1 Quintic (5th order) spline

$$
\begin{aligned}
s_i(x) &= a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 + e_i(x-x_i)^4 + f_i(x-x_i)^5 \\
\dot{s}_i(x) &= b_i + 2c_i(x-x_i) + 3d_i(x-x_i)^2 + 4e_i(x-x_i)^3 + 5f_i(x-x_i)^4 \\
\ddot{s}_i(x) &= 2c_i + 6d_i(x-x_i) + 12e_i(x-x_i)^2 + 20f_i(x-x_i)^3 \\
\dddot{s}_i(x) &= 6d_i + 24e_i(x-x_i) + 60f_i(x-x_i)^2 \\
\ddddot{s}_i(x) &= 24e_i + 120f_i(x-x_i)
\end{aligned}
$$

2 Constraints

$$
\begin{aligned}
s_i(x_i) &= y_i \\
s_i(x_i) &= s_{i-1}(x_i) \\
\dot{s}_i(x_i) &= \dot{s}_{i-1}(x_i) \\
\ddot{s}_i(x_i) &= \ddot{s}_{i-1}(x_i) \\
\dddot{s}_i(x_i) &= \dddot{s}_{i-1}(x_i) \\
\ddddot{s}_i(x_i) &= \ddddot{s}_{i-1}(x_i)
\end{aligned}
$$

3 Derivation

$$
\begin{aligned}
h_i &= x_{i+1} - x_i \\
s_{i+1}(x_i) = a_{i+1} &= a_i + b_i h_i + c_i h_i^2 + d_i h_i^3 + e_i h_i^4 + f_i h_i^5 \\
\dot{s}_{i+1}(x_i) = b_{i+1} &= b_i + 2c_i h_i + 3d_i h_i^2 + 4e_i h_i^3 + 5f_i h_i^4 \\
\ddot{s}_{i+1}(x_i) = 2c_{i+1} &= 2c_i + 6d_i h_i + 12e_i h_i^2 + 20f_i h_i^3 \\
\dddot{s}_{i+1}(x_i) = 6d_{i+1} &= 6d_i + 24e_i h_i + 60f_i h_i^2 \\
\ddddot{s}_{i+1}(x_i) = 24e_{i+1} &= 24e_i + 120f_i h_i
\end{aligned}
$$

$$
\begin{aligned}
120 f_i h_i &= 24e_{i+1} - 24e_i \\
f_i &= \frac{e_{i+1} - e_i}{5h_i}
\end{aligned}
$$

$$
\begin{aligned}
6d_{i+1} &= 6d_i + 24e_i h_i + 60\frac{e_{i+1} - e_i}{5h_i} h_i^2 \\
6d_{i+1} &= 6d_i + 12e_i h_i + 12e_{i+1} h_i
\end{aligned}
$$

$$
\begin{aligned}
2c_{i+1} &= 2c_i + 6d_i h_i + 12e_i h_i^2 + 20\frac{e_{i+1} - e_i}{5h_i} h_i^3 \\
2c_{i+1} &= 2c_i + 6d_i h_i + 8e_i h_i^2 + 4e_{i+1} h_i^2
\end{aligned}
$$

$$b_{i+1} = b_i + 2c_ih_i + 3d_ih_i^2 + 4e_ih_i^3 + 5\frac{e_{i+1} - e_i}{5h_i}h_i^4$$
$$b_{i+1} = b_i + 2c_ih_i + 3d_ih_i^2 + 3e_ih_i^3 + e_{i+1}h_i^3$$

$$a_{i+1} = a_i + b_ih_i + c_ih_i^2 + d_ih_i^3 + e_ih_i^4 + \frac{e_{i+1} - e_i}{5h_i}h_i^5$$
$$a_{i+1} = a_i + b_ih_i + c_ih_i^2 + d_ih_i^3 + \frac{4}{5}e_ih_i^4 + \frac{e_{i+1}}{5}h_i^4$$

$$b_ih_i = a_{i+1} - a_i - c_ih_i^2 - d_ih_i^3 - \frac{4}{5}e_ih_i^4 - \frac{e_{i+1}}{5}h_i^4$$
$$b_i = \frac{a_{i+1}}{h_i} - \frac{a_i}{h_i} - c_ih_i - d_ih_i^2 - \frac{4}{5}e_ih_i^3 - \frac{e_{i+1}}{5}h_i^3$$

