Exam 2

Tuesday, April 4, 2023 2:58 PM

I follow the honor code!

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1. $$\frac{d^2u}{dx^2} + \frac{\rho\omega^2}{E}x = 0$$

where $u(x=0) = 0$ & $\frac{du}{dx}(x=L) = \frac{M\omega^2 L}{EA}$

We first obtain the weak form:

$$\int_{x_1}^{x_2} \left(\frac{d^2u}{dx^2} + \frac{\rho\omega^2 x}{E}\right) w = 0$$

$$\int_{x_1}^{x_2} \frac{d^2u}{dx^2} w + \frac{\rho\omega^2 x}{E} w = 0$$

$$\int_{x_1}^{x_2} \frac{d^2u}{dx^2} w - \frac{du}{dx}\frac{dw}{dx} + \frac{\rho\omega^2 x}{E} w = 0$$

$$\int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{du}{dx} w\right] - \frac{du}{dx}\frac{dw}{dx} + \frac{\rho\omega^2 x}{E} w = 0$$

$$\left.\frac{du}{dx} w\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{du}{dx}\frac{dw}{dx} + \int_{x_1}^{x_2} \frac{\rho\omega^2 x}{E} w = 0$$ we now have 3 terms

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$$\left.\frac{du}{dx} w\right|_{x_1}^{x_2} = \left.\frac{du}{dx}\right|_{x_2} w(x_2) - \left.\frac{du}{dx}\right|_{x_1} w(x_1)$$

$$= [w(x_1) \quad w(x_2)] \begin{bmatrix} -\left.\frac{du}{dx}\right|_{x_1} \\ \left.\frac{du}{dx}\right|_{x_2} \end{bmatrix}$$

$$\int_{x_1}^{x_2} \frac{du}{dx}\frac{dw}{dx} dx$$

$$= \int_{x_1}^{x_2} \left(N_1' u(x_1) + N_2' u(x_2)\right)\left(N_1' w(x_1) + N_2' w(x_2)\right)$$

$$= [w(x_1) \; w(x_2)] \int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} [N_1' + N_2'] \begin{bmatrix} u(x_1) \\ u(x_2) \end{bmatrix}$$

$$\int_{x_1}^{x_2} \frac{\rho\omega^2 x}{E} w = \int_{x_1}^{x_2} \frac{\rho\omega^2}{E}[N_1 w(x_1) + N_2 w(x_2)] x \, dx$$

$$\int_{x_1} \frac{\rho w x}{E} w = \int_{x_1} \frac{\rho \omega^2}{E}\left[N_1 w(x_1) + N_2 w(x_2)\right] x \, dx$$

$$= \left[w(x_1) \;\; w(x_2)\right] \frac{\rho\omega^2}{E} \int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx$$

piecing everything together, we then have:

$$\frac{du}{dx} w \Big|_{x_1}^{x_2} - \int_{x}^{x_2} \frac{du}{dx}\frac{dw}{dx} + \int_{x_1}^{x_2} \frac{\rho \omega^2 x}{E} w = 0$$

$$\cancel{\left[w(x_1) \;\; w(x_2)\right]} \begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{du}{dx}\Big|_{x_2} \end{bmatrix} - \cancel{\left[w(x_1) \;\; w(x_2)\right]} \int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} \left[N_1' + N_2'\right] \begin{bmatrix} u(x_1) \\ u(x_2) \end{bmatrix} + \cancel{\left[w(x_1) \;\; w(x_2)\right]} \frac{\rho\omega^2}{E} \int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx = 0$$

$$\underbrace{\int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} \left[N_1' + N_2'\right]}_{K} \begin{bmatrix} u(x_1) \\ u(x_2) \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{du}{dx}\Big|_{x_2} \end{bmatrix} + \frac{\rho\omega^2}{E} \int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx}_{\text{Local Forces}}$$

$$\frac{du}{dx}\Big|_{x_2} = \frac{M\omega^2 L}{EA} \quad \text{from BC}$$

since domain is $x \in [0, 1]$

$$\begin{bmatrix} N_1 = \frac{x_2 - x}{x_2 - x_1} & N_2 = \frac{x - x_1}{x_2 - x_1} \\ N_1' = x_2 & N_2' = -x_1 \end{bmatrix}$$

