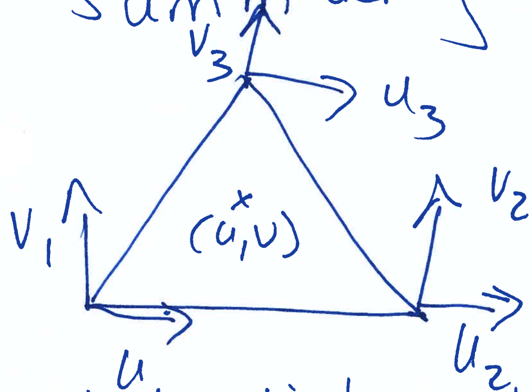
Summary: $\qquad q = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$ $\qquad$ Last lecture



Arbitrary displacement inside triangle.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} q$$

$$\underline{\varepsilon} = B\,\underline{q} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

We showed that the Galerkin reduces to the principal of virtual work

Our derivation gives:

$$\underbrace{\left(\int_{\Omega^e} B^T D B \, d\Omega\right)}_{\text{local stiffness}} q = \underbrace{\left(\int_{\Omega^e} N^T f \, d\Omega + \int_{\Gamma^e} N^T t \, d\Gamma\right)}_{\text{local force}}$$

$$\underset{6\times 6}{K} \; \underset{6\times 1}{q} = \underset{6\times 1}{f}$$

Matlab code: use same code

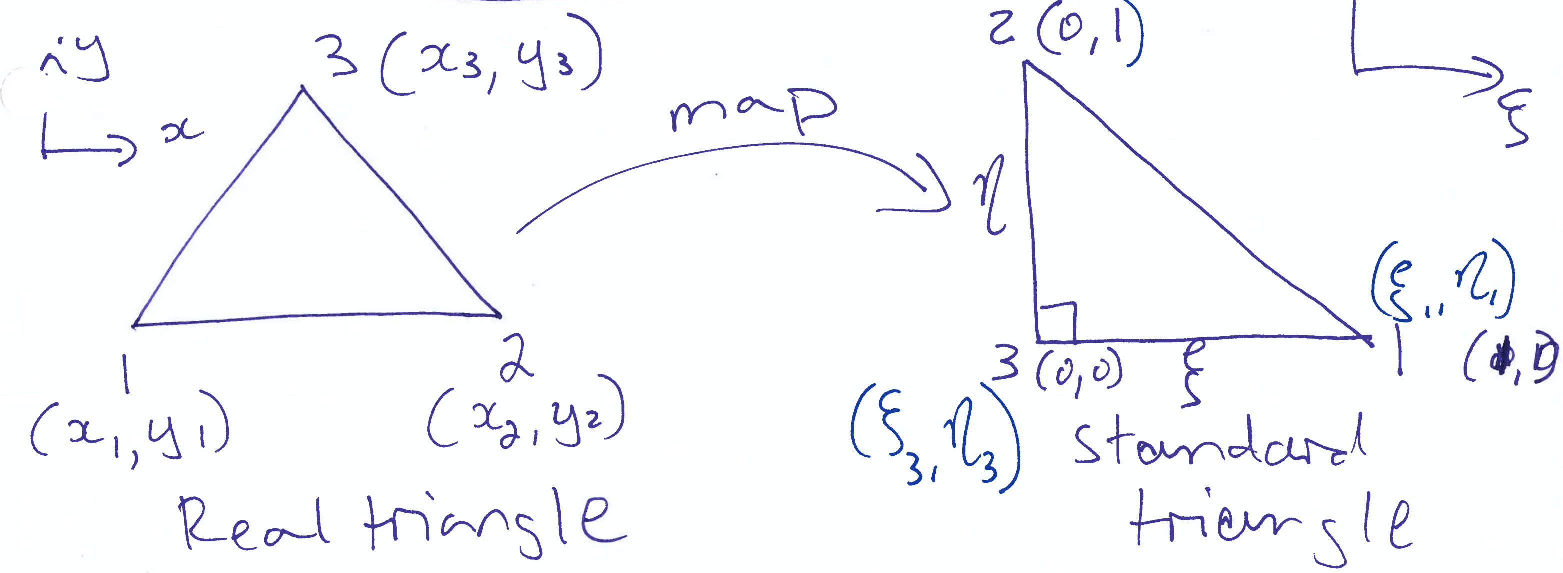
nne = 3

dof = 2

- No symbolic integration
- we will obtain numerical expressions for integrals

Today's class: What is B, D, N?

