdash\div)}{\overline{\Gamma'}, \overline{A} \vdash \div C, \overline{\Delta'}}\,(cut) \qquad \dfrac{\overline{\Gamma'}, C \vdash \overline{A}, \overline{\Delta'}}{\overline{\Gamma'} \vdash \overline{A}, \div C, \overline{\Delta'}}\,(\vdash\div)}{\Gamma, \overline{\Gamma'} \vdash \div C, \Delta, \overline{\Delta'}}\,(cut)$$

If the last rule applied in the derivation of the left premiss is $(W\vdash)$.

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,(W\vdash) \quad \dfrac{\overline{\Gamma'}, B \vdash A, \overline{\Delta'}}{\overline{\Gamma'} \vdash A, \div B, \overline{\Delta'}}\,(\vdash\div)}{\Gamma, \overline{\Gamma'} \vdash \div B, \Delta, \overline{\Delta'}}\,(cut) \quad \rightsquigarrow \quad \dfrac{\Gamma \vdash \Delta}{\Gamma, \overline{\Gamma'} \vdash \div B, \Delta, \overline{\Delta'}}\,(W)^*$$

II.c.2. If the last rule applied in the derivation of the right premiss is $(\vdash \div)_B$

$$\dfrac{\dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\,(\neg\vdash) \quad \dfrac{\Gamma', \overline{B} \vdash \neg A, \Delta'}{\Gamma' \vdash \neg A, \div\overline{B}, \Delta'}\,(\vdash\div)_B}{\Gamma, \Gamma' \vdash \div\overline{B}, \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\,(\neg\vdash) \qquad \Gamma', \overline{B} \vdash \neg A, \Delta'}{\Gamma, \Gamma', \overline{B} \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div\overline{B}, \Delta, \Delta'}\,(\vdash\div)$$

Also in the other cases we apply the (cut)-rule before applying the last rule in the derivation of the right premiss:

$$\dfrac{\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,(\wedge\vdash) \quad \dfrac{\Gamma', \overline{C} \vdash A \wedge B, \Delta'}{\Gamma' \vdash \div\overline{C}, A \wedge B, \Delta'}\,(\vdash\div)_C}{\Gamma, \Gamma' \vdash \div\overline{C}, \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma', \overline{C} \vdash A, B, \Delta'}{\Gamma' \vdash A \wedge B, \div\overline{C}, \Delta'}\,(\wedge\vdash) \qquad \Gamma', \overline{C} \vdash A \wedge B, \Delta'}{\Gamma, \Gamma', \overline{C} \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div\overline{C}, \Delta, \Delta'}\,(\vdash\div)_C$$

, and

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma, \div A \vdash \Delta}\,(W\vdash) \quad \dfrac{\Gamma', \overline{B} \vdash \div A, \Delta'}{\Gamma' \vdash \div A, \div\overline{B}, \Delta'}\,(\vdash\div)_B}{\Gamma,\Gamma' \vdash \div B, \Delta, \Delta'}\,(cut) \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\Gamma\vdash\Delta}{\Gamma,\div A\vdash\Delta}\,(W\vdash)\quad \Gamma',\overline{B}\vdash \div A,\Delta'}{\Gamma,\Gamma',\overline{B}\vdash\Delta,\Delta'}\,(cut)}{\Gamma,\Gamma'\vdash\div\overline{B},\Delta,\Delta'}\,(\vdash\div)_B$$

II.c.3. If the *cut-formula* is principal in only the left premiss and the last rule applied in the derivation of the right premiss is neither $(\vdash\div)$ nor $(\vdash\div)_A$, then we can apply the (cut) rule before this rule. We give some examples. The derivation of the left premiss is left unchanged and is not shown explicitly:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, A\vdash\Delta\end{array} \quad \dfrac{\Gamma', X\vdash A, Y, \Delta'}{\Gamma', X'\vdash A, Y', \Delta'}\,(R)}{\Gamma,\Gamma', X'\vdash Y',\Delta,\Delta'}\,(cut) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, A\vdash\Delta\end{array} \quad \Gamma', X\vdash A, Y,\Delta'}{\Gamma,\Gamma', X\vdash Y,\Delta,\Delta'}\,(cut)}{\Gamma,\Gamma', X'\vdash Y',\Delta,\Delta'}\,(R)$$

We show some examples of this:

- $$\dfrac{\begin{array}{c}\vdots\\ \Gamma, A\vdash\Delta\end{array} \quad \dfrac{\Gamma'\vdash A, \div B, \div C, \Delta'}{\Gamma'\vdash A, \div(B\wedge C), \Delta'}\,(\vdash\div\wedge)}{\Gamma,\Gamma'\vdash\div(B\wedge C),\Delta,\Delta'}\,(cut) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, A\vdash\Delta\end{array} \quad \Gamma'\vdash A,\div B,\div C,\Delta'}{\Gamma,\Gamma'\vdash\div B,\div C,\Delta,\Delta'}\,(cut)}{\Gamma,\Gamma'\vdash\div(B\wedge C),\Delta,\Delta'}\,(\vdash\div\wedge)$$

- $$\dfrac{\begin{array}{c}\vdots\\ \Gamma, A\vdash\Delta\end{array} \quad \dfrac{\Gamma'\vdash A, \div\neg B\wedge\div\neg C, \Delta'}{\Gamma'\vdash A, \div\neg(B\wedge C), \Delta'}\,(\vdash\div\neg\wedge)}{\Gamma,\Gamma'\vdash\div\neg(B\wedge C),\Delta,\Delta'}\,(cut) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, A\vdash\Delta\end{array} \quad \Gamma'\vdash A,\div\neg B\wedge\div\neg C,\Delta'}{\Gamma,\Gamma'\vdash\div\neg B\wedge\div\neg C,\Delta,\Delta'}\,(cut)}{\Gamma,\Gamma'\vdash\div\neg(B\wedge C),\Delta,\Delta'}\,(\vdash\div\neg\wedge)$$

- $$\dfrac{\begin{array}{c}\vdots\\ \Gamma, A\vdash\Delta\end{array} \quad \dfrac{\Gamma',\div B\vdash A,\Delta' \quad \Gamma'\vdash A,\div C,\Delta'}{\Gamma',\div(B\wedge\div C)\vdash A,\Delta'}\,(\div\wedge\vdash)}{\Gamma,\Gamma',\div(B\wedge\div C)\vdash\Delta,\Delta'}\,(cut)$$

$$\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, A\vdash\Delta\end{array} \quad \Gamma',\div B\vdash A,\Delta'}{\Gamma,\Gamma',\div B\vdash\Delta,\Delta'}\,(cut) \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma, A\vdash\Delta\end{array} \quad \Gamma'\vdash A,\div C,\Delta'}{\Gamma,\Gamma'\vdash\div C,\Delta,\Delta'}\,(cut)}{\Gamma,\Gamma',\div(B\wedge\div C)\vdash\Delta,\Delta'}\,(\div\wedge\vdash)$$

