ACM-ICPC 代码模板

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The real voyage of algorithm consists,
not in seeking new landscapes,
but in having new eyes.

Contents

1 头文件

1.1 头文件 (全)

```
#include<bits/stdc++.h>
using namespace std;
typedef long long ll;
typedef pair<int, int> pii;
#define iofast ios::sync_with_stdio(false),cin.tie(0)
    ,cout.tie(0)
#define lowbit(x) (x&(-x))
#define inv(x) qpow(x,mod-2)
#define ctz(x) __builtin_ctz(x) //末尾0个数
#define clz(x) __builtin_clz(x) //前导0个数
#define popcount(x) __builtin_popcount(x) //1的个数
#define ffs(x) __builtin_ffs(x) //最后一个1的位置
#define int128 __int128_t
const int iINF = 0x3f3f3f3f;
const ll llINF = 0x3f3f3f3f3f3f3f3f;
template<typename T>
void read(T& x)
{
    x = 0;
    int flag = 1;
    char c = getchar();
    while(!isdigit(c)){
        if (c == '-')
            flag = -1;
        c = getchar();
    }
    while(isdigit(c)){
        x = (x << 3) + (x << 1) + (c ^ 48);
        c = getchar();
    }
    x *= flag;
}

template<typename T,typename ...Arg>
void read(T& x,Arg& ...args)
{
    read(x);
    read(args...);
}
const ll mod = 998244353;
ll qpow(ll a,ll b)
{
    ll ans = 1;
    a %= mod;
    for (; b;b>>=1){
        if(b&1)
            ans = ans * a % mod;
        a = a * a % mod;
    }
    return ans % mod;
}

template<typename T>
void write(T x, char c = '\0') {
    if (x < 0) {
        x = -x;
        putchar('-');
    }
    if (x > 9)
        write(x / 10);
    putchar(x % 10 + '0');
    if (c != '\0')
        putchar(c);
}
/*---------------------------------------*/

void solve()
{

}

/*---------------------------------------*/

int main()
{
    iofast;
    int t = 1;
    cin >> t;
    while (t--)
        solve();

    return 0;
}
```

1.2 头文件 (赛)

```
#include<bits/stdc++.h>
using namespace std;
using ll = long long;
using pii = pair<int, int>;
#define range(x) (x).begin(), (x).end()

const ll mod = 998244353;

/*---------------------------------------*/

void solve()
{

}

/*---------------------------------------*/

int main()
{
    ios::sync_with_stdio(false);
    cin.tie(0);
    cout.tie(0);
    int t = 1;
    cin >> t;
    while (t--)
        solve();

    return 0;
}
```

2 图论

请注意, 图论算法后续省略建图过程, 默认链式前向星存图

2.1 建图

```
const int N = 100005;
const int M = 200005;

int head[N], ver[M], Next[M], edge[M];
int tot;
void add(int x,int y,int z)
{
    ver[++tot] = y;
    Next[tot] = head[x];
    edge[tot] = z;
    head[x] = tot;
}
```

2.2 SPFA 算法

```
/*寻找负环时添加cnt数组,并将队列替换为栈*/
/*注意队列操作与栈操作的替换*/

/*计算差分约束时
如果求的是最小值,则应该求最长路,如果求的是最大值,则应该求最短路
负环即无解
把每个x[i] ≤ x[j] + C[k]不等式转化为一条从x[j]走到x[i]长度为C[k]的边
从0号点向x[i]<=C[k]的i点连边
*/
//int cnt[N];
//stack<int>st;
int d[N];
bool v[N];
queue<int>q;

void spfa(int s)
{
    memset(d,0x3f,sizeof d);
    memset(v,0,sizeof v);
    //memset(cnt,0,sizeof cnt); 负环cnt数组初始化
    d[s]=0;
    v[s]=1;
    q.push(s);
    while(!q.empty()){
        int x=q.front();
        q.pop();
        v[x]=0;
        for(int i=head[x];i;i=Next[i]){
            int y=ver[i];
            int z=edge[i];
            if(d[y]>d[x]+z){
                d[y]=d[x]+z;

                /*负环操作
                cnt[y]=cnt[x]+1;
                if(cnt[y]>=n+1)
                    return true;
                    */

                if(!v[y]){
                    q.push(y);
                    v[y]=1;
                }
            }
        }
    }

    return;
}
```

2.3 Dijestra 算法

```
int d[N];
bool v[N];
typedef pair<int,int> pii;
priority_queue<pii,vector<pii>,greater<pii>>q;

void dij(int s)
{
    //初始化
    memset(d,0x3f,sizeof d);
    memset(v,0,sizeof v);
    while(!q.empty())
        q.pop();

    q.push({0,s});
    d[s]=0;
    while(!q.empty()){
        auto [dist,x]=q.top();
        q.pop();
        if(v[x])
            continue;
        v[x]=1;

        for(int i=head[x];i;i=Next[i]){
            int y=ver[i];
            int z=edge[i];
            if(d[y]>dist+z){
                d[y]=dist+z;
                q.push({d[y],y});
            }
        }

    }

    return;
}
```

2.4 Floyd 算法

```
int d[305][305];
for(int i=1;i<=m;i++){
    int x,y,z;
    cin>>x>>y>>z;
    d[x][y]=min(d[x][y],z);

    /*传递闭包
    d[x][y]=d[y][x]=1;
    */
}

for(int k=1;k<=n;k++)
    for(int i=1;i<=n;i++)
        for(int j=1;j<=n;j++){
            d[i][j]=min(d[i][j],d[i][k]+d[k][j]);

            /*传递闭包
```

```
        d[i][j]|=d[i][k]&d[k][j];
        */
    }
```

2.5 Kruscal 最小生成树

```
/*-----并查集代码省略-----*/
typedef pair<int,pair<int,int>> e;
priority_queue<e,vector<e>,greater<e>>q;
int kruscal()
{
    int ans=0;
    while(!q.empty()){
        int x=q.top().second.first;
        int y=q.top().second.second;
        int z=q.top().first;
        q.pop();

        //get()与merge()均为并查集操作
        if(get(x)==get(y))
            continue;

        ans+=z;
        merge(x,y);

    }

    return ans;
}
int main()
{

    /*----初始化并查集省略----*/

    for(int i=1;i<=k;i++){
        int x,y,z;
        cin>>x>>y>>z;
        add(x,y,z);
        add(y,x,z);
        q.push({z,{x,y}});
    }

    int ans=Kruscal();
    return 0;
}
```

2.6 Boruvka 最小生成树

```
const int N = 200005;
const int M = 2 * N;

int fa[N];
int get(int x)
{
    if (x == fa[x])
        return x;
    return fa[x] = get(fa[x]);
}

void merge(int x, int y)
{
    int a = get(x);
    int b = get(y);
    if (a == b)
        return;
    fa[b] = a;
    return;
}

struct node {
    int u, v, w;
};
vector<node> e;
ll boruvka(const vector<node> &e, int n, int m)
{
    for (int i = 1; i <= n; ++i)
        fa[i] = i;
    vector<int> id(n + 5, -1);
    int cnt = n - 1;
    bool flag = true;
    ll ans = 0;
    while (flag) {
        flag = false;
        for (int i = 0; i < m; ++i) {
            int x = get(e[i].u), y = get(e[i].v);
            if (x != y) {
                if (id[x] == -1 || e[id[x]].w > e[i].w)
                    id[x] = i;
                if (id[y] == -1 || e[id[y]].w > e[i].w)
                    id[y] = i;
            }
        }
        for (int i = 1; i <= n; ++i)
            if (get(i) == i && id[i] != -1 && get(e[id[
             i]].u) != get(e[id[i]].v)) {
                merge(e[id[i]].u, e[id[i]].v);
                ans += e[id[i]].w;
                // cout << id[i] << '\n';
                --cnt;
                flag = true;
            }
        for (int i = 1; i <= n; ++i)
            id[i] = -1;
    }

    if (!cnt)
        return ans;
    else
        return -1;
}
```

2.7 Kruscal 重构树

与 Kruscal 算法类似，其中每次操作建立虚点将合并的两个点集结合。
虚点的权值即为边权
两点间的 lca 所代表的点权即为两点路径中需要经过的最大边权最小值

```
//记得初始化 rt[i] 为 i
//两点间路径最大值最小即为 lca 的 tag 值
//要是求两点间路径最小值最大就把边权从大到小排序
//省略了并查集以及lca的代码
//最终的森林以 0 号点为根
int tag[N], rt[N];
int n;
int cnt;
```

```cpp
typedef pair<int, pair<int, int>> e;
vector<e>v;
void kruscal()
{
    sort(range(v));
    cnt = n;
    for (const auto& [z, c] : v) {
        const auto& [x, y] = c;
        int a = get(x), b = get(y);
        if (a == b)
            continue;
        ++cnt;
        tag[cnt] = z;
        add(rt[a], cnt);
        add(rt[b], cnt);
        add(cnt, rt[a]);
        add(cnt, rt[b]);
        fa[b] = a;
        rt[a] = cnt;
    }
    for (int i = 1;i <= n;i++) {
        if (i == get(i))
            add(0, rt[i]), add(rt[i], 0);
    }
    tag[0] = -1; //-1 表示不连通
}
```

2.8 树哈希

```cpp
//hah中保存以该点为子树的哈希值
//map不要清空
int id;
int hah[N];
map<vector<int>, int>mp;
void dfs(int x, int fa)
{
    vector<int>temp;
    for (int i = head[x];i;i = Next[i]) {
        int y = ver[i];
        if (y == fa)
            continue;
        dfs(y, x);
        temp.push_back(hah[y]);
    }

    sort(range(temp));
    if (!mp[temp])
        mp[temp] = ++id;
    hah[x] = mp[temp];
}
```

2.9 虚树

```cpp
//建立的虚树中只含有询问点以及他们的LCA
//在解决询问点的总数不大且只需要用到LCA和被询问点时使用
const int N = 100005;
const int M = 2 * N;

int head[N], ver[M], Next[M];
int tot;
int low[N], dfn[N];
void add(int x,int y)
{
    ver[++tot] = y;
    Next[tot] = head[x];
    head[x] = tot;
}

/*----求LCA过程省略,请自行补充----*/

vector<int> v;
stack<int> st;
int s;
bool cmp(int a,int b)
{
    return dfn[a] < dfn[b];
}
void build()
{
    sort(v.begin(), v.end(),cmp);
    int sz = v.size();
    for (int i = sz - 2; ~i;i--)
        v.emplace_back(lca(v[i], v[i + 1]));

    sort(v.begin(), v.end(),cmp);
    v.erase(unique(v.begin(), v.end()), v.end());

    s = v[0];
    while(!st.empty())
        st.pop();
    sz = v.size();
    for (int i = 0; i < sz;i++){
        int u = v[i];
        while(!st.empty()&&low[st.top()]<dfn[u])
            st.pop();

        if(!st.empty()){
            add(u, st.top());
            add(st.top(), u);
        }
        st.push(u);
    }

    return;
}

//标记是否是被询问的节点
map<int, int> mp;
void clr()
{
    mp.clear();
    for(auto c:v)
        head[c] = 0;
    tot = 0;
    v.clear();
}

//求dfs序以及low数组
//low数组中存储点x的子树内的 最大 dfs序
int tim;
void dfs(int x,int fa)
{
    dfn[x] = ++tim;
    low[x] = dfn[x];
    for (int i = head[x]; i;i=Next[i]){
        int y = ver[i];
        if(y==fa)
```

```
            continue;

        dfs(y, x);
        low[x] = max(low[x], low[y]);
    }

    return;
}

int dp(int x,int fa)
{
    //树形dp部分
}
void solve()
{
    int n;
    cin >> n;

    for (int i = 1; i < n;i++){
        int u, v;
        cin >> u >> v;

        add(u, v);
        add(v, u);
    }

    dfs(1, 0);
    bfs(1);
    memset(head, 0, sizeof head);
    tot = 0;

    int q;
    cin >> q;
    while(q--){
        clr();
        int k;
        cin >> k;

        //一次询问k个点
        for (int i = 1; i <= k;i++){
            int x;
            cin >> x;
            v.emplace_back(x);
            mp[x] = 1;
        }

        build();

        //树形dp
        cout << dp(s, 0) << '\n';
    }
}
```

2.10 倍增求树上 lca

```
/*----注意修改循环中k的大小---*/
const int N=500005;
queue<int>q;
int d[N];
int f[N][20];

//x指根节点编号
void bfs(int x)
{
    d[x]=1;
    q.push(x);
    while(!q.empty()){
        int u=q.front();
        q.pop();
        for(int i=head[u];i;i=Next[i]){
            int y=ver[i];
            if(d[y])
                continue;
            d[y]=d[u]+1;
            q.push(y);
            f[y][0]=u;
            for(int k=1;k<=15;k++){
                f[y][k]=f[f[y][k-1]][k-1];
            }
        }
    }
    return;
}

int lca(int x,int y)
{
    if(d[x]<d[y])
        swap(x,y);

    for(int k=15;k>=0;k--)
        if(d[f[x][k]]>=d[y])
            x=f[x][k];

    if(x==y)
        return x;

    for(int k=15;k>=0;k--)
        if(f[x][k]!=f[y][k])
            x=f[x][k],y=f[y][k];

    return f[x][0];


}
```

2.11 O(1) lca

```
int euler[M], tim, a[M];
void dfs(int x, int fa)
{
    euler[++tim] = x;
    a[x] = tim;
    for (int i = head[x];i;i = Next[i]) {
        int y = ver[i];
        if (y == fa)
            continue;
        dfs(y, x);
        euler[++tim] = x;
    }
}
int f[21][M];
void pre(int rt)
{
    tim = 0;
    dfs(rt, 0);
    int n = tim;
    for (int i = 1;i <= n;++i)
        f[0][i] = a[euler[i]];
```

```cpp

    int t = log2(n) + 1;
    for (int i = 1;i < t;++i)
        for (int j = 1;j <= n - (1 << i) + 1;++j)
            f[i][j] = min(f[i - 1][j], f[i - 1][j + (1
                << (i - 1))]);
}

int lca(int x, int y)
{
    int l = a[x], r = a[y];
    if (l > r)
        swap(l, r);
    int k = log2(r - l + 1);
    return euler[min(f[k][l], f[k][r - (1 << k) + 1])
        ];
}
```

2.12 树链剖分

```cpp
const int N=100005;
int sz[N],son[N],dep[N],fa[N];

void dfs1(int x)
{
    sz[x]=1;
    for(int i=head[x];i;i=Next[i]){
        int y=ver[i];
        if(y==fa[x])
            continue;
        dep[y]=dep[x]+1;
        fa[y]=x;
        dfs1(y);

        sz[x]+=sz[y];
        if(sz[y]>sz[son[x]])
            son[x]=y;
    }

    return;
}

int dfn[N],top[N],a[N],w[N];
int tim;
void dfs2(int x,int t)
{
    dfn[x]=++tim;
    top[x]=t;
    a[tim]=w[x];
    if(son[x])
        dfs2(son[x],t);

    for(int i=head[x];i;i=Next[i]){
        int y=ver[i];

        if(y==fa[x]||y==son[x])
            continue;

        dfs2(y,y);
    }

    return;
}

//询问与修改大致相同,仅需将区间操作改为区间询问即可
void change_path(int u,int v,int x)
{

    while(top[u]!=top[v]){
        if(dep[top[u]]<dep[top[v]])
            swap(u,v);

        //区间操作[dfn[top[u]],dfn[u]];
        change(1,dfn[top[u]],dfn[u],x);
        u=fa[top[u]];
    }

    if(dep[u]>dep[v])
        swap(u,v);

    //区间操作[dfn[u],dfn[v]];
    change(1,dfn[u],dfn[v],x);
    return;
}

int main()
{
    //树链剖分操作
    dep[1]=1;
    dfs1(1);
    dfs2(1,1);
}
```

2.13 有向图强连通分量

```cpp
const int N=100005;
int dfn[N],low[N];
int tim;
stack<int>st;
int sz[N],cnt; //强连通分量的数目以及大小
bool v[N];
int id[N]; //每个点所属的强连通分量编号

void tarjan(int x)
{
    dfn[x]=low[x]=++tim;
    st.push(x),v[x]=1;
    for(int i=head[x];i;i=Next[i]){
        int y=ver[i];

        if(!dfn[y]){
            tarjan(y);
            low[x]=min(low[y],low[x]);
        }
        else if(v[y])
            low[x]=min(low[x],dfn[y]);
    }

    if(low[x]==dfn[x]){
        ++cnt;
        int y;
        do{
            y=st.top();
            st.pop();

            v[y]=0;
            id[y]=cnt;
            sz[cnt]++;
```

```
        }while(y!=x);
    }
    return;
}

int main()
{
    //使用方法
    for(int i=1;i<=n;i++)
        if(!dfn[i])
            tarjan(i);

}
```

