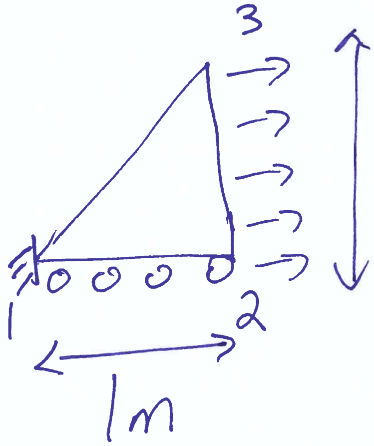
Example



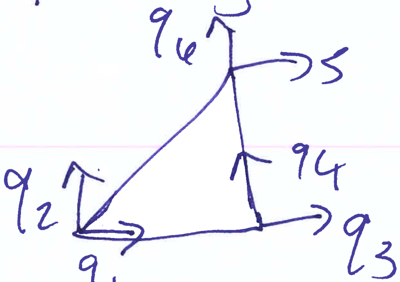
Find displacements & stresses

$E = 70\,GPa$

$\nu = 0$

Plane stress problem

Triangle element



$$q = \begin{bmatrix} \cancel{q_1} \\ \cancel{q_2} \\ q_3 \\ \cancel{q_4} \\ q_5 \\ q_6 \end{bmatrix}$$

$\rightarrow$ 3 DOF problem

$$D = 70e^{9} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Use coordinates to find B matrix

$$B = \frac{1}{x_{13} y_{23} - x_{23} y_{13}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$q_1 = 0 \qquad q_2 = 0 \qquad q_4 = 0$

Need coordinates, so

	x	y
node1	0	0
node2	1	0
node3	1	1

Cancel in B matrix (instead of K matrix)

- can do this ahead of time to reduce calculation

Reduce B matrix to 3 x 3

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

(deleted earlier)

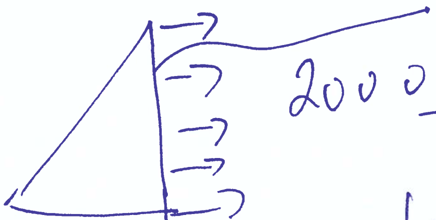
- do this only for hand calculation, use condensation in code

stiffness matrix

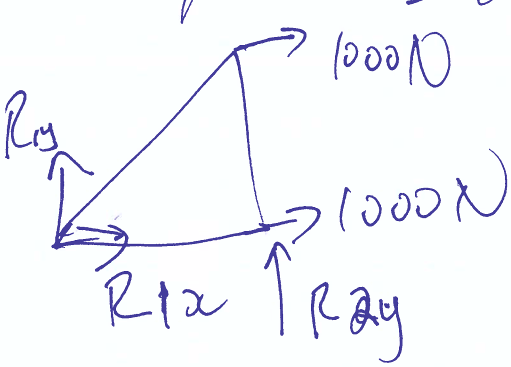
$$K = B^T D B \left(\overset{\frac{1}{2}}{A_e}\ \overset{1}{h}\right) = \frac{70e9}{4} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Force vector

area $= 1 \times 1 = 1 m^2$



Total force = traction x area
= 2000N



$$\text{local force} = \begin{bmatrix} \cancel{R_{1x}} \\ \cancel{R_{1y}} \\ 1000 \\ \cancel{R_{2y}} \\ 1000 \\ 0 \end{bmatrix}$$

Solve

$$K \begin{bmatrix} q_3 \\ q_5 \\ q_6 \end{bmatrix} = f = \begin{bmatrix} 1000 \\ 1000 \\ 0 \end{bmatrix}$$

$$q_3 = u_2 = 0.0571 \ \mu m$$

$$q_5 = u_3 = 0.1143 \ \mu m$$

$$q_6 = V_3 = 0$$

y displacement of 3rd node

Stress

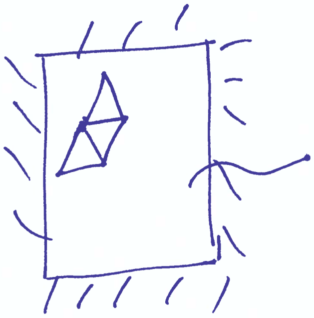
$$\underset{}{\sigma} = \underset{3\times3}{D}\ \underset{3\times6}{B}\ \underset{6\times1}{q} = 3\times1$$

$$= \begin{bmatrix} 4000 \\ 0 \\ 2000 \end{bmatrix} N/m^2$$

How to model thermal forces

$\Delta T, \alpha$ given

↳ coefficient of thermal expansion



ΔT

temp. raised.