Cut-formula principal only in right premiss or in neither premiss

When the cut-formula is principal only in the right premiss or not principal in either, we seperate them, primarily on the case of the last rule applied in the derivation of the *right* premiss, into the following cases:

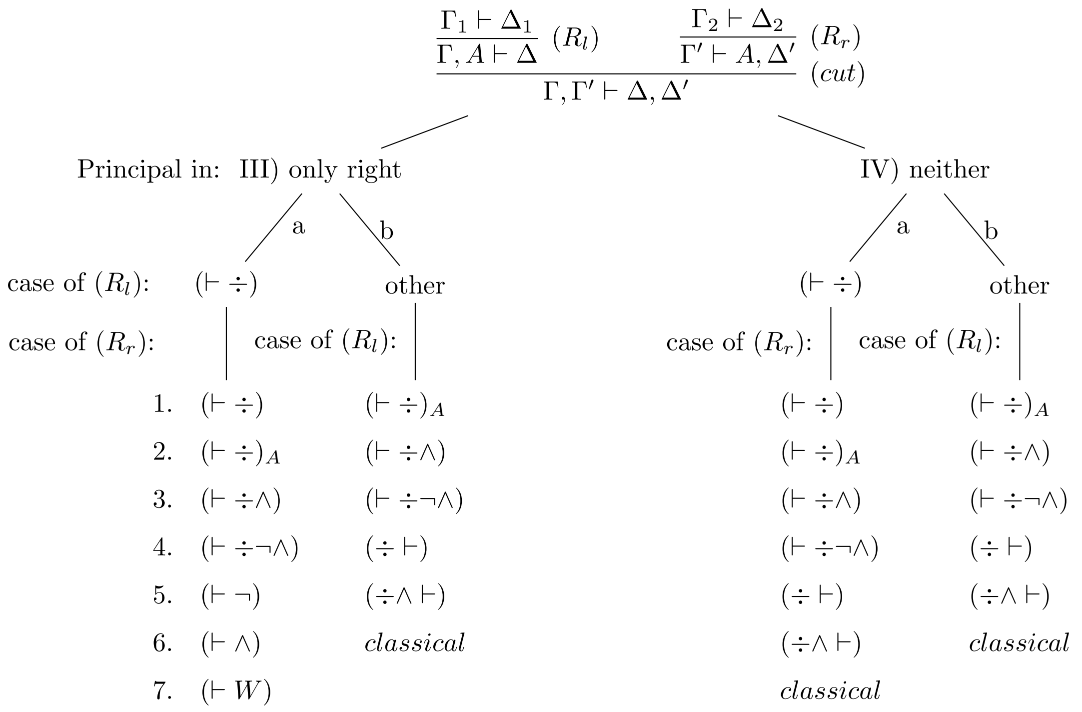

Figure 4.2: Proof map

III. We consider the cases where the cut-formula is principal in only the right premiss. In all the following cases, the cases of most interest is perhaps the cases where the last rule applied in the derivation of the left premiss is the context dependent $(\vdash\div)$ rule.

III.a. We consider the cases where the last rule applied in the derivation of the right premiss is $(\vdash\div)$, with *clopen*(Γ', Δ'). In all these cases the rule applied to infer the left premiss can be postponed to after cut:

III.a.1. If the last rule applied in the derivation of the left premiss is $(\vdash\div)$ with *clopen* context, i.e., *clopen*(), we have

$$\dfrac{\dfrac{\overline{\Gamma}, \div\overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \div\overline{A} \vdash \div B, \overline{\Delta}}(\vdash\div) \quad \dfrac{\overline{\Gamma'}, \overline{A} \vdash \overline{\Delta'}}{\overline{\Gamma'} \vdash \div\overline{A}, \overline{\Delta'}}(\vdash\div)}{\overline{\Gamma}, \overline{\Gamma'} \vdash \div B, \overline{\Delta}, \overline{\Delta'}}(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\overline{\Gamma}, \div\overline{A}, B \vdash \overline{\Delta} \quad \dfrac{\overline{\Gamma'}, \overline{A} \vdash \overline{\Delta'}}{\overline{\Gamma'} \vdash \div\overline{A}, \overline{\Delta'}}(\vdash\div)}{\overline{\Gamma}, \overline{\Gamma'}, B \vdash \overline{\Delta}, \overline{\Delta'}}(cut)}{\overline{\Gamma}, \overline{\Gamma'} \vdash \div B, \overline{\Delta}, \overline{\Delta'}}(\vdash\div)$$

III.a.2. If the last rule applied in the derivation of the left premiss is $(\vdash\div)$ with *clopen* context, and also *clopen*(A).

$$\dfrac{\dfrac{\overline{\Gamma}, \div\overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \div\overline{A} \vdash \div B, \overline{\Delta}}(\vdash\div) \quad \dfrac{\Gamma', \overline{A} \vdash \Delta'}{\Gamma' \vdash \div\overline{A}, \Delta'}(\vdash\div)_A}{\overline{\Gamma}, \Gamma' \vdash \div B, \overline{\Delta}, \Delta'}(cut) \quad \rightsquigarrow \quad \dfrac{\Gamma', \overline{A} \vdash \Delta' \quad \dfrac{\dfrac{\overline{\Gamma}, \div\overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \div\overline{A} \vdash \div B, \overline{\Delta}}(\vdash\div)}{\overline{\Gamma} \vdash \overline{A}, \div B, \overline{\Delta}}(L.4.3.2)}{\overline{\Gamma}, \Gamma' \vdash \div B, \overline{\Delta}, \Delta'}(cut)$$

III.a.3. If the last rule applied in the derivation of the right premiss is $(\vdash \div\wedge)$.

$$\cfrac{\cfrac{\overline{\Gamma}, \div(A \wedge B), C \vdash \overline{\Delta}}{\overline{\Gamma}, \div(A \wedge B) \vdash \div C, \overline{\Delta}}\,(\vdash \div) \qquad \cfrac{\Gamma' \vdash \div A, \div B, \Delta'}{\Gamma' \vdash \div(A \wedge B), \Delta'}\,(\vdash \div\wedge)}{\overline{\Gamma}, \Gamma' \vdash \div C, \overline{\Delta}, \Delta'}\,(cut)$$

We apply Lemma 4.3.2 to the premiss of the $(\vdash \div)$-rule, and obtain the sequents $\overline{\Gamma}, \div A, C \vdash \overline{\Delta}$ and $\overline{\Gamma}, \div B, C \vdash \overline{\Delta}$:

$$\cfrac{\cfrac{\overline{\Gamma}, \div B, C \vdash \overline{\Delta}}{\overline{\Gamma}, \div B \vdash \div C, \overline{\Delta}}\,(\vdash \div) \qquad \cfrac{\cfrac{\overline{\Gamma}, \div A, C \vdash \overline{\Delta}}{\overline{\Gamma}, \div A \vdash \div C, \overline{\Delta}}\,(\vdash \div) \qquad \Gamma' \vdash \div A, \div B, \Delta'}{\overline{\Gamma}, \Gamma' \vdash \div B, \div C, \overline{\Delta}, \Delta'}\,(cut)}{\overline{\Gamma}, \Gamma' \vdash \div C, \overline{\Delta}, \Delta'}\,(cut)$$

where both applications of (cut) is permitted by the induction hypothesis as the cut formulae are of lower complexity.

