Linear Programming

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Contents

Lecture 7: Simplex Algorithm

27 Sep. 08:00

1 Worry-Free Simplex Algorithm

As previously seen. Simplex Algorithm:

1. Start with a basic feasible partition
   (a) i. Compute $\overline{x}_\beta := A_\beta^{-1} b \geq 0$
   ii. Compute $\overline{c}'_\eta := c'_\eta - c'_\beta A_\beta^{-1} A_\eta$
   (b) If $\overline{c}_\eta \geq 0$, then *STOP*. $\overline{x}$ is optimal.
   (c) Otherwise, choose η_j such that $\overline{c}_{\eta_j} < 0$.
   (d) Define $i^* := \underset{i:\overline{a}_{i,\eta_j}>0}{\arg\min}\{\frac{\overline{x}_{\rho_i}}{\overline{a}_{i,\eta_j}}\}$
   (e) If i^* is undefined, then *STOP*. (P) is unbounded.

2. Swap β_i^* out of β and η_j out of η. *GOTO 1.*

Problem. How do we start with a basic feasible partition?

Answer. We consider the so-called *Phase One Problem.*

1.1 Phase one problem

$$\begin{aligned} \min \ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned} \tag{P}$$

$$\begin{aligned} \min \ & x_{n+1} \\ & Ax + A_{n+1}x_{n+1} = b \\ & x \geq 0, x_{n+1} \geq 0 \end{aligned} \tag{Φ}$$

1. If min value of x_{n+1} in Φ is 0, then we get a feasible solution of (P).
2. If min value of x_{n+1} in Φ is > 0, then there is no feasible solution of (P).
   - How do we get an initial basic feasible solution for Φ
   - Need a basic feasible solution.

Solution:

1. Start with a basic solution of (P), $\tilde{\beta}, \tilde{\eta}$ is the basic partition.
2. If its feasible($\overline{x}_{\tilde{\beta}}$) then we just use $\tilde{\beta}$ and $\tilde{\eta}$ for β and η
3. Otherwise, set $A_{n+1} = -A_{\beta}^{-1}\vec{1}$. If $\eta_j = n+1$

$$\overline{z} : \overline{z}_{\tilde{\eta}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \qquad \overline{z}_{\beta} := -A_{\tilde{\beta}}^{-1}(A_{n+1}) = \vec{1}$$

and

$$\vec{x} \to \vec{x} + \lambda\vec{z} \geq \vec{0}.$$

Example.

$$\vec{x}_{\tilde{\beta}} + \lambda\vec{z}_{\tilde{\beta}} = \begin{pmatrix} 7 \\ 0 \\ 3 \\ -5 \\ 6 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

, then

$$i^* = \arg\min_{i:\vec{x}_{\tilde{\beta}} < 0} \{-\vec{x}_{\tilde{\beta}}\}.$$

Problem. What if $x_{n+1} = 0$?

Intuition. Just stop right before $x_{n+1} = 0$, let other variable do that.

1.2 Non degeneracy hypothesis

$$
\begin{aligned}
&\vec{x}_{\beta_i} > 0 \text{ for all } i \text{ at every iteration}\\
\implies &\overline{\lambda} \neq 0\\
\implies &\text{objective value decrease at each iteration.}\\
\implies &\text{algorithm must (because a finite \# of basis)}
\end{aligned}
$$

$$
\begin{aligned}
\min\ & c^T x\\
& Ax = b + B \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \epsilon^3 \\ \vdots \\ \epsilon^m \end{pmatrix}\\
& x \geq 0
\end{aligned}
$$

where ϵ is an arbitrarily small *indeterminate*.

Remark.

$$\epsilon \neq 0.$$

Observe. polynomial in ϵ:

$$
\begin{aligned}
p(\epsilon) &= p_0 + p_1\epsilon + p_2\epsilon^2 + \cdots + p_m\epsilon^m.\\
\vec{x}_\beta &= A_\beta^{-1}\left(b + B\begin{pmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^m \end{pmatrix}\right) = A_\beta^{-1}b + A_\beta^{-1}B\begin{pmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^m \end{pmatrix}
\end{aligned}
$$

Definition 1. Let K be the minimal index with $p_K \neq 0$.

- If $p_K < 0$, then $p(\epsilon) < 0$
- If $p_K > 0$, then $p(\epsilon) > 0$
- If $p_K = 0$, namely $p_0 = p_1 = \cdots = p_m = 0$, then $p(\epsilon) = 0$

Note.

$$
\begin{aligned}
p(\epsilon) &= p_0 + p_1\epsilon + p_2\epsilon^2 + \cdots + p_m\epsilon^m.\\
q(\epsilon) &= q_0 + q_1\epsilon + q_2\epsilon^2 + \cdots + q_m\epsilon^m.
\end{aligned}
$$

with K_p and K_q. Then K_{p+q} depends on K_p and K_q.