2.14 弦图最大势算法

```
//团：完全子图
//极大团：不是其他团子图的图
//团数：最大团的点数
//单纯点：设 N(v) 表示 v 的邻域，一个点称为单纯点当且仅当 v + N(v) 的导出子图为一个团
//完美消除序列：一个点的序列满足 vi 在 {vi,...,vn} 的导出子图中为一个单纯点
//最小染色：用最少的颜色给点染色是的所有边连接的两点颜色不同
//最大独立集：最大的点集使得点集中任意两点都没有边直接相连
//最小团覆盖：用最少的团覆盖所有的点
//弦图的色数/团数：按照完美消除序列从后往前依次给每个点染色，给每个点染上可以染的最小颜色。只需要求色数时，只需要将最大label+1即可
//最大独立集：完美消除序列从前往后，选择所有没有与已经选择的点有直接连边的点
//最大独立集 = 最小团覆盖
struct Edge {
    int to,next;
    Edge() {}
    Edge(int to,int next):to(to),next(next) {}
};
struct MCS{
    Edge edge[N<<1];
    int head[N],tot;
    int n,m;
    bool vis[N];
    int id[N];//编号
    int label[N];//与多少标号点相邻
    int order[N];//完美消除序列
    vector<int> V[N];

    void init(int n,int m) {
        this->n=n;
        this->m=m;
        for(int i=1; i<=n; i++)
            V[i].clear();
        memset(head,-1,sizeof(head));
        memset(order,0,sizeof(order));
        memset(label,0,sizeof(label));
        memset(vis,0,sizeof(vis));
        memset(id,0,sizeof(id));
        tot=0;
    }

    void addEdge(int x,int y) {
        edge[tot].to=y;
        edge[tot].next=head[x];
        head[x]=tot++;
    }

    void mcs() {
        for(int i=1; i<=n; i++)
            V[0].push_back(i);
        int maxx=0;
        int now=0;
        for(int i=1; i<=n; i++) { //从前往后
            bool flag=false;
            while(!flag) {
                for(int j=V[maxx].size()-1; j>=0; j--) { //从后往前
                    if(vis[V[maxx][j]])
                        V[maxx].pop_back();
                    else {
                        flag=true;
                        now=V[maxx][j];
                        break;
                    }
                }
                if(!flag)
                    maxx--;
            }
            vis[now]=true;
            //逆序存放
            order[n-i+1]=now;
            id[now]=n-i+1;

            for(int j=head[now]; j!=-1; j=edge[j].next) {
                int to=edge[j].to;
                if(!vis[to]) {
                    label[to]++;
                    V[label[to]].push_back(to);
                    maxx=max(maxx,label[to]);
                }
            }
        }
    }
    int bucket[N];//桶
    int judge() { //判断是否是弦图
        memset(vis,0,sizeof(vis));
        memset(bucket,0,sizeof(bucket));
        for(int i=n; i>0; i--) {
            int cnt=0;

            int ret=1;
            for(int j=head[order[i]]; j!=-1; j=edge[j].next)
                if(id[edge[j].to]>i)
                    vis[bucket[++cnt]=edge[j].to]=1;

            for(int j=head[bucket[1]]; j!=-1; j=edge[j].next) {
                int to=edge[j].to;
                if(to!=bucket[1]&&vis[to]) {
                    if(vis[to]) {
                        ret++;
                        vis[to]++;
                    }
                }
            }
            for(int j=1; j<=cnt; j++)
                vis[bucket[j]]=0;
```

```
            if(cnt&&ret!=cnt)
                return false;
        }
        return true;
    }
    int getMaximumClique() { //计算最大团、最小着色
        int res=0;
        for(int i=1; i<=n; i++)
            res=max(res,label[i]+1);
        return res;
    }
    int getMaximumIndependentSet() { //计算最大独立集、最小团覆盖
        memset(vis,0,sizeof(vis));
        int res=0;
        for(int i=1; i<=n; i++) {
            if(!vis[order[i]]) {
                res++;
                vis[order[i]]=true;
                for(int j=head[order[i]]; j!=-1; j=edge[j].next)
                    vis[edge[j].to]=true;
            }
        }
        return res;
    }
}mcs;
int main() {
    int n,m;
    while(scanf("%d%d",&n,&m)!=EOF&&(n+m)) {
        mcs.init(n,m);
        for(int i=1; i<=m; i++) {
            int x,y;
            scanf("%d%d",&x,&y);
            mcs.addEdge(x,y);
            mcs.addEdge(y,x);
        }
        mcs.mcs();

        if(!mcs.judge())//若不为弦图
            printf("No\n");
        else { //若为弦图
            printf("Yes\n");

            int res1=mcs.getMaximumClique();//最大团、最小着色
            int res2=mcs.getMaximumIndependentSet();//最大独立集、最小团覆盖
            printf("The maximum clique:%d\n",res1);
            printf("The maximum independent set:%d\n",res2);
        }
    }
    return 0;
}
```

2.15 2-SAT

连边方式:
$a \cup b \Rightarrow$!a→b 与!b→a
$a \cap b \Rightarrow$!a→a 与!b→b
a if $b \Rightarrow$ b→a 与!a→!b

若 a 与!a 位于同一个强连通分量, 则无解
否则一定有解, 且 a 的值取 0 和 1 中拓扑序靠后的值 (tarjan 算法求出的强连通分量编号为拓扑序逆序)

2.16 边双连通分量

```
vector<vector<int>> ebcc;
int dfn[N], low[N], tim;
bool tag[M]; // 是否是割边
void tarjan(int x, int id)
{
    dfn[x] = low[x] = ++tim;
    for (int i = head[x]; ~i; i = Next[i]) {
        int y = ver[i];
        if (!dfn[y]) {
            tarjan(y, i);
            if (dfn[x] < low[y])
                tag[i] = tag[i ^ 1] = true;
            low[x] = min(low[x], low[y]);
        }
        else if (i != (id ^ 1))
            low[x] = min(low[x], dfn[y]);
    }
}

int belong[N];
void dfs(int x, int id)
{
    belong[x] = id;
    ebcc.back().push_back(x);
    for (int i = head[x]; ~i; i = Next[i]) {
        int y = ver[i];
        if (belong[y] || tag[i])
            continue;
        dfs(y, id);
    }
}

void pre(int n, int m)
{
    ebcc.clear();
    tim = 0;
    tot = -1;

    for (int i = 1; i <= n; ++i) {
        head[i] = -1;
        belong[i] = 0;
        dfn[i] = 0;
    }

    for (int i = 0; i <= 2 * m; ++i)
        tag[i] = 0;
}

void work(int n)
{
    for (int i = 1; i <= n; ++i)
        if (!dfn[i])
            tarjan(i, -1);

    for (int i = 1; i <= n; ++i)
        if (!belong[i]) {
            ebcc.push_back(vector<int>());
            dfs(i, ebcc.size());
        }
}
```

2.17 点双连通分量

```
vector<vector<int>> vbcc;
int tim;
int dfn[N], low[N];
stack<int> stk;

void tarjan(int x, int fa)
{
    bool flag = false;
    dfn[x] = low[x] = ++tim;
    stk.push(x);

    for (int i = head[x]; i; i = Next[i]) {
        int y = ver[i];

        if (!dfn[y]) {
            flag = true;
            tarjan(y, x);
            low[x] = min(low[x], low[y]);

            if (low[y] >= dfn[x]) {
                vbcc.push_back(vector<int>());

                while (stk.top() != y) {
                    vbcc.back().push_back(stk.top());
                    stk.pop();
                }
                vbcc.back().push_back(y);
                stk.pop();

                vbcc.back().push_back(x);
            }
        }
        else if (y != fa)
            low[x] = min(low[x], dfn[y]);
    }
    if (!fa & !flag)
        vbcc.push_back({x});
}

void pre(int n)
{
    tot = tim = 0;
    for (int i = 1; i <= n; ++i)
        head[i] = dfn[i] = 0;
}

void work(int n)
{
    for (int i = 1; i <= n; ++i)
        if (!dfn[i]) {
            while (!stk.empty())
                stk.pop();
            tarjan(i, 0);
        }
}
```

2.18 拓扑排序

```
//使用前注意预处理出个点的入度
//答案存放在a数组中,共有cnt个点
//若 cnt<n 则说明图中有环
const int N=100005;
const int M=200005;
int n;
int a[N],cnt;
queue<int>q;
int deg[N]; //存储入度
void topsort()
{
    for(int i=1;i<=n;i++)
        if(deg[i]==0)
            q.push(i);

    while(!q.empty()){
        int x=q.front();
        q.pop();
        a[++cnt]=x;
        for(int i=head[x];i;i=Next[i]){
            int y=ver[i];
            deg[y]--;
            if(!deg[y])
                q.push(y);
        }
    }

    return;
}
```

2.19 匈牙利算法

```
//时间复杂度 O(NM)
//正确性:当一个节点成为匹配点后,至多因为找到增广路而更换匹配对象,并不会变为非匹配点
const int N=10005;
bool v[N];
int match[N];
bool dfs(int x)
{
    for(int i=head[x];i;i=Next[i]){
        int y=ver[i];
        if(v[y])
            continue;

        v[y]=1;
        if(!match[y]||dfs(match[y])){
            match[y]=x;
            return true;
        }

    }

    return false;
}

int main()
{
    int ans=0;
    for(int i=1;i<=n;i++){
        memset(v,0,sizeof v); //注意清空
        if(dfs(i))
            ans++;
    }
}
```

2.20 二分图最大权匹配 (KM 算法)

```
//最终匹配一定要是完美匹配，可以通过初始化负无穷来判断
//本模板通过添加虚点与虚边构建完全图，再计算需要计算的点的
    费用和
//最终匹配中满足 lx + ly = a
//匹配方案存在 match 数组中
//最大取正权，最小取负权
int n;
const int N = 505;
int a[N][N];
int posx[N], posy[N], pre[N];
int lx[N], ly[N], matchx[N], matchy[N];
int slack[N], d;
bool visx[N], visy[N];
queue<int> q;
int nl, nr; //左右的点数

void aug(int v)
{
    while (v) {
        int t = matchx[pre[v]];
        matchx[pre[v]] = v;
        matchy[v] = pre[v];
        v = t;
    }
}

void bfs(int s)
{
    for (int i = 1; i <= nr; i++)
        slack[i] = iINF;

    memset(visx, 0, sizeof visx);
    memset(visy, 0, sizeof visy);
    memset(pre, 0, sizeof pre);

    while (!q.empty())
        q.pop();
    q.push(s);

    while (1) {
        while (!q.empty()) {
            int u = q.front();
            q.pop();
            visx[u] = 1;
            for (int v = 1; v <= nr; v++) {
                if (!visy[v]) {
                    if (lx[u] + ly[v] - a[u][v] < slack[
                        v]) {
                        slack[v] = lx[u] + ly[v] - a[u][v
                            ];
                        pre[v] = u;
                    }
                    if (!slack[v]) {
                        visy[v] = 1;
                        if (!matchy[v]) {
                            aug(v);
                            return;
                        }
                        else
                            q.push(matchy[v]);
                    }

            }
        }
    }
    d = iINF;

    for (int i = 1; i <= nr; i++)
        if (!visy[i])
            d = min(d, slack[i]);

    if (d == iINF)
        break;

    for (int i = 1; i <= nl; i++)
        if (visx[i])
            lx[i] -= d;
    for (int i = 1; i <= nr; i++)
        if (visy[i])
            ly[i] += d;
        else
            slack[i] -= d;

    for (int i = 1; i <= nr; i++) {
        if (!visy[i] && !slack[i]) {
            visy[i] = 1;
            if (!matchy[i]) {
                aug(i);
                return;
            }
            else
                q.push(matchy[i]);
        }
    }

  }

}

ll km()
{
    for (int i = 1; i <= nl; i++)
        for (int j = 1; j <= nr; j++)
            lx[i] = max(lx[i], a[i][j]);

    for (int i = 1; i <= nl; i++)
        bfs(i);

    ll ans = 0;
    for (int i = 1;i <= nl;i++)
        ans += lx[i];
    for (int i = 1;i <= nr;i++)
        ans += ly[i];
    return abs(ans);
}

void init()
{
    for (int i = 0; i <= n; i++) {
        posx[i] = -1;
        posy[i] = -1;
        matchy[i] = matchx[i] = 0;
        lx[i] = 0;
        ly[i] = 0;
    }

    //将 a 置为负无穷
    memset(a, 0xcf, sizeof a);
```

```
}
```

2.21 一般图匹配（带花树）

```
//匹配点存在 match 数组中
//复杂度跑不满，1000数据量完全可以
//使用时直接跑调用接口即可
const int N = 1005;
const int M = 100005;
int head[N], ver[M], Next[M];
int tot;
void add(int x, int y)
{
   ver[++tot] = y;
   Next[tot] = head[x];
   head[x] = tot;
}
int n, m;

int fa[N];
int get(int x)
{
   if (x == fa[x])
      return x;
   return fa[x] = get(fa[x]);
}
int tim;
int dfn[N], match[N], pre[N];
int lca(int x, int y)
{
   tim++;
   x = get(x);
   y = get(y);
   while (dfn[x] != tim) {
      dfn[x] = tim;
      x = get(pre[match[x]]);
      if (y)
         swap(x, y);
   }
   return x;
}

int vis[N]; //1:黑 2:白
queue<int>q;
void blossom(int x, int y, int z)
{
   while (get(x) != z) {
      pre[x] = y;
      y = match[x];
      if (vis[y] == 2) {
         vis[y] = 1;
         q.push(y);
      }

      if (x == get(x))
         fa[x] = z;
      if (y == get(y))
         fa[y] = z;

      x = pre[y];
   }
}

bool bfs(int s)
{
   for (int i = 1;i <= n;i++)
      vis[i] = pre[i] = 0, fa[i] = i;
   while (!q.empty())
      q.pop();

   q.push(s);
   vis[s] = 1;
   while (!q.empty()) {
      int x = q.front();
      q.pop();
      for (int i = head[x];i;i = Next[i]) {
         int y = ver[i];
         if (get(x) == get(y) || vis[y] == 2)
            continue;
         if (!vis[y]) {
            vis[y] = 2;
            pre[y] = x;

            if (!match[y]) {
               int last;
               while (y) {
                  last = match[pre[y]];
                  match[y] = pre[y];
                  match[pre[y]] = y;
                  y = last;
               }
               return true;
            }
            vis[match[y]] = 1;
            q.push(match[y]);
         }
         else {
            int now = lca(x, y);
            blossom(x, y, now);
            blossom(y, x, now);
         }
      }
   }

   return false;
}

//调用接口
int cal()
{
   int ans = 0;
   for (int i = 1;i <= n;i++)
      if (!match[i] && bfs(i))
         ans++;
   return ans;
}
void init()
{
   tot = tim = 0;
   for (int i = 1;i <= n;i++)
      head[i] = dfn[i] = match[i] = 0;
}
```

2.22 Dinic 最大流

```
//最大流==最小割
//无向边的反边不为零,边权与正边相同
const int N=100;
```

```
const int M=25010;
const int INF=0x7fffffff;
int s,t;

int head[N],cur[N],d[N],ver[M],edge[M],Next[M];
int tot=-1;

void add(int x,int y,int z)
{
    ver[++tot]=y;
    Next[tot]=head[x];
    edge[tot]=z;
    head[x]=tot;

    ver[++tot]=x;
    Next[tot]=head[y];
    edge[tot]=0;
    head[y]=tot;
    return;
}


queue<int>q;
bool bfs()
{
    memset(d,-1,sizeof d);
    while(!q.empty())
        q.pop();

    d[s]=0;
    q.push(s);
    cur[s]=head[s];

    while(!q.empty()){
        int x=q.front();
        q.pop();

        for(int i=head[x];~i;i=Next[i]){
            int y=ver[i];
            int z=edge[i];

            if(d[y]!=-1||!z)
                continue;

            d[y]=d[x]+1;
            cur[y]=head[y];
            if(y==t)
                return true;

            q.push(y);
        }
    }

    return false;
}

int dfs(int u,int limit)
{
    if(u==t)
        return limit;
    int flow=0;
    for(int i=cur[u];~i&&flow<limit;i=Next[i]){
        cur[u]=i;
        int y=ver[i];
        int z=edge[i];

            if(d[y]!=d[u]+1||!z)
                continue;

            int use=dfs(y,min(z,limit-flow));
            if(!use)
                d[y]=0;
            edge[i]-=use;
            edge[i^1]+=use;
            flow+=use;

    }

    return flow;
}
int dinic()
{
    int ans=0,flow;
    while(bfs())
        while(flow=dfs(s,INF))
            ans+=flow;

    return ans;
}
int main()
{
    //注意初始化head为-1
    memset(head,-1,sizeof head);
    t=n+1; //点数 +1

    /*---加边---*/

    int ans=dinic();
}
```