$$k = \int_{x_1}^{x_2} \begin{bmatrix} N_1'^2 & N_1' N_2' \\ N_1' N_2' & N_2'^2 \end{bmatrix}$$

$$\int_{x_1}^{x_2} N_1' \, dx = \int_{x_1}^{x_2} \left(\frac{-1}{x_2 - x_1}\right)^2 dx = \frac{1}{x_2 - x_1}$$

$$\int_{x_1}^{x_2} N_1' N_2' \, dx = \int^{x_2} \left(-\frac{1}{\;}\right)\left(\frac{1}{\;}\right) = \frac{-1}{\;}$$

$$\int_{x_1}^{x_2} N_1' N_2' \, dx = \int_{x_1}^{x_2} \left(-\frac{1}{x_2 - x_1}\right)\left(\frac{1}{x_2 - x_1}\right) = \frac{-1}{x_2 - x_1}$$

$$\int_{x_1}^{x_2} N_1'^2 \, dx = \int_{x_1}^{x_2} \left(\frac{1}{x_2 - x_1}\right)^2 dx = \frac{1}{x_2 \, x_1}$$

$$k = \frac{1}{1-0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{du}{dx}\Big|_{x_2} \end{bmatrix} + \frac{\rho \omega^2}{E}\int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx$$

$$= \begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{M\omega^2 L}{EA} \end{bmatrix} + \frac{\rho \omega^2}{E}\begin{bmatrix} \frac{x_2^3}{6} - \frac{x_2 x_1^2}{2} + \frac{x_1^3}{3} \\ \frac{x_2^3}{3} - \frac{x_1 x_2^2}{2} + \frac{x_1^3}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{M\omega^2 L}{EA} \end{bmatrix} + \frac{\rho \omega^2}{E}\begin{bmatrix} \frac{1}{6} - 0 + 0 \\ \frac{1}{3} - 0 + 0 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{du}{dx}\Big|_{x_1} \\ \frac{M\omega^2 L}{EA} \end{bmatrix} + \frac{\rho \omega^2}{E}\begin{bmatrix} \frac{1}{6} - 0 + 0 \\ \frac{1}{3} - 0 + 0 \end{bmatrix}$$

$$u_2 = \frac{M\omega^2 L}{EA} + \frac{\rho \omega^2}{E}\left(\frac{1}{3}\right)$$

$$= \frac{(1 \cdot 0.01^2 \cdot 0.1)(1000)^2(1)}{70E9 \cdot (0.01)^2} + \frac{(2700)(1000)^2}{70E9}\left(\frac{1}{3}\right)$$

$$= 0.0167 \text{ m}$$

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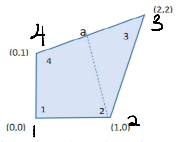
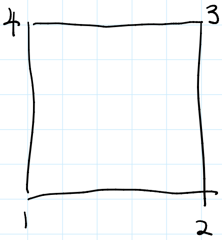
3)

1. A single four noded quadrilateral element is shown below:
(a) Compute the Jacobian determinant in terms of natural coordinates ($\xi,\eta$)
(b) Compute the derivative $dN_2/dy$ at node 2
(c) Compute the integral $\iint N_1 N_2 dA$ using two point integration rule
(d) Compute the integral $\int N_2 dl$ along the line joining nodes 2 and 3 using one point integration
(e) Compute the integral $\int N_2 dl$ along the line joining node 2 and the mid point of line 3-4 as shown in the figure.




a)




$$N_1 = \frac{(\eta-1)(\xi-1)}{4}$$

$$N_2 = \frac{(\xi+1)(\eta-1)}{4}$$

$$N_3 = \frac{(\xi+1)(\eta+1)}{4}$$

$$N_4 = (\xi-1)(\eta+1)/4$$

$$J = \frac{dN}{d\xi} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

from code "Quad.m"

$$J = \begin{bmatrix} \frac{\eta+3}{4} & \frac{\eta+1}{4} \\ \frac{\xi+1}{4} & \frac{\xi+3}{4} \end{bmatrix}$$

b) At node 2: $[\xi, \eta] = [1, -1]$

$$\frac{\partial N}{\partial x} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix}$$

$$J \cdot \frac{\partial N}{\partial x} = \frac{\partial N}{\partial \xi}$$

$$\frac{\partial N}{\partial x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0.5 & -1 & 0.5 & 0 \end{bmatrix}$$

$$\frac{\partial N_2}{\partial y} = -1$$

c)

$$\iint N_1 N_2 dA$$

integral points $\left( \left[-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right], \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right], \left[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right], \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \right)$

$$\iint N_1 N_2 \, dA = \sum_{i=1}^{4} w N_1(\text{int point}_i) N_2(\text{int point}_i) \det J_i$$

where $w = 1$.