Triangle element 2D

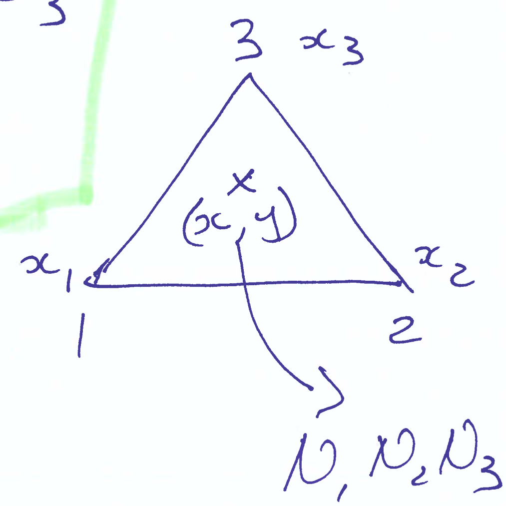


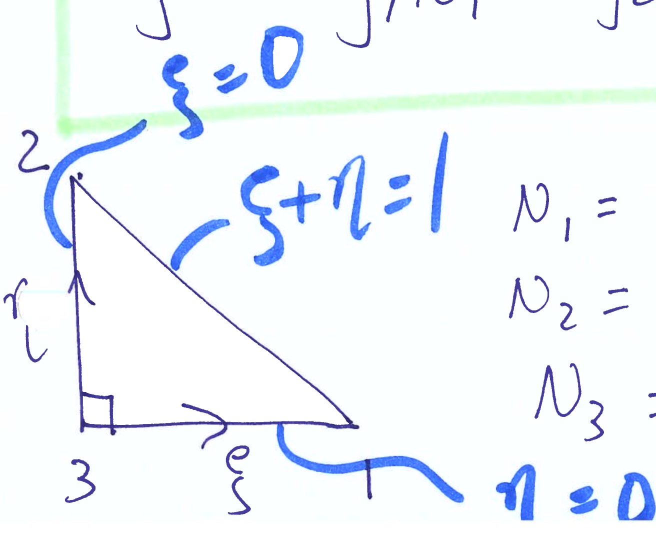
Every element is mapped to a 'standard' triangle of edge length '1'

<u>Define</u>

$$\text{coord} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \quad (\text{matrix})$$

$$\boxed{\begin{aligned} x &= x_1 N_1 + x_2 N_2 + x_3 N_3 \\ y &= y_1 N_1 + y_2 N_2 + y_3 N_3 \end{aligned}}$$





