Quadratic Residues

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1 Find all positive integers n that are quadratic residues modulo all primes greater than n.

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2 The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

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3 Let p be an odd prime number. Show that the smallest positive quadratic nonresidue of p is smaller than $\sqrt{p} + 1$.

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4 Let M be an integer, and let p be a prime with $p > 25$. Show that the set $\{M, M + 1, \cdots, M + 3\lfloor\sqrt{p}\rfloor - 1\}$ contains a quadratic non-residue to modulus p.

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5 Let p be an odd prime and let Z_p denote (the field of) integers modulo p. How many elements are in the set

$$\{x^2 : x \in Z_p\} \cap \{y^2 + 1 : y \in Z_p\}?$$

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6 Let a, b, c be integers and let p be an odd prime with

$$p \not| a \text{ and } p \not| b^2 - 4ac.$$

Show that

$$\sum_{k=1}^{p} \left(\frac{ak^2 + bk + c}{p} \right) = -\left(\frac{a}{p} \right).$$
