Exam 2

Tuesday, April 4, 2023 2:58 PM

I follow the honor code!

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1. $\frac{d^2u}{dx^2} + \frac{\rho\omega^2}{E}x = 0$

where $u(x=0) = 0$ & $\frac{du}{dx}(x=L) = \frac{M\omega^2 L}{EA}$

We first obtain the weak form:

$$\int_{x_1}^{x_2} \left(\frac{d^2u}{dx^2} + \frac{\rho\omega^2 x}{E}\right) w = 0$$

$$\int_{x_1}^{x_2} \frac{d^2u}{dx^2} w + \frac{\rho\omega^2 x}{E} w = 0$$

$$\int_{x_1}^{x_2} \frac{d^2u}{dx^2} w - \frac{du}{dx}\frac{dw}{dx} + \frac{\rho\omega^2 x}{E} w = 0$$

$$\int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{du}{dx} w\right] - \frac{du}{dx}\frac{dw}{dx} + \frac{\rho\omega^2 x}{E} w = 0$$

$$\left.\frac{du}{dx} w\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{du}{dx}\frac{dw}{dx} + \int_{x_1}^{x_2} \frac{\rho\omega^2 x}{E} w = 0$$ we now have 3 terms

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$$\left.\frac{du}{dx} w\right|_{x_1}^{x_2} = \left.\frac{du}{dx}\right|_{x_2} w(x_2) - \left.\frac{du}{dx}\right|_{x_1} w(x_1)$$

$$= [w(x_1) \quad w(x_2)] \begin{bmatrix} -\left.\frac{du}{dx}\right|_{x_1} \\ \left.\frac{du}{dx}\right|_{x_2} \end{bmatrix}$$

$$\int_{x_1}^{x_2} \frac{du}{dx}\frac{dw}{dx}\, dx$$

$$= \int_{x_1}^{x_2} \left(N_1' u(x_1) + N_2' u(x_2)\right)\left(N_1' w(x_1) + N_2' w(x_2)\right)$$

$$= [w(x_1) \; w(x_2)] \int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} [N_1' + N_2'] \begin{bmatrix} u(x_1) \\ u(x_2) \end{bmatrix}$$

$$\int_{x_1}^{x_2} \frac{\rho\omega^2 x}{E} w = \int_{x_1}^{x_2} \frac{\rho\omega^2}{E}[N_1 w(x_1) + N_2 w(x_2)]\, x\, dx$$

$$\int_{x_1} \frac{\rho\omega x}{E} w = \int_{x_1} \frac{\rho\omega^2}{E}\left[N_1 w(x_1) + N_2 w(x_2)\right] x \, dx$$

$$= \left[w(x_1) \;\; w(x_2)\right] \frac{\rho\omega^2}{E} \int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx$$

piecing everything together, we then have:

$$\frac{du}{dx} w \Big|_{x_1}^{x_2} - \int_{x}^{x_2} \frac{du}{dx}\frac{dw}{dx} + \int_{x_1}^{x_2} \frac{\rho\omega^2 x}{E} w = 0$$

$$\cancel{\left[w(x_1) \;\; w(x_2)\right]} \begin{bmatrix} -\frac{du}{dx}\big|_{x_1} \\ \frac{du}{dx}\big|_{x_2} \end{bmatrix} - \cancel{\left[w(x_1) \;\; w(x_2)\right]} \int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} \left[N_1' + N_2'\right] \begin{bmatrix} u(x_1) \\ u(x_2) \end{bmatrix} + \cancel{\left[w(x_1) \;\; w(x_2)\right]} \frac{\rho\omega^2}{E} \int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx = 0$$

$$\underbrace{\int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} \left[N_1' + N_2'\right]}_{K} \begin{bmatrix} u(x_1) \\ u(x_2) \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{du}{dx}\big|_{x_1} \\ \frac{du}{dx}\big|_{x_2} \end{bmatrix} + \frac{\rho\omega^2}{E} \int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx}_{\text{Local Forces}}$$

$$\frac{du}{dx}\Big|_{x_2} = \frac{M\omega^2 L}{EA} \quad \text{from BC}$$

since domain is $x \in [0, 1]$

$$\begin{bmatrix} N_1 = \frac{x_2 - x}{x_2 - x_1} & N_2 = \frac{x - x_1}{x_2 - x_1} \\ N_1' = x_2 & N_2' = -x_1 \end{bmatrix}$$

$$k = \int_{x_1}^{x_2} \begin{bmatrix} N_1'^2 & N_1' N_2' \\ N_1' N_2' & N_2'^2 \end{bmatrix}$$

$$\int_{x_1}^{x_2} N_1' \, dx = \int_{x_1}^{x_2} \left(\frac{-1}{x_2 - x_1}\right)^2 dx = \frac{1}{x_2 - x_1}$$

