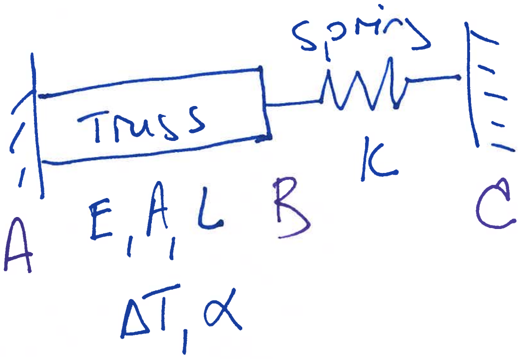
$$\boxed{\begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} F_1 - EA\alpha\Delta T \\ F_2 + EA\alpha\Delta T \end{bmatrix}}$$

elemental matrix

Example

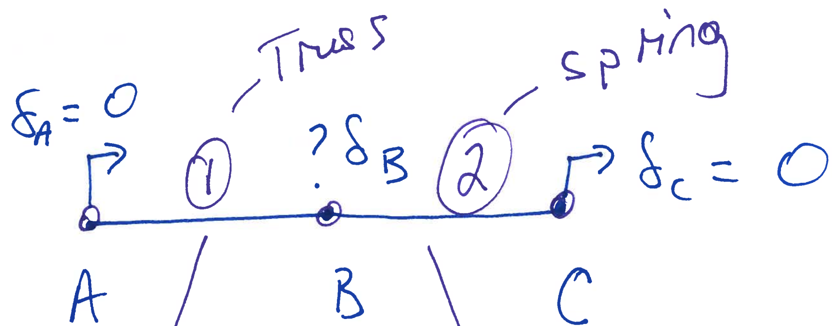


Truss (only) is heated by ΔT

Given E, A, L, α

Find displacement at B

Mesh



~~Assemble~~ Element stiffness matrix

$$\begin{array}{c|cc} & A & B \\ \hline A & \frac{EA}{L} & -\frac{EA}{L} \\ B & -\frac{EA}{L} & \frac{EA}{L} \end{array}$$

$$\begin{array}{c|cc} & B & C \\ \hline B & K & -K \\ C & -K & K \end{array}$$

force

$$\begin{bmatrix} R_A - EA\alpha\Delta T \\ F_B^{(1)} + EA\alpha\Delta T \end{bmatrix}$$

force

$$R_A = F_A^{(1)}$$

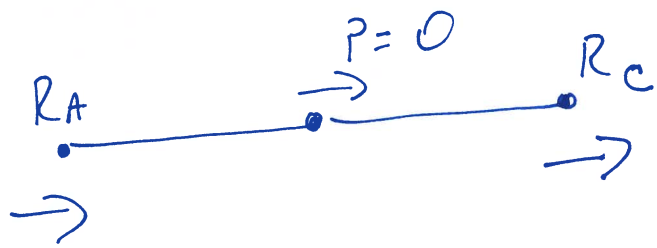
$$R_c = F_c^{(2)}$$

$$\begin{array}{c} B \\ C \end{array}\begin{bmatrix} F_B^{(2)} \\ R_c \end{bmatrix}$$

Assemble Global Matrix

no net external force on node B

$$\begin{array}{cccc} & A & B & C \\ A & \frac{EA}{L} & -\frac{EA}{L} & 0 \\ B & -\frac{EA}{L} & \frac{EA}{L}+K & -K \\ C & 0 & -K & K \end{array} \quad \begin{bmatrix} \delta_A \\ \delta_B \\ \delta_c \end{bmatrix} = \begin{bmatrix} R_A - EA\alpha\Delta T \\ \cancel{\boxed{F_B^{(1)} + F_B^{(2)}}} + EA\alpha\Delta T \\ R_c \end{bmatrix}$$



$$\begin{bmatrix} R_A \\ P \\ R_c \end{bmatrix} + \begin{bmatrix} -EA\alpha\Delta T \\ EA\alpha\Delta T \\ 0 \end{bmatrix}$$

