Machine Intelligence

Lecture 3: Constraint satisfaction problems

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Tentative course overview

Topics:

- Introduction
- Search-based methods
- **Constraint satisfaction problems**
- Logic-based knowledge representation
- Representing domains endowed with uncertainty.
- Bayesian networks
- Machine learning
- Planning
- Multi-agent systems

Features and Possible Worlds

Features and Variables

Describing the world (environment) by features:

Name (**Algebraic Variable**)	Domain
Symbol_on_square_1	$\{1, 2, 3, 4, 5, 6, 7, 8, \textit{empty}\}$
Robot_battery	$\{\textit{full}, \textit{half}, \textit{empty}\}$
Robot_position	$\{r131, \ldots, 0111\}$
Coffee_ready	$\{\textit{true}, \textit{false}\}$
No._of_undelivered_packages	$\{1, 2, 3, \ldots\}$
Temperature	$[-25, 40]$

- We will be mostly concerned with (algebraic) variables that have a *finite* domain.
- Special interest: **boolean** variable with domain $\{\textit{true}, \textit{false}\}$
- Numerical variables can be approximated:

$$\begin{array}{ccc} \{1,2,3,\ldots\} & \mapsto & \{1,2,3,4,5,>5\} \\ [-25,40] & \mapsto & \{-25,-24,\ldots,-1,0,1,\ldots 40\} \end{array}$$

Possible Worlds

From variables to possible worlds

A **possible world** for a set of variables is an assignment of a value to each variable.

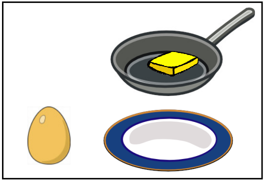
Connection with state spaces

The set of all possible worlds for a given set of variables defines a state space (we can also call a possible world simply a state).

Variables:

egg	{*whole*, *broken*}
butter_in	{*pan*, *plate*, *table*}
egg_in	{*pan*, *plate*, *table*}

One out of $2 \cdot 3 \cdot 3 = 18$ possible worlds:

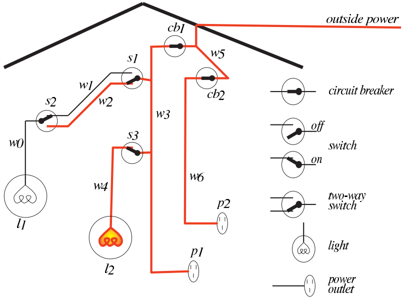


egg=whole
butter_in=pan
egg_in=table

Variables:

S_1_*pos*	{*up*, *down*}	S_1_*st*	{*ok*, *broken*, *short*}
S_2_*pos*	{*up*, *down*}	S_2_*st*	{*ok*, *broken*, *short*}
W_1_*st*	{*ok*, *broken*}	W_1_*current*	{*yes*, *no*}
...	...	...	...

One out of many possible worlds:



S_1_*pos* = *down*
S_1_*st* = *ok*
S_2_*pos* = *up*
S_2_*st* = *ok*
W_1_*st* = *ok*
W_1_*current* = *no*
...

Example: Schedule

Variables:

Teacher_MI	$\{PD, MJ, TDN\}$
Time_MI	{*Mo_m*, *Mo_a*, ..., *Fr_m*, *Fr_a*}
Room_MI	$\{0.2.12, 0.2.13, 0.2.90\}$
Teacher_AD	$\{PD, MJ, TDN\}$
Time_AD	{*Mo_m*, *Mo_a*, ..., *Fr_m*, *Fr_a*}
Room_AD	$\{0.2.12, 0.2.13, 0.2.90\}$

One possible world:

Mo	Tue	Wed	Thu	Fr
	MI, TDN, 0.2.13			
			AD, PD, 0.2.90	

Teacher_MI=TDN
Time_MI=*Tue_m*
Room_MI=0.2.13
Teacher_AD=PD
Time_AD=*Thu_a*
Room_AD=0.2.90

Constraint Satisfaction Problems

A **constraint** is a condition on the values of variables in a possible world.

Extensional Constraint Specification

Explicitly list all allowed (or disallowed) combination of values:

Teacher_MI	*Time_MI*	*Room_MI*	*Teacher_AD*	*Time_AD*	*Room_AD*
PD	*Mo_m*	0.2.12	PD	*Mo_a*	0.2.12
PD	*Mo_m*	0.2.12	PD	*Mo_a*	0.2.13
...	...	...	...	...	...

Not on the list of allowed possible worlds:

Teacher_MI	*Time_MI*	*Room_MI*	*Teacher_AD*	*Time_AD*	*Room_AD*
PD	*Mo_m*	0.2.12	PD	*Mo_m*	0.2.12
PD	*Mo_m*	0.2.12	MJ	*Mo_m*	0.2.12
...	...	...	...	...	...

