Homework 4

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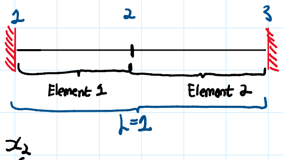
Problem 1. Solve the problem described by the following equation (strong form) using Galerkin finite element approach:

$\frac{d^2u}{dx^2} = -x$, 0<x<1; u(0) = 0, u(1) = 0

Derive the element stiffness matrix and force vector for a 2 noded one-dimensional element. Use two linear elements, write down the final assembled matrix and report the value of u at x=0.5.

We start with the weak form & work forward.



$$\int_{x_1}^{x_2} \left(\frac{\partial^2 u}{\partial x^2} + x\right) w \, dx = 0$$

using u-v rule:

$$u'w' = (u'w)' - u''w$$

$$\int_{x_1}^{x_2} \left(\left((u'w)' - u'w'\right) + xw\right) dx = 0$$

By divergence theorem:

$$\int_{x_1}^{x_2} (u'w)' \, dx = u'w\Big|_{x_1}^{x_2}$$

This then lead us to the reduced weak form:

$$\int_{x_1}^{x_2} \left(\left(u'w\Big|_{x_1}^{x_2}\right) - u'w + xw \right) dx = 0$$

$$u'w\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} (u'w) dx + \int_{x_1}^{x_2} (xw) dx = 0$$

We then have:

$$u = N_1 u_1 + N_2 u_2 \qquad w = N_1 w_1 + N_2 w_2$$

$$u' = \frac{\partial u}{\partial x} = N_1' u_1 + N_2' u_2 \qquad w' = N_1' w_1 + N_2' w_2$$

We examine the first term

First Term:

$$u'w = u'(x_2)w(x_2) - u'(x_1)w(x_1)$$

$$= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} -u_1' \\ u_2' \end{bmatrix}$$

Second Term:

$$\int_{x_1}^{x_2} (u'w) dx = \int_{x_1}^{x_2} (N_1' u_1 + N_2' u_2)(N_1 w_1 + N_2 w_2) \, dx$$

$$= [w_1 \quad w_2] \left(\int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} [N_1' \quad N_2'] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right)$$

Third Term:

$$\int_{x_1}^{x_2} x w \, dx = \int_{x_1}^{x_2} x (N_1 w_1 + N_2 w_2) \, dx$$

$$= [w_1 \quad w_2] \left[\int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx \right]$$

re-assembling the terms, we then have:

$$0 = \cancel{[w_1 \quad w_2]} \left(\begin{bmatrix} -u_2' \\ u_2' \end{bmatrix} - \left(\int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} [N_1' \quad N_2'] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) + \left[\int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx \right] \right)$$

$$= \left(\begin{bmatrix} -u_2' \\ u_2' \end{bmatrix} - \left(\int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} [N_1' \quad N_2'] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) + \left[\int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx \right] \right)$$

$$\left(\int_{x_1}^{x_2} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} [N_1' \quad N_2'] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) = \left(\begin{bmatrix} -u_2' \\ u_2' \end{bmatrix} + \left[\int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx \right] \right)$$

$$\underbrace{\int_{x_1}^{x_2} \begin{bmatrix} N_1'^2 & N_1' N_2' \\ N_2' N_1' & N_2'^2 \end{bmatrix}}_{\text{Local Stiffness Matrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\left(\begin{bmatrix} -u_2' \\ u_2' \end{bmatrix} + \left[\int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} x \, dx \right] \right)}_{\text{Local Force Vector}}$$

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \int_{x_1}^{x_2} \overset{1}{N_1'^2} \, dx & \int_{x_1}^{x_2} \overset{1}{N_1'} \overset{1}{N_2'} \, dx & 0 \\ \int_{x_1}^{x_2} \overset{1}{N_2'} \overset{1}{N_1'} \, dx & \int_{x_1}^{x_2} \overset{1}{N_2'^2} \, dx + \int_{x_2}^{x_3} \overset{2}{N_1'^2} \, dx & \int_{x_2}^{x_3} \overset{2}{N_1'} \overset{2}{N_2'} \, dx \\ 0 & \int_{x_2}^{x_3} \overset{2}{N_1'} \overset{2}{N_2'} \, dx & \int_{x_2}^{x_3} \overset{2}{N_2'^2} \, dx \end{bmatrix} \end{array} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \int_{x_1}^{x_2} x \overset{1}{N_1} \, dx \\ \int_{x_1}^{x_2} x \overset{1}{N_2} \, dx + \int_{x_2}^{x_3} x \overset{2}{N_1} \, dx \\ \int_{x_2}^{x_3} x \overset{2}{N_1} \, dx \end{bmatrix} + \begin{bmatrix} -\overset{1}{u_1'} \\ \overset{1}{u_2'} - \overset{2}{u_1'} \\ \overset{2}{u_2'} \end{bmatrix}$$

Based on the boundary condition, we know that:

$$-u_1^{(1)\prime} = \frac{R_1}{A}$$

$$-u_2^{(1)\prime} = \frac{R_3}{A}$$

In addition:

$$\left[\begin{array}{ll} N_1^{(2)} = \dfrac{x_{i+1}-x}{x_{i+1}-x_i} & N_1^{(2)\prime} = \dfrac{-1}{L} = \dfrac{-1}{\left(\frac{1}{2}\right)} = -2 \\ N_2^{(2)} = \dfrac{x-x_i}{x_{i+1}-x_i} & N_2^{(2)\prime} = \dfrac{1}{L} = \dfrac{1}{\left(\frac{1}{2}\right)} = 2 \end{array}\right]$$

We may now calculate the displacement at $x=0.5$ or u_2.

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \int_{x_1}^{x_2} N_1^{(1)\prime 2}\,dx & \int_{x_1}^{x_2} N_1^{(1)\prime} N_2^{(1)\prime}\,dx & 0 \\ \int_{x_1}^{x_2} N_2^{(1)\prime} N_1^{(1)\prime}\,dx & \int_{x_1}^{x_2} N_2^{(1)\prime 2}\,dx + \int_{x_2}^{x_3} N_1^{(2)\prime 2}\,dx & \int_{x_2}^{x_3} N_1^{(2)\prime} N_2^{(2)\prime}\,dx \\ 0 & \int_{x_2}^{x_3} N_1^{(2)\prime} N_2^{(2)\prime}\,dx & \int_{x_2}^{x_3} N_2^{(2)\prime 2}\,dx \end{bmatrix} \end{array} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \int_{x_1}^{x_2} x N_1^{(1)}\,dx \\ \int_{x_1}^{x_2} x N_2^{(1)}\,dx + \int_{x_2}^{x_3} x N_1^{(2)}\,dx \\ \int_{x_2}^{x_3} x N_1^{(2)}\,dx \end{bmatrix} + \begin{bmatrix} -u_1^{(1)\prime} \\ u_2^{(1)\prime} - u_1^{(2)\prime} \\ u_2^{(2)\prime} \end{bmatrix}$$

The resulting equation is:

$$\left(\int_{x_1}^{x_2} N_2^{(1)\prime 2}\,dx + \int_{x_2}^{x_3} N_1^{(2)\prime 2}\,dx \right) u_2 = \int_{x_1}^{x_2} x N_2^{(1)}\,dx + \int_{x_2}^{x_3} x N_1^{(2)}\,dx$$

$$\left[\begin{array}{l} (2+2)u_2 = \dfrac{1}{12} + \dfrac{1}{6} \\ \qquad u_2 = \dfrac{1}{16} = 0.0625 \end{array}\right]$$

$2x^2$

$\frac{2}{3}x^3$

$\frac{2}{3} -$