III.a.4. When the last rule applied in the derivation of the right premiss is $(\vdash \div\neg\wedge)$

$$\cfrac{\cfrac{\overline{\Gamma}, \div\neg(A \wedge B), C \vdash \overline{\Delta}}{\overline{\Gamma}, \div\neg(A \wedge B) \vdash \div C, \overline{\Delta}}\,(\vdash \div) \qquad \cfrac{\Gamma' \vdash \div\neg A \wedge \div\neg B, \Delta'}{\Gamma' \vdash \div\neg(A \wedge B), \Delta'}\,(\vdash \div\neg\wedge)}{\overline{\Gamma}, \Gamma' \vdash \div C, \overline{\Delta}, \Delta'}\,(cut)$$

We apply Lemma 4.3.1 to the premiss of the $(\vdash \div\neg\wedge)$-rule obtaining the sequents $\Gamma' \vdash \div\neg A, \Delta'$ and $\Gamma' \vdash \div\neg B, \Delta'$. We can now apply cut as follows; in both cases the complexity is less than that of the topical case:

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{\Gamma}, \div\neg(A \wedge B), C \vdash \overline{\Delta}}{\overline{\Gamma}, \div\neg A, \div\neg B, C \vdash \overline{\Delta}}\,(L.4.3.2)}{\overline{\Gamma}, \div\neg A, \div\neg B \vdash \div C, \overline{\Delta}}\,(\vdash \div) \qquad \Gamma' \vdash \div\neg A, \Delta'}{\overline{\Gamma}, \Gamma', \div\neg B \vdash \div C, \overline{\Delta}, \Delta'}\,(cut) \qquad \Gamma' \vdash \div\neg B, \Delta'}{\overline{\Gamma}, \Gamma' \vdash \div C, \overline{\Delta}, \Delta'}\,(cut)$$

III.a.5 If the last rule applied is $(\vdash \neg)$ we have

$$\cfrac{\cfrac{\overline{\Gamma}, \neg\overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \neg\overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \cfrac{\Gamma', \overline{A} \vdash \Delta'}{\Gamma' \vdash \neg\overline{A}, \Delta'}\,(\vdash \neg)}{\overline{\Gamma}, \Gamma \vdash \div B, \overline{\Delta}, \Delta'}\,(cut) \quad \rightsquigarrow \quad \cfrac{\Gamma', A \vdash \Delta' \qquad \cfrac{\cfrac{\overline{\Gamma}, \neg\overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \neg\overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div)}{\overline{\Gamma} \vdash \div B, \overline{A}, \overline{\Delta}}\,(L.4.3.1)}{\overline{\Gamma}, \Gamma \vdash \div B, \overline{\Delta}, \Delta'}\,(cut)$$

III.a.6 If the last rule applied in the derivation of the right premiss is $(\vdash \wedge)$, we have:

$$\cfrac{\cfrac{\overline{\Gamma}, \overline{A} \wedge \overline{B}, C \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \wedge \overline{B} \vdash \div C, \overline{\Delta}}\,(\vdash \div) \qquad \cfrac{\Gamma' \vdash \overline{A}, \Delta' \quad \Gamma' \vdash \overline{B}, \Delta'}{\Gamma' \vdash \overline{A} \wedge \overline{B}, \Delta'}\,(\vdash \wedge)}{\overline{\Gamma}, \Gamma' \vdash \div C, \overline{\Delta}, \Delta'}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma}, \overline{A} \wedge \overline{B}, C \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \wedge \overline{B} \vdash \div C, \overline{\Delta}}\,(\vdash \div)}{\overline{\Gamma}, \overline{A}, \overline{B} \vdash \div C, \overline{\Delta}}\,(L.4.3.1) \qquad \Gamma' \vdash \overline{A}, \Delta'}{\overline{\Gamma}, \Gamma', \overline{B} \vdash \div C, \overline{\Delta}, \Delta'}\,(cut) \qquad \Gamma' \vdash \overline{B}, \Delta'}{\overline{\Gamma}, \Gamma' \vdash \div C, \overline{\Delta}, \Delta'}\,(cut)$$

III.a.7 And finally, if the last rule applied in the derivation of the right premiss is $(\vdash W)$, we have:

$$\dfrac{\dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \dfrac{\Gamma' \vdash \Delta'}{\Gamma' \vdash \overline{A}, \Delta'}\,(\vdash W)}{\overline{\Gamma}, \Gamma' \vdash \div B, \overline{\Delta}, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\Gamma' \vdash \Delta'}{\overline{\Gamma}, \Gamma' \vdash \div B, \overline{\Delta}, \Delta'}\,(W)^*$$

III.b In all the cases where the last rule applied in the derivation of the left premiss is not $(\vdash \div)$ we handle the case without taking the derivation of the right premiss into consideration. The cases here are thus the cases where the cut-formula is not principal in the left premiss. We can apply the last rule applied in the derivation of the right premiss after applying *cut*,

$$\dfrac{\dfrac{\Gamma, A, X \vdash Y, \Delta}{\Gamma, A, X' \vdash Y', \Delta}\,(R_l) \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\overline{\Gamma}, \Gamma' \vdash \div B, \overline{\Delta}, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma', A, X \vdash Y, \Delta' \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', X \vdash Y, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', X' \vdash Y', \Delta, \Delta'}\,(R_l)$$

III.b.1 When the last rule applied in the derivation of the left premiss is $(\vdash \div)_B$ we have

$$\dfrac{\dfrac{\Gamma, A, \overline{B} \vdash \Delta}{\Gamma, A \vdash \div\overline{B}, \Delta}\,(\vdash \div)_B \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \div\overline{B}, \Delta, \Delta'}\,(cut) \qquad \dfrac{\dfrac{\Gamma, A, \overline{B} \vdash \Delta \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', \overline{B} \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div\overline{B}, \Delta, \Delta'}\,(\vdash \div)_B$$

