

**Diophantine Equations**
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by Peter

- 1** One of Euler's conjectures was disproved in the 1980s by three American Mathematicians when they showed that there is a positive integer  $n$  such that

$$n^5 = 133^5 + 110^5 + 84^5 + 27^5.$$

 Find the value of  $n$ .

- 2** The number 21982145917308330487013369 is the thirteenth power of a positive integer. Which positive integer?

- 3** Does there exist a solution to the equation

$$x^2 + y^2 + z^2 + u^2 + v^2 = xyzuv - 65$$

 in integers with  $x, y, z, u, v$  greater than 1998?

- 4** Find all pairs  $(x, y)$  of positive rational numbers such that  $x^2 + 3y^2 = 1$ .

- 5** Find all pairs  $(x, y)$  of rational numbers such that  $y^2 = x^3 - 3x + 2$ .

- 6** Show that there are infinitely many pairs  $(x, y)$  of rational numbers such that  $x^3 + y^3 = 9$ .

- 7** Determine all pairs  $(x, y)$  of positive integers satisfying the equation

$$(x + y)^2 - 2(xy)^2 = 1.$$

- 8** Show that the equation

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integral solutions.

- 9** Determine all integers  $a$  for which the equation

$$x^2 + axy + y^2 = 1$$

 has infinitely many distinct integer solutions  $x, y$ .

- 10** Prove that there are unique positive integers  $a$  and  $n$  such that

$$a^{n+1} - (a+1)^n = 2001.$$

**11** Find all  $(x, y, n) \in \mathbb{N}^3$  such that  $\gcd(x, n+1) = 1$  and  $x^n + 1 = y^{n+1}$ .

**12** Find all  $(x, y, z) \in \mathbb{N}^3$  such that  $x^4 - y^4 = z^2$ .

**13** Find all pairs  $(x, y)$  of positive integers that satisfy the equation

$$y^2 = x^3 + 16.$$

**14** Show that the equation  $x^2 + y^5 = z^3$  has infinitely many solutions in integers  $x, y, z$  for which  $xyz \neq 0$ .

**15** Prove that there are no integers  $x$  and  $y$  satisfying  $x^2 = y^5 - 4$ .

**16** Find all pairs  $(a, b)$  of different positive integers that satisfy the equation  $W(a) = W(b)$ , where  $W(x) = x^4 - 3x^3 + 5x^2 - 9x$ .

**17** Find all positive integers  $n$  for which the equation

$$a + b + c + d = n\sqrt{abcd}$$

has a solution in positive integers.

**18** Determine all positive integer solutions  $(x, y, z, t)$  of the equation

$$(x+y)(y+z)(z+x) = xyzt$$

for which  $\gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1$ .

**19** Find all  $(x, y, z, n) \in \mathbb{N}^4$  such that  $x^3 + y^3 + z^3 = nx^2y^2z^2$ .

**20** Determine all positive integers  $n$  for which the equation

$$x^n + (2+x)^n + (2-x)^n = 0$$

has an integer as a solution.

**21** Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except  $a = b = c = n = 0$ .

**22** Find all integers  $a, b, c, x, y, z$  such that

$$a + b + c = xyz, \quad x + y + z = abc, \quad a \geq b \geq c \geq 1, \quad x \geq y \geq z \geq 1.$$

**23** Find all  $(x, y, z) \in \mathbb{Z}^3$  such that  $x^3 + y^3 + z^3 = x + y + z = 3$ .

**24** Prove that if  $n$  is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n.$$

has a solution in integers  $(x, y)$ , then it has at least three such solutions. Show that the equation has no solutions in integers when  $n = 2891$ .

**25** What is the smallest positive integer  $t$  such that there exist integers  $x_1, x_2, \dots, x_t$  with

$$x_1^3 + x_2^3 + \dots + x_t^3 = 2002^{2002} \quad ?$$

**26** Solve in integers the following equation

$$n^{2002} = m(m+n)(m+2n) \cdots (m+2001n).$$

**27** Prove that there exist infinitely many positive integers  $n$  such that  $p = nr$ , where  $p$  and  $r$  are respectively the semi-perimeter and the inradius of a triangle with integer side lengths.

**28** Let  $a, b, c$  be positive integers such that  $a$  and  $b$  are relatively prime and  $c$  is relatively prime either to  $a$  or  $b$ . Prove that there exist infinitely many triples  $(x, y, z)$  of distinct positive integers such that

$$x^a + y^b = z^c.$$

**29** Find all pairs of integers  $(x, y)$  satisfying the equality

$$y(x^2 + 36) + x(y^2 - 36) + y^2(y - 12) = 0.$$

**30** Let  $a, b, c$  be given integers,  $a > 0$ ,  $ac - b^2 = p$  a squarefree positive integer. Let  $M(n)$  denote the number of pairs of integers  $(x, y)$  for which  $ax^2 + bxy + cy^2 = n$ . Prove that  $M(n)$  is finite and  $M(n) = M(p^k \cdot n)$  for every integer  $k \geq 0$ .

