Additive Number Theory

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by Peter

1 Show that any integer can be expressed as a sum of two squares and a cube.

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2 Show that each integer n can be written as the sum of five perfect cubes (not necessarily positive).

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3 Prove that infinitely many positive integers cannot be written in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11},$$

where $x_1, x_2, x_3, x_4, x_5 \in \mathbb{N}$.

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4 Determine all positive integers that are expressible in the form

$$a^2 + b^2 + c^2 + c,$$

where a, b, c are integers.

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5 Show that any positive rational number can be represented as the sum of three positive rational cubes.

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6 Show that every integer greater than 1 can be written as a sum of two square-free integers.

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7 Prove that every integer $n \geq 12$ is the sum of two composite numbers.

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8 Prove that any positive integer can be represented as an aggregate of different powers of 3, the terms in the aggregate being combined by the signs $+$ and $-$ appropriately chosen.

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9 The integer 9 can be written as a sum of two consecutive integers: 9=4+5. Moreover it can be written as a sum of (more than one) consecutive positive integers in exactly two ways, namely 9=4+5= 2+3+4. Is there an integer which can be written as a sum of 1990 consecutive integers and which can be written as a sum of (more than one) consecutive positive integers in exactly 1990 ways?

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10 For each positive integer $n,\ S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n),\ n^2$ can be written as the sum of k positive squares. - Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$. - Find an integer n such that $S(n) = n^2 - 14$. - Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

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11 For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4, 2+2, 2+1+1, 1+1+1+1.$$

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

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12 The positive function $p(n)$ is defined as the number of ways that the positive integer n can be written as a sum of positive integers. Show that, for all positive integers $n \geq 2$,

$$2^{\lfloor \sqrt{n} \rfloor} < p(n) < n^{3\lfloor \sqrt{n} \rfloor}.$$

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13 Let $a_1 = 1$, $a_2 = 2$, a_3, a_4, $\cdots$ be the sequence of positive integers of the form $2^\alpha 3^\beta$, where α and β are nonnegative integers. Prove that every positive integer is expressible in the form

$$a_{i_1} + a_{i_2} + \cdots + a_{i_n},$$

where no summand is a multiple of any other.

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14 Let n be a non-negative integer. Find all non-negative integers a, b, c, d such that

$$a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n.$$

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15 Find all integers $m > 1$ such that m^3 is a sum of m squares of consecutive integers.

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16 Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers.

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17 Let p be a prime number of the form $4k+1$. Suppose that r is a quadratic residue of p and that s is a quadratic nonresidue of p. Show that $p = a^2 + b^2$, where

$$a = \frac{1}{2} \sum_{i=1}^{p-1} \left(\frac{i(i^2 - r)}{p} \right), b = \frac{1}{2} \sum_{i=1}^{p-1} \left(\frac{i(i^2 - s)}{p} \right).$$

Here, $\left(\frac{k}{p} \right)$ denotes the Legendre Symbol.

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18 Let p be a prime with $p \equiv 1 \pmod 4$. Let a be the unique integer such that

$$p = a^2 + b^2, \ a \equiv -1 \pmod 4, \ b \equiv 0 \pmod 2)$$

Prove that

$$\sum_{i=0}^{p-1} \left(\frac{i^3 + 6i^2 + i}{p} \right) = 2 \left(\frac{2}{p} \right),$$

where $\left(\frac{k}{p}\right)$ denotes the Legendre Symbol.

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19 Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \leq \sqrt{n}$, then p divides ab. Determine all such n.

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20 If an integer n is such that $7n$ is the form $a^2 + 3b^2$, prove that n is also of that form.

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21 Let A be the set of positive integers of the form $a^2 + 2b^2$, where a and b are integers and $b \neq 0$. Show that if p is a prime number and $p^2 \in A$, then $p \in A$.

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22 Show that an integer can be expressed as the difference of two squares if and only if it is not of the form $4k + 2$ $(k \in \mathbb{Z})$.

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23 Show that there are infinitely many positive integers which cannot be expressed as the sum of squares.

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24 Show that any integer can be expressed as the form $a^2 + b^2 - c^2$, where $a, b, c \in \mathbb{Z}$.

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25 Let a and b be positive integers with $\gcd(a, b) = 1$. Show that every integer greater than $ab - a - b$ can be expressed in the form $ax + by$, where $x, y \in \mathbb{N}_0$.

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26 Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where $x, y, z \in \mathbb{N}_0$

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27 Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

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28 Prove that any positive integer can be represented as a sum of Fibonacci numbers, no two of which are consecutive.

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29 Show that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.

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30 Let $a_1, a_2, a_3, \cdots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where $i, j,$ and k are not necessarily distinct. Determine a_{1998}.

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31 A finite sequence of integers $a_0, a_1, \cdots, a_n$ is called quadratic if for each $i \in \{1, 2, \cdots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$. - Prove that for any two integers b and c, there exists a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$. - Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

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32 A composite positive integer is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite positive integer is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

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33 Let $a_1, a_2, \cdots, a_k$ be relatively prime positive integers. Determine the largest integer which cannot be expressed in the form

$$x_1 a_2 a_3 \cdots a_k + x_2 a_1 a_3 \cdots a_k + \cdots + x_k a_1 a_2 \cdots a_{k-1}$$

for some nonnegative integers $x_1, x_2, \cdots, x_k$.

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34 If n is a positive integer which can be expressed in the form $n = a^2 + b^2 + c^2$, where a, b, c are positive integers, prove that for each positive integer k, n^{2k} can be expressed in the form $A^2 + B^2 + C^2$, where A, B, C are positive integers.

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35 Prove that every positive integer which is not a member of the infinite set below is equal to the sum of two or more distinct numbers of the set

$$\{3, -2, 2^2 3, -2^3, \cdots, 2^{2k} 3, -2^{2k+1}, \cdots\} = \{3, -2, 12, -8, 48, -32, 192, \cdots\}.$$

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36 Let k and s be odd positive integers such that

$$\sqrt{3k-2} - 1 \le s \le \sqrt{4k}.$$

Show that there are nonnegative integers t, u, v, and w such that

$$k = t^2 + u^2 + v^2 + w^2, \quad \text{and} \quad s = t + u + v + w.$$

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37 Let $S_n = \{1, n, n^2, n^3, \cdots\}$, where n is an integer greater than 1. Find the smallest number $k = k(n)$ such that there is a number which may be expressed as a sum of k (possibly repeated) elements in S_n in more than one way. (Rearrangements are considered the same.)

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38 Find the smallest possible n for which there exist integers $x_1, x_2, \cdots, x_n$ such that each integer between 1000 and 2000 (inclusive) can be written as the sum (without repetition), of one or more of the integers $x_1, x_2, \cdots, x_n$.

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39 In how many ways can 2^n be expressed as the sum of four squares of natural numbers?

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40 Show that - infinitely many perfect squares are a sum of a perfect square and a prime number, - infinitely many perfect squares are not a sum of a perfect square and a prime number.

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41 The famous conjecture of Goldbach is the assertion that every even integer greater than 2 is the sum of two primes. Except 2, 4, and 6, every even integer is a sum of two positive composite integers: $n = 4 + (n-4)$. What is the largest positive even integer that is not a sum of two odd composite integers?

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42 Prove that for each positive integer K there exist infinitely many even positive integers which can be written in more than K ways as the sum of two odd primes.

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43 A positive integer n is abundant if the sum of its proper divisors exceeds n. Show that every integer greater than 89×315 is the sum of two abundant numbers.
