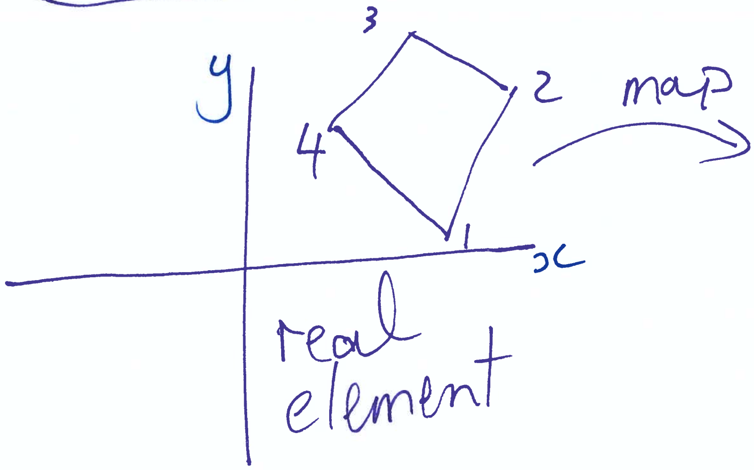
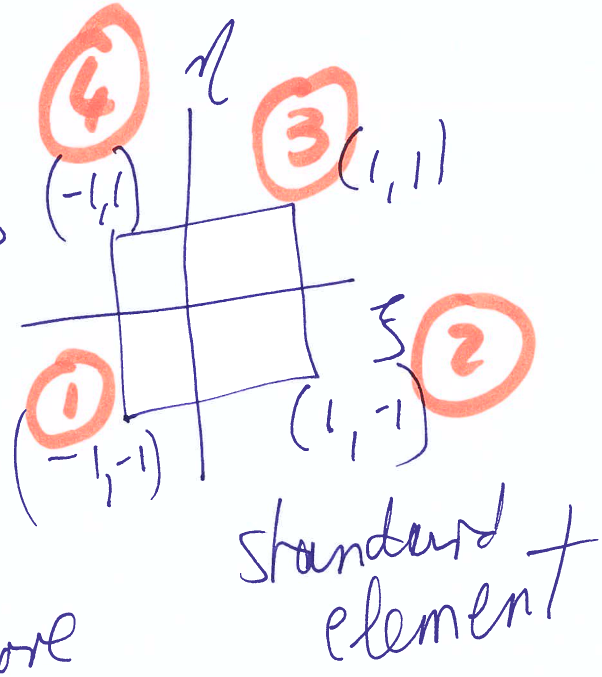
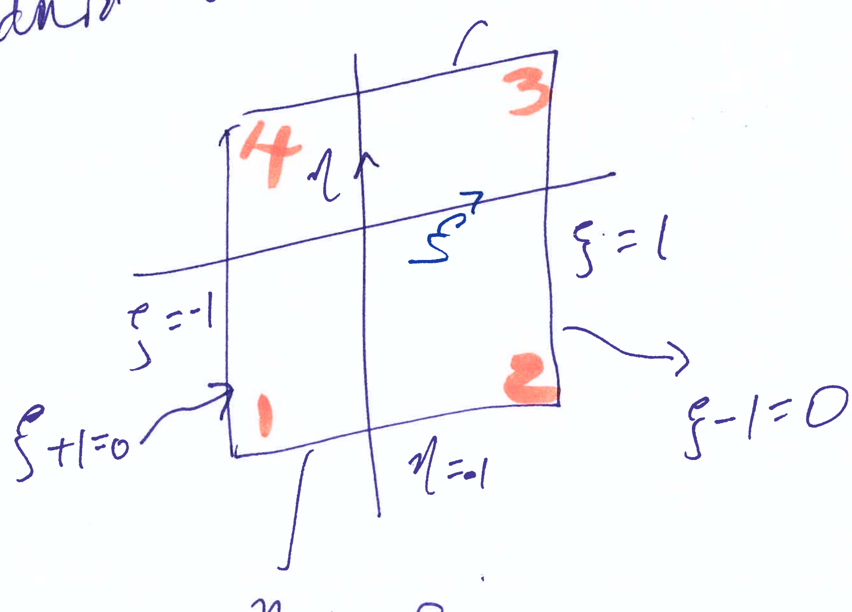
Quad elements





Element is mapped to a square of area '4' ($-1 \le x \le +1$, $-1 \le y \le 1$)

→ ccw numbering, numbering is ~~standard~~ (unchanged) for standard element.



