Boundary Conditions

- static condensation (harder to implement, more general)
- Penalty method (easy to implement, restrictive)

Until now, we have applied zero displacement B.C.'s ($u_2 = 0$) by deleting row & column 2 of global matrices.

Now let's say $u_2 = \delta$ given

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1N} \\ \cancel{K_{21}} & \cancel{K_{22}} & \cancel{K_{23}} & \cdots & \cancel{K_{2N}} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3N} \\ \vdots & \vdots & & & \vdots \\ K_{N1} & K_{N2} & \cdots & \cdots & K_{NN} \end{bmatrix} \begin{bmatrix} u_1 \\ \boxed{u_2} = \delta \\ u_3 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \cancel{f_2} \\ \vdots \\ f_N \end{bmatrix}$$

Serial implementation

$$\begin{bmatrix} K_{11} & K_{13} & K_{14} & \cdots & K_{1N} \\ K_{31} & K_{33} & K_{34} & \cdots & K_{3N} \\ \vdots & \vdots & & & \\ K_{N1} & K_{N3} & \cdots & \cdots & K_{NN} \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 - K_{12}u_2 \\ f_3 - K_{32}u_2 \\ \vdots \\ f_N - K_{N2}u_2 \end{bmatrix}$$

Penalty method (approximation method to implement B.C.)

* Modify the P.E. *

Add $\frac{1}{2} c (u_2 - \delta)^2$ term where c is a very large ~~term~~ number

$$\Pi = \Pi^{(a)} + \Pi^{(b)} + \dots \frac{1}{2} c (u_2 - \delta)^2$$

Minimize Π

$$\frac{\partial \Pi}{\partial u_2} = \frac{\partial \Pi^{(a)}}{\partial u_2} + \frac{\partial \Pi^{(b)}}{\partial u_2} + \dots c(u_2 - \delta)$$

Global System

$$\begin{bmatrix} \frac{\partial \Pi}{\partial u_1} \\ \frac{\partial \Pi}{\partial u_2} \\ \vdots \end{bmatrix} = \begin{bmatrix} & & & \\ k_{21} & k_{22}+c & k_{23} & \\ & & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} - \begin{bmatrix} f_1 \\ f_2 + c\delta \\ \vdots \\ f_N \end{bmatrix}$$

Add c to k_{22} term & add $c\delta$ to the f_2 term in global matrix/vector

General contraint

$$u_2 \cos\theta + u_3 \sin\theta = \delta$$

$$\Pi = \Pi^{(a)} + \Pi^{(b)} + \dots + \frac{1}{2} c (u_2 \cos\theta + u_3 \sin\theta - \delta)^2$$

$$\frac{\partial \Pi}{\partial u_2} = \frac{\partial \Pi^a}{\partial u_2} + \cdots + C(u_2 \cos\theta + u_3 \sin\theta - \delta)\cos\theta$$

$$\frac{\partial \Pi}{\partial u_3} = \frac{\partial \Pi^{a}}{\partial u_3} + \cdots + C(u_2 \cos\theta + u_3 \sin\theta - \delta)\sin\theta$$

$$\begin{bmatrix} \frac{\partial \Pi}{\partial u_1} \\ \frac{\partial \Pi}{\partial u_2} \\ \frac{\partial \Pi}{\partial u_3} \\ \vdots \end{bmatrix} = \begin{bmatrix} & & \\ & K_{22} + C\cos^2\theta & K_{23} + C\sin\theta\cos\theta \\ & K_{32} + C\sin\theta\cos\theta & K_{33} + C\sin^2\theta \\ & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix}$$

$$- \begin{bmatrix} f_1 \\ f_2 + C\delta\cos\theta \\ f_3 + C\delta\sin\theta \\ \vdots \end{bmatrix}$$

C is a choice made by the user
– it is a guestimate

Usually $C \approx 10^4 \times K_{max}$

Galerkin method in 1D

So far in Aero 510

The Galerkin method is very general → CFD, etc.

- Direct
- P.E. minimization (elastic)
- Galerkin (principle of virtual work)

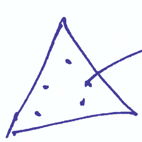
all these methods use the same code

These methods generate 'local' stiffness matrix, 'local' force vector.

Some definitions

Strong Form

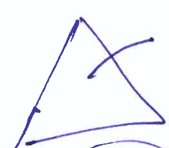
- Differential equation
- valid at every point on the domain



$$\frac{\partial^2 u}{\partial x^2} = 0$$

Weak Form

- weighted average of the strong form over an element



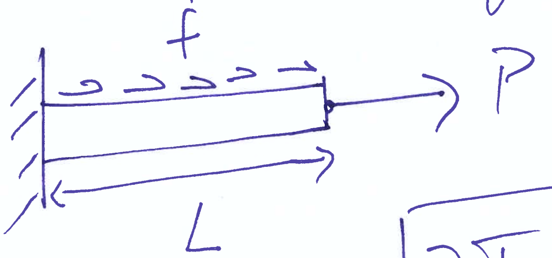
$$\int \frac{\partial^2 u}{\partial x^2} \, w \, dV = 0$$

weight

FEM

As you refine your grid, the solution to the weak form will converge on the strong form.

Example of derivation of 'local' stiffness matrix & 'local' force vector



strong form: $\boxed{\frac{\partial \sigma}{\partial x} + f = 0}$ 1D equilibrium equation

Strong

Solve $\Rightarrow$ $\boxed{E\frac{\partial^2 u}{\partial x^2} + f = 0}$ at every point

such that $u = 0$ @ $x = 0$
$F(L) = P$ @ $x = L$

Weak

Solve $\int_{x_1}^{x_2} \left(E\frac{\partial^2 u}{\partial x^2} + f \right) w \, dx = 0$ for each element

such that
$u = 0$ @ $x = 0$
$F(L) = P$ @ $x = L$

FEM

$u = u_1 N_1 + u_2 N_2$
$w = w_1 N_1 + w_2 N_2$
for each element

In Galerkin method, the weight function is interpolated using the shape functions

Weak Form for 1 element

$$\int_{x_1}^{x_2} (Eu'' + f) w \, dx = 0$$

$$\int_{x_1}^{x_2} (E(u''w) + fw) dx = 0$$

Notation

$$u'' = \frac{\partial^2 u}{\partial x^2} = N_1'' u_1 + N_2'' u_2$$

where

$$N_1'' = \frac{\partial^2 N_1}{\partial x^2}$$

Apply the u-v rule of differentiation

$$u'w' = (u'w)' - u''w$$

$$\frac{\partial^2 u}{\partial x^2} w = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} w\right) - \frac{\partial u}{\partial x}\frac{\partial w}{\partial x}$$

$$= \int_{x_1}^{x_2} \left(E\left((u'w)' - u'w'\right) + fw\right) dx = 0$$

Divergence Theorem

$$\int_{x_1}^{x_2} (u'w)' \, dx = u'w \Big|_{x_1}^{x_2}$$

integral ↗ ↑ derivative

$$= u'(x_2) w(x_2) - u'(x_1) w(x_1)$$

Now we have the reduced weak form

$$\boxed{= Eu'w \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} Eu'w' dx + \int_{x_1}^{x_2} fw \, dx}$$

only has 1st derivatives