2.23 最小费用最大流

```
//注意与最大流的区别
const int N=5005;
const int M=100005;
const int INF=0x3f3f3f3f;

int head[N],ver[M],Next[M],edge[M];
int w[M];
int tot=-1;
void add(int x,int y,int z,int d)
{
    ver[++tot]=y;
    edge[tot]=z;
    w[tot]=d;
    Next[tot]=head[x];
    head[x]=tot;
    ver[++tot]=x;
    edge[tot]=0;
    w[tot]=-d;
    Next[tot]=head[y];
    head[y]=tot;
}

int s,t;
int incf[N];
int d[N],pre[N];
bool v[N];
queue<int>q;
```

```
bool spfa()
{
    memset(d,0x3f,sizeof d);
    memset(incf,0,sizeof incf);
    while(!q.empty())
        v[q.front()]=0,q.pop();

    d[s]=0;
    incf[s]=INF;
    q.push(s);
    v[s]=1;
    while(!q.empty()){
        int x=q.front();
        q.pop();
        v[x]=0;
        for(int i=head[x];~i;i=Next[i]){
            int y=ver[i];
            int z=edge[i];
            int d1=w[i];

            if(z&&d[y]>d[x]+d1){
                d[y]=d[x]+d1;
                pre[y]=i;
                incf[y]=min(z,incf[x]);
                if(!v[y]){
                    q.push(y);
                    v[y]=1;
                }
            }
        }
    }

    return incf[t]>0;
}

void SFPA(int& flow,int& cost)
{
    flow=cost=0;
    while(spfa()){
        int now=incf[t];
        flow+=now;
        cost+=now*d[t];
        for(int i=t;i!=s;i=ver[pre[i]^1]){
            edge[pre[i]]-=now;
            edge[pre[i]^1]+=now;
        }
    }
    return;
}
int main()
{
    memset(head,-1,sizeof head);
    int n,m;
    cin>>n>>m>>s>>t;

    for(int i=1;i<=m;i++){
        int u,v,c,w;
        cin>>u>>v>>c>>w;
        add(u,v,c,w);
    }
    int flow,cost;
    SFPA(flow,cost);
    cout<<flow<<' '<<cost<<'\n';
    return 0;
}
```

2.24 原始对偶费用流

```
//费用流卡常专用，亲测好用
const int N = 5005;
const int M = 100005;

int head[N], ver[M], Next[M];
ll edge[M], w[M];
int tot = -1;
void add(int x, int y, int z, int d)
{
    ver[++tot] = y;
    edge[tot] = z;
    w[tot] = d;
    Next[tot] = head[x];
    head[x] = tot;
    ver[++tot] = x;
    edge[tot] = 0;
    w[tot] = -d;
    Next[tot] = head[y];
    head[y] = tot;
}
int s, t;
ll delta;
ll dist[N];
void Reduce()
{
    for (int i = 0;i <= tot;++i) {
        int x = ver[i ^ 1], y = ver[i];
        w[i] += dist[y] - dist[x];
    }

    delta += dist[s];
}

bool vis[N];
bool BellmanFord()
{
    queue<int>q;
    memset(dist, 0x3f, sizeof dist);
    dist[t] = 0;
    q.push(t);
    vis[t] = 1;
    while (!q.empty()) {
        int x = q.front();
        q.pop();
        vis[x] = 0;
        for (int i = head[x];~i;i = Next[i]) {
            int y = ver[i];
            ll z = w[i ^ 1];
            if (edge[i ^ 1] && dist[y] > dist[x] + z) {
                dist[y] = dist[x] + z;
                if (!vis[y]) {
                    vis[y] = 1;
                    q.push(y);
                }
            }
        }
    }
    return dist[s] != llINF;
}

priority_queue<pair<ll, ll>, vector<pair<ll, ll>>,
    greater<pair<ll, ll>>>q;
bool Dijkstra()
```

```cpp
{
    memset(dist, 0x3f, sizeof dist);
    memset(vis, 0, sizeof vis);
    dist[t] = 0;
    q.push({ dist[t],t });

    while (!q.empty()) {
        auto [dis, x] = q.top();
        q.pop();
        if (vis[x])
            continue;
        vis[x] = 1;

        for (int i = head[x];~i;i = Next[i]) {
            int y = ver[i];
            ll z = w[i ^ 1];
            if (edge[i ^ 1] && dist[y] > dist[x] + z) {
                dist[y] = dist[x] + z;
                q.push({ dist[y],y });
            }
        }
    }
    return dist[s] != llINF;
}

ll dfs(int x, ll flow)
{
    if (x == t || !flow)
        return flow;
    vis[x] = 1;
    ll res = flow;
    for (int i = head[x];~i;i = Next[i]) {
        int y = ver[i];
        ll z = w[i];
        if (!vis[y] && edge[i] && !z) {
            ll temp = dfs(y, min(res, edge[i]));
            edge[i] -= temp;
            edge[i ^ 1] += temp;
            res -= temp;
        }
    }
    return flow - res;
}

void Augment(ll& flow,ll& cost)
{
    ll cur = 0;
    while (memset(vis, 0, sizeof vis),cur = dfs(s,
        llINF)) {
        flow += cur;
        cost += delta * cur;
    }
    return;
}

void PrimalDual(ll& flow,ll& cost)
{
    flow = 0, cost = 0;
    if (!BellmanFord())
        return;
    Reduce();
    Augment(flow, cost);
    while (Dijkstra()) {
        Reduce();
        Augment(flow, cost);
    }
}

void init()
{
    memset(vis, 0, sizeof vis);
    memset(head, -1, sizeof head);
    tot = -1;
    delta = 0;
}
void solve()
{
    init();
    int n, m;
    cin >> n >> m >> s >> t;

    for (int i = 1;i <= m;i++) {
        int x, y;
        ll z, d;
        cin >> x >> y >> z >> d;
        add(x, y, z, d);
    }
    ll flow, cost;
    PrimalDual(flow, cost);
    cout << flow << ' ' << cost << '\n';
    return;
}
```

2.25 朱刘算法

```
```

2.26 二分图性质

最小边覆盖 = 最大独立集 = n - 最大匹配
最小点覆盖 = 最大匹配
最大独立集 = n - 最小点覆盖 = n - 最大匹配
DAG 的最小路径覆盖 = 将每个点拆为 i 与 i1，边 i->j 变为 i->j1。求新图的最小边覆盖即可

3 数学

3.1 快速幂

```cpp
//多项式快速幂与之类似,将a当作多项式做NTT即可
const ll mod = 998244353;
ll qpow(ll a,ll b)
{
    ll ans = 1;
    a %= mod;
    for (; b;b>>=1){
        if(b&1)
            ans = ans * a % mod;
        a = a * a % mod;
    }
    return ans % mod;
}
```

3.2 整除分块

```
//下取整
for(int l = 1, r; l <= n; l = r + 1)
{
    r = n / (n / l);
    /*----操作----*/
}

//上取整
for(int l = 1, r; l <= n; l = r + 1)
{

    r = ((n+l-1)/l==1? n:(n-1)/((n+l-1)/l-1));
    /*----操作----*/
}
```

3.3 Eratosthenes 筛法

```
//筛出1-n之间的质数
const int N=100005;
bool v[N];
vector<int>prime;
void primes(int n)
{
    memset(v,0,sizeof v);
    for(int i=2;i<=n;i++){
        if(v[i])
            continue;
        prime.emplace_back(i);
        for(int j=1;j<=n/i;j++)
            v[i*j]=1;
    }
    return;
}
```

3.4 线性筛

```
const int N=500005;
int v[N],prime[N];
int cnt;
void primes(int n)
{
    memset(v,0,sizeof v);
    cnt=0;
    for(int i=2;i<=n;i++){
        if(!v[i]){
            v[i]=i;
            prime[++cnt]=i;
        }
        for(int j=1;j<=cnt;j++){
            if(prime[j]>v[i]||prime[j]>n/i)
                break;
            v[i*prime[j]]=prime[j];
        }
    }
    return;
}
```

3.5 质因数分解

```
//时间复杂度O(sqrt(n))
//p中存质因数,c中存幂次
int cnt;
int p[N],c[N];
void divide(int n)
{
    cnt=0;
    for(int i=2;i<=sqrt(n);i++)
        if(n%i==0){
            p[++cnt]=i;
            c[cnt]=0;
            while(n%i==0){
                n/=i;
                c[cnt]++;
            }
        }

    if(n>1)
        p[++cnt]=n,c[cnt]=1;

    return;
}
```

3.6 Pollard's Rho 质因数分解

```
//要重写qpow函数
#define int128 __int128_t
const int128 tag = 1;

int prime[12] = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29,
    31, 37};
ll qpow(ll a,ll b,ll mod)
{
    ll ans = 1;
    a %= mod;
    for (; b;b>>=1){
        if(b&1)
            ans = (tag * ans * a) % mod;
        a = (tag * a * a) % mod;
    }
    return ans % mod;
}

//判断是否是素数
bool check(ll a,ll p)
{
    ll d = p - 1;
    ll get = qpow(a, d, p);
    if(get!=1)
        return 1;
    while((d&1)^1){
        d >>= 1;
        if((get=qpow(a,d,p))==p-1)
            return 0;
        else if(get!=1)
            return 1;
    }

    return 0;
}
bool Miller_Rabin(ll x)
{
```

```cpp
    if(x>40){
        for (int i = 0; i < 12;i++)
            if(check(prime[i],x))
                return 0;
        return 1;
    }

    for (int i = 0; i < 12;i++)
        if(x==prime[i])
            return 1;
    return 0;
}

//结果存在factor中
//key为底数,val为指数
map<ll, int> factor;
ll pollard_rho(ll n,ll c)
{
    if(n==4)
        return 2;
    ll x = rand() % (n - 1) + 1;
    ll y = x;
    x = (tag * x * x + c) % n;
    y = (tag * y * y + c) % n;
    y = (tag * y * y + c) % n;
    for (int lim = 1; x != y;lim=min(lim<<1,128)){
        ll cnt = 1;
        for (int i = 0; i < lim;i++){
            cnt = tag * cnt * abs(x - y) % n;
            if(!cnt)
                break;
            x = (tag * x * x + c) % n;
            y = (tag * y * y + c) % n;
            y = (tag * y * y + c) % n;
        }
        ll d = __gcd(cnt, n);
        if(d!=1)
            return d;
    }

    return n;
}

void findFac(ll n)
{
    if(Miller_Rabin(n)){
        factor[n]++;
        return;
    }
    ll p = n;
    while(p>=n)
        p = pollard_rho(p, rand() % (n - 1) + 1);
    findFac(p);
    findFac(n / p);
}
void solve()
{
    factor.clear();
    ll x;

    //使用前先判断是否是素数
    //注意1的问题
    if(Miller_Rabin(x)){
        cout << "Prime\n";
        return;
    }

    findFac(x);
}
```

3.7 1-N 正约数集合

```cpp
//时间复杂度O(nlogn)
const int N=500005;
vector<int>factor[N];
int main()
{
    for(int i=1;i<=n;i++)
        for(int j=1;j<=n/i;j++)
            factor[i*j].emplace_back(i);
}
```

3.8 欧拉函数

```cpp
int phi(int n)
{
    int ans=n;
    for(int i=2;i<=sprt(n);i++)
        if(n%i==0){
            ans=(ans/i)*(i-1);
            while(n%i==0)
                n/=i;
        }

    if(n>1)
        ans=(ans/n)*(n-1);
    return ans;
}
```

3.9 2-N 欧拉函数

```cpp
//时间复杂度O(nlogn)
const int N=200005;
int phi[N];
void euler(int n)
{
    for(int i=2;i<=n;i++)
        phi[i]=i;
    for(int i=2;i<=n;i++)
        if(phi[i]==i)
            for(int j=i;j<=n;j+=i)
                phi[j]=(phi[j]/i)*(i-1);

    return;
}
```

3.10 扩展欧几里得算法

计算 $ax + by = \gcd(a, b)$ 的通解
方程 $ax + by = c$ 的通解可表示为:
$x = \frac{c}{d}x_0 + k\frac{b}{d}, y = \frac{c}{d}y_0 - k\frac{a}{d}$ $(k \in \mathbb{Z})$

```cpp
//返回a与b的gcd,并将通解通过引用传出
int exgcd(int a, int b, int& x, int& y)
{
```

```
    if (!b) {
        x = 1, y = 0;
        return a;
    }
    int d = exgcd(b, a % b, x, y);
    int z = x;
    x = y;
    y = z - y * (a / b);

    return d;
}

// x 的最小非负整数解
int Exabs(int a, int b, int c, int& x, int& y)
{
    int gd = exgcd(a, b, x, y);
    x *= c / gd;

    int t = b / gd;
    t = abs(t);
    x = (x % t + t) % t;

    y = (c - a * x) / b;
    return gd;
}
```

3.11 类欧几里得算法

定义以下内容:

$f(a,b,c,n) = \sum_{i=0}^{n} \left\lfloor \frac{ai+b}{c} \right\rfloor$

$g(a,b,c,n) = \sum_{i=0}^{n} i \left\lfloor \frac{ai+b}{c} \right\rfloor$

$h(a,b,c,n) = \sum_{i=0}^{n} \left\lfloor \frac{ai+b}{c} \right\rfloor^2$

$m = \left\lfloor \frac{an+b}{c} \right\rfloor$

那么有:

$f(a,b,c,n) = nm - f(c, c-b-1, a, m-1)$

$g(a,b,c,n) = \frac{1}{2}[mn(n+1) - h(c, c-b-1, a, m-1) - f(c, c-b-1, a, m-1)]$

$h(a,b,c,n) = nm(m+1) - 2g(c, c-b-1, a, m-1) - 2f(c, c-b-1, a, m-1) - f(a,b,c,n)$

```
constexpr ll inv2 = qpow(2, mod - 2);
constexpr ll inv6 = qpow(6, mod - 2);

struct data_t {
    ll f = 0, g = 0, h = 0;
};

data_t calc(ll n, ll a, ll b, ll c)
{
    ll ac = a / c, bc = b / c;
    ll m = (a * n + b) / c;
    ll n1 = n + 1, n21 = n * 2 + 1;

    data_t d;
    if (!a) {
        d.f = bc * n1 % mod;
        d.g = bc * n % mod * n1 % mod * inv2 % mod;
        d.h = bc * bc % mod * n1 % mod;
        return d;
    }
    if (a >= c || b >= c) {
        d.f = n * n1 % mod * inv2 % mod * ac % mod +
              bc * n1 % mod;
        d.g = ac * n % mod * n1 % mod * n21 % mod *
             inv6 % mod +
              bc * n % mod * n1 % mod * inv2 % mod;
        d.h = ac * ac % mod * n % mod * n1 % mod * n21
              % mod * inv6 % mod +
              bc * bc % mod * n1 % mod +
              ac * bc % mod * n % mod * n1 % mod;

        d.f %= mod;
        d.g %= mod;
        d.h %= mod;

        data_t e = calc(n, a % c, b % c, c);

        d.h += e.h +
               2 * bc * e.f % mod +
               2 * ac * e.g % mod;
        d.g += e.g;
        d.f += e.f;

        d.f %= mod;
        d.g %= mod;
        d.h %= mod;
        return d;
    }

    data_t e = calc(m - 1, c, c - b - 1, a);
    d.f = n * m % mod - e.f;
    d.f = (d.f % mod + mod) % mod;
    d.g = m * n % mod * n1 % mod - e.h - e.f;
    d.g = (d.g * inv2 % mod + mod) % mod;
    d.h = n * m % mod * (m + 1) % mod - 2 * e.g - 2 *
        e.f - d.f;
    d.h = (d.h % mod + mod) % mod;
    return d;
}
```

3.12 扩展中国剩余定理

```
//时间复杂度 O(nlogn)
//通解是 last + k*lm
ll Mod[5], X[5];
ll excrt(int n)
{
    ll lm = Mod[1], last = X[1];
    for (int i = 2;i <= n;i++) {
        ll lma = ((X[i] - last) % Mod[i] + Mod[i]) %
            Mod[i];
        ll x, y, k = lm;
        ll gd = exgcd(lm, Mod[i], x, y);
        if (lma % gd)
            return -1;

        ll b = Mod[i] / gd;
        ll c = lma / gd;
        x = (x * c % b + b) % b;
        lm = lm * b;
        last = (last + k * x) % lm;
    }
    return (last % lm + lm) % lm;
}
```