thermal strain $\varepsilon^{Th} = \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix}$

no shear $\gamma_{xy}^{thermal}$

$$\sigma = D(\varepsilon - \varepsilon^{Th})$$

$$= D\left(\varepsilon - \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix}\right) = D\left(Bq - \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix}\right)$$

$$\int_{\Omega^e} \tilde{\varepsilon}^T \sigma \, dV = \int_{\Omega^e} \tilde{\varepsilon}^T D \left(Bq - \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix}\right) d\Omega$$

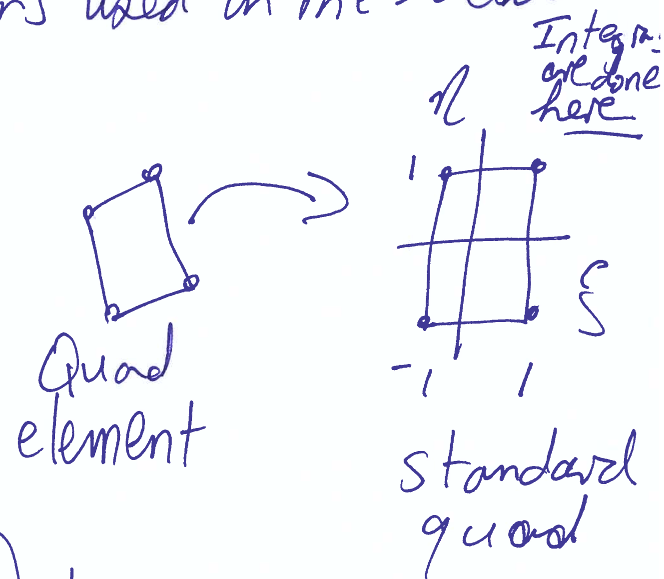
$$= \cancel{\tilde{q}^T} \left(\int_{\Omega^e} B^T D B \right) q = \cancel{\tilde{q}^T} \underbrace{\left(\int B^T D \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix} d\Omega \right)}_{f^{thermal}}$$

(this term goes to the force side of eqn)

$$f^{thermal} = +\int_{\Omega^e} B^T D \begin{bmatrix}\alpha\Delta T\\ \alpha\Delta T\\ 0\end{bmatrix} d\Omega$$

Numerical Integration

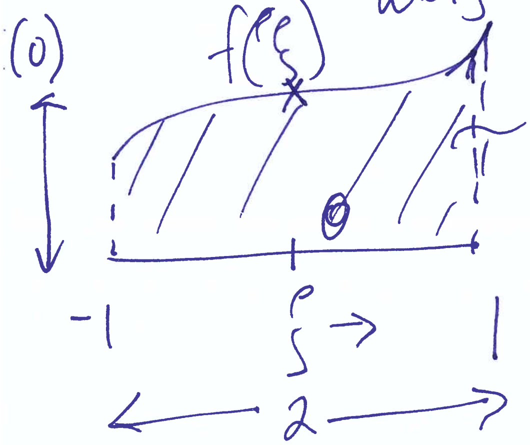
With numerical integration techniques, you can directly integrate the functions used in the local stiffnes & local force



$$\boxed{I = \int_{-1}^{1} f(\xi)\,d\xi}$$

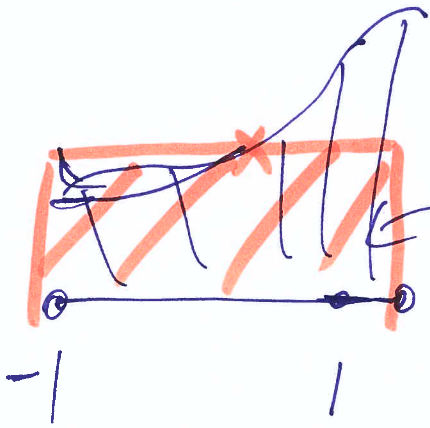
$$= w_1 f(\xi_1) + w_2 f(\xi_2) + \dots$$

