The Geometry of Numbers

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1 Does there exist a convex pentagon, all of whose vertices are lattice points in the plane, with no lattice point in the interior?

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2 Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

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3 Prove no three lattice points in the plane form an equilateral triangle.

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4 The sidelengths of a polygon with 1994 sides are $a_i = \sqrt{i^2+4}\ \ (i=1,2,\cdots,1994)$. Prove that its vertices are not all on lattice points.

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5 A triangle has lattice points as vertices and contains no other lattice points. Prove that its area is $\frac{1}{2}$.

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6 Let R be a convex region symmetrical about the origin with area greater than 4. Show that R must contain a lattice point different from the origin.

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7 Show that the number $r(n)$ of representations of n as a sum of two squares has π as arithmetic mean, that is

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} r(m) = \pi.$$

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8 Prove that on a coordinate plane it is impossible to draw a closed broken line such that - coordinates of each vertex are rational, - the length of its every edge is equal to 1, - the line has an odd number of vertices.

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9 Prove that if a lattice parallellogram contains an odd number of lattice points, then its centroid.

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10 Prove that if a lattice triangle has no lattice points on its boundary in addition to its vertices, and one point in its interior, then this interior point is its center of gravity.

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11 Prove that if a lattice parallelogram contains at most three lattice points in addition to its vertices, then those are on one of the diagonals.

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12 Find coordinates of a set of eight non-collinear planar points so that each has an integral distance from others.