3.13 BSGS 算法

```cpp
int BSGS(int a, int n, int mod, int st = 1)
{
	unordered_map<int, int> mp;
	int m = ceill(sqrtl(mod));
	int s = 1;
	for (int i = 0; i < m; ++i) {
		mp[1ll * s * n % mod] = i;
		s = 1ll * s * a % mod;
	}
	int temp = s;
	s = st;
	for (int i = 0; i <= m; ++i) {
		if (mp.find(s) != mp.end() && i * m >= mp[s])
			return i * m - mp[s];
		s = 1ll * s * temp % mod;
	}
	return -1;
}

int exBSGS(int a, int n, int mod)
{
	a %= mod, n %= mod;
	if (n == 1 || mod == 1)
		return 0;
	int cnt = 0;
	int d, now = 1;
	while ((d = gcd(a, mod)) ^ 1) {
		if (n % d)
			return -1;
		cnt++;
		n /= d;
		mod /= d;
		now = (1ll * now * a / d) % mod;
		if (now == n)
			return cnt;
	}

	int ans = BSGS(a, n, mod, now);
	if (ans == -1)
		return -1;
	else
		return ans + cnt;
}

//固定模数与原根，O(P^0.75/log^0.5 P)预处理，O(logP)查询
//查询的是a^x=n中的a，返回的是最小x
//先init，然后调用get即可，solve不需要知道
struct BSGS_O {
	__gnu_pbds::gp_hash_table<int, int> mp;
	int g, mod, block, tag;

	vector<int> h, p, v;
	int h0, sz;
	int solve(int x)
	{
		int w = x;
		for (int i = 0; i <= mod / block; ++i) {
			if (mp.find(w) != mp.end())
				return i * block + mp[w];
			w = 1ll * w * tag % mod;
		}
		return -1;
	}

	void init(int _g, int _mod)
	{
		g = _g, mod = _mod, block = sqrtl(1ll * mod * sqrtl(mod) / logl(mod));
		mp.clear();
		tag = qpow(qpow(g, block, mod), mod - 2, mod);

		int w = 1;
		for (int i = 0; i < block; ++i) {
			mp[w] = i;
			w = 1ll * w * g % mod;
		}

		sz = sqrtl(mod) + 1;
		h.resize(sz + 5), p.resize(sz + 5), v.resize(sz + 5);
		h0 = solve(mod - 1);

		h[1] = 0;
		int idx = 0;
		for (int i = 2; i <= sz; ++i) {
			if (!v[i]) {
				p[++idx] = i;
				h[i] = solve(i);
			}
			for (int j = 1; j <= idx && p[j] <= sz / i; ++j) {
				h[i * p[j]] = (h[i] + h[p[j]]) % (mod - 1);
				v[i * p[j]] = true;
				if (i % p[j] == 0)
					break;
			}
		}
	}

	int get(int x)
	{
		if (x <= sz)
			return h[x];
		int now = mod / x, r = mod % x;
		if (r < x - r)
			return ((h0 + get(r)) % (mod - 1) - h[now] + mod - 1) % (mod - 1);
		else
			return (get(x - r) - h[now + 1] + mod - 1) % (mod - 1);
	}
} bsgs;
```

3.14 矩阵运算

```cpp
//注意构造函数传参
struct matrix {
	vector<vector<ll>>m;
	int r, c;
	matrix(int _r, int _c)
	{
		r = _r;
		c = _c;
		m.resize(r);
		for (auto& cc : m)
```

```
        cc.resize(c);
    }
};

matrix operator*(const matrix& a, const matrix& b)
{
    matrix c(a.r, b.c);
    for (int i = 0;i < a.r;++i)
        for (int j = 0;j < b.c;++j)
            for (int k = 0;k < a.c;++k)
                (c.m[i][j] += a.m[i][k] * b.m[k][j] % mod) %= mod;
    return c;
}

matrix matrix_pow(matrix a,ll b)
{
    matrix ans(a.r, a.c);
    for (int i = 0;i < a.r;i++)
        ans.m[i][i] = 1;

    for (; b;b >>= 1) {
        if (b & 1)
            ans = ans * a;
        a = a * a;
    }
    return ans;
}
```

3.15 高斯消元

```
//矩阵求逆则构造 n*2n 的矩阵 (A,In),将其化简为行最简型,右半部分则为逆矩阵
//O(n^3)
long double c[N][N],b[N];
void Gauss(int n)
{
    for (int i = 1; i <= n;i++){
        for (int j = i; j <= n;j++){

            //找非零元
            if(c[j][i]){
                for (int k = 1; k <= n;k++)
                    swap(c[i][k], c[j][k]);
                swap(b[i], b[j]);
                break;
            }
        }

        for (int j = 1; j <= n;j++){
            if(i==j)
                continue;

            auto rate = c[j][i] / c[i][i];
            for (int k = i; k <= n;k++)
                c[j][k] -= c[i][k] * rate;

            b[j] -= b[i] * rate;
        }
    }

    //b[i]中存储的即为答案
    for(int i = 1; i <= n;i++)
        b[i]/=c[i][i];
}
```

3.16 线性基

```
//每次使用记得清空
//区间线性基需要保存以位置 i 为右端点的前缀线性基，此外，还要记录每个向量的位置，更新时尽可能靠右
ll p[64];
bool flag;
void insert(ll x)
{
    ll use = 1ll << 62;
    for (int i = 62;i >= 0;i--) {
        if (use & x) {
            if (p[i])
                x ^= p[i];
            else {
                p[i] = x;
                return;
            }
        }
        use >>= 1;
    }
    flag = 1;
    return;
}

ll ask_max()
{
    ll ans = 0;
    for (int i = 62;i >= 0;i--)
        if ((ans ^ p[i]) > ans)
            ans ^= p[i];
    return ans;
}

ll ask_min()
{
    if (flag)
        return 0;
    for (int i = 0;i <= 62;i++)
        if (p[i])
            return p[i];
    return -1;
}

void rebuild()
{
    for(int i = 62;~i;--i)
        for(int j = i - 1;~j;--j)
            if(p[i] & (1 << j))
                p[i] ^= p[j];
    tot = 0;
    for(int i = 0;i <= 62;++i)
        if(p[i])
            d[++tot] = p[i];
}

int cmp(ll a, ll b)
{
    for (int i = 62;~i;--i) {
        bool xa = (a & (1 << i)), xb = (b & (1 << i));
        if (xa && xb == 0)
            return 1;
```

```
        else if (xa == 0 && xb)
            return -1;
        else if (xa && xb)
            return 0;
    }
    return 1;
}

ll ask_rk(ll x)
{
    ll rk = 1;
    ll temp = 0;
    for (int i = tot;i;--i) { //tot 指rebuild后的维数
        int flag = cmp (x ^ temp, d[i]);
        if (flag == 1) {
            return (rk += qpow(2, i) - 1 + mod) %= mod;
        }
        else if (now < 0)
            continue;
        else {
            ll ttmp = temp ^ d[i];
            if (ttmp > btt)
                continue;
            else
                (rk += qpow(2, i - 1)) %= mod, temp =
                    ttmp;
        }
    }
    return rk;
}

bool ask_check(ll x)
{
    if (!x)
        return flag;

    ll use = 1ll << 62;
    for (int i = 62;i >= 0;i--) {
        if (x & use) {
            if (!p[i])
                return false;
            else
                x ^= p[i];
        }
        use >>= 1;
    }
    return true;
}

ll ask_kth(ll k) //记得rebuild
{
    if (flag)
        k--;
    if (!k)
        return 0;

    ll ans = 0;
    ll use = 1ll << 62;
    int cnt = 0;
    for (int i = 62;i >= 0;i--)
        cnt += bool(p[i]);

    if ((1ll << cnt) - 1 < k)
        return -1;

    ll now = 1ll << (cnt - 1);
    for (int i = 62;i >= 0;i--) {
        if (p[i]) {
            if (bool(ans & use) != bool(k & now))
                ans ^= p[i];
            now >>= 1;
        }
        use >>= 1;
    }

    return ans;
}

/*---------区间线性基*----------*/
pii tag[2 * N][32];
void insert(int p, int x)
{
    int pos = p;
    for (int i = 30;~i;--i) {
        int use = (1 << i);
        if (!(x & use)) {
            tag[p][i] = tag[p - 1][i];
            continue;
        }

        if (!tag[p - 1][i].first) {
            tag[p][i].first = x;
            tag[p][i].second = pos;
            x = 0;
        }
        else if (pos > tag[p - 1][i].second) {
            tag[p][i].first = x;
            tag[p][i].second = pos;
            pos = tag[p - 1][i].second;
        }
        else
            tag[p][i] = tag[p - 1][i];
        x = x ^ tag[p - 1][i].first;
    }
}
ll ask(int l, int r)
{
    ll ans = 0;
    for (int i = 30;~i;--i) {
        if (tag[r][i].second < l)
            continue;
        if ((ans ^ tag[r][i].first) > ans)
            ans ^= tag[r][i].first;
    }
    return ans;
}
```

3.17 Lucas 定理

$O(p * \log_p n)$

```
//C代表组合数,求取过程省略
ll lucas(ll n,ll m)//递归lucas函数
{
    if(m==0)
        return 1ll;
    return lucas(n/mod,m/mod)*C(n%mod,m%mod)%mod;
}
```

3.18 莫比乌斯函数

```
const int N = 50005;
bool vis[N];
int prime[N], mo[N];
void get_mo()
{
    int cnt = 0;
    vis[1] = 1;
    mo[1] = 1;
    for (int i = 2;i <= 50000;i++) {
        if (!vis[i]) {
            prime[++cnt] = i;
            mo[i] = -1;
        }
        for (int j = 1;j <= cnt && 1ll * i * prime[j]
         <= 50000;j++) {
            vis[i * prime[j]] = 1;
            mo[i * prime[j]] = (i % prime[j] ? -mo[i] :
                0);
            if (i % prime[j] == 0)
                break;
        }
    }

}
```

3.19 莫比乌斯反演

第一种形式:
如果有 $F(n)=\sum_{d|n} f(d)$, 则 $f(n)=\sum_{d|n}\mu(d)F(\frac{n}{d})$
第二种形式:
如果有 $F(n)=\sum_{n|d} f(d)$, 则 $f(n)=\sum_{n|d}\mu(\frac{d}{n})F(d)$
套路与难点: 构造适当的 $f(n)$ 与 $F(n)$, 从而套用整除分块等技巧

3.20 0/1 分数规划

0/1 分数规划模型是指, 给定整数 $a_1,a_2,\cdots,a_n$ 以及 $b_1,b_2,\cdots,b_n$ 求一组解 $x_i(1\leqslant i\leqslant n, x_i=0$ 或 $1)$, 使下式最大化:

$$\frac{\sum_{i=1}^{n} a_i * x_i}{\sum_{i=1}^{n} b_i * x_i}$$

即从给定的 n 对整数 a_i,b_i 中选出若干对, 使得选出的数对的 a 之和与 b 之和的商最大

二分答案, 当二分的值为 mid 时, 我们就计算 $\sum_{i=1}^{n}(a_i-mid*b_i)*x_i$ 的最大值若非负, 则令 l=mid, 否则令 r=mid
最大值显然为 $a_i-mid*b_i$ 中所有正数的和

3.21 容斥原理

集合 S 不具有性质 $P_1,P_2,\cdots,P_m$ 的物体个数:

$$|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_m}| = |S| - \sum_{i=1}^{m}|A_i| + \sum_{i,j:i<j}|A_i\cap A_j| - \sum_{i,j,k:i<j<k}|A_i\cap A_j\cap A_k| + \cdots + (-1)^m|A_1\cap A_2\cap\cdots\cap A_m|$$

推论: 至少具有性质 $P_1,P_2,\cdots,P_m$ 之一的物体个数:

$$|A_1\cup A_2\cup\cdots\cup A_m| = \sum_{i=1}^{m}|A_i| - \sum_{i,j:i<j}|A_i\cap A_j| + \sum_{i,j,k:i<j<k}|A_i\cap A_j\cap A_k| + \cdots + (-1)^{m+1}|A_1\cap A_2\cap\cdots\cap A_m|$$

其中 A_i 表示 S 中具有性质 P_i 的集合,$\overline{A_i}$ 表示 A_i 在 S 中的补集

3.22 多项式反演

```
```

3.23 Min25 筛

对于 Min25 筛, 其要求求和的函数满足以下性质:
1. 积性函数, 即对于任意的 $gcd(x,y)=1$, 都有 $f(x)f(y)=f(xy)$
2. $f(p),p\in\mathbb{P}$ 能被表示为项数比较小的多项式
3. $f(p^c),p\in\mathbb{P}$ 能够快速求值

Min25 筛基本步骤:
1. 将待求函数在质数情况下的表达式转化成 $\sum i^k$ 的形式
2. 线性筛出 $1\sim\sqrt{n}$ 之间的质数, 并求出其 k 次方前缀和
3. 对于形如 $\lfloor\frac{n}{x}\rfloor$ 的数, 算出 $g(\lfloor\frac{n}{x}\rfloor,0)$, 注意去掉 1^k
4. 套递归式计算 $g(n,j)$, 这里 g 可以用滚动数组
5. 递归计算 $S(n,x)$, 无需记忆化, 答案即为 $S(n,0)+1$

该模板以 $\sum_{i=1}^{n} f(i)$, 其中 $f(p^k)=p^k$ 为例

```
//当题目有高次项时,仅需在标记位置做出添加或修改即可
//sp数组为质数高次前缀和
//具体方式为添加新的g和sp数组,当作新的项处理,原式结果即为
    项的线性组合
//g数组中存储的为 1^m + 2^m +... 的前tn项和,在该模板中给
    出的是m=1的情况
//其他部位模仿模板添加即可
const int N = 1000005;
const ll inv2 = 500000004;
const ll mod = 1000000007;
int cnt;
int v[N], prime[N];
void primes(int n)
{
    for (int i = 2; i <= n;i++){
        if(!v[i]){
            v[i] = i;
            prime[++cnt] = i;
        }
        for (int j = 1; j <= cnt;j++){
            if(prime[j]>v[i]||prime[j]>n/i)
                break;
            v[i * prime[j]] = prime[j];
        }
    }
    return;
}

//题目的f(x)函数,其中x代指 p^k
ll f(ll x)
{
    return x % mod;
}

ll sp[N], g[N];
ll w[N];
int idx1[N], idx2[N];
ll n;
int sqrn;
ll S(ll x,ll y)
```

```cpp
{
    if(prime[y]>=x)
        return 0;
    int k;
    if(x<=sqrn)
        k = idx1[x];
    else
        k = idx2[n / x];
    ll ans = g[k] - sp[y] + mod; //记得修改
    ans %= mod;

    for (int i = y + 1; i <= cnt;i++){
        if(1ll*prime[i]*prime[i]>x)
            break;

        ll pe = prime[i];
        for (int e = 1; pe <= x;e++,pe*=prime[i]){
            ll nx = pe % mod;
            ans = (ans + f(nx) * (S(x / pe, i) + (e !=
                 1)) % mod) % mod;
        }
    }

    return ans;
}

void solve()
{
    read(n);
    sqrn = sqrt(n);

    primes(sqrn);

    //注意可能的添加
    for (int i = 1; i <= cnt;i++)
        sp[i] = (sp[i - 1] + prime[i]) % mod;

    int tot = 0;
    for (ll l = 1, r, tn; l <= n;l=r+1){
        r = n / (n / l);
        w[++tot] = n / l; //注意不要取模
        tn = w[tot] % mod;

        //注意可能的添加
        g[tot] = tn * (tn + 1) % mod * inv2 % mod - 1;
        if(w[tot]<=sqrn)
            idx1[w[tot]] = tot;
        else
            idx2[n / w[tot]] = tot;
    }

    for (int i = 1; i <= cnt;i++)
        for (int j = 1; j <= tot;j++){
            if (1ll * prime[i] * prime[i] > w[j])
                break;

            int k;
            if(w[j]/prime[i]<=sqrn)
                k = idx1[w[j] / prime[i]];
            else
                k = idx2[n / (w[j] / prime[i])];

            //注意可能的添加
            (g[j] -= 1ll * prime[i] * (g[k] - sp[i - 1]
                 + mod) % mod) %= mod;
            (g[j] += mod) %= mod;
        }

    ll ans = S(n, 0) + 1; \\计算答案
    ans %= mod;
}
```

3.24 杜教筛

```cpp
//求欧拉函数前缀和
const int maxn = 5e6 + 5;
const ll inv = inv(2);

bool IsPrime[maxn];
int Tot;
ll Prime[maxn];
int Phi[maxn];
ll PrefixPhi[maxn];

void Sieve()
{
    for (int i = 2; i < maxn; ++i) IsPrime[i] = true;
    Phi[1] = 1;
    for (ll i = 2; i < maxn; ++i) {
        if (IsPrime[i]) {
            Phi[i] = i - 1;
            Prime[Tot++] = i;
        }
        for (ll j = 0; j < Tot && i * Prime[j] < maxn;
             ++j) {
            IsPrime[i * Prime[j]] = false;
            if (i % Prime[j] == 0) {
                Phi[i * Prime[j]] = Phi[i] * Prime[j];
                break;
            }
            Phi[i * Prime[j]] = Phi[i] * Phi[Prime[j]];
        }
    }
    for (int i = 1; i < maxn; ++i) PrefixPhi[i] = (
        PrefixPhi[i - 1] + Phi[i]) % mod;
}

map<ll, ll> SumPhi;

ll SigmaPhi(ll Key)
{
    if (Key < maxn) return PrefixPhi[Key];
    if (SumPhi[Key]) return SumPhi[Key];
    ll Ans = Key % mod * (Key % mod + 1) % mod * inv %
         mod;
    for (ll l = 2, tp; l <= Key; l = tp + 1) {
        tp = Key / (Key / l);
        Ans = (Ans - (tp - l + 1) % mod * SigmaPhi(Key
             / l) % mod + mod) % mod;
    }
    SumPhi[Key] = Ans;
    return SumPhi[Key];
}
```

