Recursive Sequences

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1   Let $P(x)$ be a nonzero polynomial with integer coefficients. Let $a_0 = 0$ and for $i \geq 0$
             define $a_{i+1} = P(a_i)$. Show that $\gcd(a_m, a_n) = a_{\gcd(m,n)}$ for all $m, n \in \mathbb{N}$.

2   An integer sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 1, \ a_{n+1} = a_n + \lfloor \sqrt{a_n} \rfloor.$$

             Show that a_n is a square if and only if $n = 2^k + k - 2$ for some $k \in \mathbb{N}$.

3   Let $f(n) = n + \lfloor \sqrt{n} \rfloor$. Prove that, for every positive integer m, the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \cdots$$

             contains at least one square of an integer.

4   The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 1, \ a_2 = 2, \ a_3 = 24, \ a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}} \quad (n \geq 4).$$

             Show that a_n is an integer for all n, and show that $n | a_n$ for every $n \in \mathbb{N}$.

5   Show that there is a unique sequence of integers $\{a_n\}_{n \geq 1}$ with

$$a_1 = 1, \ a_2 = 2, \ a_4 = 12, \ a_{n+1}a_{n-1} = a_n^2 \pm 1 \ (n \geq 2).$$

6   The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 1, \ a_{n+1} = 2a_n + \sqrt{3a_n^2 + 1}.$$

             Show that a_n is an integer for every n.

7   Prove that the sequence $\{y_n\}_{n \geq 1}$ defined by

$$y_0 = 1, \ y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right)$$

             consists only of integers.

8 The Bernoulli sequence $\{B_n\}_{n \geq 0}$ is defined by

$$B_0 = 1, \ B_n = -\frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k \ \ (n \geq 1)$$

Show that for all $n \in \mathbb{N}$,

$$(-1)^n B_n - \sum \frac{1}{p},$$

is an integer where the summation is done over all primes p such that $p | 2k - 1$.

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9 An integer sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 2, \ a_{n+1} = \left\lfloor \frac{3}{2} a_n \right\rfloor.$$

Show that it has infinitely many even and infinitely many odd integers.

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10 An integer sequence satisfies $a_{n+1} = a_n{}^3 + 1999$. Show that it contains at most one square.

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11 Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, n \geq 4.$$

Determine $a_{14^{14}}$.

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12 Let k be a fixed positive integer. The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = k + 1, a_{n+1} = a_n^2 - ka_n + k.$$

Show that if $m \neq n$, then the numbers a_m and a_n are relatively prime.

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13 The sequence $\{x_n\}$ is defined by

$$x_0 \in [0, 1], \ x_{n+1} = 1 - |1 - 2x_n|.$$

Prove that the sequence is periodic if and only if x_0 is irrational.

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14 Let x_1 and x_2 be relatively prime positive integers. For $n \geq 2$, define $x_{n+1} = x_n x_{n-1} + 1$.- Prove that for every $i > 1$, there exists $j > i$ such that $x_i{}^i$ divides $x_j{}^j$. - Is it true that x_1 must divide $x_j{}^j$ for some $j > 1$?

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15 For a given positive integer k denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{1991}(2^{1990})$.

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16   Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

17   A sequence of integers, $\{a_n\}_{n\geq 1}$ with $a_1 > 0$, is defined by

$$a_{n+1} = \frac{a_n}{2} \quad \text{if} \quad n \equiv 0 \pmod 4,$$

$$a_{n+1} = 3a_n + 1 \quad \text{if} \quad n \equiv 1 \pmod 4,$$

$$a_{n+1} = 2a_n - 1 \quad \text{if} \quad n \equiv 2 \pmod 4,$$

$$a_{n+1} = \frac{a_n + 1}{4} \quad \text{if} \quad n \equiv 3 \pmod 4.$$

Prove that there is an integer m such that $a_m = 1$.

18   Given is an integer sequence $\{a_n\}_{n\geq 0}$ such that $a_0 = 2$, $a_1 = 3$ and, for all positive integers $n \geq 1$, $a_{n+1} = 2a_{n-1}$ or $a_{n+1} = 3a_n - 2a_{n-1}$. Does there exist a positive integer k such that $1600 < a_k < 2000$?

19   A sequence with first two terms equal 1 and 24 respectively is defined by the following rule: each subsequent term is equal to the smallest positive integer which has not yet occurred in the sequence and is not coprime with the previous term. Prove that all positive integers occur in this sequence.

20   Each term of a sequence of natural numbers is obtained from the previous term by adding to it its largest digit. What is the maximal number of successive odd terms in such a sequence?

