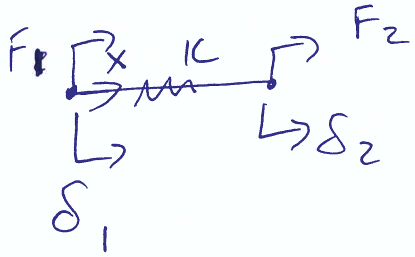
# 2D Trusses

## 1D Truss

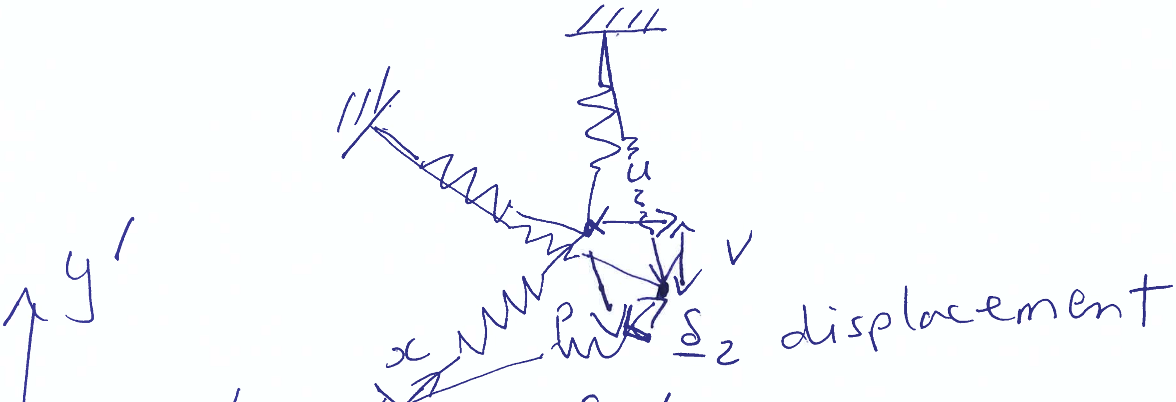


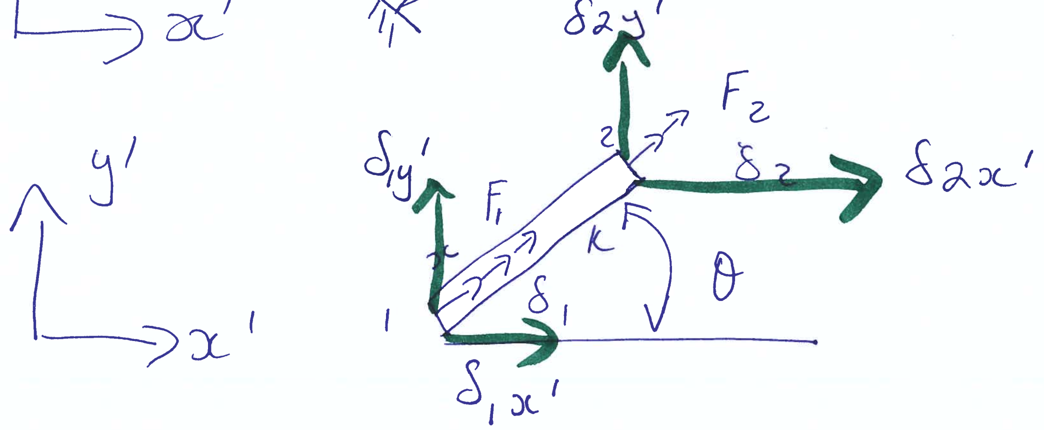

$$F_2 = k(\delta_2 - \delta_1)$$

$$F_1 = -F_2 \quad \text{(equilibrium)}$$

$$F_1 = k(\delta_1 - \delta_2)$$

## 2D Truss

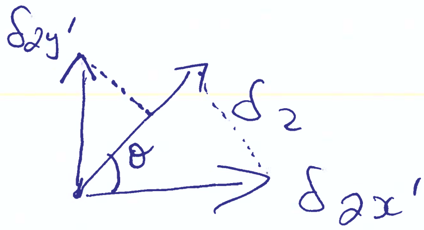



$$\boxed{\begin{aligned} F_2 &= k(\delta_2 - \delta_1) \\ F_1 &= k(\delta_1 - \delta_2) \end{aligned}}$$

