Miscellaneous Problems

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1 a) Two positive integers are chosen. The sum is revealed to logician A, and the sum of squares is revealed to logician B. Both A and B are given this information and the information contained in this sentence. The conversation between A and B goes as follows: B starts

B: ' I can't tell what they are.'
A: ' I can't tell what they are.'
B: ' I can't tell what they are.'
A: ' I can't tell what they are.'
B: ' I can't tell what they are.'
A: ' I can't tell what they are.'
B: ' Now I can tell what they are.'

What are the two numbers?

b) When B first says that he cannot tell what the two numbers are, A receives a large amount of information. But when A first says that he cannot tell what the two numbers are, B already knows that A cannot tell what the two numbers are. What good does it do B to listen to A?

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2 It is given that 2^{333} is a 101-digit number whose first digit is 1. How many of the numbers 2^k, $1 \le k \le 332$, have first digit 4?

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3 Is there a power of 2 such that it is possible to rearrange the digits giving another power of 2?

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4 If x is a real number such that $x^2 - x$ is an integer, and for some $n \ge 3$, $x^n - x$ is also an integer, prove that x is an integer.

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5 Suppose that both $x^3 - x$ and $x^4 - x$ are integers for some real number x. Show that x is an integer.

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6 Suppose that x and y are complex numbers such that

$$\frac{x^n - y^n}{x - y}$$

are integers for some four consecutive positive integers n. Prove that it is an integer for all positive integers n.

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7 Let n be a positive integer. Show that

$$\sum_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}$$

is an odd integer.

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8 The set $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ contains arithmetic progressions of various lengths. For instance, $\frac{1}{20}$, $\frac{1}{8}$, $\frac{1}{5}$ is such a progression of length 3 and common difference $\frac{3}{40}$. Moreover, this is a maximal progression in S since it cannot be extended to the left or the right within S ($\frac{11}{40}$ and $\frac{-1}{40}$ not being members of S). Prove that for all $n \in \mathbb{N}$, there exists a maximal arithmetic progression of length n in S.

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9 Suppose that

$$\prod_{n=1}^{1996} (1 + nx^{3^n}) = 1 + a_1 x^{k_1} + a_2 x^{k_2} + \cdots + a_m x^{k_m}$$

where a_1, a_2,..., a_m are nonzero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}.

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10 Let p be an odd prime. Show that there is at most one non-degenerate integer triangle with perimeter $4p$ and integer area. Characterize those primes for which such triangle exist.

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11 For each positive integer n, prove that there are two consecutive positive integers each of which is the product of n positive integers greater than 1.

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12 Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

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13 The sum of the digits of a natural number n is denoted by $S(n)$. Prove that $S(8n) \geq \frac{1}{8} S(n)$ for each n.

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14 Let p be an odd prime. Determine positive integers x and y for which $x \leq y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is nonnegative and as small as possible.

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15 Let $\alpha(n)$ be the number of digits equal to one in the dyadic representation of a positive integer n. Prove that - the inequality $\alpha(n^2) \le \frac{1}{2}\alpha(n)(1+\alpha(n))$ holds, - equality is attained for infinitely $n \in \mathbb{N}$, - there exists a sequence $\{n_i\}$ such that $\lim_{i\to\infty} \frac{\alpha({n_i}^2)}{\alpha(n_i)} = 0$.

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16 Show that if a and b are positive integers, then

$$\left(a+\frac{1}{2}\right)^n + \left(b+\frac{1}{2}\right)^n$$

is an integer for only finitely many positive integer n.

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17 Determine the maximum value of m^2+n^2, where m and n are integers satisfying $m, n \in \{1, 2, ..., 1981\}$ and $(n^2-mn-m^2)^2 = 1$.

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18 Denote by S the set of all primes p such that the decimal representation of $\frac{1}{p}$ has the fundamental period of divisible by 3. For every $p \in S$ such that $\frac{1}{p}$ has the fundamental period $3r$ one may write

$$\frac{1}{p} = 0.a_1a_2\cdots a_{3r}a_1a_2\cdots a_{3r}\cdots,$$

where $r = r(p)$. For every $p \in S$ and every integer $k \ge 1$ define

$$f(k,p) = a_k + a_{k+r(p)} + a_{k+2r(p)}.$$

- Prove that S is finite. - Find the highest value of $f(k,p)$ for $k \ge 1$ and $p \in S$.