III.b.2 If the last rule applied in the derivation of the left premiss is $(\vdash \div\wedge)$, we have

$$\dfrac{\dfrac{\Gamma, A \vdash \div B, \div C, \Delta}{\Gamma, A \vdash \div(B \wedge C), \Delta}\,(\vdash \div\wedge) \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \div(B \wedge C), \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash \div B, \div C, \Delta \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \div B, \div C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div(B \wedge C), \Delta, \Delta'}\,(\vdash \div\wedge)$$

III.b.3 If the last rule applied in the derivation of the left premiss is $(\vdash \div\neg\wedge)$ we have

$$\dfrac{\dfrac{\Gamma, A \vdash \div\neg B \wedge \div\neg C, \Delta}{\Gamma, A \vdash \div\neg(B \wedge C), \Delta}\,(\vdash \div\neg\wedge) \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \div(B \wedge C), \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash \div\neg B \wedge \div\neg C, \Delta \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \div\neg B \wedge \div\neg C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div\neg(B \wedge C), \Delta, \Delta'}\,(\vdash \div\wedge)$$

III.b.4 If the last rule applied in the derivation of the left premiss is $(\div \vdash)$ we have

$$\dfrac{\dfrac{\Gamma, A, \div B \vdash B, \Delta}{\Gamma, A, \div B \vdash \Delta}\,(\div \vdash) \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', \div B \vdash \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, \div B \vdash B, \Delta \qquad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', \div B \vdash B, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', \div B \vdash \Delta, \Delta'}\,(\div \vdash)$$

III.b.5 If the last rule applied in the derivation of the left premiss is $(\div\wedge\vdash)$ we have

$$\dfrac{\dfrac{\Gamma, A, \div B \vdash \Delta \quad \Gamma, A \vdash \div C, \Delta}{\Gamma, A, \div(B \wedge \div C) \vdash \Delta}\,(\div\wedge\vdash) \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', \div(B \wedge \div C) \vdash \Delta, \Delta'}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, \div B \vdash \Delta \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', \div B \vdash \Delta, \Delta'}\,(cut) \quad \dfrac{\Gamma, A \vdash \div C, \Delta \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \div C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', \div(B \wedge \div C) \vdash \Delta, \Delta'}\,(\div\wedge\vdash)$$

III.b.6 For all cases where the last rule applied in the left premiss is a classical rule, we can apply the (cut)-rule to the premiss/premises of this rule and apply the topical rule afterwards:

- $$\dfrac{\dfrac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash B, \Delta}\,(\vdash W) \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash \Delta \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(\neg\vdash)$$

- $$\dfrac{\dfrac{\Gamma, A \vdash B, \Delta}{\Gamma, A, \neg B \vdash \Delta}\,(\neg\vdash) \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', \neg B \vdash \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash B, \Delta \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', \neg B \vdash \Delta, \Delta'}\,(\neg\vdash)$$

- $$\dfrac{\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \vdash \neg B, \Delta}\,(\vdash\neg) \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \neg B, \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, B \vdash \Delta \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', B \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \neg B, \Delta, \Delta'}\,(\vdash\neg)$$

- $$\dfrac{\dfrac{\Gamma, A, B, C \vdash \Delta}{\Gamma, A, B \wedge C \vdash \Delta}\,(\wedge\vdash) \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', B \wedge C \vdash \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, B, C \vdash \Delta \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma', B, C \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', B \wedge C \vdash \Delta, \Delta'}\,(\wedge\vdash)$$

- $$\dfrac{\dfrac{\Gamma, A \vdash B, \Delta \quad \Gamma, A \vdash C, \Delta}{\Gamma, A \vdash B \wedge C, \Delta}\,(\vdash\wedge) \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \neg B, \Delta, \Delta'}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash B, \Delta \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(cut) \quad \dfrac{\Gamma, A \vdash C, \Delta \quad \begin{matrix}\vdots \\ \Gamma' \vdash A, \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash B \wedge C, \Delta, \Delta'}\,(\vdash\wedge)$$

IV We now show the cases for when the cut-formula is not principal in either premiss. We seperate this case in two sub-cases, IV.a) where the last rule applied in the derivation of the left premiss is the context sensitive $(\vdash\div)$-rule, and IV.b) where the last rule applied in the derivation of the left premiss is any other rule.

IV.a For the case where last rule applied in the derivation of the left premiss is $(\vdash\div)$, we show explicitly most of the cases of rules which might have been the last rule applied in the derivation of the right premiss with, A, the cut-formula, parametric. We show all the cases where the last rule applied in the derivation of the right premiss is one involving $\div$, and leave out some cases of classical rules(IV.a.7).

IV.a.1 When the last rule applied in the derivation of the right premiss is $(\vdash\div)$, with A, the cut-formula, not principal, we postpone the application of the $(\vdash\div)$-rule (in the derivation of either premiss) until after the application of (cut). Which we choose is not important, since, in either case, we obtain a sequent with all formulae clopen except for B(or C) so that we can apply $(\vdash\div)$ to this formula.

$$\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div B,\overline{\Delta}}\,(\vdash\div)\quad\dfrac{\overline{\Gamma},\overline{A},C\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div C,\overline{\Delta}}\,(\vdash\div)}{\overline{\Gamma},\overline{\Gamma'}\vdash\div B,\div C,\overline{\Delta},\overline{\Delta'}}\,(cut)\quad\rightsquigarrow\quad\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}\quad\dfrac{\overline{\Gamma'},\overline{A},C\vdash\overline{\Delta'}}{\overline{\Gamma'},\overline{A}\vdash\div C,\overline{\Delta'}}\,(\vdash\div)}{\overline{\Gamma},\overline{\Gamma'},B\vdash\div C,\overline{\Delta},\overline{\Delta'}}\,(cut)}{\overline{\Gamma},\overline{\Gamma'}\vdash\div C,\div B,\overline{\Delta},\overline{\Delta'}}\,(\vdash\div)$$

IV.a.2 When the last rule applied in the derivation of the right premiss is $(\vdash\div)_C$ we have

$$\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div B,\overline{\Delta}}\,(\vdash\div)\quad\dfrac{\Gamma',\overline{A},\overline{C}\vdash\Delta}{\Gamma',\overline{A}\vdash\div\overline{C},\Delta}\,(\vdash\div)_C}{\overline{\Gamma},\Gamma'\vdash\div B,\div\overline{C},\overline{\Delta},\Delta'}\,(cut)\quad\rightsquigarrow\quad\dfrac{\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div B,\overline{\Delta}}\,(\vdash\div)\quad\Gamma',\overline{A},\overline{C}\vdash\Delta'}{\overline{\Gamma},\Gamma',\overline{C}\vdash\div B,\overline{\Delta},\Delta'}\,(cut)}{\overline{\Gamma},\Gamma'\vdash\div C,\div B,\overline{\Delta},\overline{\Delta'}}\,(\vdash\div)_C$$

