Regression

Machine Learning for Behavioral Data

March 6, 2023

Today's Topic

Week	Lecture/Lab
1	Introduction
2	Data Exploration
3	**Regression**
4	Classification
5	Model Evaluation
6	Time Series Prediction
7	Time Series Prediction
8	Spring Break

Complete pipeline for one use case:

- **Data exploration**
- **Prediction**
- **Model evaluation**

Getting ready for today's lecture…

- **If not done yet**: clone the repository containing the Jupyter notebook and data for today's lecture into your Noto workspace.
- SpeakUp room for today's lecture:

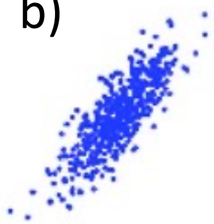
https://go.epfl.ch/speakup-mlbd

Short quiz about the past…

- Which of the four graphs have the following properties:

High Pearson’s Correlation, High Mutual Information

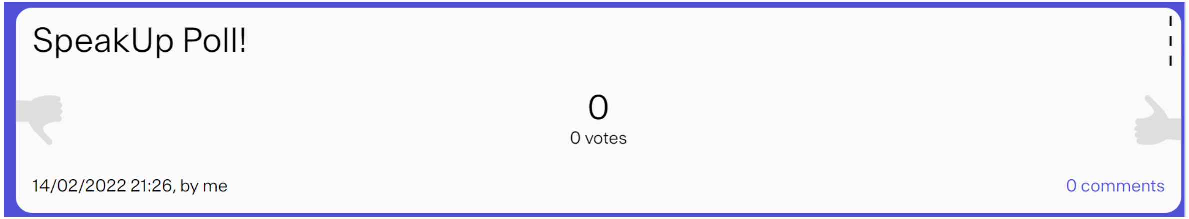
a) 

b) 

c) None

d) 

e) 

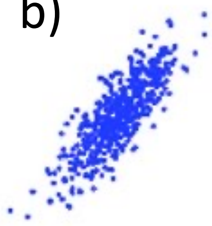


Short quiz about the past…

- Which of the four graphs have the following properties:

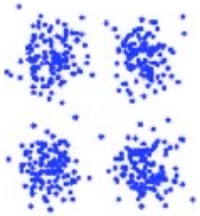
High Pearson’s Correlation, Low Mutual Information

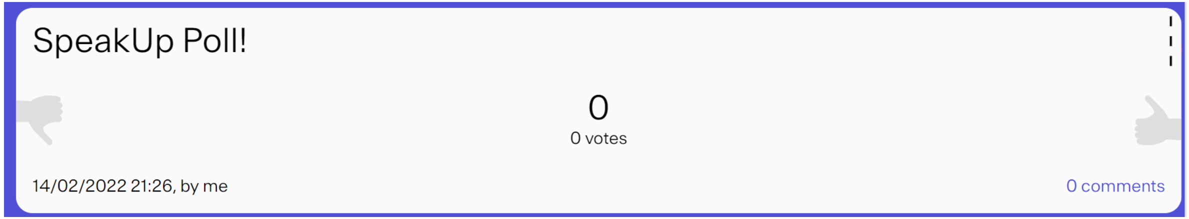
a) 

b) 

c) None

d) 

e) 

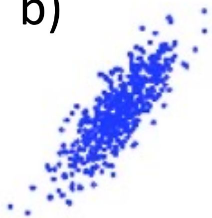


Short quiz about the past…

- Which of the four graphs have the following properties:

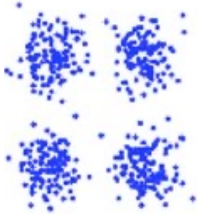
Low Pearson's Correlation, Low Mutual Information

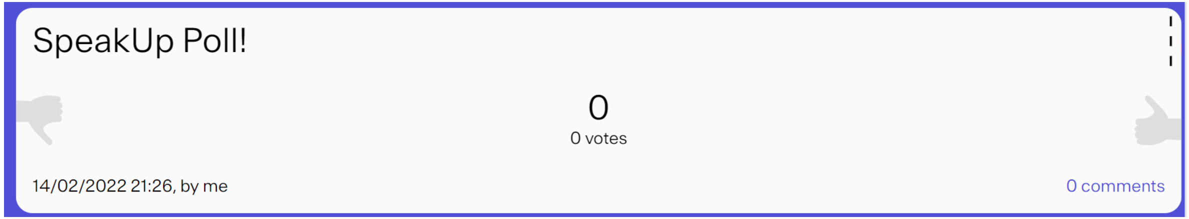
a) 

b) 

c) None

d) 

e) 



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Today’s Use Case: Flipped Classroom Course

- Participants: 288 EPFL students of a course taught in *flipped classroom* mode with a duration of 10 weeks
- Structure:
  - Preparation: watch videos (and solve simple quizzes) on **new content** at home as a preparation for the lecture
  - Lecture: discuss open questions and solve more complex tasks
  - Lab session: solve paper-an-pen assignments
- Data: clickstream data (all interactions of the student with the system)

Agenda

- **Linear Regresssion**
- Generalized Linear Models
- Mixed-Effect Models
- Performance Metrics
- Regression for Time-Series

Idea

- In regression, a single aspect of the data (output variable) is modeled by some combination of other aspects of the data (input variables)

More formal

- In regression, a single aspect of the data (output variable) is modeled by some combination of other aspects of the data (input variables)
- Given: N data points $(y_n, \boldsymbol{x}_n)$, where y_n is the n'th output variable and $\boldsymbol{x}_n$ is a D-dimensional vector of input variables
- Goal: $y_n \approx f(\boldsymbol{x}_n)$

