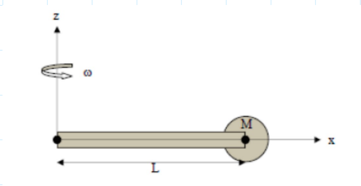
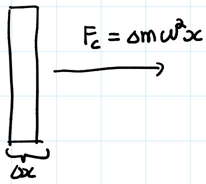


a) We have a constant internal body force due to rotational motion and a point force due to added mass at the tip.

We first derive the internal force expression:

$$\Delta F_c = \Delta m \omega^2 x$$
$$= \rho A \Delta x \cdot \omega^2 x$$
$$\frac{\Delta F_c}{\Delta x} = \rho A \omega^2 x$$



Taking $\lim_{\Delta x \to 0}$,

$$\boxed{\frac{dF_c}{dx} = \rho A \omega^2 x} \quad (1)$$

We then derive the point mass formula:

$$M_{bar} = \rho V$$
$$= \rho (L \cdot t^2)$$
$$M_{point} = 0.1\, M_{bar} = 0.1 \rho (L \cdot t^2)$$
$$\boxed{\begin{aligned} F_{point} &= M_{point} \cdot \omega^2 L \\ &= 0.1 \rho L \cdot t^2 \cdot \omega^2 \cdot L \end{aligned}} \quad (2)$$

We then recall the Rayleigh-Ritz formula:

$$\Pi = \frac{1}{2}\int_V \underline{\varepsilon}\,\underline{\tau}^T\, dV - \int_S \underline{u}^T \underline{t}\, ds - \int_V \underline{u}^T \underline{b}\, dV - P\delta$$

For our case, we have:

$$\Pi = \underbrace{\frac{1}{2}\int_0^L \underline{\varepsilon}\,\underline{\tau}^T\, dV}_{\text{Term 1}} - \underbrace{F_{point}\, u\big|_{x=L}}_{\text{Term 2}} - \underbrace{\int_0^L \underline{u}^T \underline{b}\, dV}_{\text{Term 3}}$$

Our guess is:

$$u = a + bx + cx^2 + dx^2$$

we know $u = 0$ @ $x = 0$.

$$\boxed{0 = a} \quad (3)$$

$$u(x) = bx + cx^2 + dx^2$$

$$\text{Term 1} = \frac{1}{2}\int_0^L EA\left(\frac{\partial u(x)}{\partial x}\right)^2 dx$$

$$= 3.5E6\, b^2 + (7E6) bc + (7E6) bd + \frac{14E6\, c^2}{3} + (1.05E7) cd + 6.3E6\, d^2$$

Term 2 $= 0.1\rho L \cdot t^2 \cdot \omega^2 \cdot L\,(b+c+d)$

Term 3 $= \int_0^L \underline{u}^T F_c \, dx$

$$= \int_0^L u(x)\left(\rho A \omega^2 x\right) dx$$

$$= 90000b + 67500c + 54000d$$

$$\Pi = \text{Term 1} - \text{Term 2} - \text{Term 3}$$

We then minimize π w.r.t b, c, d

$$\frac{\partial \Pi}{\partial b} = 0 \quad (4)$$

$$\frac{\partial \Pi}{\partial c} = 0 \quad (5)$$

$$\frac{\partial \Pi}{\partial d} = 0 \quad (6)$$

We plug this into MATLAB symbolic expression and solve for coefficients:

$$\begin{bmatrix} b = +0.0231 \\ c = 0 \\ d = -0.0064 \end{bmatrix}$$

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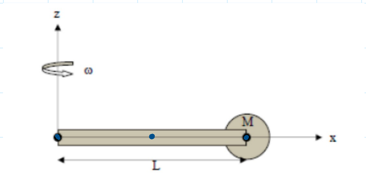
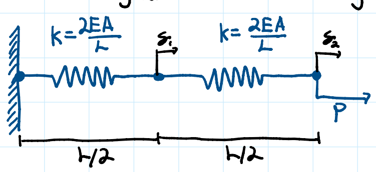
b)



Using three nodes / two elements, perhaps we can draw the new system as the following:



$m_{x=0} = 0\,kg \qquad m_{x=0.5m} = \frac{1}{2}\rho A L \qquad m_{x=1m} = \rho A L$

We do this because cross-sectional area is constant and the increase in mass is linear.

