Floor Function and Fractional Part Function

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by Peter

1 Let α be the positive root of the equation $x^2 = 1991x + 1$. For natural numbers m and n define

$$m * n = mn + \lfloor \alpha m \rfloor \lfloor \alpha n \rfloor.$$

Prove that for all natural numbers p, q, and r,

$$(p * q) * r = p * (q * r).$$

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2 Prove that for any positive integer n,

$$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor.$$

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3 Prove that for any positive integer n,

$$\left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n+4}{8} \right\rfloor + \left\lfloor \frac{n+8}{16} \right\rfloor + \cdots = n.$$

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4 Show that for all positive integers n,

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor = \lfloor \sqrt{4n+3} \rfloor.$$

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5 Find all real numbers α for which the equality

$$\lfloor \sqrt{n} + \sqrt{n+\alpha} \rfloor = \lfloor \sqrt{4n+1} \rfloor$$

holds for all positive integers n.

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6 Prove that for all positive integers n,

$$\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \rfloor = \lfloor \sqrt{9n+8} \rfloor.$$

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7 Prove that for all positive integers n,

$$\lfloor \sqrt[3]{n} + \sqrt[3]{n+1} \rfloor = \lfloor \sqrt[3]{8n+3} \rfloor.$$

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8 Prove that $\lfloor \sqrt[3]{n}+\sqrt[3]{n+1}+\sqrt[3]{n+2}\rfloor = \lfloor \sqrt[3]{27n+26}\rfloor$ for all positive integers n.

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9 Show that for all positive integers m and n,

$$\gcd(m,n) = m+n-mn+2\sum_{k=0}^{m-1}\left\lfloor \frac{kn}{m}\right\rfloor.$$

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10 Show that for all primes p,

$$\sum_{k=1}^{p-1}\left\lfloor \frac{k^3}{p}\right\rfloor = \frac{(p+1)(p-1)(p-2)}{4}.$$

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11 Let p be a prime number of the form $4k+1$. Show that

$$\sum_{i=1}^{p-1}\left(\left\lfloor \frac{2i^2}{p}\right\rfloor - 2\left\lfloor \frac{i^2}{p}\right\rfloor\right) = \frac{p-1}{2}.$$

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12 Let $p=4k+1$ be a prime. Show that

$$\sum_{i=1}^{k}\left\lfloor \sqrt{ip}\right\rfloor = \frac{p^2-1}{12}.$$

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13 Suppose that $n\geq 2$. Prove that

$$\sum_{k=2}^{n}\left\lfloor \frac{n^2}{k}\right\rfloor = \sum_{k=n+1}^{n^2}\left\lfloor \frac{n^2}{k}\right\rfloor.$$

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14 Let a,b,n be positive integers with $\gcd(a,b)=1$. Prove that

$$\sum_{k}\left\{\frac{ak+b}{n}\right\} = \frac{n-1}{2},$$

where k runs through a complete system of residues modulo m.

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15 Find the total number of different integer values the function

$$f(x) = \lfloor x\rfloor + \lfloor 2x\rfloor + \left\lfloor \frac{5x}{3}\right\rfloor + \lfloor 3x\rfloor + \lfloor 4x\rfloor$$

takes for real numbers x with $0\leq x\leq 100$.

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16 Prove or disprove that there exists a positive real number u such that $\lfloor u^n \rfloor - n$ is an even integer for all positive integer n.

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17 Determine all real numbers a such that

$$4\lfloor an \rfloor = n + \lfloor a \lfloor an \rfloor \rfloor \text{ for all } n \in \mathbb{N}.$$

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18 Do there exist irrational numbers $a, b > 1$ and $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for any positive integers m and n?

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19 Let a, b, c, and d be real numbers. Suppose that $\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor + \lfloor nd \rfloor$ for all positive integers n. Show that at least one of $a + b$, $a - c$, $a - d$ is an integer.

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20 Find all integer solutions of the equation

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor = 1001.$$