3.25 快速数论变换 (NTT)

```cpp
namespace ntt
```

```
{
static constexpr int N = 262150;
static constexpr int mod = 998244353;
static constexpr int g = 3;
static int gk[N], rev[N];
static unsigned long long t[N];

int qpow(int x, long long n)
{
    int ans = 1;
    for (; n; n >>= 1) {
        if (n & 1)
            ans = 1ll * ans * x % mod;
        x = 1ll * x * x % mod;
    }
    return ans;
}
void NTT(vector<int> &a, int n)
{
    a.resize(n);
    for (int i = 0; i < n; ++i)
        t[i] = a[rev[i]];

    for (int i = 1; i < n; i <<= 1) {
        for (int k = i & (1 << 19); k; --k)
            if (t[k] >= mod * 9ull)
                t[k] -= mod * 9ull;
        for (int j = 0, p = i << 1; j < n; j += p)
            for (int k = 0; k < i; ++k) {
                const int x = t[i + j + k] * gk[i + k] %
                    mod;
                t[i + j + k] = t[k + j] + mod - x;
                t[k + j] += x;
            }
    }
    for (int i = 0; i < n; ++i)
        a[i] = t[i] % mod;
}

int com_size(int n, int m)
{
    auto sz = 2 << __lg(n + m - 1);
    int cnt = __builtin_ctz(sz);
    for (static int i = 1; i < sz; i <<= 1) {
        gk[i] = 1;
        const int w = qpow(g, mod / i / 2);
        for (int j = 1; j < i; ++j)
            gk[i + j] = 1ll * gk[i + j - 1] * w % mod;
    }
    for (int i = 0; i < sz; ++i)
        rev[i] = (rev[i >> 1] >> 1) | ((i & 1) << (cnt
             - 1));

    return sz;
}

void mult(vector<int> &a, vector<int> b)
{
    auto n = com_size(a.size(), b.size());
    int inv = mod - (mod - 1) / n;
    NTT(a, n), NTT(b, n);
    for (int i = 0; i < n; ++i)
        a[i] = 1ll * a[i] * b[i] % mod * inv % mod;
    NTT(a, n);
    if (n)
        reverse(a.begin() + 1, a.begin() + n);
}

class poly : public vector<int> {
public:
    using vector<int>::vector;
    poly operator*=(const poly &t)
    {
        auto &a = *this;
        if (a.empty() || t.empty())
            return {0};
        mult(a, t);
        while (!a.empty() && a.back() == 0)
            a.pop_back();
        return a;
    }
    poly operator*(const poly &t)
    {
        return poly(*this) *= t;
    }
};
}
using namespace ntt;
```

3.26 快速傅立叶变换 (FFT)

```
namespace fft
{
using db = double;
using cp = complex<db>;
using poly = vector<db>;

static constexpr db pi = acosl(-1);
static constexpr int N = 2097160;
static cp wn[N];
static int rev[N];

void FFT(vector<cp> &a, int n, int opt)
{
    a.resize(n);
    for (int i = 0; i < n; ++i)
        if (rev[i] < i)
            swap(a[rev[i]], a[i]);
    wn[0] = {1, 0};
    for (int i = 1; i < n; i <<= 1) {
        cp wi = {cosl(pi / i), opt * sinl(pi / i)};
        for (int k = i - 2; k >= 0; k -= 2)
            wn[k + 1] = (wn[k] = wn[k >> 1]) * wi;
        for (int j = 0, p = i << 1; j < n; j += p)
            for (int k = 0; k < i; ++k) {
                cp x = a[j + k], y = wn[k] * a[j + k + i
                    ];
                a[j + k] = x + y, a[j + k + i] = x - y;
            }
    }

    if (opt == -1)
        for (int i = 0; i < n; ++i)
            a[i] /= n;
}

int com_size(int n, int m)
{
    if (!n || !m)
```

```
      return 0;
   auto sz = 2 << __lg(n + m - 1);
   int cnt = __builtin_ctz(sz);
   for (int i = 0; i < sz; ++i)
      rev[i] = (rev[i >> 1] >> 1) | ((i & 1) << (cnt - 1));

   return sz;
}

void mult(poly &a, poly b)
{
   vector<cp> use;
   auto n = com_size(a.size(), b.size());
   use.resize(max(a.size(), b.size()));

   for (int i = 0; i < a.size(); ++i)
      use[i].real(a[i]);
   for (int i = 0; i < b.size(); ++i)
      use[i].imag(b[i]);

   FFT(use, n, 1);
   for (int i = 0; i < n; ++i)
      use[i] = use[i] * use[i];
   FFT(use, n, -1);

   a.resize(n);
   for (int i = 0; i < n; ++i)
      a[i] = use[i].imag() / 2;
}

poly &operator*=(poly &a, const poly &t)
{
   mult(a, t);
   return a;
}
poly operator*(poly a, const poly &t)
{
   return a *= t;
}
}
using namespace fft;
```

3.27 任意模多项式 (MTT)

```
namespace mtt
{
   using Type = double;
   using cp = complex<Type>;
   using poly = vector<int>;

   static constexpr Type pi = acosl(-1);
   static constexpr int N = 262150;
   static constexpr int B = 1 << 15;

   static constexpr int mod = 1000000007;
   static cp wn[N];
   static int rev[N];

   void DIF(vector<cp> &f, int n)
   {
      for (int l = n >> 1, r = n; l; r = l, l >>= 1)
         {
         for (int k = 0; k < l; ++k) {
            cp x = f[k], y = f[k + l];
            f[k] = x + y, f[k + l] = x - y;
         }
         for (int j = r, idx = 1; j < n; j += r, ++idx)
            for (int k = j; k < j + l; ++k) {
               cp x = f[k], y = f[k + l] * wn[idx];
               f[k] = x + y, f[k + l] = x - y;
            }
      }
   }

   void DIT(vector<cp> &f, int n)
   {
      for (int l = 1, r = 2; l < n; l = r, r <<= 1)
         {
         for (int k = 0; k < l; ++k) {
            cp x = f[k], y = f[k + l];
            f[k] = x + y, f[k + l] = x - y;
         }
         for (int j = r, idx = 1; j < n; j += r, ++idx)
            for (int k = j; k < j + l; ++k) {
               cp x = f[k], y = f[k + l];
               f[k] = x + y, f[k + l] = (x - y) * conj(wn[idx]);
            }
      }
   }

   int com_size(int n, int m)
   {
      if (!n || !m)
         return 0;
      auto sz = 2 << __lg(n + m - 1);
      int cnt = __builtin_ctz(sz);
      for (int i = 0; i < sz; ++i)
         rev[i] = (rev[i >> 1] >> 1) | ((i & 1) << (cnt - 1));
      wn[0] = {1};
      int t = (__lg(sz) >> 1);
      int len = (1 << t);
      vector<cp> bas(len << 1);
      const Type p0 = acosl(-1) / len, p1 = p0 / len;

      int p = 0;
      for (int i = 0, j = (len * 3) >> 1; i < len;
            p -= len - (j >> __builtin_ctz(++i))) {
         bas[i] = polar((Type)1, p * p0);
         bas[i | len] = polar((Type)1, p * p1);
      }

      for (int i = 1; i < (sz >> 1); ++i)
         wn[i] = bas[i & (len - 1)] * bas[len | (i >> t)];
      return sz;
   }

   void mult(poly &a, const poly &b)
   {
      int n = com_size(a.size(), b.size());
      vector<cp> f(n), g(n);
      for (int i = 0; i < a.size(); ++i)
         f[i] = {a[i] / B, a[i] % B};
```

```cpp
    for (int i = 0; i < b.size(); ++i)
        g[i] = {b[i] / B, b[i] % B};
    DIF(f, n), DIF(g, n);

    Type fx = .5L / n;
    for (int i = 0; i < min(2, n); ++i) {
        cp p = f[i], r = g[i] * fx;
        f[i] = (p + conj(p)) * r;
        g[i] = (conj(p) - p) * r;
    }

    for (int k = 2, m = 3; k < n; k <<= 1, m <<=
        1)
        for (int i = k, j = k << 1; --j, i < m; ++i
            ) {
            cp p = f[i], q = f[j], r = g[i] * fx, s
                = g[j] * fx;
            f[i] = (p + conj(q)) * r;
            g[i] = (conj(q) - p) * r;
            f[j] = (q + conj(p)) * s;
            g[j] = (conj(p) - q) * s;
        }
    DIT(f, n), DIT(g, n);

    a.resize(n);
    for (int i = 0; i < n; ++i) {
        int aa = (ll)roundl(f[i].real()) % mod * B
            * B % mod;
        int bb = (ll)roundl(f[i].imag()) % mod * B
            % mod;
        int cc = (ll)roundl(-g[i].imag()) % mod * B
             % mod;
        int dd = (ll)roundl(g[i].real()) % mod;
        a[i] = (1ll * aa + bb + cc + dd) % mod;
    }
}

poly &operator*=(poly &a, const poly &t)
{
    mult(a, t);
    return a;
}

poly operator*(poly a, const poly &t)
{
    return a *= t;
}

int qpow(int a, long long b)
{
    int ans = 1;
    for (; b; b >>= 1) {
        if (b & 1)
            ans = 1ll * ans * a % mod;
        a = 1ll * a * a % mod;
    }
    return ans;
}
poly inv(const poly &f, int n)
{
    if (n == 1)
        return {qpow(f[0], mod - 2)};
    poly h = inv(f, (n + 1) >> 1);
    auto g = f * h * h;
    for (auto &c : h)
        (c += c) %= mod;
    g.resize(n);
    h.resize(n);

    for (int i = 0; i < n; ++i)
        h[i] = (h[i] + mod - g[i]) % mod;
    return h;
}
};

using namespace mtt;
```

3.28 快速沃尔什变换 (FWT)

```cpp
//NTT可以处理下标相加，而FWT主要是处理下标的其他逻辑运算
//用法与NTT相同
//记得初始化bit与tot
//不需要蝴蝶变换
int bit,tot;
void fwt_or(int a[],int op)
{
    for(int mid=1;mid<tot;mid<<=1){
        int len=mid<<1;
        for(int i=0;i<tot;i+=len)
            for(int j=0;j<mid;j++)
                (a[mid+i+j]+=a[i+j]*op+mod)%=mod;
    }
    return;
}

void fwt_and(int a[],int op)
{
    for(int mid=1;mid<tot;mid<<=1){
        int len=mid<<1;
        for(int i=0;i<tot;i+=len)
            for(int j=0;j<mid;j++)
                (a[i+j]+=a[i+j+mid]*op+mod)%=mod;
    }
    return;
}

void fwt_xor(int a[],int op)
{
    if(op==-1)
        op=qpow(2,mod-2);
    for(int mid=1;mid<tot;mid<<=1){
        int len=mid<<1;
        for(int i=0;i<tot;i+=len)
            for(int j=0;j<mid;j++){
                int x=a[i+j],y=a[mid+i+j];
                a[i+j]=(x+y)%mod;
                a[mid+i+j]=(x-y+mod)%mod;
                a[i+j]=op*a[i+j]%mod;
                a[mid+i+j]=a[mid+i+j]*op%mod;
            }
    }

    return;
}
```

3.29 多项式求逆

```cpp
poly operator-=(const poly &t)
{
    auto &a = *this;
    a.resize(max(a.size(), t.size()));
    for (int i = 0; i < t.size(); ++i)
        a[i] = (a[i] - t[i] < 0 ? a[i] - t[i] + mod : a[i] - t[i]);
    while (!a.empty() && a.back() == 0)
        a.pop_back();
    return a;
}
poly operator-(const poly &t)
{
    return poly(*this) -= t;
}

poly mod_xk(int k)
{
    return poly(begin(), begin() + min(k, (int)size()));
}

// 多项式求逆
poly inv(int n)
{
    if (n == 1)
        return {qpow(at(0), mod - 2)};
    poly h = inv((n + 1) >> 1);
    return ((poly{2} - (mod_xk(n) * h).mod_xk(n)) * h).mod_xk(n);
}
```

3.30 多项式 ln

```cpp
static int frac(int x)
{
    static int F[N];
    static bool init = false;
    if (!init) {
        F[0] = 1;
        for (int i = 1; i < N; i++)
            F[i] = 1ll * F[i - 1] * i % mod;
        init = true;
    }
    return F[x];
}

static int ifrac(int x)
{
    static int F[N];
    static bool init = false;
    if (!init) {
        F[N - 1] = qpow(frac(N - 1), mod - 2);
        for (int i = N - 2; i >= 0; i--) {
            F[i] = 1ll * F[i + 1] * (i + 1) % mod;
        }
        init = true;
    }
    return F[x];
}

// 多项式求导
poly deriv(int k = 1)
{
    auto &a = *this;
    if (k > size())
        return {0};
    poly f(a.size() - k);
    for (int i = k; i < a.size(); ++i)
        f[i - k] = 1ll * a[i] * frac(i) % mod * ifrac(i - 1) % mod;
    return f;
}

// 多项式积分
poly integr()
{
    auto &a = *this;
    poly f(a.size() + 1);
    for (int i = 1; i <= a.size(); ++i)
        f[i] = 1ll * a[i - 1] * qpow(i, mod - 2) % mod;
    return f;
}

// 求ln(F(x)), 必须满足 F(0) = 1
poly ln(int n)
{
    return (deriv() * inv(n)).integr().mod_xk(n);
}
```

3.31 多项式 exp

```cpp
poly operator+=(const poly &t)
{
    auto &a = *this;
    a.resize(max(a.size(), t.size()));
    for (int i = 0; i < t.size(); ++i)
        a[i] = (a[i] + t[i] >= mod ? a[i] + t[i] - mod : a[i] + t[i]);
    while (!a.empty() && a.back() == 0)
        a.pop_back();
    return a;
}
poly operator+(const poly &t)
{
    return poly(*this) += t;
}

poly exp(int n)
{
    if (n == 1)
        return {1};
    poly g = exp((n + 1) >> 1);
    return (g * (poly{1} - g.ln(n) + mod_xk(n)).mod_xk(n)).mod_xk(n);
}
```

3.32 多项式快速幂

```cpp
poly operator*=(const int &t)
{
    auto &a = *this;
    for (auto &c : a)
        c = 1ll * c * t % mod;
```

```
    return a;
}
poly operator*(const int &t)
{
    return poly(*this) *= t;
}

// 求(F(x))^k mod (x^n)的值
poly pow(int k, int n) // k < mod
{
    auto &a = *this;
    int cnt = 0, tag = 0;
    if (all_of(a.begin(), a.end(), [](auto &p) {
        return p == 0; }))
        return {0};
    else {
        for (const auto &c : a)
            if (!c)
                cnt++;
            else {
                tag = c;
                break;
            }
    }
    if (1ll * k * cnt >= n)
        return {0};
    return ((poly{qpow(tag, mod - 2)} * div_xk(cnt)).
        ln(n) * k).exp(n).mul_xk(1ll * k * cnt).
        mod_xk(n) * qpow(tag, k);
}

poly pow(const string &s, int n) // k >= mod
{
    auto &a = *this;
    int k = 0, k1 = 0;
    int cnt = 0, tag = 0;
    for (const auto &c : s)
        k = (k * 10ll % mod + c - '0') % mod, k1 = (k1
            * 10ll % (mod - 1) + c - '0') % (mod - 1)
            ;
    if (all_of(a.begin(), a.end(), [](auto &p) {
        return p == 0; }))
        return {0};
    else {
        for (const auto &c : a)
            if (!c)
                cnt++;
            else {
                tag = c;
                break;
            }
    }

    if (cnt && s.size() > 7)
        return {0};
    else if (1ll * cnt * k >= n)
        return {0};
    else
        return ((poly{qpow(tag, mod - 2)} * div_xk(cnt
            )).ln(n) * k).exp(n).mul_xk(1ll * k * cnt)
            .mod_xk(n) * qpow(tag, k1);
}
```