weights



area under the curve between $-1 < \xi < 1$

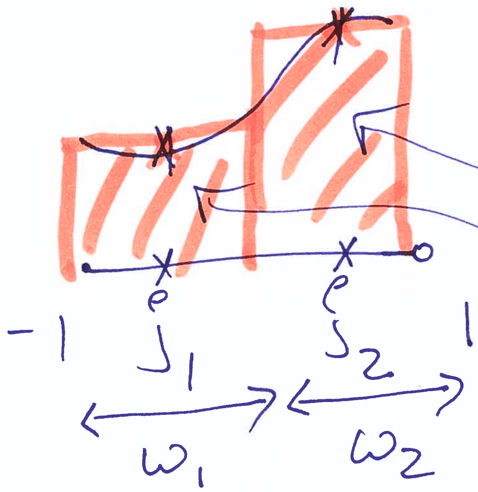
Midpoint $\xi = 0$



red shaded area $= 2 \times f(\xi = 0)$

integration point

One point integration



Two point integration

$$\int_{-1}^{1} f(\xi)\, d\xi = w_1 f(\xi_1) + w_2 f(\xi_2)$$

General rule

n points $\rightarrow$ $2n-1$ order polynomial for $f(\xi)$ exactly integrated

$n = 1$ $\rightarrow$ linear $f(\xi)$ is exactly integrated $\quad f(\xi) = a + b\xi$

$n = 2$ $\rightarrow$ cubic $f(\xi)$ is exactly integrated $\quad f(\xi) = a\xi^3 + b\xi^2 + c\xi + d$

Prove

1 point ($\xi = 0$), weight $= 2$

Proof, 1 point ($\xi = 0$), weight = 2

Try $\boxed{f(\xi) = a_0 + a_1\xi}$

$$\int_{-1}^{1} (a_0 + a_1\xi)\,d\xi = a_0\xi + \frac{a_1\xi^2}{2}\Big|_{-1}^{1}$$

$$= 2a_0$$

weight

$f(\xi = 0)$

Try $f(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$

$$\int_{-1}^{1} (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3)\,d\xi = \boxed{2a_0 + \frac{2a_2}{3}}$$

$$= w_1 f(\xi_1) + w_2 f(\xi_2)$$

$$\cancel{= w_1 f(\xi_1) + w_2 f(\xi_2)}$$

$$= w_1(a_0 + a_1\xi_1 + a_2\xi_1^2 + a_3\xi_1^3) +$$
$$w_2(a_0 + a_1\xi_2 + a_2\xi_2^2 + a_3\xi_2^3)$$

$$= (w_1 + w_2)a_0 + (w_1\xi_1 + w_2\xi_2)a_1 + (w_1\xi_1^2 + w_2\xi_2^2)a_2 + (w_1\xi_1^3 + w_2\xi_2^3)a_3$$

$$= 2a_0 + \frac{2a_2}{3}$$

$$\left.\begin{aligned} w_1 + w_2 &= 2 \\ w_1\xi_1 + w_2\xi_2 &= 0 \\ w_1\xi_1^2 + w_2\xi_2^2 &= \frac{2}{3} \\ w_1\xi_1^3 + w_2\xi_2^3 &= 0 \end{aligned}\right\}$$

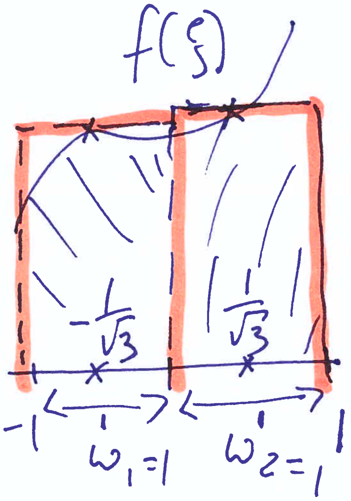
Solve these for w_1, w_2, ξ_1, ξ_2

Solution

$$w_1 = w_2 = 1$$

$$\xi_1 = -\xi_2 = \frac{1}{\sqrt{3}}$$

Gauss points



$$\int_{-1}^{1} f(\xi)\, d\xi = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

cubic

irrational

In code, Store 10 decimal places

Table 7.1 in Belegundu textbook

Interpolation point table

n	w	ξ
$n = 1$	$w_1 = 2$	$\xi_1 = 0$
$n = 2$	$w_1 = 1$, $w_2 = 1$	$\xi_1 = -\frac{1}{\sqrt{3}}$, $\xi_2 = \frac{1}{\sqrt{3}}$
$\vdots$		

Example:

$$I = \int_{-1}^{1} \left(3e^{x} + x^2 + \frac{1}{x+2}\right) dx \quad \text{exact} = 8.816$$