$$p(\epsilon) - q(\epsilon) \geq 0? \qquad \text{Then } p(\epsilon) \geq q(\epsilon).$$

Problem. Where does this ϵ thing links with the Simplex algorithm? (d)

1.3 Perturbed Problem

Suppose

$$\overset{\text{value of some basic variable}}{p(\epsilon)} = p_0 + p_1\epsilon + p_2\epsilon^2 + \cdots + p_m\epsilon^m,$$

Feasible for perturbed problem means $p(\epsilon) \geq \vec{0}$. $\implies p(0) = p_0 \geq 0$.

$$\begin{aligned} \min\ & c^T x \\ & Ax = b + \cancel{B}\vec{\epsilon} \\ & x \geq 0 \end{aligned}$$

Find an initial feasible basis β, η for unperturbed problem, $B := A_\beta$,

$$\vec{x}_\beta = A_\beta^{-1}(b + A_\beta\vec{\epsilon})$$

$$= \underbrace{A_\beta^{-1}b}_{\geq\vec{0}} + \vec{\epsilon}$$

$$= \vec{x}_\beta + \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \epsilon^3 \\ \vdots \\ \epsilon^4 \end{pmatrix} = \begin{pmatrix} \vec{x}_{\beta_1} + \epsilon \\ \vec{x}_{\beta_2} + \epsilon^2 \\ \vdots \vec{x}_{\beta_m} + \epsilon^m \end{pmatrix} \geq \vec{0}.$$

Claim: Perturbed problem is non-degenerate. $\implies$ some later basis $\tilde{\beta}$

$$\vec{x}_{\tilde{\beta}} := A_{\tilde{\beta}}^{-1}(b + A_\beta\vec{\epsilon}) = A_{\tilde{\beta}}^{-1}b + A_{\tilde{\beta}}^{-1}A_\beta\vec{\epsilon}$$

$(\vec{x}_{\tilde{\beta}_i} \overset{?}{=} 0)$

$$i^{\text{th}} \text{ element of } A_{\tilde{\beta}}^{-1}A_\beta \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \epsilon^3 \\ \vdots \\ \epsilon^m \end{pmatrix}$$

$$\underbrace{i^{\text{th}} \text{ row of } A_{\tilde{\beta}}^{-1}A_\beta}_{=\vec{0}} \text{ dot } \vec{\epsilon}\ \lightning$$

because $A_{\tilde{\beta}}^{-1}A_\beta$ is invertible($A_\beta^{-1}A_{\tilde{\beta}}$ is the inverse)

Lecture 8: Practical Simplex Algorithm

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1.4 A_β^{-1} in Reality

Note. In reality, we don't really calculate A_β^{-1}, since in order to calculate

$$A_\beta x_\beta = b,$$

we do not use

$$\overline{x}_\beta = A_\beta^{-1} b.$$

Instead, we use *LU-Factorization.* And since after applying pivot change, there is only a column change in A_β^{-1}, we can use the previous result to calculate the new $\overline{x}_\beta$ much faster.

1.5 Why *Simplex*?

For a standard form problem

$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

But instead, we consider

$$\begin{aligned} \min\ & z \\ & z - c'x = 0 \iff (c'x = z) \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

As previously seen. Our picture is in $\mathbb{R}^{n-m}$, but we consider *Dantzig picture*, which is in $\mathbb{R}^{m+1}$

1.5.1 Column geometry

Plot columns:

$$\underbrace{\begin{pmatrix} c_1 \\ A_1 \end{pmatrix} \begin{pmatrix} c_2 \\ A_2 \end{pmatrix} \cdots \begin{pmatrix} c_n \\ A_n \end{pmatrix}}_{n \text{ points in } \mathbb{R}^{m+1}}$$

The requirement line is

$$\begin{pmatrix} z \\ b \end{pmatrix}.$$

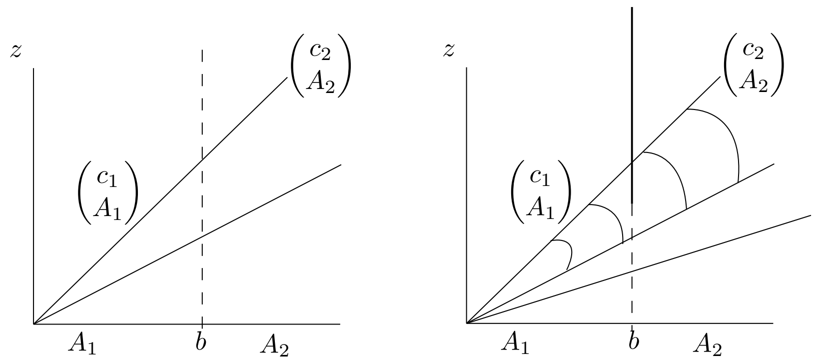

Figure 1: column-geometry

1.6 Simplices(plural of simplex)

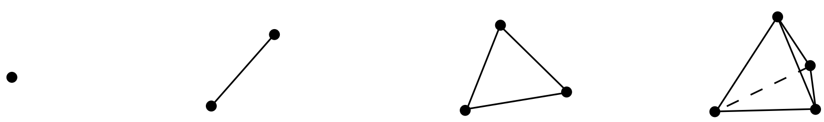

Figure 2: Simplex shapes

Example. Example of a simplex:

$$\{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0\}.$$

Which is $n-1$ dimensional simplex in $\mathbb{R}^n$ with n standard unit vectors are the corners.

