Introduction to Multiresolution Analysis (MRA)

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Outline

Introduction and Example

Multiresolution Analysis

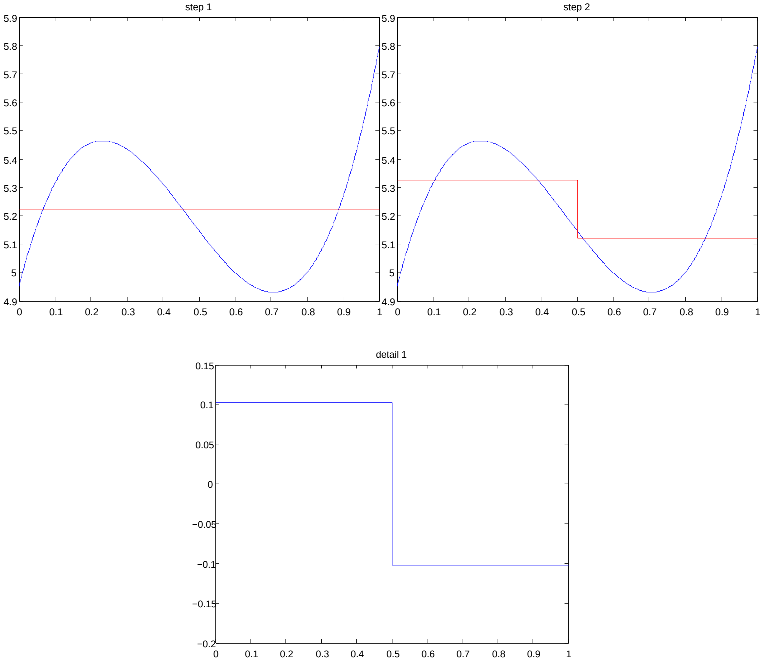
Discrete Wavelet Transform (DWT)

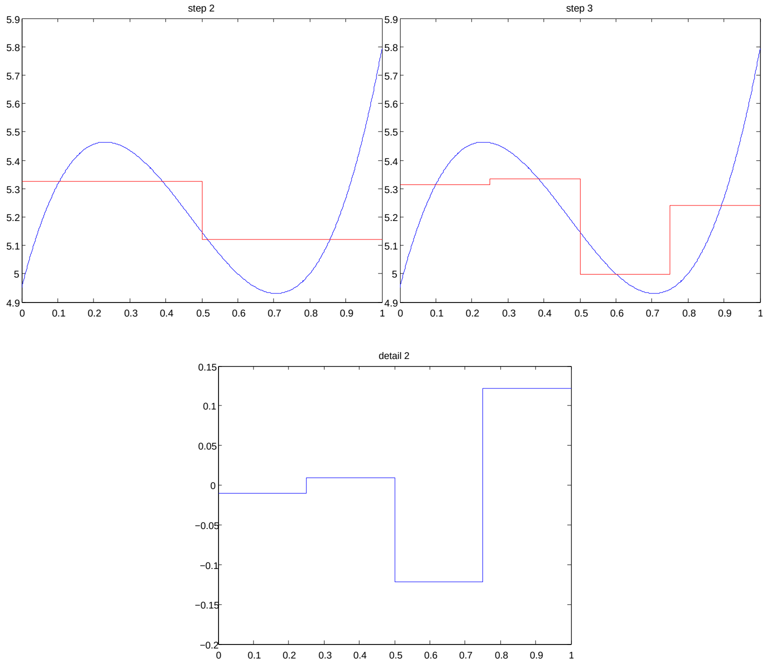
Finite Calculation

goal approximation of functions (e.g. signals, images, orbitals)