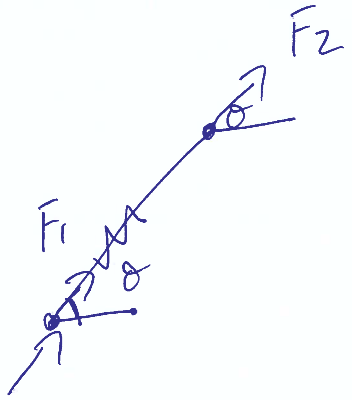
Force is only along the truss

$$\delta_2 = \delta_{2x'} \cos\theta + \delta_{2y'} \sin\theta$$




Similarly,

$$\delta_1 = \delta_{1x'} \cos\theta + \delta_{1y'} \sin\theta$$




Force on node 2 in $x'y'$ coordinate system

$$\underline{F_2} = \begin{bmatrix} F_2\cos\theta \\ F_2\sin\theta \end{bmatrix} = \begin{bmatrix} F_{2x'} \\ F_{2y'} \end{bmatrix}$$

Similarly

$$\underline{F_1} = \begin{bmatrix} F_1\cos\theta \\ F_1\sin\theta \end{bmatrix} = \begin{bmatrix} F_{1x'} \\ F_{1y'} \end{bmatrix}$$

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}}_{T \text{ matrix}} \begin{bmatrix} \delta_{1x'} \\ \delta_{1y'} \\ \delta_{2x'} \\ \delta_{2y'} \end{bmatrix}$$

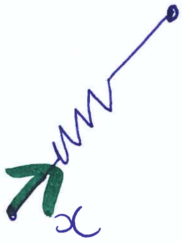
Eq. 1.

$$\begin{bmatrix} F_{1x'} \\ F_{1y'} \\ F_{2x'} \\ F_{2y'} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}}_{T^T} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Eq. 2.

$T$ : Transformation matrix

$T^T$ : transpose of $T$



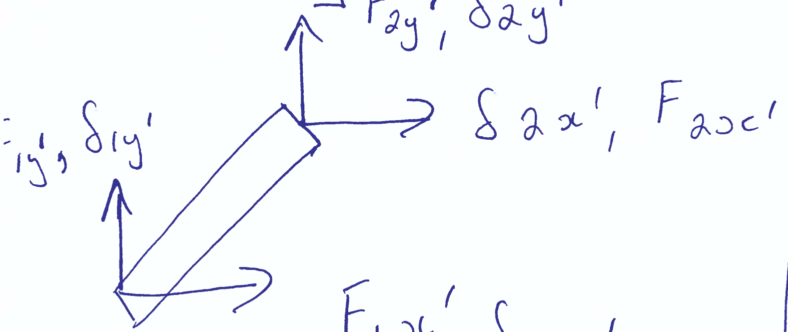

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{k \text{ matrix}} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

Using eq. 1.

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = K\, T \begin{bmatrix} \delta_{1x'} \\ \delta_{1y'} \\ \delta_{2x'} \\ \delta_{2y'} \end{bmatrix}$$

Using eq. 2.

$$\begin{bmatrix} F_{1x'} \\ F_{1y'} \\ F_{2x'} \\ F_{2y'} \end{bmatrix} = T^T \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = T^T K\, T \begin{bmatrix} \delta_{1x'} \\ \delta_{1y'} \\ \delta_{2x'} \\ \delta_{2y'} \end{bmatrix}$$




$$\begin{bmatrix} F_{1x'} \\ F_{1y'} \\ F_{2x'} \\ F_{2y'} \end{bmatrix} = \underbrace{\left[ T^T K\, T \right]}_{\substack{\text{new transformed} \\ \text{local stiffness matrix} \\ (4 \times 4) \text{ matrix}}} \begin{bmatrix} \delta_{1x'} \\ \delta_{1y'} \\ \delta_{2x'} \\ \delta_{2y'} \end{bmatrix}$$