Usage

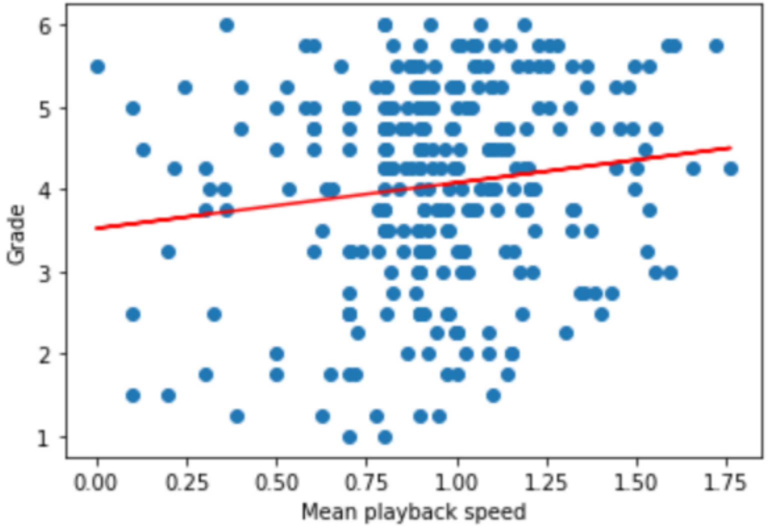
- *Prediction*: predict the output for a new (unseen) input vector $\boldsymbol{x}$
- *Interpretation*: analyze the relationships between the variables (what effect the input variables have on the output variable)

Example | Mean playback speed

x-axis: Mean playback speed of videos

y-axis: Course grade

Each point is one student



Students who watch the videos faster tend to have better grades.

Linear Regression

The output variable y_n with $n = 1, \dots, N$ is modeled by a **linear** combination of the input variables $x_{n,d}$ with $d = 1, \dots, D$.

$$\boxed{y_n = \beta_0 + \beta_1 x_{n,1} + \cdots + \beta_D x_{n,D} + \epsilon_n}$$

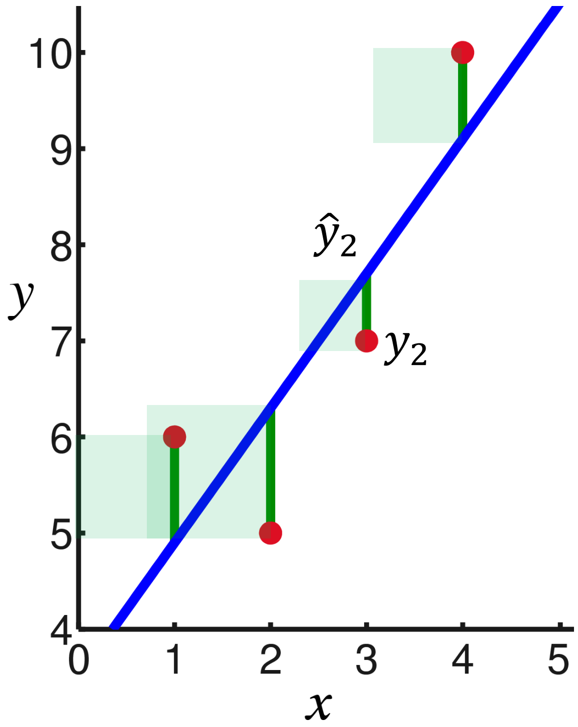
where ϵ_n are error terms that should be as small as possible and $\epsilon_n \sim \mathrm{N}(0, \sigma^2)$.

Goal: find optimal parameters

Find parameters $\hat{\beta}$ that minimize

$$\boxed{\sum_{n=1}^{N} \left(y_n - \widetilde{\boldsymbol{x}}_n^T \cdot \widehat{\boldsymbol{\beta}}\right)^2}$$

with $\widetilde{\boldsymbol{x}}_n = \begin{bmatrix} 1 \\ x_{n,1} \\ \ldots \\ x_{n,D} \end{bmatrix}$ and $\widehat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \ldots \\ \hat{\beta}_D \end{bmatrix}$



$$SS_{tot} = \sum_i (y_i - \bar{y})^2 \qquad SS_{res} = \sum_i (y_i - f(x_i))^2$$

Fitting the parameters

$$grade = \beta_0 + \beta_1 \cdot time\_in\_problem + \beta_2 \cdot percentage\_correct$$

Formula: grade~ch_time_in_prob_sum+wa_num_subs_perc_correct

Family: gaussian   Estimator: OLS

Std-errors: non-robust   CIs: standard 95%   Inference: parametric

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$$

Number of observations: 288   R^2: 0.110   R^2_adj: 0.104 $= 1 - \frac{(1-R^2)(n-1)}{(n-k-1)}$