We calculate in MATLAB, & obtain:

$$\iint N_1 N_2 \, dA = 0.0972$$

d) $\int N_2 \, dl$ along $L_{23}$

$$\int N_2 \det J^* \, d\eta$$

$$J = \begin{bmatrix} \frac{\eta+3}{4} & \frac{\eta+1}{4} \\ \frac{\xi+1}{4} & \frac{\xi+3}{4} \end{bmatrix}$$

$$\det J^* = \sqrt{\left(\frac{\xi+1}{4}\right)^2 + \left(\frac{\xi+3}{4}\right)^2}$$

1 point integration,

int point $= [1, 0]$

$$\int N_2 \det(J)^* \, d\eta = 2 N_2(\text{intpoint}) \det(J^*)$$

$$= 0.6250$$

e) The coordinate are:

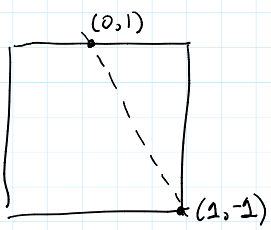
$a = (1, 1.5)$

$b = (1, 0)$

The equation of line passing through is

$x = 1$.

In reference coordinate:




$\eta = -\xi + 1$

By formula, we know

$\int \quad \int^{\xi_b} \quad dl$

By formula, we know

$$\int_{l_{a\to b}} N_1 \, dl = \int_{\xi_a}^{\xi_b} N_2 \frac{dl}{d\xi} \, d\xi$$

$$\frac{dl}{d\xi} = \sqrt{\left(\frac{\partial x}{\partial \xi} + \frac{\partial x}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2}$$

From $J$

$\frac{\partial \eta}{\partial \xi}$ from line:

$$\eta = -\xi + 1$$
$$d\eta = -d\xi$$
$$\frac{d\eta}{d\xi} = -1$$

The midpoint of the line in reference coordinate is:

$$(0.5, 0)$$

$$J = \begin{bmatrix} 0.75 & 0.25 \\ 0.375 & 0.875 \end{bmatrix}$$

We can the proceed with 1 point integration:

$$2 \cdot N_2(\text{int point}) \sqrt{\left(\frac{\partial x}{\partial \xi} + \frac{\partial x}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2}$$

$$= 2 \cdot 0.375 \sqrt{(0.75 + 0.375(-1))^2 + (0.25 + 0.875(-1))^2}$$

$$= 0.5467$$

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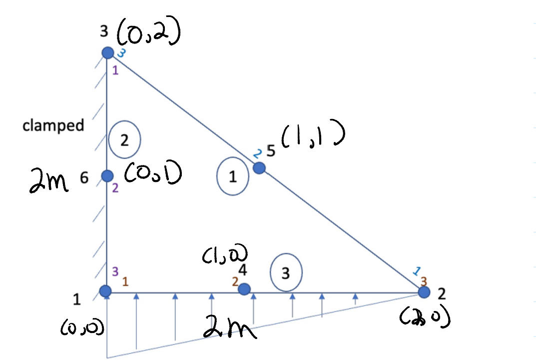
c)




a) We have here a 6-noded element.

