Divisibility Theory

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1 Show that if x, y, z are positive integers, then $(xy+1)(yz+1)(zx+1)$ is a perfect square if and only if $xy+1$, $yz+1$, $zx+1$ are all perfect squares.

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2 Find infinitely many triples (a, b, c) of positive integers such that a, b, c are in arithmetic progression and such that $ab+1$, $bc+1$, and $ca+1$ are perfect squares.

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3 Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

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4 If a, b, c are positive integers such that

$$0 < a^2+b^2-abc \le c,$$

show that a^2+b^2-abc is a perfect square.

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5 Let x and y be positive integers such that xy divides x^2+y^2+1. Show that

$$\frac{x^2+y^2+1}{xy} = 3.$$

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6 - Find infinitely many pairs of integers a and b with $1 < a < b$, so that ab exactly divides a^2+b^2-1. - With a and b as above, what are the possible values of

$$\frac{a^2+b^2-1}{ab}?$$

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7 Let n be a positive integer such that $2+2\sqrt{28n^2+1}$ is an integer. Show that $2+2\sqrt{28n^2+1}$ is the square of an integer.

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8 The integers a and b have the property that for every nonnegative integer n the number of $2^n a+b$ is the square of an integer. Show that $a=0$.

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9 Prove that among any ten consecutive positive integers at least one is relatively prime to the product of the others.

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10 Let n be a positive integer with $n \geq 3$. Show that

$$n^{n^{n^n}} - n^{n^n}$$

is divisible by 1989.

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11 Let a, b, c, d be integers. Show that the product

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 12.

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12 Let k, m, and n be natural numbers such that $m+k+1$ is a prime greater than $n+1$. Let $c_s = s(s+1)$. Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \cdots (c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 \cdots c_n$.

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13 Show that for all prime numbers p,

$$Q(p) = \prod_{k=1}^{p-1} k^{2k-p-1}$$

is an integer.

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14 Let n be an integer with $n \geq 2$. Show that n does not divide $2^n - 1$.

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15 Suppose that $k \geq 2$ and $n_1, n_2, \cdots, n_k \geq 1$ be natural numbers having the property

$$n_2 \mid 2^{n_1} - 1, n_3 \mid 2^{n_2} - 1, \cdots, n_k \mid 2^{n_{k-1}} - 1, n_1 \mid 2^{n_k} - 1.$$

Show that $n_1 = n_2 = \cdots = n_k = 1$.

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16 Determine if there exists a positive integer n such that n has exactly 2000 prime divisors and $2^n + 1$ is divisible by n.

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17 Let m and n be natural numbers such that

$$A = \frac{(m+3)^n + 1}{3m}$$

is an integer. Prove that A is odd.

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18 Let m and n be natural numbers and let $mn+1$ be divisible by 24. Show that $m+n$ is divisible by 24.

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19 Let $f(x) = x^3 + 17$. Prove that for each natural number $n \geq 2$, there is a natural number x for which $f(x)$ is divisible by 3^n but not 3^{n+1}.

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20 Determine all positive integers n for which there exists an integer m such that $2^n - 1$ divides $m^2 + 9$.

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21 Let n be a positive integer. Show that the product of n consecutive positive integers is divisible by $n!$

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22 Prove that the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

is not divisible by 5 for any integer $n \geq 0$.

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23 (Wolstenholme's Theorem) Prove that if

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

is expressed as a fraction, where $p \geq 5$ is a prime, then p^2 divides the numerator.

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24 Let $p > 3$ is a prime number and $k = \lfloor \frac{2p}{3} \rfloor$. Prove that

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

is divisible by p^2.

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25 Show that $\binom{2n}{n} \mid \text{lcm}(1, 2, \cdots, 2n)$ for all positive integers n.

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26 Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

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27 Show that the coefficients of a binomial expansion $(a+b)^n$ where n is a positive integer, are all odd, if and only if n is of the form $2^k - 1$ for some positive integer k.

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28 Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of positive integers (m, n) with $n \geq m \geq 1$.

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29 For which positive integers k, is it true that there are infinitely many pairs of positive integers (m, n) such that

$$\frac{(m+n-k)!}{m!\,n!}$$

is an integer?

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30 Show that if $n \geq 6$ is composite, then n divides $(n-1)!$.

