Rational Numbers

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1 Suppose that a rectangle with sides a and b is arbitrarily cut into n squares with sides $x_1, \ldots, x_n$. Show that $\frac{x_i}{a} \in \mathbb{Q}$ and $\frac{x_i}{b} \in \mathbb{Q}$ for all $i \in \{1, \cdots, n\}$.

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2 Find all x and y which are rational multiples of π with $0 < x < y < \frac{\pi}{2}$ and $\tan x + \tan y = 2$.

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3 Let α be a rational number with $0 < \alpha < 1$ and $\cos(3\pi\alpha) + 2\cos(2\pi\alpha) = 0$. Prove that $\alpha = \frac{2}{3}$.

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4 Suppose that $\tan \alpha = \frac{p}{q}$, where p and q are integers and $q \neq 0$. Prove the number $\tan \beta$ for which $\tan 2\beta = \tan 3\alpha$ is rational only when $p^2 + q^2$ is the square of an integer.

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5 Prove that there is no positive rational number x such that

$$x^{\lfloor x \rfloor} = \frac{9}{2}.$$

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6 Let x, y, z non-zero real numbers such that xy, yz, zx are rational. - Show that the number $x^2 + y^2 + z^2$ is rational. - If the number $x^3 + y^3 + z^3$ is also rational, show that x, y, z are rational.

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7 If x is a positive rational number, show that x can be uniquely expressed in the form

$$x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots,$$

where $a_1 a_2, \cdots$ are integers, $0 \leq a_n \leq n - 1$ for $n > 1$, and the series terminates. Show also that x can be expressed as the sum of reciprocals of different integers, each of which is greater than 10^6.

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8 Find all polynomials W with real coefficients possessing the following property: if $x + y$ is a rational number, then $W(x) + W(y)$ is rational.

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9 Prove that every positive rational number can be represented in the form

$$\frac{a^3 + b^3}{c^3 + d^3}$$

for some positive integers a, b, c, and d.

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10 The set S is a finite subset of $[0,1]$ with the following property: for all $s \in S$, there exist $a, b \in S \cup \{0,1\}$ with $a, b \neq s$ such that $s = \frac{a+b}{2}$. Prove that all the numbers in S are rational.

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11 Let $S = \{x_0, x_1, \cdots, x_n\} \subset [0,1]$ be a finite set of real numbers with $x_0 = 0$ and $x_1 = 1$, such that every distance between pairs of elements occurs at least twice, except for the distance 1. Prove that all of the x_i are rational.

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12 Does there exist a circle and an infinite set of points on it such that the distance between any two points of the set is rational?

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13 Prove that numbers of the form

$$\frac{a_1}{1!} + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots,$$

where $0 \leq a_i \leq i-1$ $(i = 2, 3, 4, \cdots)$ are rational if and only if starting from some i on all the a_i's are either equal to 0 (in which case the sum is finite) or all are equal to $i-1$.

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14 Let k and m be positive integers. Show that

$$S(m,k) = \sum_{n=1}^{\infty} \frac{1}{n(mn+k)}$$

is rational if and only if m divides k.

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15 Find all rational numbers k such that $0 \leq k \leq \frac{1}{2}$ and $\cos k\pi$ is rational.

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16 Prove that for any distinct rational numbers a, b, c, the number

$$\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2}$$

is the square of some rational number.