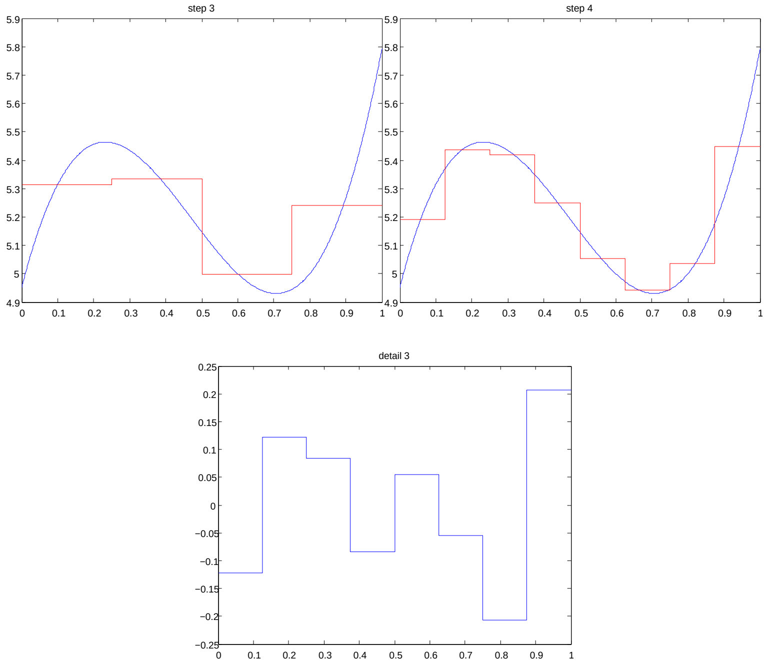
idea coarse approximation (trend) $+$ fine improvement (detail) with detail $<<$ trend

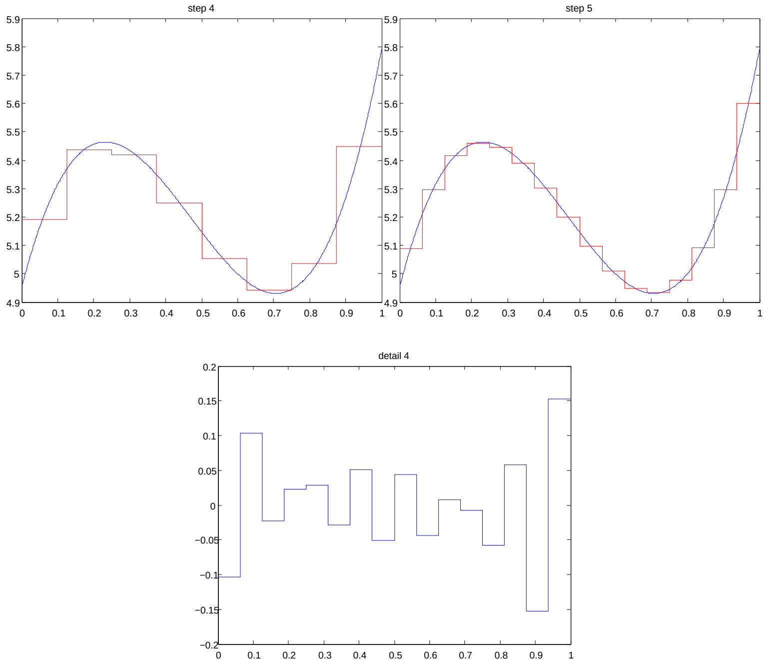
imagination building a house. start with big pieces and fill in with middle sized and at the end with little pieces

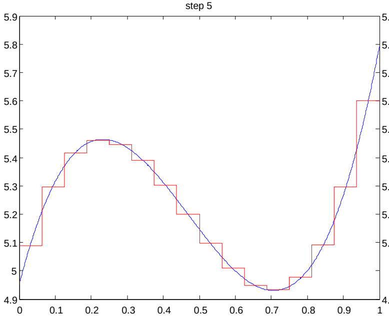

step 1
step 2
detail 1
5.9
5.8
5.7
5.6
5.5
5.4
5.3
5.2
5.1
5
4.9
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