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31 Show that there exist infinitely many positive integers n such that n^2+1 divides $n!$.

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32 Let a and b be natural numbers such that

$$\frac{a}{b} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that a is divisible by 1979.

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33 Let $a, b, x \in \mathbb{N}$ with $b > 1$ and such that $b^n - 1$ divides a. Show that in base b, the number a has at least n non-zero digits.

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34 Let $p_1, p_2, \cdots, p_n$ be distinct primes greater than 3. Show that

$$2^{p_1 p_2 \cdots p_n} + 1$$

has at least 4^n divisors.

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35 Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p-2$ such that neither $a^{p-1} - 1$ nor $(a+1)^{p-1} - 1$ is divisible by p^2.

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36 Let n and q be integers with $n \geq 5$, $2 \leq q \leq n$. Prove that $q-1$ divides $\left\lfloor \frac{(n-1)!}{q} \right\rfloor$.

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37 If n is a natural number, prove that the number $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.

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38 Let p be a prime with $p > 5$, and let $S = \{p - n^2 | n \in \mathbb{N}, n^2 < p\}$. Prove that S contains two elements a and b such that $a|b$ and $1 < a < b$.

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39 Let n be a positive integer. Prove that the following two statements are equivalent. - n is not divisible by 4 - There exist $a, b \in \mathbb{Z}$ such that $a^2 + b^2 + 1$ is divisible by n.

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40 Determine the greatest common divisor of the elements of the set

$$\{n^{13} - n \mid n \in \mathbb{Z}\}.$$

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41 Show that there are infinitely many composite numbers n such that $3^{n-1} - 2^{n-1}$ is divisible by n.

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42 Suppose that $2^n + 1$ is an odd prime for some positive integer n. Show that n must be a power of 2.

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43 Suppose that p is a prime number and is greater than 3. Prove that $7^p - 6^p - 1$ is divisible by 43.

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44 Suppose that $4^n + 2^n + 1$ is prime for some positive integer n. Show that n must be a power of 3.

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45 Let $b, m, n \in \mathbb{N}$ with $b > 1$ and $m \neq n$. Suppose that $b^m - 1$ and $b^n - 1$ have the same set of prime divisors. Show that $b + 1$ must be a power of 2.

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46 Let a and b be integers. Show that a and b have the same parity if and only if there exist integers c and d such that $a^2 + b^2 + c^2 + 1 = d^2$.

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47 Let n be a positive integer with $n > 1$. Prove that

$$\frac{1}{2} + \cdots + \frac{1}{n}$$

is not an integer.

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48 Let n be a positive integer. Prove that

$$\frac{1}{3} + \cdots + \frac{1}{2n+1}$$

is not an integer.

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49 Prove that there is no positive integer n such that, for $k = 1, 2, \cdots, 9$, the leftmost digit of $(n+k)!$ equals k.

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50 Show that every integer $k > 1$ has a multiple less than k^4 whose decimal expansion has at most four distinct digits.

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51 Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.

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52 Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a and b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

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53 Suppose that $x, y,$ and z are positive integers with $xy = z^2 + 1$. Prove that there exist integers $a, b, c,$ and d such that $x = a^2 + b^2$, $y = c^2 + d^2$, and $z = ac + bd$.

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54 A natural number n is said to have the property P, if whenever n divides $a^n - 1$ for some integer a, n^2 also necessarily divides $a^n - 1$. - Show that every prime number n has the property P. - Show that there are infinitely many composite numbers n that possess the property P.

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55 Show that for every natural number n the product

$$\left(4 - \frac{2}{1}\right)\left(4 - \frac{2}{2}\right)\left(4 - \frac{2}{3}\right)\cdots\left(4 - \frac{2}{n}\right)$$

is an integer.

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56 Let $a, b,$ and c be integers such that $a + b + c$ divides $a^2 + b^2 + c^2$. Prove that there are infinitely many positive integers n such that $a + b + c$ divides $a^n + b^n + c^n$.

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57 Prove that for every $n \in \mathbb{N}$ the following proposition holds: $7|3^n + n^3$ if and only if $7|3^n n^3 + 1$.

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58 Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \geq \frac{3k}{4}$. Let n be a composite integer. Prove that - if $n = 2p_k$, then n does not divide $(n-k)!$, - if $n > 2p_k$, then n divides $(n-k)!$.

