COMP3251
Lecture 10: Bellman-Ford Algorithm
(Chapter 4.6)

Recap: Single-Source Shortest Path

Given a directed graph with edge lengths and a source vertex *s*, find the length of the shortest path from *s* to *v* for every vertex *v*.

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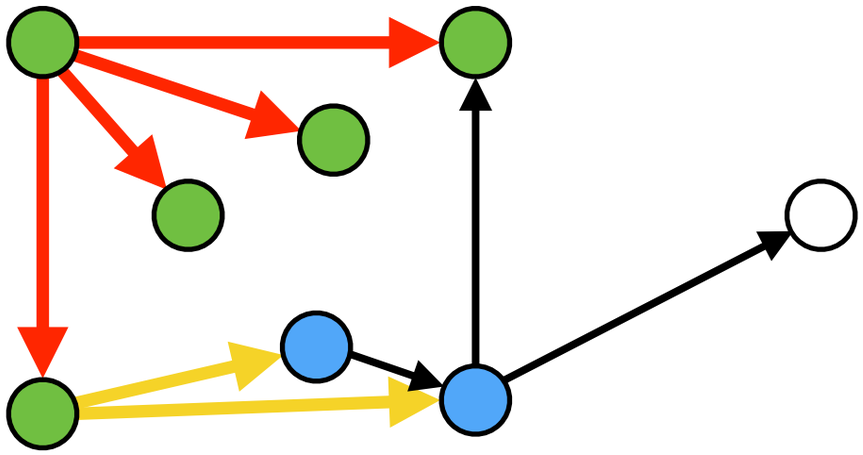

Breadth First Search

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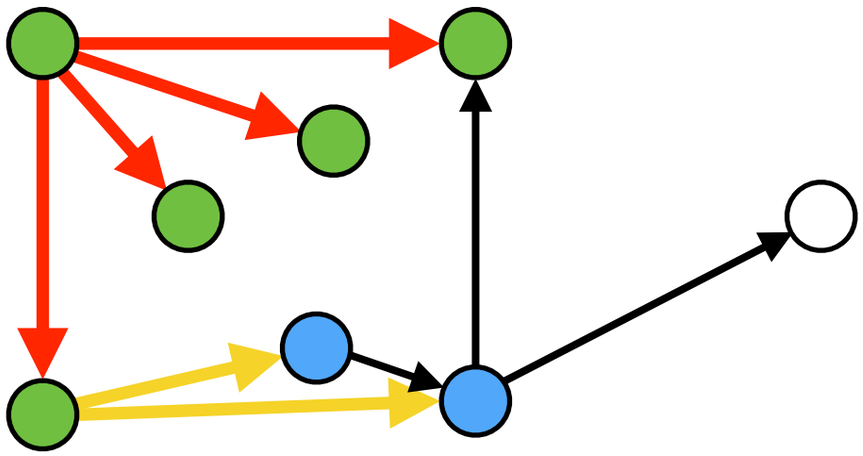

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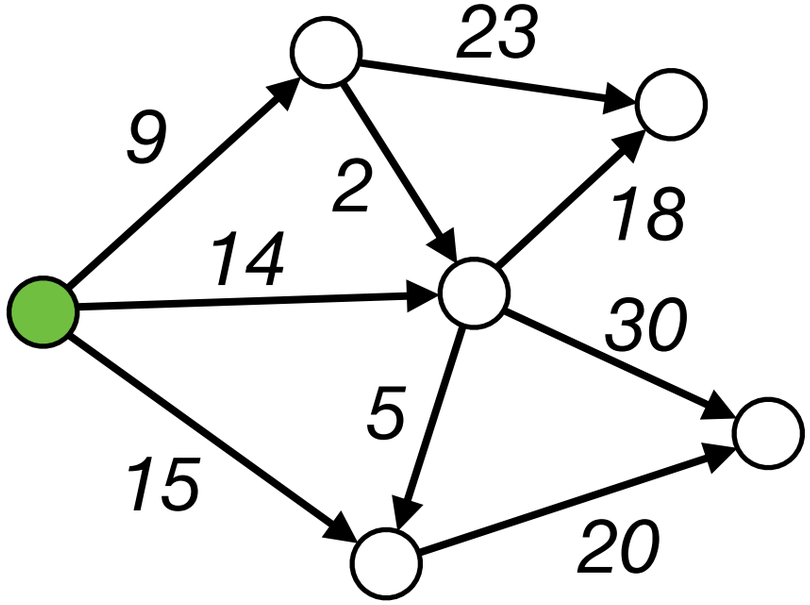
Positive Lengths



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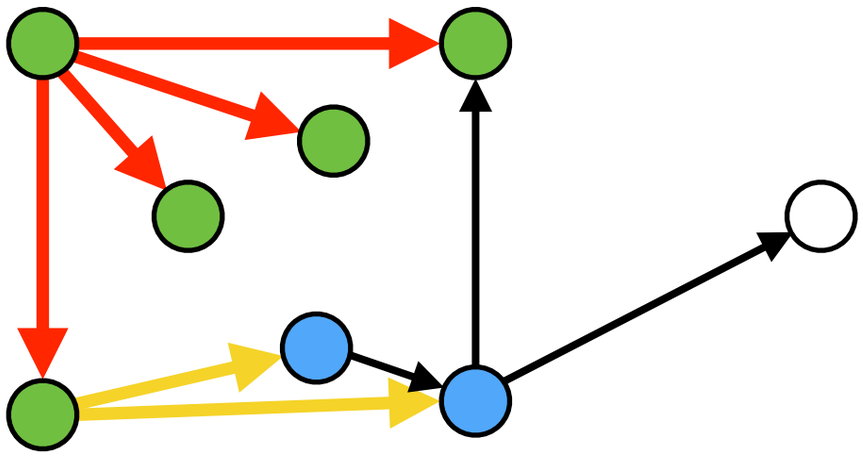

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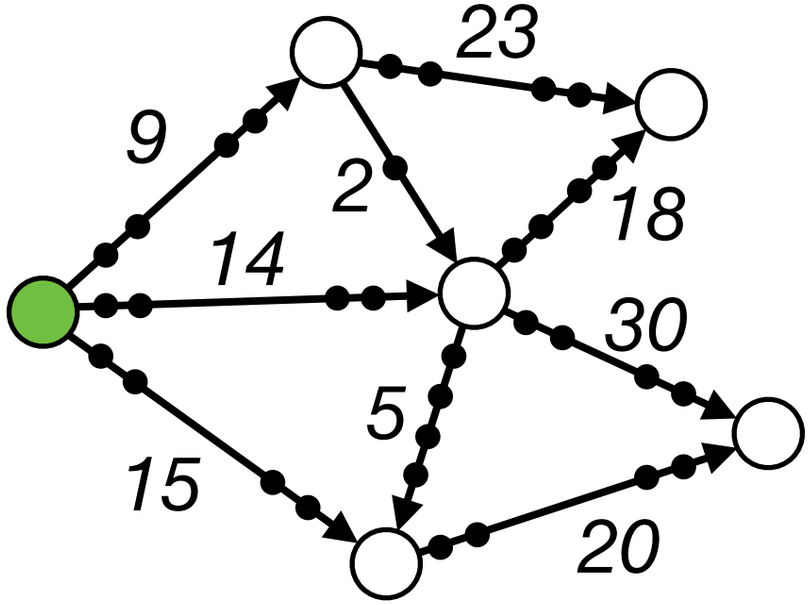

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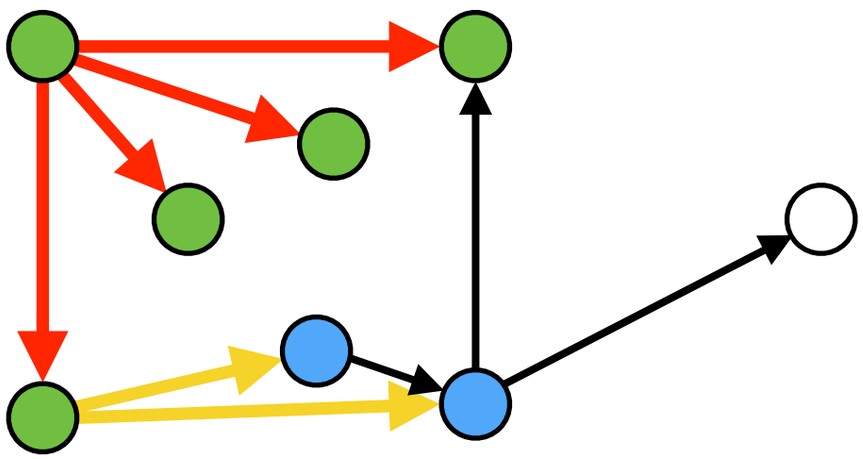

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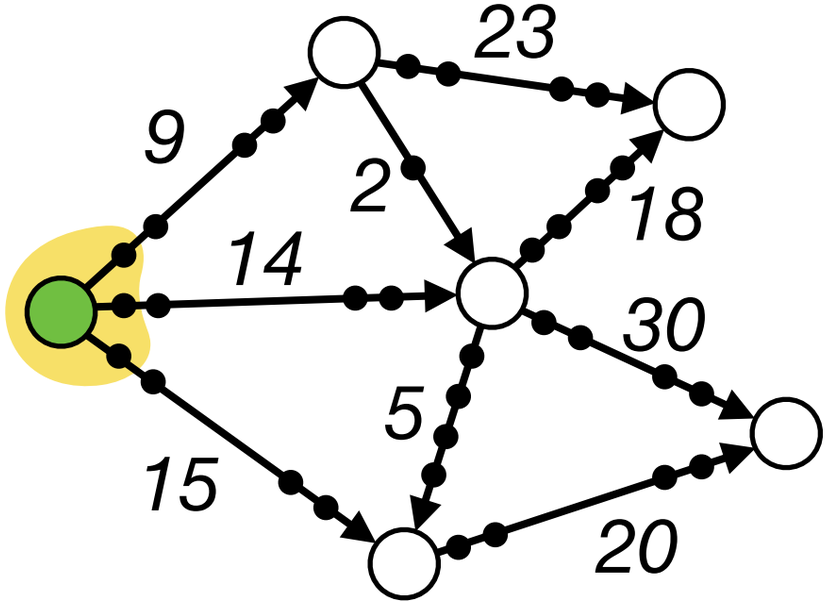

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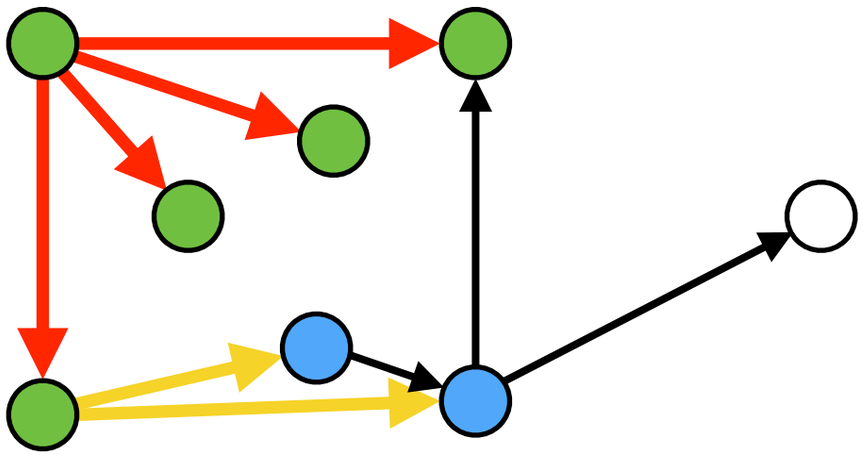

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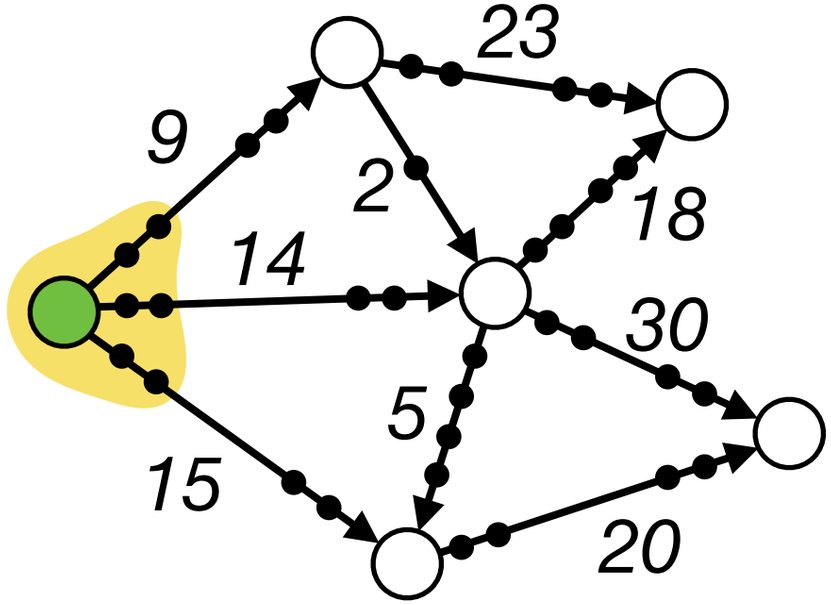