Intensional Constraint Specification

Use logical expressions:

$$Teacher\_AD = Teacher\_MI \rightarrow Time\_AD \neq Time\_MI$$
$$Time\_AD = Time\_MI \rightarrow Room\_AD \neq Room\_MI$$

A1	*A2*	1	*A4*	*A5*	*A6*	*A7*	*A8*	*A9*
B1	*B2*	2	*B4*	3	*B6*	*B7*	*B8*	4
C1	*C2*	*C3*	5	*C5*	*C6*	6	*C8*	7
5	*D2*	*D3*	1	4	*D6*	*D7*	*D8*	*D9*
E1	7	*E3*	*E4*	*E5*	*E6*	*E7*	2	*E9*
F1	*F2*	*F3*	*F4*	7	8	*F7*	*F8*	9
8	*G2*	7	*G4*	*G5*	9	*G7*	*G8*	*G9*
4	*H2*	*H3*	*H4*	6	*H6*	3	*H8*	*H9*
I1	*I2*	*I3*	*I4*	*I5*	*I6*	5	*I8*	*I9*

Constraints:

$$A1 = 2 \lor A2 = 2 \lor A4 = 2 \lor A5 = 2 \lor A6 = 2 \lor A7 = 2 \lor A8 = 2 \lor A9 = 2$$
$$A1 = 3 \lor A2 = 3 \lor A4 = 3 \lor A5 = 3 \lor A6 = 3 \lor A7 = 3 \lor A8 = 3 \lor A9 = 3$$
$$\ldots$$
$$A1 = 3 \lor A2 = 3 \lor B1 = 3 \lor B2 = 3 \lor C1 = 3 \lor C2 = 3 \lor C3 = 3$$
$$\ldots$$

Constraint Satisfaction Problem

A **Constraint Satisfaction Problem (CSP)** is given by

- a set of variables
- a set of constraints (usually intensional)

A **solution** to a CSP consists of a possible world that satisfies all the constraints (also called a **model** of the constraints).

Other tasks:

- Determine the number of models of the constraints
- Find an *optimal* model (given also a value function on possible worlds).
- ...

CSP as State Space Problem

A CSP can be represented as a state space problem:

- States are all partial assignments of values to variables that are consistent with the constraints
- For a state s: select some variable V not assigned a value in s, and let the neighbors of s be all states that assign a value to V (if any exist).
- The start state is the state that does not assign any values
- A goal state is a state that assigns values to all variables

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Solving the CSP

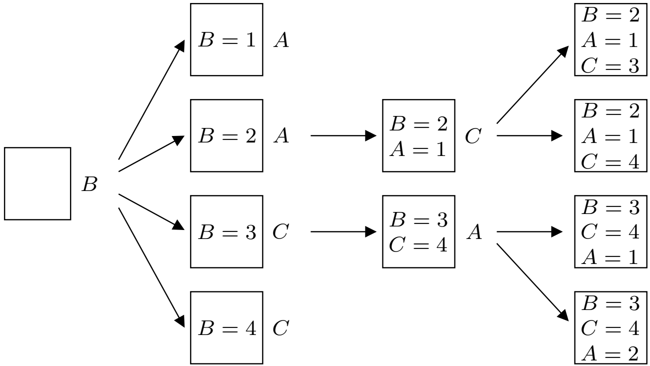
- A solution to the state space problem is a path with a goal state at the end: a solution to the CSP problem
- To solve the state space problem need only be able to:
  - enumerate all partial assignments that assign a value to one more variable than s
  - check whether a partial assignment is consistent with the constraints

  (that is sufficient to implement the *get_neighbors* and *goal* functions needed in the generic search algorithm)

Variables A, B, C; all with domain $\{1, 2, 3, 4\}$.

Constraints: $A < B$, $B < C$.

State Space Graph (showing at each node which variable was selected to generate the neighbors):



- The state space graph is a tree (= search tree)

Consistency Algorithms

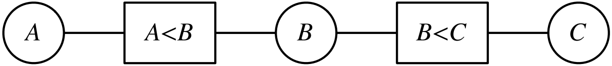
Idea

Example: Variables A, B, C; all with domain $\{1, 2, 3, 4\}$.

Constraints: $A < B$, $B < C$.

Observation: There is no solution with $A = 4$.

Approach to solving CSPs: iteratively eliminate value assignments that cannot be part of a solution.



The **constraint network** for a CSP consists of

- One (oval) node for each variable X
- One (rectangular) node for each constraint c
- An (undirected) arc $\langle X, c\rangle$ between every constraint and every variable involved in the constraint

With each variable node X is associated a **(reduced) domain** D_X:

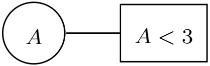
- Initially the domain of the variable
- Reduced by successively deleting values that cannot be part of a solution

An arc $\langle X, c\rangle$ is **arc consistent**, if

- for all $x \in D_X$ there exists values $y_1, \ldots, y_k$ for the other variables involved in c, such that $x, y_1, \ldots, y_k$ is consistent with c.

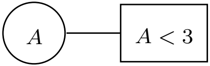
A constraint network is **arc consistent**, if all its arcs are arc consistent.

Examples



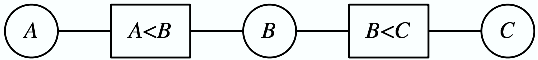
$D_A = \{1, 2, 3, 4\}$

Not arc consistent



$D_A = \{1, 2\}$

Arc consistent

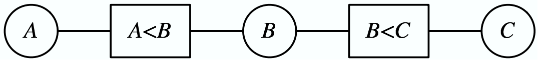


$D_A = \{1, 2\}$

$D_B = \{2, 3\}$

$D_C = \{3, 4\}$

Arc Consistency Examples

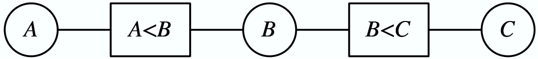


$D_A =$
$\{1, 2\}$

$D_B =$
$\{2, 3\}$

$D_C =$
$\{3, 4\}$

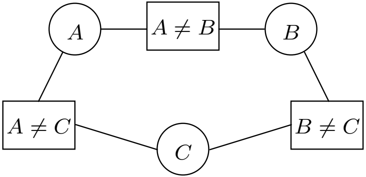
Arc consistent. Not every combination of values from D_A, D_B, D_C is a solution!