Log-likelihood: -449.516   AIC: 905.031   BIC: 916.020

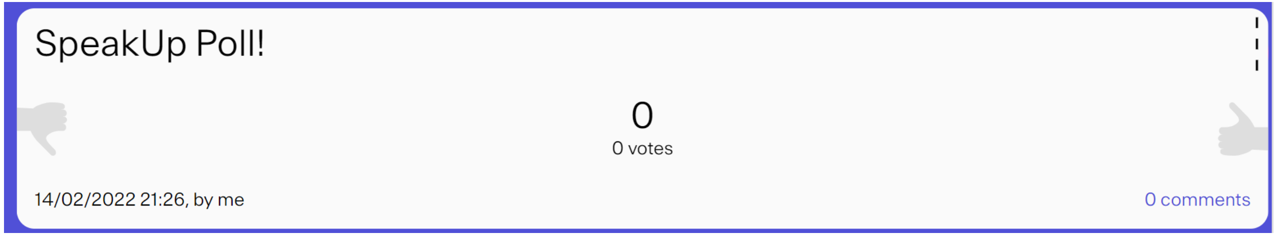
$p(y|x)$ $= 2k - 2LL$ $= k \cdot \ln(n) - 2LL$

Fixed effects:

	Estimate	2.5_ci	97.5_ci	SE	DF	T-stat	P-val	Sig
Intercept	3.410119	3.148091	3.672148	0.133123	285	25.616335	0.000000	***
ch_time_in_prob_sum	0.000157	0.000094	0.000220	0.000032	285	4.921856	0.000001	***
wa_num_subs_perc_correct	0.716132	0.035683	1.396581	0.345700	285	2.071542	0.039208	*

Influence of input variables

$$grade = 3.4 + 0.000016 \cdot time\_in\_problem + 0.72 \cdot percentage\_correct$$



Which of the input variables has the largest impact on $grade$?

a) $time\_in\_problem$

b) $percentage\_correct$

c) I don't know

Different units of measurements

$$grade = 3.2 + 0.000016 \cdot time\_in\_problem + 0.72 \cdot percentage\_correct$$

increase in $time\_in\_problem$ by 1s -> increase of grade by 0.000016

increase in $percentage\_correct$ by 1 percentage point -> increase of grade by 0.72

Transformation: Z-Scores

$$\boxed{\tilde{x}_{n,d} = \frac{x_{n,d} - \bar{\boldsymbol{x}}_d}{\sigma(\boldsymbol{x}_d)}} \qquad \begin{array}{l} d = 1, \ldots, D \\ n = 1, \ldots, N \end{array}$$

➡ Standardization via z-score: $\tilde{x}_{n,d}$ denotes the distance between the raw feature $x_{n,d}$ and the sample mean $\bar{\boldsymbol{x}}_d$ (in units of the standard deviation)

Transformation: Example

$$grade = 4.05 + 0.35 \cdot time\_in\_problem + 0.15 \cdot percentage\_correct$$

Example in Jupyter Notebook

Transformation: Summary

- Lets us compare the impact of input variables with different scales/units of measurements (e.g., time in problem in *seconds* and percentage correct)
- Reduces interpretability of individual input variables

Interpretation: Caveat

$$bodyfat = -45.95 + 0{,}99 \cdot abdomen - 0{,}33 \cdot weight$$

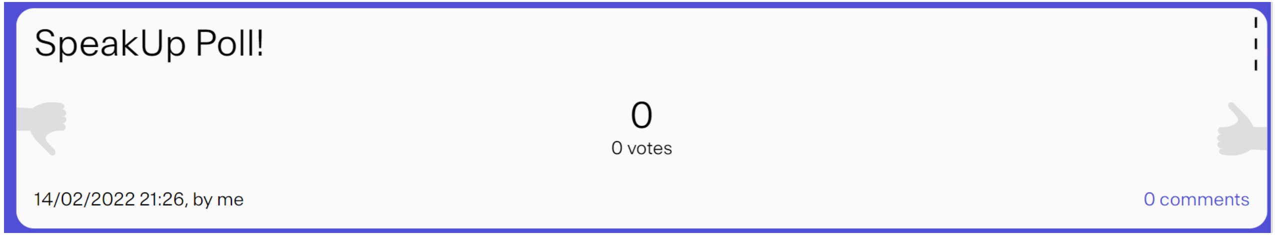
Can I conclude that heavier people (higher weight) have a lower bodyfat percentage?

Interpretation: Caveat

$$bodyfat = -45.95 + 0{,}99 \cdot abdomen - 0{,}33 \cdot weight$$

Can I conclude that heavier people (higher weight) have a lower bodyfat percentage?

a) Yes
b) No
c) I don't know



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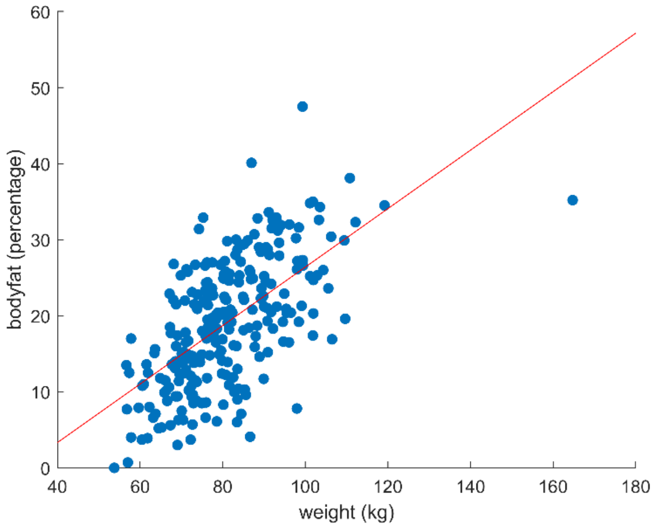
Interpretation: Caveat

- There is a positive correlation between $weight$ and $bodyfat$ ($r = 0.61, p < .001$).