$$\Pi = \text{internal work} - \text{external work}$$

$$\text{Internal Work} = \frac{1}{2}\left(\frac{2EA}{L}\right)\delta_1^2 + \frac{1}{2}\left(\frac{2EA}{L}\right)(\delta_1 + \delta_2)^2$$

$$\text{External Work} = \left(m_{x=..}\,\omega^2\left(\tfrac{L}{2}+\delta_1\right) + m_{x=..}\,\omega^2(L+\delta_1+\delta_2\right) + \left(0.1\rho L \cdot t^2 \cdot \omega^2 \cdot L\right)(\delta_1+\delta_2)$$

Internal Work $= \frac{1}{2}\left(\frac{2EA}{L}\right)\delta_1^2 + \frac{1}{2}\left(\frac{2EA}{L}\right)(\delta_1+\delta_2)^2$

External Work $= \underbrace{\left(m_{L=0.5}\,\omega^2\left(\frac{L}{2}+\delta_1\right) + m_{L=1m}\,\omega^2(L+\delta_1+\delta_2)\right)}_{\text{Body Force}} + \underbrace{\left(0.1\rho L\cdot t^2\cdot\omega^2\cdot L\right)(\delta_1+\delta_2)}_{\text{Point Force}}$

$$\Pi = \frac{1}{2}\left(\frac{2EA}{L}\right)\delta_1^2 + \frac{1}{2}\left(\frac{2EA}{L}\right)(\delta_1+\delta_2)^2 - \left(m_{L=0.5}\,\omega^2(\delta_1) + m_{L=1}\,\omega^2(\delta_1+\delta_2)\right) - \left(0.1\rho L\cdot t^2\cdot\omega^2\cdot L\right)(\delta_1+\delta_2)$$

We then take derivate with respect to both δ_1 & δ_2 and set them to 0:

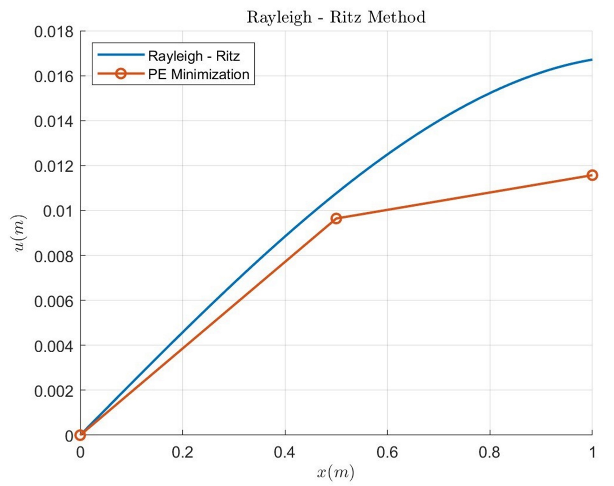
$$\frac{\partial\Pi}{\partial\delta_1} = 0 = 28000000\,\delta_1 + 14000000\,\delta_2 - 432000$$

$$\frac{\partial\Pi}{\partial\delta_2} = 0 = 14000000\,\delta_1 + 14000000\,\delta_2 - 297000$$

We put this into matrix form:

$$\begin{bmatrix} 28000000 & 14000000 \\ 14000000 & 14000000 \end{bmatrix}\begin{bmatrix}\delta_1 \\ \delta_2\end{bmatrix} = \begin{bmatrix} 432000 \\ 297000 \end{bmatrix}$$

$$\boxed{\begin{bmatrix}\delta_1 \\ \delta_2\end{bmatrix} = \begin{bmatrix} 0.0094 \\ 0.0116 \end{bmatrix} m}$$



We see that for both Rayleigh-Ritz method and potential energy minimization, the trends look approximately the same. The discrepancy between the values however diverges further along the bar. With more points, we should expect a better alignment of the PE Minimization with regards to the Rayleigh-Ritz method.

In addition, I am a bit unsure whether my calculation for PE Minimization is correct, as I would have expected the PE Minimization to match up more with Rayleigh-Ritz. The point of unsure lies in the displacement between delta_1 and delta_2. Intuitively I think that the potential energy of the springs would depend on the combined displacement of both delta_1 and delta_2, but I have seen other sources where the displacement is actually delta_2 - delta_1. Regardless, I believe my solution procedure for both parts are approximately correct. I have also attached my code for conveniences.

Part C) was left blank due to message from Professor Goulbourne that we should leave the part until the next homework.