$$b_{i+1} = \frac{a_{i+2}}{h_{i+1}} - \frac{a_{i+1}}{h_{i+1}} - c_{i+1}h_{i+1} - d_{i+1}h_{i+1}^2 - \frac{4}{5}e_{i+1}h_{i+1}^3 - \frac{e_{i+2}}{5}h_{i+1}^3$$

$$\frac{a_{i+2}}{h_{i+1}} - \frac{a_{i+1}}{h_{i+1}} - c_{i+1}h_{i+1} - d_{i+1}h_{i+1}^2 - \frac{4}{5}e_{i+1}h_{i+1}^3 - \frac{e_{i+2}}{5}h_{i+1}^3 =$$
$$\frac{a_{i+1}}{h_i} - \frac{a_i}{h_i} - c_ih_i - d_ih_i^2 - \frac{4}{5}e_ih_i^3 - \frac{e_{i+1}}{5}h_i^3 + 2c_ih_i + 3d_ih_i^2 + 3e_ih_i^3 + e_{i+1}h_i^3$$

$$\frac{a_{i+2}}{h_{i+1}} - \frac{a_{i+1}}{h_{i+1}} - c_{i+1}h_{i+1} - d_{i+1}h_{i+1}^2 - \frac{4}{5}e_{i+1}h_{i+1}^3 - \frac{e_{i+2}}{5}h_{i+1}^3 =$$
$$\frac{a_{i+1}}{h_i} - \frac{a_i}{h_i} + c_ih_i + 2d_ih_i^2 + 2\frac{1}{5}e_ih_i^3 + \frac{4}{5}e_{i+1}h_i^3$$

4 Putting it in matrix form

$$\frac{a_{i+2}}{h_{i+1}} - \frac{a_{i+1}}{h_{i+1}} - \frac{a_{i+1}}{h_i} + \frac{a_i}{h_i} =$$
$$+c_ih_i + 2d_ih_i^2 + 2\frac{1}{5}e_ih_i^3 + \frac{4}{5}e_{i+1}h_i^3 + c_{i+1}h_{i+1} + d_{i+1}h_{i+1}^2 + \frac{4}{5}e_{i+1}h_{i+1}^3 + \frac{e_{i+2}}{5}h_{i+1}^3$$

$$\frac{a_{i+2}}{h_{i+1}} - \frac{a_{i+1}}{h_{i+1}} - \frac{a_{i+1}}{h_i} + \frac{a_i}{h_i} =$$
$$c_ih_i + c_{i+1}h_{i+1} + 2d_ih_i^2 + d_{i+1}h_{i+1}^2 + 2\frac{1}{5}e_ih_i^3 + \frac{4}{5}e_{i+1}h_i^3 + \frac{4}{5}e_{i+1}h_{i+1}^3 + \frac{e_{i+2}}{5}h_{i+1}^3$$

$$
\begin{aligned}
6d_{i+1} &= 6d_i + 12e_i h_i + 12e_{i+1} h_i \\
d_{i+1} &= d_i + 2e_i h_i + 2e_{i+1} h_i
\end{aligned}
$$

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2h_0 & 2h_0 & 0 & 0 & 0 \\
0 & 1 & 2h_0 & 2h_0 + 2h_1 & 2h_1 & 0 & 0 \\
0 & 1 & 2h_0 & 2h_0 + 2h_1 & 2h_1 + h_2 & 2h_2 & 0
\end{bmatrix}
\begin{bmatrix} c_0 \\ d_0 \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}
=
\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}
$$

$$
c_{i+1} = c_i + 3d_i h_i + 4e_i h_i^2 + 2e_{i+1} h_i^2
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 4h_0^2 & 2h_0^2 & 0 & 0 & 0 \\
1 & 0 & 4h_0^2 & 2h_0^2 + 4h_1^2 & 2h_1^2 & 0 & 0 \\
1 & 0 & 4h_0^2 & 2h_0^2 + 4h_1^2 & 2h_1^2 + 4h_2^2 & 2h_2^2 & 0 \\
1 & 0 & 4h_0^2 & 2h_0^2 + 4h_1^2 & 2h_1^2 + 4h_2^2 & 2h_2^2 + 4h_3^2 & 2h_3^2
\end{bmatrix}
\begin{bmatrix} c_0 \\ d_0 \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}
+
\begin{bmatrix} 0 \\ 3h_i d_0 \\ 3h_i(d_0 + d_1) \\ 3h_i(d_0 + d_1 + d_2) \\ 3h_i(d_0 + d_1 + d_2 + d_3) \end{bmatrix}
=
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}
$$