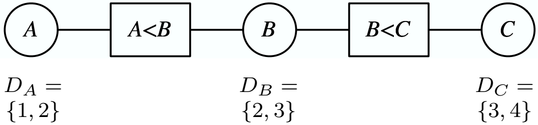
$D_A = \{1, 2\}$ $D_B = \{2, 3\}$ $D_C = \{3, 4\}$

Arc consistent. Not every combination of values from D_A, D_B, D_C is a solution!

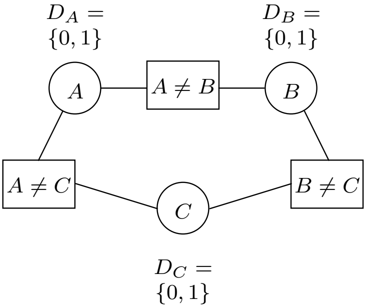
$D_A = \{0, 1\}$ $D_B = \{0, 1\}$



$D_C = \{0, 1\}$



Arc consistent. Not every combination of values from D_A, D_B, D_C is a solution!



Arc consistent. There exists no solution!

Algorithm Outline

1. *To-do-arcs*= all arcs in constraint network // Potentially inconsistent arcs
2. **while** *To-do-arcs* $\neq \emptyset$
3. select and delete one arc $\langle X, c\rangle$ from *To-do-arcs*
4. make arc consistent by deleting values from D_X, if necessary
5. **if** values were deleted: add all other arcs $\langle Z, c'\rangle$ ($c \neq c'$, $X \in dom(c')$) to *To-do-arcs*

Generalized Arc Consistency Algorithm

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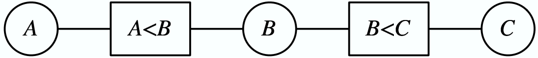
Example



$D_A = \{1,2,3,4\}$ $D_B = \{1,2,3,4\}$ $D_C = \{1,2,3,4\}$

- *To-do-arcs* $= \{\langle A, A<B\rangle, \langle B, A<B\rangle, \langle B, B<C\rangle, \langle C, B<C\rangle\}$
- Selecting $\langle A, A<B\rangle$: For $A=4$, no value of B satisfies $4<B$.
  - Remove $\langle A, A<B\rangle$ from *To-do-arcs*.
  - Update D_A.

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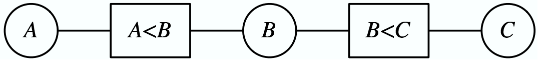
Example



$D_A =$ $\{1, 2, 3\}$

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$D_C =$ $\{1, 2, 3, 4\}$

- *To-do-arcs* $= \{\langle B, A < B\rangle, \langle B, B < C\rangle, \langle C, B < C\rangle\}$

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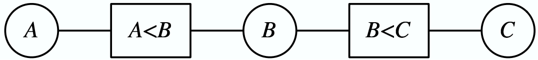

$D_A = \{1, 2, 3\}$ $D_B = \{1, 2, 3, 4\}$ $D_C = \{1, 2, 3, 4\}$

- *To-do-arcs* $= \{\langle B, A < B\rangle, \langle B, B < C\rangle, \langle C, B < C\rangle\}$
- Selecting $\langle B, A < B\rangle$: $B = 1$ can be pruned.
  - Remove $\langle B, A < B\rangle$ from *To-do-arcs*.
  - Update D_B.

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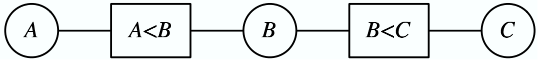
Example



$D_A =$ $\{1, 2, 3\}$ $D_B =$ $\{2, 3, 4\}$ $D_C =$ $\{1, 2, 3, 4\}$

- *To-do-arcs* $= \{\langle B, B < C\rangle, \langle C, B < C\rangle\}$

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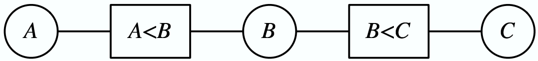

$D_A = \{1, 2, 3\}$ $D_B = \{2, 3, 4\}$ $D_C = \{1, 2, 3, 4\}$

- *To-do-arcs* $= \{\langle B, B < C \rangle, \langle C, B < C \rangle\}$
- Selecting $\langle B, B < C \rangle$: $B = 4$ can be pruned.
  - Add $\langle A, A < B \rangle$ to *To-do-arcs* and remove $\langle B, B < C \rangle$.
  - Update D_B.

