AEROSP 510 HW7

Andi Zhou

March 29, 2023

1 Introduction

We have the following quadrilateral element:

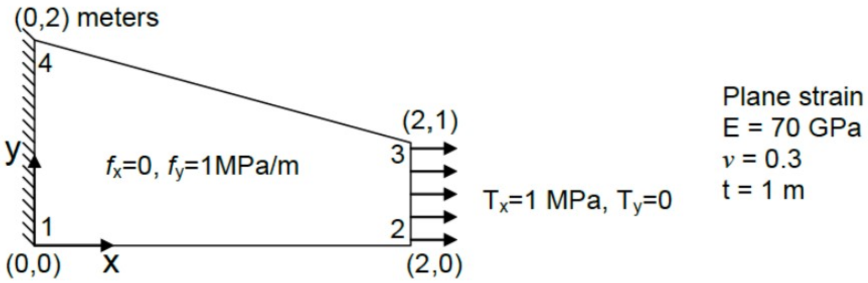

Figure 1: Sample Quadrilateral Element

we are interested in finding properties such as the shape function, the Jacobian matrix, etc. We are also given the following area quadrature integration points:

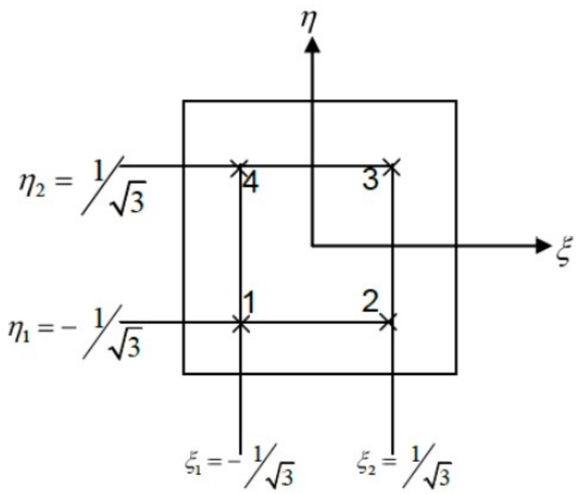

Figure 2: Area integral

We use the above reference points to conduct numerical integration within an element.

2 Question 1

For this question, we compute the shape function matrix, N, its derivative $\frac{\partial N}{\partial \vec{\xi}}$ and the Jacobian determinant $\det(J)$ for the reference coordinate $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$. The shape function are of the following:

$$
\begin{aligned}
N_1 &= \frac{(\eta-1)(\xi-1)}{4} \\
N_2 &= \frac{(\xi+1)(\eta-1)}{-4} \\
N_3 &= \frac{(\xi+1)(\eta+1))}{4} \\
N_4 &= \frac{(\xi-1)(\eta+1)}{-4}
\end{aligned} \tag{1}
$$

whereas the matrix N and $\frac{\partial N}{\partial \vec{\xi}}$ are specified as follow:

$$
\begin{aligned}
N(\xi,\eta) &= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \\
\frac{\partial N}{\partial \vec{\xi}} &= \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix}
\end{aligned} \tag{2}
$$

and we calculate Jacobian J with

$$
J = \frac{\partial N}{\partial \vec{\xi}} \times \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{3}
$$

Since we know the reference point, we tabulate our answers as follows:

N_1	0.1667
N_2	0.6220
N_3	0.1667
N_4	0.0447
$(\frac{\partial N_1}{\partial \xi}, \frac{\partial N_1}{\partial \eta})$	(-0.3943, -0.1057)
$(\frac{\partial N_2}{\partial \xi}, \frac{\partial N_2}{\partial \eta})$	(0.3943, -0.3943)
$(\frac{\partial N_3}{\partial \xi}, \frac{\partial N_3}{\partial \eta})$	(0.1057, 0.3943)
$(\frac{\partial N_4}{\partial \xi}, \frac{\partial N_4}{\partial \eta})$	(-0.1057, 0.1057)
$\det(J)$	0.6057

3 Question 2

In this question, we are interested in the body force term, which we calculate using the following equation:

$$
f^e = t_e \iint_A N^T f dxdy \tag{4}
$$

where $f = [0,1]^T$ MPa/m and

$$
N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{5}
$$

We use the 2×2 integration rule where we plug in our reference coordinates specified and sum them up with an individual weight of $w = 1$.

We obtain the following body force term:

$$
f = \begin{bmatrix} 0 \\ 833333.33 \\ 0 \\ 666666.66 \\ 0 \\ 666666.66 \\ 0 \\ 833333.33 \end{bmatrix} N \tag{6}
$$

4 Question 3

For question 3, we follow the equation:

$$T^e = t_e \int_{-1}^{1} N^T T \det(J) d\eta \tag{7}$$

where $T = [T_x, T_y]^T$ and N remains identical to that of the last question. However this is a line integral and we integral with the following quadrature points:

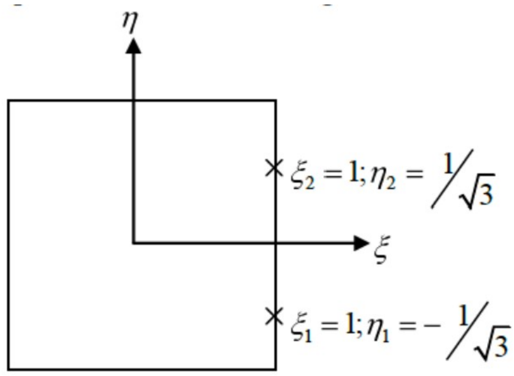

Figure 3: Line integral quadrature points

The integration formula remains the same as we sub in these two reference coordinates into the above equation and sum up all values withe an individual weight of $w = 1$. We obtain the following traction force term:

$$f_T = \begin{bmatrix} 0 \\ 0 \\ 500000 \\ 0 \\ 500000 \\ 0 \\ 0 \\ 0 \end{bmatrix} N \tag{8}$$

5 Question 4

For the last question, we are asked to calculate the stiffness matrix, where we use the following formula:

$$K^e = \iint_V B^T D B dV \tag{9}$$

where:

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \tag{10}$$

$$D = \frac{E}{1 - v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \tag{11}$$

We again perform the 2×2 quadrature integration rule, where we use the previously specified quadrature points, substitute them into the above equation, and sum up all contributions with an individual weight of $w = 1$.

We obtain the following stiffness matrix K^e:

$$K^e = \begin{bmatrix} 27.0833 & 10.4167 & -15.8654 & -1.7628 & -12.9808 & -9.4551 & 1.7628 & 0.8013 \\ 10.4167 & 29.1667 & -0.3205 & 2.8846 & -9.2949 & -12.9808 & -0.8013 & -19.0705 \\ -15.8654 & -0.3205 & 24.8397 & -8.3333 & 4.0064 & -0.6410 & -12.9808 & 9.2949 \\ -1.7628 & 2.8846 & -8.3333 & 22.7564 & 0.6410 & -12.6603 & 9.4551 & -12.9808 \\ -12.9808 & -9.2949 & 4.0064 & 0.6410 & 24.8397 & 8.3333 & -15.8654 & 0.3205 \\ -9.4551 & -12.9808 & -0.6410 & -12.6603 & 8.3333 & 22.7564 & 1.7628 & 2.8846 \\ 1.7628 & -0.8013 & -12.9808 & 9.4551 & -15.8654 & 1.7628 & 27.0833 & -10.4167 \\ 0.8013 & -19.0705 & 9.2949 & -12.9808 & 0.3205 & 2.8846 & -10.4167 & 29.1667 \end{bmatrix} GPa \tag{12}$$