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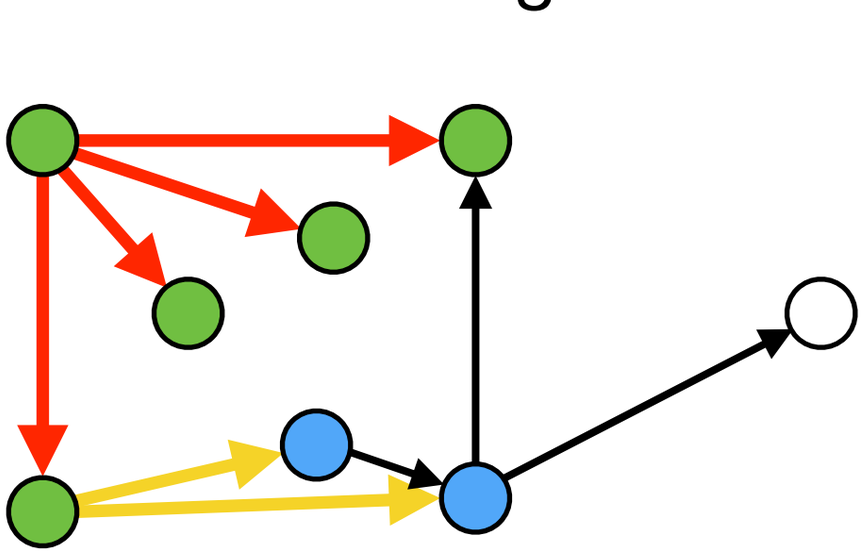

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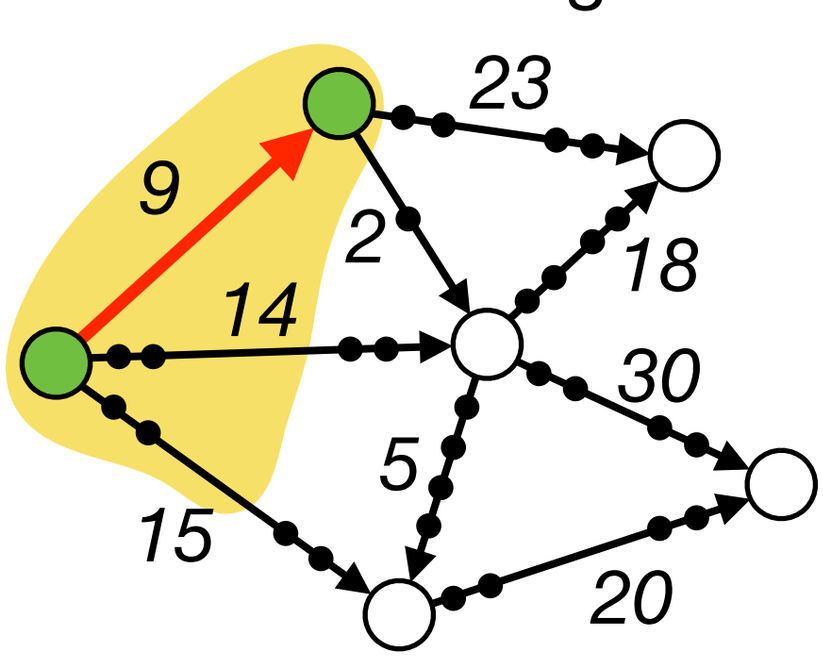

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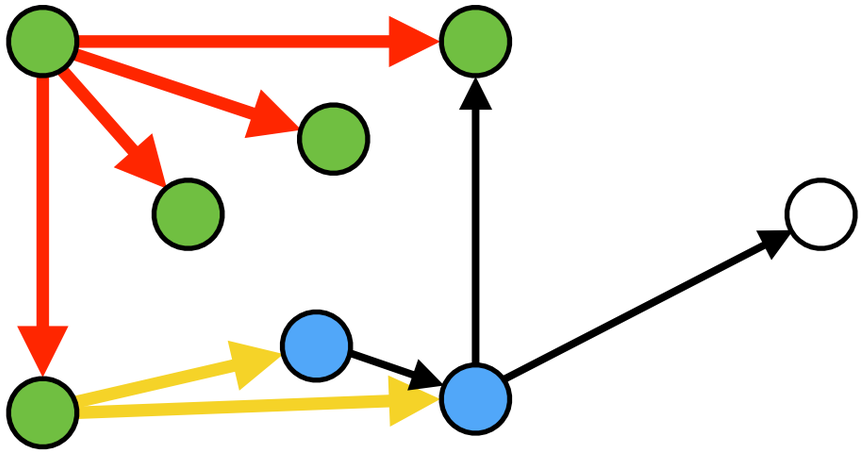
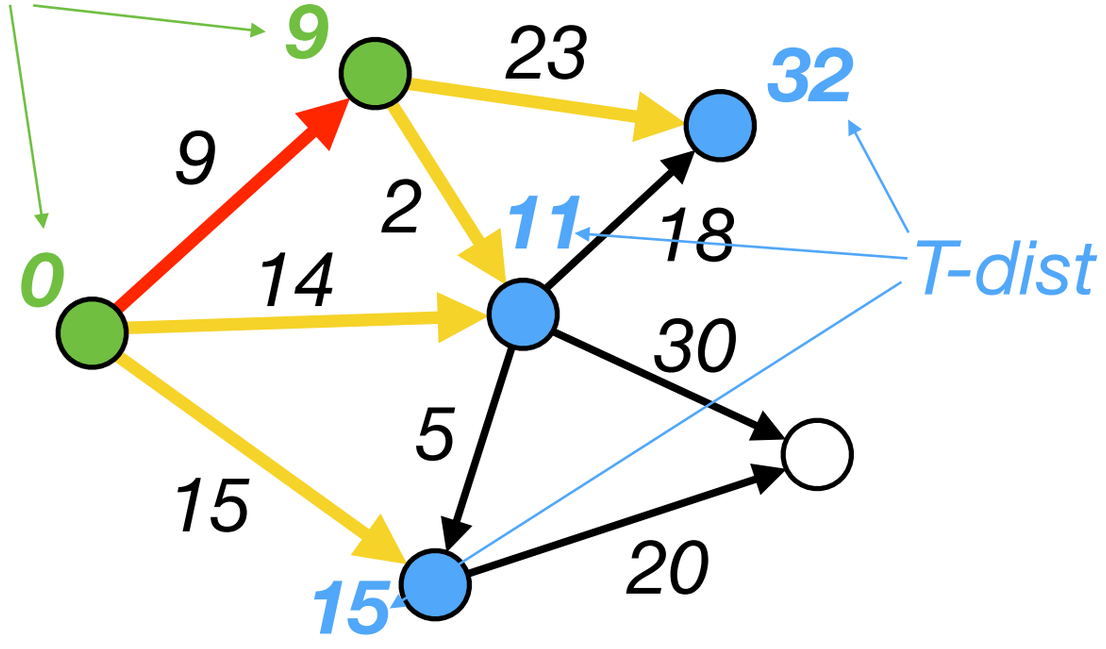


Breadth First Search

distance

Positive Lengths



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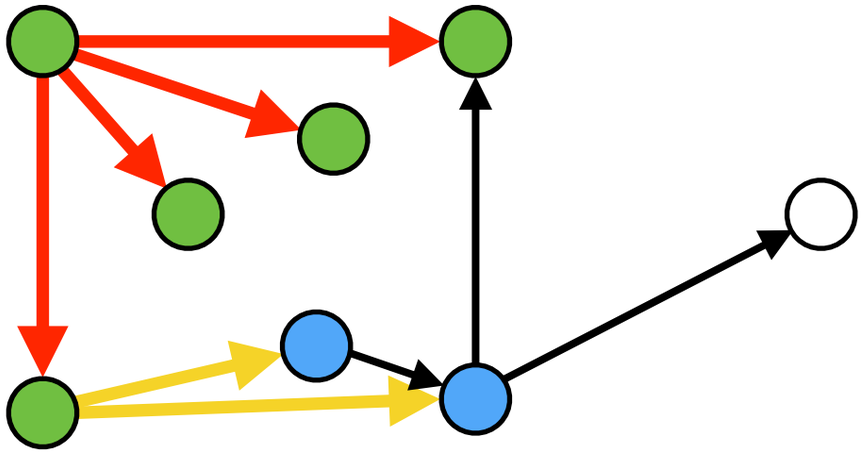

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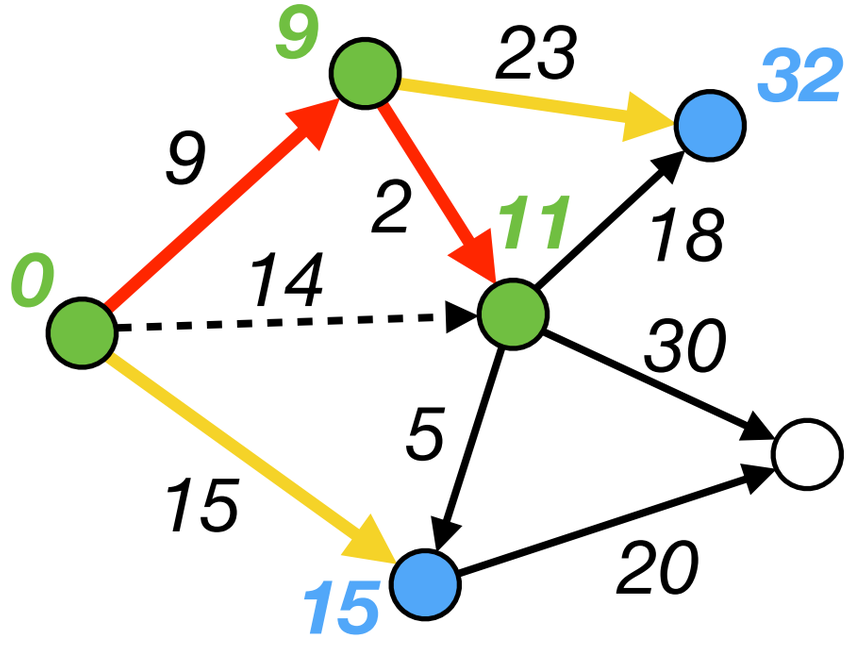

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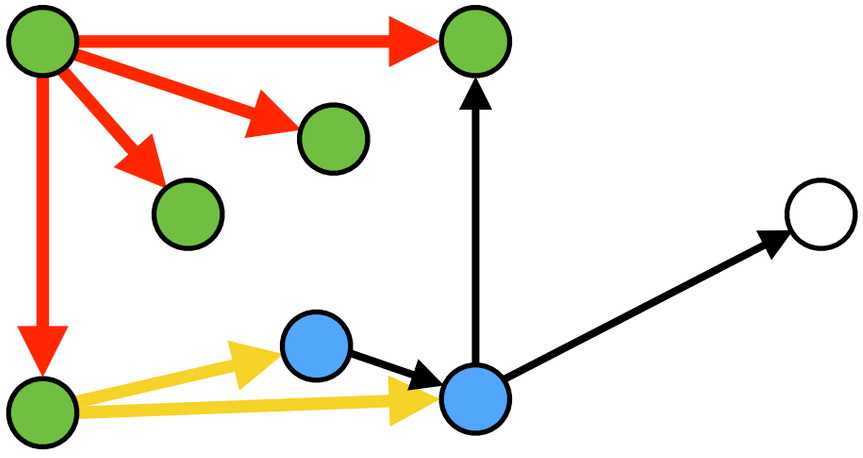

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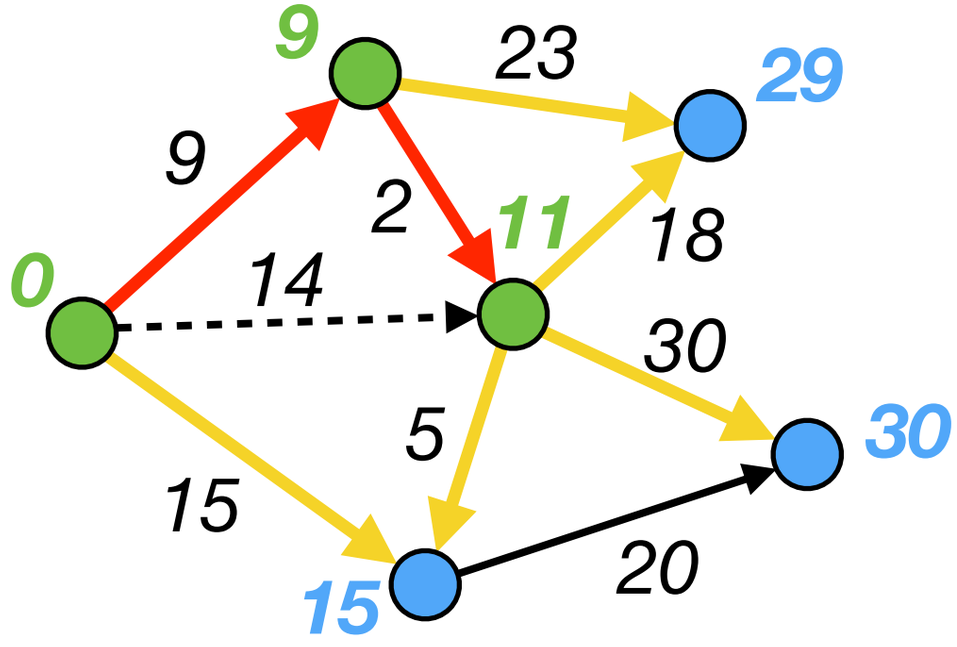
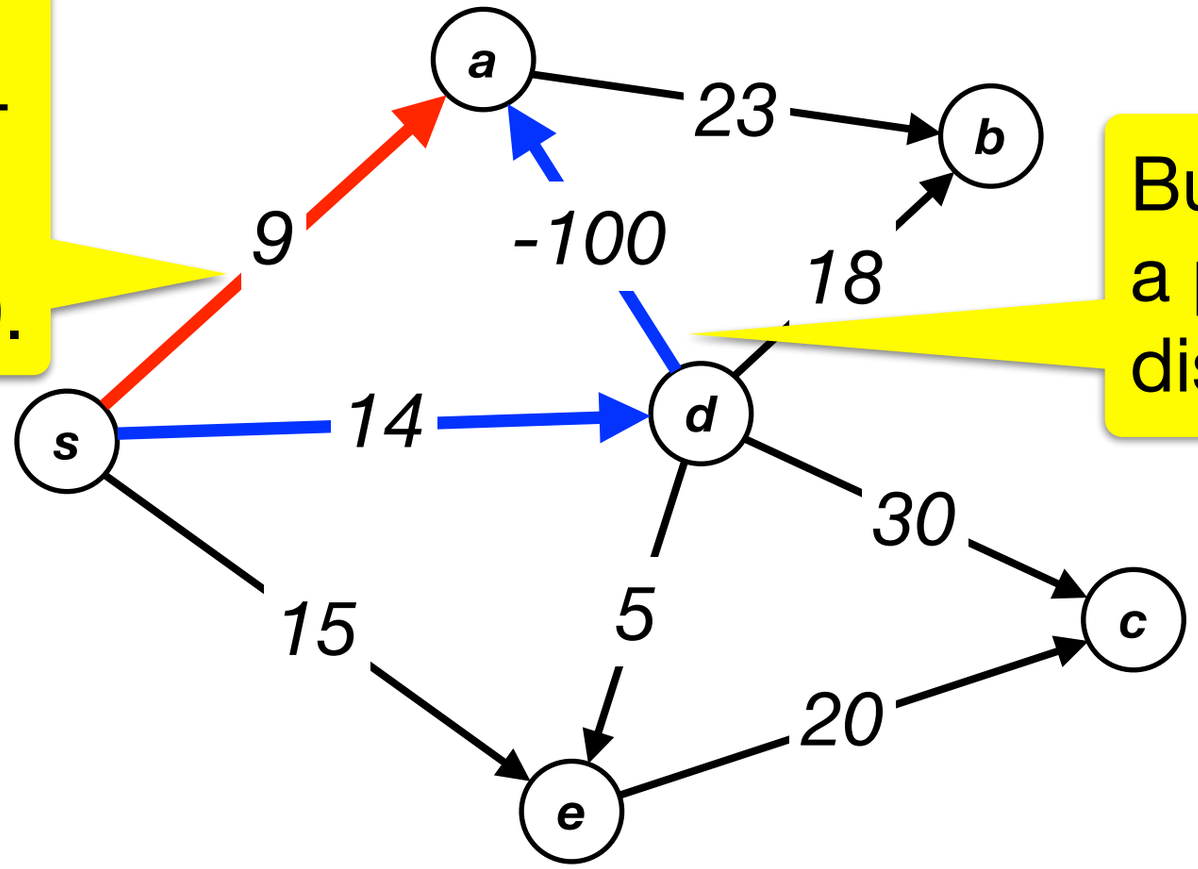