Generalized Arc Consistency Algorithm

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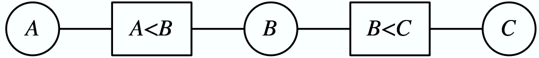
Example



$D_A = \{1, 2, 3\}$ $D_B = \{2, 3\}$ $D_C = \{1, 2, 3, 4\}$

- *To-do-arcs* $= \{\langle A, A < B\rangle, \langle C, B < C\rangle\}$

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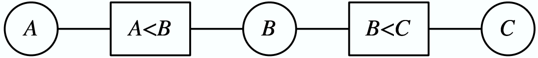
Example



$D_A = \{1, 2, 3\}$ $D_B = \{2, 3\}$ $D_C = \{1, 2, 3, 4\}$

- *To-do-arcs* $= \{\langle A, A < B \rangle, \langle C, B < C \rangle\}$
- Selecting $\langle A, A < B \rangle$: $A = 3$ can be pruned.
  - Remove $\langle A, A < B \rangle$ from *To-do-arcs*.
  - Update D_A.

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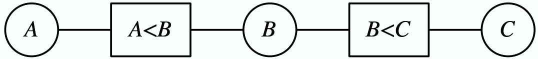
Example



$D_A = \{1,2\}$

$D_B = \{2,3\}$

$D_C = \{1,2,3,4\}$

- *To-do-arcs* $= \{\langle C, B < C\rangle\}$

Generalized Arc Consistency Algorithm

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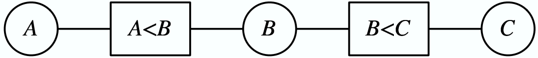
Example



$D_A = \{1,2\}$ $D_B = \{2,3\}$ $D_C = \{1,2,3,4\}$

- *To-do-arcs* $= \{\langle C, B < C\rangle\}$
- Selecting $\langle C, B < C\rangle$: $C = 1$ and $C = 2$ can be pruned.
  - Remove $\langle B, B < C\rangle$ from *To-do-arcs*.
  - Update D_C.

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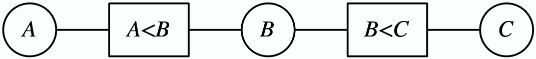
Example



$D_A = \{1, 2\}$ $D_B = \{2, 3\}$ $D_C = \{3, 4\}$

- *To-do-arcs* $= \{\}$
- DONE!

Algorithm Outline

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Algorithm Outcomes

Algorithm is guaranteed to terminate. Result independent of order in which arcs are processed. Possible cases at termination:

- $D_X = \emptyset$ for some X: CSP has no solution
- D_X contains exactly one value for each X: CSP has unique solution, given by the D_X values.
- Other: if the CSP has a solution, then the solution can only consist of current D_X values.

Variable Elimination

Idea

- Arc Consistency: simplify problem by eliminating values
- Variable Elimination: simplify problem by eliminating variables

Relational Operations

Variable Elimination operates on extensional (table) representations of constraints:

$A < B$:

A	B
1	2
1	3
1	4
2	3
2	4
3	4

$B < C$:

B	C
1	2
1	3
1	4
2	3
2	4
3	4

Algorithm requires **projection** and **join** operations on tables.

Projection of a table:

Course	Year	Student	Grade
cs322	2008	fran	77
cs111	2009	billie	88
cs111	2009	jess	78
cs444	2008	fran	83
cs322	2009	jordan	92

$$\xrightarrow{\pi_{\{Student, Year\}}}$$

Student	Year
fran	2008
billie	2009
jess	2009
jordan	2009

Given two tables r_1, r_2 for variables *vars*$_1$, *vars*$_2$. The **join** is the table $r_3 = r_1 \bowtie r_2$ for variables *vars*$_1 \cup$ *vars*$_2$ that

- contains all tuples, which restricted to *vars*$_1$ are in r_1, and restricted to *vars*$_2$ are in r_2.

Example

Course	Year	Student	Grade
cs322	2008	fran	77
cs111	2009	billie	88
cs111	2009	jess	78
cs444	2008	fran	83
cs322	2009	jordan	92

$\bowtie$

Course	Year	TA
cs322	2008	yuki
cs111	2009	sam
cs111	2009	chris
cs322	2009	yuki

=

Course	Year	Student	Grade	TA
cs322	2008	fran	77	yuki
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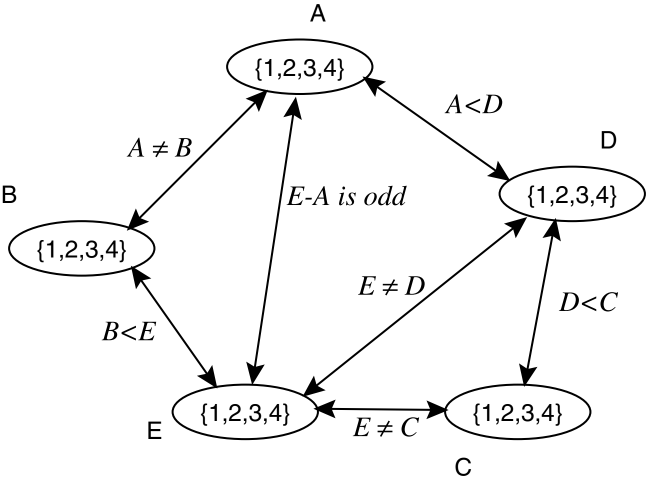
cs111	2009	jess	78	chris
cs322	2009	jordan	92	yuki

