

Diophantine Equations

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by Peter

- 1** One of Euler's conjectures was disproved in the 1980s by three American Mathematicians when they showed that there is a positive integer n such that

$$n^5 = 133^5 + 110^5 + 84^5 + 27^5.$$

Find the value of n .

- 2** The number 21982145917308330487013369 is the thirteenth power of a positive integer.
Which positive integer?

- 3** Does there exist a solution to the equation

$$x^2 + y^2 + z^2 + u^2 + v^2 = xyzuv - 65$$

in integers with x, y, z, u, v greater than 1998?

- 4** Find all pairs (x, y) of positive rational numbers such that $x^2 + 3y^2 = 1$.

- 5** Find all pairs (x, y) of rational numbers such that $y^2 = x^3 - 3x + 2$.

- 6** Show that there are infinitely many pairs (x, y) of rational numbers such that $x^3 + y^3 = 9$.

- 7** Determine all pairs (x, y) of positive integers satisfying the equation

$$(x + y)^2 - 2(xy)^2 = 1.$$

- 8** Show that the equation

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integral solutions.

- 9** Determine all integers a for which the equation

$$x^2 + axy + y^2 = 1$$

has infinitely many distinct integer solutions x, y .

- 10** Prove that there are unique positive integers a and n such that

$$a^{n+1} - (a + 1)^n = 2001.$$

11 Find all $(x, y, n) \in \mathbb{N}^3$ such that $\gcd(x, n+1) = 1$ and $x^n + 1 = y^{n+1}$.

12 Find all $(x, y, z) \in \mathbb{N}^3$ such that $x^4 - y^4 = z^2$.

13 Find all pairs (x, y) of positive integers that satisfy the equation

$$y^2 = x^3 + 16.$$

14 Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y, z for which $xyz \neq 0$.

15 Prove that there are no integers x and y satisfying $x^2 = y^5 - 4$.

16 Find all pairs (a, b) of different positive integers that satisfy the equation $W(a) = W(b)$, where $W(x) = x^4 - 3x^3 + 5x^2 - 9x$.

17 Find all positive integers n for which the equation

$$a + b + c + d = n\sqrt{abcd}$$

has a solution in positive integers.

18 Determine all positive integer solutions (x, y, z, t) of the equation

$$(x+y)(y+z)(z+x) = xyzt$$

for which $\gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1$.

19 Find all $(x, y, z, n) \in \mathbb{N}^4$ such that $x^3 + y^3 + z^3 = nx^2y^2z^2$.

20 Determine all positive integers n for which the equation

$$x^n + (2+x)^n + (2-x)^n = 0$$

has an integer as a solution.

21 Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

- 22** Find all integers a, b, c, x, y, z such that

$$a + b + c = xyz, \quad x + y + z = abc, \quad a \geq b \geq c \geq 1, \quad x \geq y \geq z \geq 1.$$

- 23** Find all $(x, y, z) \in \mathbb{Z}^3$ such that $x^3 + y^3 + z^3 = x + y + z = 3$.

- 24** Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n.$$

has a solution in integers (x, y) , then it has at least three such solutions. Show that the equation has no solutions in integers when $n = 2891$.

- 25** What is the smallest positive integer t such that there exist integers x_1, x_2, \dots, x_t with

$$x_1^3 + x_2^3 + \dots + x_t^3 = 2002^{2002} ?$$

- 26** Solve in integers the following equation

$$n^{2002} = m(m+n)(m+2n)\cdots(m+2001n).$$

- 27** Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semi-perimeter and the inradius of a triangle with integer side lengths.

- 28** Let a, b, c be positive integers such that a and b are relatively prime and c is relatively prime either to a or b . Prove that there exist infinitely many triples (x, y, z) of distinct positive integers such that

$$x^a + y^b = z^c.$$

- 29** Find all pairs of integers (x, y) satisfying the equality

$$y(x^2 + 36) + x(y^2 - 36) + y^2(y - 12) = 0.$$

- 30** Let a, b, c be given integers, $a > 0$, $ac - b^2 = p$ a squarefree positive integer. Let $M(n)$ denote the number of pairs of integers (x, y) for which $ax^2 + bxy + cy^2 = n$. Prove that $M(n)$ is finite and $M(n) = M(p^k \cdot n)$ for every integer $k \geq 0$.