Interpretation: Caveat

- There is a positive correlation between $weight$ and $bodyfat$ $(r = 0.61, p < .001)$.
  - ➢ $weight$ only has a negative coefficient β in the context of $abdomen$, i.e. for fixed $abdomen$ predictor
  - ➢ a predictor can only be interpreted **in the context** of the other predictors in the model

What means linear?

$$y_n = \beta_0 + \beta_1 x_{n,1} + \cdots + \beta_D x_{n,D} + \epsilon_n$$

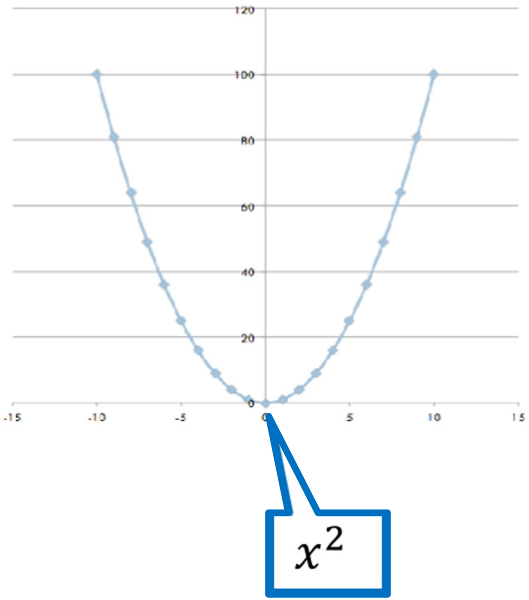
Linear in the **parameters** -> we can apply arbitrary functions to the raw input variables, e.g.,

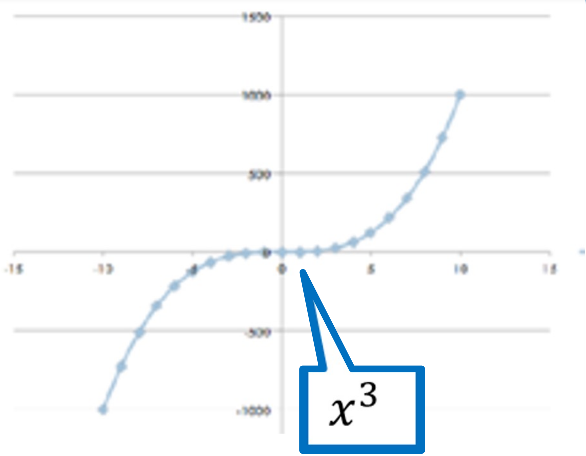
- logarithms, exponentials
- polynomials
- inverse

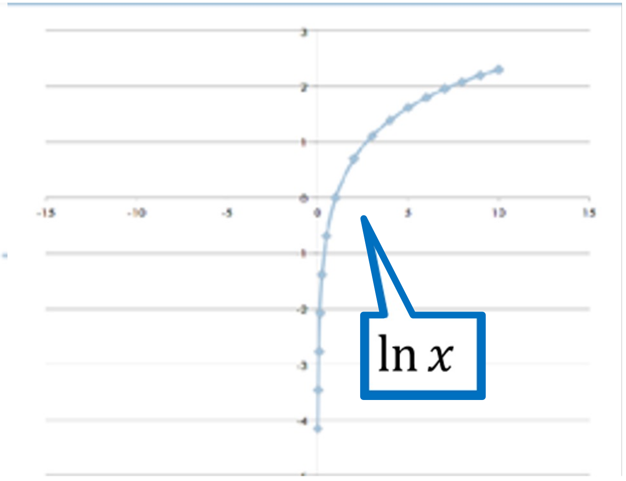
(time in problem)^2

What means linear

Different transformations



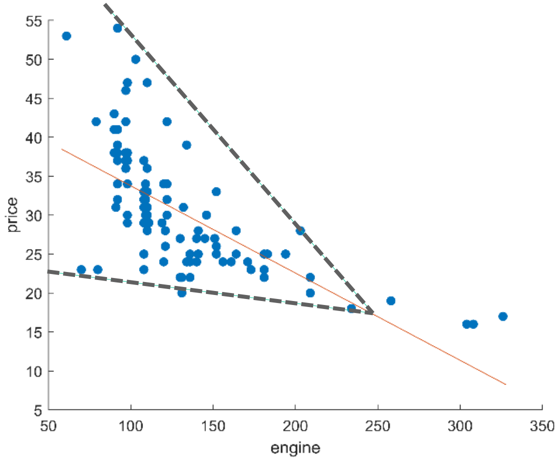


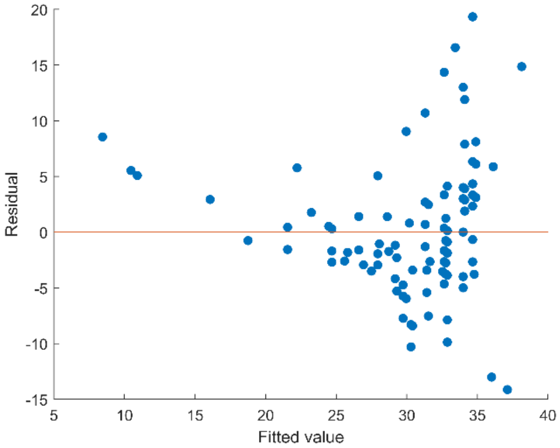


Restrictions of linear models

For some cases, linear regression models are not appropriate:

- the variance of $\boldsymbol{y}$ depends on the mean





Assumption for statistics (t-test, chi, etc.): $\epsilon \sim \mathrm{N}(0, \sigma^2)$