Variable Elimination Algorithm

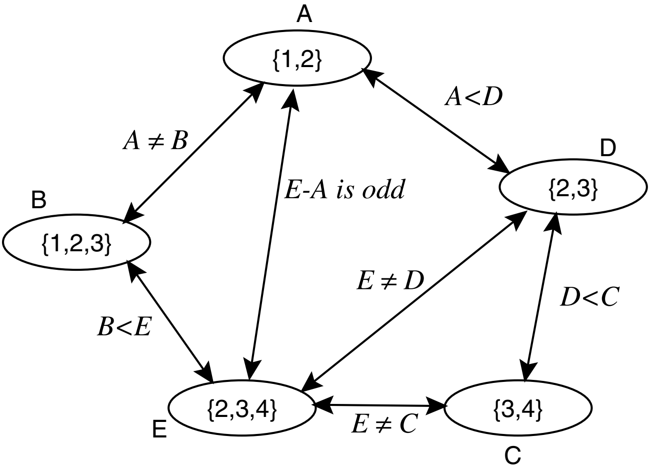
Algorithm Outline

1. **Input:** C: set of constraints on variables *vars*
2. **while** C contains more than one element
3.   select a variable $X \in$ *vars*
4.   delete X from *vars*
5.   remove all constraints involving X from C and construct their join
6.   **if** *vars* is not empty
7.   project the join onto the variables other than X
8.   add the (projected) join to C

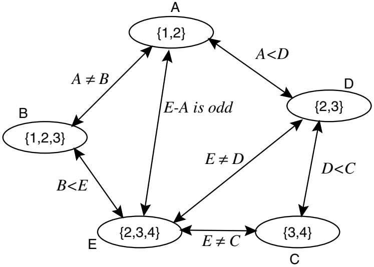
Intuition: the constraint constructed in line 5. summarizes the effect that all the constraints involving X have on variables other than X.


A
{1,2,3,4}
A<D
D
{1,2,3,4}
A ≠ B
B
{1,2,3,4}
E-A is odd
E ≠ D
D<C
B<E
{1,2,3,4}
E
E ≠ C
{1,2,3,4}
C

...now arc-consistent



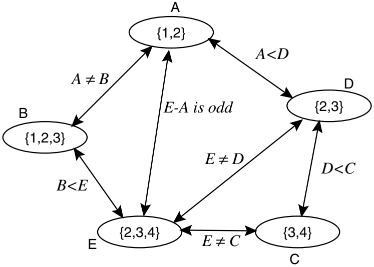
Example: eliminating C



$r_1 : C \neq E$	C	E
	3	2
	3	4
	4	2
	4	3

$r_2 : C > D$	C	D
	3	2
	4	2
	4	3

Example: eliminating C



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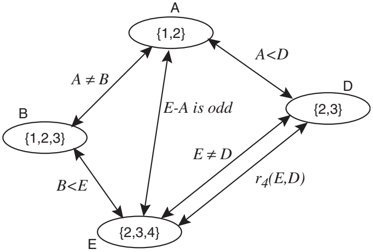
	4	2
	4	3

$r_2 : C > D$	C	D
	3	2
	4	2
	4	3

$r_3 : r_1 \bowtie r2$	C	D	E
	3	2	2
	3	2	4
	4	2	2
	4	2	3
	4	3	2
	4	3	3

Example: eliminating C



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	4	2
	4	3

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	3	2
	4	2
	4	3

$r_3 : r_1 \bowtie r2$	C	D	E
	3	2	2
	3	2	4
	4	2	2
	4	2	3
	4	3	2
	4	3	3

$r_4 : \pi_{\{D,E\}} r_3$	D	E
	2	2
	2	3
	2	4
	3	2
	3	3

$\hookrightarrow$ new constraint

Properties

- The algorithm terminates
- The CSP has a solution if and only if the final constraint is non-empty
- The set of all solutions can be generated by joining the final constraint with the intermediate "summarizing" constraints generated in line 5.
- Algorithm operates on extensional constraint representations, therefore
  - constraints must not contain too many tuples (initial and constructed constraints)
- Worst case: VE is not more efficient than enumerating all possible worlds and checking whether they are solutions.

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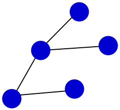
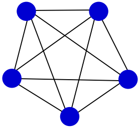
Constraint Graph

Consider the graph where
- there is one node for each variable
- two variables are connected when they appear together in one constraint


sparse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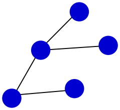
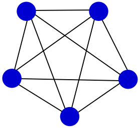


sparse



dense

Then: VE will work better if the constraint graph is sparsely connected

Local Search

Systematic vs. Local Search

So far: all methods systematically explored the state space (possible worlds).

Problem: Time and space when search space is large.

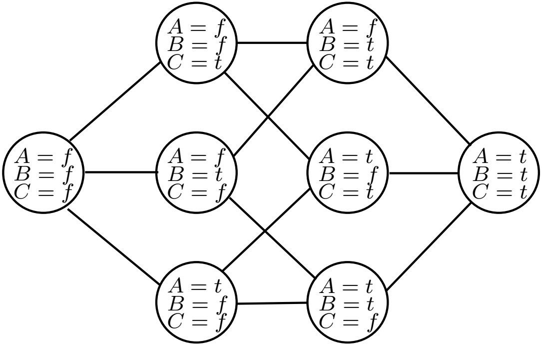
Local Search approach:

- explore state space without 'bookkeeping' (where have we been? what still needs to be explored?).
- no success/termination guarantees
- in practice, often the only thing that works