external force added at the end

The next step is to apply the Boundary Conditions by deleting the row & columns corresponding to zero displacements

i.e. delete row 1, column 1 ← $\delta_A = 0$

delete row 3, column 3 ← $\delta_c = 0$

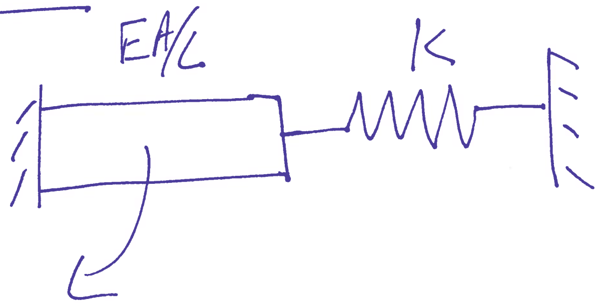
$$\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} & 0 \\ -\frac{EA}{L} & \frac{EA}{L}+K & -K \\ 0 & -K & K \end{bmatrix} \begin{bmatrix} \delta_A \\ \delta_B \\ \delta_C \end{bmatrix} = \begin{bmatrix} R_A - EA\alpha\Delta T \\ EA\alpha\Delta T \\ R_C \end{bmatrix}$$

That leaves us with:

$$\left(\frac{EA}{L} + K\right)\delta_B = EA\alpha\Delta T$$

$$\delta_B = \frac{EA\alpha\Delta T}{\frac{EA}{L} + K}$$

Stress



$$\sigma^{TRUSS} = \frac{E(\delta_B - \delta_A)}{L} - E\alpha\Delta T$$

Reaction Force (R_A & R_C)

Go back to original FE equations

$$\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} & 0 \\ -\frac{EA}{L} & \frac{EA}{L}+K & -K \\ 0 & -K & K \end{bmatrix} \begin{bmatrix} 0 \\ \frac{EA\alpha\Delta T}{\frac{EA}{L}+K} \\ 0 \end{bmatrix} = \begin{bmatrix} R_A - EA\alpha\Delta T \\ EA\alpha\Delta T \\ R_C \end{bmatrix}$$

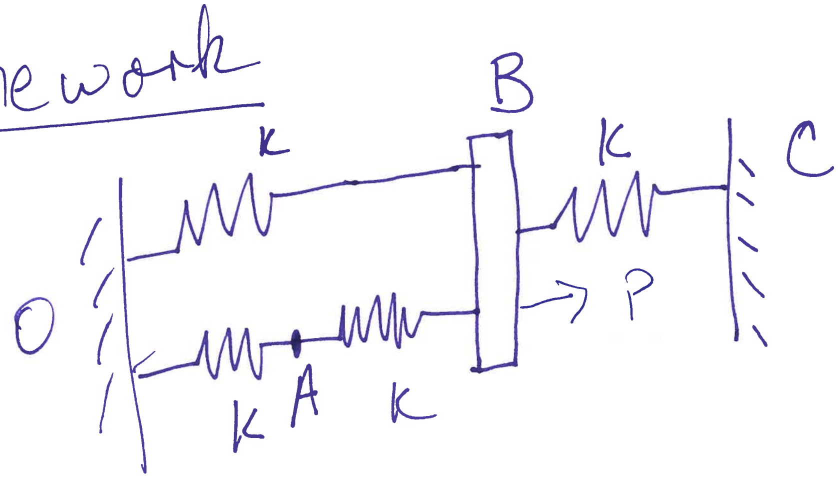
Expanding Equation 1

$$-\frac{EA}{L}\left(\frac{EA\alpha\Delta T}{\frac{EA}{L}+K}\right) = R_A - EA\alpha\Delta T$$