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59 Suppose that n has (at least) two essentially distinct representations as a sum of two squares. Specifically, let $n = s^2 + t^2 = u^2 + v^2$, where $s \geq t \geq 0$, $u \geq v \geq 0$, and $s > u$. Show that $\gcd(su - tv, n)$ is a proper divisor of n.

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60 Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t, the number $at + b$ is a triangular number if and only if t is a triangular number.

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61 For any positive integer $n > 1$, let $p(n)$ be the greatest prime divisor of n. Prove that there are infinitely many positive integers n with

$$p(n) < p(n+1) < p(n+2).$$

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62 Let $p(n)$ be the greatest odd divisor of n. Prove that

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\frac{p(k)}{k} > \frac{2}{3}.$$

63 There is a large pile of cards. On each card one of the numbers $1, 2, \cdots, n$ is written. It is known that the sum of all numbers of all the cards is equal to $k \cdot n!$ for some integer k. Prove that it is possible to arrange cards into k stacks so that the sum of numbers written on the cards in each stack is equal to $n!$.

64 The last digit of the number $x^2 + xy + y^2$ is zero (where x and y are positive integers). Prove that two last digits of this numbers are zeros.

65 Clara computed the product of the first n positive integers and Valerid computed the product of the first m even positive integers, where $m \geq 2$. They got the same answer. Prove that one of them had made a mistake.

66 (Four Number Theorem) Let $a, b, c,$ and d be positive integers such that $ab = cd$. Show that there exists positive integers p, q, r, s such that

$$a = pq, \quad b = rs, \quad c = ps, \quad d = qr.$$

67 Prove that $\binom{2n}{n}$ is divisible by $n + 1$.

68 Suppose that $S = \{a_1, \cdots, a_r\}$ is a set of positive integers, and let S_k denote the set of subsets of S with k elements. Show that

$$\text{lcm}(a_1, \cdots, a_r) = \prod_{i=1}^{r} \prod_{s \in S_i} \gcd(s)^{\left((-1)^i\right)}.$$

69 Prove that if the odd prime p divides $a^b - 1$, where a and b are positive integers, then p appears to the same power in the prime factorization of $b(a^d - 1)$, where $d = \gcd(b, p - 1)$.

70 Suppose that $m = nq$, where n and q are positive integers. Prove that the sum of binomial coefficients

$$\sum_{k=0}^{n-1} \binom{\gcd(n, k)q}{\gcd(n, k)}$$

is divisible by m.

71 Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

72 Determine all pairs (n, p) of nonnegative integers such that - p is a prime, - $n < 2p$, - $(p-1)^n + 1$ is divisible by n^{p-1}.

73 Determine all pairs (n, p) of positive integers such that - p is a prime, $n > 1$, - $(p-1)^n + 1$ is divisible by n^{p-1}.

74 Find an integer n, where $100 \leq n \leq 1997$, such that

$$\frac{2^n + 2}{n}$$

is also an integer.

75 Find all triples (a, b, c) of positive integers such that $2^c - 1$ divides $2^a + 2^b + 1$.

76 Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1) \quad \text{is a divisor of} \quad abc - 1.$$

77 Find all positive integers, representable uniquely as

$$\frac{x^2 + y}{xy + 1},$$

where x and y are positive integers.

78 Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

79 Determine all pairs of integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

80 Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that
$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$
is itself an integer.

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81 Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a + 1)^n$.

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82 Which integers can be represented as
$$\frac{(x + y + z)^2}{xyz}$$
where x, y, and z are positive integers?

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83 Find all $n \in \mathbb{N}$ such that $\lfloor \sqrt{n} \rfloor$ divides n.

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84 Determine all $n \in \mathbb{N}$ for which - n is not the square of any integer, - $\lfloor \sqrt{n} \rfloor^3$ divides n^2.

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85 Find all $n \in \mathbb{N}$ such that 2^{n-1} divides $n!$.

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86 Find all positive integers (x, n) such that $x^n + 2^n + 1$ divides $x^{n+1} + 2^{n+1} + 1$.

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87 Find all positive integers n such that $3^n - 1$ is divisible by 2^n.

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88 Find all positive integers n such that $9^n - 1$ is divisible by 7^n.