- 31** Determine all integer solutions of the system

$$\begin{aligned} 2uv - xy &= 16, \\ xv - yu &= 12. \end{aligned}$$

- 32** Let n be a natural number. Solve in whole numbers the equation

$$x^n + y^n = (x - y)^{n+1}.$$

- 33** Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7$, where n is integer?

- 34** Are there integers m and n such that $5m^2 - 6mn + 7n^2 = 1985$?

- 35** Find all cubic polynomials $x^3 + ax^2 + bx + c$ admitting the rational numbers a, b and c as roots.

- 36** Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c) .

- 37** Prove that for each positive integer n there exist odd positive integers x_n and y_n such that $x_n^2 + 7y_n^2 = 2^n$.

- 38** Suppose that p is an odd prime such that $2p + 1$ is also prime. Show that the equation $x^p + 2y^p + 5z^p = 0$ has no solutions in integers other than $(0, 0, 0)$.

- 39** Let A, B, C, D, E be integers, $B \neq 0$ and $F = AD^2 - BCD + B^2E \neq 0$. Prove that the number N of pairs of integers (x, y) such that

$$Ax^2 + Bxy + Cx + Dy + E = 0,$$

satisfies $N \leq 2d(|F|)$, where $d(n)$ denotes the number of positive divisors of positive integer n .

- 40** Determine all pairs of rational numbers (x, y) such that

$$x^3 + y^3 = x^2 + y^2.$$

- 41** Suppose that $A = 1, 2$, or 3 . Let a and b be relatively prime integers such that $a^2 + Ab^2 = s^3$ for some integer s . Then, there are integers u and v such that $s = u^2 + Av^2$, $a = u^3 - 3Avu^2$, and $b = 3u^2v - Av^3$.

42 Find all integers a for which $x^3 - x + a$ has three integer roots.

43 Find all solutions in integers of $x^3 + 2y^3 = 4z^3$.

44 For all $n \in \mathbb{N}$, show that the number of integral solutions (x, y) of

$$x^2 + xy + y^2 = n$$

is finite and a multiple of 6.

45 Show that there cannot be four squares in arithmetical progression.

46 Let a, b, c, d, e, f be integers such that $b^2 - 4ac > 0$ is not a perfect square and $4acf + bde - ae^2 - cd^2 - fb^2 \neq 0$. Let

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$$

Suppose that $f(x, y) = 0$ has an integral solution. Show that $f(x, y) = 0$ has infinitely many integral solutions.

47 Show that the equation $x^4 + y^4 + 4z^4 = 1$ has infinitely many rational solutions.

48 Solve the equation $x^2 + 7 = 2^n$ in integers.

49 Show that the only solutions of the equation $x^3 - 3xy^2 - y^3 = 1$ are given by $(x, y) = (1, 0), (0, -1), (-1, 1), (1, -3), (-3, 2), (2, 1)$.

50 Show that the equation $y^2 = x^3 + 2a^3 - 3b^2$ has no solution in integers if $ab \neq 0, a \not\equiv 1 \pmod{3}$, 3 does not divide b , a is odd if b is even, and $p = t^2 + 27u^2$ has a solution in integers t, u if $p|a$ and $p \equiv 1 \pmod{3}$.

51 Prove that the product of five consecutive positive integers is never a perfect square.

52 Do there exist two right-angled triangles with integer length sides that have the lengths of exactly two sides in common?

53 Suppose that a, b , and p are integers such that $b \equiv 1 \pmod{4}, p \equiv 3 \pmod{4}$, p is prime, and if q is any prime divisor of a such that $q \equiv 3 \pmod{4}$, then $q^p|a^2$ and p does not divide $q - 1$ (if $q = p$, then also $q|b$). Show that the equation

$$x^2 + 4a^2 = y^p - b^p$$

has no solutions in integers.

- 54** Show that the number of integral-sided right triangles whose ratio of area to semi-perimeter is p^m , where p is a prime and m is an integer, is $m + 1$ if $p = 2$ and $2m + 1$ if $p \neq 2$.
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- 55** Given that

$$34! = 95232799cd96041408476186096435ab000000_{(10)},$$

determine the digits a, b, c , and d .