$T^T K T$ : $[4\times2][2\times2][2\times4]$

: $4\times4$ matrix

stiffness matrix for 2D trusses

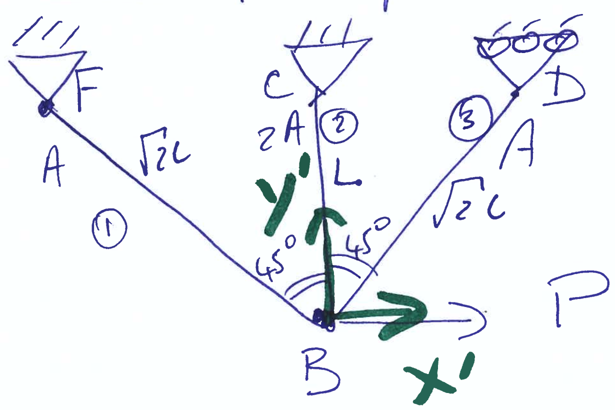
$$T^T K T = k\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

$s$ : $\sin\theta$

$c$ : $\cos\theta$

$k$ : $EA/L$

Example of a 2D Truss




Trusses are at $45^\circ$

3 Trusses ①, ②, ③

① : $E, A, \sqrt{2}L \Rightarrow K_1 = \dfrac{EA}{\sqrt{2}L}$

② : $E, 2A, L \Rightarrow K_2 = \dfrac{2EA}{L}$

③ : $E, A, \sqrt{2}L \Rightarrow K_3 = \dfrac{EA}{\sqrt{2}L}$

# Step 1 : Mesh

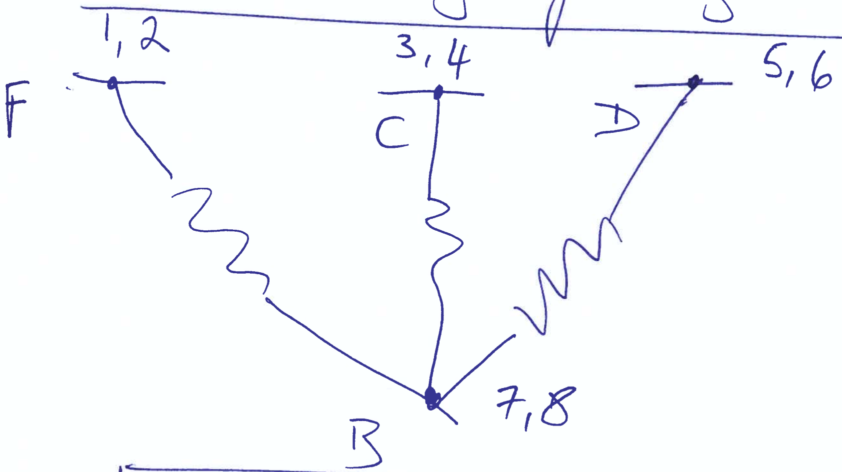
Coordinate matrix

$$\begin{array}{c|cc} & x' & y' \\ \hline B & 0 & 0 \\ F & -L & L \\ C & 0 & L \\ D & L & L \end{array}$$

connectivity matrix

$$\begin{array}{c|cc} & \text{node 1} & \text{node 2} \\ \hline 1 & B & F \\ 2 & B & C \\ 3 & B & D \end{array}$$