Equations of those 4 lines

Shape function for node 1 is the product of the equations of lines passing through nodes 2, 3, 4 (and normalized) such that $N_1 = 1$ @ node 1

$$N_1 = \frac{(\eta - 1)(\xi - 1)}{4}$$

$N_1 = 1$ @ node 1

@ $\xi = -1$

$\eta = 1$

Similarly, — Lines passing through, 1, 3, 4

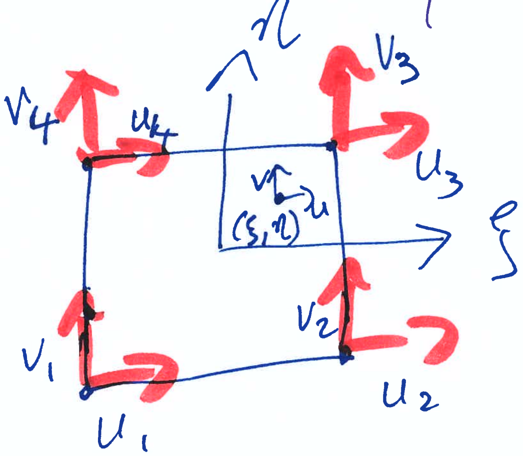
$$N_2 = \frac{(\xi + 1)(\eta - 1)}{-4}$$

$= 1$

@ $\xi = 1$

$\eta = -1$

$$N_3 = \frac{(\xi + 1)(\eta + 1)}{4} \qquad N_4 = \frac{(\xi - 1)(\eta + 1)}{-4}$$



$$q = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

@ (ξ, η)

$$\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}}_{\textbf{N matrix}} \overbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}}^{q}$$

a function of ξ, η

Quad elements

B matrix

$$\varepsilon = B q$$

$$\underset{3\times 8}{B} = \underset{3\times 2}{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}} \underset{2\times 8}{N}$$

$$B = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & \cdots & 0 \\ 0 & N_1 & 0 & N_2 \cdots & N_4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

Principle of virtual work

$$\int_{V^e} \tilde{\varepsilon}^T \sigma \, dV = \int_{V^e} w^T f \, dV + \int_S w^T t \, dS$$

$\tilde{\varepsilon} = B\tilde{q}$ $\qquad u = Nq$

$\sigma = DBq$ $\qquad \varepsilon = Bq$

$w = N\tilde{q}$

$$\cancel{\tilde{q}^T}\underbrace{\left(\int_{V^e} B^T D B \, dV\right)}_{\text{local stiffness}} q = \underbrace{\cancel{\tilde{q}^T}\int_{V^e} N^T f \, dV + \cancel{\tilde{q}^T}\int_{S^e} N^T t \, dS}_{\text{local force}}$$

$$K = \int_{V^e} B^T D B \, dV = \int_{-1}^{1}\int_{-1}^{1} \overbrace{B^T D B \, t^e \det J}^{f(\xi,\eta)} \, d\xi \, d\eta$$

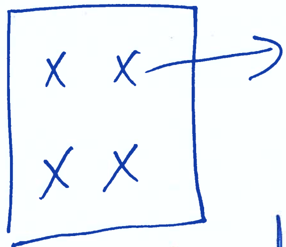
↑ thickness

$$f(\xi,\eta) = B^T D B \, t^e \det J \qquad 8\times 8$$

$$= \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi \, d\eta$$

2point integration

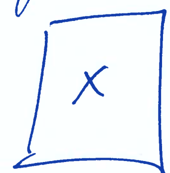
$$= f\left(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right) + f\left(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)$$



integration points

standard element

1 point integration $= 4f(0,0)$



Body force term

$$f^{b.f.} = \int_{V^e} N^T f \, dV = \int_{-1}^{1}\int_{-1}^{1} \boxed{N^T \begin{pmatrix} f_x \\ f_y \end{pmatrix} t^e \det J} \, d\xi \, d\eta$$