IV.a.3 If the last rule applied in the derivation of the right premiss is $(\vdash\div\wedge)$

$$\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div B,\overline{\Delta}}\,(\vdash\div)\quad\dfrac{\Gamma',\overline{A}\vdash\div C,\div D,\Delta}{\Gamma',\overline{A}\vdash\div(C\wedge D),\Delta}\,(\vdash\div\wedge)}{\overline{\Gamma},\Gamma'\vdash\div B,\div(C\wedge D),\overline{\Delta},\Delta'}\,(cut)$$

$$\rightsquigarrow\quad\dfrac{\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div B,\overline{\Delta}}\,(\vdash\div)\quad\Gamma',\overline{A}\vdash\div C,\div D,\Delta}{\overline{\Gamma},\Gamma'\vdash\div B,\div C,\div D,\overline{\Delta},\Delta'}\,(cut)}{\overline{\Gamma},\Gamma'\vdash\div B,\div(C\wedge D),\overline{\Delta},\overline{\Delta'}}\,(\vdash\div\wedge)$$

IV.a.4 If the last rule applied in the derivation of the right premiss is $(\vdash\div\neg\wedge)$ we have

$$\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div B,\overline{\Delta}}\,(\vdash\div)\quad\dfrac{\Gamma',\overline{A}\vdash\div\neg C\wedge\div\neg D,\Delta}{\Gamma',\overline{A}\vdash\div\neg(C\wedge D),\Delta}\,(\vdash\div\neg\wedge)}{\overline{\Gamma},\Gamma'\vdash\div B,\div\neg(C\wedge D),\overline{\Delta},\Delta'}\,(cut)$$

$$\rightsquigarrow\quad\dfrac{\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div B,\overline{\Delta}}\,(\vdash\div)\quad\Gamma',\overline{A}\vdash\div\neg C\wedge\div\neg D,\Delta}{\overline{\Gamma},\Gamma'\vdash\div B,\div\neg C\wedge\div\neg D,\overline{\Delta},\Delta'}\,(cut)}{\overline{\Gamma},\Gamma'\vdash\div B,\div\neg(C\wedge D),\overline{\Delta},\overline{\Delta'}}\,(\vdash\div\neg\wedge)$$

IV.a.5 If the last rule applied in the derivation of the right premiss is $(\div\vdash)$ we have

$$\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div B,\overline{\Delta}}\,(\vdash\div)\quad\dfrac{\Gamma',\overline{A},\div C\vdash C,\Delta'}{\Gamma',\overline{A},\div C\vdash\Delta'}\,(\div\vdash)}{\overline{\Gamma},\Gamma',\div C\vdash\div B,\overline{\Delta},\Delta'}\,(cut)\quad\rightsquigarrow\quad\dfrac{\dfrac{\dfrac{\overline{\Gamma},\overline{A},B\vdash\overline{\Delta}}{\overline{\Gamma},\overline{A}\vdash\div B,\overline{\Delta}}\,(\vdash\div)\quad\Gamma',\overline{A},\div C\vdash C,\Delta'}{\overline{\Gamma},\Gamma',\div C\vdash C,\div B,\overline{\Delta},\Delta'}\,(cut)}{\Gamma,\Gamma',\div C\vdash\div B,\overline{\Delta},\Delta'}\,(\vdash\div\wedge)$$

IV.a.6 If the last rule applied in the derivation of the right premiss is $(\div\wedge\vdash)$

$$\dfrac{\dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \dfrac{\Gamma', \overline{A}, \div C \vdash \Delta' \quad \Gamma', \overline{A} \vdash \div D, \Delta'}{\Gamma', \overline{A}, \div(C \wedge \div D) \vdash \Delta'}\,(\div\wedge \vdash)}{\overline{\Gamma}, \Gamma', \div(C \wedge \div D) \vdash \div B, \overline{\Delta}, \Delta'}\,(cut) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \Gamma', \overline{A}, \div C \vdash \Delta'}{\Gamma, \Gamma', \div C \vdash \div B, \overline{\Delta}, \Delta'}\,(cut) \qquad \dfrac{\dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \Gamma', \overline{A} \vdash \div D, \Delta'}{\Gamma, \Gamma' \vdash \div D, \div B, \overline{\Delta}, \Delta'}\,(cut)}{\Gamma, \Gamma', \div(C \wedge \div D) \vdash \div B, \overline{\Delta}, \Delta'}\,(\div\wedge \vdash)}{}$$

IV.a.7 For any other case of the last rule applied in the derivation of the right premiss, this rule is a classical rule. In these cases, the last rule applied in the derivation of the right premiss can be applied after we apply the (cut)-rule, as is done in the previous five cases. We show only the cases for $(\vdash \neg)$ and $(\vdash \wedge)$, a rule with one premiss and a rule with two premises:

- $$\dfrac{\dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \dfrac{\Gamma', C \vdash \overline{A}, \Delta'}{\Gamma' \vdash \neg C, \overline{A}, \Delta'}\,(\vdash \neg)}{\overline{\Gamma}, \Gamma' \vdash \div B, \neg C, \overline{\Delta}, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \Gamma', C \vdash \overline{A}, \Delta'}{\overline{\Gamma}, \Gamma', C \vdash \div B, \overline{\Delta}, \Delta'}\,(cut)}{\overline{\Gamma}, \Gamma' \vdash \div B, \neg C, \overline{\Delta}, \Delta'}\,(\vdash \neg)$$
- $$\dfrac{\dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \dfrac{\Gamma' \vdash C, \overline{A}, \Delta' \quad \Gamma' \vdash D, \overline{A}, \Delta'}{\Gamma' \vdash C \wedge D, \overline{A}, \Delta'}}{\overline{\Gamma}, \Gamma' \vdash \div B, C \wedge D, \overline{\Delta}, \Delta'}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \Gamma' \vdash C, \overline{A}, \Delta'}{\overline{\Gamma}, \Gamma' \vdash \div B, C, \overline{\Delta}, \Delta'}\,(cut) \qquad \dfrac{\dfrac{\overline{\Gamma}, \overline{A}, B \vdash \overline{\Delta}}{\overline{\Gamma}, \overline{A} \vdash \div B, \overline{\Delta}}\,(\vdash \div) \qquad \Gamma' \vdash C, \overline{A}, \Delta'}{\overline{\Gamma}, \Gamma' \vdash \div B, C, \overline{\Delta}, \Delta'}\,(cut)}{\overline{\Gamma}, \Gamma' \vdash \div B, C \wedge D, \overline{\Delta}, \Delta'}\,(\vdash \wedge)$$

where, in this last case, both applications of the (cut) rule are allowed by the induction hypothesis as the complexity of the cut-formula is unchanged, the length of the derivation of the left premiss is unchanged, and the length of the derivation of the right premiss is shorter.