## Numbering of degrees of freedom (dof)




1 : x - displacement of F

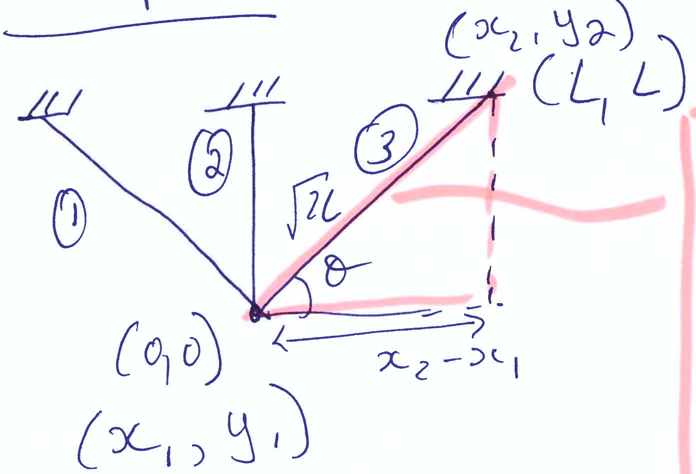
2 : y - displacement of F

| | | |
|---|---|---|
| B | 7 | 8 |
| C | 3 | 4 |
| D | 5 | 6 |
| F | 1 | 2 |
| | x-dof | y-dof |

← $u_{Dx}, u_{Dy}$

← $u_{Fx}, u_{Fy}$

# Step 2 assemble global matrix




$$c = \cos\theta = \frac{x_2 - x_1}{\sqrt{2}L} = \frac{L-0}{\sqrt{2}L}$$

$$s = \sin\theta = \frac{y_2 - y_1}{\sqrt{2}L} = \frac{L-0}{\sqrt{2}L}$$

$$T^T K T$$

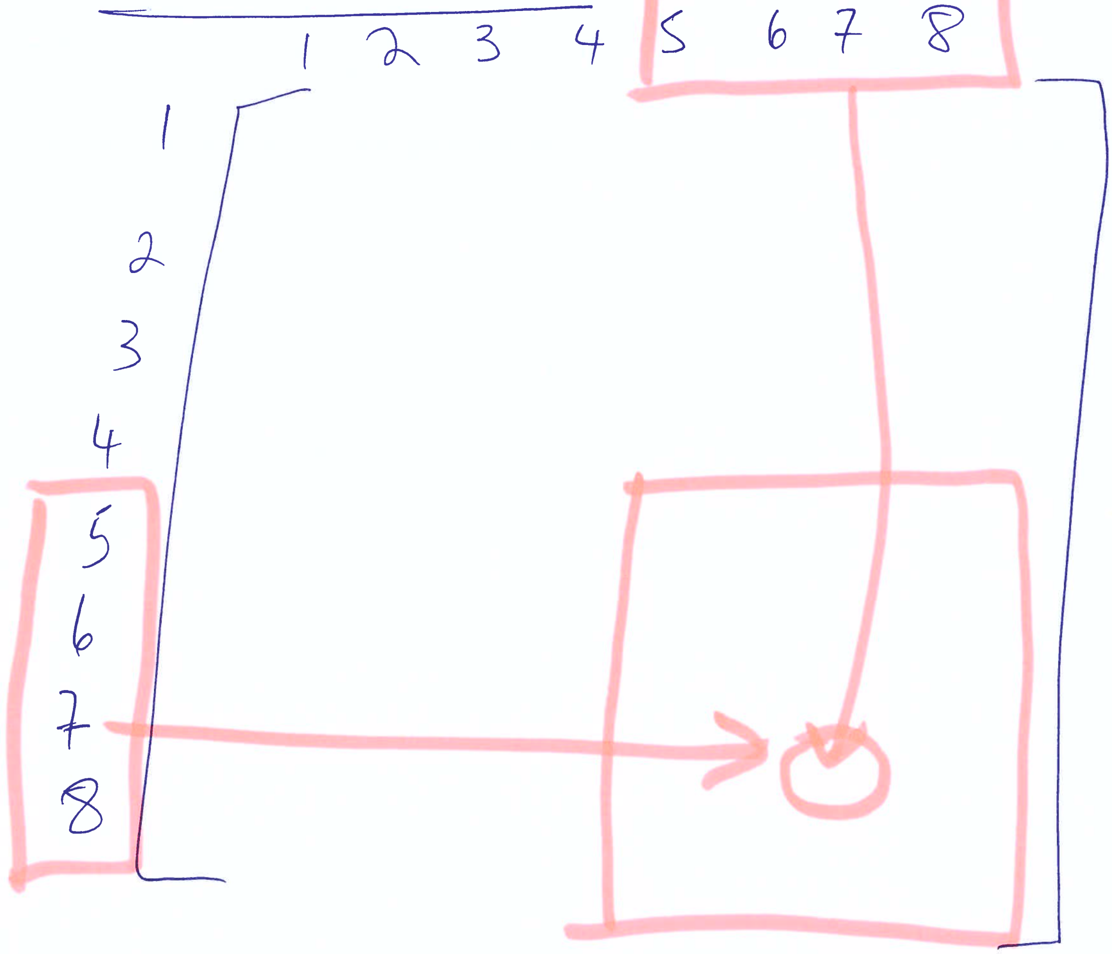
$$T = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \qquad K = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}$$