$$\int_{x_1}^{x_2} N_1' N_2' \, dx = \int^{x_2} \left(-\frac{1}{\;}\right)\left(\frac{1}{\;}\right) = \frac{-1}{\;}$$

$$\int_{x_1}^{x_2} N_1' N_2' \, dx = \int_{x_1}^{x_2} \left(-\frac{1}{x_2-x_1}\right)\left(\frac{1}{x_2-x_1}\right) = \frac{-1}{x_2-x_1}$$

$$\int_{x_1}^{x_2} N_1'^2 \, dx = \int_{x_1}^{x_2} \left(\frac{1}{x_2-x_1}\right)^2 dx = \frac{1}{x_2 x_1}$$

$$k = \frac{1}{1-0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{du}{dx}\Big|_{x_2} \end{bmatrix} + \frac{\rho\omega^2}{E}\int_{x_1}^{x_2}\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx$$

$$= \begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{M\omega^2 L}{EA} \end{bmatrix} + \frac{\rho\omega^2}{E}\begin{bmatrix} \frac{x_2^3}{6} - \frac{x_2 x_1^2}{2} + \frac{x_1^3}{3} \\ \frac{x_2^3}{3} - \frac{x_1 x_2^2}{2} + \frac{x_1^3}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{M\omega^2 L}{EA} \end{bmatrix} + \frac{\rho\omega^2}{E}\begin{bmatrix} \frac{1}{6} - 0 + 0 \\ \frac{1}{3} - 0 + 0 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{M\omega^2 L}{EA} \end{bmatrix} + \frac{\rho\omega^2}{E}\begin{bmatrix} \frac{1}{6} - 0 + 0 \\ \frac{1}{3} - 0 + 0 \end{bmatrix}$$

$$u_2 = \frac{M\omega^2 L}{EA} + \frac{\rho\omega^2}{E}\left(\frac{1}{3}\right)$$

$$= \frac{(1 \cdot 0.01^2 \cdot 0.1)(1000)^2(1)}{70E9 \cdot (0.01)^2} + \frac{(2700)(1000)^2}{70E9}\left(\frac{1}{3}\right)$$

$$= 0.0167 \text{ m}$$

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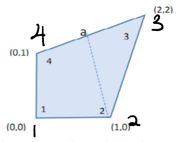
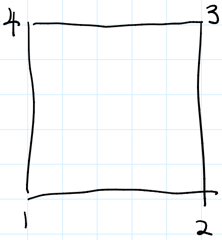
3)

1. A single four noded quadrilateral element is shown below:
(a) Compute the Jacobian determinant in terms of natural coordinates (ξ,η)
(b) Compute the derivative dN_2/dy at node 2
(c) Compute the integral $\iint N_1 N_2 dA$ using two point integration rule
(d) Compute the integral $\int N_2 dl$ along the line joining nodes 2 and 3 using one point integration
(e) Compute the integral $\int N_2 dl$ along the line joining node 2 and the mid point of line 3-4 as shown in the figure.



a)



$$N_1 = \frac{(\eta-1)(\xi-1)}{4}$$

$$N_2 = \frac{(\xi+1)(\eta-1)}{4}$$

$$N_3 = \frac{(\xi+1)(\eta+1)}{4}$$

$$N_4 = (\xi-1)(\eta+1)/4$$

$$J = \frac{dN}{d\xi} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

from code "Quad.m"

$$J = \begin{bmatrix} \frac{\eta+3}{4} & \frac{\eta+1}{4} \\ \frac{\xi+1}{4} & \frac{\xi+3}{4} \end{bmatrix}$$

b) At node 2: $[\xi, \eta] = [1, -1]$

$$\frac{\partial N}{\partial x} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix}$$

$$J \cdot \frac{\partial N}{\partial x} = \frac{\partial N}{\partial \xi}$$

$$\frac{\partial N}{\partial x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0.5 & -1 & 0.5 & 0 \end{bmatrix}$$

$$\frac{\partial N_2}{\partial y} = -1$$

c)

$$\iint N_1 N_2 \, dA$$

integral points $\left(\left[\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right], \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right], \left[\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right], \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \right)$

$\iint N_1 N_2 \, dA =$ [illegible]

$$\iint N_1 N_2 \, dA = \sum_{i=1}^{4} w N_1(\text{int point}_i) N_2(\text{int point}_i) \det J_i$$

where $w = 1$.

