Primes and Composite Numbers

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1 Prove that the number $512^3 + 675^3 + 720^3$ is composite.

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2 Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that $ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime.

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3 Find the sum of all distinct positive divisors of the number 104060401.

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4 Prove that 1280000401 is composite.

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5 Prove that $\frac{5^{125}-1}{5^{25}-1}$ is a composite number.

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6 Find a factor of $2^{33} - 2^{19} - 2^{17} - 1$ that lies between 1000 and 5000.

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7 Show that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for all $n \in \mathbb{N}_0$.

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8 Show that for all integer $k > 1$, there are infinitely many natural numbers n such that $k \cdot 2^{2^n} + 1$ is composite.

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9 Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$). Is it possible after 1996 such steps to have numbers a, b, c and d such that the numbers $|bc - ad|$, $|ac - bd|$ and $|ab - cd|$ are primes?

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10 Represent the number $989 \cdot 1001 \cdot 1007 + 320$ as a product of primes.

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11 In 1772 Euler discovered the curious fact that $n^2 + n + 41$ is prime when n is any of $0, 1, 2, \cdots, 39$. Show that there exist 40 consecutive integer values of n for which this polynomial is not prime.

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12 Show that there are infinitely many primes.

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13 Find all natural numbers n for which every natural number whose decimal representation has $n - 1$ digits 1 and one digit 7 is prime.

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14 Prove that there do not exist polynomials P and Q such that

$$\pi(x) = \frac{P(x)}{Q(x)}$$

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for all $x \in \mathbb{N}$.

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15 Show that there exist two consecutive squares such that there are at least 1000 primes between them.

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16 Prove that for any prime p in the interval $\left]n, \frac{4n}{3}\right]$, p divides

$$\sum_{j=0}^{n} \binom{n}{j}^4.$$

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17 Let a, b, and n be positive integers with $\gcd(a, b) = 1$. Without using Dirichlet's theorem, show that there are infinitely many $k \in \mathbb{N}$ such that $\gcd(ak + b, n) = 1$.

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18 Without using Dirichlet's theorem, show that there are infinitely many primes ending in the digit 9.

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19 Let p be an odd prime. Without using Dirichlet's theorem, show that there are infinitely many primes of the form $2pk + 1$.

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20 Verify that, for each $r \geq 1$, there are infinitely many primes p with $p \equiv 1 \pmod{2^r}$.

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21 Prove that if p is a prime, then $p^p - 1$ has a prime factor that is congruent to 1 modulo p.

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22 Let p be a prime number. Prove that there exists a prime number q such that for every integer n, $n^p - p$ is not divisible by q.

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23 Let $p_1 = 2, p_2 = 3, p_3 = 5, \cdots, p_n$ be the first n prime numbers, where $n \geq 3$. Prove that

$$\frac{1}{{p_1}^2} + \frac{1}{{p_2}^2} + \cdots + \frac{1}{{p_n}^2} + \frac{1}{p_1 p_2 \cdots p_n} < \frac{1}{2}.$$

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24 Let p_n again denote the nth prime number. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges.

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25 Prove that $\ln n \geq k \ln 2$, where n is a natural number and k is the number of distinct primes that divide n.

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26 Find the smallest prime which is not the difference (in some order) of a power of 2 and a power of 3.

27 Prove that for each positive integer n, there exist n consecutive positive integers none of which is an integral power of a prime number.

28 Show that $n^{\pi(2n)-\pi(n)} < 4^n$ for all positive integer n.

29 Let s_n denote the sum of the first n primes. Prove that for each n there exists an integer whose square lies between s_n and s_{n+1}.

30 Given an odd integer $n > 3$, let k and t be the smallest positive integers such that both $kn+1$ and tn are squares. Prove that n is prime if and only if both k and t are greater than $\frac{n}{4}$

31 Suppose n and r are nonnegative integers such that no number of the form $n^2+r-k(k+1)$ $(k \in \mathbb{N})$ equals to -1 or a positive composite number. Show that $4n^2+4r+1$ is 1, 9, or prime.

32 Let $n \geq 5$ be an integer. Show that n is prime if and only if $n_i n_j \neq n_p n_q$ for every partition of n into 4 integers, $n = n_1 + n_2 + n_3 + n_4$, and for each permutation (i, j, p, q) of $(1, 2, 3, 4)$.

33 Prove that there are no positive integers a and b such that for all different primes p and q greater than 1000, the number $ap+bq$ is also prime.

34 Let p_n denote the nth prime number. For all $n \geq 6$, prove that

$$\pi\left(\sqrt{p_1 p_2 \cdots p_n}\right) > 2n.$$

35 There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001!+2$, $1001!+3$, $\cdots$, $1001!+1001$. Does there exist a block of 1000 consecutive positive integers containing exactly five prime numbers?

36 Prove that there are infinitely many twin primes if and only if there are infinitely many integers that cannot be written in any of the following forms:

$$6uv+u+v, \quad 6uv+u-v, \quad 6uv-u+v, \quad 6uv-u-v,$$

for some positive integers u and v.

37 It's known that there is always a prime between n and $2n-7$ for all $n \geq 10$. Prove that, with the exception of 1, 4, and 6, every natural number can be written as the sum of distinct primes.

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38 Prove that if $c > \frac{8}{3}$, then there exists a real number θ such that $\lfloor \theta^{c^n} \rfloor$ is prime for every positive integer n.

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39 Let c be a nonzero real number. Suppose that $g(x) = c_0 x^r + c_1 x^{r-1} + \cdots + c_{r-1} x + c_r$ is a polynomial with integer coefficients. Suppose that the roots of $g(x)$ are $b_1, \cdots, b_r$. Let k be a given positive integer. Show that there is a prime p such that $p > \max(k, |c|, |c_r|)$, and moreover if t is a real number between 0 and 1, and j is one of $1, \cdots, r$, then

$$|(\ c^r \ b_j g(tb_j) \ )^p e^{(1-t)b}| < \frac{(p-1)!}{2r}.$$

Furthermore, if

$$f(x) = \frac{e^{rp-1} x^{p-1} (g(x))^p}{(p-1)!}$$

then

$$\left| \sum_{j=1}^{r} \int_0^1 e^{(1-t)b_j} f(tb_j) dt \right| \leq \frac{1}{2}.$$

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40 Prove that there do not exist eleven primes, all less than 20000, which form an arithmetic progression.

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41 Show that n is prime iff

$$\lim_{r\to\infty} \lim_{s\to\infty} \lim_{t\to\infty} \sum_{u=0}^{s} \left(1 - \left(\cos \frac{(u!)^r \pi}{n}\right)^{2t}\right) = n$$

PS : I posted it because it's in the PDF file but not here ...

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