Note. $m+1$ points of a simplex of dimension.

1.6.1 Simplicial cones

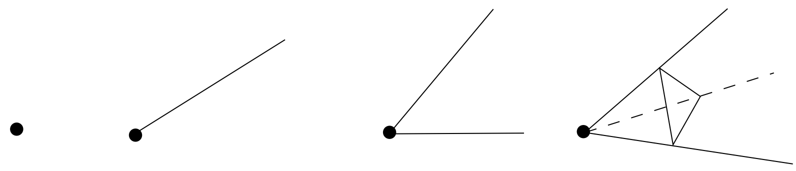

Figure 3: Simplicial Cones

2 Combined: Complete Simplex Algorithm

Now, we have the standard form problem (P):

$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned} \qquad (P)$$

and the *Phase one problem*(Φ)(Getting started): $A_{n+1} = -A_{\tilde{\beta}}\vec{1}$:

- First pivot is special
- Last pivot is special

:

$$\begin{aligned} \min\ & x_{n+1} \\ & Ax + A_{n+1}x_{n+1} = b \\ & x \geq 0, x_{n+1} \geq 0 \end{aligned} \qquad (\Phi)$$

with the perturbed problem (P_ϵ)(making sure we stop):

$$\begin{aligned} \min\ & c^T x \\ & Ax = b + B\vec{\epsilon} \\ & x \geq 0 \end{aligned} \qquad (P_\epsilon)$$

3 Duality

Consider the standard problem and its duality:

$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \max\ & y^T b \\ & y^T A \leq c^T \end{aligned}$$

As previously seen. Weak duality theorem: If $\hat{x}$ is feasible for P, and $\hat{y}$ is feasible for D, then

$$c^T \hat{x} \geq \hat{y}' b.$$

Moreover, the equality holds if and only if $\hat{x}$ and $\hat{y}$ are optimal.

Theorem 1. Weak optimal Basic Theorem: If we have a basic partition β, η, and we also have $\overline{x}_\beta \geq \vec{0}$($\overline{x}$ is feasible for P) and $\overline{c}_\eta \geq \vec{0}$($\overline{y}$ is feasible for D)

$$\implies \overline{x}\ \&\ \overline{y} \text{ are optimal.}$$

Now, we have so-called *strong optimal basis theorem.*

Theorem 2. Strong Optimal Basis Theorem: If (P) has a feasible solution, and if (P) is not unbounded, then there exist a basic partition β, η such that $\overline{x}$ and $\overline{y}$ are optimal, and

$$c'\overline{x} = \overline{y}' b.$$

Note. The proof is based on the *mathematical complete* version of Simplex Algorithm.

Theorem 3. Strong Duality Theorem: If P has a feasible solution and P is not unbounded, then there exist optimal solutions $\hat{x}$ and $\hat{y}$ with

$$x'\hat{x} = \hat{y}' b.$$

Simplex Algorithm	P\ D	optimal solution	infeasible	unbounded
$\overline{c}_{\eta \geq \vec{0}}$ Stop	optimal solution	$\surd$	$\times$	$\times$
optimal $x_{n+1} \in \Phi$ is positive	infeasible	$\times$	$\surd$	$\surd$
$\overline{A}_{\eta_j} \leq \vec{0}$ Stop	unbounded	$\times$	$\surd$	$\times$

Table 1: Comparison between P and D

$\widetilde{b}$

Lecture 9: Duality

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3.1 Complementary

Solutions $\hat{x}$ to (P) and $\hat{y}$ to (D) are *complementary* if

$$m+n \text{ equations} \begin{cases} \underbrace{(c_j - \hat{y}' A_{\cdot j})}_{=0 \text{ for } j\in\beta} \underbrace{\hat{x}_j}_{\substack{=0 \\ \text{for} \\ j\in\eta}} = 0, & j = 1 \cdots n \\ \\ \hat{y}_i \underbrace{(A_{i\cdot}\hat{x} - b_i)}_{=0 \text{ for } \overline{x}} = 0, & i = 1 \cdots m \end{cases}.$$

Now, suppose we have a basic partition β, η such that

$$\overline{x} : \overline{x}_\beta = A_\beta^{-1} b, \ \overline{x}_\eta = \vec{0}$$
$$\overline{y} : \overline{y} = c_\beta^T A_\beta^{-1}$$

Theorem 4. If $\overline{x}$ and $\overline{y}$ are basic solutions for β, η, then $\overline{x}$ and $\overline{y}$ are complementary.