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89 Determine all pairs (a, b) of integers for which $a^2 + b^2 + 3$ is divisible by ab.

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90 Determine all pairs (x, y) of positive integers with $y|x^2 + 1$ and $x^2|y^3 + 1$.

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91 Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

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92 Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.

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93 Find the largest positive integer n such that n is divisible by all the positive integers less than $\sqrt[3]{n}$.

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94 Find all $n \in \mathbb{N}$ such that $3^n - n$ is divisible by 17.

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95 Suppose that a and b are natural numbers such that

$$p = \frac{b}{4}\sqrt{\frac{2a-b}{2a+b}}$$

is a prime number. What is the maximum possible value of p?

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96 Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2 \cdots, d_{16}$ such that $1 = d_1 < d_2 < \cdots < d_{16} = n$, $d_6 = 18$, and $d_9 - d_8 = 17$.

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97 Suppose that n is a positive integer and let

$$d_1 < d_2 < d_3 < d_4$$

be the four smallest positive integer divisors of n. Find all integers n such that

$$n = {d_1}^2 + {d_2}^2 + {d_3}^2 + {d_4}^2.$$

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98 Let n be a positive integer with $k \geq 22$ divisors $1 = d_1 < d_2 < \cdots < d_k = n$, all different. Determine all n such that

$${d_7}^2 + {d_{10}}^2 = \left(\frac{n}{d_{22}}\right)^2.$$

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99 Let $n \geq 2$ be a positive integer, with divisors

$$1 = d_1 < d_2 < \cdots < d_k = n \ .$$

Prove that

$$d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$$

is always less than n^2, and determine when it divides n^2.

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100 Find all positive integers n such that n has exactly 6 positive divisors $1 < d_1 < d_2 < d_3 < d_4 < n$ and $1 + n = 5(d_1 + d_2 + d_3 + d_4)$.

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101 Find all composite numbers n having the property that each proper divisor d of n has $n - 20 \leq d \leq n - 12$.

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102 Determine all three-digit numbers N having the property that N is divisible by 11, and $\frac{N}{11}$ is equal to the sum of the squares of the digits of N.

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103 When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

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104 A wobbly number is a positive integer whose $digits$ in base 10 are alternatively non-zero and zero the units digit being non-zero. Determine all positive integers which do not divide any wobbly number.

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105 Find the smallest positive integer n such that - n has exactly 144 distinct positive divisors, - there are ten consecutive integers among the positive divisors of n.

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106 Determine the least possible value of the natural number n such that $n!$ ends in exactly 1987 zeros.

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107 Find four positive integers, each not exceeding 70000 and each having more than 100 divisors.

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108 For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Determine all triples (x, y, z) of distinct positive integers satisfying - x, y, z are in arithmetic progression, - $p(xyz) \leq 3$.

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109 Find all positive integers a and b such that

$$\frac{a^2+b}{b^2-a} \text{ and } \frac{b^2+a}{a^2-b}$$

are both integers.

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110 For each positive integer n, write the sum $\sum_{m=1}^{n} 1/m$ in the form p_n/q_n, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

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111 Find all natural numbers n such that the number $n(n+1)(n+2)(n+3)$ has exactly three different prime divisors.

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112 Prove that there exist infinitely many pairs (a, b) of relatively prime positive integers such that

$$\frac{a^2-5}{b} \text{ and } \frac{b^2-5}{a}$$

are both positive integers.

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113 Find all triples (l, m, n) of distinct positive integers satisfying

$$\gcd(l, m)^2 = l + m,\ \gcd(m, n)^2 = m + n, \text{ and } \gcd(n, l)^2 = n + l.$$

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114 What is the greatest common divisor of the set of numbers

$$\{16^n + 10n - 1 \mid n = 1, 2, \cdots\}?$$

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115 Does there exist a 4-digit integer (in decimal form) such that no replacement of three of its digits by any other three gives a multiple of 1992?

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116 What is the smallest positive integer that consists base 10 of each of the ten digits, each used exactly once, and is divisible by each of the digits 2 through 9?

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117 Find the smallest positive integer n such that

$$2^{1989} \mid m^n - 1$$

for all odd positive integers $m > 1$.

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118 Determine the highest power of 1980 which divides

$$\frac{(1980n)!}{(n!)^{1980}}.$$