(Another) state space graph representation for CSPs:

- Nodes are possible worlds
- Neighbors are possible worlds that differ in the value of exactly one variable

State space graph for 3 boolean variables:



Algorithm Outline

1. Select some node in state space graph as *current state*
2. **while** *current state* is not a solution
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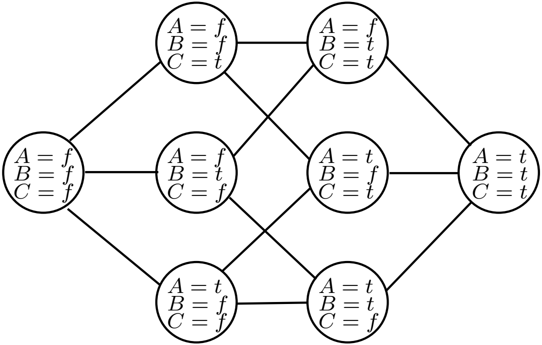

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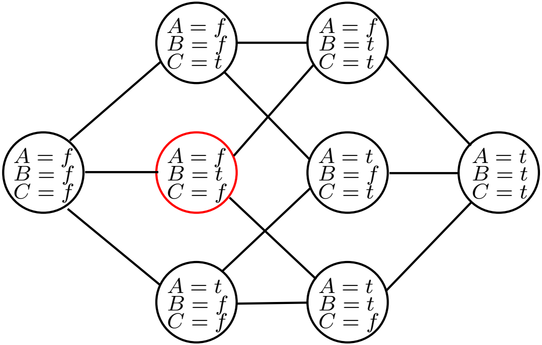

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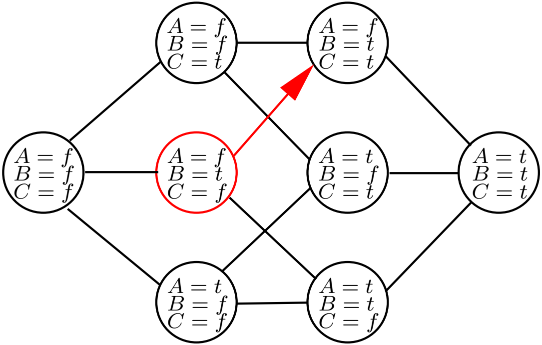

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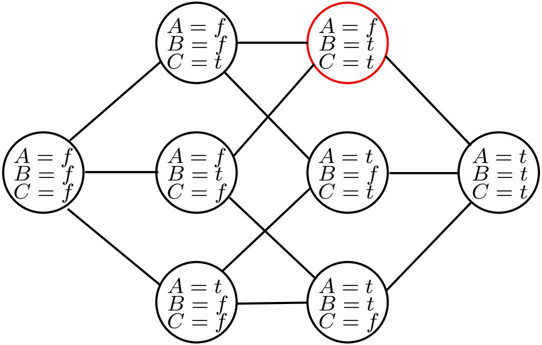

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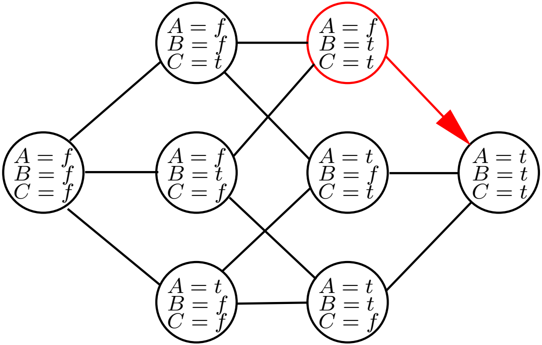

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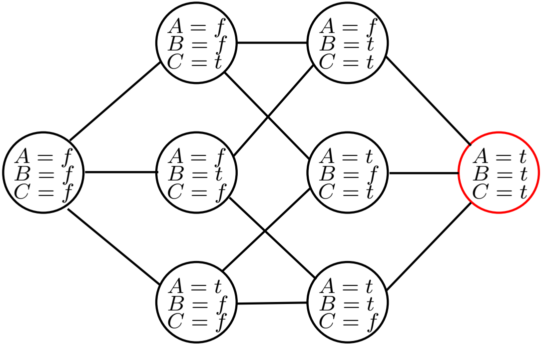

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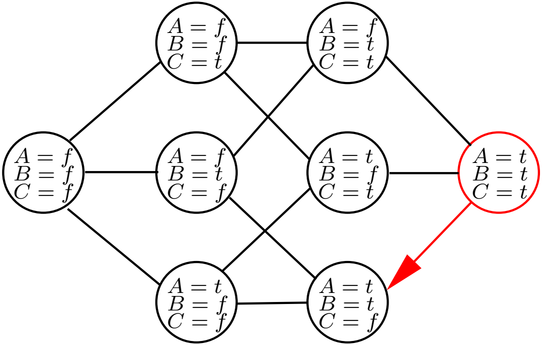

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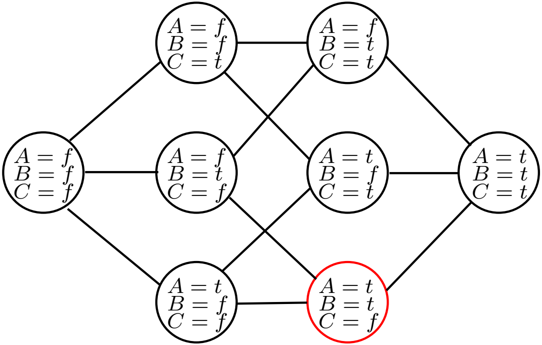