Theorem 5. If $\hat{x}$ and $\hat{y}$ are complementary, then

$$c^T \hat{x} = \hat{y}^T b.$$

Note.

$$c^T a_\beta^{-1} b = \overline{y}^T b, \qquad c^T (A_\beta^{-1} b) = c_\beta^T \overline{x}_\beta = c^T \overline{x}$$

Proof. We show that

$$c^T \hat{x} = \hat{y}^T b = 0.$$

We have

$$
\begin{aligned}
c^T\hat{x} = \hat{y}^T b &= (c^T - \underbrace{\hat{y}^T A)\hat{x} + \hat{y}^T(A\hat{x}}_{\text{added terms}} - b) \\
&= \sum_{j=1}^{n} \underbrace{(c_j - \hat{y}^T A_{\cdot j})x_j}_{=0 \text{ for } i=1\ldots n} + \sum_{i=1}^{m} \underbrace{\hat{y}_i(A_{i\cdot}\hat{x} - b_i)}_{=0 \text{ for } i=1\ldots m} \\
&= 0.
\end{aligned}
$$

■

> **Theorem 6.** Weak Complementary Slackness Theorem: If $\hat{x}$ and $\hat{y}$ are feasible and complementary, then they are optimal.

Proof. Follows from weak duality and complementary solutions having equal objective value. ■

> **Theorem 7.** Strong Complementary Slackness Theorem: If $\hat{x}$ and $\hat{y}$ are optimal, then $\hat{x}$ and $\hat{y}$ are complementary.

Proof. Recall that

$$
\overbrace{\sum_{j=1}^{n} \underbrace{(c_j - \hat{y}^T A_{\cdot j})}_{\geq 0 \text{ for each } j} \quad \underbrace{x_j}_{\geq 0 \text{ for each } j}}^{\cancel{\geq 0} \implies =0 \text{ for each } j} + \overbrace{\sum_{i=1}^{m} \underbrace{\hat{y}_i(A_{i\cdot}\hat{x} - b_i)}_{=0 \text{ for each } i}}^{=0} \underbrace{= 0 = c^T\hat{x} - \hat{y}^T b}_{\substack{\text{if } \hat{x} \text{ and } \hat{y} \text{ are optimal} \\ \text{since same object value}}}
$$

■

Now consider a general linear programming problem

$$
\begin{aligned}
\min\ & c_P^T x_P + c_N^T x_N + c_u^T x_u \\
& A_{Gp}x_P + A_{GN}x_N + A_{Gu}x_u \geq b_G \\
& A_{Lp}x_P + A_{LN}x_N + A_{Lu}x_u \leq b_L \\
& A_{Ep}x_P + A_{EN}x_N + A_{Eu}x_u = b_E \\
& x_P \geq 0, x_N \leq 0, x_u \text{ unrestricted.}
\end{aligned}
$$

We first turn this into a standard form problem:

1. $\widetilde{x}_N := -x_N$:

$$
\begin{aligned}
\min\ & c_P^T x_P + c_N^T x_N + c_u^T x_u \\
& A_{Gp}x_P - A_{GN}x_N + A_{Gu}x_u \geq b_G \\
& A_{Lp}x_P - A_{LN}x_N + A_{Lu}x_u \leq b_L \\
& A_{Ep}x_P - A_{EN}x_N + A_{Eu}x_u = b_E \\
& x_P \geq 0, x_N \leq 0, x_u \text{ unrestricted}
\end{aligned}
$$

2. $x_u = \widetilde{x}_u - \widetilde{\widetilde{x}}_u$, where $\widetilde{x}_u, \widetilde{\widetilde{x}}_u \geq 0$:

$$\begin{aligned} \min\ & c_P^T x_P + c_N^T x_N + c_u^T \widetilde{x}_u - c_u \widetilde{\widetilde{x}}_u \\ & A_{Gp} x_P - A_{GN} x_N + A_{Gu} \widetilde{x}_u - A_{Gu} \widetilde{\widetilde{x}}_u \geq b_G \\ & A_{Lp} x_P - A_{LN} x_N + A_{Lu} \widetilde{x}_u - A_{Lu} \widetilde{\widetilde{x}}_u \leq b_L \\ & A_{Ep} x_P - A_{EN} x_N + A_{Eu} \widetilde{x}_u - A_{Eu} \widetilde{\widetilde{x}}_u = b_E \\ & x_P \geq 0, x_N \leq 0, \widetilde{x}_u \geq 0, \widetilde{\widetilde{x}}_u \geq 0 \end{aligned}$$