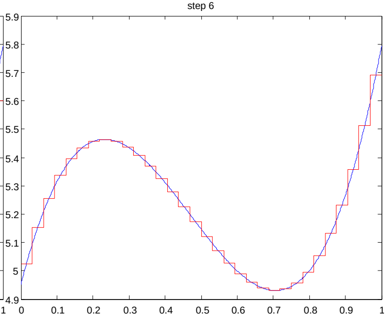

step 2
step 3
detail 2

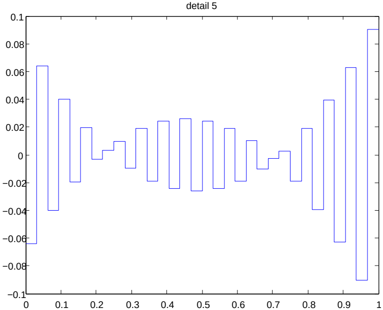
Introduction and Example



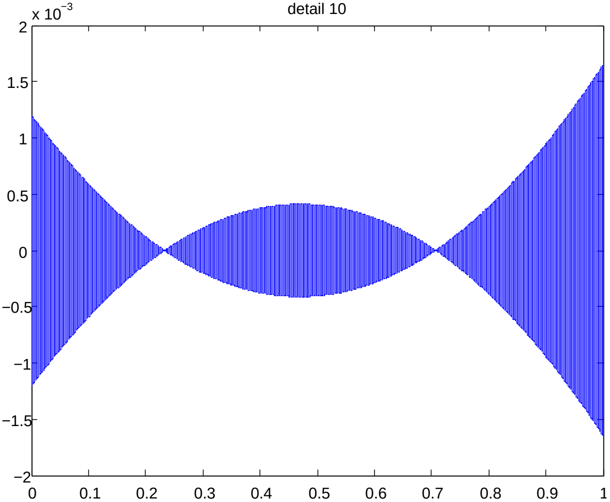

step 4
step 5
detail 4


step 5
5.9
5.8
5.7
5.6
5.5
5.4
5.3
5.2
5.1
5
4.9
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1


step 6
5.9
5.8
5.7
5.6
5.5
5.4
5.3
5.2
5.1
5
4.9
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

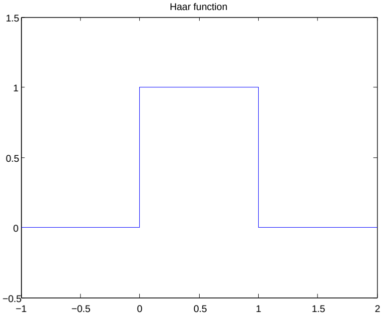

detail 5
0.1
0.08
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
−0.1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Details

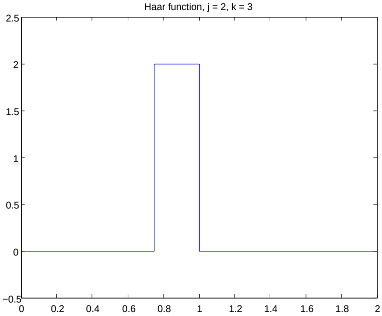


In the example a very simple set of basis functions is used:

$$\varphi(x) = \mathbf{1}_{[0,1)}(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{else} \end{cases}$$



$$\varphi_k^j(x) = 2^{j/2}\mathbf{1}_{[2^{-j}k, 2^{-j}(k+1))}(x) = \begin{cases} 2^{j/2}, & x \in [2^{-j}k, 2^{-j}(k+1)) \\ 0, & \text{else} \end{cases}$$
$$= 2^{j/2}\varphi(2^j x - k).$$

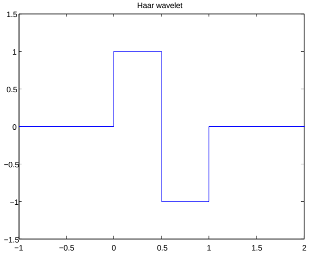


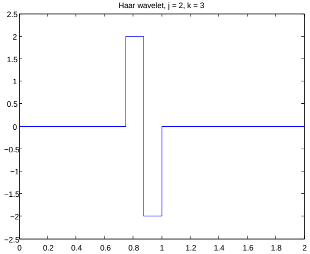
The details are given by the functions

$$\psi_k^j(x) = 2^{j/2}\psi(2^j x - k)$$

with

$$\psi(x) = \begin{cases} 1, & x \in [0, 0.5) \\ -1, & x \in [0.5, 1) \\ 0, & \text{else.} \end{cases}$$

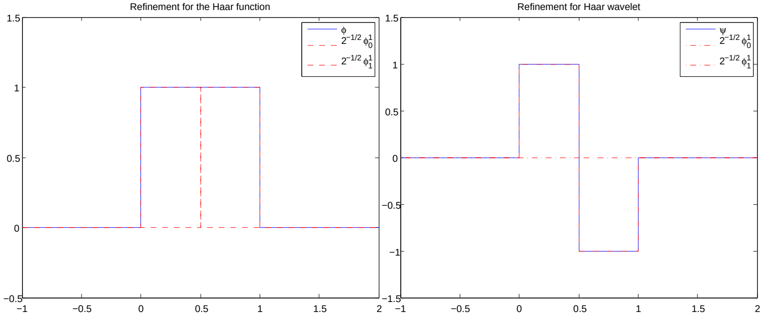




Refinement Equation

φ is called *Haar function* and ψ is called the *Haar wavelet*. They satisfy the so-called *refinement equations* :

$$\varphi_k^j(x) = 2^{-1/2}(\varphi_{2k}^{j+1}(x) + \varphi_{2k+1}^{j+1}(x))$$
$$\psi_k^j(x) = 2^{-1/2}(\varphi_{2k}^{j+1}(x) - \varphi_{2k+1}^{j+1}(x))$$



