Functional Equations

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1 Prove that there is a function f from the set of all natural numbers into itself such that $f(f(n)) = n^2$ for all $n \in \mathbb{N}$.

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2 Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$:

$$m|n \Longleftrightarrow f(m)|f(n).$$

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3 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(n+1) > f(f(n)).$$

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4 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(f(n))) + f(f(n)) + f(n) = 3n.$$

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5 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(m) + f(n)) = m + n.$$

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6 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f^{(19)}(n) + 97f(n) = 98n + 232.$$

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7 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(n)) + f(n) = 2n + 2001 \text{ or } 2n + 2002.$$

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8 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2001.$$

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9 Find all functions $f:\mathbb{N}_0 \to \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$:

$$f(f(n))+f(n)=2n+6.$$

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10 Find all functions $f:\mathbb{N}_0 \to \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$:

$$f(m+f(n))=f(f(m))+f(n).$$

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11 Find all functions $f:\mathbb{N}_0 \to \mathbb{N}_0$ such that for all $m,n \in \mathbb{N}_0$:

$$mf(n)+nf(m)=(m+n)f(m^2+n^2).$$

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12 Find all functions $f:\mathbb{N} \to \mathbb{N}$ such that for all $m,n \in \mathbb{N}$: - $f(2)=2$, - $f(mn)=f(m)f(n)$, - $f(n+1)>f(n)$.

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13 Find all functions $f:\mathbb{Z} \to \mathbb{Z}$ such that for all $m \in \mathbb{Z}$:

$$f(f(m))=m+1.$$

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14 Find all functions $f:\mathbb{Z} \to \mathbb{Z}$ such that for all $m \in \mathbb{Z}$: - $f(m+8)\le f(m)+8$, - $f(m+11)\ge f(m)+11$.

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15 Find all functions $f:\mathbb{Z} \to \mathbb{Z}$ such that for all $m,n \in \mathbb{Z}$:

$$f(m+f(n))=f(m)-n.$$

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16 Find all functions $f:\mathbb{Z} \to \mathbb{Z}$ such that for all $m,n \in \mathbb{Z}$:

$$f(m+f(n))=f(m)+n.$$

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17 Find all functions $h:\mathbb{Z} \to \mathbb{Z}$ such that for all $x,y \in \mathbb{Z}$:

$$h(x+y)+h(xy)=h(x)h(y)+1.$$

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18 Find all functions $f:\mathbb{Q}\to\mathbb{R}$ such that for all $x,y\in\mathbb{Q}$:

$$f(xy)=f(x)f(y)-f(x+y)+1.$$

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19 Find all functions $f:\mathbb{Q}^{+}\to\mathbb{Q}^{+}$ such that for all $x,y\in\mathbb{Q}$:

$$f\left(x+\frac{y}{x}\right)=f(x)+\frac{f(y)}{f(x)}+2y,\ x,y\in\mathbb{Q}^{+}.$$

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20 Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that for all $x,y\in\mathbb{Q}$:

$$f(x+y)+f(x-y)=2(f(x)+f(y)).$$

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22 Find all functions $f:\mathbb{Q}^{+}\to\mathbb{Q}^{+}$ such that for all $x\in\mathbb{Q}^{+}$: - $f(x+1)=f(x)+1$, - $f(x^2)=f(x)^2$.

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23 Let $\mathbb{Q}^{+}$ be the set of positive rational numbers. Construct a function $f:\mathbb{Q}^{+}\to\mathbb{Q}^{+}$ such that

$$f(xf(y))=\frac{f(x)}{y}$$

for all $x,y\in\mathbb{Q}^{+}$.

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24 A function f is defined on the positive integers by

$$\begin{cases} f(1) &= 1, \\ f(3) &= 3, \\ f(2n) &= f(n), \\ f(4n+1) &= 2f(2n+1)-f(n), \\ f(4n+3) &= 3f(2n+1)-2f(n), \end{cases}$$

for all positive integers n. Determine the number of positive integers n, less than or equal to 1988, for which $f(n)=n$.

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25 Consider all functions $f:\mathbb{N}\to\mathbb{N}$ satisfying $f(t^2f(s))=s(f(t))^2$ for all s and t in N. Determine the least possible value of $f(1998)$.

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26 The function $f:\mathbb{N}\to\mathbb{N}_0$ satisfies for all $m,n\in\mathbb{N}$:

$$f(m+n)-f(m)-f(n)=0 \text{ or } 1,\ \ f(2)=0,\ \ f(3)>0,\ \ \text{and } f(9999)=3333.$$

Determine $f(1982)$.

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27 Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that for all $m,n\in\mathbb{N}$:

$$f(f(m)+f(n))=m+n.$$

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28 Find all surjective functions $f:\mathbb{N}\to\mathbb{N}$ such that for all $n\in\mathbb{N}$:

$$f(n)\geq n+(-1)^n.$$

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29 Find all functions $f:\mathbb{Z}\setminus\{0\}\to\mathbb{Q}$ such that for all $x,y\in\mathbb{Z}\setminus\{0\}$:

$$f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{2},\quad x,y\in\mathbb{Z}\setminus\{0\}$$

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30 Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that for all $n\in\mathbb{N}$:

$$f(f(f(n)))+f(f(n))+f(n)=3n.$$

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31 Find all strictly increasing functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$f(f(n))=3n.$$

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32 Find all functions $f:\mathbb{Z}^2\to\mathbb{R}^+$ such that for all $i,j\in\mathbb{Z}$:

$$f(i,j)=\frac{f(i+1,j)+f(i,j+1)+f(i-1,j)+f(i,j-1)}{4}.$$

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33 Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that for all $x,y,z\in\mathbb{Q}$:

$$f(x+y+z)+f(x-y)+f(y-z)+f(z-x)=3f(x)+3f(y)+3f(z).$$

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34 Show that there exists a bijective function $f:\mathbb{N}_0\to\mathbb{N}_0$ such that for all $m,n\in\mathbb{N}_0$:

$$f(3mn+m+n)=4f(m)f(n)+f(m)+f(n).$$
