

Combinatorial Number Theory
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1 Suppose all the pairs of a positive integers from a finite collection

$$A = \{a_1, a_2, \dots\}$$

are added together to form a new collection

$$A^* = \{a_i + a_j \mid 1 \leq i < j \leq n\}.$$

For example, $A = \{2, 3, 4, 7\}$ would yield $A^* = \{5, 6, 7, 9, 10, 11\}$ and $B = \{1, 4, 5, 6\}$ would give $B^* = \{5, 6, 7, 9, 10, 11\}$. These examples show that it's possible for different collections A and B to generate the same collections A^* and B^* . Show that if $A^* = B^*$ for different sets A and B , then $|A| = |B|$ and $|A| = |B|$ must be a power of 2.

2 Let p be a prime. Find all positive integers k such that the set $\{1, 2, \dots, k\}$ can be partitioned into p subsets with equal sum of elements.

3 Prove that the set of integers of the form $2^k - 3$ ($k = 2, 3, \dots$) contains an infinite subset in which every two members are relatively prime.

4 The set of positive integers is partitioned into finitely many subsets. Show that some subset S has the following property: for every positive integer n , S contains infinitely many multiples of n .

5 Let M be a positive integer and consider the set

$$S = \{n \in \mathbb{N} \mid M^2 \leq n < (M+1)^2\}.$$

 Prove that the products of the form ab with $a, b \in S$ are distinct.

6 Let S be a set of integers such that - there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a-2, b-2) = 1$, - if $x, y \in S$, then $x^2 - y \in S$. Prove that $S = \mathbb{Z}$.

7 Show that for each $n \geq 2$, there is a set S of n integers such that $(a-b)^2$ divides ab for every distinct $a, b \in S$.

8 Let a and b be positive integers greater than 2. Prove that there exists a positive integer k and a finite sequence n_1, \dots, n_k of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \leq i \leq k$).

- 9** Let n be an integer, and let X be a set of $n + 2$ integers each of absolute value at most n . Show that there exist three distinct numbers $a, b, c \in X$ such that $c = a + b$.
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- 10** Let $m \geq 2$ be an integer. Find the smallest integer $n > m$ such that for any partition of the set $\{m, m + 1, \dots, n\}$ into two subsets, at least one subset contains three numbers a, b, c such that $c = a^b$.
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- 11** Let $S = \{1, 2, 3, \dots, 280\}$. Find the smallest integer n such that each n -element subset of S contains five numbers which are pairwise relatively prime.
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- 12** Let m and n be positive integers. If x_1, x_2, \dots, x_m are positive integers whose arithmetic mean is less than $n + 1$ and if y_1, y_2, \dots, y_n are positive integers whose arithmetic mean is less than $m + 1$, prove that some sum of one or more x 's equals some sum of one or more y 's.
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- 13** Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, \dots, n - 1\}$ is colored either blue or white. It is given that - for each $i \in M$, both i and $n - i$ have the same color, - for each $i \in M, i \neq k$, both i and $|i - k|$ have the same color. Prove that all numbers in M have the same color.
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- 14** Let p be a prime number, $p \geq 5$, and k be a digit in the p -adic representation of positive integers. Find the maximal length of a non constant arithmetic progression whose terms do not contain the digit k in their p -adic representation.
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- 15** Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5 , no three of which are consecutive terms of an arithmetic progression?
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- 16** Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different?
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- 17** Find the maximum number of pairwise disjoint sets of the form
- $$S_{a,b} = \{n^2 + an + b \mid n \in \mathbb{Z}\},$$
- with $a, b \in \mathbb{Z}$.
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- 18** Let p be an odd prime number. How many p -element subsets A of $\{1, 2, \dots, 2p\}$ are there, the sum of whose elements is divisible by p ?
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- 19** Let $m, n \geq 2$ be positive integers, and let a_1, a_2, \dots, a_n be integers, none of which is a multiple of m^{n-1} . Show that there exist integers e_1, e_2, \dots, e_n , not all zero, with $|e_i| < m$ for all i , such that $e_1 a_1 + e_2 a_2 + \dots + e_n a_n$ is a multiple of m^n .
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- 20** Determine the smallest integer $n \geq 4$ for which one can choose four different numbers a, b, c , and d from any n distinct integers such that $a + b - c - d$ is divisible by 20.
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- 21** A sequence of integers a_1, a_2, a_3, \dots is defined as follows: $a_1 = 1$, and for $n \geq 1$, a_{n+1} is the smallest integer greater than a_n such that $a_i + a_j \neq 3a_k$ for any i, j , and k in $\{1, 2, 3, \dots, n+1\}$, not necessarily distinct. Determine a_{1998} .
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- 22** Prove that for each positive integer n , there exists a positive integer with the following properties: - it has exactly n digits, - none of the digits is 0, - it is divisible by the sum of its digits.
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- 23** Let k, m, n be integers such that $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset S of the set $\{1, 2, \dots, k\}$ such that no n distinct elements of S add up to m .
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- 24** Find the number of subsets of $\{1, 2, \dots, 2000\}$, the sum of whose elements is divisible by 5.
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- 25** Let A be a non-empty set of positive integers. Suppose that there are positive integers b_1, \dots, b_n and c_1, \dots, c_n such that - for each i the set $b_i A + c_i = \{b_i a + c_i \mid a \in A\}$ is a subset of A , - the sets $b_i A + c_i$ and $b_j A + c_j$ are disjoint whenever $i \neq j$. Prove that
- $$\frac{1}{b_1} + \dots + \frac{1}{b_n} \leq 1.$$
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- 26** A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called historic if $\{z - y, y - x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.
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- 27** Let p and q be relatively prime positive integers. A subset $S \subseteq \mathbb{N}_0$ is called ideal if $0 \in S$ and, for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S . Determine the number of ideal subsets of \mathbb{N}_0 .
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- 28** Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that, for any two integers x, y taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.
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- 29** Let A be a set of N residues $(\text{mod } N^2)$. Prove that there exists a set B of N residues $(\text{mod } N^2)$ such that the set $A + B = \{a + b \mid a \in A, b \in B\}$ contains at least half of all the residues $(\text{mod } N^2)$.
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- 30** Determine the largest positive integer n for which there exists a set S with exactly n numbers such that - each member in S is a positive integer not exceeding 2002, - if $a, b \in S$ (not necessarily different), then $ab \notin S$.