Restrictions of linear models

For some cases, linear regression models are not appropriate:

- the variance of $\boldsymbol{y}$ depends on the mean
- the range of $\boldsymbol{y}$ is restricted

$$\#bycicles = -2291 + 83 \cdot maxTemp - 13 \cdot minTemp - 890 \cdot precipitation$$

➡ prediction $\hat{y}$ can be negative…

Agenda

- Linear Regresssion
- **Generalized Linear Models**
- Mixed-Effect Models
- Performance Metrics
- Regression for Time-Series

Generalized Linear Models

A generalized linear model is composed of a **linear predictor**

$$\pi_n = \beta_0 + \beta_1 x_{n,1} + \cdots + \beta_D x_{n,D}$$

and a **link function**

$$g(\mu_n) = \pi_n$$

with $\mu_n = \mathrm{E}[Y|\mathrm{X} = \boldsymbol{x}_n]$

conditional expectation

Generalized Linear Models

Conditional expectation: the mean μ_n depends on the values of independent variables $\boldsymbol{x}_n$

is composed of a **linear predictor**

$x_{n,1}$ +

Each y_n represents the realization of the random variable Y, which is distributed according to a specific probability distribution

and a ... **unction**

$$g(\mu_n) = \pi_n$$

with $\mu_n = \mathrm{E}[Y|\mathrm{X} = \boldsymbol{x}_n]$

$\hat{\hat{}}$ y_n

Generalized Linear Models

A generalized linear model is composed of a **linear predictor**

$$\pi_n = \beta_0 + \beta_1 x_{n,1} + \cdots$$

In practice (for parameter fitting): observed values y_n are assumed to represent μ_n

and a **link function**

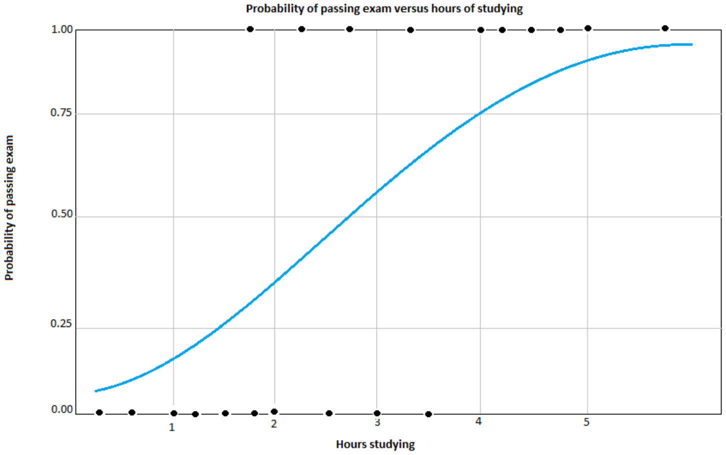
$$g(\mu_n) = \pi_n$$

with $\mu_n = \mathrm{E}[Y | \mathrm{X} = \boldsymbol{x}_n]$

Logistic Regression

In logistic regression, the link function is

$$g(\mu_n) = \log(\frac{\mu_n}{1-\mu_n})$$



and therefore (for fitting)

$$\log(\frac{y_n}{1-y_n}) = \beta_0 + \beta_1 x_{n,1} + \cdots + \beta_D x_{n,D}$$

$$y_n = \frac{1}{1+e^{-\beta x}}$$

Support : $\{0, 1\}$

Poisson Regression

In Poisson regression, the link function is

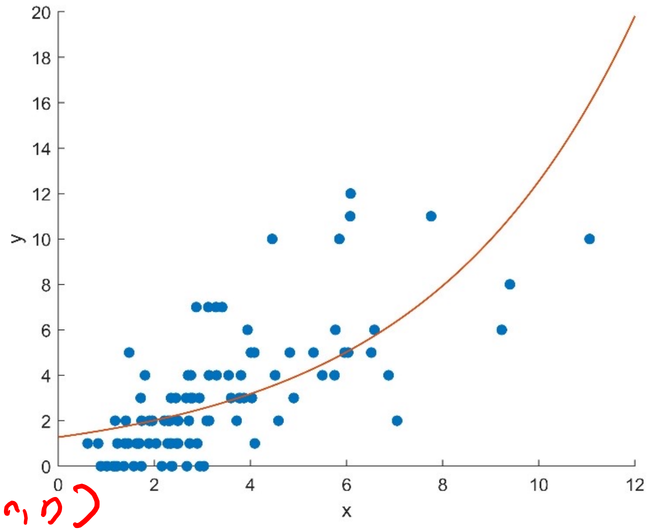
$$g(\mu_n) = \log(\mu_n)$$

and therefore (for fitting)

$$\log(\mu_n) = \beta_0 + \beta_1 x_{n,1} + \cdots + \beta_D x_{n,D}$$

$\Downarrow$

$$y_n = e^{(\beta_0 + \beta_1 x_{n,1} + \dots + \beta_D x_{n,D})}$$



support: 0, 1, 2, ...,

Linear Regression as a special case

For the linear regression, the link function is

$$g(\mu_n) = \mu_n$$

and therefore (for fitting)

