Irrational Numbers

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by Peter

1 Find the smallest positive integer n such that

$$0 < \sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor < 0.00001.$$

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2 Prove that for any positive integers a and b

$$\left| a\sqrt{2} - b \right| > \frac{1}{2(a+b)}.$$

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3 Prove that there exist positive integers m and n such that

$$\left| \frac{m^2}{n^3} - \sqrt{2001} \right| < \frac{1}{10^8}.$$

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4 Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that

$$\left| a + b\sqrt{2} + c\sqrt{3} \right| > \frac{1}{10^{21}}.$$

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5 Let a, b, c be integers, not all equal to 0. Show that

$$\frac{1}{4a^2 + 3b^2 + 2c^2} \le |\sqrt[3]{4}a + \sqrt[3]{2}b + c|.$$

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6 Prove that for any irrational number ξ, there are infinitely many rational numbers $\frac{m}{n}$ $((m, n) \in \mathbb{Z} \times \mathbb{N})$ such that

$$\left| \xi - \frac{n}{m} \right| < \frac{1}{\sqrt{5}m^2}.$$

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7 Show that π is irrational.

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8 Show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is irrational.

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9 Show that $\cos \frac{\pi}{7}$ is irrational.

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10 Show that $\frac{1}{\pi}\arccos\left(\frac{1}{\sqrt{2003}}\right)$ is irrational.

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11 Show that $\cos 1^{\circ}$ is irrational.

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12 An integer-sided triangle has angles $p\theta$ and $q\theta$, where p and q are relatively prime integers. Prove that $\cos\theta$ is irrational.

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13 It is possible to show that $\csc\frac{3\pi}{29}-\csc\frac{10\pi}{29}=1.999989433...$. Prove that there are no integers j, k, n with odd n satisfying $\csc\frac{j\pi}{n}-\csc\frac{k\pi}{n}=2$.

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14 For which angles θ, with θ a rational number of degrees, is $\tan^{2}\theta+\tan^{2}2\theta$ is irrational?

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15 Prove that for any $p, q \in \mathbb{N}$ with $q > 1$ the following inequality holds:

$$\left|\pi-\frac{p}{q}\right| \geq q^{-42}.$$

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16 For each integer $n \geq 1$, prove that there is a polynomial $P_{n}(x)$ with rational coefficients such that $x^{4n}(1-x)^{4n}=(1+x)^{2}P_{n}(x)+(-1)^{n}4^{n}$. Define the rational number a_{n} by

$$a_{n}=\frac{(-1)^{n-1}}{4^{n-1}}\int_{0}^{1}P_{n}(x)\;dx,\;\;n=1,2,\cdots.$$

Prove that a_{n} satisfies the inequality

$$\left|\pi-a_{n}\right|<\frac{1}{4^{5n-1}},\;\;n=1,2,\cdots.$$

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17 Suppose that $p, q \in \mathbb{N}$ satisfy the inequality

$$\exp(1)\cdot(\sqrt{p+q}-\sqrt{q})^{2}<1.$$

Show that $\ln\left(1+\frac{p}{q}\right)$ is irrational.

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18 Show that the cube roots of three distinct primes cannot be terms in an arithmetic progression.

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19 Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with a rational area.

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20 You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:

$$\begin{array}{lll} A & B & C \\ 10 & 1010 & 20 \\ 100 & 1100100 & 400 \\ 1000 & 1111101000 & 13000 \\ \vdots & \vdots & \vdots \end{array}.$$

Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.

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21 Prove that if α and β are positive irrational numbers satisfying $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then the sequences

$$\lfloor\alpha\rfloor, \lfloor 2\alpha\rfloor, \lfloor 3\alpha\rfloor, \cdots$$

and

$$\lfloor\beta\rfloor, \lfloor 2\beta\rfloor, \lfloor 3\beta\rfloor, \cdots$$

together include every positive integer exactly once.

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22 For a positive real number α, define

$$S(\alpha)=\{\lfloor n\alpha\rfloor \mid n=1,2,3,\cdots\}.$$

Prove that $\mathbb{N}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

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23 Let $f(x)=\prod_{n=1}^{\infty}\left(1+\frac{x}{2^n}\right)$. Show that at the point $x=1$, $f(x)$ and all its derivatives are irrational.

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24 Let $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers such that

$$a_{n+1}^2 = a_n + 1, \quad n \in \mathbb{N}.$$

Show that the sequence contains an irrational number.

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25 Show that $\tan\left(\frac{\pi}{m}\right)$ is irrational for all positive integers $m \geq 5$.

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26 Prove that if $g \geq 2$ is an integer, then two series

$$\sum_{n=0}^{\infty}\frac{1}{g^{n^2}} \text{ and } \sum_{n=0}^{\infty}\frac{1}{g^{n!}}$$

both converge to irrational numbers.

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27 Let $1 < a_1 < a_2 < \cdots$ be a sequence of positive integers. Show that

$$\frac{2^{a_1}}{a_1!} + \frac{2^{a_2}}{a_2!} + \frac{2^{a_3}}{a_3!} + \cdots$$

is irrational.

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28 Do there exist real numbers a and b such that - $a + b$ is rational and $a^n + b^n$ is irrational for all $n \in \mathbb{N}$ with $n \geq 2$? - $a + b$ is irrational and $a^n + b^n$ is rational for all $n \in \mathbb{N}$ with $n \geq 2$?

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29 Let $p(x) = x^3 + a_1 x^2 + a_2 x + a_3$ have rational coefficients and have roots r_1, r_2, and r_3. If $r_1 - r_2$ is rational, must r_1, r_2, and r_3 be rational?

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30 Let $\alpha = 0.d_1 d_2 d_3 \cdots$ be a decimal representation of a real number between 0 and 1. Let r be a real number with $|r| < 1$. - If α and r are rational, must $\sum_{i=1}^{\infty} d_i r^i$ be rational? - If $\sum_{i=1}^{\infty} d_i r^i$ and r are rational, α must be rational?