- 31** Prove that, for any integer $a_1 > 1$, there exist an increasing sequence of positive integers a_1, a_2, a_3, \dots such that

$$a_1 + a_2 + \dots + a_n \mid a_1^2 + a_2^2 + \dots + a_n^2$$

for all $n \in \mathbb{N}$.

- 32** An odd integer $n \geq 3$ is said to be nice if and only if there is at least one permutation a_1, \dots, a_n of $1, \dots, n$ such that the n sums $a_1 - a_2 + a_3 - \dots - a_{n-1} + a_n$, $a_2 - a_3 + a_3 - \dots - a_n + a_1$, $a_3 - a_4 + a_5 - \dots - a_1 + a_2, \dots$, $a_n - a_1 + a_2 - \dots - a_{n-2} + a_{n-1}$ are all positive. Determine the set of all 'nice' integers.

- 33** Assume that the set of all positive integers is decomposed into r disjoint subsets A_1, A_2, \dots, A_r $A_1 \cup A_2 \cup \dots \cup A_r = \mathbb{N}$. Prove that one of them, say A_i , has the following property: There exist a positive integer m such that for any k one can find numbers a_1, \dots, a_k in A_i with $0 < a_{j+1} - a_j \leq m$ ($1 \leq j \leq k-1$).

- 34** Determine for which positive integers k , the set

$$X = \{1990, 1990 + 1, 1990 + 2, \dots, 1990 + k\}$$

can be partitioned into two disjoint subsets A and B such that the sum of the elements of A is equal to the sum of the elements of B .

- 35** Let $n \geq 3$ be a prime number and $a_1 < a_2 < \dots < a_n$ be integers. Prove that a_1, \dots, a_n is an arithmetic progression if and only if there exists a partition of $\{0, 1, 2, \dots\}$ into sets A_1, A_2, \dots, A_n such that

$$a_1 + A_1 = a_2 + A_2 = \dots = a_n + A_n,$$

where $x + A$ denotes the set $\{x + a \mid a \in A\}$.

- 36** Let a and b be non-negative integers such that $ab \geq c^2$ where c is an integer. Prove that there is a positive integer n and integers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ such that

$$x_1^2 + \dots + x_n^2 = a, \quad y_1^2 + \dots + y_n^2 = b, \quad x_1 y_1 + \dots + x_n y_n = c$$

- 37** Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in the decimal representation is k .