$g(\xi, \eta)$

assume constant

linear in ξ & η

$= 4\, g(0,0)$ for 1 point integration

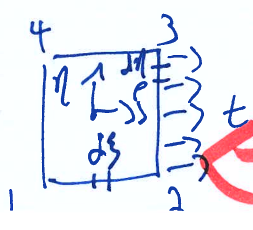
$= g\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + g\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

for 2 point integration

Traction term

if traction is on 2-3 or 1-4

$$f^{tr} = \int_{S^e} N^T t \, ds = \int_{-1}^{1} N^T \begin{pmatrix} t_x \\ t_y \end{pmatrix} t^e \det J^* \, d\eta$$

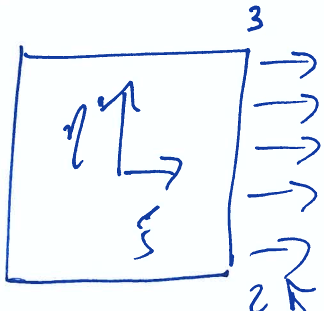


$\xi = -1$ for this surface

$$f^{tr} = \int_{\xi=-1}^{1} N^T \begin{pmatrix} t_x \\ t_y \end{pmatrix} t^e \det J^* \, d\xi$$

if traction is on 1-2 or 4-3 surface

N^T is a function of ξ & η but one of those is known on the surface

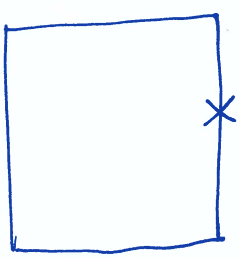


η goes from -1 to 1

Plug in $\xi = 1$ in N^T for 2-3

$\xi = 1$

Using integration points



$\times \rightarrow \xi = 1$

$\eta = 0$

$$f^{tr} = \int_{-1}^{1} \underbrace{N^T \begin{pmatrix} t_x \\ t_y \end{pmatrix} t^e \det J^*}_{h(\xi,\eta)} \, d\eta = \int_{-1}^{1} h(\xi,\eta) \, d\eta$$

$$= 2\,h(1,0)$$

weight

$\xi = 1$ on 2-3

integration point

We still need to find B, det J, det J*

You have to write a code that takes in the integration point (ξ, η) and element coordinates and RETURNS N matrix, 'dNdx' matrix, J matrix, det J

$$N = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

known (inputs) integration points

$$= \left[\frac{(1-\xi)(1-\eta)}{4}, \ldots \right]$$

$$dNd\xi = \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix}}_{\text{input for } J} = \begin{bmatrix} -\frac{(1-\eta)}{4} & \ldots \\ -\frac{(1-\xi)}{4} & \ldots \end{bmatrix}$$

ξ, η are known

$$J = dNd\xi \times \underbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}}_{\text{coord. matrix (input)}}$$

2x2 matrix

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$$dNdx = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix} ?$$

2x4 matrix

$$= \underbrace{\begin{bmatrix} J^{-1} \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} dNd\xi \end{bmatrix}}_{2\times 4}$$

Returns N, $dNdx$, J, $\det J$

Matlab code

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$ real element

function $[N, dNdx, J, \det J] = \text{element}(\xi, \eta, \text{coord})$

$$N = \begin{bmatrix} \frac{(1-\xi)(1-\eta)}{4} & \cdots \end{bmatrix}$$

$$dNd\xi = \begin{bmatrix} \frac{-(1-\eta)}{4} & \cdots \\ \frac{-(1-\xi)}{4} & \cdots \end{bmatrix}$$

$J = dNd\xi \times coord$

$dNdx = inv(J) \times dNd\xi$

$$\longrightarrow = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} \end{bmatrix}$$

How to find the local stiffness matrix for element e
for $(\xi, \eta) \Rightarrow (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

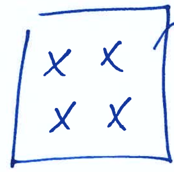
- Call this function that returns
$dNdx$, $\det J$ at integration point (ξ, η)

$$B = \begin{bmatrix} dNdx(1,1) & 0 \\ 0 & dNdx(2,1) & \cdots \\ dNdx(2,1) & dNdx(1,1) \end{bmatrix} \longleftarrow \text{type}$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \cdots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} \end{bmatrix}$$

local stiffness = local stiffness + $B^T D B \det J \, t^e$

end



loop over the 4 integration points