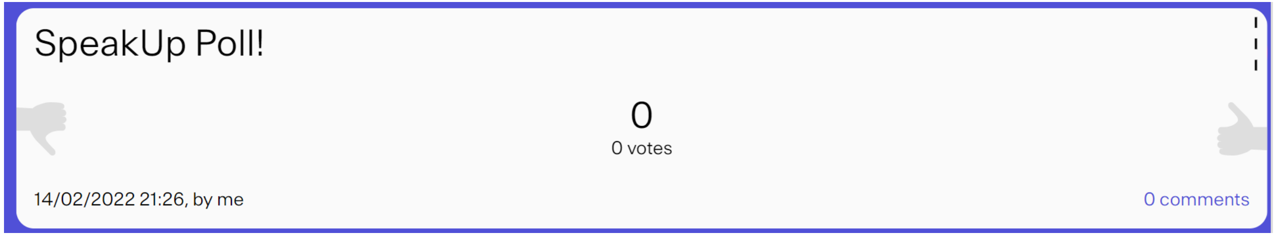
$$y_n = \beta_0 + \beta_1 x_{n,1} + \cdots + \beta_D x_{n,D}$$

Example

What type of model would you use for the following tasks?

1. Predict the **number of awards** earned by students at one high school. Predictors include the type of program in which the student was enrolled (e.g., vocational, general or academic) and the score on their final exam in math.

(a) Generalized Linear Model (b) Logistic Regression (c) Poisson Regression



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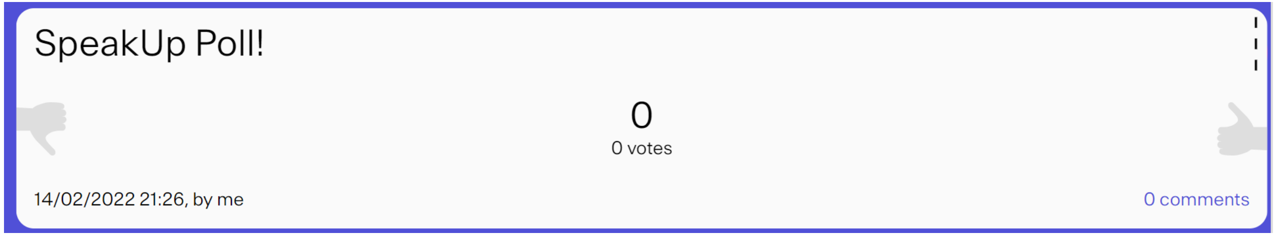
Example

What type of model would you use for the following tasks?

2. Predict whether a student will **solve a task correctly**. Predictors include the difficulty of the task and the number of tasks the student has already solved.

(a) ~~Generalized Linear Model~~ (b) Logistic Regression (c) Poisson Regression

Linear Regression



Example

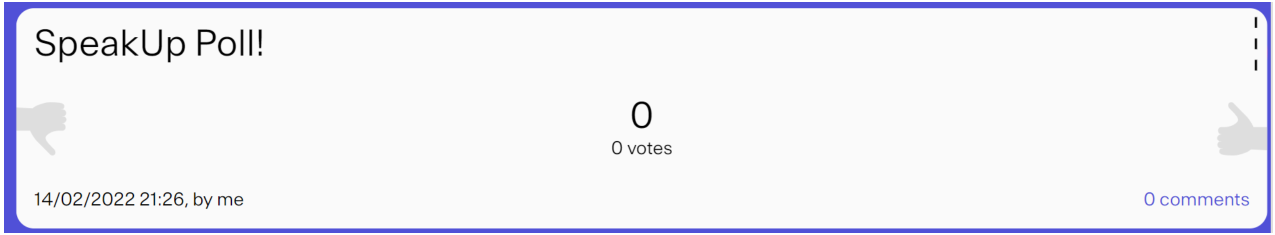
What type of model would you use for the following tasks?

3. Predict the **profit (in $)** of a company based on their advertising budget on Youtube.

(a) Linear Regression

(b) Logistic Regression

(c) Poisson Regression



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Agenda

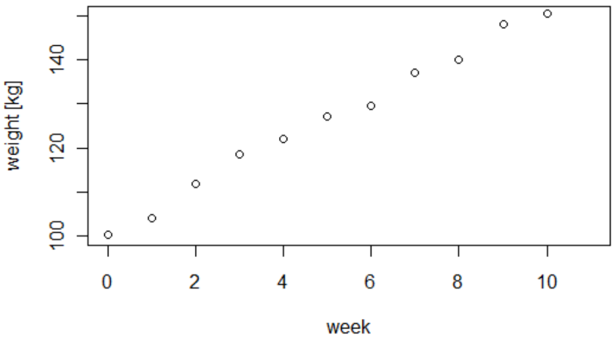
- Linear Regresssion
- Generalized Linear Models
- **Mixed-Effect Models**
- Performance Metrics
- Regression for Time-Series

Why mixed-effect models?