IV.b When the last rule applied in the derivation of the left premiss is not $(\vdash \div)$ we disciminate further on the case of which rule is the last applied in the derivation of the left premiss. The cases here are identical to the cases from III.b.i, as those cases where all handled without assumptions about the derivation of the right premiss (i.e.,III.b and IV.b are cases where the *cut-formula is not principal in the left premiss*).

IV.b.1 If the last rule applied in the derivation of the left premiss is $(\vdash \div)_B$ with *clopen* principal formula

$$\dfrac{\dfrac{\Gamma, A, \overline{B} \vdash \Delta}{\Gamma, A \vdash \div\overline{B}, \Delta}\,(\vdash \div)_B \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \div\overline{B}, \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, \overline{B} \vdash \Delta \quad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma', \overline{B} \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div\overline{B}, \Delta, \Delta'}\,(\vdash \div)_B$$

IV.b.2 If the last rule applied in the derivation of the left premiss is $(\vdash \div\wedge)$ we have

$$\dfrac{\dfrac{\Gamma, A \vdash \div B, \div C, \Delta}{\Gamma, A \vdash \div(B \wedge C), \Delta}\,(\vdash \div\wedge) \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \div(B \wedge C), \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash \div B, \div C, \Delta \quad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \div B, \div C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div(B \wedge C), \Delta, \Delta'}\,(\vdash \div\wedge)$$

IV.b.3 If the last rule applied in the derivation of the left premiss is $(\vdash \div\neg\wedge)$

$$\dfrac{\dfrac{\Gamma, A \vdash \div\neg B \wedge \div\neg C, \Delta}{\Gamma, A \vdash \div\neg(B \wedge C), \Delta}\,(\vdash \div\neg\wedge) \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \div(B \wedge C), \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash \div\neg B \wedge \div\neg C, \Delta \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \div\neg B \wedge \div\neg C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \div\neg(B \wedge C), \Delta, \Delta'}\,(\vdash \div\wedge)$$

IV.b.4 if the last rule applied in the derivation of the left premiss is $(\div\vdash)$ we have

$$\dfrac{\dfrac{\Gamma, A, \div B \vdash B, \Delta}{\Gamma, A, \div B \vdash \Delta}\,(\div\vdash) \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma', \div B \vdash \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, \div B \vdash B, \Delta \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma', \div B \vdash B, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', \div B \vdash \Delta, \Delta'}\,(\div\vdash)$$

IV.b.5 If the last rule applied in the derivation of the left premiss is $(\div\wedge\vdash)$ we have

$$\dfrac{\dfrac{\Gamma, A, \div B \vdash \Delta \qquad \Gamma, A \vdash \div C, \Delta}{\Gamma, A, \div(B \wedge \div C) \vdash \Delta}\,(\div\wedge\vdash) \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma', \div(B \wedge \div C) \vdash \Delta, \Delta'}\,(cut)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, \div B \vdash \Delta \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma', \div B \vdash \Delta, \Delta'}\,(cut) \qquad \dfrac{\Gamma, A \vdash \div C, \Delta \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \div C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', \div(B \wedge \div C) \vdash \Delta, \Delta'}\,(\div\wedge\vdash)$$

IV.b.6 If the last rule applied in the derivation of the left premiss is a classical rule, we have the following cases:

- $$\dfrac{\dfrac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash B, \Delta}\,(\vdash W) \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash \Delta \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(\neg\vdash)$$

- $$\dfrac{\dfrac{\Gamma, A \vdash B, \Delta}{\Gamma, A, \neg B \vdash \Delta}\,(\neg\vdash) \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma', \neg B \vdash \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash B, \Delta \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', \neg B \vdash \Delta, \Delta'}\,(\neg\vdash)$$

- $$\dfrac{\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \vdash \neg B, \Delta}\,(\vdash\neg) \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \neg B, \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, B \vdash \Delta \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma', B \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash \neg B, \Delta, \Delta'}\,(\vdash\neg)$$

- $$\dfrac{\dfrac{\Gamma, A, B, C \vdash \Delta}{\Gamma, A, B \wedge C \vdash \Delta}\,(\wedge\vdash) \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma', B \wedge C \vdash \Delta, \Delta'}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, B, C \vdash \Delta \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma', B, C \vdash \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma', B \wedge C \vdash \Delta, \Delta'}\,(\wedge\vdash)$$

- $$\dfrac{\dfrac{\Gamma, A \vdash B, \Delta \qquad \Gamma, A \vdash C, \Delta}{\Gamma, A \vdash B \wedge C, \Delta}\,(\vdash\wedge) \qquad \begin{array}{c}\vdots\\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash \neg B, \Delta, \Delta'}\,(cut)$$

$$
\rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \vdash B, \Delta \quad \begin{array}{c}\vdots \\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,(cut) \quad \dfrac{\Gamma, A \vdash C, \Delta \quad \begin{array}{c}\vdots \\ \Gamma' \vdash A, \Delta'\end{array}}{\Gamma, \Gamma' \vdash C, \Delta, \Delta'}\,(cut)}{\Gamma, \Gamma' \vdash B \wedge C, \Delta, \Delta'}\,(\vdash \wedge)
$$

We have now shown that the (*cut*) rule is admissible in LK[S5] provided Conjecture 4.3.3 holds.

4.5 LK[S5]s - S5 with the subformula property

In the proposed system for which we have attempted to give a proof for admissibility of the *cut*-rule, there is one rule which does has the undesirable property of having a more complex premiss than conclusion; the $(\vdash \div\neg\wedge)$. We replace this rule with a slightly different rule which has the subformula property:

$$\frac{\Gamma \vdash \div\neg A, \Delta \quad \Gamma \vdash \div\neg B, \Delta}{\Gamma \vdash \div\neg(A \wedge B), \Delta} \ (\vdash \div\neg\wedge)'$$

This new rule has the subformula property. We show that a reasoning system indentical to the proposed system but with the $(\vdash \div\neg\wedge)$-rule replaced with this new rule by showing that each of these systems will have the other formulation of the rule admissible. We will refer to the system obtained by exchaning $(\vdash \div\neg\wedge)$ by $(\vdash \div\neg\wedge)'$ as LK[S5]s, appending an 's' for subformula.