We define the shape functions

$$N_1 = 1 - 3\xi + 2\xi^2 - 3\eta + 4\xi\eta + 2\eta^2$$
$$N_2 = -\xi + 2\xi^2$$

$N_1 = 1-3\xi+2\xi^2-3\eta+4\xi\eta+2\eta^2$

$N_2 = -\xi+2\xi^2$

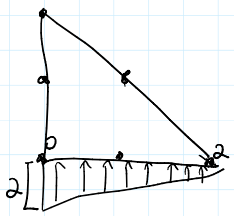
$N_3 = -\eta+2\eta$

$N_4 = 4\xi-4\xi^2-4\xi\eta$

$N_5 = 4\xi\eta$

$N_6 = 4\eta-4\xi\eta-4\eta^2$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix}$$



$$T = \begin{bmatrix} 0 \\ x+2 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 \\ (x_1N_1+x_2N_2+x_3N_3+x_4N_4+x_5N_5+x_6N_6)+2 \end{bmatrix}$$

$$f_{traction} = \int_{\Gamma^e} N^T T t^e \det J^* \cdot \sqrt{d\xi^2+d\eta^2}$$

We are on line $\eta=0$

$$\sqrt{d\xi^2+0} = d\xi$$

$$\boxed{f_{traction} = \int_{\Gamma^e} N^T T(1) \det J^* \, d\xi}$$

$$N^T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ \cancel{N_3} & \cancel{0} \\ \cancel{0} & \cancel{N_3} \\ N_4 & 0 \\ 0 & N_4 \\ \cancel{N_5} & \cancel{0} \\ \cancel{0} & \cancel{N_5} \\ \cancel{N_6} & \cancel{0} \\ \cancel{0} & \cancel{N_6} \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ 0 & 0 \\ 0 & 0 \\ N_4 & 0 \\ 0 & N_4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b) There are 3 points along the line, therefore, the function is quadratic and we need 2 points to integrate.

c) int point at
$[\frac{1}{3}, 0]$ & $[\frac{2}{3}, 0]$

c) int point at
$\left[\frac{1}{3}, 0\right]$ & $\left[\frac{2}{3}, 0\right]$

We compute with MATLAB & obtain

refer to Q2.m

$$F = \begin{bmatrix} 0 \\ 21.3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2.67 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Extra Credit: T
F
T

```
clc
clear
close all

E = 70E9;
w = 1000;
L = 1;
A = 0.01^2;
rho = 2700;
M = 0.1.*rho*L*A

M*w^2*L./(E*A) + rho*w^2./E*(1/3)
```

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```
clc
clear
close all
```

# 6 noded Tri

```
syms xi eta


N_1 = 1-3*xi+2*xi^2-3*eta+4*eta*xi+2*eta^2;
N_2 = -xi + 2*xi^2;
N_3 = -eta + 2*eta;
N_4 = 4*xi-4*xi^2-4*xi*eta;
N_5 = 4*xi*eta;
N_6 = 4*eta-4*xi*eta - 4*eta^2;
N_mat = [N_1, N_2, N_3, N_4, N_5, N_6];

N = [N_1 0 N_2 0 N_3 0 N_4 0 N_5 0 N_6 0
         0 N_1 0 N_2 0 N_3 0 N_4 0 N_5 0 N_6 ];
coords = [0,0;
          2,0;
          0,2;
          1,0;
          1,1;
          0,1];



intpoint = [1/3, 0;
            2/3, 0];

F = zeros(2*6,1);
for i = 1:2
    detJstar = element_tri(intpoint(i,1),intpoint(i,2),coords);

    N = [N_1 0 N_2 0 0 0 N_4 0 0 0 0 0
         0 N_1 0 N_2 0 0 0 N_4 0 0 0 0 ];
    N = subs(N,[xi,eta],[intpoint(i,1),intpoint(i,2)]);
    N = double(N);

    T = [0;(coords(1,1)*N_1 + coords(2,1)*N_2 + coords(3,1)*N_3 +
 coords(4,1)*N_4 + coords(5,1)*N_5 + coords(6,1)*N_6)];
    T = subs(T,[xi,eta],[intpoint(i,1),intpoint(i,2)]);
    T = double(T);

    temp = N'*T.*detJstar;
    F = F + temp;

end

function [detJstar] = element_tri(xi_num, eta_num, coords) %hw6, p1
    syms xi eta
```

```
    N_1 = 1-3*xi+2*xi^2-3*eta+4*eta*xi+2*eta^2;
    N_2 = -xi + 2*xi^2;
    N_3 = -eta + 2*eta;
    N_4 = 4*xi-4*xi^2-4*xi*eta;
    N_5 = 4*xi*eta;
    N_6 = 4*eta-4*xi*eta - 4*eta^2;
    N_mat = [N_1, N_2, N_3, N_4, N_5, N_6];



    dNdxi = [diff(N_mat,xi); diff(N_mat,eta)];
    dNdxi = subs(dNdxi,[xi,eta],[xi_num,eta_num]);
    dNdxi = double(dNdxi);