3. Adding slack variables:

$$\begin{aligned} \min\ & c_P^T x_P + c_N^T x_N + c_u^T \widetilde{x}_u - c_u \widetilde{\widetilde{x}}_u \\ & A_{Gp} x_P - A_{GN} x_N + A_{Gu} \widetilde{x}_u - A_{Gu} \widetilde{\widetilde{x}}_u - A_G = b_G \\ & A_{Lp} x_P - A_{LN} x_N + A_{Lu} \widetilde{x}_u - A_{Lu} \widetilde{\widetilde{x}}_u + t_L = b_L \\ & A_{Ep} x_P - A_{EN} x_N + A_{Eu} \widetilde{x}_u - A_{Eu} \widetilde{\widetilde{x}}_u = b_E \\ & x_P \geq 0, x_N \leq 0, \widetilde{x}_u \geq 0, \widetilde{\widetilde{x}}_u \geq 0, A_G \geq 0, t_L \geq 0 \end{aligned}$$

With *Dual variables* y_G, y_L, y_E, we have

$$\begin{aligned} \max\ & y_E^T b_G \quad + y_L^T b_L \quad + y_E^T b_E \\ & y_G^T A_{GP} + y_L^T A_{LP} + y_E^T A_{EP} \leq c_P^T \\ & -y_G^T A_{GN} - y_L^T A_{LN} - y_E^T A_{EN} \leq -c_N^T. \\ & y_G^T A_{Gu} \ + y_L^T A_{Lu} \ + y_E^T A_{Eu} \leq c_u^T \\ & -y_G^T A_{Gu} \ - y_L^T A_{Lu} \ - y_E^T A_{Eu} \leq -c_u^T \end{aligned}$$

y inequality

Lecture 10: Duality

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As previously seen. Complementary: we have

$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

with the dual

$$\begin{aligned} \max\ & y^T b \\ & y^T A \leq c^T \end{aligned}$$

Then the complementary means that

$$\begin{aligned} & \underbrace{(c_j - \hat{y}' A_{\cdot j})}_{\geq 0} \underbrace{\hat{x}_j}_{\geq 0} = 0 \text{ for } j = 1 \dots n \\ & \hat{y}_i \underbrace{(A_{i \cdot} \hat{x} - b_i)}_{=0} = 0 \text{ for } i = 1 \dots m. \end{aligned}$$

As previously seen. The production problem: The primal:

$$\begin{aligned} \max\ & c^T x \\ & Ax \leq b \\ & x \geq \vec{0} \end{aligned}$$

- n products activities
- c_j = per-unit revenue for activity $j = 1 \dots n$
- b_i = resource endowment for resource $i (i = 1 \dots m)$
- a_{ij} = amount of resource i consumed by activity j

$$\begin{aligned} \min\ & y^T b \\ & y^T A \geq \vec{c} \\ & y \geq \vec{0} \end{aligned}$$

where

$$y^T A_{\cdot j} \geq c_j (\sum_{i=1}^{m} y_i a_{ij} \geq c_j).$$

Note. We have

min	max
$\geq$	≥ 0
$\leq$	≤ 0
$=$	unrestricted
≥ 0	$\leq$
≤ 0	$\geq$
unrestricted	$=$

Table 2: situations

$$\begin{aligned} &\hat{y}' A_{\cdot j} - c_j \hat{x}_j = 0 \text{ for } j = 1 \dots n \\ &\hat{y}_i (b_i - A_{i \cdot} x = 0) \text{ for } i = 1 \dots m \end{aligned}$$

Note. For feasible solutions of P and D, at most one of $\hat{y} A_{\cdot j} - c_j$ and $\hat{x}_j$ is positive for $j = 1 \dots n$; while at most one of $b_i - A_{\cdot j} \hat{x}$ and $\hat{y}_j$ is positive for $i = 1 \dots m$;

Problem. We are looking for a way to find out the upper bound of $c^T x$ from the dual.

Since

$$c^T x \underset{?}{\leq} \underbrace{y^T A}_{\geq c^T} \underbrace{x}_{\geq \vec{0}} \leq \underbrace{y^T}_{\geq \vec{0}} b \iff \sum_{i=1}^{m} y_i (\sum_{j=1}^{n} a_{ij} x_j) \leq \sum_{i=1}^{m} y_i b_i.$$

Observe. We want
$$c^T \le y^T A \implies c^T x \le y^T A x$$

Now, return to the standard form problem, we have
$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \ge 0 \end{aligned}$$

with the dual
$$\begin{aligned} \max\ & y^T b \\ & y^T A \le c^T \end{aligned}$$

with y unrestricted.