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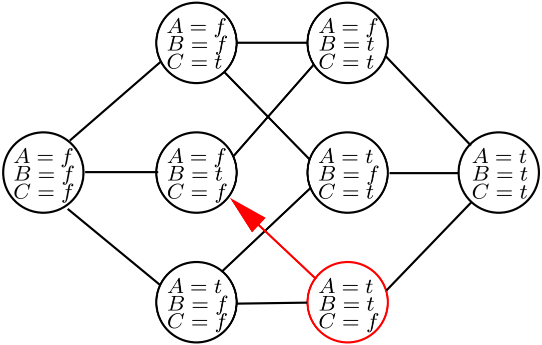

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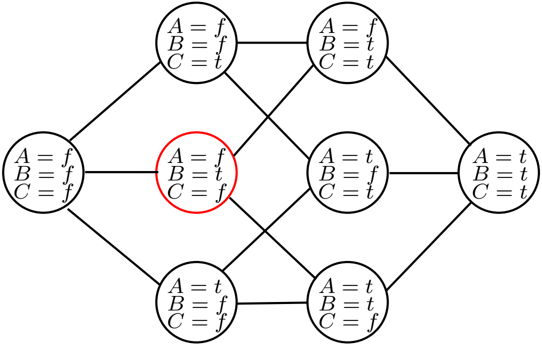

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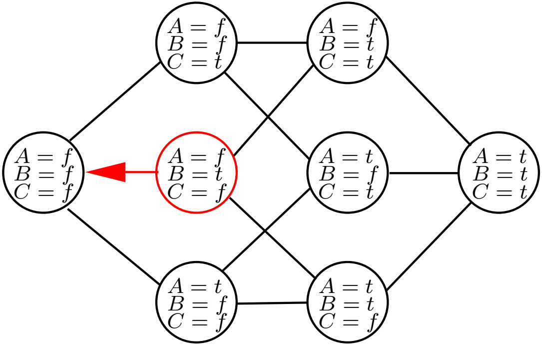


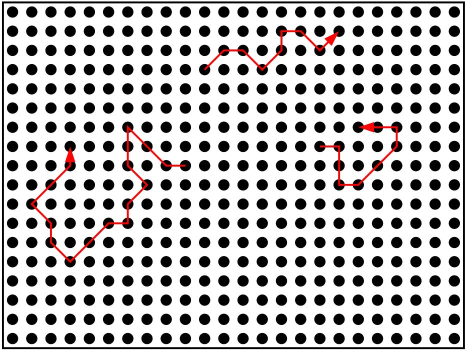
Random Search

- Make choices in line 1. and 3. completely random
- “Random walk”
- Unlikely to find a solution if state space large with only few solutions

Greedy Search or **Hill Climbing**:

- Use an *evaluation function* on states
- Example for evaluation function: number of constraints not satisfied by state
- Always choose neighbor with minimal evaluation function value
- Terminates when all neighbors have higher value than current state: current state is a **local minimum**.

Possible greedy search paths starting from different states:



Escaping Local Minima

Problem

- Search terminates with local minimum of evaluation function. This may not be a solution to the CSP.

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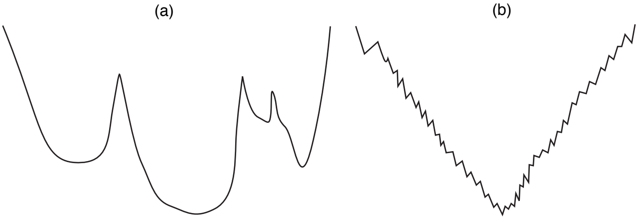
Solution Approaches

- Random restarts: repeat greedy search with several randomly chosen initial states
- Random moves: combine greedy moves with random steps



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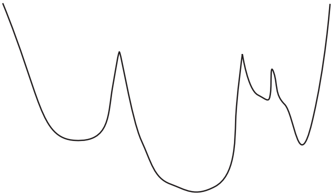
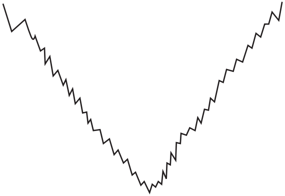
Solution Approaches

- Random restarts: repeat greedy search with several randomly chosen initial states
- Random moves: combine greedy moves with random steps
- **Example (a)**: Small number of random restarts will find global minimum
- **Example (b)**: Make random move when local minimum reached

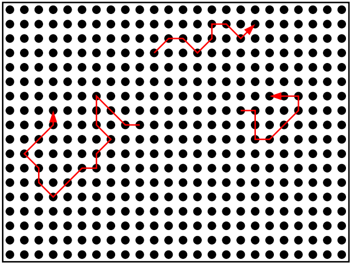
(a)



(b)



- Maintain an assignment of a value to each variable.
- At each step, select a “neighbor” of the current assignment (e.g., one that improves some heuristic value).
- Stop when a satisfying assignment is found, or return the best assignment found.



Requires:

- What is a neighbor?
- Which neighbor should be selected?

Most improving step

Principle

Select the variable-value pair that gives the highest improvement.