- Useful when we are dealing with correlated samples
  - Grouping of subjects (e.g., students within a classroom)
  - Repeated measurements on each subject over time (e.g., student in flipped classroom course over 10 weeks)

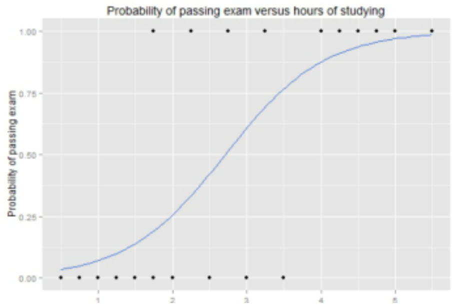
Generalized Linear Models

- Example 1: strength gain by weight training



$$y_n = \beta_0 + \beta_1 x_{n,1}$$

- Example 2: probability of passing exam of a course c depending on the hours studied

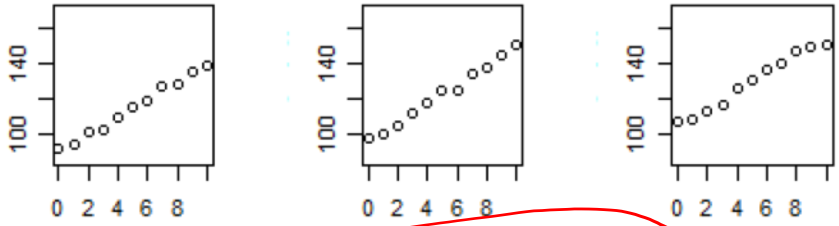


$$\log\left(\frac{y_n}{1-y_n}\right) = \beta_0 + \beta_1 x_{n,1}$$

“Fixed” Effects

Generalized Linear Mixed Effects Model

- Example 1: strength gain by weight training
  - Each person has individual starting strength



$$y_n = \beta_0 + u_n + \beta_1 x_{n,1} \qquad u_n \sim \mathrm{N}(0, \sigma_u^2)$$

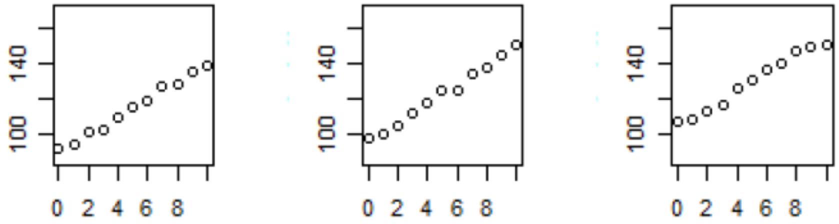
Random intercept

"Fixed" Effects

"Random" Effect

Generalized Linear Mixed Effects Model

- Example 1: strength gain by weight training
  - Each person has individual starting strength



$$y_n = \beta_0 + u_n + \beta_1 x_{n,1} \qquad u_n \sim \mathrm{N}(0, \sigma_u^2)$$

Fitting the parameters:

- Fixed effects only: linear least squares
- Mixed effects: maximum likelihood estimation

"Fixed" Effects + "Random" Effect = "Mixed" Effects

Generalized Linear Mixed Effects Model

- In our case, students come from different origins and we assume that students from the same origin are more similar (same education system)
- We therefore use origin ($category$) as a proxy for prior knowledge and add a random intercept to the model

$$\boxed{passed \sim 1|category + percentage\_correct}$$

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Usage

- *Interpretation*: analyze the relationships between the variables (what effect the input variables have on the output variable)
- *Prediction*: predict the output for a new (unseen) input vector $\boldsymbol{x}$

Regression: R^2

$$\boxed{R^2 = 1 - \frac{SS_{res}}{SS_{tot}}}$$

where $SS_{res} = \sum_i (y_i - f(x_i))^2$ and $SS_{tot} = \sum_i (y_i - \bar{y})^2$

- Can be interpreted as the fraction of explained variability of the data
- Often used when the goal is *interpretation*
- Often used in the fields of Psychometrics, Learning Sciences, Psychology, etc.

Regression: MAE and RMSE

- Mean absolute error: $$MAE = \frac{1}{N}\sum_{i=1}^{N}|y_i - f(x_i)|$$
- Root mean squared error: $$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(y_i - f(x_i))^2}$$
- Often used when the goal is *prediction*
- **RMSE is largely preferred to MAE**

Hypothetical Example

- Given:
  - Student giving correct answers 70% of the time
  - Model A: predicts correct 70% of the time
  - Model B: predicts 100% correctness

MAE: Model B is better

- 70% of the time the student gives a correct answer (response = 1)
  - Model A: absolute error = 0.3
  - Model B: absolute error = 0.0
- 30% of the time the student answers wrong (response = 0)
  - Model A: absolute error = 0.7
  - Model B: absolute error = 1.0
- $MAE_A = 0.42, \; MAE_B = 0.30$

RMSE: Model A is better

- $RMSE_A = 0.21$
- $RMSE_B = 0.30$

- $\boldsymbol{RMSE}$ **penalizes large errors heavier**

Agenda

- Linear Regresssion
- Generalized Linear Models
- Mixed-Effect Models
- Performance Metrics
- **Regression for Time-Series**

