Linear Recurrences

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by Peter

1 An integer sequence $\{a_n\}_{n\geq 1}$ is defined by

$$a_0 = 0,\ a_1 = 1,\ a_{n+2} = 2a_{n+1} + a_n$$

Show that 2^k divides a_n if and only if 2^k divides n.

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2 The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $\gcd(F_m, F_n) = F_{\gcd(m,n)}$ for all $m, n \in \mathbb{N}$.

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3 The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn-1} - F_{n-1}^m$ is divisible by F_n^2 for all $m \geq 1$ and $n > 1$.

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4 The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn} - F_{n+1}^m + F_{n-1}^m$ is divisible by F_n^3 for all $m \geq 1$ and $n > 1$.

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5 The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{2n-1}^2 + F_{2n+1}^2 + 1 = 3F_{2n-1}F_{2n+1}$ for all $n \geq 1$.

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6 Prove that no Fibonacci number can be factored into a product of two smaller Fibonacci numbers, each greater than 1.

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7 Let m be a positive integer. Define the sequence $\{a_n\}_{n\geq 0}$ by

$$a_0 = 0,\ a_1 = m,\ a_{n+1} = m^2 a_n - a_{n-1}.$$

Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

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8 Let $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ be two sequences defined recursively as follows

$$x_0 = 1,\ x_1 = 4,\ x_{n+2} = 3x_{n+1} - x_n,$$

$$y_0 = 1,\ y_1 = 2,\ y_{n+2} = 3y_{n+1} - y_n.$$

- Prove that ${x_n}^2 - 5{y_n}^2 + 4 = 0$ for all non-negative integers. - Suppose that a, b are two positive integers such that $a^2 - 5b^2 + 4 = 0$. Prove that there exists a non-negative integer k such that $a = x_k$ and $b = y_k$.

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9 Let $\{u_n\}_{n\ge 0}$ be a sequence of positive integers defined by

$$u_0 = 1,\ u_{n+1} = au_n + b,$$

where $a, b \in \mathbb{N}$. Prove that for any choice of a and b, the sequence $\{u_n\}_{n\ge 0}$ contains infinitely many composite numbers.

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10 The sequence $\{y_n\}_{n\ge 1}$ is defined by

$$y_1 = y_2 = 1,\quad y_{n+2} = (4k-5)y_{n+1} - y_n + 4 - 2k.$$

Determine all integers k such that each term of this sequence is a perfect square.

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11 Let the sequence $\{K_n\}_{n\ge 1}$ be defined by

$$K_1 = 2, K_2 = 8, K_{n+2} = 3K_{n+1} - K_n + 5(-1)^n.$$

Prove that if K_n is prime, then n must be a power of 3.

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12 The sequence $\{a_n\}_{n\ge 1}$ is defined by

$$a_1 = 1,\ a_2 = 12,\ a_3 = 20,\ a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n.$$

Prove that $1 + 4a_na_{n+1}$ is a square for all $n \in \mathbb{N}$.

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13 The sequence $\{x_n\}_{n\ge 1}$ is defined by

$$x_1 = x_2 = 1,\ x_{n+2} = 14x_{n+1} - x_n - 4.$$

Prove that x_n is always a perfect square.
