

More Sequences

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- 1 Show that the sequence $\{a_n\}_{n \geq 1}$ defined by $a_n = \lfloor n\sqrt{2} \rfloor$ contains an infinite number of integer powers of 2.

- 2 Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence a_1, a_2, a_3, \dots become periodic after a finite number of terms?

- 3 Let $n > 6$ be an integer and a_1, a_2, \dots, a_k be all the natural numbers less than n and relatively prime to n . If

$$a_2 - a_1 = a_3 - a_2 = \dots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

- 4 Show that if an infinite arithmetic progression of positive integers contains a square and a cube, it must contain a sixth power.

- 5 Prove that there exist two strictly increasing sequences a_n and b_n such that $a_n(a_n + 1)$ divides $b_n^2 + 1$ for every natural n .

- 6 Let $\{a_n\}$ be a strictly increasing positive integers sequence such that $\gcd(a_i, a_j) = 1$ and $a_{i+2} - a_{i+1} > a_{i+1} - a_i$. Show that the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{a_i}$$

converges.

- 7 Let $\{n_k\}_{k \geq 1}$ be a sequence of natural numbers such that for $i < j$, the decimal representation of n_i does not occur as the leftmost digits of the decimal representation of n_j . Prove that

$$\sum_{k=1}^{\infty} \frac{1}{n_k} \leq \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{9}.$$

- 8 An integer sequence $\{a_n\}_{n \geq 1}$ is given such that

$$2^n = \sum_{d|n} a_d$$

for all $n \in \mathbb{N}$. Show that a_n is divisible by n for all $n \in \mathbb{N}$.

9 Let q_0, q_1, \dots be a sequence of integers such that

- a) for any $m > n$, $m - n$ is a factor of $q_m - q_n$,
- b) item $|q_n| \leq n^{10}$ for all integers $n \geq 0$.

Show that there exists a polynomial $Q(x)$ satisfying $q_n = Q(n)$ for all n .

10 Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence n_1, n_2, \dots, n_k of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \leq i < k$).

11 The infinite sequence of 2's and 3's

$$\begin{aligned} & 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, \\ & 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 2, \dots \end{aligned}$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n , the n th term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m .

12 The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_n = 1 + 2^2 + 3^3 + \cdots + n^n.$$

Prove that there are infinitely many n such that a_n is composite.

13 One member of an infinite arithmetic sequence in the set of natural numbers is a perfect square. Show that there are infinitely many members of this sequence having this property.

14 Let a be the common difference and $an + b$ the sequence. If $ak + b = m^2$ for some k, m , then ∞ many squares are given by

$$(na + m)^2 = n^2a^2 + 2nam + m^2 = (k + an^2 + 2mn) \cdot a + b.$$

This immediately generalizes to any powers.

15 In the sequence 00, 01, 02, 03, \dots , 99 the terms are rearranged so that each term is obtained from the previous one by increasing or decreasing one of its digits by 1 (for example, 29 can be followed by 19, 39, or 28, but not by 30 or 20). What is the maximal number of terms that could remain on their places?

- 16 Does there exist positive integers $a_1 < a_2 < \dots < a_{100}$ such that for $2 \leq k \leq 100$, the greatest common divisor of a_{k-1} and a_k is greater than the greatest common divisor of a_k and a_{k+1} ?

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- 17 Suppose that a and b are distinct real numbers such that

$$a - b, a^2 - b^2, \dots, a^k - b^k, \dots$$

are all integers. Show that a and b are integers.