3.33 多项式开根

```
int BSGS(int a, int n)
{
    unordered_map<int, int> mp;
    int m = ceill(sqrtl(mod));
    int s = n;
    for (int i = 0; i < m; ++i) {
        mp[s] = i;
        s = 1ll * s * a % mod;
    }
    int temp = qpow(a, m);
    s = 1;
    for (int i = 0; i <= m; ++i) {
        if (mp.find(s) != mp.end() && i * m >= mp[s])
            return i * m - mp[s];
        s = 1ll * s * temp % mod;
    }
    return -1;
}
poly sqrt_k(int k, int n) // f[0]是mod下k次剩余
{
    poly f(*this);
    int s = f[0], inv = qpow(s, mod - 2);
    for (auto &c : f)
        c = 1ll * c * inv % mod;
    f = f.ln(n);
    int p = qpow(k, mod - 2);
    for (auto &c : f)
        c = 1ll * c * p % mod;
    f = f.exp(n);

    s = BSGS(qpow(g, k), s);
    s = qpow(g, s);
    s = min(s, mod - s); //求字典序最小解，按需变化

    for (auto &c : f)
        c = 1ll * c * s % mod;
    return f;
}
```

3.34 多项式三角函数

```
poly cos(int n)
{
    poly f(*this);
    int img = qpow(g, (mod - 1) >> 2);
    f = (f * img).exp(n);
    poly g = f.inv(n);
    int inv = qpow(2, mod - 2);
    poly h(n);
    f.resize(n), g.resize(n);
    for (int i = 0; i < n; ++i)
        h[i] = 1ll * inv * (f[i] + g[i]) % mod;
    return h;
}

poly sin(int n)
{
    poly f(*this);
    int img = qpow(g, (mod - 1) >> 2);
    f = (f * img).exp(n);
    poly g = f.inv(n);
    int inv = qpow(img << 1, mod - 2);
```

```
    poly h(n);
    f.resize(n), g.resize(n);
    for (int i = 0; i < n; ++i)
        h[i] = 1ll * inv * (f[i] - g[i] + mod) % mod;
    return h;
}
```

3.35 常用组合公式

$$\sum_{k=1}^{n}(2k-1)^2 = \frac{n(4n^2-1)}{3}$$
$$\sum_{k=1}^{n}k^3 = \left(\frac{n(n+1)}{2}\right)^2$$
$$\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$$
$$\sum_{k=1}^{n}k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$
$$\sum_{k=1}^{n}k^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$
$$\sum_{k=1}^{n}k(k+1) = \frac{n(n+1)(n+2)}{3}$$
$$\sum_{k=1}^{n}k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$
$$\sum_{k=1}^{n}k(k+1)(k+2)(k+3) = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$$

错排 (排列中所有的元素都在不正确的位置上的排列种数) 公式:
$D_n = n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!}\right) = (n-1)(D_{n-2}-D_{n-1})$

3.36 Bertrand 猜想

对任意 $n > 3$, 都存在 $n < p < 2 \times n$, 其中 p 为质数

3.37 威尔逊定理

p 为素数时 (充要条件)

$$(p-1)! \equiv -1(mod \quad p)$$

3.38 最小二乘法

$$k = \frac{\sum_{i=1}^{n} x_i y_i - n \cdot \overline{xy}}{\sum_{i=1}^{n} x_i^2 - n \cdot (\overline{x})^2},\ b = \overline{y} - k\overline{x}$$

3.39 数相关结论

10^9 内所有数的最多因子个数为 1344 个 10^{18} 内最多有 103680 个
10^7 内所有数的因子大小的和最大约为 5×10^7

3.40 卡特兰数

通项: $C_n = \frac{C_{2n}^{n}}{n+1}$

3.41 斯特林数

表示将 n 个不同元素构成 m 个圆排列的数目

3.42 第二类斯特林数

表示将 n 个不同元素分成 m 个集合的方案数, 不能有空集合
$S(n,m) = \frac{1}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n$

3.43 复数操作

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

```
std::conj(z) //返回复数z的共轭
std::norm(z) //返回复数z的模平方
std::abs(z) //返回复数z的模
std::arg(z) //返回复数z的相位(角度)
z.real(),z.imag() //获取实部和虚部
z.real(value),z.imag(value) //修改为value
std::exp,log,pow,sqrt //指数、对数、幂和平方根
```

3.44 康托展开

```
//O(nlogn)
//add与ask函数为树状数组操作
int n;
ll cantor()
{
    ll ans = 0;
    for (int i = 1; i <= n;i++){
        ans = (ans + (ask(a[i]) - 1) * fac[n - i] %
            mod) % mod;
        add(a[i], -1);
    }

    return ans;
}
void solve()
{
    cin >> n;
    fac[0]=1;
    for(int i=1;i<n;++i)
        fac[i] = fac[i - 1] * i % mod;

    for (int i = 1; i <= n;++i)
        add(i, 1);
    for (int i = 1; i <= n;++i)
        cin >> a[i];

    cout << cantor() + 1 << '\n';
    return ;
}
```

3.45 逆康托展开

```
//排名为x的排列
vector<int> incantor(int x,int n){
    x--;
    vector<int> res;
    int cnt;
    for(int i = 0;i < n; ++i){
        cnt = x / fact[n - i - 1];
        x %= fact[n - i - 1];

        /*------寻找第cnt+1大数的大小,记作ans--------*/
        /*-----可使用权值线段树二分或者树状数组倍增-----*/
```

```
        res.emplace_back(ans);
    }
    return res;
}
```

3.46 生成函数

指数型生成函数适用于解决多重集选择排列问题。
常用替换式:
$\sum_{i\geq 0}\frac{x^n}{n!}=e^x$
$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\frac{e^x+e^{-x}}{2}$
$\frac{x^m}{m!}$ 前的系数即为选 m 个元素排列的值

3.47 自适应辛普森积分

```
const long double eps = 1e-8L; //设置为精度 + 2
typedef long double ld;
ld a, b, c, d, l, r;
ld f(ld x)
{
    //修改为待积分函数
    return (c * x + d) / (a * x + b);
}

ld simpson(ld l, ld r, ld eps = eps)
{
    ld mid = (l + r) / 2;
    return (f(l) + 4 * f(mid) + f(r)) * (r - l) / 6;
}

ld integr(ld l, ld r, ld eps = eps)
{
    ld mid = (l + r) / 2;
    ld sum = simpson(l, r);
    ld suml = simpson(l, mid);
    ld sumr = simpson(mid, r);

    if (abs(suml + sumr - sum) <= 15 * eps)
        return suml + sumr + (suml + sumr - sum) / 15;
    return integr(l, mid, eps / 2) + integr(mid, r,
         eps / 2);
}
```

3.48 行列式求值

```
const int N = 605;
int a[N][N];
ll cal(int n)
{
    ll ans = 1, w = 1;
    for (int i = 1; i <= n; ++i)
        for (int j = i + 1; j <= n; ++j) {
            while (a[i][i]) {
                int div = a[j][i] / a[i][i];
                for (int k = i; k <= n; ++k)
                    a[j][k] = (a[j][k] - 1ll * div * a[i
                        ][k] % mod + mod) % mod;
                swap(a[i], a[j]);
                w *= -1;
            }
            swap(a[i], a[j]);
            w *= -1;
        }
    for (int i = 1; i <= n; ++i)
        (ans *= a[i][i]) %= mod;
    ans *= w;
    return (mod + ans) % mod;
}
```

4 数据结构

4.1 并查集

```
int fa[N];
int get(int x)
{
    if(x==fa[x])
        return x;
    return fa[x]=get(fa[x]);
}

void merge(int x,int y)
{
    int a=get(x);
    int b=get(y);
    if(a==b)
        return;
    fa[b]=a;
    return;
}

//初始化
for(int i=1;i<=n;i++)
    fa[i]=i;
```

4.2 并查集跳跃

```
//当每个元素的操作次数有上限时，此方法可以保证不重复操作
for (int i = 1;i <= n + 1;i++)
    fa[i] = i;
if(EQUAL)
    fa[i] = get(i + 1);
for (int i = get(l);i <= r;i = get(i + 1)) {
    /*操作*/
    if (EQUAL)
        fa[i] = get(i + 1);
}
```

4.3 可持久化并查集

4.4 树状数组

```
//树状数组主要是用来处理前缀和
//除前缀和外,所有前缀性质都可以维护
//如前缀最值等操作
#define lowbit(x) (x&(-x))
const int N=200005;
```

```cpp
int c[N];
void add(int p,int x)
{
	for(int i=p;i<=N;i+=lowbit(i))
		c[i]+=x; //或其他操作
	return;
}

int ask(int p)
{
	int ans=0;
	for(int i=p;i;i-=lowbit(i))
		ans+=c[i]; //或其他操作

	return ans;
}
```

4.5 线段树

```cpp
//以区间和为例
const int N=100010;
typedef long long ll;

int a[N];
struct{
	int l,r;
	ll add,sum;
}tr[N*4];
void pushup(int p)
{
	tr[p].sum=tr[p<<1].sum+tr[p<<1|1].sum;
	return;
}
void build(int p,int l,int r)
{
	if(l==r){
		tr[p].l=tr[p].r=l;
		tr[p].sum=a[l];
		return;
	}

	int mid=(l+r)>>1;
	tr[p].l=l;
	tr[p].r=r;
	build(p<<1,l,mid);
	build(p<<1|1,mid+1,r);
	pushup(p);
	return;
}
void pushdown(int p)
{
	if(tr[p].l==tr[p].r)
		return;
	tr[p<<1].add+=tr[p].add;
	tr[p<<1|1].add+=tr[p].add;

	tr[p<<1].sum+=1ll*(tr[p<<1].r-tr[p<<1].l+1)*tr[p].add;
	tr[p<<1|1].sum+=1ll*(tr[p<<1|1].r-tr[p<<1|1].l+1)*tr[p].add;
	tr[p].add=0;
	return;
}
void change(int p,int l,int r,int x)
{
	pushdown(p);
	if(l<=tr[p].l&&r>=tr[p].r){
		//打标记,计算该线段修改后的值
		tr[p].add+=x;
		tr[p].sum+=1ll*x*(tr[p].r-tr[p].l+1);
		return;
	}

	int mid=(tr[p].r+tr[p].l)>>1;
	if(l<=mid)
		change(p<<1,l,r,x);
	if(r>mid)
		change(p<<1|1,l,r,x);

	pushup(p);
	return;
}

ll ask(int p,int l,int r)
{
	pushdown(p);
	ll sum=0;
	if(l<=tr[p].l&&r>=tr[p].r)
		return tr[p].sum;

	int mid=(tr[p].l+tr[p].r)>>1;
	if(l<=mid)
		sum+=ask(p<<1,l,r);
	if(r>mid)
		sum+=ask(p<<1|1,l,r);

	return sum;

}
```

4.6 李超线段树

```cpp
//支持添加线性函数，询问每个横坐标中值最大的函数编号
//注意精度，能用 int 尽量用
//本题以编号最小为例，使用时仅需要add线段
const int N = 100005;
const double eps = 1e-9;
struct node {
	int l, r, id;
}tr[N << 2];

int cmp(double x,double y)
{
	if (x - y > eps)
		return 1;
	if (y - x > eps)
		return -1;
	return 0;
}
int cnt;
struct line {
	double k, b;
}seg[N];

double f(int id, int x)
{
	return seg[id].k * x + seg[id].b;
}
```

```

void build(int p, int l, int r)
{
    if (l == r) {
        tr[p].l = tr[p].r = l;
        return;
    }
    tr[p].l = l;
    tr[p].r = r;
    int mid = (l + r) >> 1;
    build(p << 1, l, mid);
    build(p << 1 | 1, mid + 1, r);
    return;
}

void update(int p, int id)
{
    int& v = tr[p].id;
    int u = id;
    int mid = (tr[p].l + tr[p].r) >> 1;
    if (cmp(f(u, mid), f(v, mid)) == 1)
        swap(u, v);
    int tagl = cmp(f(u, tr[p].l), f(v, tr[p].l));
    int tagr = cmp(f(u, tr[p].r), f(v, tr[p].r));

    //其中 u < v 含义为保留标号最小
    if (tagl == 1 || (!tagl && u < v))
        update(p << 1, u);
    if (tagr == 1 || (!tagr && u < v))
        update(p << 1 | 1, u);
    return;
}

void change(int p, int l, int r, int u)
{
    if (l <= tr[p].l && r >= tr[p].r) {
        update(p, u);
        return;
    }
    int mid = (tr[p].l + tr[p].r) >> 1;
    if (l <= mid)
        change(p << 1, l, r, u);
    if (r > mid)
        change(p << 1 | 1, l, r, u);
}
void add(int x0, int y0, int x1, int y1)
{
    cnt++;
    if (x0 == x1) {
        seg[cnt].k = 0;
        seg[cnt].b = max(y0, y1);
    }
    else {
        seg[cnt].k = 1. * (y1 - y0) / (x1 - x0);
        seg[cnt].b = y0 - seg[cnt].k * x0;
    }
    change(1, x0, x1, cnt);
}

//ask 时会遍历所有区间，可以进行标号最小操作等询问
//以标号最小为例
typedef pair<double, int> pdi;
pdi get_max(pdi a, pdi b)
{
    if (cmp(a.first, b.first) == 1)
        return a;
    if (cmp(a.first, b.first) == -1)
        return b;
    if (a.second > b.second)
        return b;
    else
        return a;
}
pdi ask(int p, int x)
{
    //注意编号
    pdi now = { f(tr[p].id, x) ,tr[p].id };
    if (tr[p].l == tr[p].r)
        return now;

    int mid = (tr[p].l + tr[p].r) >> 1;
    if (x <= mid && x >= tr[p].l)
        return get_max(now, ask(p << 1, x));
    else
        return get_max(now, ask(p << 1 | 1, x));
}
```

4.7 动态开点李超线段树

```
//注意tot是用来记录直线的
//同时注意第0条直线的问题，必要时置为负无穷或正无穷
//默认最高的线以及最小的编号
const int N = 100005, INF = 40005;
const double eps = 1e-8;
int root, node_idx, tot;
struct Segs
{
    double k, b;
}segs[N];
struct Node
{
    int l, r;
    int id;
}tr[N << 6];

bool cmp(double a)
{
    return fabs(a) <= eps ? 0 : a < 0 ? -1 : 1;
}

double f(int id, int x)
{
    return segs[id].k * x + segs[id].b;
}

//同等情况编号小优先
//改大小优先级就改 f1 和 f2 的比较，改编号优先级就改 id1 和 id2 的比较
bool judge(int id1, int id2, int x)
{
    double f1 = f(id1, x), f2 = f(id2, x);
    return cmp(f1 - f2) ? f1 < f2 : id1 > id2;
}

//id指的是想要插入的直线的id
//modify(root,1,n,l,r,id);
void modify(int& u, int l, int r, int x, int y, int id)
{
```

```
    if (!u)
        u = ++node_idx;
    if (x <= l && y >= r) {
        if (judge(id, tr[u].id, l) && judge(id, tr[u].
            id, r))
            return;
        if (judge(tr[u].id, id, l) && judge(tr[u].id,
            id, r)) {
            tr[u].id = id;
            return;
        }
        int mid = l + r >> 1;
        if (judge(tr[u].id, id, mid))
            swap(tr[u].id, id);
        if (judge(tr[u].id, id, l))
            modify(tr[u].l, l, mid, x, y, id);
        if (judge(tr[u].id, id, r))
            modify(tr[u].r, mid + 1, r, x, y, id);
        return;
    }
    else {
        int mid = (l + r) >> 1;
        if (x <= mid)
            modify(tr[u].l, l, mid, x, y, id);
        if (y > mid)
            modify(tr[u].r, mid + 1, r, x, y, id);
    }
}

//得到最高的线的 id
int query(int u, int l, int r, int x)
{
    if (l == x && x == r)
        return tr[u].id;
    else {
        int mid = (l + r) >> 1, res = 0;
        if (x <= mid)
            res = query(tr[u].l, l, mid, x);
        if (x > mid)
            res = query(tr[u].r, mid + 1, r, x);
        if (judge(res, tr[u].id, x))
            res = tr[u].id;
        return res;
    }
}
```