$N_1 = \xi$

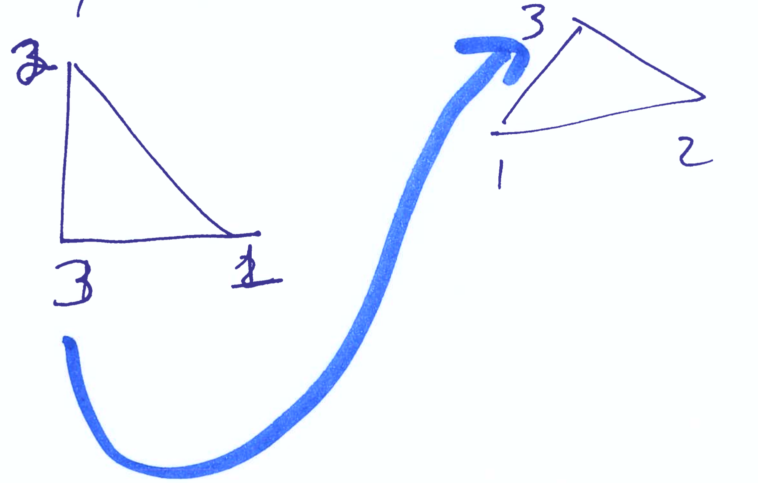
$N_2 = \eta$

$N_3 = 1 - \xi - \eta$

Note

$$x = x_1 \xi + x_2 \eta + x_3 (1 - \xi - \eta)$$

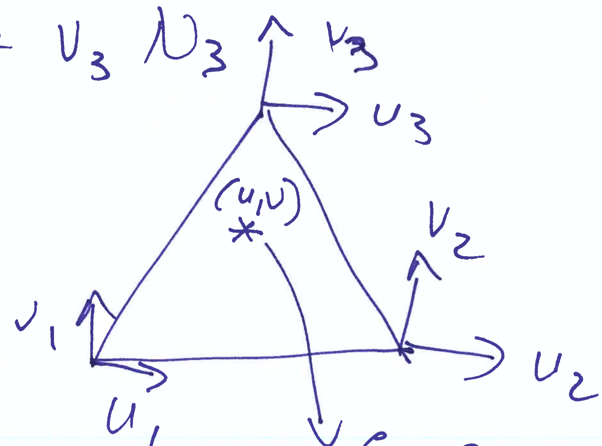
Plug in $\xi = 0, \eta = 0 \Rightarrow x = x_3$



Displacements

$$u = u_1 N_1 + u_2 N_2 + u_3 N_3$$

$$v = v_1 N_1 + v_2 N_2 + v_3 N_3$$



We still need to find ξ, η @ (x, y)

$\xi = ?$

$\eta = ?$

To find ξ & η, solve

$$\boxed{\begin{aligned} x &= x_1 \xi + x_2 \eta + x_3 (1 - \xi - \eta) \\ y &= y_1 \xi + y_2 \eta + y_3 (1 - \xi - \eta) \end{aligned}}$$

We have 2 eqtns & 2 unknowns

$\eta = ?$ $\xi = ?$

$$N_1 = \xi, \quad N_2 = \eta, \quad N_3 = 1 - \xi - \eta$$

$$\begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} & \dfrac{\partial N_2}{\partial \xi} & \dfrac{\partial N_3}{\partial \xi} \\[2ex] \dfrac{\partial N_1}{\partial \eta} & \dfrac{\partial N_2}{\partial \eta} & \dfrac{\partial N_3}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\underline{\varepsilon} = B\underline{q} = \nabla_s N q$$

$$= \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 \\[2ex] 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} \\[2ex] \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

To find $\dfrac{\partial N_1}{\partial x}$

Let's write

$$\frac{\partial N_1}{\partial \xi} = \frac{\partial N_1}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_1}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial N_1}{\partial \eta} = \frac{\partial N_1}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_1}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} \\[2ex] \dfrac{\partial N_1}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}_{\text{Jacobian matrix } (J)} \begin{bmatrix} \dfrac{\partial N_1}{\partial x} \\[2ex] \dfrac{\partial N_1}{\partial y} \end{bmatrix}$$

$$x = x_1 \overset{N_1}{\hat{\xi}} + x_2 \overset{N_2}{\hat{\eta}} + x_3 \overset{N_3}{\overbrace{(1-\xi-\eta)}}$$

$$\frac{\partial x}{\partial \xi} = x_1 - x_3 = x_{13}$$

(new notation)

$$\frac{\partial x}{\partial \eta} = x_2 - x_3 = x_{23}$$

$$\begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} \\[2ex] \dfrac{\partial N_1}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \dfrac{\partial N_1}{\partial x} \\[2ex] \dfrac{\partial N_1}{\partial y} \end{bmatrix}$$