Naive approach

- Linearly scan all variables and for each value of each variable determine the improvement (how many fewer constrains are violated).

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Alternative

- Maintain a priority queue with variable-value pairs not part of the current assignment.
- Weight$\langle X, v\rangle =$ *eval(current assignment)* $-$ *eval(current assignment but with* $X = v$).
- If X is given a new value, update the weight of all pairs participating in a changed constraint.

Principle

- First: choose variable
- Second: choose state

Data structure

- Maintain priority queue of variables; weight is the number of participating conflicts.
- After selecting a variable, pick the value minimizes the number of conflicts.
- Update weights of variables that participate in a conflict that is changed.

Algorithm

- Pick a variable at random and a new value at random.
- If it is an improvement, adopt it.
- If it isn't an improvement, adopt it probabilistically depending on a temperature parameter, T.
  - With current assignment n and proposed assignment n' we move to n' with probability

$$e^{(h(n')-h(n))/T}$$

- Reduce the temperature.

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Probability of accepting a change

Temperature	1-worse	2-worse	3-worse
10	0.91	0.81	0.74
1	0.37	0.14	0.05
0.25	0.02	0.0003	0.000005
0.1	0.00005	0	0

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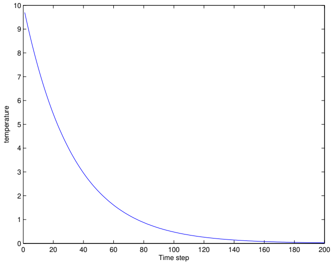

$$y = 10 \cdot 0.97^x$$

Propositional Logic Basics

Previously ...

Intensional representation of constraints:

$$A < B$$
$$\textit{Teacher\_AD} = \textit{Teacher\_MI} \;\rightarrow\; \textit{Time\_AD} \neq \textit{Time\_MI}$$
$$\ldots$$

CSP algorithms (arc-consistency algorithm) need to perform certain operations:

- *test* whether a certain value for one variable is *consistent* with a given constraint (and certain values for other variables)

To implement this:

- need a **formal language for representing constraints**

Propositional Logic

- provides a formal language for representing constraints on *binary variables*.

Atomic Propositions

Boolean variables are now seen as **atomic propositions**. Convention: start with lowercase letter.

Constraints	Logic
$A =$ *true*	a
$A =$ *false*	$\neg a$

Propositions

Using **logical connectives** more complex propositions are constructed:

$\neg p$	**not** p
$(p \wedge q)$	p **and** q
$(p \vee q)$	p **or** q
$(p \rightarrow q)$	p **implies** q

A *set* of propositions is also called a **Knowledge Base**

Example

"If **it rains** I'll **take my umbrella**, or I'll **stay home**"

Propositional Logic: Syntax

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Example

“If **it rains** I’ll **take my umbrella**, or I’ll **stay home**”

$$\textit{rains} \rightarrow (\textit{umbrella} \vee \textit{home})$$

An **interpretation** π for a set of atomic propositions $a_1, a_2, \ldots, a_n$ is an assignment of a truth value to each proposition (= possible world when atomic propositions seen as boolean variables):

$$\pi(a_i) \in \{\textit{true}, \textit{false}\}$$

An interpretation defines a truth value for all propositions:

$\pi(p)$	$\pi(\neg p)$
true	*false*
false	*true*

$\pi(p)$	$\pi(q)$	$\pi(p \wedge q)$
true	*true*	*true*
true	*false*	*false*
false	*true*	*false*
false	*false*	*false*

$\pi(p)$	$\pi(q)$	$\pi(p \vee q)$
true	*true*	*true*
true	*false*	*true*
false	*true*	*true*
false	*false*	*false*

$\pi(p)$	$\pi(q)$	$\pi(p \rightarrow q)$
true	*true*	*true*
true	*false*	*false*
false	*true*	*true*
false	*false*	*true*

Models

A **model** of a proposition (a knowledge base) is an interpretation in which the proposition (all the propositions in the knowledge base) is true.

Propositions as constraints: a model is a possible world that satisfies the constraint.

Logical consequence

A proposition g is a **logical consequence** of a knowledge base *KB*, if every model of *KB* is a model of g. Written:

$$\mathit{KB} \models g$$

(whenever *KB* is true, then g also is true).

Example

$\mathit{KB}=\{\mathit{man} \rightarrow \mathit{mortal}, \mathit{man}\}$. Then

$$\mathit{KB} \models \mathit{mortal}$$

Simple Example

$$KB = \begin{cases} p \leftarrow q. \\ q. \\ r \leftarrow s. \end{cases}$$

	p	q	r	s
I_1	*true*	*true*	*true*	*true*
I_2	*false*	*false*	*false*	*false*
I_3	*true*	*true*	*false*	*false*
I_4	*true*	*true*	*true*	*false*
I_5	*true*	*true*	*false*	*true*

Model?

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I_3	*true*	*true*	*false*	*false*	is a model of KB
I_4	*true*	*true*	*true*	*false*	is a model of KB
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Which of p, q, r, q logically follow from KB?

Simple Example

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I_4	*true*	*true*	*true*	*false*	is a model of KB
I_5	*true*	*true*	*false*	*true*	not a model of KB

Which of p, q, r, q logically follow from KB?

$$KB \models p,\ KB \models q,\ KB \not\models r,\ KB \not\models s$$