Multiresolution Analysis

Given a function

$$\varphi \in L_2(\mathbb{R}) = \{f \mid \|f\|_2 = \Big(\int_{\mathbb{R}} |f(x)|^2 dx\Big)^{1/2} < \infty\}.$$

We consider the shifts and dilatations of φ :

$$\varphi_k^j(x) = 2^{j/2}\varphi(2^j x - k), \quad j,k \in \mathbb{Z}.$$

We write

$$V_j = \overline{\operatorname{span}\{\varphi_k^j \mid k \in \mathbb{Z}\}}.$$

If every $f \in L_2(\mathbb{R})$ can be arbitrarily accurately approximated by φ_k^j's, i.e.

$$\overline{\bigcup_{j \geq j_0}^{\infty} V_j} = L_2(\mathbb{R})$$

holds and φ fulfills a refinement equation

$$\varphi = \sum_{k \in \mathbb{Z}} h_k \varphi_k^1$$

then we say that φ or the $V_j's$, respectively, build a multi resolution analysis (MRA).

Orthonormal Wavelets

Because of the refinement equation it holds $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$. Therefore there exists the orthogonal space W_j of V_j in V_{j+1}. Therefore

$$V_j \perp W_j, \qquad V_j \oplus W_j = V_{j+1}.$$

W_j is called the *detail space* or the *wavelet space* for V_j. A function ψ that satisfies

1. $\int_{\mathbb{R}} \psi(x)dx = 0$
2. $\{\psi(\cdot - k) \mid k \in \mathbb{Z}\}$ is an orthonormal basis of W_0

is called (orthonormal) wavelet or mother wavelet for the function φ. φ is also called scaling function or generator function or father wavelet. The Haar wavelet is a wavelet for the Haar function, for example.

If the translations of ψ are not orthonormal we need *biorthogonal wavelets*. But we do not go into details for this case.

Multi Levels

Let J be the level at which we want to approximate, i.e. we project into the space V_J. Then we have

$$\begin{aligned} V_J =& V_{J-1} \oplus W_{J-1} \\ =& V_{J-2} \oplus W_{J-2} \oplus W_{J-1} \\ & \vdots \\ =& V_0 \oplus \bigoplus_{j=0}^{J-1} W_j . \end{aligned}$$

If we go to infinity we have

$$L_2(\mathbb{R}) = \overline{V_0 \oplus \bigoplus_{j=0}^{\infty} W_j}.$$

We can imagine that we start at the very coarse level 0 and improve the result by successively adding the finer becoming details.

Filters

Because of the fact that $V_j \subset V_{j+1}$ and $W_j \subset V_{j+1}$ in a MRA we have the refinement equations

$$\varphi_k^j = \sum_l h_l \varphi_{2k+l}^{j+1}$$
$$\psi_k^j = \sum_l g_l \varphi_{2k+l}^{j+1}.$$

$(h_l)_{l\in\mathbb{Z}}$ and $(g_l)_{l\in\mathbb{Z}}$ are called filters. If φ has compact support h has finite length. If additionally ψ has compact support g has also finite length.

Reconstruction

If $\{\varphi_k^j \mid k \in \mathbb{Z}\}$ and $\{\psi_k^j \mid k \in \mathbb{Z}\}$ are orthonormal bases, i.e.

$$\begin{aligned} \langle \varphi_k^j, \varphi_l^j \rangle = \int_{\mathbb{R}} \varphi_k^j(x) \varphi_l^j(x) dx =& \delta_{k,l} \\ \langle \psi_k^j, \psi_l^j \rangle =& \delta_{k,l} \end{aligned}$$

we have the reconstruction

$$\varphi_k^{j+1} = \sum_l h_{k-2l} \varphi_l^j + \sum_l g_{k-2l} \psi_l^j.$$

There is also a very simple correlation between g and h:

$$g_k = (-1)^{1-k} h_{1-k}$$

Introduction and Example

Multiresolution Analysis

Discrete Wavelet Transform (DWT)

Finite Calculation

Orthogonal Projection

Assume that $\{\varphi_k \mid k \in \mathbb{Z}\}$ and $\{\psi_k \mid k \in \mathbb{Z}\}$ are orthonormal. Given $f \in L_2(\mathbb{R})$. We consider the orthogonal projections P_J onto V_J and Q_J onto W_J. Let

$$\lambda_k^j = \langle \varphi_k^j, f \rangle = \int_{\mathbb{R}} \varphi_k^j(x) f(x) dx \text{ and}$$
$$\mu_k^j = \langle \psi_k^j, f \rangle.$$