Dijkstra Algorithm

Today: Negative Edge Lengths

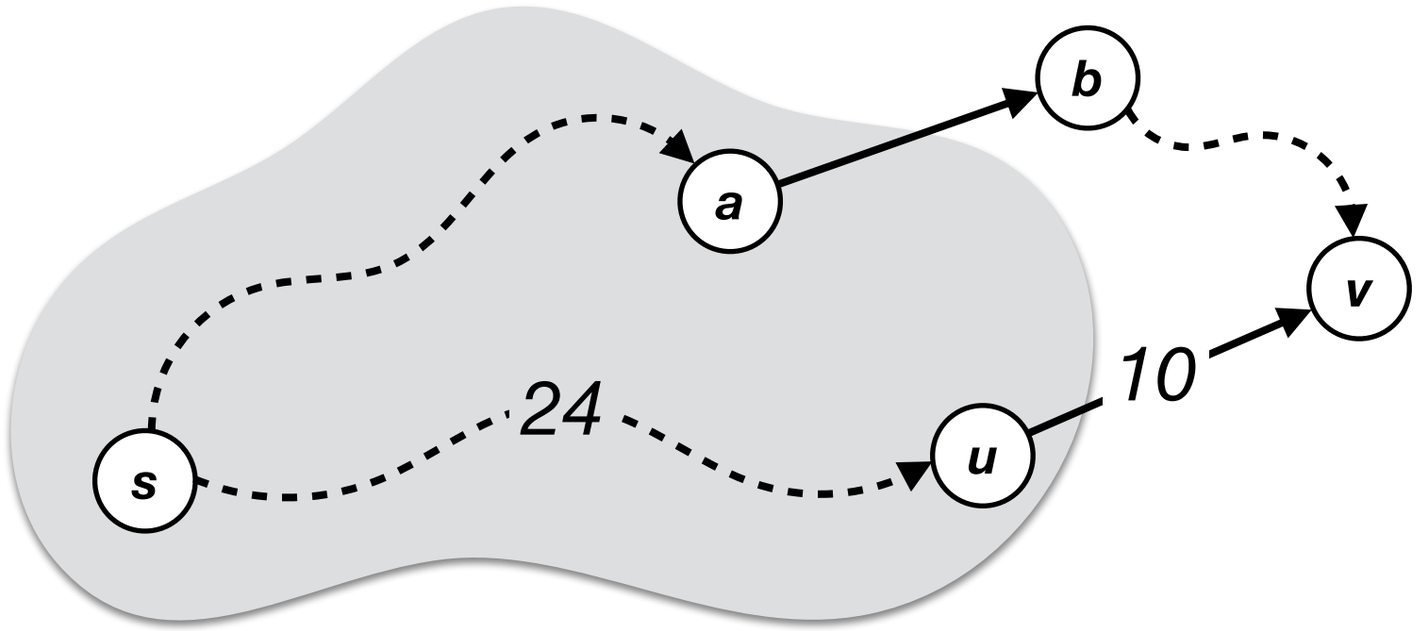
- Negative lengthes are natural in some applications.
- The correctness of Dijkstra depends on the fact that no edge has negative weight.

Dijkstra will add this edge to SPT at step 1, and assert dist(*a*) = 9.



But there is a path with distance -86

What goes wrong in the proof of Dijkstra's correctness with negative edges?



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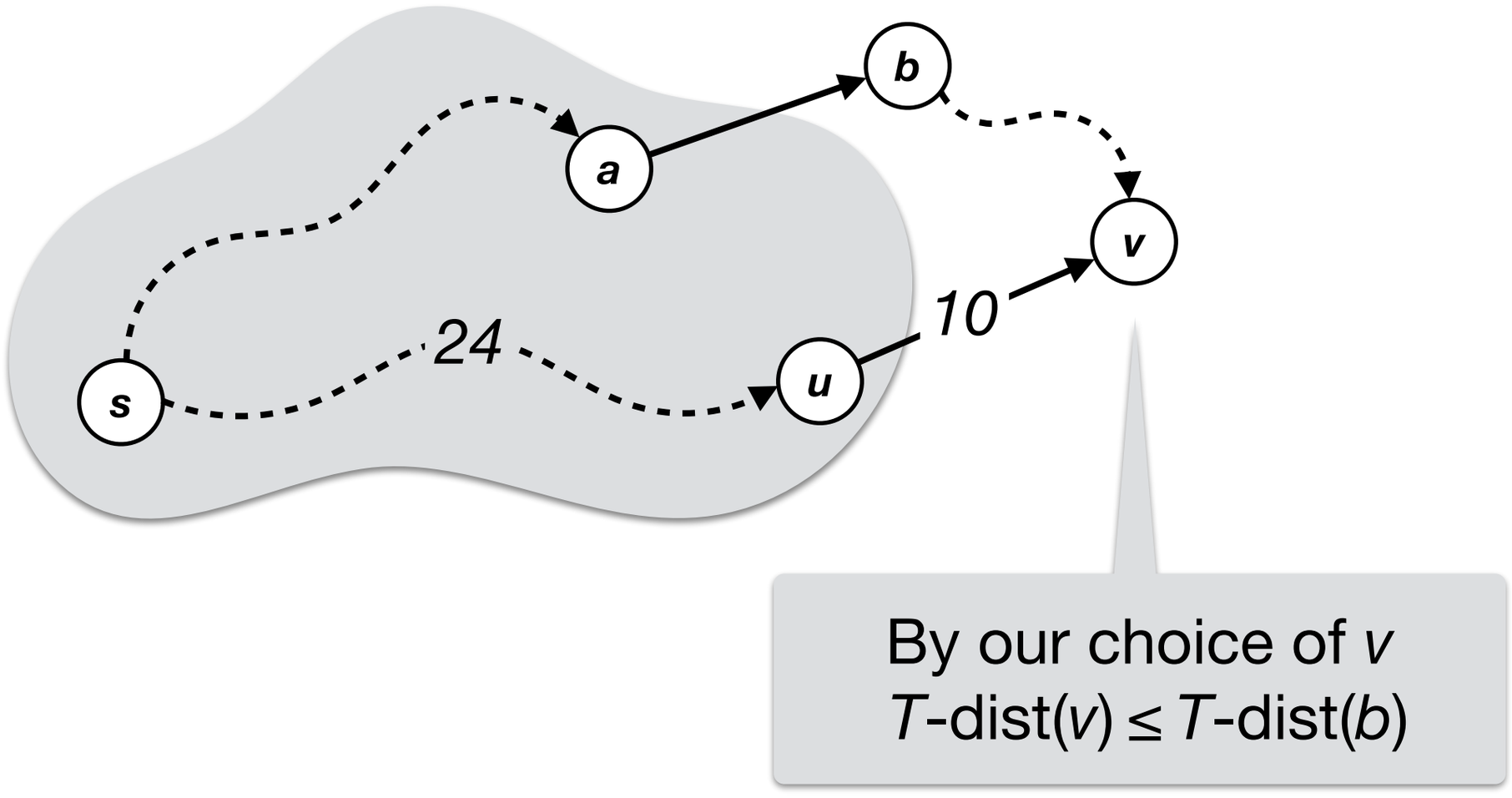
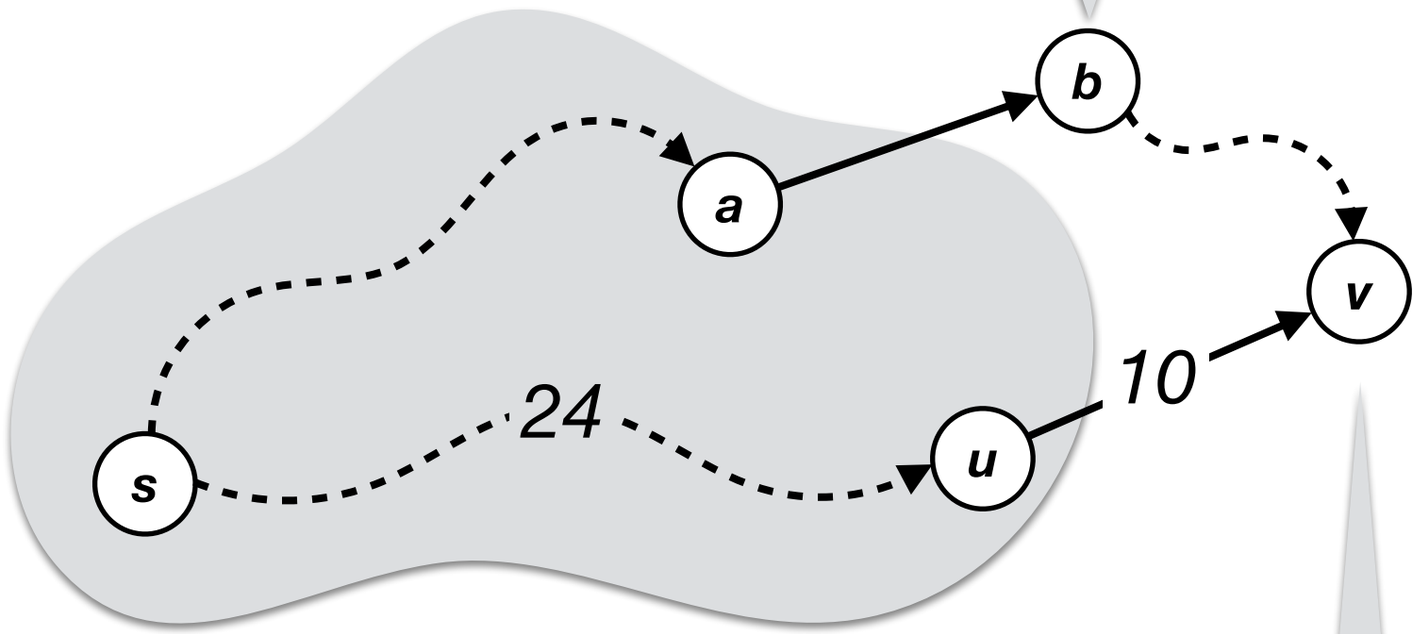


What goes wrong in the proof of Dijkstra's correctness with negative edges?

Since going through u is the shortest T-path from s to v, any alternative path from s to v must go through some vertex $b \notin T$, and T-dist(b) ≥ T-dist(v).



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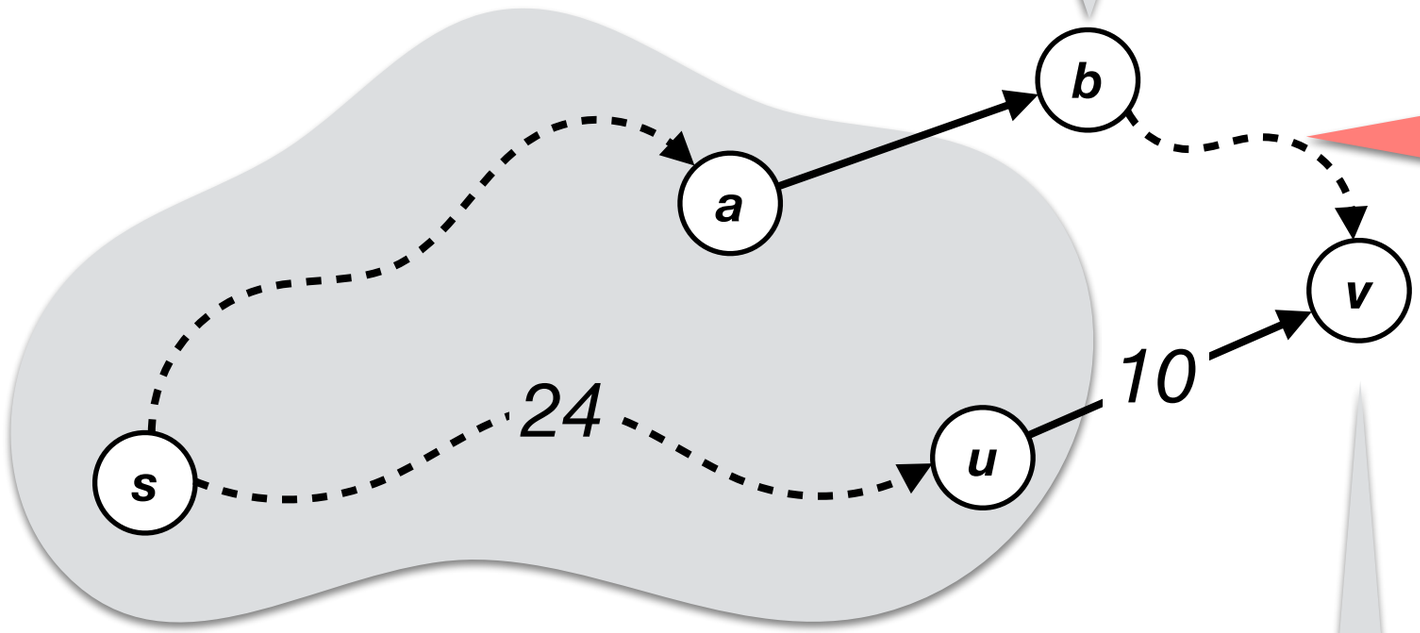


The length of this part from b to v is positive. (all edges have positive length.)