In this section we describe a new system, LK[S5]s (Table 4.2), which, in terms of provable sequents, is equivalent to the system for which we give a decidability argument. The system exchanges only one rule; the $(\vdash \div\neg\wedge)$-rule.

$$\frac{}{p \vdash p} \ (ax)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \ (\neg \vdash) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \ (\vdash \neg)$$

$$\frac{\Gamma A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \ (\wedge \vdash) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B \Delta} \ (\vdash \wedge)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \ (W \vdash) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \ (\vdash W)$$

$$\frac{\Gamma, \div A \vdash A, \Delta}{\Gamma, \div A \vdash \Delta} \ (\div \vdash) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \div A, \Delta} \ (\vdash \div) \qquad \text{for } clopen(\Gamma, \Delta) \text{ or } clopen(A)$$

$$\frac{\Gamma, \div A \vdash \Delta \quad \Gamma \vdash \div B, \Delta}{\Gamma, \div(A \wedge \div B) \vdash \Delta} \ (\div\wedge \vdash) \qquad \frac{\Gamma \vdash \div A, \div B, \Delta}{\Gamma \vdash \div(A \wedge B), \Delta} \ (\vdash \div\wedge)$$

$$\frac{\Gamma \vdash \div\neg A, \Delta \quad \Gamma \vdash \div\neg B, \Delta}{\Gamma \vdash \div\neg(A \wedge B), \Delta} \ (\vdash \div\neg\wedge)'$$

Table 4.2: The LK[S5]s system

A usual definition of the *subformula property* for rules is that a rule, (R), has the subformula property if all active formulae, $A_{l_1}, \ldots, A_{,l_n}, A_{r_1}, \ldots, A_{r_m}$ are subformulae of the principal formula A,

$$\frac{\Gamma, A_{l_1}, \ldots, A_{l_n} \vdash A_{r_1}, \ldots, A_{r_m}, \Delta}{\Gamma, A \vdash \Delta} \ (R) \quad \text{or} \quad \frac{\Gamma, A_{l_1}, \ldots, A_{l_n} \vdash A_{r_1}, \ldots, A_{r_m}, \Delta}{\Gamma \vdash A, \Delta} \ (R)$$

where A_i is a subformula of A if $A_i \in Sub(A)$ and the definition of $Sub(A)$ is the straightforward inductive definition:

- $A \in Sub(A)$

- if $A = \neg A'$ or $A = \div A'$, then $\{A'\} \cup Sub(A') \subseteq Sub(A)$, and
- if $A = A_1 \wedge A_2$ then both $\{A_1, A_2\} \cup Sub(A_1) \cup Sub(A_2) \subseteq Sub(A)$ (and similarly for other possible binary connectives)

By such a definition, some rules(specifically, the rules for which two connectives are explicit in the principal formula) of the proposed system would not satisfy the subformula property; for example, the principal formula in the $(\vdash \div\wedge)$-rule, would have subformulae as shown below:

$$\frac{\Gamma \vdash \div A, \div B, \Delta}{\Gamma \vdash \div(A \wedge B), \Delta}\ (\vdash \div\wedge)$$

$Sub(\div(A \wedge B)) = \{\div(A \wedge B), A \wedge B, A, B\} \cup Sub(A) \cup Sub(B)$, and, thus, neither of the active formulae, $\div A$ nor $\div B$, is in $Sub(\div(A \wedge B))$. However, we can define the subformula property for rules differetly.

We say that a rule has the subformula if all active formulae has complexity not greater than that of the principal formula and all formulae in all premises are identifiably connected to some formula in the conclusion, i.e. that formulae occuring in a premiss must also occur (in some modified form) in the conclusion. This definition of the subformula property for rules includes all the rules system we have proposed.

The (*cut*)-rule does not satisfy this definition of the subformula property; there is a formula occuring in a premiss which does not occur in the conclusion, namely the cut-formula.

We first show that the two reasoning systems LK[S5] and LK[S5]s are equivalent in terms of which sequents are provable. We do this by showing that the new rule, $(\vdash \div\neg\wedge)'$ is admissible in LK[S5], and that the previous rule, $(\vdash \div\neg\wedge)$, is admissible in LK[S5]s.

Lemma 4.5.1.

1. *The rule* $\dfrac{\Gamma \vdash \div\neg A, \Delta \quad \Gamma \vdash \div\neg B, \Delta}{\Gamma \vdash \div\neg(A \wedge B), \Delta}$ *is admissible in the LK[S5] reasoning system*
2. *The rules* $\dfrac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash A, \Delta}$ *and* $\dfrac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash B, \Delta}$ *are admissible and the length of the derivation of the conclusion is no longer than the length of the derivation of the premiss in either case.*
3. *The rule* $\dfrac{\Gamma \vdash \div\neg A \wedge \div\neg B, \Delta}{\Gamma \vdash \div\neg(A \wedge B), \Delta}$ *is admissible in the LK[S5]s reasoning system and the length of the derivation of the conclusion is no longer than the length of the derivation of the premiss.*

Proof.

1. Given derivations of the premises, δ_1 and δ_2, we apply the $(\vdash \wedge)$ rule and then $(\vdash \div\neg\wedge)$

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \delta_1 \\ \Gamma \vdash \div\neg A, \Delta\end{matrix} \quad \begin{matrix}\vdots\ \delta_2 \\ \Gamma \vdash \div\neg B, \Delta\end{matrix}}{\Gamma \vdash \div\neg A \wedge \div\neg B, \Delta}\ (\vdash \wedge)}{\Gamma \vdash \div\neg(A \wedge B), \Delta}\ (\vdash \div\neg\wedge)$$

2. The proofs of admissibility of these rules are identical to the proofs of admissibility of the same rules in the LK[S5] system given in Lemma 4.3.1, save for the cases involving the $(\vdash \div\neg\wedge)$-rule. We show the new case for only the first rule. In Lemma 4.3.1 we proved the claim by induction on the length of the derivation of the premiss, in a derivation, δ in which $A \wedge B$ was not principal and the last rule applied in the derivation was $(\vdash \div\neg\wedge)$ we had

$$\dfrac{\overset{\vdots\ \delta}{\Gamma \vdash A \wedge B, \div\neg C \wedge \div\neg D, \Delta}}{\Gamma \vdash A \wedge B, \div\neg(C \wedge D), \Delta}\ (\vdash \div\neg\wedge) \quad \leadsto^{IH} \quad \dfrac{\overset{\vdots\ \delta}{\Gamma \vdash A, \div\neg C \wedge \div\neg D, \Delta}}{\Gamma \vdash A, \div\neg(C \wedge D), \Delta}\ (\vdash \div\neg\wedge)$$

