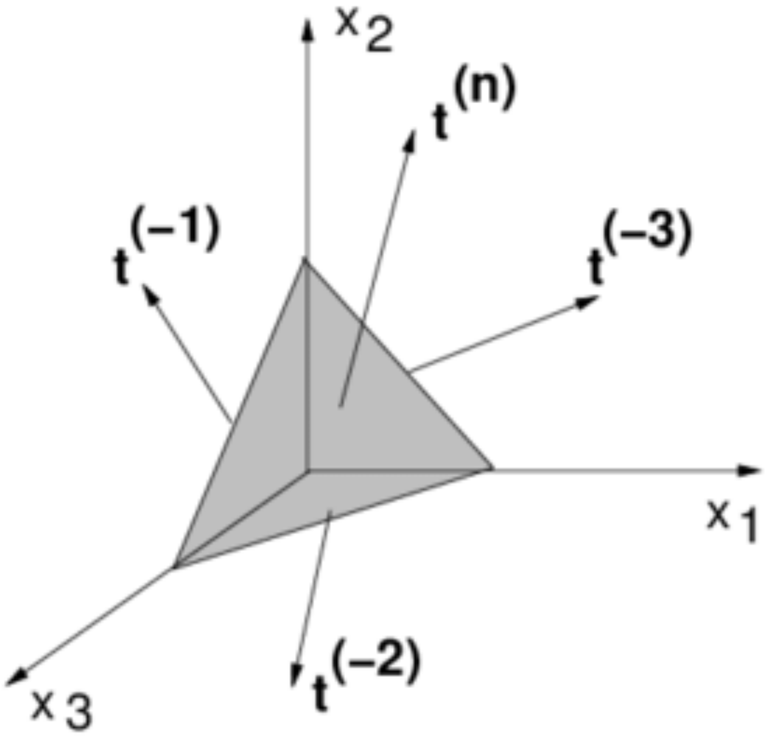
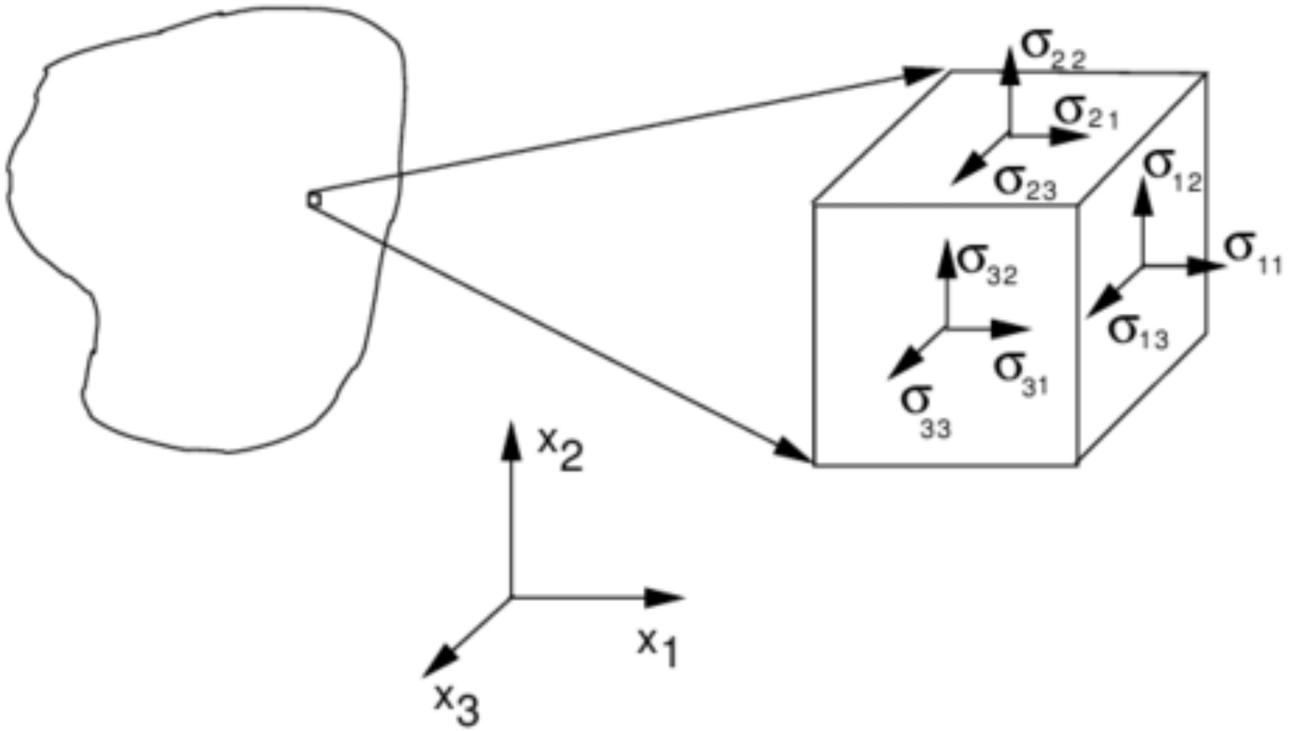
SOLID MECHANICS