4.8 主席树

```
//以区间第K大数为例
//idx根节点编号,每次修改都会建立一个新的根节点
const int N=100005;
int idx ,root[N];
struct{
    int l,r,cnt;
}tr[N*4+N*17];
int a[N];

int build(int l,int r)
{
    int p=++idx;
    if(l==r)
        return p;

    int mid=(l+r)>>1;
    tr[p].l=build(l,mid);
    tr[p].r=build(mid+1,r);

    return p;
}

int change(int p,int l,int r,int x)
{
    int q=++idx;
    tr[q]=tr[p];
    if(l==r){
        tr[q].cnt++;
        return q;
    }

    int mid=(l+r)>>1;
    if(x<=mid)
        tr[q].l=change(tr[p].l,l,mid,x);
    else
        tr[q].r=change(tr[p].r,mid+1,r,x);

    tr[q].cnt=tr[tr[q].l].cnt+tr[tr[q].r].cnt;

    return q;
}

int ask(int p,int q,int l,int r,int k)
{
    if(l==r)
        return l;

    int cnt=tr[tr[q].l].cnt-tr[tr[p].l].cnt;

    int mid=(l+r)>>1;
    if(cnt>=k)
        return ask(tr[p].l,tr[q].l,l,mid,k);
    else
        return ask(tr[p].r,tr[q].r,mid+1,r,k-cnt);

}
int main()
{
    /*---主席树操作---*/
    root[0]=build(0,num.size()-1);

    //在第i-1代树上添加a[i],得到第i代树
    for(int i=1;i<=n;i++)
        root[i]=change(root[i-1],0,num.size()-1,a[i]);

    ans=ask(root[l-1],root[r],0,num.size()-1,k);
    return 0;
}
```

4.9 动态开点线段树

```
```

4.10 线段树分裂与合并

```
```

4.11 Splay

```cpp
const int N = 2000005;
int rt, tot, fa[N], ch[N][2], val[N], cnt[N], sz[N];
void maintain(int x)
{
    sz[x] = sz[ch[x][0]] + sz[ch[x][1]] + cnt[x];
    return;
}
bool get(int x)
{
    return x == ch[fa[x]][1];
}
void clear(int x)
{
    ch[x][0] = ch[x][1] = fa[x] = val[x] = sz[x] = cnt
        [x] = 0;
}
void rotate(int x)
{
    int f = fa[x], gf = fa[f], id = get(x);
    ch[f][id] = ch[x][id ^ 1];
    if (ch[x][id ^ 1])
        fa[ch[x][id ^ 1]] = f;
    ch[x][id ^ 1] = f;
    fa[f] = x;
    fa[x] = gf;
    if (gf)
        ch[gf][f == ch[gf][1]] = x;
    maintain(f);
    maintain(x);
}

void splay(int x)
{
    int f = fa[x];
    while (f) {
        if (fa[f])
            rotate(get(x) == get(f) ? f : x);
        rotate(x);
        f = fa[x];
    }
    rt = x;
}
int pre()
{
    int cur = ch[rt][0];
    if (!cur)
        return cur;
    while (ch[cur][1])
        cur = ch[cur][1];
    splay(cur);
    return cur;
}
int nxt()
{
    int cur = ch[rt][1];
    if (!cur)
        return cur;
    while (ch[cur][0])
        cur = ch[cur][0];
    splay(cur);
    return cur;
}

//插入元素
void insert(int x)
{
    if (!rt) {
        val[++tot] = x;
        cnt[tot]++;
        rt = tot;
        maintain(rt);
        return;
    }
    int cur = rt, f = 0;
    while (true) {
        if (val[cur] == x) {
            cnt[cur]++;
            maintain(cur);
            maintain(f);
            splay(cur);
            break;
        }
        f = cur;
        cur = ch[cur][val[cur] < x];
        if (!cur) {
            val[++tot] = x;
            cnt[tot]++;
            fa[tot] = f;
            ch[f][val[f] < x] = tot;
            maintain(tot);
            maintain(f);
            splay(tot);
            break;
        }
    }
}

//询问 x 的排名 (比 x 小的数的个数 + 1)
int ask_rank(int x)
{
    int ans = 0, cur = rt;
    while (cur) {
        if (x < val[cur])
            cur = ch[cur][0];
        else {
            ans += sz[ch[cur][0]];
            if (x == val[cur]) {
                splay(cur);
                return ans + 1;
            }
            ans += cnt[cur];
            cur = ch[cur][1];
        }
    }
    return ans + 1;
}

//删除值为 x 的元素
void del(int x)
{
    ask_rank(x);
    if (cnt[rt] > 1) {
        cnt[rt]--;
        maintain(rt);
    }
    else if (!ch[rt][0] && !ch[rt][1]) {
        clear(rt);
        rt = 0;
    }
    else if (!ch[rt][0]) {
```

```
      int cur = rt;
      rt = ch[rt][1];
      fa[rt] = 0;
      clear(cur);
   }
   else if (!ch[rt][1]) {
      int cur = rt;
      rt = ch[rt][0];
      fa[rt] = 0;
      clear(cur);
   }
   else {
      int cur = rt, x = pre();
      fa[ch[cur][1]] = x;
      ch[x][1] = ch[cur][1];
      clear(cur);
      maintain(rt);
   }
}

//查询第 k 大的元素，若没有则为排名小于 k 的最大数
int ask_kth(int k)
{
   int cur = rt;
   while (true) {
      if (ch[cur][0] && k <= sz[ch[cur][0]])
         cur = ch[cur][0];
      else {
         k -= cnt[cur] + sz[ch[cur][0]];
         if (k <= 0) {
            splay(cur);
            return val[cur];
         }
         cur = ch[cur][1];
      }
   }
}

//查询 x 的前驱
int ask_pre(int x)
{
   insert(x);
   int ans = val[pre()];
   del(x);
   return ans;
}

//查询 x 的后继
int ask_nxt(int x)
{
   insert(x);
   int ans = val[nxt()];
   del(x);
   return ans;
}
```

4.12 AC 自动机

4.13 分块

```
//分块中的预处理
//区间处理时，可以先特判做右端点在同一块中，后从pos[L]+1到pos[R]-1处理，最后再处理两端的小段
int st[N], ed[N];
int pos[N];
int block, cnt;
void init(int n)
{
   block = sqrt(n);
   cnt = n / block + bool(n % block);

   for (int i = 1;i <= cnt;i++) {
      st[i] = (i - 1) * block + 1;
      ed[i] = i * block;
   }

   ed[cnt] = n;
   for (int i = 1;i <= n;i++)
      pos[i] = (i - 1) / block + 1;
}
```

4.14 莫队

```
//程序基本上只需要添加辅助数组以及编写add与del函数即可
const int N=50000;
int a[N],belong[N];
struct query{
   int l,r,id;
}q[N];

int cmp(const query& a,const query& b)
{
   if(belong[a.l]^belong[b.l])
      return belong[a.l]<belong[b.l];
   else if(belong[a.l]&1)
      return a.r<b.r;
   else
      return a.r>b.r;
}

int now; //记录当前答案
void add(int pos)
{
   //添加第pos位后的答案
   //操作省略
}

void del(int pos)
{
   //删除第pos位的答案
   //操作省略
}

int main()
{
   int n,m; //n为数据个数，m为询问数
   read(n,m);
   int sz=sqrt(n);
   int bnum=ceil(1.*n/sz);
   for(int i=1;i<=bnum;i++)
      for(int j=(i-1)*sz+1;j<=i*sz;j++)
         belong[j]=i;

   //读入原始数据
   for(int i=1;i<=n;i++)
```

```
      read(a[i]);

   for(int i=1;i<=m;i++){
      read(q[i].l,q[i].r);
      q[i].id=i;
   }

   sort(q+1,q+m+1,cmp);
   int l=1,r=0;

   for(int i=1;i<=m;i++){
      int ql=q[i].l,qr=q[i].r;
      while(l<ql)
         del(l++);
      while(l>ql)
         add(--l);
      while(r<qr)
         add(++r);
      while(r>qr)
         del(r--);

      ans[q[i].id]= now;
   }

   for(int i=1;i<=m;i++)
      cout<<ans[i]<<'\n';
   return 0;
}
```

4.15 点分治

4.16 CDQ 分治

4.17 LCT 动态树

4.18 哈希

```
//h1 与 h2 分别储存两次的哈希值
//set 用来判重
//模数与底数选取质数
//hash[l,r]=hash[r]-hash[l-1]*(base^(r-l+1))

const ll mod1 = 1297151213;
const ll mod2 = 1126300213;
const ll base = 233251;
ll h1[N];
ll h2[N];
ll bs1[N];
ll bs2[N];
void get(const string& s)
{
   int n = s.size();
   bs1[0] = bs2[0] = 1;
   for (int i = 1;i <= n;i++) {
      bs1[i] = bs1[i - 1] * base % mod1;
      bs2[i] = bs2[i - 1] * base % mod2;
   }
   for (int j = 0; j < n; j++) {
      h1[j + 1] = (h1[j] * base + s[j]) % mod1;
      h2[j + 1] = (h2[j] * base + s[j]) % mod2;
   }
}

ll ask(int l, int r)
{
   pair<ll, ll>ans;
   ans.first = (h1[r] - h1[l - 1] * bs1[r - l + 1] %
       mod1 + mod1) % mod1;
   ans.second = (h2[r] - h2[l - 1] * bs2[r - l + 1] %
        mod2 + mod2) % mod2;
   return (ans.first * mod2 + ans.second);
}

__gnu_pbds::gp_hash_table<ll, int> mp;
```

4.19 KMP 模式匹配

```
//f[i]表示B中以i结尾的子串与A的前缀能够匹配的最长长度
//Next[i]表示A中以i结尾的非前缀子串与A的前缀能够匹配的最
    长长度
//下标从1开始
//[f[i]==n] 时即为B在A中第一次出现
const int N=100005;
string a,b;
int Next[N],f[N];
void pre()
{
   Next[1]=0;
   for(int i=2,j=0;i<=n;i++){
      while(j>0&&a[i]!=a[j+1])
         j=Next[j];
      if(a[i]==a[j+1])
         j++;
      Next[i]=j;
   }

   for(int i=1,j=0;i<=m;i++){
      while(j>0&&(j==n||b[i]!=a[j+1]))
         j=Next[j];
      if(b[i]==a[j+1])
         j++;
      f[i]=j;
   }
}
```

4.20 扩展 KMP 算法

4.21 manacher 算法

```
//N的大小为字符串二倍
//p[i]中存的是以i为中心字符的回文串半径(中心字符不算)
//p[i]-1即为回文串长度
const int N=20000005;
int p[N];
void manacher(string& s)
{
```

```
    int l=0,r=0;
    int n=s.size();
    string use="|";
    for(int i=0;i<n;i++)
        use+="#",use+=s[i];
    use+="#^";

    for(int i=1;i<use.size();i++){
        if(i<=r)
            p[i]=min(p[l+r-i],r-i+1);
        while(use[i+p[i]]==use[i-p[i]])
            p[i]++;
        if(p[i]+i-1>r)
            l=i-p[i]+1,r=i+p[i]-1;
    }

    return;
}
```

4.22 Trie 树

```
//以字符串出现次数为例
int tr[100005][30];
int cnt[100005];
int tot;
void insert(const string& s)
{
    int p=0;
    for(const auto& c:s){
        if(!tr[p][c-'a'])
            tr[p][c-'a']=++tot,p=tot;
        else
            p=tr[p][c-'a'];
    }

    cnt[p]++;
}

int ask(const string& s)
{
    int p=0;
    for(const auto& c:s){
        if(!tr[p][c-'a'])
            return 0;
        else
            p=tr[p][c-'a'];
    }

    return cnt[p];
}
```

4.23 可持久化 Trie 树

```
```

4.24 后缀数组

定义编号为 i 的后缀与编号为 j 的后缀的最长前缀长度为 $LCP(i,j)$
$LCP(i,j) = min_{i+1\leqslant p\leqslant j} height[p]$

字符串中不同子串的数目为每一个后缀的长度减去其 height 之和

判断子串：
跑出 sa，然后从最小的后缀开始，一个个往后枚举，记录下当前匹配到的位置，如果匹配不上就下一个后缀，否则位置向后移一位。如果枚举完了后缀还没有完全匹配则不是原串子串。

两串的最长公共子串：将两串拼接，求出 sa 和 height。枚举 sa，对于每个串找到其后第一个“起点在后一个串上的后缀”，求出 LCP 后取最大

```
const int N = 1000010;

int n, m;
//rk数组存放编号为i的后缀的排名
//sa数组存放排名为i的后缀的编号
//height数组存放排名为i的后缀与排名为i-1的后缀的最长相同
    前缀长度
int sa[N], x[N], y[N], c[N], rk[N], height[N];

void get_sa(const string& s)
{
    for (int i = 1; i <= n; i ++ )
        c[x[i] = s[i]] ++ ;
    for (int i = 2; i <= m; i ++ )
        c[i] += c[i - 1];
    for (int i = n; i; i -- )
        sa[c[x[i]] -- ] = i;

    for (int k = 1; k <= n; k <<= 1){
        int num = 0;
        for (int i = n - k + 1; i <= n; ++i )
            y[ ++ num] = i;
        for (int i = 1; i <= n; ++i )
            if (sa[i] > k)
                y[ ++ num] = sa[i] - k;
        for (int i = 1; i <= m; ++i )
            c[i] = 0;
        for (int i = 1; i <= n; ++i )
            ++c[x[i]] ;
        for (int i = 2; i <= m; ++i )
            c[i] += c[i - 1];
        for (int i = n; i; --i ){
            sa[c[x[y[i]]] -- ] = y[i];
            y[i] = 0;
        }

        swap(x, y);
        x[sa[1]] = 1;
        num = 1;
        for (int i = 2; i <= n; ++i )
            x[sa[i]] = (y[sa[i]] == y[sa[i - 1]] && y[
                sa[i] + k] == y[sa[i - 1] + k]) ? num :
                 ++ num;
        if (num == n)
            break;
        m = num;
    }

    return;
}

void get_height(const string& s)
{
    for (int i = 1; i <= n; i ++ )
        rk[sa[i]] = i;
    for (int i = 1, k = 0; i <= n; i ++ ){
        if (rk[i] == 1)
```

```
            continue;

        if (k)
            k -- ;
        int j = sa[rk[i] - 1];
        while (i + k <= n && j + k <= n && s[i + k] ==
             s[j + k])
            k ++ ;

        height[rk[i]] = k;
    }
    return;
}

int main()
{
    string s;
    cin>>s;
    n = s.size();

    //m为字符元素的最大值
    m = 122;

    s.insert(s.begin(),'&');
    get_sa(s);
    get_height(s);
}
```

4.25 后缀自动机

```
//注意修改 insert 中的标识字符
//tot 和 last 的初始值为 1
//np 代表的是前缀所在的等价类
//若计算每个节点处出现的次数，则将取消注释处的注释后dfs
const int N = 100005;
struct node {
    int ch[26];
    int len, fa;
}sam[N << 1];
int tot = 1, last = 1;
//int f[N << 1];
void insert(char cc)
{
    int c = cc - 'A';
    int p = last;
    int np = last = ++tot;
//  f[np] = 1;
    sam[np].len = sam[p].len + 1;
    memset(sam[np].ch, 0, sizeof sam[np].ch);
    for (;p && !sam[p].ch[c];p = sam[p].fa)
        sam[p].ch[c] = np;
    if (!p)
        sam[np].fa = 1;
    else {
        int q = sam[p].ch[c];
        if (sam[q].len == sam[p].len + 1)
            sam[np].fa = q;
        else {
            int nq = ++tot;
            sam[nq] = sam[q];
            sam[nq].len = sam[p].len + 1;
            sam[q].fa = sam[np].fa = nq;
            for (;p && sam[p].ch[c] == q;p = sam[p].fa)
                sam[p].ch[c] = nq;
            }
        }
    }
}
void solve()
{
    int n, m;
    cin >> n >> m;
    string s;
    cin >> s;
    memset(sam[1].ch, 0, sizeof sam[1].ch);
    tot = last = 1;
    for (const auto& c : s)
        insert(c);

}
```

4.26 回文自动机

4.27 lyndon 分解

4.28 笛卡尔树

4.29 Dance Links 精确覆盖

```
//选法作为行，限制作为列
//精确覆盖是指从中选取一些行，使得每一列有且仅有一个1
//注意，只能解决限制为1的问题
//与网络流较为类似，关键在于如何构建矩阵
//ans中存选哪些行
const int N = 5510;

int n, m;
int l[N], r[N], u[N], d[N], s[N], row[N], col[N], idx
    ;
int ans[N], top;

void init()
{
    for (int i = 0; i <= m; i ++ )
    {
        l[i] = i - 1, r[i] = i + 1;
        u[i] = d[i] = i;
    }
    l[0] = m, r[m] = 0;
    idx = m + 1;
}

void add(int& hh, int& tt, int x, int y)
{
    row[idx] = x, col[idx] = y, s[y] ++ ;
    u[idx] = y, d[idx] = d[y], u[d[y]] = idx, d[y] =
        idx;
    r[hh] = l[tt] = idx, r[idx] = tt, l[idx] = hh;
    tt = idx ++ ;
}