$$
\begin{bmatrix}
0 & 0 & 2\tfrac{1}{5}h_0^3 & \tfrac{4}{5}h_0^3 + \tfrac{4}{5}h_1^3 & \tfrac{1}{5}h_1^3 & 0 & 0 \\
0 & 0 & 0 & 2\tfrac{1}{5}h_1^3 & \tfrac{4}{5}h_1^3 + \tfrac{4}{5}h_2^3 & \tfrac{1}{5}h_2^3 & 0 \\
0 & 0 & 0 & 0 & 2\tfrac{1}{5}h_2^3 & \tfrac{4}{5}h_2^3 + \tfrac{4}{5}h_3^3 & \tfrac{1}{5}h_3^3
\end{bmatrix}
\begin{bmatrix} c_0 \\ d_0 \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}
+
\begin{bmatrix} c_0 h_0 + c_1 h_1 + 2d_0 h_0^2 + d_1 h_1^2 \\ c_1 h_1 + c_2 h_2 + 2d_1 h_1^2 + d_2 h_2^2 \\ c_2 h_2 + c_3 h_3 + 2d_2 h_2^2 + d_3 h_3^2 \end{bmatrix}
=
\begin{bmatrix} \frac{y_2}{h_1} - \\ \frac{y_3}{h_2} - \\ \frac{y_4}{h_3} - \end{bmatrix}
$$

5 Boundary conditions

$$
\begin{aligned}
\dot{s}_0 &= \dot{y}_0 \\
\ddot{s}_0 &= \ddot{y}_0 \\
\dot{s}_n &= \dot{y}_n \\
\ddot{s}_n &= \ddot{y}_n
\end{aligned}
$$

$$
\begin{aligned}
\dot{y}_0 = b_0 &= \frac{a_1}{h_0} - \frac{a_0}{h_1} - c_0 h_0 - d_0 h_0^2 - \frac{4}{5} e_0 h_0^3 - \frac{e_1}{5} h_0^3 \\
\frac{a_1}{h_0} - \frac{a_0}{h_1} - \dot{y}_0 &= c_0 h_0 + d_0 h_0^2 + \frac{4}{5} e_0 h_0^3 + \frac{e_1}{5} h_0^3
\end{aligned}
$$

$$\ddot{y}_0 = 2c_0$$

$$\begin{aligned}
\dot{s}_n = b_n &= b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}h_{n-1}^2 + 3e_{n-1}h_{n-1}^3 + e_n h_{n-1}^3 \\
b_{n-1} &= \frac{a_n}{h_{n-1}} - \frac{a_{n-1}}{h_{n-1}} - c_{n-1}h_{n-1} - d_{n-1}h_{n-1}^2 - \frac{4}{5}e_{n-1}h_{n-1}^3 - \frac{e_n}{5}h_{n-1}^3 \\
\dot{y}_n &= \frac{a_n}{h_{n-1}} - \frac{a_{n-1}}{h_{n-1}} + c_{n-1}h_{n-1} + 2d_{n-1}h_{n-1}^2 + 2\frac{1}{5}e_{n-1}h_{n-1}^3 + \frac{4}{5}e_n h_{n-1}^3 \\
c_{n-1}h_{n-1} + 2d_{n-1}h_{n-1}^2 + 2\frac{1}{5}e_{n-1}h_{n-1}^3 + \frac{4}{5}e_n h_{n-1}^3 &= \dot{y}_n - \frac{a_n}{h_{n-1}} + \frac{a_{n-1}}{h_{n-1}}
\end{aligned}$$

$$\begin{bmatrix} h_0 & h_0^2 & \frac{4}{5}h_0^3 & \frac{1}{5}h_0^3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\frac{1}{5}h_3^3 & \frac{4}{5}h_3^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ d_0 \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_3h_3 + 2d_3h_3^2 \\ 2c_4 \end{bmatrix} = \begin{bmatrix} \frac{a_1}{h_0} - \frac{a_0}{h_1} - \dot{y}_0 \\ \ddot{y}_0 \\ \dot{y}_3 - \frac{a_4}{h_3} + \frac{a_3}{h_3} \\ \ddot{y}_4 \end{bmatrix}$$