- 31 Determine all integer solutions of the system

$$2uv - xy = 16,$$

$$xv - yu = 12.$$

- 32 Let  $n$  be a natural number. Solve in whole numbers the equation

$$x^n + y^n = (x - y)^{n+1}.$$

- 33 Does there exist an integer such that its cube is equal to  $3n^2 + 3n + 7$ , where  $n$  is integer?

- 34 Are there integers  $m$  and  $n$  such that  $5m^2 - 6mn + 7n^2 = 1985$ ?

- 35 Find all cubic polynomials  $x^3 + ax^2 + bx + c$  admitting the rational numbers  $a$ ,  $b$  and  $c$  as roots.

- 36 Prove that the equation  $a^2 + b^2 = c^2 + 3$  has infinitely many integer solutions  $(a, b, c)$ .

- 37 Prove that for each positive integer  $n$  there exist odd positive integers  $x_n$  and  $y_n$  such that  $x_n^2 + 7y_n^2 = 2^n$ .

- 38 Suppose that  $p$  is an odd prime such that  $2p + 1$  is also prime. Show that the equation  $x^p + 2y^p + 5z^p = 0$  has no solutions in integers other than  $(0, 0, 0)$ .

- 39 Let  $A, B, C, D, E$  be integers,  $B \neq 0$  and  $F = AD^2 - BCD + B^2E \neq 0$ . Prove that the number  $N$  of pairs of integers  $(x, y)$  such that

$$Ax^2 + Bxy + Cx + Dy + E = 0,$$

satisfies  $N \leq 2d(|F|)$ , where  $d(n)$  denotes the number of positive divisors of positive integer  $n$ .

- 40 Determine all pairs of rational numbers  $(x, y)$  such that

$$x^3 + y^3 = x^2 + y^2.$$

- 41 Suppose that  $A = 1, 2$ , or  $3$ . Let  $a$  and  $b$  be relatively prime integers such that  $a^2 + Ab^2 = s^3$  for some integer  $s$ . Then, there are integers  $u$  and  $v$  such that  $s = u^2 + Av^2$ ,  $a = u^3 - 3Avu^2$ , and  $b = 3u^2v - Av^3$ .

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- 42** Find all integers  $a$  for which  $x^3 - x + a$  has three integer roots.
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- 43** Find all solutions in integers of  $x^3 + 2y^3 = 4z^3$ .
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- 44** For all  $n \in \mathbb{N}$ , show that the number of integral solutions  $(x, y)$  of
- $$x^2 + xy + y^2 = n$$
- is finite and a multiple of 6.
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- 45** Show that there cannot be four squares in arithmetical progression.
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- 46** Let  $a, b, c, d, e, f$  be integers such that  $b^2 - 4ac > 0$  is not a perfect square and  $4acf + bde - ae^2 - cd^2 - fb^2 \neq 0$ . Let
- $$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$$
- Suppose that  $f(x, y) = 0$  has an integral solution. Show that  $f(x, y) = 0$  has infinitely many integral solutions.
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- 47** Show that the equation  $x^4 + y^4 + 4z^4 = 1$  has infinitely many rational solutions.
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- 48** Solve the equation  $x^2 + 7 = 2^n$  in integers.
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- 49** Show that the only solutions of the equation  $x^3 - 3xy^2 - y^3 = 1$  are given by  $(x, y) = (1, 0), (0, -1), (-1, 1), (1, -3), (-3, 2), (2, 1)$ .
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- 50** Show that the equation  $y^2 = x^3 + 2a^3 - 3b^2$  has no solution in integers if  $ab \neq 0$ ,  $a \not\equiv 1 \pmod{3}$ , 3 does not divide  $b$ ,  $a$  is odd if  $b$  is even, and  $p = t^2 + 27u^2$  has a solution in integers  $t, u$  if  $p|a$  and  $p \equiv 1 \pmod{3}$ .
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- 51** Prove that the product of five consecutive positive integers is never a perfect square.
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- 52** Do there exist two right-angled triangles with integer length sides that have the lengths of exactly two sides in common?
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- 53** Suppose that  $a, b$ , and  $p$  are integers such that  $b \equiv 1 \pmod{4}$ ,  $p \equiv 3 \pmod{4}$ ,  $p$  is prime, and if  $q$  is any prime divisor of  $a$  such that  $q \equiv 3 \pmod{4}$ , then  $q^p | a^2$  and  $p$  does not divide  $q - 1$  (if  $q = p$ , then also  $q|b$ ). Show that the equation
- $$x^2 + 4a^2 = y^p - b^p$$
- has no solutions in integers.
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- 54** Show that the number of integral-sided right triangles whose ratio of area to semi-perimeter is  $p^m$ , where  $p$  is a prime and  $m$  is an integer, is  $m + 1$  if  $p = 2$  and  $2m + 1$  if  $p \neq 2$ .
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- 55** Given that
- $$34! = 95232799cd96041408476186096435ab000000_{(10)},$$
- determine the digits  $a, b, c$ , and  $d$ .
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- 56** Prove that the equation  $\prod_{cyc}(x_1 - x_2) = \prod_{cyc}(x_1 - x_3)$  has a solution in natural numbers where all  $x_i$  are different.
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- 57** Show that the equation  $\binom{n}{k} = m^l$  has no integral solution with  $l \geq 2$  and  $4 \leq k \leq n - 4$ .
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- 58** Solve in positive integers the equation  $10^a + 2^b - 3^c = 1997$ .
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- 59** Solve the equation  $28^x = 19^y + 87^z$ , where  $x, y, z$  are integers.
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- 60** Show that the equation  $x^7 + y^7 = 1998^z$  has no solution in positive integers.
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- 61** Solve the equation  $2^x - 5 = 11^y$  in positive integers.
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- 62** Solve the equation  $7^x - 3^y = 4$  in positive integers.
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- 63** Show that  $|12^m - 5^n| \geq 7$  for all  $m, n \in \mathbb{N}$ .
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- 64** Show that there is no positive integer  $k$  for which the equation
- $$(n - 1)! + 1 = n^k$$
- is true when  $n$  is greater than 5.
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- 65** Determine all pairs  $(x, y)$  of integers such that
- $$(19a + b)^{18} + (a + b)^{18} + (19b + a)^{18}$$
- is a nonzero perfect square.
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- 66** Let  $b$  be a positive integer. Determine all 2002-tuples of non-negative integers  $(a_1, a_2, \dots, a_{2002})$  satisfying
- $$\sum_{j=1}^{2002} a_j^{a_j} = 2002b^b.$$
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67 Is there a positive integer  $m$  such that the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers  $a, b, c$ ?