    J = dNdxi*coords;
    detJstar = det(J);
%     detJstar = sqrt(J(1,1)^2 + J(1,2)^2);
end
```

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```
clc
clear
close all

coords = [0,0;
          1,0;
          2,2;
          0,1];
element(1,-1,coords)

intpoint = [-1/sqrt(3), -1./sqrt(3);
            -1/sqrt(3), 1/sqrt(3);
            1/sqrt(3), -1/sqrt(3);
            1/sqrt(3), 1/sqrt(3)];
out = 0;
for i = 1:4
    [Ni, J, B] = element(intpoint(i,1), intpoint(i,2), coords);
    temp = Ni(1).*Ni(2).*det(J);
    out = out + temp;
end

%Line Integral
[Ni, J, B] = element(1, 0, coords);
out = Ni(2).*det(J);

out = 2*out;

%Q3e
intpoint = [0.5,0];
[Ni, J, B] = element(0.5, 0, coords);

2.*0.375.*sqrt((0.75-0.375)^2 + (0.25 - 0.875)^2)



%Q2;
```

# Functions

```
function [Ni, J, B] = element(xi, eta, coords) %hw6, p1
    Ni = 0.25*[(1-xi)*(1-eta), (1+xi)*(1-eta), (1+xi)*(1+eta), (1-
xi)*(1+eta)];
    N = [Ni(1) 0 Ni(2) 0 Ni(3) 0 Ni(4) 0
         0 Ni(1) 0 Ni(2) 0 Ni(3) 0 Ni(4) ];

    dNdxi = 0.25*[ eta-1 1-eta 1+eta -1-eta ; xi-1 -1-xi xi+1 1-xi ];
    J = dNdxi*coords;
    dN = J \ dNdxi;
    B = [dN(1, 1) 0 dN(1, 2) 0 dN(1, 3) 0 dN(1, 4) 0
        0 dN(2, 1) 0 dN(2, 2) 0 dN(2, 3) 0 dN(2, 4)
        dN(2, 1) dN(1, 1) dN(2, 2) dN(1, 2) dN(2, 3) dN(1, 3) dN(2, 4) dN(1,
 4)
```

```
    ];
end
```

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```
clc
clear
close all

n_1 = [1,-0.2];
n_2 = [4,0];
n_3 = [1,0.2];


nu = 0.3;
E  = 70E9;

D = E./(1-nu^2).*[1, nu, 0;
                nu, 1, 0;
                0, 0, (1-nu)./2];
t = 10/1E3;

syms xi eta

N_1 = xi;
N_2 = eta;
N_3 = 1-xi-eta;

N = [N_1, 0, N_2, 0, N_3, 0;
     0, N_1, 0, N_2, 0, N_3];

rho = 2700;
w = 1000;
```

# Computation B, K

```
x = [n_1(1);n_2(1);n_3(1)];
y = [n_1(2);n_2(2);n_3(2)];

A = (1/2).*(x(1).*(y(2)-y(3)) + x(2).*(y(3)-y(1)) + x(3).*(y(1)-y(2)));


x_13 = x(1) - x(3);
y_23 = y(2) - y(3);

x_23 = x(2) - x(3);
y_13 = y(1) - y(3);

B = [y(2)-y(3), 0, y(3)-y(1), 0, y(1)-y(2), 0;
     0, x(3)-x(2), 0, x(1)-x(3), 0, x(2)-x(1);
     x(3)-x(2), y(2)-y(3), x(1)-x(3), y(3)-y(1), x(2)-x(1), y(1)-y(2)];
B = 1./(x_13.*y_23 - x_23.*y_13).*B;

K = B'*D*B.*t.*A;
```

## Change the following for force calculations

```
F_b = [rho.*w.^2.*(N_1.*x(1) + N_2.*x(2) + N_3.*x(3));
       0];

F = double(int(int(N'*F_b.*(2.*A).*t,eta,0,1-xi),xi,0,1))

K(3:4,3:4)\F(3:4)
```

```
F =

     9450000
           0
    13500000
           0
     9450000
           0


ans =

    0.2632
         0
```

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