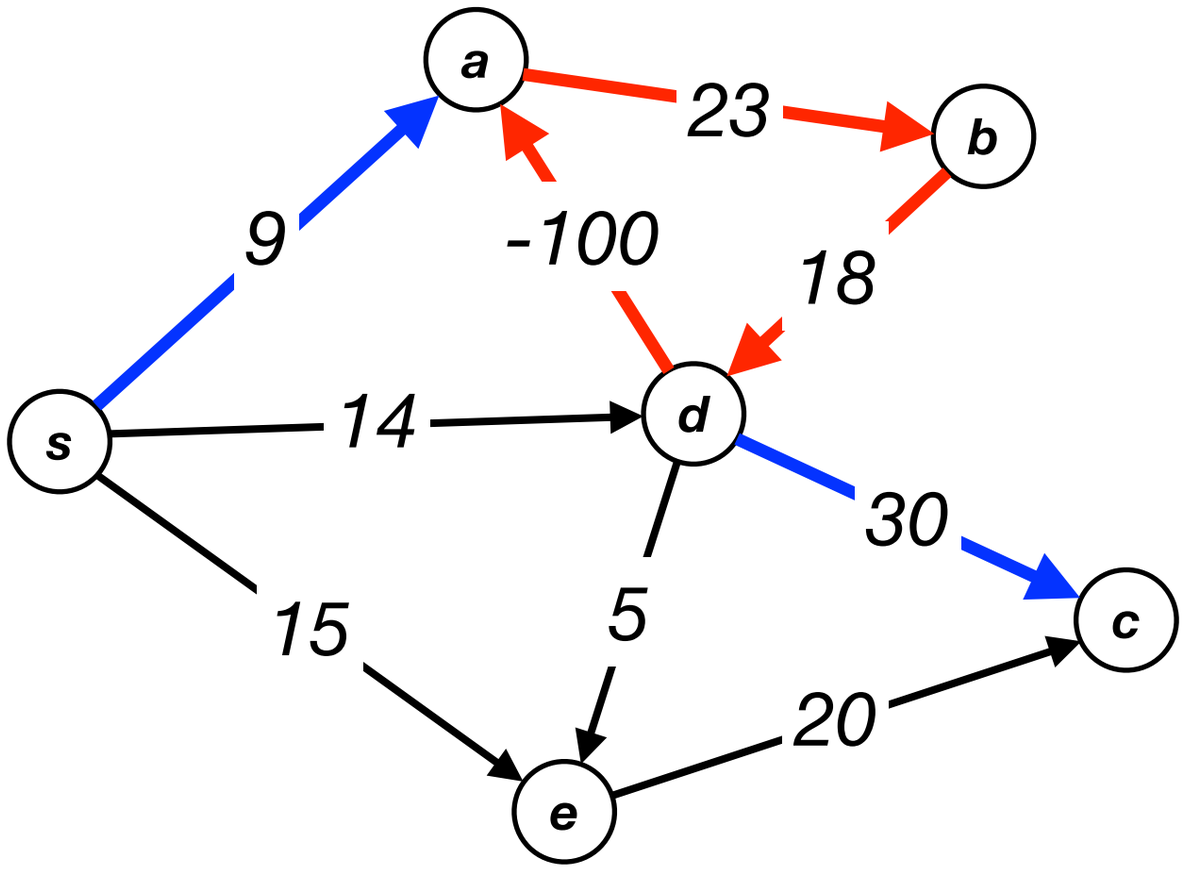
The red part is no longer true.

By our choice of v
T-dist(v) ≤ T-dist(b)

Negative cycles

Note: "shortest path" is not well defined if the input graph has negative cycles.

- From *s* to *c*, we can traverse the red cycle as many times as we want to get a path with distance as small as we want.



This lecture: Bellman-Ford algorithm

For graphs that have negative edges, **but do not have negative cycle**, we can use the Bellman-Ford algorithm to solve the single source shortest path problem.

Negative Lengths??

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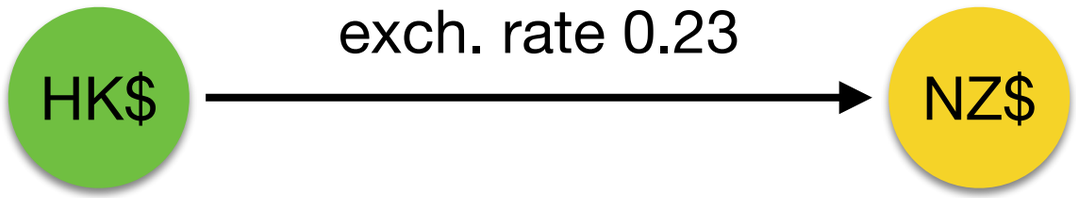
- Suppose Zhiyi will go to New Zealand in reading week and needs some New Zealand dollars



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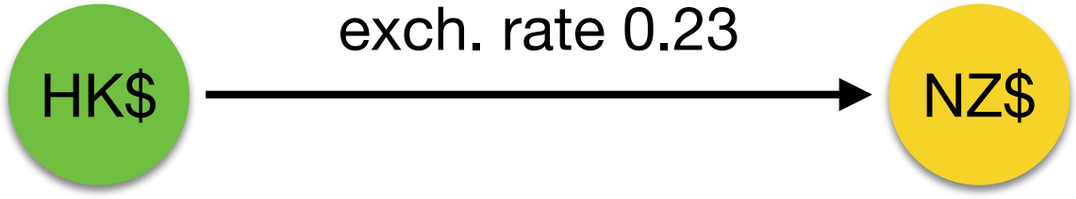

- Or:

with $19.36 \times 0.012 = 0.23232$

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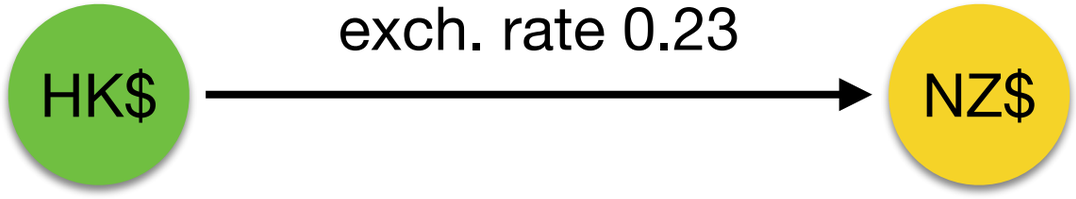

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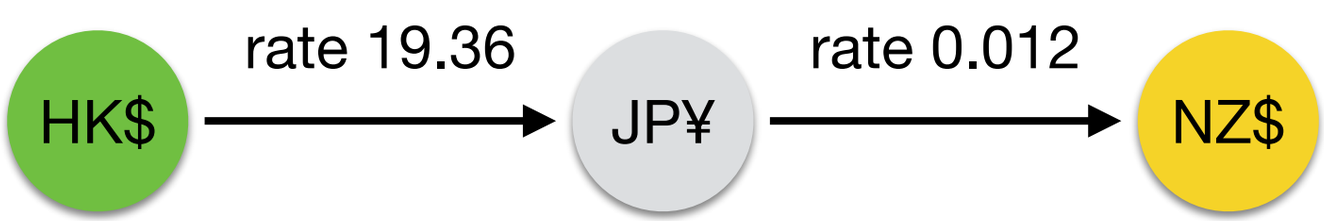

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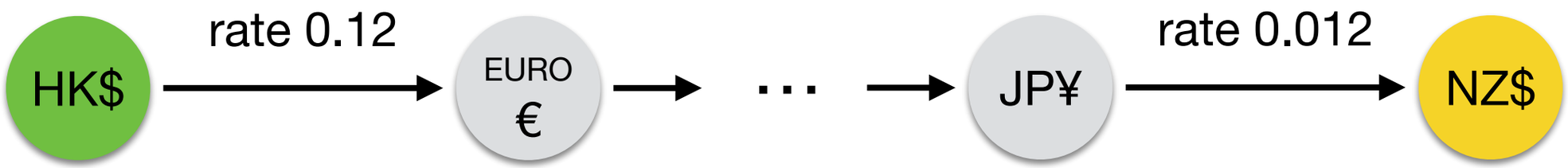

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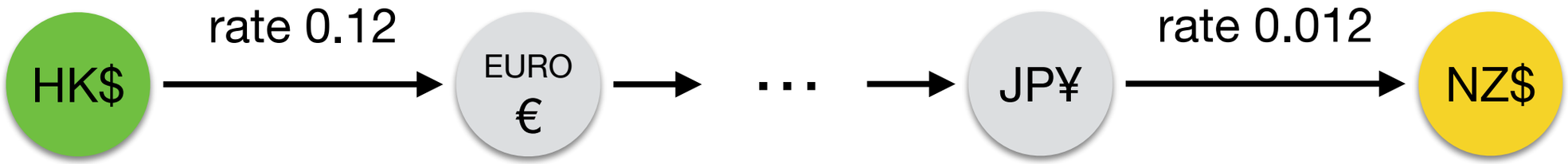
“Reduction” to Shortest Path



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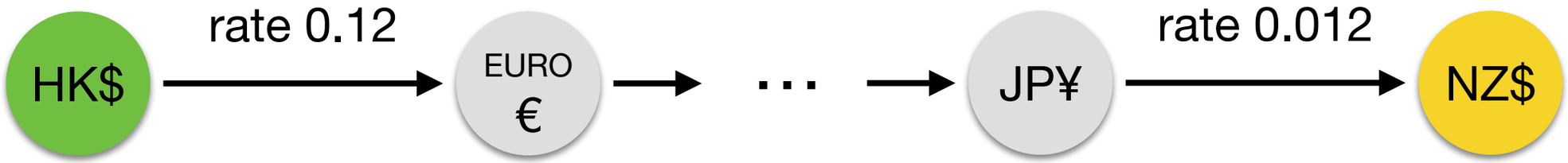

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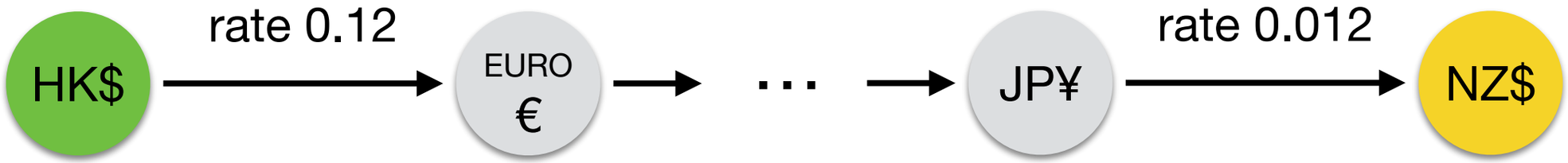

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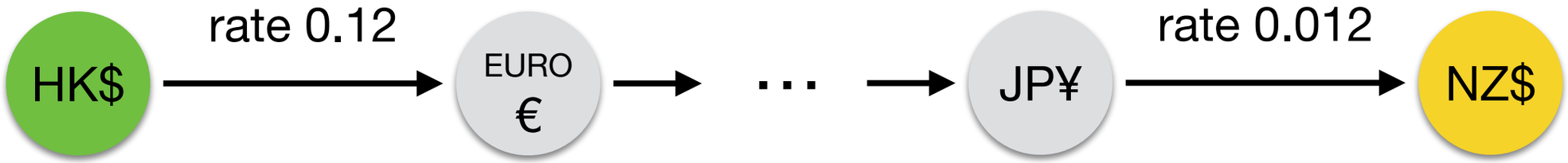

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$$\log(abc) = \log(a) + \log(b) + \log(c)$$

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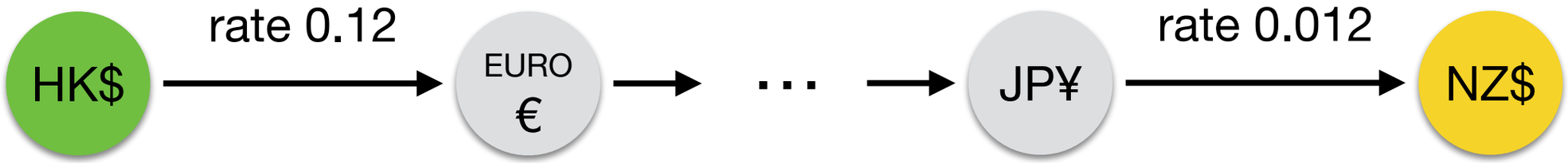