Then it holds

$$f_J = P_J f = \sum_k \lambda_k^J \varphi_k^J \in V_J$$
$$Q_J f = (P_{J+1} - P_J) f = \sum_k \mu_k^J \psi_k^J \in W_J.$$

We can easily get the coefficients in the coarser levels and the detail spaces by using successively the synthese equation.

$$
\begin{aligned}
f_J &= \sum_k \lambda_k^J \varphi_k^J \\
&= \sum_k \lambda_k^J \big(\sum_l h_{k-2l} \varphi_l^{J-1} + \sum_l g_{k-2l} \psi_l^{J-1}\big) \\
&= \sum_l \lambda_l^{J-1} \varphi_l^{J-1} + \sum_l \mu_l^{J-1} \psi_l^{J-1} \\
&\vdots \\
&= \sum_l \lambda_l^0 \varphi_l^0 + \sum_{j=0}^{J-1} \sum_l \mu_l^j \psi_l^j
\end{aligned}
$$

The occurring coefficients are given by

$$\lambda_l^j = \sum_k \lambda_k^{j+1} h_{k-2l} \text{ and} \tag{1}$$

$$\mu_l^j = \sum_k \lambda_k^{j+1} g_{k-2l}, \quad j = 0, \ldots, J-1. \tag{2}$$

(1) and (2) can be written as

$$\begin{pmatrix} \underline{\lambda}^j \\ \underline{\mu}^j \end{pmatrix} = T \underline{\lambda}^{j+1}.$$

Discrete Inverse Wavelet Tansform (DIWT)

Given the coefficients λ_k^j and μ_k^j we get the cofficients λ_k^{j+1} back by

$$\lambda_k^{j+1} = \sum_l h_{k-2l} \lambda_l^j + \sum_l g_{k-2l} \mu_l^j .$$

This describes again a linear transformation

$$\underline{\lambda}^{j+1} = T^{-1} \begin{pmatrix} \underline{\lambda}^j \\ \underline{\mu}^j \end{pmatrix} = T^T \begin{pmatrix} \underline{\lambda}^j \\ \underline{\mu}^j \end{pmatrix} .$$

Note: Because of the D(I)WT it is not really necessary to know explicitly the functions φ and ψ. It is sufficient to know the filters h and g.

Schemes

DWT	DIWT
$$\begin{array}{ccc} \underline{\lambda}^J & & \\ \downarrow & \searrow & \\ \underline{\lambda}^{J-1} & & \underline{\mu}^{J-1} \\ \downarrow & \searrow & \\ \underline{\lambda}^{J-2} & & \underline{\mu}^{J-2} \\ \downarrow & \searrow & \\ \vdots & & \vdots \\ \downarrow & \searrow & \\ \underline{\lambda}^1 & & \underline{\mu}^1 \\ \downarrow & \searrow & \\ \underline{\lambda}^0 & & \underline{\mu}^0 \end{array}$$	$$\begin{array}{ccc} \underline{\lambda}^0 & & \underline{\mu}^0 \\ \downarrow & \swarrow & \\ \underline{\lambda}^1 & & \underline{\mu}^1 \\ \downarrow & \swarrow & \\ \underline{\lambda}^2 & & \underline{\mu}^2 \\ \downarrow & \swarrow & \\ \vdots & & \vdots \\ \downarrow & \swarrow & \\ \underline{\lambda}^{J-1} & & \underline{\mu}^{J-1} \\ \downarrow & \swarrow & \\ \underline{\lambda}^J & & \end{array}$$

In the previous chapter we still had infinite many coefficients λ_k^j, μ_k^j. For practical calculations we have to make their size finite. There are mainly two ways to do this.

1. **Zeropadding:** You consider only a finite region and assume that all coefficients out of this region are zero.
2. **Periodizing:** You assume that your data is periodic and you calculate only on one period.

Then the D(I)WT is a finite transform if the filters are finite. The number of arithmetic operations for one transformation is $\mathcal{O}(N)$ if N is the size of the input data. To transform on J levels you have $\mathcal{O}(J \cdot N)$ arithmetic operations.

References

S. Mallat, A Wavelet Tour of Signal Processing, 2nd. ed., Academic Press, 1999