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19 Determine all pairs (a,b) of real numbers such that $a\lfloor bn\rfloor = b\lfloor an\rfloor$ for all positive integer n.

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20 Let n be a positive integer that is not a perfect cube. Define real numbers a, b, c by

$$a = \sqrt[3]{n},\;\; b = \frac{1}{a-\lfloor a\rfloor},\;\; c = \frac{1}{b-\lfloor b\rfloor}.$$

Prove that there are infinitely many such integers n with the property that there exist integers r, s, t, not all zero, such that $ra+sb+tc = 0$.

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21 Find, with proof, the number of positive integers whose base-n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left.

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22 The decimal expression of the natural number a consists of n digits, while that of a^3 consists of m digits. Can $n+m$ be equal to 2001?

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23 Observe that

$$\frac{1}{1}+\frac{1}{3}=\frac{4}{3},\quad 4^2+3^2=5^2,$$
$$\frac{1}{3}+\frac{1}{5}=\frac{8}{15},\quad 8^2+15^2=17^2,$$
$$\frac{1}{5}+\frac{1}{7}=\frac{12}{35},\quad 12^2+35^2=37^2.$$

State and prove a generalization suggested by these examples.

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24 A number n is called a Niven number, named for Ivan Niven, if it is divisible by the sum of its digits. For example, 24 is a Niven number. Show that it is not possible to have more than 20 consecutive Niven numbers.

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26 Prove that there does not exist a natural number which, upon transfer of its initial digit to the end, is increased five, six or eight times.

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27 Which integers have the following property? If the final digit is deleted, the integer is divisible by the new number.

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28 Let A be the set of the 16 first positive integers. Find the least positive integer k satisfying the condition: In every k-subset of A, there exist two distinct $a, b \in A$ such that a^2+b^2 is prime.

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29 What is the rightmost nonzero digit of $1000000!$?

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30 For how many positive integers n is

$$\left(1999+\frac{1}{2}\right)^n+\left(2000+\frac{1}{2}\right)^n$$

an integer?

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31 Is there a 3×3 magic square consisting of distinct Fibonacci numbers (both f_1 and f_2 may be used; thus two 1s are allowed)?

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32 Alice and Bob play the following number-guessing game. Alice writes down a list of positive integers $x_1, \cdots, x_n$, but does not reveal them to Bob, who will try to determine the numbers by asking Alice questions. Bob chooses a list of positive integers $a_1, \cdots, a_n$ and asks Alice to tell him the value of $a_1x_1+\cdots+a_nx_n$. Then Bob chooses another list of positive integers $b_1, \cdots, b_n$ and asks Alice for $b_1x_1+\cdots+b_nx_n$. Play continues in this way until Bob is able to determine Alice's numbers. How many rounds will Bob need in order to determine Alice's numbers?

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33 Four consecutive even numbers are removed from the set

$$A=\{1,2,3,\cdots,n\}.$$

If the arithmetic mean of the remaining numbers is 51.5625, which four numbers were removed?

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34 Let S_n be the sum of the digits of 2^n. Prove or disprove that $S_{n+1}=S_n$ for some positive integer n.

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35 Counting from the right end, what is the 2500th digit of $10000!$?

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36 For every natural number n, denote $Q(n)$ the sum of the digits in the decimal representation of n. Prove that there are infinitely many natural numbers k with $Q(3^k)>Q(3^{k+1})$.

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37 Let n and k are integers with $n>0$. Prove that

$$-\frac{1}{2n}\sum_{m=1}^{n-1}\cot\frac{\pi m}{n}\sin\frac{2\pi km}{n}=\begin{cases}\frac{k}{n}-\lfloor\frac{k}{n}\rfloor-\frac{1}{2} & \text{if } k|n\\ 0 & \text{otherwise}\end{cases}.$$

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38 The function $\mu:\mathbb{N}\to\mathbb{C}$ is defined by

$$\mu(n)=\sum_{k\in R_n}\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right),$$

where $R_n=\{k\in\mathbb{N}|1\le k\le n,\gcd(k,n)=1\}$. Show that $\mu(n)$ is an integer for all positive integer n.