We calculate in MATLAB, & obtain:

$$\iint N_1 N_2 \, dA = 0.0972$$

d) $\int N_2 \, dl$ along L_{23}

$$\int N_2 \det J^* \, d\eta$$

$$J = \begin{bmatrix} \frac{\eta+3}{4} & \frac{\eta+1}{4} \\ \frac{\xi+1}{4} & \frac{\xi+3}{4} \end{bmatrix}$$

$$\det J^* = \sqrt{\left(\frac{\xi+1}{4}\right)^2 + \left(\frac{\xi+3}{4}\right)^2}$$

1 point integration,

int point $= [1, 0]$

$$\int N_2 \det(J)^* \, d\eta = 2 N_2(\text{intpoint}) \det(J^*)$$

$$= 0.6250$$

e) The coordinate are:

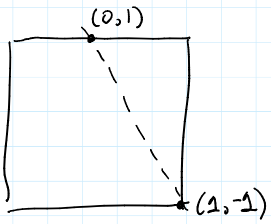
$a = (1, 1.5)$

$b = (1, 0)$

The equation of line passing through is

$x = 1$.

In reference coordinate:



$\eta = -\xi + 1$

By formula, we know

By formula, we know

$$\int_{l_{a\to b}} N_1 \, dl = \int_{\xi_a}^{\xi_b} N_2 \frac{dl}{d\xi}\, d\xi$$

$$\frac{dl}{d\xi} = \sqrt{\left(\frac{\partial x}{\partial \xi} + \frac{\partial x}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}+\frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2}$$

From J

$\frac{\partial \eta}{\partial \xi}$ from line:

$$\eta = -\xi + 1$$
$$d\eta = -d\xi$$
$$\frac{d\eta}{d\xi} = -1$$

The midpoint of the line in reference coordinate is:

$$(0.5, 0)$$

$$J = \begin{bmatrix} 0.75 & 0.25 \\ 0.375 & 0.875 \end{bmatrix}$$

We can the proceed with 1 point integration:

$$2 \cdot N_2(\text{int point}) \sqrt{\left(\frac{\partial x}{\partial \xi} + \frac{\partial x}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}+\frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2}$$

$$= 2 \cdot 0.375 \sqrt{(0.75 + 0.375(-1))^2 + (0.25 + 0.875(-1))^2}$$

$$= 0.5467$$

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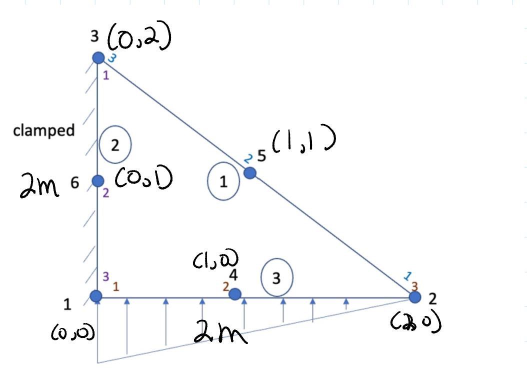
c)



a) We have here a 6-noded element.

We define the shape functions

$$N_1 = 1 - 3\xi + 2\xi^2 - 3\eta + 4\xi\eta + 2\eta^2$$
$$N_2 = -\xi + 2\xi^2$$

$N_1 = 1-3\xi+2\xi^2-3\eta+4\xi\eta+2\eta^2$

$N_2 = -\xi+2\xi^2$

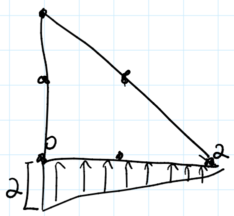
$N_3 = -\eta+2\eta$

$N_4 = 4\xi-4\xi^2-4\xi\eta$

$N_5 = 4\xi\eta$

$N_6 = 4\eta-4\xi\eta-4\eta^2$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix}$$



$$T = \begin{bmatrix} 0 \\ x+2 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 \\ (x_1N_1+x_2N_2+x_3N_3+x_4N_4+x_5N_5+x_6N_6)+2 \end{bmatrix}$$

$$f_{traction} = \int_{\Gamma^e} N^T\, T\, t^e \det J^* \cdot \sqrt{d\xi^2+d\eta^2}$$

We are on line $\eta=0$

$$\sqrt{d\xi^2+0} = d\xi$$

$$\left[f_{traction} = \int_{\Gamma^e} N^T T (1) \det J^* \, d\xi \right]$$

$$N^T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ \cancel{N_3} & \cancel{0} \\ \cancel{0} & \cancel{N_3} \\ N_4 & 0 \\ 0 & N_4 \\ \cancel{N_5} & \cancel{0} \\ \cancel{0} & \cancel{N_5} \\ \cancel{N_6} & \cancel{0} \\ \cancel{0} & \cancel{N_6} \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ 0 & 0 \\ 0 & 0 \\ N_4 & 0 \\ 0 & N_4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b) There are 3 points along the line, therefore, the function is quadratic and we need 2 points to integrate.

c) int point at

$[\frac{1}{3}, 0]$ & $[\frac{2}{3}, 0]$

c) int point at
$\left[\frac{1}{3}, 0\right]$ & $\left[\frac{2}{3}, 0\right]$

We compute with MATLAB & obtain

refer to Q2.m

$$F = \begin{bmatrix} 0 \\ 21.3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2.67 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Extra Credit: T
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