```

```
void remove(int p)
{
    r[l[p]] = r[p], l[r[p]] = l[p];
    for (int i = d[p]; i != p; i = d[i])
        for (int j = r[i]; j != i; j = r[j]){
            s[col[j]] -- ;
            u[d[j]] = u[j], d[u[j]] = d[j];
        }
}

void resume(int p)
{
    for (int i = u[p]; i != p; i = u[i])
        for (int j = l[i]; j != i; j = l[j]){
            u[d[j]] = j, d[u[j]] = j;
            s[col[j]] ++ ;
        }
    r[l[p]] = p, l[r[p]] = p;
}

bool dfs()
{
    if (!r[0])
        return true;
    int p = r[0];
    for (int i = r[0]; i; i = r[i])
        if (s[i] < s[p])
            p = i;
    remove(p);
    for (int i = d[p]; i != p; i = d[i]){
        ans[ ++ top] = row[i];
        for (int j = r[i]; j != i; j = r[j])
            remove(col[j]);

        if (dfs())
            return true;
        for (int j = l[i]; j != i; j = l[j])
            resume(col[j]);
        top -- ;
    }
    resume(p);
    return false;
}

void solve()
{
    cin >> n >> m;
    init();
    for (int i = 1; i <= n;i++){

        //每次插入新行时都需要执行
        int hh = idx, tt = idx;
        for (int j = 1; j <= m;j++){
            int x;
            cin >> x;

            //只有1需要插入
            if(x)
                add(hh, tt, i, j);
        }
    }

    if(dfs()){
        for (int i = 1; i <= top;i++)
            cout << ans[i] << " \n"[i == top];
    }
    else
        cout << "No Solution!\n";

    return ;
}
```

4.30 Dance Links 重复覆盖

```
//重复覆盖解决的是选出行的数量最小问题，并且可以重复覆盖
//需保证答案较小，因为基于IDA*算法
const int N = 10010;

int n, m;
int l[N], r[N], u[N], d[N], col[N], row[N], s[N], idx
    ;
int ans[N];
bool st[110];

void init()
{
    for (int i = 0; i <= m; i ++ ){
        l[i] = i - 1, r[i] = i + 1;
        col[i] = u[i] = d[i] = i;
        s[i] = 0;
    }
    l[0] = m, r[m] = 0;
    idx = m + 1;
}

void add(int& hh, int& tt, int x, int y)
{
    row[idx] = x, col[idx] = y, s[y] ++ ;
    u[idx] = y, d[idx] = d[y], u[d[y]] = idx, d[y] =
        idx;
    r[hh] = l[tt] = idx, r[idx] = tt, l[idx] = hh;
    tt = idx ++ ;
}

int h()
{
    int cnt = 0;
    memset(st, 0, sizeof st);
    for (int i = r[0]; i; i = r[i]){
        if (st[col[i]])
            continue;
        cnt ++ ;
        st[col[i]] = true;
        for (int j = d[i]; j != i; j = d[j])
            for (int k = r[j]; k != j; k = r[k])
                st[col[k]] = true;
    }
    return cnt;
}

void remove(int p)
{
    for (int i = d[p]; i != p; i = d[i]){
        r[l[i]] = r[i];
        l[r[i]] = l[i];
    }
}

void resume(int p)
```

```
{
    for (int i = u[p]; i != p; i = u[i]){
        r[l[i]] = i;
        l[r[i]] = i;
    }
}

bool dfs(int k, int depth)
{
    if (k + h() > depth)
        return false;
    if (!r[0])
        return true;
    int p = r[0];
    for (int i = r[0]; i; i = r[i])
        if (s[p] > s[i])
            p = i;

    for (int i = d[p]; i != p; i = d[i]){
        ans[k] = row[i];
        remove(i);
        for (int j = r[i]; j != i; j = r[j])
            remove(j);
        if (dfs(k + 1, depth))
            return true;
        for (int j = l[i]; j != i; j = l[j])
            resume(j);
        resume(i);
    }
    return false;
}

void solve()
{
    cin >> n >> m;
    init();
    for (int i = 1; i <= n; i ++ ){
        int hh = idx, tt = idx;
        for (int j = 1; j <= m; j ++ ){
            int x;
            cin >> x;
            if (x)
                add(hh, tt, i, j);
        }
    }

    int depth = 0;
    while (!dfs(0, depth))
        depth ++ ;
    cout << depth << '\n';
    for (int i = 0; i < depth; i ++ )
        cout << ans[i] << '\n';
    return;
}
```

5 动态规划

5.1 0/1 背包

```
const int N=105;
const int M=100005;
int f[M];
int v[N],w[N];
int main()
{
    memset(f,0,sizeof f);
    //memset(f,0xcf,sizeof f);
    f[0]=0;
    for(int i=1;i<=n;i++)
        for(int j=m;j>=v[i];j--)
            f[j]=max(f[j],f[j-v[i]]+w[i]);
}
```

5.2 完全背包

```
const int N=105;
const int M=100005;
int f[M];
int v[N],w[N];
int main()
{
    memset(f,0,sizeof f);
    //memset(f,0xcf,sizeof f);
    f[0]=0;
    for(int i=1;i<=n;i++)
        for(int j=v[i];j<=m;j++)
            f[j]=max(f[j],f[j-v[i]]+w[i]);
}
```

5.3 多重背包

```
```

5.4 分组背包

```
//时间复杂度: O(n*m)
//objs表示物品组集合
for (const auto &obj : objs)
    for (int j = m; ~j; --j)
        for (const auto &[w, val] : obj)
            if (w <= j)
                dp[j] = max(dp[j], dp[j - w] + val);
```

5.5 回退背包

```
//适用于计算方案数问题
//退去第 x 个物品后满足总价值为 i 的方案数
//01背包
for(int i = w[x];i <= m;++i)
    dp[i] -= dp[i - w[x]];

//多重背包
for(int i = m;i >= w[x];--i)
    dp[i] -= dp[i - w[x]];
```

5.6 高维前缀和 SOSDP

```
//相当于每一位是 0/1 的 bit 维前缀和
for (int i= 0; i < (1 << bit); ++i)
    f[i] = a[i];
for (int i= 0; i < bit; ++i){
    for (int mask= 0; mask < (1 << bit); ++mask) {
```

```
        if (mask & (1 << i))
            f[mask] += f[mask ^ (1 << i)];
    }
}
```

5.7 状压 DP

```
//取出x的第i位
y=(x>>(i-1))&1;
//将x第i位取反
x^=1<<(i-1);
//将x第i位变为1
x|=1<<(i-1);
//将x第i位变为0
x&= ~(1<<(i-1));
//将x最靠右的1变成0
x=x&(x-1);
//取出x最靠右的1
y=x&(~x);
//把最靠右的0变成1
x|=(x+1);
//判断是否有两个连续的1, n个连续的1与之类似
if(x&(x<<1))
    cout<<"YES\n";
//枚举子集
for(int i=sta;i;i=((i-1)&sta)){
    //i即为子集
}
```

5.8 四边形不等式优化 DP

5.9 斜率优化 DP

6 博弈

6.1 Nim 游戏

6.2 反 Nim 游戏

6.3 威佐夫博弈

威佐夫博弈：有两堆各若干物品，两个人轮流从任意一堆中至少取出一个或者从两堆中取出同样多的物品，规定每次至少取一个，至多不限，最后取光者胜。
结论：若较小堆石子数为两堆差值的 1.618 倍下取整，则先手必败
注意：我们一般使用 $\frac{\sqrt{5}+1}{2}$ 来代替 1.618

6.4 SG 函数

能到达的局面 SG 值为其后继状态 SG 值的最小未出现的非负整数

7 杂项算法

7.1 离散化

```
vector<int> nums;
sort(nums.begin(),nums.end());
nums.erase(unique(nums.begin(),nums.end()),nums.end()
    );

int get(int x)
{
    return lower_bound(nums.begin(),nums.end(),x)-nums
        .begin();
}
```

7.2 二分

```
//>= x的数中最小的一个
while(l<r){
    int mid=(l+r)>>1;
    if(a[mid]>=x)
        r=mid;
    else
        l=mid+1;
}

//<= x的数中最大的一个
while(l<r){
    int mid=(l+r+1)>>1;
    if(a[mid]<=x)
        l=mid;
    else
        r=mid-1;
}

//结果储存在l中
```

7.3 三分

```
//整数三分
int l = 1,r = 100;
while(l < r) {
    int lmid = l + (r - l) / 3;
    int rmid = r - (r - l) / 3;
    lans = f(lmid),rans = f(rmid);
    // 求凹函数的极小值
    if(lans <= rans)
        r = rmid - 1;
    else
        l = lmid + 1;
    // 求凸函数的极大值
    if(lans <= rans)
        l = lmid + 1;
    else
        r = rmid - 1;
}
// 求凹函数的极小值
cout << min(lans,rans) << endl;
// 求凸函数的极大值
cout << max(lans,rans) << endl;


```

```
//浮点三分
const double EPS = 1e-9;
while(r - l > EPS) {
    double lmid = l + (r - l) / 3;
    double rmid = r - (r - l) / 3;
    lans = f(lmid),rans = f(rmid);
    // 求凹函数的极小值
    if(lans <= rans)
        r = rmid;
    else
        l = lmid;
    // 求凸函数的极大值
    if(lans <= rans)
        l = lmid;
    else
        r = rmid;
}
// 输出 l 或 r 都可
cout << l << endl;
```

7.4 倍增

7.5 ST 表

```
//解决可重复问题
const int N=100005;
int f[20][N];
void pre(int n)
{
    for (int i = 1;i <= n;++i)
        f[0][i] = a[i];

    int t = log2(n) + 1;
    for (int i = 1;i < t;++i)
        for (int j = 1;j <= n - (1 << i) + 1;++j)
            f[i][j] = min(f[i - 1][j], f[i - 1][j + (1
                << (i - 1))]);
}

int ask(int l,int r)
{
    int k = log2(r - l + 1);
    return min(f[k][l], f[k][r - (1 << k) + 1]);
}
```

7.6 启发式合并

每次合并均将小集合合并至大集合中
时间复杂度 O(nlogn)

7.7 dsu on tree

```
const int N=100005;
const int M=200005;
/*---建树操作省略---*/
int sz[N],son[N];
ll sum,cnt[N];
int mx=0;
bool v[N];
void dfs_son(int x)
{
    v[x]=1;
    sz[x]=1;
    for(int i=head[x];i;i=Next[i]){
        int y=ver[i];
        if(v[y])
            continue;
        dfs_son(y);
        sz[x]+=sz[y];
        if(sz[y]>sz[son[x]])
            son[x]=y;
    }

    return;
}

ll ans[N];
void update(int x,int father,int flag,int pson)
{
    /*维护并统计答案
    此处以出现次数最多元素编号之和为例
    int color=c[x];
    cnt[color]+=flag;
    if(cnt[color]>mx)
        mx=cnt[color],sum=color;
    else if(cnt[color]==mx)
        sum+=color;
    */
    for(int i=head[x];i;i=Next[i]){
        int y=ver[i];
        if(y==father||y==pson)
            continue;
        update(y,x,flag,pson);
    }

    return;
}

void dfs(int x,int father,int op)
{
    for(int i=head[x];i;i=Next[i]){
        int y=ver[i];
        if(y==father||y==son[x])
            continue;

        dfs(y,x,0);
    }

    if(son[x])
        dfs(son[x],x,1);
    update(x,father,1,son[x]);

    ans[x]=sum;
    if(!op)
        update(x,father,-1,0),sum=mx=0;

    return;

}
int main()
{
    //主要操作过程在dfs()中实现
    //dfs_son()仅为预处理
```

```
    dfs_son(1);
    dfs(1,0,1);
}
```

7.8 欧拉降幂

$$a^n \equiv \begin{cases} a^{n \mod \varphi(m)}, & \text{if } \gcd(a,m)=1 \\ a^{\min(n, n \mod \varphi(m)+\varphi(m))}, & \text{if } \gcd(a,m)\neq 1 \end{cases}$$

7.9 切比雪夫距离与曼哈顿距离转化

切比雪夫距离：$max(|x_1 - x_2|, |y_1 - y_2|)$
切比雪夫距离转换为曼哈顿距离：坐标变换为 $(x+y, x-y)$ 反之：坐标变换为 $(\frac{x+y}{2}, \frac{x-y}{2})$

7.10 高精度加法

```
string add(string a, string b)
{
    reverse(range(a));
    reverse(range(b));

    int del = 0;
    int n = max(a.size(), b.size());
    string ans;
    for (int i = 0; i < n; ++i) {
        int sum = del;
        if (i < a.size())
            sum += a[i] - '0';
        if (i < b.size())
            sum += b[i] - '0';
        ans.push_back(sum % 10 + '0');
        del = sum / 10;
    }
    if (del)
        ans.push_back(del + '0');
    reverse(range(ans));
    return ans;
}
```

7.11 高精度减法

```
string minu(string a, string b)
{
    bool flag = false;
    if (b.size() > a.size() || (a.size() == b.size()
        && b > a)) {
        swap(a, b);
        flag = true;
    }
    reverse(range(a));
    reverse(range(b));
    int del = 0;
    int n = a.size();
    string ans;
    for (int i = 0; i < n; ++i) {
        int now = a[i] - del - (i < b.size() ? b[i] :
            '0');
        del = now < 0;
        now += del * 10;
        ans.push_back(now + '0');
    }
    while (ans.size() > 1 && ans.back() == '0')
        ans.pop_back();
    reverse(range(ans));
    if (flag)
        ans = "-" + ans;
    return ans;
}
```

7.12 卡常指令

```
#pragma GCC optimize("Ofast")
#pragma GCC target("sse3","sse2","sse")
#pragma GCC target("avx","sse4","sse4.1","sse4.2","
    ssse3")
#pragma GCC target("f16c")
#pragma GCC optimize("inline","fast-math","unroll-
    loops","no-stack-protector")
```

7.13 数论公式总结

$$[n=1] = \sum_{d|n} \mu(d)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} f(i) = \sum_{i=1}^{n}\sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} f(j)$$

$$gcd(i,j) = \sum_{d|i \wedge d|j} \phi(d)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}[gcd(i,j)=x] = 2 \times pre\phi(\lfloor \frac{n}{x} \rfloor) - 1$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}[gcd(i,j)=x] = \sum_{d=1}^{\lfloor \frac{n}{x} \rfloor}\left(\mu(d)\lfloor \frac{n}{dx} \rfloor \lfloor \frac{m}{dx} \rfloor\right)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} gcd(i,j) = \sum_{d=1}^{n} \phi(d) \lfloor \frac{n}{d} \rfloor^2$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} gcd(i,j) = \sum_{d=1}^{n} \phi(d) \lfloor \frac{n}{d} \rfloor \lfloor \frac{m}{d} \rfloor$$

$$gcd(x^a - y^a, x^b - y^b) = x^{gcd(a,b)} - y^{gcd(a,b)}$$

$$\sum_{i=1}^{n} i[gcd(i,n) = 1] = \frac{[n=1]+n\phi(n)}{2}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{i} j[gcd(i,j)=1] = \frac{1}{2}\sum_{i=1}^{n} i\phi(i) + \frac{[n \geq 1]}{2}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}[lcm(i,j)=n] = \prod_{i=1}(2c_i+1)[n = p_1^{c_1}p_2^{c_2}\cdots]$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} d(i \times j) = \sum_{d=1}^{n} \mu(d) \times pred(\lfloor \frac{n}{d} \rfloor) \times pred(\lfloor \frac{m}{d} \rfloor)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{\lfloor\frac{m}{i}\rfloor} ij\mu(i) = 1$$

$$\sum_{d|n}\frac{\mu(d)}{d} = \frac{\phi(n)}{n}$$

$$\mu(n) = \sum_{d|n}\mu(d)\times d\times\phi(\frac{n}{d})$$

$$\sum_{d|n}\phi(d) = n$$

$$\sum_{i=1}^{n}[\sigma(i)\%2 = 1] = \sqrt{n} + \sqrt{\frac{n}{2}}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma(gcd(i,j)) = \sum_{i=1}^{n}\sigma(x)\sum_{d=1}^{\lfloor\frac{n}{x}\rfloor}\sum_{d=1}^{\lfloor\frac{n}{x}\rfloor}\left(\mu(d)\lfloor\frac{n}{dx}\rfloor\lfloor\frac{m}{dx}\rfloor\right)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma(gcd(i,j))[\sigma\le a] = \sum_{k=1}^{n}\lfloor\frac{n}{k}\rfloor\lfloor\frac{m}{k}\rfloor g(k)$$

$$g(k) = \textstyle\sum_{d|k}\sigma(d)\mu(\lfloor\frac{k}{d}\rfloor)[\sigma(d)\le a]$$

$$f(n) = \sum_{i=1}^{\sqrt{n}}\mu(i)\lfloor\frac{n}{i^2}\rfloor$$

$$\sum_{i=1}^{n}\frac{1}{i} = ln(n) + r + \frac{1}{2n}$$

$$r = 0.57721566490153286060651209$$

其中 σ 代表因数和，$f(n)$ 代表 $[1,n]$ 范围内所有数不含平方因子的数的数量，ϕ 代表欧拉函数，pre 代表前缀和