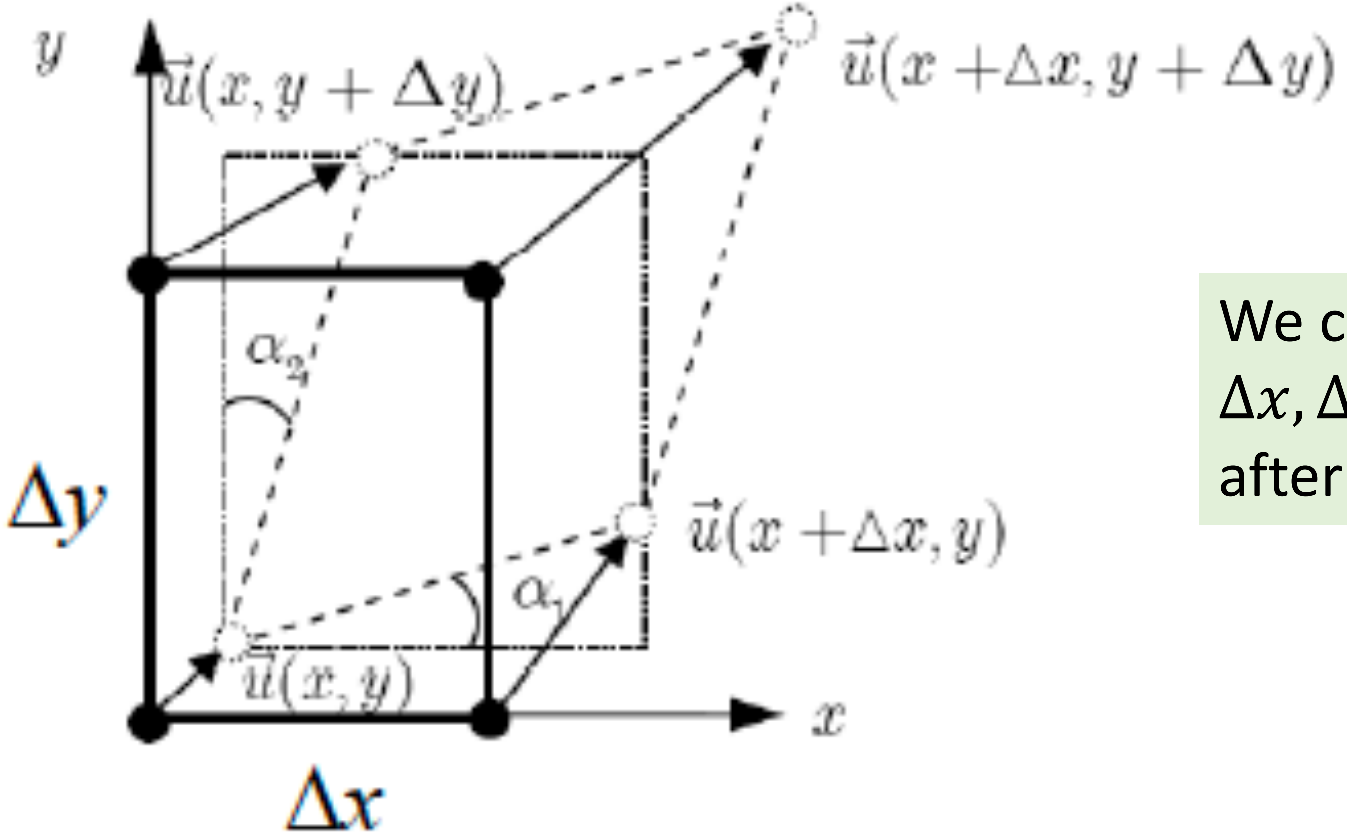
REVIEW OF UNDERGRAD SOLID MECHANICS (PREREQUISITES)