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- **Maximization vs. Minimization**

$$\max\ a + b + c \quad \Leftrightarrow \quad \min\ (-a) + (-b) + (-c)$$

Putting together

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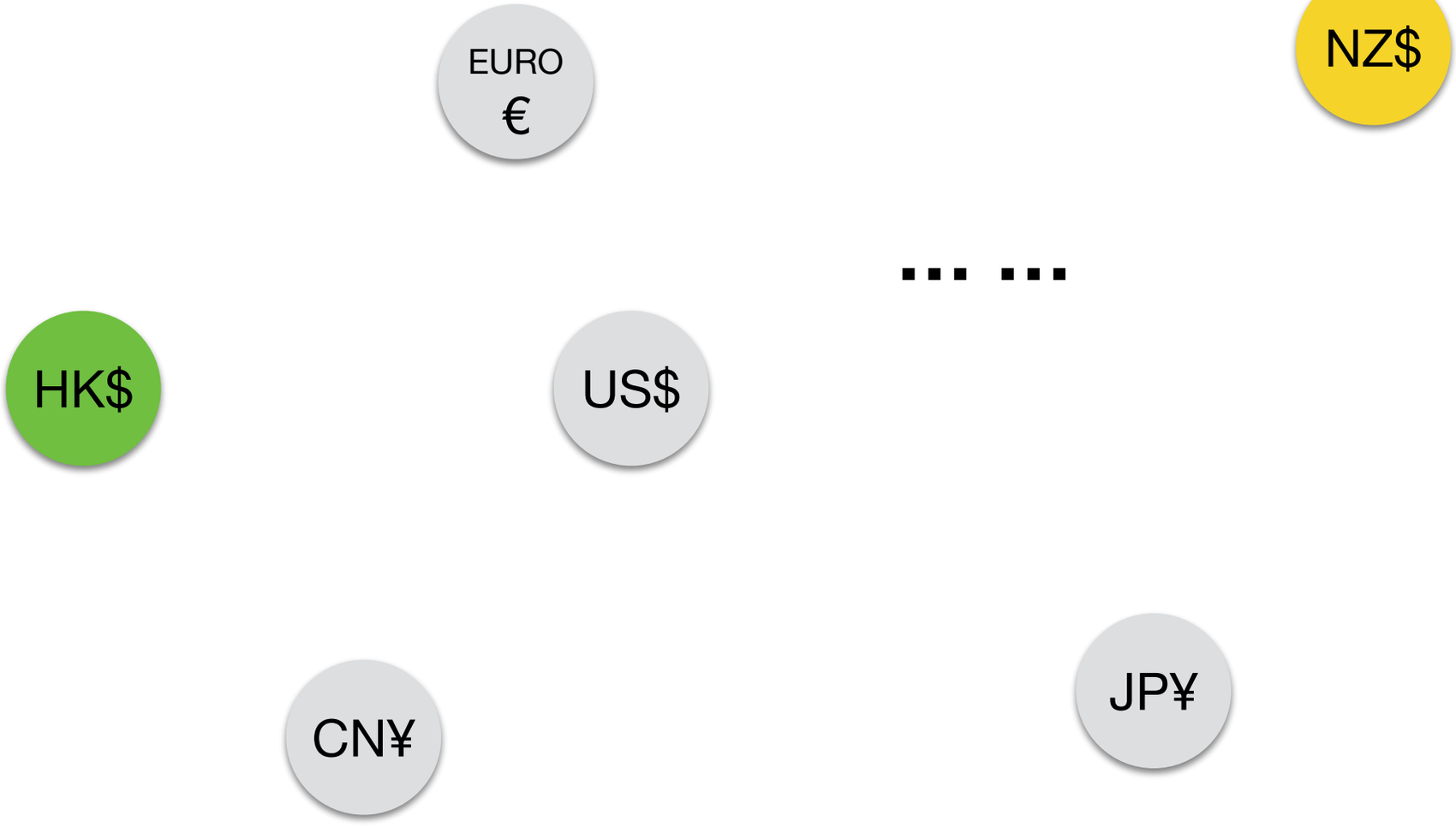
- Let there be a vertex for each currency



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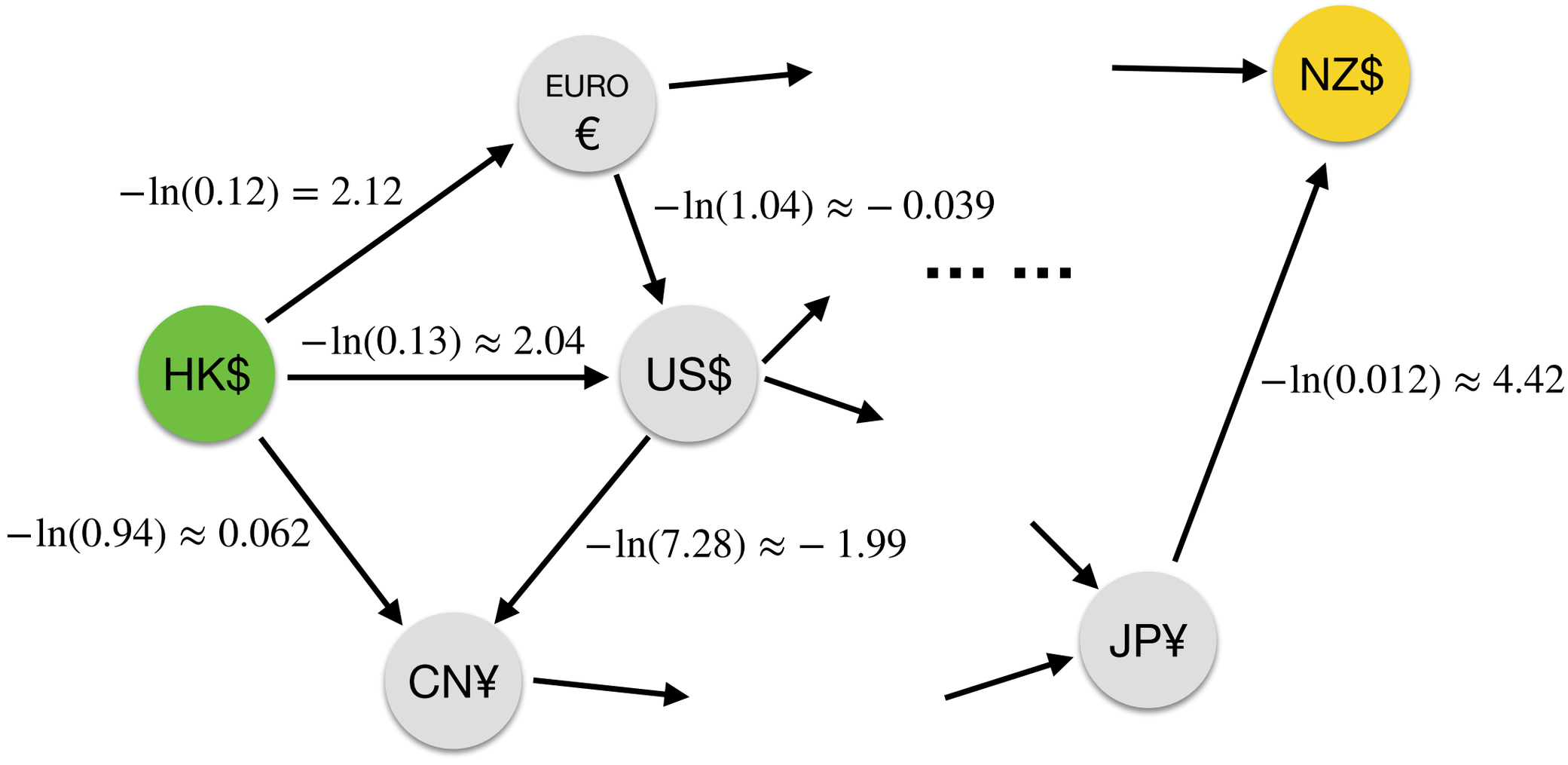
- Let there be a vertex for each currency
- Let there be a directed edge between each pair of currencies with the **negative logarithm** of exchange rate as **"length"**



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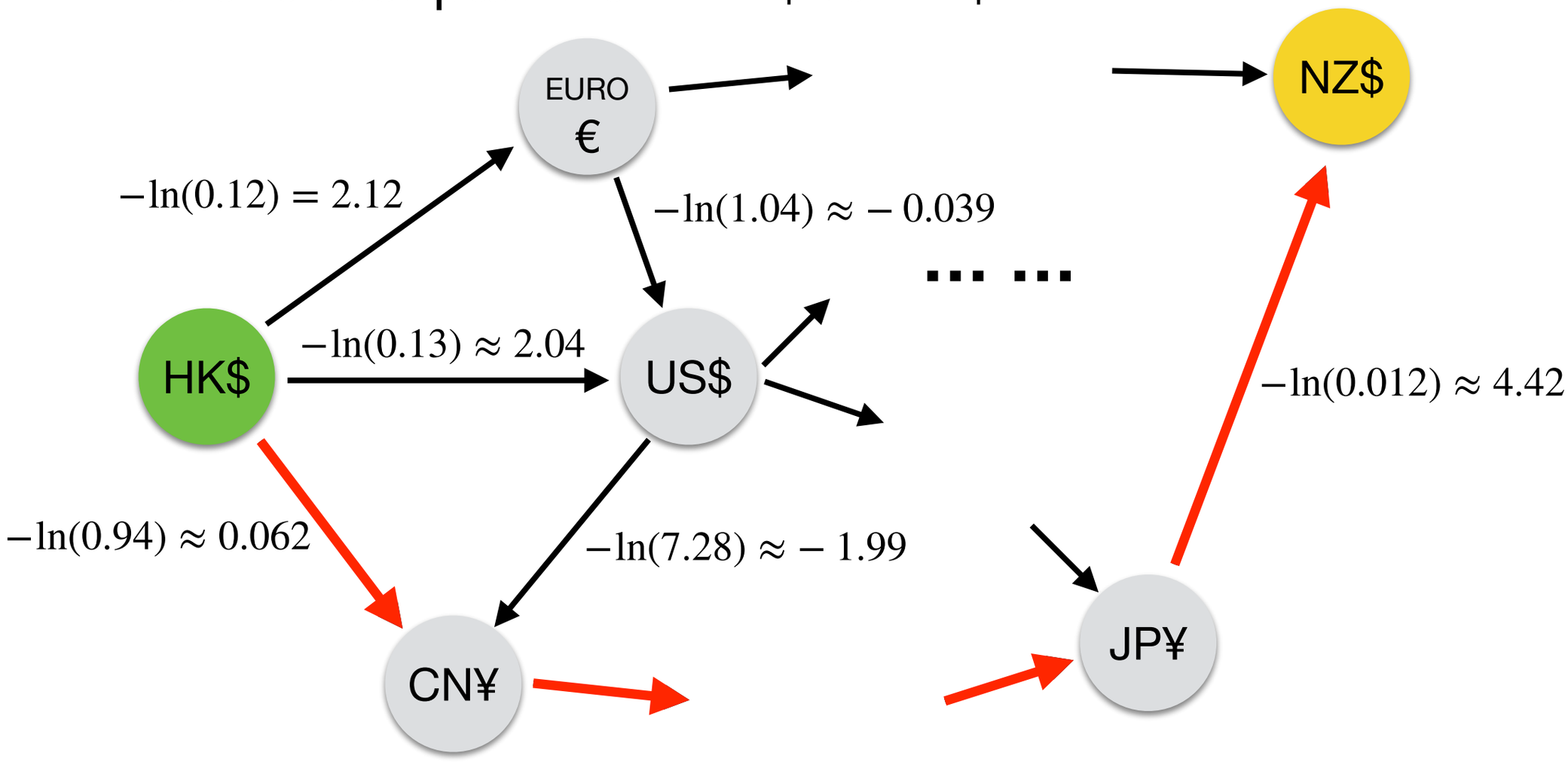

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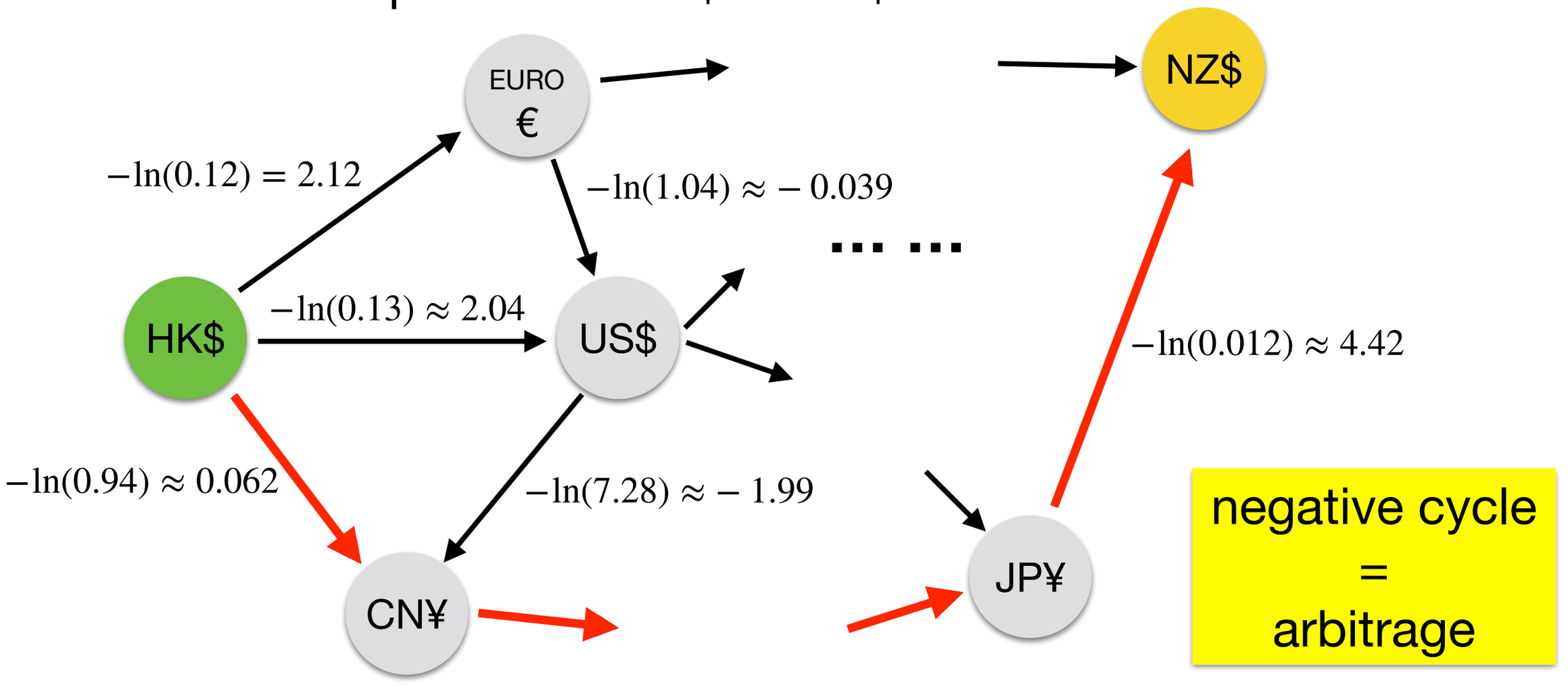