Then we have
$$c^T x \underset{?}{\ge} \underbrace{y^T A}_{\le c^T} \underbrace{x}_{\ge 0} = y^T b$$

since
$$y^T A x \ge c^T x.$$

Example. Consider the following linear programming problem:
$$\begin{aligned} \max\ & c^T x + d^T z \\ & Ax \ge b \\ & Bx - Fz = g \\ & x \le 0, z \text{ unrestricted} \end{aligned}$$

Then the dual is(with variable y, w)
$$\begin{aligned} \min\ & y^T b\ + w^T g \\ & y^T A + w^T B \le c^T \\ & \quad - w^T F = d^T \\ & y \le 0, w \text{ unrestricted.} \end{aligned}$$

Or, from
$$\begin{aligned} & y^T A + w^T B \le c^T \\ & (y^T A + w^T B) x \ge c^T x\text{'} \end{aligned}$$

hence
$$\begin{aligned} & \overbrace{y^T}^{\le 0} (Ax \ge b) \\ + \ & w^T (Bx - Fz = g) \end{aligned}$$

$$c^T x + d^T z \overset{\text{want}}{\le} \underbrace{y^T A x + w^T B x - w^T F z}_{\underbrace{(y^T A + w^T B)}_{\le c^T} x \underbrace{-(w^T F) z}_{= d^T}} \overset{\text{want}}{\le} y^T b + w^T g$$

Remark. Think about what if all are equal sign?(both in constraints and variables, namely unrestricted) Rethink about it

3.2 Geometrically Understanding of Duality

Lemma 1. Farkas' Lemma: Let (I) and (II) being

$$\begin{aligned} (I) \qquad & Ax = b \\ & x \geq 0 \\ (II) \qquad & y^T b \geq 0 \\ & y^T A \leq 0 \end{aligned}$$

for any data A and b, exactly one of (I) or (II) has a solution.

Note. Recall that the *LP Duality*

$$\begin{aligned} (P) \qquad & \min\ c^T x \\ & \quad\ \ Ax = b \\ & \quad\ \ x \geq 0 \\ (D) \qquad & \max\ y^T b \geq 0 \\ & \quad\ \ y^T A \leq c^T. \end{aligned}$$

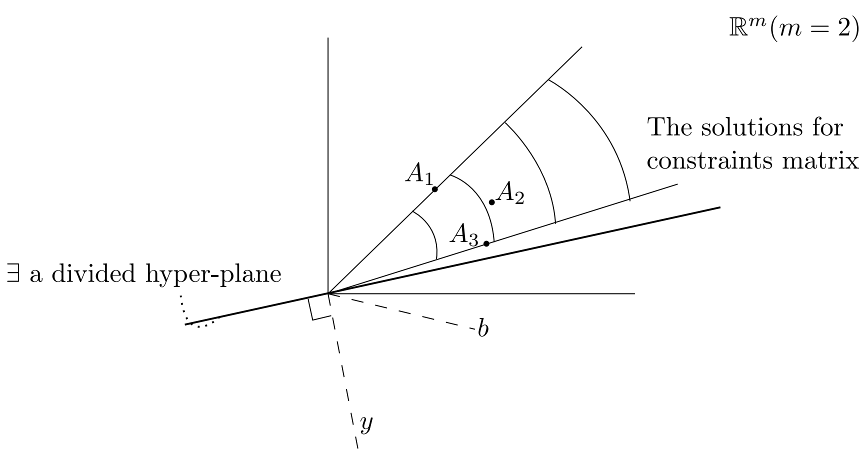

Figure 4: Farkas Lemma - Geometrically point of view with $\mathbb{R}^m, m = 2$

Proof. 1. Step 1: (I) and (II) can't both have solutions for the same A, b. Suppose $\hat{x}$ solves (I) and $\hat{y}$ solves (II). Then we have

$$0 \geq \underbrace{\hat{y}' A}_{\geq \vec{0}} \underbrace{\hat{x}}_{\geq \vec{0}} = \hat{y}' b \ ↯$$

2. Step 2: Show that if (I) has no solution, then (II) has a solution.

■

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4 Farkas Lemma

Besides this, we also have so-called *Gauss' Lemma*.

Lemma 2. Gauss' Lemma: Exactly one of the following has a solution:

$$\begin{aligned} (I) \qquad & Ax = b \\ (II) \qquad & y^T A \geq 0 \\ & y^T b \neq 0 \end{aligned}$$

This just follows from the Gauss elimination. By doing the elimination, there are two cases:

1. The system has no solution.
2. There is a(some) solution(s).

For second case, it's just $Ax = b$ is solvable. For the fist case, we see that

$$\begin{pmatrix} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \\ \end{pmatrix} \begin{pmatrix} \\ a \\ \\ \end{pmatrix}$$

where $a \neq 0$.

Proof of Farkas Lemma.

Proof. 1. I & II can't both have solutions. Suppose $\hat{x}$ solves I and $\hat{y}$ solves II. Then we hve

$$\hat{y}'(\hat{Ax} = b) \implies \underbrace{(\hat{y}' A)}_{\geq \vec{0}} \underbrace{\hat{x}}_{\geq \vec{0}} = \hat{y} b > 0 \lightning$$

2. At least one of I or II has a solution $\cong$ If I has no solution, then II has a solution. Assume that I has no solution, which means that P is infeasible with P being

$$\begin{aligned} \max \ & \vec{0}^T x \\ & Ax = b \\ (P) \quad & x \geq 0 \end{aligned}$$

with

$$\begin{aligned} &\max\ y^T b \\ (D)\quad & y^T A \leq \vec{0} \end{aligned}$$

But this means that D is infeasible or unbounded. Which means that D can't be infeasible, because $y = \vec{0}$ is a feasible solution $\implies$ D is unbounded $\implies$ there exist a feasible solution $\widetilde{y}$ to D with positive objective.