$$k_3 = \frac{EA}{\sqrt{2}L}$$

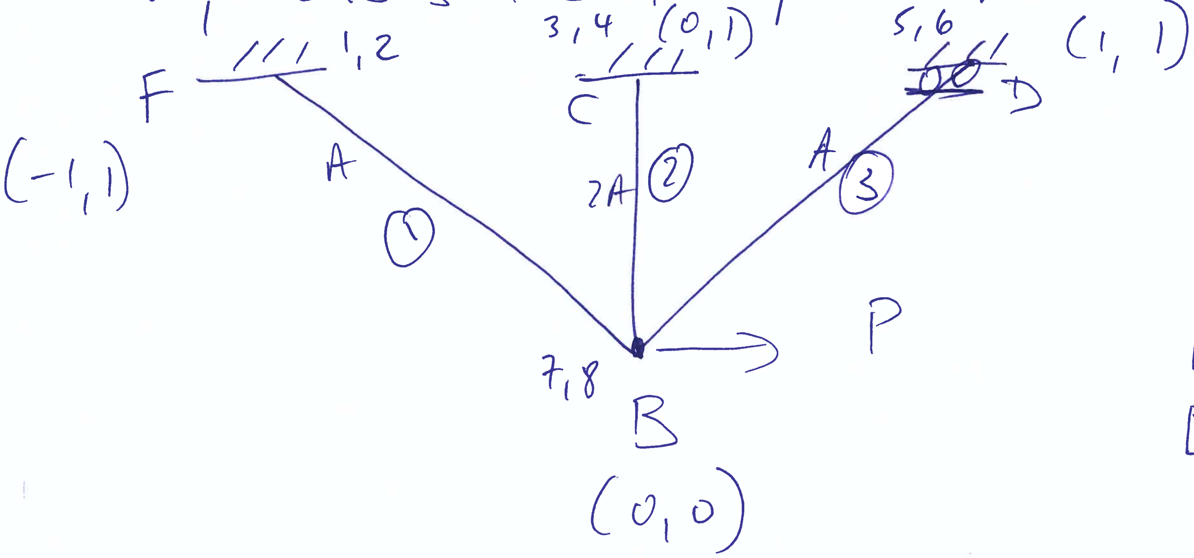
$$T^T K T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{EA}{\sqrt{2}L} & -\frac{EA}{\sqrt{2}L} \\ -\frac{EA}{\sqrt{2}L} & \frac{EA}{\sqrt{2}L} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{array}{ll} B & \begin{matrix} 7 \\ 8 \end{matrix} \\ D & \begin{matrix} 5 \\ 6 \end{matrix} \end{array} = \frac{EA}{\sqrt{2}L} \begin{array}{c} \begin{matrix} 7 & \;\;8 & \;\;5 & \;\;6 \end{matrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Global matrix




Now, let's solve the problem




$P = 10^3\ N$

$A = 10^{-2}\ m^2$

$E = 10^7\ Pa$

$L = 1$

Find displacement $u_{Bx}$, $u_{By}$, $u_{Dx}$