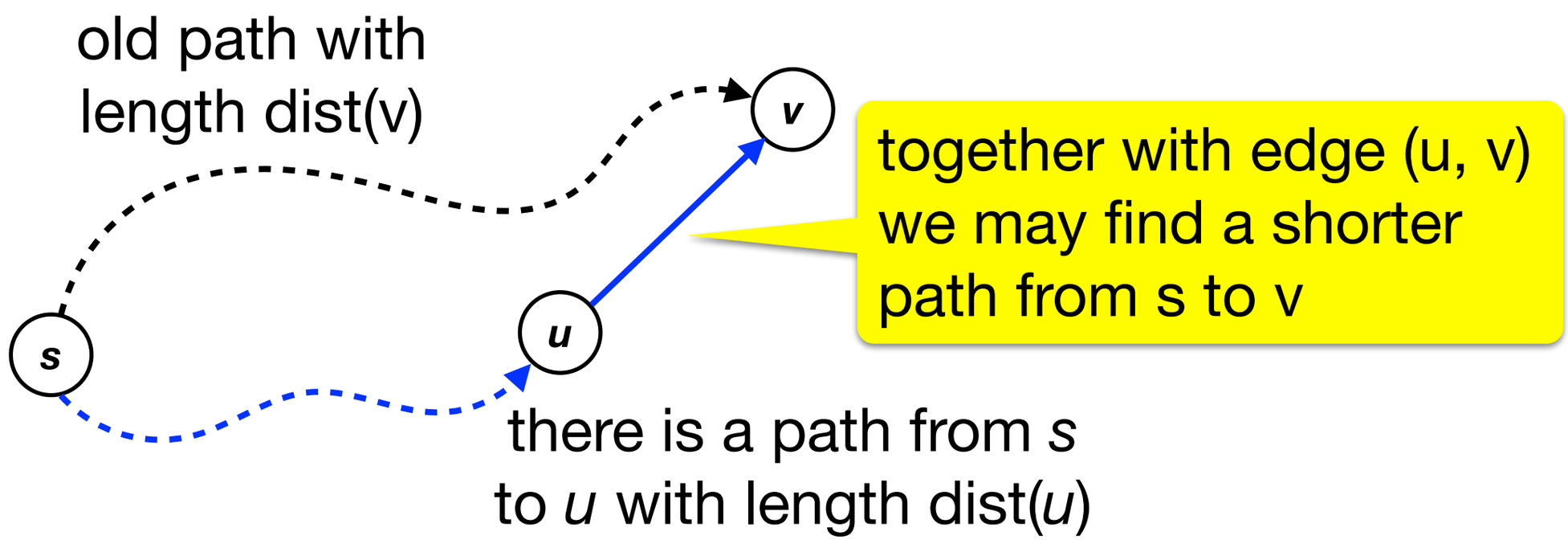
Another view on Dijkstra

Basic step: update *(u, v)* for an edge *(u, v)* $\in$ *E* as follows

$$\text{dist}(v) = \min \{ \text{dist}(v), \text{dist}(u) + L(u, v) \}$$

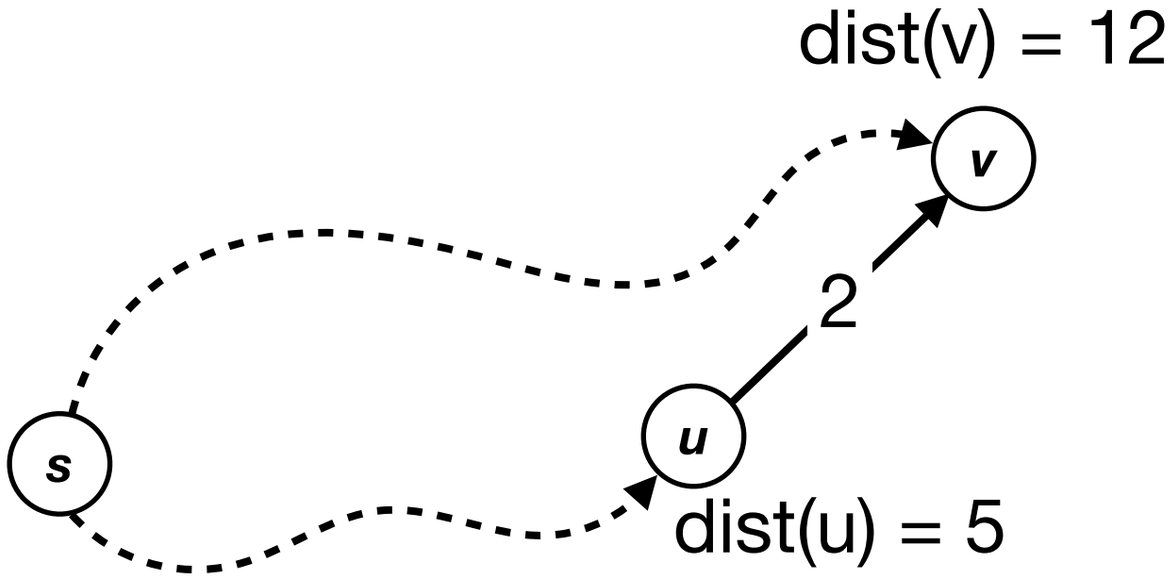
Intuition:

- dist(*v*) is the length of the shortest path we have found so far from *s* to *v*.
- Whenever we find a shorter path, we update dist(*v*).



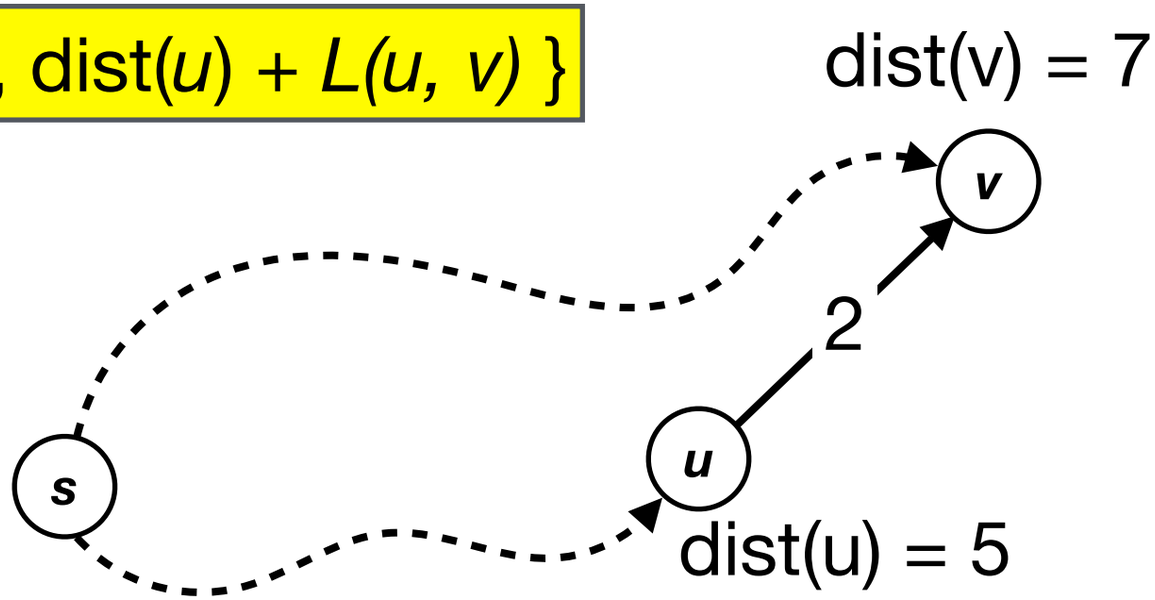
An Example

before update:



dist(*v*) = min { dist(*v*), dist(*u*) + *L(u, v)* }

after update:



Bellman-Ford algorithm

- **Without** negative weights, Dijkstra updates edges in a clever order so that the algorithm updates each edge only once.
- **With** negative weights, Dijkstra's clever order no longer works.
- Bellman-Ford returns to the "stupid" way and updates each edge multiple times:

```
1) initialize dist(s) = 0, and dist(x) = ∞ for other vertices x;
2) do
3)     for each edge (u, v) in E :
4)         dist(v) = min { dist(v), dist(u) + L(u, v) }
5) until there is no more update
```

Bellman-Ford algorithm

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- **With** negative weights, Dijkstra's clever order no longer works.
- Bellman-Ford returns to the "stupid" way and updates each edge multiple times:

```
1) initialize dist(s) = 0, and dist(x) = ∞ for other vertices x;
2) for i from 1 to |V| - 1 :
3)     for each edge (u, v) in E :
4)         dist(v) = min { dist(v), dist(u) + L(u, v) }
```

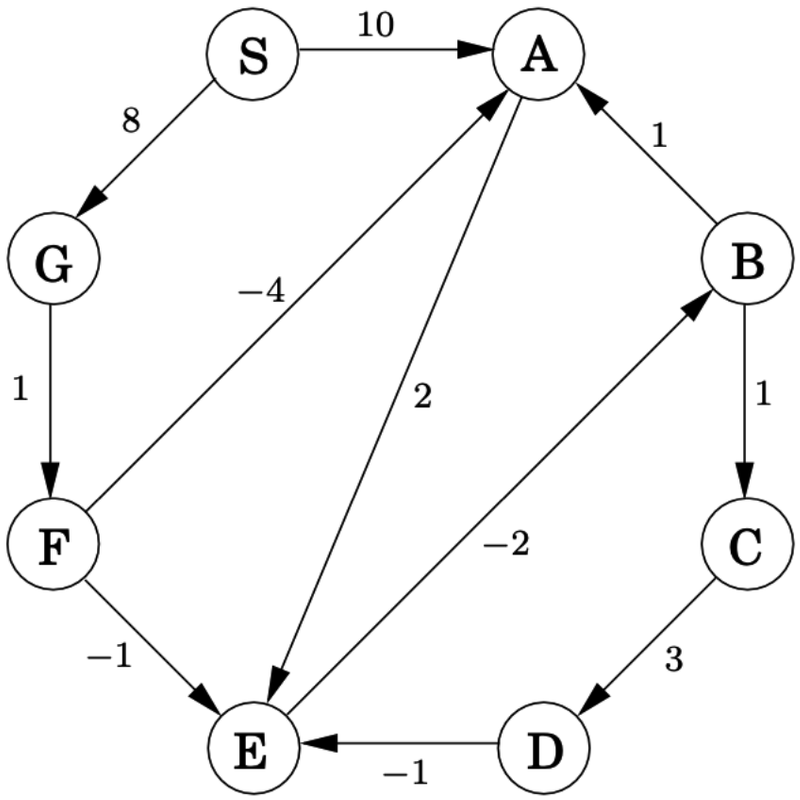
Bellman-Ford algorithm

- **Without** negative weights, Dijkstra updates edges in a clever order so that the algorithm updates each edge only once.
- **With** negative weights, Dijkstra's clever order no longer works.
- Bellman-Ford returns to the "stupid" way and updates each edge multiple times:

1) **initialize** dist(s) = *0*, and dist(x) = ∞ for other vertices x;
2) **for** i from *1* to $|V| - 1$ **:**
3) **for** each edge (u, v) in E **:**
4) dist(v) = min { dist(v), dist(u) + $L(u, v)$ }