21   In the sequence $1, 0, 1, 0, 1, 0, 3, 5, \cdots$, each member after the sixth one is equal to the last digit of the sum of the six members just preceeding it. Prove that in this sequence one cannot find the following group of six consecutive members:

$$0, 1, 0, 1, 0, 1$$

22   Let a, and b be odd positive integers. Define the sequence $\{f_n\}_{n\geq 1}$ by putting $f_1 = a$, $f_2 = b$, and by letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for sufficiently large n and determine the eventual value as a function of a and b.

23   Define
$$\begin{cases} d(n,0) = d(n,n) = 1 & (n \geq 0), \\ md(n,m) = md(n-1,m) + (2n-m)d(n-1, m-1) & (0 < m < n). \end{cases}$$

Prove that $d(n, m)$ are integers for all $m, n \in \mathbb{N}$.

24   Let k be a given positive integer. The sequence x_n is defined as follows: $x_1 = 1$ and x_{n+1} is the least positive integer which is not in $\{x_1, x_2, ..., x_n, x_1 + k, x_2 + 2k, ..., x_n + nk\}$. Show that there exist real number a such that $x_n = \lfloor an \rfloor$ for all positive integer n.

25   Let $\{a_n\}_{n \geq 1}$ be a sequence of positive integers such that

$$0 < a_{n+1} - a_n \leq 2001 \ \text{ for all } \ n \in \mathbb{N}.$$

Show that there are infinitely many pairs (p, q) of positive integers such that $p > q$ and $a_q \mid a_p$.

26   Let p be an odd prime p such that $2h \neq 1 \pmod{p}$ for all $h \in \mathbb{N}$ with $h < p - 1$, and let a be an even integer with $a \in]\frac{p}{2}, p[$. The sequence $\{a_n\}_{n \geq 0}$ is defined by $a_0 = a$, $a_{n+1} = p - b_n \ (n \geq 0)$, where b_n is the greatest odd divisor of a_n. Show that the sequence $\{a_n\}_{n \geq 0}$ is periodic and find its minimal (positive) period.

27   Let $p \geq 3$ be a prime number. The sequence $\{a_n\}_{n \geq 0}$ is defined by $a_n = n$ for all $0 \leq n \leq p - 1$, and $a_n = a_{n-1} + a_{n-p}$ for all $n \geq p$. Compute $a_{p^3} \pmod{p}$.

28   Let $\{u_n\}_{n \geq 0}$ be a sequence of integers satisfying the recurrence relation $u_{n+2} = u_{n+1}^2 - u_n \ (n \in \mathbb{N})$. Suppose that $u_0 = 39$ and $u_1 = 45$. Prove that 1986 divides infinitely many terms of this sequence.

29   The sequence $\{a_n\}_{n \geq 1}$ is defined by $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{4a_n} \ (n \in \mathbb{N}).$$

Prove that $\sqrt{\frac{2}{2a_n^2 - 1}}$ is a positive integer for $n > 1$.

30   Let k be a positive integer. Prove that there exists an infinite monotone increasing sequence of integers $\{a_n\}_{n \geq 1}$ such that

$$a_n \text{ divides } a_{n+1}^2 + k \ \text{ and } \ a_{n+1} \text{ divides } a_n^2 + k$$

for all $n \in \mathbb{N}$.

31   Each term of an infinite sequence of natural numbers is obtained from the previous term by adding to it one of its nonzero digits. Prove that this sequence contains an even number.

32   In an increasing infinite sequence of positive integers, every term starting from the 2002-th term divides the sum of all preceding terms. Prove that every term starting from some term is equal to the sum of all preceding terms.

33   The sequence $\{x_n\}_{n\geq 1}$ is defined by

$$x_1 = 2, x_{n+1} = \frac{2 + x_n}{1 - 2x_n} \ \ (n \in \mathbb{N}).$$

Prove that

a) $x_n \neq 0$ for all $n \in \mathbb{N}$,

b) $\{x_n\}_{n\geq 1}$ is not periodic.

34   The sequence of integers $\{x_n\}_{n\geq 1}$ is defined as follows:

$$x_1 = 1, \ \ x_{n+1} = 1 + x_1^2 + \cdots + x_n^2 \ (n = 1, 2, 3 \cdots).$$

Prove that there are no squares of natural numbers in this sequence except x_1.

35   The first four terms of an infinite sequence S of decimal digits are $1, 9, 8, 2$, and succeeding terms are given by the final digit in the sum of the four immediately preceding terms. Thus S begins $1, 9, 8, 2, 0, 9, 9, 0, 8, 6, 3, 7, 4, \cdots$. Do the digits $3, 0, 4, 4$ ever come up consecutively in S?