↑ Jacobian matrix

* J is function of coordinates

$$J^{-1} \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} \\[2ex] \dfrac{\partial N_1}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} \\[2ex] \dfrac{\partial N_1}{\partial y} \end{bmatrix}$$

Generalize

$$J^{-1}\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{bmatrix}$$

$$J^{-1}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \checkmark$$

After plugging in J in terms of x_{13}, x_{23} etc

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

$$B = \frac{1}{x_{13}y_{23} - x_{23}y_{13}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

B is constant

local stiffness $= \int_{\Omega^e} B^T \underbrace{D}_{\text{elastic constants}} B \, d\Omega$

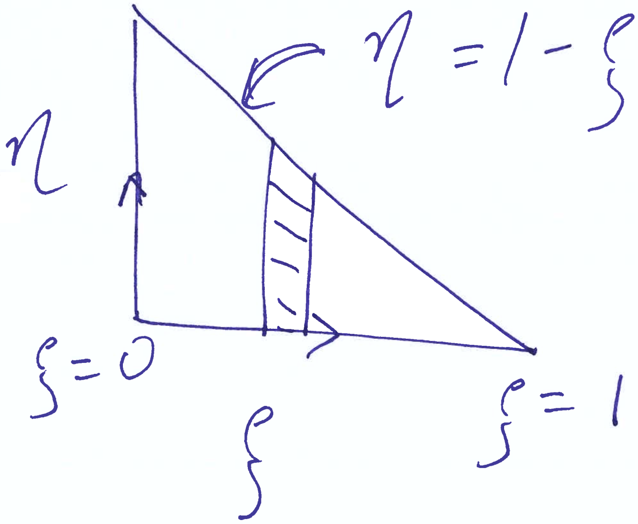
$$= B^T D B \underbrace{\int_{\Omega^e} d\Omega}_{\text{volume of the element}}$$

real element

$$\boxed{K = B^T D B \,(\underbrace{t}_{\text{thickness of plate}} \overbrace{A}^{\text{area of real triangle}})}$$

local force

$$f^{\text{local body force}} = \int_{\Omega^e} \overbrace{\begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}}^{N^T} \begin{bmatrix} f_x \\ f_y \end{bmatrix} d\Omega$$



Integrate over the standard triangle

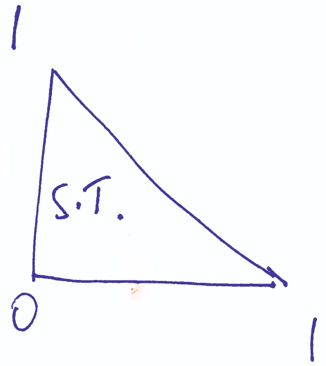
Use $\det J$ = determinant of Jacobian matrix J

Integrate over the standard triangle

$$f^{local} = \int_{\xi=0}^{1}\int_{\eta=0}^{1-\xi} \begin{bmatrix} \xi & 0 \\ 0 & \xi \\ \eta & 0 \\ 0 & \eta \\ 1-\xi-\eta & 0 \\ 0 & 1-\xi-\eta \end{bmatrix} \underbrace{\begin{bmatrix} f_x \\ f_y \end{bmatrix}}_{\text{in terms of } \xi \,\&\, \eta} t \det J \, d\eta \, d\xi$$

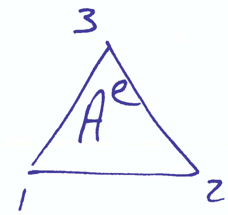
$$\underbrace{d\Omega}_{\text{real triangle volume element}} = \underbrace{t}_{\text{thickness}} \det(J) \underbrace{d\xi \, d\eta}_{\text{standard triangle area element}}$$

$\det J$ = ratio of volumes of real triangle to standard triangle



$\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$$\det J = \frac{A^e}{\left(\frac{1}{2}\right)} = 2A^e$$



$\det J > 0$ if numbering ccw