Where we could apply the induction hypothesis to the premiss of the $(\vdash \div\neg\wedge)$-rule as its derivation was shorter, and then applied the $(\vdash \div\neg\wedge)$-rule obtaining the required sequent. In the proof of admissibility for the corresponding rule in LK[S5]s this case would be (in a proof by induction on the length of the derivation of the premiss, as in LK[S5]):

$$\dfrac{\overset{\vdots\ \delta_1}{\Gamma \vdash A \wedge B, \div\neg C, \Delta} \quad \overset{\vdots\ \delta_2}{\Gamma \vdash A \wedge B, \div\neg D, \Delta}}{\Gamma \vdash A \wedge B, \div\neg(C \wedge D), \Delta}\ (\vdash \div\neg\wedge)' \quad \leadsto^{IH} \quad \dfrac{\overset{\vdots\ \delta_1}{\Gamma \vdash A, \div\neg C, \Delta} \quad \overset{\vdots\ \delta_2}{\Gamma \vdash A, \div\neg D, \Delta}}{\Gamma \vdash A, \div\neg(C \wedge D), \Delta}\ (\vdash \div\neg\wedge)'$$

where we apply the induction hypothesis to both premisses of the rule.

3. Given a derivation of the premiss, δ, we apply the rules for which we just showed admissibility and obtain derivations δ_1: $\Gamma \vdash \div\neg A, \Delta$ and δ_2: $\Gamma \vdash \div\neg B, \Delta$, which serve as premisses to the $(\vdash \div\neg\wedge)'$-rule giving the required conclusion:

$$\dfrac{\dfrac{\overset{\vdots\ \delta}{\Gamma \vdash \div\neg A \wedge \div\neg B, \Delta}}{\Gamma \vdash \div\neg A, \Delta}\ (L.4.5.1) \quad \dfrac{\overset{\vdots\ \delta}{\Gamma \vdash \div\neg A \wedge \div\neg B, \Delta}}{\Gamma \vdash \div\neg B, \Delta}\ (L.4.5.1)}{\Gamma \vdash \div\neg(A \wedge B), \Delta}\ (\vdash \div\neg\wedge)'$$

□

4.6 Decidability

By the complexity of a sequent we will refer to the sum of complexities of formulae in that sequent, and define it as follows, for sequent $\Gamma \vdash \Delta$,

$$|\Gamma \vdash \Delta| = \sum_{\gamma \in \Gamma} |\gamma| + \sum_{\delta \in \Delta} |\delta|$$

There are two rules in LK[S5]s which do not have less complex premises than conclusion, these are the $(\div \vdash)$ and $(\vdash \div\wedge)$ rules. In the former, the complexity is increased, and in the latter the complexity of the premiss is the same as that of the conclusion.

We answer the question of the decidability of the $\vdash$-relation, that is to answer decisively, yes or no, whether a given sequent, $S = \Gamma \vdash \Delta$, can be proven in this system. We construct a tree of possible proofs and show that the search for a proof will terminate with either a positive or negative answer. For any given sequent, S, there are a certain number for rules which could have been applied to obtain the given sequent; for each formula, ϕ, in S, there are only a few, at most 4, rules which might have ϕ principal. We construct a tree where the nodes are sequences of sequents, with the sequence containing only the sequent S as the root, for each formula, ϕ in S, for each possible rule, (R), which can have ϕ principal, we let the node representing the collection containing the premisses of (R) instead of its conclusion be a child of the current node. For example, at some point, let the sequents $S_1, \ldots, S_i, \ldots, S_n$ be the sequents which we would be required to prove for some node. Let S_i be a sequent containing, for example, some formula of the form $A \wedge B$ in the right hand side. The following illustration shows the two nodes constructed for the two rules which can be applied and have $A \wedge B$ principal.

$$
\begin{tikzcd}
 & & S_1; \dots; \Gamma \vdash A \wedge B, \Delta; \dots; S_n \arrow[dll, "\text{from formulae left of } \phi"'] \arrow[ddl, "(\vdash W)"'] \arrow[ddr, "(\vdash \wedge)"] \arrow[drr, "\text{from formulae right of } \phi"] & & \\
\dots & & & & \dots \\
 & S_1; \dots; \Gamma \vdash \Delta; \dots; S_n & & S_1; \dots; \Gamma \vdash A, \Delta; \Gamma \vdash B, \Delta; \dots; S_n &
\end{tikzcd}
$$

The ellipses in leftmost and rightmost nodes in the illustration refer to not only formulae left/right of $A \wedge B$ in the sequent S_i, but also to formulae in sequents from $S_1; \dots; S_{i-1}$ and $S_{i+1}; \dots; S_n$, respectively (if there are such sequents).

If at some point there are no rules which can be applied in this way, the branch terminates and this node is called a leaf. If a sequence of contains only instances of (ax), then the branch terminates. All the rules, except $(\div \vdash)$ and $(\vdash \div\wedge)$ decrease the complexity of the premisses, so they will lead to branches which will terminate. For the exceptions, the branches will also terminate:

After an application of $(\vdash \div\wedge)$, the active formulae $\div A$ and $\div B$ are both less complex than the principal formula, $\div(A \wedge B)$. Neither of $\div A$ and $\div B$ can, by the rules of LK[S5]s be derived from $\div(A \wedge B)$ and no cycles can occur. The rules which might in turn be applied to $\div A$ (or $\div B$) are the $(\vdash W)$, $(\vdash \div)$, $(\vdash \div\wedge)$ and $(\vdash \div\neg\wedge)$ rules; all of which decrease the complexity of their premisses except for the $(\vdash \div\wedge)$-rule, but this is repeated application of the $(\vdash \div\wedge)$-rule and can not cause an infinite branch since it's active formulae are less complex for each application.

The only rule which can cause infinite branches is the $(\div \vdash)$-rule. If, when $(\div \vdash)$ is applicable to some sequence of sequents, $S_1; \dots; S_n$, and there is already an instance of $S_1; \dots; S_n$ in this branch, then such a cycling has occured and this branch is terminated. If this sequence of sequents is provable in this system, then a branch containing only instances of (ax) would be in another branch from the first such sequence. We terminate the branch after one such cycle.

If such a tree with root node S has a leaf node in which all the sequents are instances of (ax) then S is provable in this system, and otherwise, if there are no suchs leaves, then S is not provable.

4.7 Conclusion

We were not able to present a full proof of the admissibility of the (cut)-rule, in this system. However in the one problematic case of the proof of admissibility of the needed rule (Conjecture 4.3.3) we have a nearly complete proof. In attempting to falsify the claim of admissibility we have constructed some examples of derivations fitting the assumptions of the difficult case; all such derivations end in sequents that have equivalent cut-free derivations from which we easily obtain the conclusion of the conjectured rule.

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