Polynomials

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by Peter

1 Suppose $p(x) \in \mathbb{Z}[x]$ and $P(a)P(b) = -(a-b)^2$ for some distinct $a, b \in \mathbb{Z}$. Prove that $P(a) + P(b) = 0$.

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2 Prove that there is no nonconstant polynomial $f(x)$ with integral coefficients such that $f(n)$ is prime for all $n \in \mathbb{N}$.

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3 Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n-2$.

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4 A prime p has decimal digits $p_n p_{n-1} \cdots p_0$ with $p_n > 1$. Show that the polynomial $p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ cannot be represented as a product of two nonconstant polynomials with integer coefficients

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5 (Eisentein's Criterion) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a nonconstant polynomial with integer coefficients. If there is a prime p such that p divides each of a_0, a_1, $\cdots$,a_{n-1} but p does not divide a_n and p^2 does not divide a_0, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

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6 Prove that for a prime p, $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbb{Q}[x]$.

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7 Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

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8 Show that a polynomial of odd degree $2m+1$ over $\mathbb{Z}$,

$$f(x) = c_{2m+1}x^{2m+1} + \cdots + c_1 x + c_0,$$

is irreducible if there exists a prime p such that

$$p \not| c_{2m+1}, p|c_{m+1}, c_{m+2}, \cdots, c_{2m}, p^2|c_0, c_1, \cdots, c_m, \text{ and } p^3 \not| c_0.$$

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9 For non-negative integers n and k, let $P_{n,k}(x)$ denote the rational function

$$\frac{(x^n-1)(x^n-x)\cdots(x^n-x^{k-1})}{(x^k-1)(x^k-x)\cdots(x^k-x^{k-1})}.$$

Show that $P_{n,k}(x)$ is actually a polynomial for all $n, k \in \mathbb{N}$.

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10 Suppose that the integers $a_1, a_2, \cdots, a_n$ are distinct. Show that

$$(x-a_1)(x-a_2)\cdots(x-a_n)-1$$

cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

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11 Show that the polynomial x^8+98x^4+1 can be expressed as the product of two nonconstant polynomials with integer coefficients.

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12 Prove that if the integers $a_1, a_2, \cdots, a_n$ are all distinct, then the polynomial

$$(x-a_1)^2(x-a_2)^2\cdots(x-a_n)^2+1$$

cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

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13 On Christmas Eve, 1983, Dean Jixon, the famous seer who had made startling predictions of the events of the preceding year that the volcanic and seismic activities of 1980 and 1981 were connected with mathematics. The diminishing of this geological activity depended upon the existence of an elementary proof of the irreducibility of the polynomial

$$P(x)=x^{1981}+x^{1980}+12x^2+24x+1983.$$

Is there such a proof?