Time Series – Our flipped classroom case

Student i

Weeks

sessions

0	1	2	3	4	5	6	7	8	9

⋮

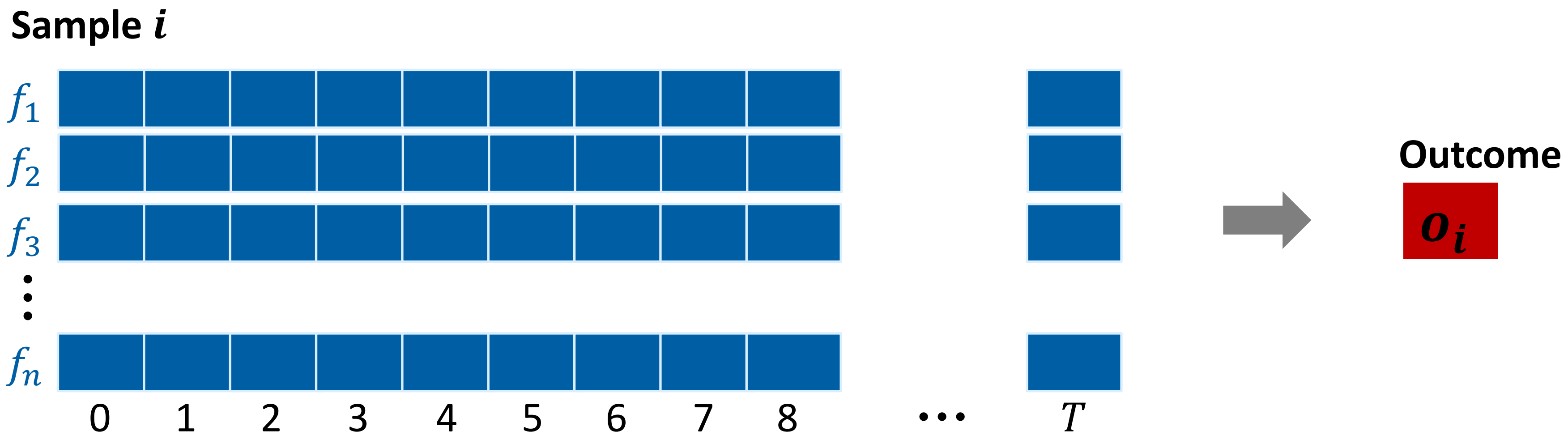
Weeks

submissions_correct

0	1	2	3	4	5	6	7	8	9

Time Series – Possible Tasks

- Prediction of a target variable after $\boldsymbol{t} < \boldsymbol{T}$ time steps, where T is the total number of time steps



Time Series – Possible Tasks

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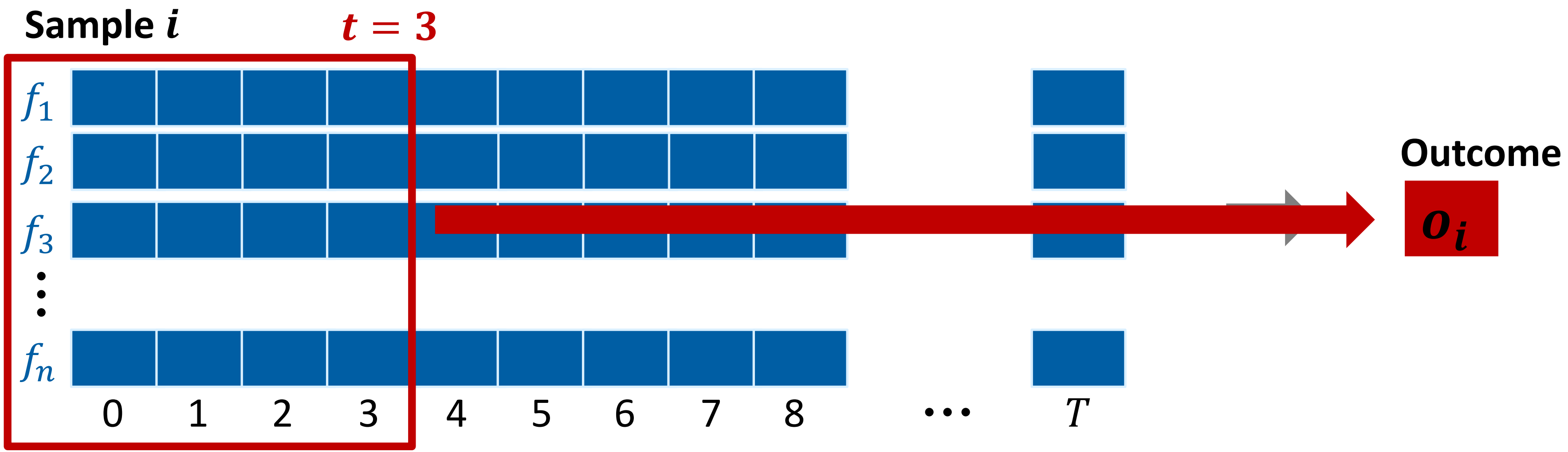

Handling Time Series Data

grade = $\beta_0 + \beta_1 \cdot$ time_problem_week1 $+ \beta_2 \cdot$ time_in_prob_week2 + ...

- Flattening
  - ➢ The number of parameters of the model depends on the number of time steps of the model
- Aggregation
  - ➢ Averaging across weeks
  - ➢ Accumulating across weeks

grade = $\beta_0 + \beta_1 \cdot$ average_time_prob $+ \beta_2 \cdot$ var(atime_in_problem)

Example – Prediction of Grade

- Prediction of grade after $t < T$ weeks
- We will try to predict after 5 weeks and after 10 weeks

$$\boxed{grade \sim (1|category) \; + \; average\_percentage\_correct \; [week \; n]}$$

Your Turn – Prediction of Passing

- Adjust the example equation to predict after week 5 and then, whether students will pass the exam
- Extension (if you have time):
  - Improve the accuracy of the model by adding more features
  - Justify, why you selected the chosen features and send us your RMSEs.

Your Turn – Feedback

Do you want feedback or have questions?
Upload your Jupyter Notebook here:
https://go.epfl.ch/notebooks-mlbd

Summary

- Linear regression is a useful framework for interpreting data and making predictions
- Caveat: be careful when interpreting the models
- Linear regression is flexible, i.e. arbitrary functions can be applied to the raw input data
- Generalized linear models are a more general framework appropriate for response variables from exponential family distributions
- Mixed models allow for capturing correlation in the data
- Modeling time series data requires some type of aggregation