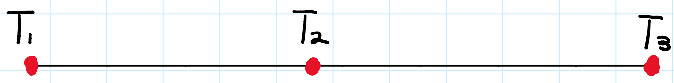
Homework 5

Thursday, February 23, 2023 9:03 PM



$$\int_{x_1}^{x_3}\left(k\frac{\partial^2 T}{\partial x^2}+Q\right)w\,dx=0$$

using u-v rule:

$$u'w' = (u'w)' - u''w$$

$$\int_{x_1}^{x_3}\left(k\left((T'w)' - T'w'\right)+Qw\right)dx = 0$$

By divergence theorem:

$$\int_{x_1}^{x_2}(T'w)'\,dx = T'w\Big|_{x_1}^{x_2}$$

This then lead us to the reduced weak form:

$$\int_{x_1}^{x_2}\left(k\left(T'w\Big|_{x_1}^{x_2}\right) - kT'w + Qw\right)dx = 0$$

$$kT'w\Big|_{x_1}^{x_2} - k\int_{x_1}^{x_2}(T'w)\,dx + \int_{x_1}^{x_2}(Tw)\,dx = 0$$

We then have:

$$u = N_1T_1 + N_2T_2 + N_3T_3 \qquad w = N_1w_1 + N_2w_2 + N_3w_3$$

$$u' = \frac{\partial u}{\partial x} = N_1'T_1 + N_2'T_2 + N_3'T_3 \qquad w' = N_1'w_1 + N_2'w_2 + N_3'w_3$$

We examine the first term

First Term:

$$ku'w\Big|_{x_1}^{x_3} = T'(x_3)w(x_3) - T'(x_1)w(x_1)$$

$$= k\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}\begin{bmatrix} -T_1' \\ 0 \end{bmatrix}$$

$$= k \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} -T_1' \\ 0 \\ T_3' \end{bmatrix}$$

Second Term:

$$k\int_{x_1}^{x_3} (u'w)\,dx = k\int_{x_1}^{x_3} k\,(N_1'T_1 + N_2'T_2 + N_3'T_3)(N_1'w_1 + N_2'w_2 + N_3'w_3)\,dx$$

$$= k\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \left(\int_{x_1}^{x_3} \begin{bmatrix} N_1' \\ N_2' \\ N_3' \end{bmatrix} \begin{bmatrix} N_1' & N_2' & N_3' \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \right)$$

Third Term:

$$\int_{x_1}^{x_2} Qw\,dx = \int_{x_1}^{x_2} Q\,(N_1w_1 + N_2w_2 + N_3w_3)\,dx$$

$$= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \left[\int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} Q\,dx \right]$$

re-assembling the terms, we then have:

$$\underbrace{k\int_{x_1}^{x_3} \begin{bmatrix} N_1'^2 & N_1'N_2' & N_1'N_3' \\ N_1'N_2' & N_2'^2 & N_2'N_3' \\ N_1'N_3' & N_2'N_3' & N_3'^2 \end{bmatrix}}_{\text{Local Stiffness}} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \underbrace{\int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} Q\,dx + k \begin{bmatrix} -T_1' \\ 0 \\ T_3' \end{bmatrix}}_{\text{Local Force Vector}}$$

remains zero for interior elements

We have a boundary condition on both sides:

$$T'(x=0) = 0$$

$$T'(x = 12.5\,\text{cm}) = -\frac{h}{k}(T-30)$$

$$T'(x=12.5\,cm) = -\frac{h}{k}\underbrace{(T-30)}_{\hookrightarrow\, T(x=12.5\,cm)}$$

$$N_1'N_3'(T_1) + N_2'N_3'(T_2) + N_3'^2(T_3) = \int_{x_1}^{x_3} N_3\,Q\,dx + kT_3'$$

$$N_1'N_3'(T_1) + N_2'N_3'(T_2) + N_3'^2(T_3) = \int_{x_1}^{x_3} N_3\,Q\,dx - \cancel{k}\left(\frac{h}{\cancel{k}}(T_3-30)\right)$$

$$N_1'N_3'(T_1) + N_2'N_3'(T_2) + N_3'^2(T_3) - hT_3 = \int_{x_1}^{x_3} N_3\,Q\,dx + h\cdot 30$$

$$\boxed{N_1'N_3'(T_1) + N_2'N_3'(T_2) + T_3\underbrace{(N_3'^2+h)} = \int_{x_1}^{x_3} N_3\,Q\,dx + \underbrace{h\cdot 30}}$$

We modify the matrix and boundary conditions as follow