■

Remark. Now, consider $\lambda\widetilde{y}$(feasible for D). Drive to $+\infty$ by increasing λ. We now see what Farkas Lemma really tells us.

$$\begin{aligned} &\max\ c^T x \\ & Ax = b && \text{feasibility} \\ (P)\quad & x \geq 0 && \Updownarrow \\ &\max\ y^T b && \text{unbounded direction} \\ (D)\quad & y^T A \leq c^T \end{aligned}$$

Suppose $\widetilde{y}$ is feasible to D and suppose $\hat{y}$ satisfies II, then

$$(\widetilde{y} + \lambda\hat{y})'A = \underbrace{\widetilde{y}^T A}_{\leq c^T} + \underbrace{\lambda}_{>0} \underbrace{\hat{y}'A}_{\leq \vec{0}} \leq c^T.$$

Furthermore, we have

$$(\widetilde{y} + \lambda\hat{y})'b = \widetilde{y}'b + \lambda\hat{y}'b \implies \infty \text{ as } \lambda \uparrow.$$

Example.

$$\begin{aligned} (I)\quad & Ax \leq b \\ (II)\quad & ? \end{aligned}$$

$$\begin{aligned} &\max\ \vec{0}^T x \\ & Ax \leq b \\ (P)\quad & x \geq 0 \\ &\max\ y^T b \\ & y^T A = \vec{0} \\ (D)\quad & y \leq \vec{0} \end{aligned}$$

$$\begin{aligned} (I)\quad & Ax \leq b \\ (II)\quad & y^T A = \vec{0} \\ & y \leq \vec{0} \\ & y^T b > 0 \end{aligned}$$

Check:

$$0 = \underbrace{\hat{y}' A}_{=\vec{0}} \hat{x} \underset{\hat{y} \leq \vec{0}}{\geq} \widetilde{y}' b > 0 \lightning$$

or,

$$\begin{aligned} Ax &\overset{y \leq \vec{0}}{\leq} b \qquad (y^T b > 0) \\ 0 &\overset{?}{\geq} \underbrace{y^T A}_{=\vec{0}} x \geq y^T b > 0 \end{aligned}$$

Example.

$$\begin{aligned} & (\min\ \vec{0}^T x + \vec{0}^T w) \\ (I) \quad & Ax + Bw = b \\ & -Fw \geq f \\ & x \geq 0,\ w \text{ unrestricted} \end{aligned}$$

with the dual variables y, w, we have

$$\begin{aligned} & \text{(Suppose I has nno solution.)} \\ (II) \quad & \cancel{\max}\ y^T b + v^T b (> 0) \\ & y^T A \leq \vec{0} \\ & y^T B - v^T F = \vec{0} \end{aligned}$$

with y unrestricted, $v \geq \vec{0}$.

$$\begin{aligned} (I) \qquad & Ax = b \\ & x \geq 0 \quad \Longleftrightarrow b \text{ is in the cone } K \\ (II) \qquad & y^T b \geq 0 \quad \Longleftrightarrow y \text{ makes an acute angle with } b. \\ & y^T A \leq 0 \quad y \text{ makes a non-acute angle with all columns of } A \end{aligned}$$

Suppose $\hat{z}$ in K, then

$$\hat{z} = A\hat{x} \text{ for some } \hat{x} \geq \vec{0}.$$

Then we have

$$y^T \hat{z} = \underbrace{y^T A}_{\leq \vec{0}^T} \underbrace{\vec{x}}_{\geq \vec{0}} \leq 0.$$

y makes a non-acute angle with everything in K. Now, suppose $\hat{y}$ solves II. Consider

$$\underbrace{\hat{y}^T}_{\text{numberes}} \underbrace{z}_{\text{variables}} = 0.$$

Now, we have the hyperplane:$\{z : \hat{y}^T z = 0\}$ separates b and K.

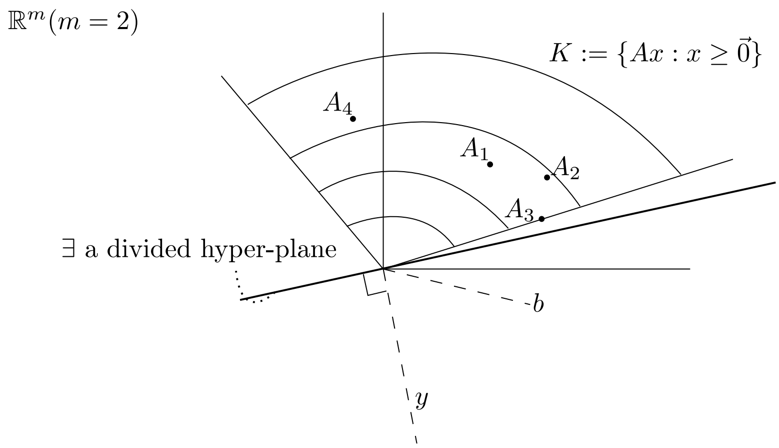

Figure 5: Farkas Lemma, $m = 2$

4.1 The big picture of Cones

$$\begin{aligned} \max \ & y^T b \\ & y^T A \leq c^T \end{aligned}$$

with the partition β, η, we see that

$$y^T A \leq c^T \implies \begin{cases} y^T A_\beta & \leq c_\beta^T \\ y^T A_\eta & \leq c_\eta^T \end{cases}.$$