- 38** Prove that for every real number M there exists an infinite arithmetical progression of positive integers such that - the common difference is not divisible by 10, - the sum of digits of each term exceeds M .
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- 39** Find the smallest positive integer n for which there exist n different positive integers a_1, a_2, \dots, a_n satisfying - $\text{lcm}(a_1, a_2, \dots, a_n) = 1985$, - for each $i, j \in \{1, 2, \dots, n\}$, $\gcd(a_i, a_j) \neq 1$, - the product $a_1 a_2 \dots a_n$ is a perfect square and is divisible by 243, and find all such n -tuples (a_1, \dots, a_n) .
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- 40** Let X be a non-empty set of positive integers which satisfies the following: - if $x \in X$, then $4x \in X$, - if $x \in X$, then $\lfloor \sqrt{x} \rfloor \in X$. Prove that $X = \mathbb{N}$.
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- 41** Prove that for every positive integer n there exists an n -digit number divisible by 5^n all of whose digits are odd.
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- 42** Let N_n denote the number of ordered n -tuples of positive integers (a_1, a_2, \dots, a_n) such that
- $$1/a_1 + 1/a_2 + \dots + 1/a_n = 1.$$
- Determine whether N_{10} is even or odd.
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- 43** Is it possible to find a set A of eleven positive integers such that no six elements of A have a sum which is divisible by 6?
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- 44** A set C of positive integers is called good if for every integer k there exist distinct $a, b \in C$ such that the numbers $a + k$ and $b + k$ are not relatively prime. Prove that if the sum of the elements of a good set C equals 2003, then there exists $c \in C$ such that the set $C - \{c\}$ is good.
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- 45** Find all positive integers n with the property that the set
- $$\{n, n+1, n+2, n+3, n+4, n+5\}$$
- can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.
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- 46** Suppose p is a prime with $p \equiv 3 \pmod{4}$. Show that for any set of $p-1$ consecutive integers, the set cannot be divided into two subsets so that the product of the members of the one set is equal to the product of the members of the other set.
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- 47** Let S be the set of all composite positive odd integers less than 79. - Show that S may be written as the union of three (not necessarily disjoint) arithmetic progressions. - Show that S cannot be written as the union of two arithmetic progressions.
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- 48** Let a_1, \dots, a_{44} be natural numbers such that

$$0 < a_1 < a_2 < \dots < a_{44} < 125.$$

Prove that at least one of the 43 differences $d_j = a_{j+1} - a_j$ occurs at least 10 times.

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- 49** Consider the set of all five-digit numbers whose decimal representation is a permutation of the digits 1, 2, 3, 4, 5. Prove that this set can be divided into two groups, in such a way that the sum of the squares of the numbers in each group is the same.
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- 50** What's the largest number of elements that a set of positive integers between 1 and 100 inclusive can have if it has the property that none of them is divisible by another?
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- 51** Prove that among 16 consecutive integers it is always possible to find one which is relatively prime to all the rest.
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- 52** Is there a set S of positive integers such that a number is in S if and only if it is the sum of two distinct members of S or a sum of two distinct positive integers not in S ?
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- 53** Suppose that the set $M = \{1, 2, \dots, n\}$ is split into t disjoint subsets M_1, \dots, M_t where the cardinality of M_i is m_i , and $m_i \geq m_{i+1}$, for $i = 1, \dots, t-1$. Show that if $n > t! \cdot e$ then at least one class M_z contains three elements x_i, x_j, x_k with the property that $x_i - x_j = x_k$.
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- 54** Let S be a subset of $\{1, 2, 3, \dots, 1989\}$ in which no two members differ by exactly 4 or by exactly 7. What is the largest number of elements S can have?
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- 55** The set M consists of integers, the smallest of which is 1 and the greatest 100. Each member of M , except 1, is the sum of two (possibly identical) numbers in M . Of all such sets, find one with the smallest possible number of elements.
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- 56** Show that it is possible to color the set of integers
- $$M = \{1, 2, 3, \dots, 1987\},$$
- using four colors, so that no arithmetic progression with 10 terms has all its members the same color.
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- 57** Prove that every selection of 1325 integers from $M = \{1, 2, \dots, 1987\}$ must contain some three numbers $\{a, b, c\}$ which are pairwise relatively prime, but that it can be avoided if only 1324 integers are selected.
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- 58** Prove that every infinite sequence S of distinct positive integers contains either an infinite subsequence such that for every pair of terms, neither term ever divides the

other, or an infinite subsequence such that in every pair of terms, one always divides the other.

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- 59** Let $a_1 < a_2 < a_3 < \dots$ be an infinite increasing sequence of positive integers in which the number of prime factors of each term, counting repeated factors, is never more than 1987. Prove that it is always possible to extract from A an infinite subsequence $b_1 < b_2 < b_3 < \dots$ such that the greatest common divisor (b_i, b_j) is the same number for every pair of its terms.
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