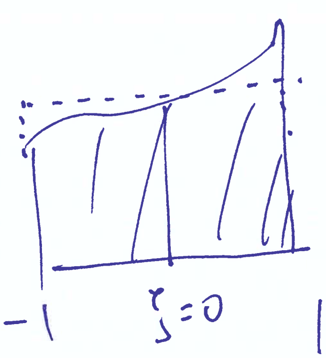
$n = 1$

$$I = \left(3e^{0} + 0^2 + \frac{1}{0+2}\right) \times 2$$

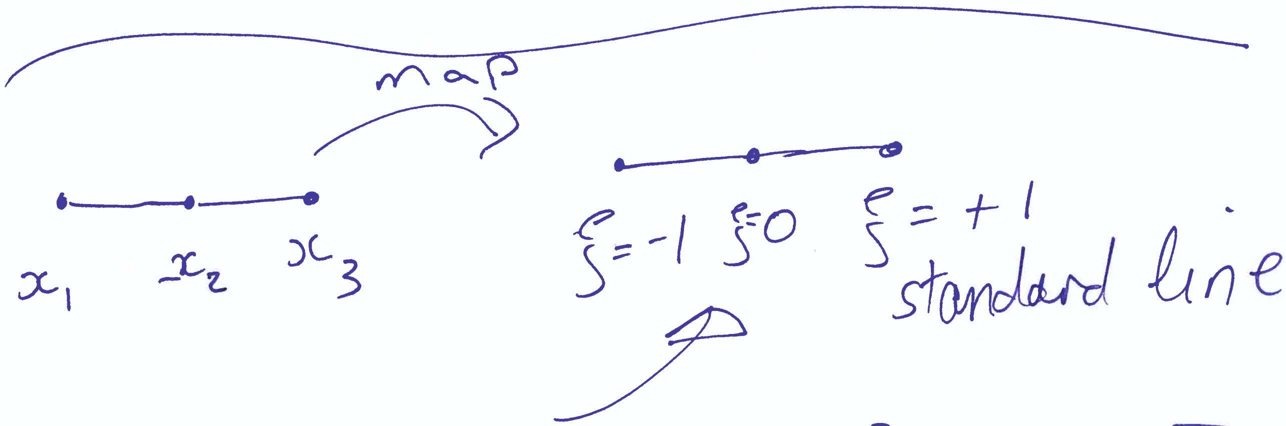
weight

@ $\xi = 0$

$= 7$



$$n = 2 \quad I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 8.786$$



$$N = \left[\frac{\xi(\xi-1)}{2},\ 1-\xi^2,\ \frac{\xi(\xi+1)}{2}\right]$$

$$\text{coord} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \frac{dN}{d\xi} = \left[\frac{2\xi-1}{2},\ -2\xi,\ \frac{2\xi+1}{2}\right]$$

$$J = \frac{dN}{d\xi}(\text{coord}) = \left[\frac{2\xi-1}{2},\ -2\xi,\ \frac{2\xi+1}{2}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \left(\frac{2\xi-1}{2}\right)x_1 - 2\xi x_2 + \frac{2\xi+1}{2}x_3$$

$$= \xi\left(\cancel{x_1 - 2x_2 + x_3}\right) + \frac{L^e}{2}$$

if mid node is at the center

where $L^e = x_3 - x_1$

If $x_2 = \frac{x_3 + x_1}{2}$

So we have

$$J = \frac{L^e}{2}$$

Recall that

$$B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi} = \frac{2}{L^e}\left[\underbrace{\frac{2\xi-1}{2}}_{\frac{dN_1}{dx}},\ \underbrace{-2\xi}_{\frac{dN_2}{dx}},\ \underbrace{\frac{2\xi+1}{2}}_{\frac{dN_3}{dx}}\right]$$

$$K = \int_{x_1}^{x_3} B^T \underbrace{D}_{E} B\, A\,dx = \int_{-1}^{1} EA\left(\frac{2}{L^e}\right)^2 \underbrace{\begin{bmatrix} (\xi-\frac{1}{2})^2 & -2\xi(\xi-\frac{1}{2}) & \xi^2-\frac{1}{4} \\ & 4\xi^2 & -2\xi(\xi+\frac{1}{2}) \\ & & (\xi+\frac{1}{2})^2 \end{bmatrix}}_{\text{quadratic in } \xi} \underbrace{\left(\frac{L^e}{2}d\xi\right)}_{\substack{\det J\, d\xi \\ = dx}}$$

Since it's a quadratic in $\xi \Rightarrow$ use 2-point integration (exact up to cubic in ξ)

$n = 2$, $\xi_1 = -\frac{1}{\sqrt{3}}$, $\xi_2 = \frac{1}{\sqrt{3}}$

weights $= 1$

$$K = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = B^T E B A \det J \Big|_{\xi = -\frac{1}{\sqrt{3}}} + B^T E B A \det J \Big|_{\xi = \frac{1}{\sqrt{3}}}$$

$$= \frac{EA}{3L^e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

Code