68 Consider the system

$$x + y = z + u,$$

$$2xy = zu.$$

Find the greatest value of the real constant  $m$  such that  $m \leq \frac{x}{y}$  for any positive integer solution  $(x, y, z, u)$  of the system, with  $x \geq y$ .

69 Determine all positive rational numbers  $r \neq 1$  such that  $r^{-1}\sqrt[r]{r}$  is rational.

70 Show that the equation  $\{x^3\} + \{y^3\} = \{z^3\}$  has infinitely many rational non-integer solutions.

71 Let  $n$  be a positive integer. Prove that the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

72 Find all pairs  $(x, y)$  of positive rational numbers such that  $x^y = y^x$ .

73 Find all pairs  $(a, b)$  of positive integers that satisfy the equation

$$a^{b^2} = b^a.$$

74 Find all pairs  $(a, b)$  of positive integers that satisfy the equation

$$a^{a^a} = b^b.$$

75 Let  $a, b$ , and  $x$  be positive integers such that  $x^{a+b} = a^b b$ . Prove that  $a = x$  and  $b = x^x$ .

76 Find all pairs  $(m, n)$  of integers that satisfy the equation

$$(m - n)^2 = \frac{4mn}{m + n - 1}.$$

- 77 Find all pairwise relatively prime positive integers  $l, m, n$  such that

$$(l + m + n) \left( \frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right)$$

is an integer.

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- 78 Let  $x, y$ , and  $z$  be integers with  $z > 1$ . Show that

$$(x+1)^2 + (x+2)^2 + \cdots + (x+99)^2 \neq y^z.$$

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- 79 Find all positive integers  $m$  and  $n$  for which

$$1! + 2! + 3! + \cdots + n! = m^2$$

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- 80 Prove that if  $a, b, c, d$  are integers such that  $d = (a + \sqrt[3]{2}b + \sqrt[3]{4}c)^2$  then  $d$  is a perfect square.
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- 81 Find a pair of relatively prime four digit natural numbers  $A$  and  $B$  such that for all natural numbers  $m$  and  $n$ ,  $|A^m - B^n| \geq 400$ .
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- 82 Find all triples  $(a, b, c)$  of positive integers to the equation

$$a!b! = a! + b! + c!.$$

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- 83 Find all pairs  $(a, b)$  of positive integers such that

$$(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 49 + 20\sqrt[3]{6}.$$

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- 84 For what positive numbers  $a$  is

$$\sqrt[3]{2 + \sqrt{a}} + \sqrt[3]{2 - \sqrt{a}}$$

an integer?

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- 85 Find all integer solutions to  $2(x^5 + y^5 + 1) = 5xy(x^2 + y^2 + 1)$ .
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- 86 A triangle with integer sides is called Heronian if its area is an integer. Does there exist a Heronian triangle whose sides are the arithmetic, geometric and harmonic means of two positive integers?
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**87** What is the smallest perfect square that ends in 9009?

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**88** (Leo Moser) Show that the Diophantine equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = 1$$

has at least one solution for every positive integers  $n$ .

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**89** Prove that the number  $99999 + 111111\sqrt{3}$  cannot be written in the form  $(A + B\sqrt{3})^2$ , where  $A$  and  $B$  are integers.

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**90** Find all triples of positive integers  $(x, y, z)$  such that

$$(x + y)(1 + xy) = 2^z.$$

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**91** If  $R$  and  $S$  are two rectangles with integer sides such that the perimeter of  $R$  equals the area of  $S$  and the perimeter of  $S$  equals the area of  $R$ , then we call  $R$  and  $S$  a friendly pair of rectangles. Find all friendly pairs of rectangles.

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