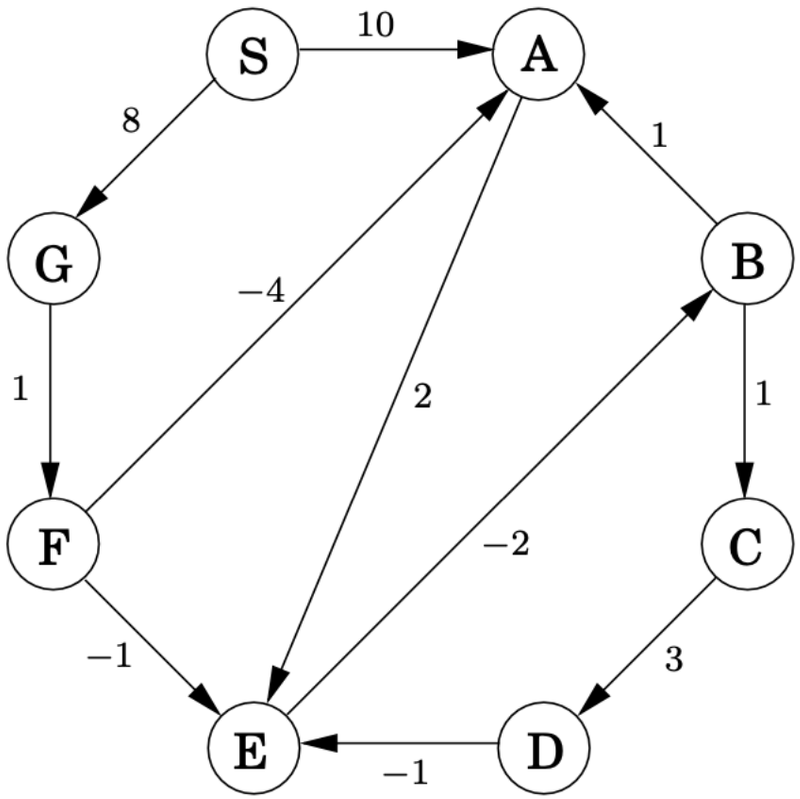
Running time: $O(|V||E|) = O(|V|^3)$

Sample Run



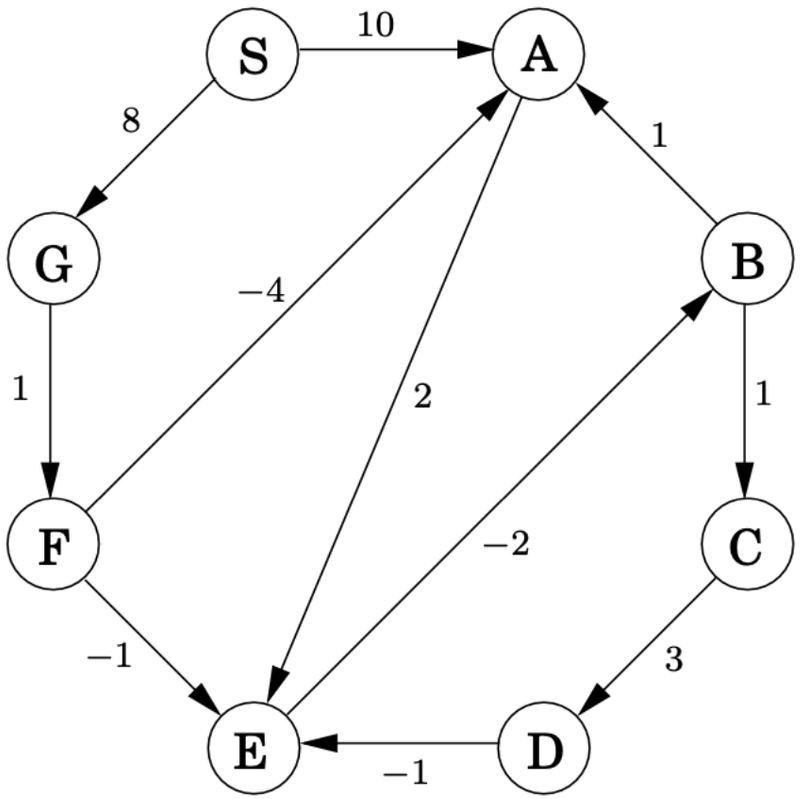
	0	1	2	3	4	5	6	7
S	0							
A	∞							
B	∞							
C	∞							
D	∞							
E	∞							
F	∞							
G	∞							

Sample Run



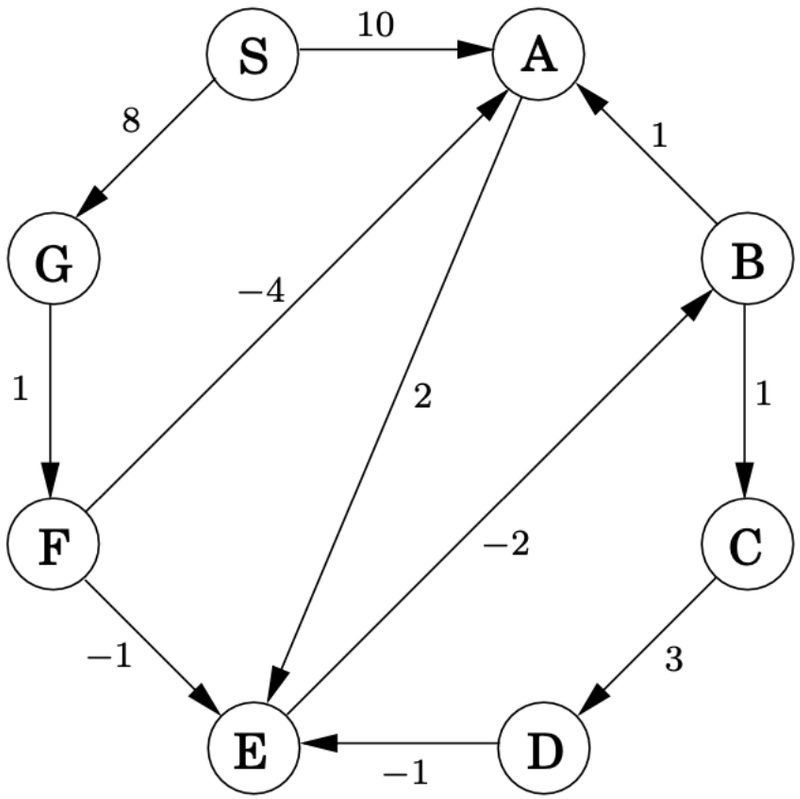
	0	1	2	3	4	5	6	7
S	0	0						
A	∞	**10**						
B	∞	∞						
C	∞	∞						
D	∞	∞						
E	∞	∞						
F	∞	∞						
G	∞	**8**						

Sample Run



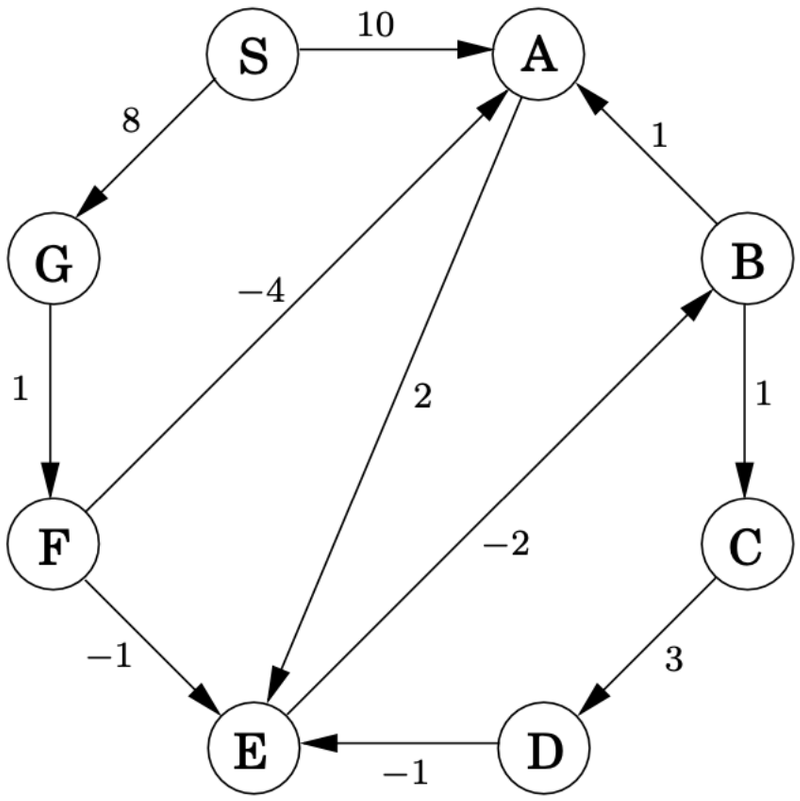
	0	1	2	3	4	5	6	7
S	0	0	0					
A	∞	10	10					
B	∞	∞	∞					
C	∞	∞	∞					
D	∞	∞	∞					
E	∞	∞	**12**					
F	∞	∞	**9**					
G	∞	8	8					

Sample Run



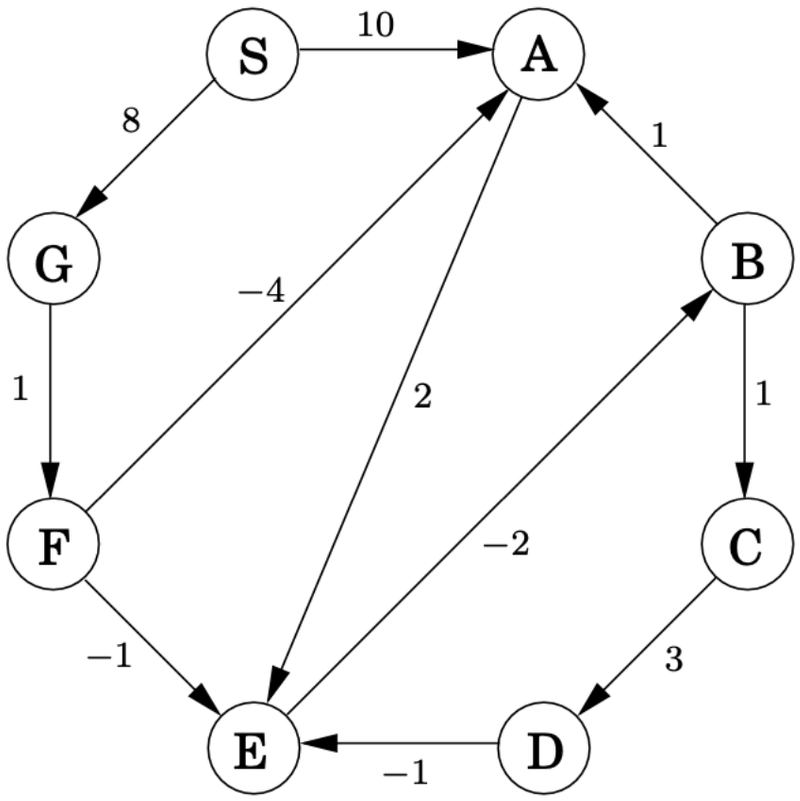
	0	1	2	3	4	5	6	7
S	0	0	0	0				
A	∞	10	10	**5**				
B	∞	∞	∞	**10**				
C	∞	∞	∞	∞				
D	∞	∞	∞	∞				
E	∞	∞	12	**8**				
F	∞	∞	9	9				
G	∞	8	8	8				

Sample Run



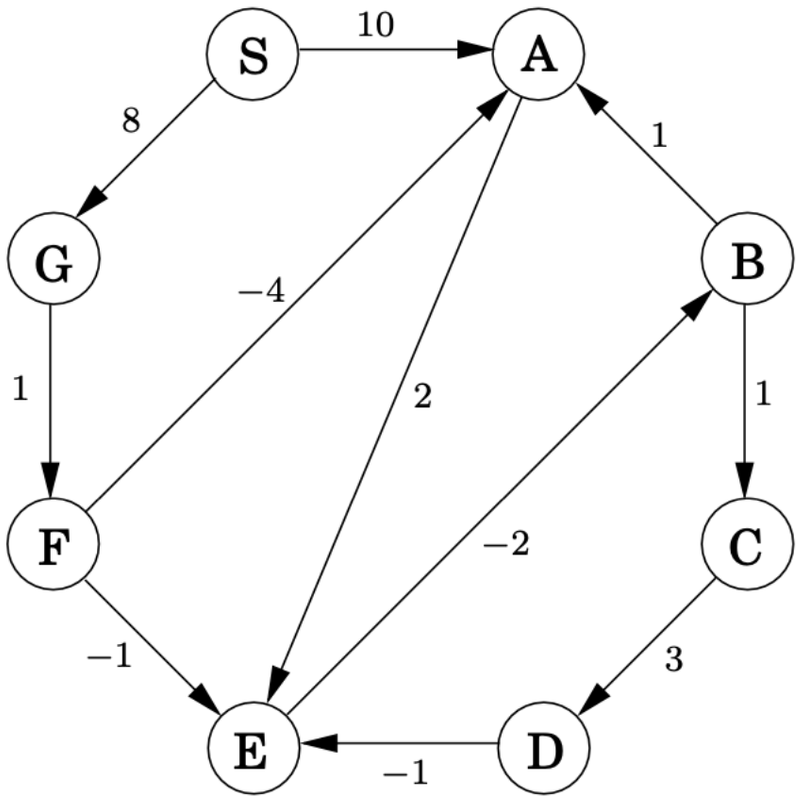
	0	1	2	3	4	5	6	7
S	0	0	0	0	0			
A	∞	10	10	5	5			
B	∞	∞	∞	10	**6**			
C	∞	∞	∞	∞	**11**			
D	∞	∞	∞	∞	∞			
E	∞	∞	12	8	**7**			
F	∞	∞	9	9	9			
G	∞	8	8	8	8			

Sample Run



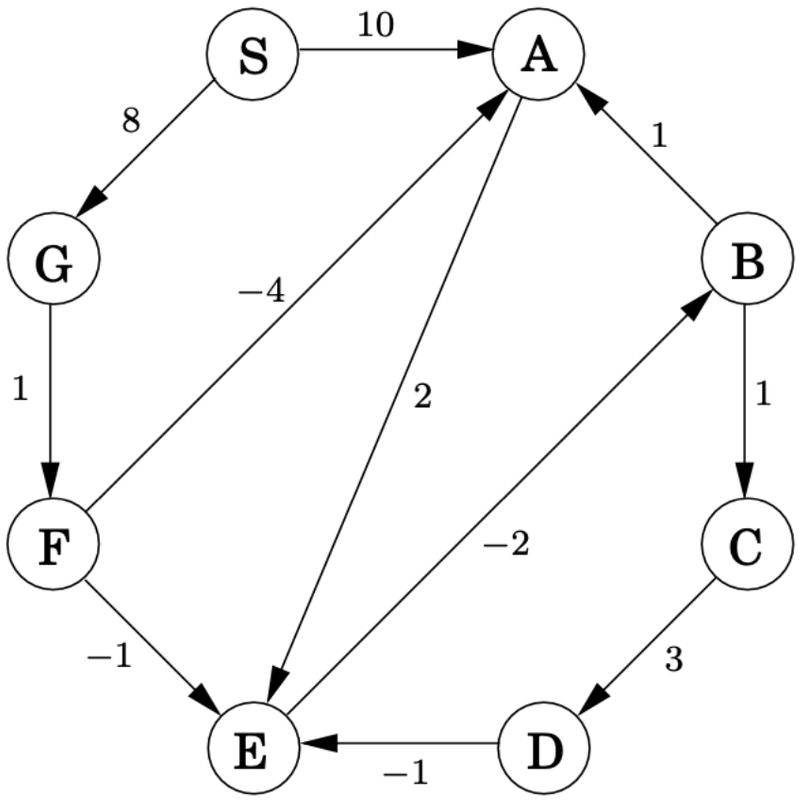
	0	1	2	3	4	5	6	7
S	0	0	0	0	0	0		
A	∞	10	10	5	5	5		
B	∞	∞	∞	10	6	**5**		
C	∞	∞	∞	∞	11	**7**		
D	∞	∞	∞	∞	∞	**14**		
E	∞	∞	12	8	7	7		
F	∞	∞	9	9	9	9		
G	∞	8	8	8	8	8		

Sample Run



	0	1	2	3	4	5	6	7
S	0	0	0	0	0	0	0	
A	∞	10	10	5	5	5	5	
B	∞	∞	∞	10	6	5	5	
C	∞	∞	∞	∞	11	7	**6**	
D	∞	∞	∞	∞	∞	14	**10**	
E	∞	∞	12	8	7	7	7	
F	∞	∞	9	9	9	9	9	
G	∞	8	8	8	8	8	8	

Sample Run



	0	1	2	3	4	5	6	7
S	0	0	0	0	0	0	0	0
A	∞	10	10	5	5	5	5	5
B	∞	∞	∞	10	6	5	5	5
C	∞	∞	∞	∞	11	7	6	6
D	∞	∞	∞	∞	∞	14	10	**9**
E	∞	∞	12	8	7	7	7	7
F	∞	∞	9	9	9	9	9	9
G	∞	8	8	8	8	8	8	8

Correctness of Bellman-Ford

Suppose that $G = (V, E)$ is a weighted directed graph without negative cycles.

