Divisor Functions

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by Peter

1 Let n be an integer with $n \geq 2$. Show that $\phi(2^n-1)$ is divisible by n.

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2 Show that for all $n \in \mathbb{N}$,

$$n = \sum_{d|n} \phi(d).$$

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3 If p is a prime and n an integer such that $1 < n \leq p$, then

$$\phi\left(\sum_{k=0}^{p-1} n^k\right) \equiv 0 \pmod{p}.$$

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4 Let m, n be positive integers. Prove that, for some positive integer a, each of $\phi(a)$, $\phi(a+1)$, $\cdots$, $\phi(a+n)$ is a multiple of m.

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5 If n is composite, prove that $\phi(n) \leq n-\sqrt{n}$.

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6 Show that if m and n are relatively prime positive integers, then $\phi(5^m-1) \neq 5^n-1$.

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7 Show that if the equation $\phi(x)=n$ has one solution, it always has a second solution, n being given and x being the unknown.

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8 Prove that for any $\delta \in [0,1]$ and any $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $\left\vert \frac{\phi(n)}{n}-\delta \right\vert < \varepsilon$.

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9 Show that the set of all numbers $\frac{\phi(n+1)}{\phi(n)}$ is dense in the set of all positive reals.

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10 Show that - if $n > 49$, then there are positive integers $a > 1$ and $b > 1$ such that $a+b=n$ and $\frac{\phi(a)}{a}+\frac{\phi(b)}{b} < 1$. - if $n > 4$, then there are $a > 1$ and $b > 1$ such that $a+b=n$ and $\frac{\phi(a)}{a}+\frac{\phi(b)}{b} > 1$.

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11 Prove that $d((n^2+1)^2)$ does not become monotonic from any given point onwards.

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12 Determine all positive integers n such that $n = d(n)^2$.

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13 Determine all positive integers k such that

$$\frac{d(n^2)}{d(n)} = k$$

for some $n \in \mathbb{N}$.

14 Find all positive integers n such that $d(n)^3 = 4n$.

15 Determine all positive integers for which $d(n) = \frac{n}{3}$ holds.

16 We say that an integer $m \geq 1$ is super-abundant if

$$\frac{\sigma(m)}{m} > \frac{\sigma(k)}{k}$$

for all $k \in \{1, 2, \cdots, m-1\}$. Prove that there exists an infinite number of super-abundant numbers.

17 Show that $\phi(n) + \sigma(n) \geq 2n$ for all positive integers n.

18 Prove that for any δ greater than 1 and any positive number ϵ, there is an n such that $\left|\frac{\sigma(n)}{n} - \delta\right| < \epsilon$.

19 Prove that $\sigma(n)\phi(n) < n^2$, but that there is a positive constant c such that $\sigma(n)\phi(n) \geq cn^2$ holds for all positive integers n.

20 Show that $\sigma(n) - d(m)$ is even for all positive integers m and n where m is the largest odd divisor of n.

21 Show that for any positive integer n,

$$\frac{\sigma(n!)}{n!} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

22 Let n be an odd positive integer. Prove that $\sigma(n)^3 < n^4$.