By solving only for β, then we have $\overline{y}^T = c_\beta^T A_\beta^{-1}$. And then, by considering the cones, we have

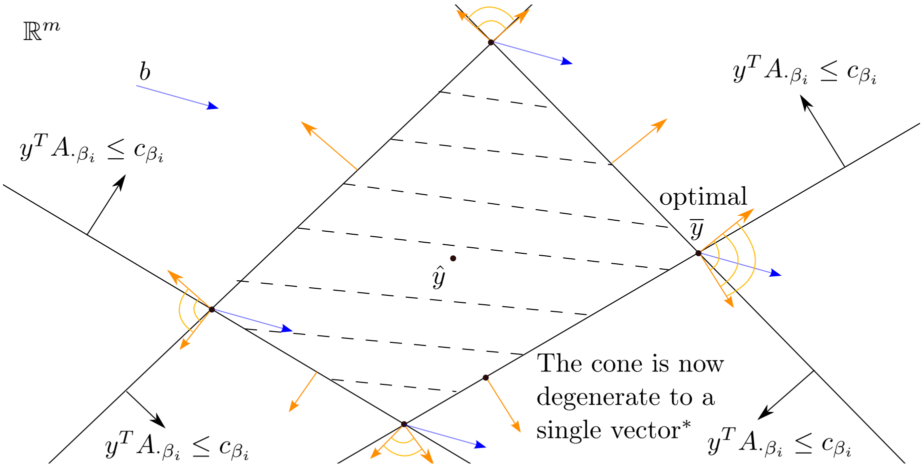

Figure 6: Optimality of Cones. (* This corresponds to the case that we run into the overlapping issue in 7)

with

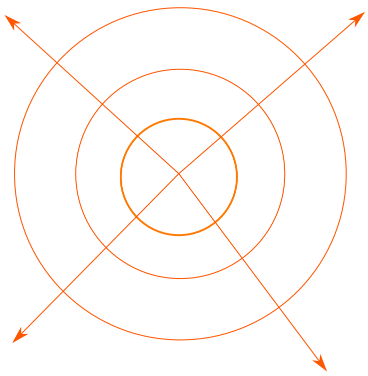

Figure 7: Cones join together

Note. Consider $b = \vec{0}(\hat{y})$. It's in every cone $\implies$ every point is optimal.

Remark. We see that each corner(extreme point) corresponds to a solution for β, while the blue vector $\vec{b}$ corresponds to the dual constraints $y^T A_\eta < c_\eta^T$. Only when the blue vector are in the region of orange sectors span by two *normal vectors* of $y^T A_{\cdot\beta_i} \le c_{\beta_i}$, the constraints are satisfied.

Example. Exercise 5.5. ~~Over~~ Strictly Complementarity.

$$
\begin{aligned}
\min\ & c^T x \\
& Ax = b \\
(P)\quad & x \geq 0 \\
\max\ & y^T b \\
(D)\quad & y^T A \leq \vec{0}
\end{aligned}
$$

As previously seen. Complementarity of $\hat{x}$ and $\hat{y}$:

$$
\begin{aligned}
(c_j - \hat{y}^T A_{\cdot j})\hat{x}_j = 0,\ \text{for } j = 1 \ldots n \\
y_i^T (A_{i]cdot}\hat{x} - b_i) = 0,\ \text{for } i = 1 \ldots m
\end{aligned}
$$

Feasible: $\hat{x}$ and $\hat{y}$ are strictly complementary if they are complementary and exactly one of

$$c_j - \hat{y}^T A_{\cdot j} \text{ and } \hat{x}_j = 0.$$

> **Theorem 8.** Strictly Complementarity. If P and D are both feasible, then for P and D there exist strictly complementary(feasible) optimal solutions.

$$
\begin{aligned}
v = \min\ & c^T x \\
& Ax = b \\
(P)\quad & x \geq 0
\end{aligned}
$$

Now, we try to find an optimal solution with

$$x_j > 0, \quad \text{fix } j.$$

If failed, then construct an optimal solution to D with

$$c_j - y^T A_{\cdot j} > 0.$$

For the first try, we formulate the following linear programming:

$$
\begin{aligned}
\max\ & x_j \\
& c^T x \leq v \\
& Ax = b \\
& x \geq 0
\end{aligned}
$$