Observation 1: For every vertex v, there is a shortest path from s to v with at most $|V| - 1$ edges.

Why?

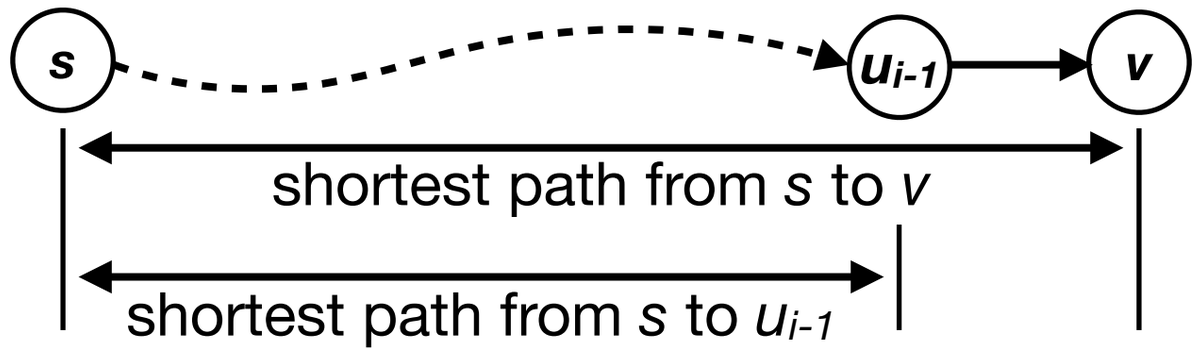
- We can assume without loss of generality that there is a shortest path that does not contain any cycle.
- All the vertices on this shortest path must be distinct; otherwise there is a cycle.
- Thus, there are at most $|V|$ vertices, and thus at most $|V| - 1$ edges on this path.

Correctness of Bellman-Ford (cont'd)

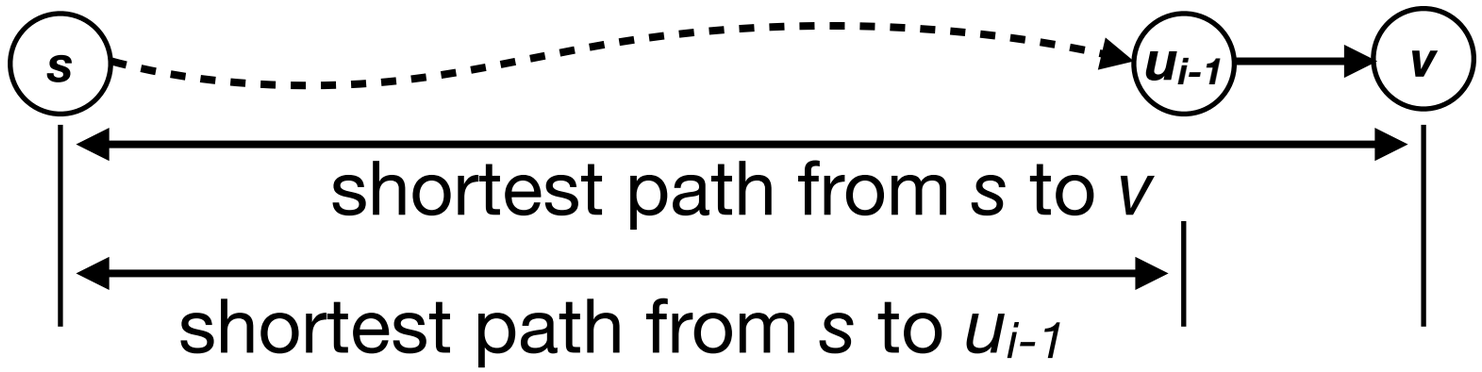
Fact: dist[*v*] is no larger than the length of the shortest path from *s* to *v* through out the algorithm.

Observation 2: For any vertex *v*, if there is a shortest path from *s* to *u* with *i* edges, then dist[*v*] = *d(s, v)* after the *i*-th iteration.

- Prove by mathematical induction.
- **Base case:** obviously true for *i = 0* because only *s* has a shortest path with no edge.
- **Inductive hypothesis:** suppose it is true for iteration *i - 1*.
- Consider a shortest path of length *i*: $P = (s, u_1, \ldots, u_{i-1}, v)$.



Correctness of Bellman-Ford (cont'd)



- Thus, there is a shortest path from s to u_{i-1} with $i - 1$ edges.
- By the inductive hypothesis, we have dist[u_{i-1}] = $d(s, u_{i-1})$ after iteration i-1.
- When the algorithm update edge (u_{i-1}, v) at iteration i,

$$\text{dist}(v) = \min \{ \text{dist}(v), \text{dist}(u_{i-1}) + L(u_{i-1}, v) \} = d(s, v)$$

- Note that dist[v] will remain the same hereafter because there are no shorter paths from s to v.

Correctness of Bellman-Ford (cont'd)

Observation 1: For every vertex *v*, there is a shortest path from *s* to *v* with at most $|V| - 1$ edges.

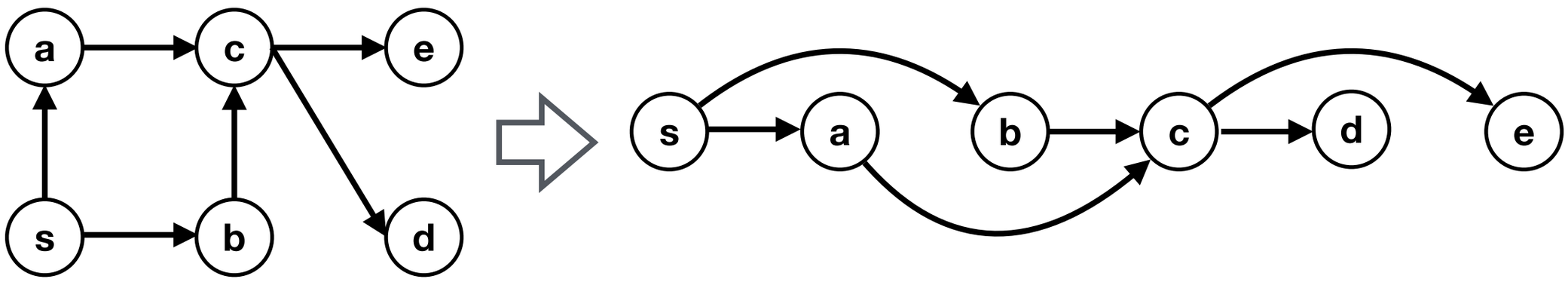
Observation 2: For any vertex *v*, if there is a shortest path from *s* to *v* with *i* edges, then dist[*v*] = *d(s, v)* after the *i*-th iteration.

- When Bellman-Ford terminates, it has iterated $|V| - 1$ times.
- By the above observations, dist[*v*] = *d(s, v)* when Bellman-Ford terminates.

Optional Topic:
Shortest Path in DAGs (Chapter 4.7)

Shortest Path in DAGs

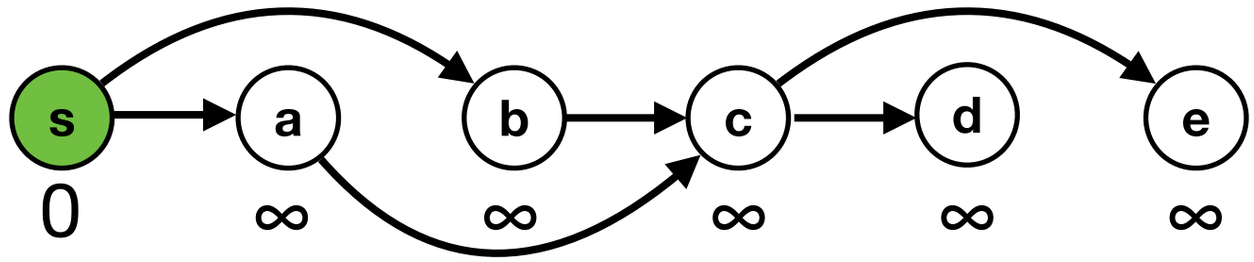
- Given a Directed Acyclic Graph (DAG) *G* with possibly negative-length edges, we can solve the single-source shortest path problem in $O(|V| + |E|)$ time.
- **Note:** Since *G* is a DAG, it does not have any cycle, and thus it does not have any negative cycle.
- **Recall:** We can linearize G in a line such that every edge is from left to right by running a DFS in $O(|V| + |E|)$ *time*.



Shortest Path in DAGs

1) **initialize** dist(s) = 0, and dist(*u*) = ∞ for all other $u \in V$
2) Linearize G;
3) **for** each $u \in V$ in linearized order **:**
4) **for** all edges *(u, v)*, update (u, v) as follows **:**

$$\text{dist}(v) = \min \{ \text{dist}(v), \text{dist}(u) + L(u, v) \}$$

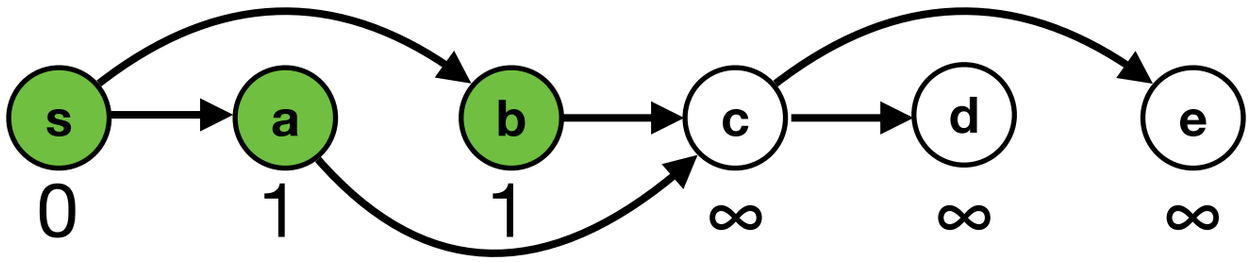


Initially, dist(s) is correct.

Shortest Path in DAGs

1) **initialize** dist(s) = 0, and dist(*u*) = ∞ for all other $u \in V$
2) Linearize G;
3) **for** each $u \in V$ in linearized order **:**
4) **for** all edges *(u, v)*, update (u, v) as follows **:**

$$\text{dist}(v) = \min \{ \text{dist}(v), \text{dist}(u) + L(u, v) \}$$

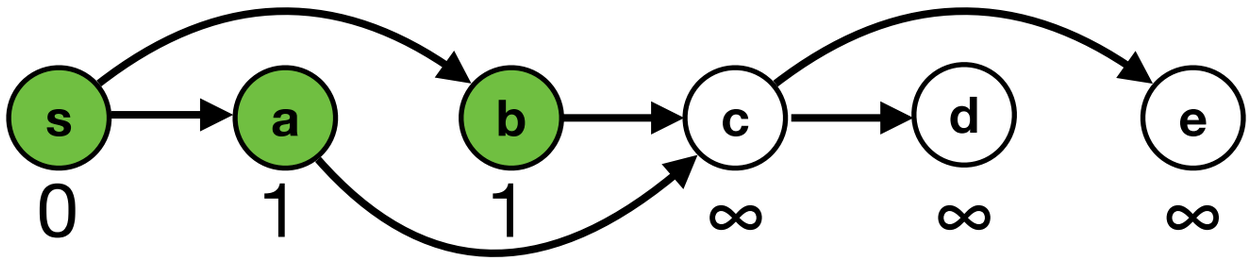


After the 1st iteration of the first for loop (step 3), the 1st vertex following *s* has correct dist(.) value.

Shortest Path in DAGs

1) **initialize** dist(s) = 0, and dist(*u*) = ∞ for all other $u \in V$
2) Linearize G;
3) **for** each $u \in V$ in linearized order **:**
4) **for** all edges *(u, v)*, update (u, v) as follows **:**

dist(*v*) = min { dist(*v*), dist(*u*) + *L(u, v)* }



After the *i*-th iteration of the first for loop (step 3), the *i*-th vertex following *s* has correct dist(.) value.