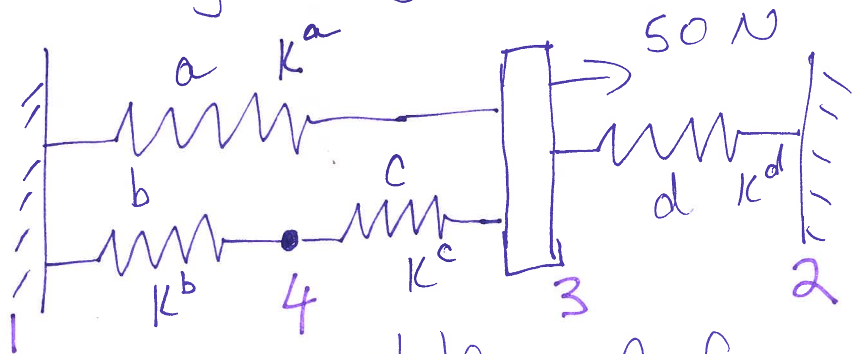
$$R_A = EA\alpha\Delta T - \frac{EA}{L}\left(\frac{EA\alpha\Delta T}{\frac{EA}{L}+K}\right)$$

Similarly solve for R_C

Homework



Assembling large structures



Four spring problem. A force 50N is applied. Assemble the global system.

First, write down the connectivity matrix

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 4 & 3 \\ 3 & 2 \end{bmatrix}$$

Now the local element matrices can be written down as:

Spring a

$$\begin{bmatrix} F_1^a \\ F_3^a \end{bmatrix} = \begin{bmatrix} K^a & -K^a \\ -K^a & K^a \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_3 \end{bmatrix}$$

Spring b

$$\begin{bmatrix} F_1^b \\ F_4^b \end{bmatrix} = \begin{bmatrix} K^b & -K^b \\ -K^b & K^b \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_4 \end{bmatrix}$$

Spring c

$$\begin{bmatrix} F_4^c \\ F_3^c \end{bmatrix} = \begin{bmatrix} K^c & -K^c \\ -K^c & K^c \end{bmatrix} \begin{bmatrix} \delta_4 \\ \delta_3 \end{bmatrix}$$

Spring d

$$\begin{bmatrix} F_3^d \\ F_2^d \end{bmatrix} = \begin{array}{c} 3 \\ 2 \end{array} \begin{bmatrix} K^d & -K^d \\ -K^d & K^d \end{bmatrix} \begin{bmatrix} \delta_3 \\ \delta_2 \end{bmatrix}$$

Now add up the forces on node 1 $(F_1^a + F_1^b)$

Add the forces on node 2 (F_2^d)

Add forces on node 3 $(F_3^a + F_3^c + F_3^d)$

Add forces on node 4 $(F_4^b + F_4^c)$

We have assembled the ~~sto~~ forces.

The final set of equations are:

This is the assembly step

$$\begin{bmatrix} F_1^a + F_1^b \\ F_2^d \\ F_3^a + F_3^c + F_3^d \\ F_4^b + F_4^c \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} K^a + K^b & 0 & -K^a & -K^b \\ 0 & K^d & -K^d & 0 \\ -K^a & -K^d & K^a + K^d + K^c & -K^c \\ -K^b & 0 & -K^c & K^c + K^b \end{bmatrix} \end{array} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}$$

Then plug in what you know about the forces and displacements

$$\begin{bmatrix} R_1 \\ R_2 \\ 50 \\ 0 \end{bmatrix} = \begin{bmatrix} K^a + K^b & 0 & -K^a & -K^b \\ 0 & K^d & -K^d & 0 \\ -K^a & -K^d & K^a + K^d + K^c & -K^c \\ -K^b & 0 & -K^c & K^c + K^b \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \delta_3 \\ \delta_4 \end{bmatrix}$$

Perform condensation:

$$\begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} K^a + K^d + K^c & -K^c \\ -K^c & K^c + K^b \end{bmatrix} \begin{bmatrix} \delta_3 \\ \delta_4 \end{bmatrix}$$

From here you solve for the two unknown displacements.