- 56** Prove that the equation $\prod_{cyc}(x_1 - x_2) = \prod_{cyc}(x_1 - x_3)$ has a solution in natural numbers where all x_i are different.
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- 57** Show that the equation $\binom{n}{k} = m^l$ has no integral solution with $l \geq 2$ and $4 \leq k \leq n - 4$.
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- 58** Solve in positive integers the equation $10^a + 2^b - 3^c = 1997$.
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- 59** Solve the equation $28^x = 19^y + 87^z$, where x, y, z are integers.
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- 60** Show that the equation $x^7 + y^7 = 1998^z$ has no solution in positive integers.
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- 61** Solve the equation $2^x - 5 = 11^y$ in positive integers.
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- 62** Solve the equation $7^x - 3^y = 4$ in positive integers.
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- 63** Show that $|12^m - 5^n| \geq 7$ for all $m, n \in \mathbb{N}$.
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- 64** Show that there is no positive integer k for which the equation

$$(n - 1)! + 1 = n^k$$

is true when n is greater than 5.

- 65** Determine all pairs (x, y) of integers such that

$$(19a + b)^{18} + (a + b)^{18} + (19b + a)^{18}$$

is a nonzero perfect square.

- 66** Let b be a positive integer. Determine all 2002-tuples of non-negative integers $(a_1, a_2, \dots, a_{2002})$ satisfying

$$\sum_{j=1}^{2002} a_j^{a_j} = 2002b^b.$$

- 67** Is there a positive integer m such that the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers a, b, c ?

- 68** Consider the system

$$x + y = z + u,$$

$$2xy = zu.$$

Find the greatest value of the real constant m such that $m \leq \frac{x}{y}$ for any positive integer solution (x, y, z, u) of the system, with $x \geq y$.

- 69** Determine all positive rational numbers $r \neq 1$ such that $\sqrt[r-1]{r}$ is rational.

- 70** Show that the equation $\{x^3\} + \{y^3\} = \{z^3\}$ has infinitely many rational non-integer solutions.

- 71** Let n be a positive integer. Prove that the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

- 72** Find all pairs (x, y) of positive rational numbers such that $x^y = y^x$.

- 73** Find all pairs (a, b) of positive integers that satisfy the equation

$$a^{b^2} = b^a.$$

- 74** Find all pairs (a, b) of positive integers that satisfy the equation

$$a^{a^a} = b^b.$$

- 75** Let a, b , and x be positive integers such that $x^{a+b} = a^b b$. Prove that $a = x$ and $b = x^x$.

- 76** Find all pairs (m, n) of integers that satisfy the equation

$$(m - n)^2 = \frac{4mn}{m + n - 1}.$$

- 77** Find all pairwise relatively prime positive integers l, m, n such that

$$(l + m + n) \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right)$$

is an integer.

- 78** Let x, y , and z be integers with $z > 1$. Show that

$$(x + 1)^2 + (x + 2)^2 + \cdots + (x + 99)^2 \neq y^z.$$

- 79** Find all positive integers m and n for which

$$1! + 2! + 3! + \cdots + n! = m^2$$

- 80** Prove that if a, b, c, d are integers such that $d = (a + \sqrt[3]{2}b + \sqrt[3]{4}c)^2$ then d is a perfect square.
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- 81** Find a pair of relatively prime four digit natural numbers A and B such that for all natural numbers m and n , $|A^m - B^n| \geq 400$.
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- 82** Find all triples (a, b, c) of positive integers to the equation

$$a!b! = a! + b! + c!.$$

- 83** Find all pairs (a, b) of positive integers such that

$$(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 49 + 20\sqrt[3]{6}.$$

- 84** For what positive numbers a is

$$\sqrt[3]{2 + \sqrt{a}} + \sqrt[3]{2 - \sqrt{a}}$$

an integer?

- 85** Find all integer solutions to $2(x^5 + y^5 + 1) = 5xy(x^2 + y^2 + 1)$.
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- 86** A triangle with integer sides is called Heronian if its area is an integer. Does there exist a Heronian triangle whose sides are the arithmetic, geometric and harmonic means of two positive integers?
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87 What is the smallest perfect square that ends in 9009?

88 (Leo Moser) Show that the Diophantine equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = 1$$

has at least one solution for every positive integers n .

89 Prove that the number $99999 + 111111\sqrt{3}$ cannot be written in the form $(A + B\sqrt{3})^2$, where A and B are integers.

90 Find all triples of positive integers (x, y, z) such that

$$(x + y)(1 + xy) = 2^z.$$

91 If R and S are two rectangles with integer sides such that the perimeter of R equals the area of S and the perimeter of S equals the area of R , then we call R and S a friendly pair of rectangles. Find all friendly pairs of rectangles.