Kinematics in 2 Dimensions

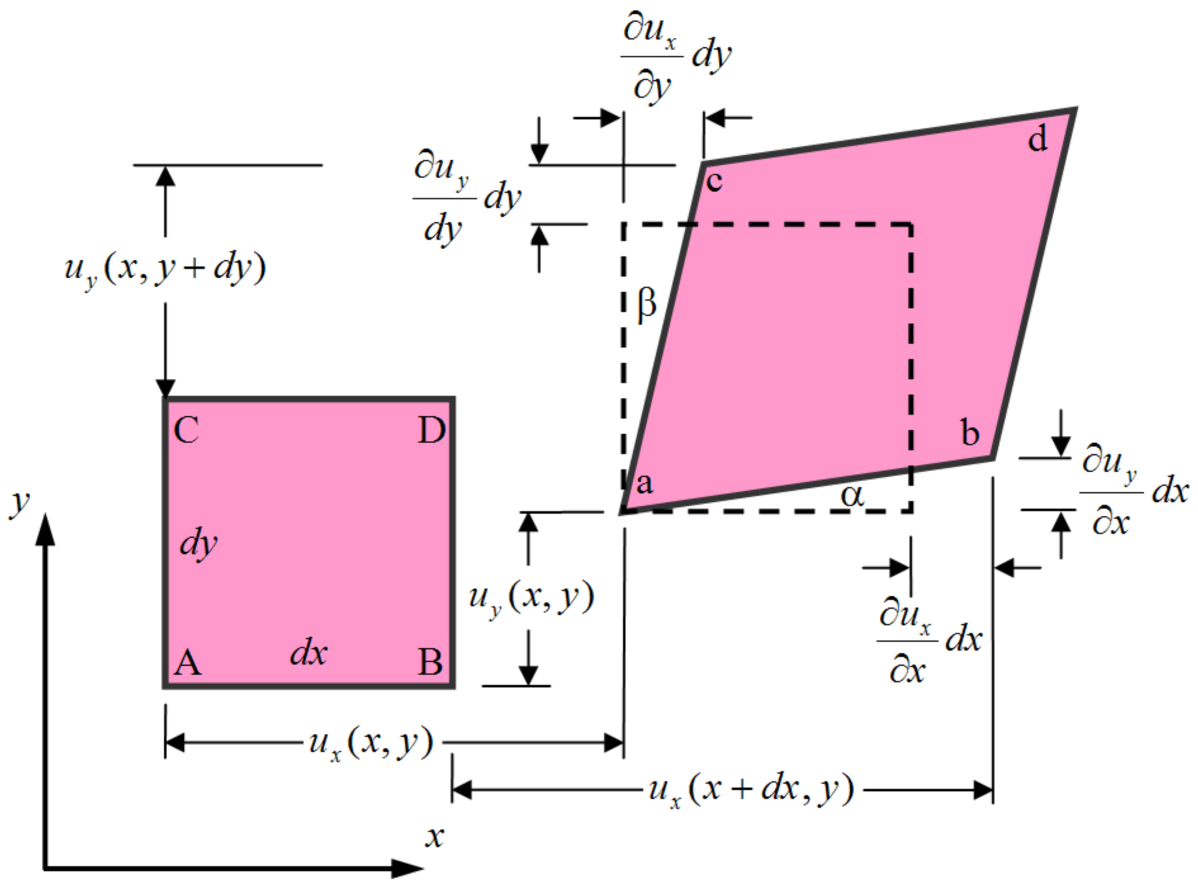
We introduced a displacement vector at a given point in 2D. It is a vector with x- and y-components. We write it using both matrix and vector notation.



$$u = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \qquad \vec{u} = u_x\vec{i} + u_y\vec{j}$$

We consider a control volume $\Delta x, \Delta y$ on the plane before and after deformation:

2D Strain: Extensional Strain Components



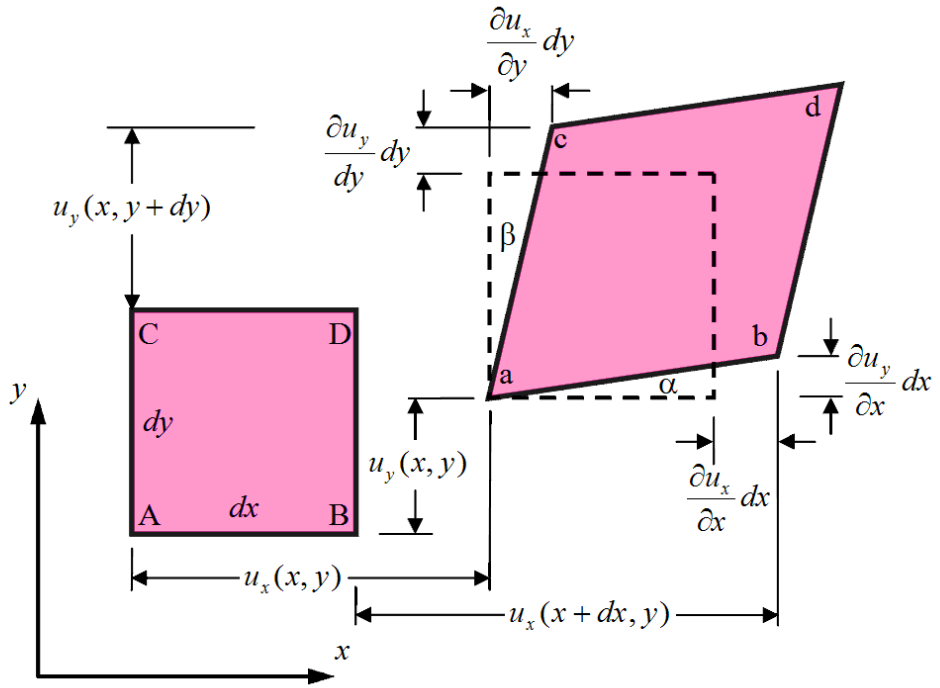
The extensional strains ε_{xx} and ε_{yy} are defined as:

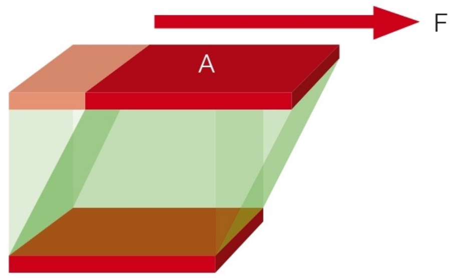
$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{u_x(x+\Delta x, y) - u_x(x, y)}{\Delta x} \equiv \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \lim_{\Delta y \to 0} \frac{u_y(x, y+\Delta y) - u_y(x, y)}{\Delta y} \equiv \frac{\partial u_y}{\partial y}$$

The extensional strains ε_{xx} and ε_{yy} represent the change in lengths of the infinitesimal line segments in the x and y directions, $\Delta x, \Delta y,$ (or dx,dy) divided by their original lengths

2D Strain: Shear Strain Components





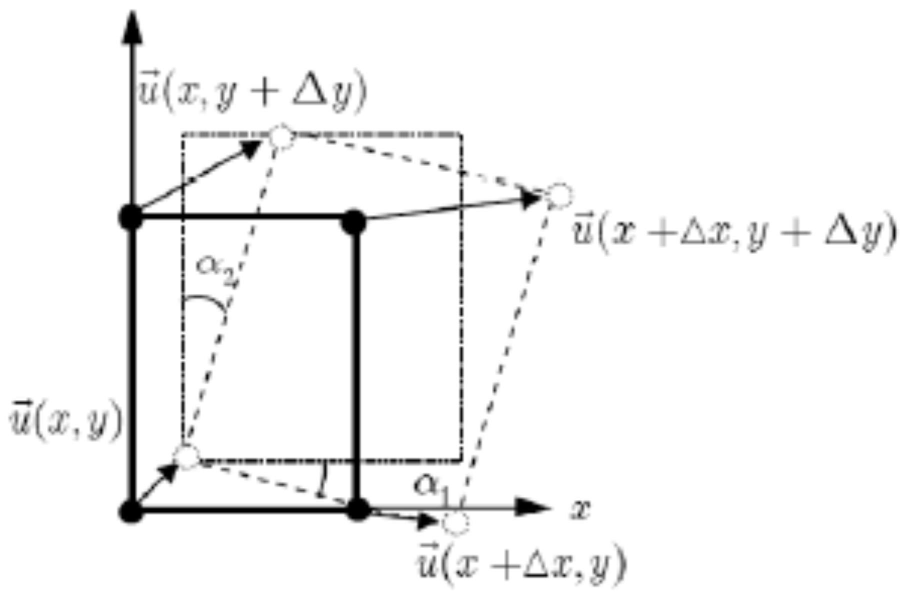
The shear strain γ_{xy} measures the change in angle $\alpha + \beta$ between unit vectors in the x and y (in radians)

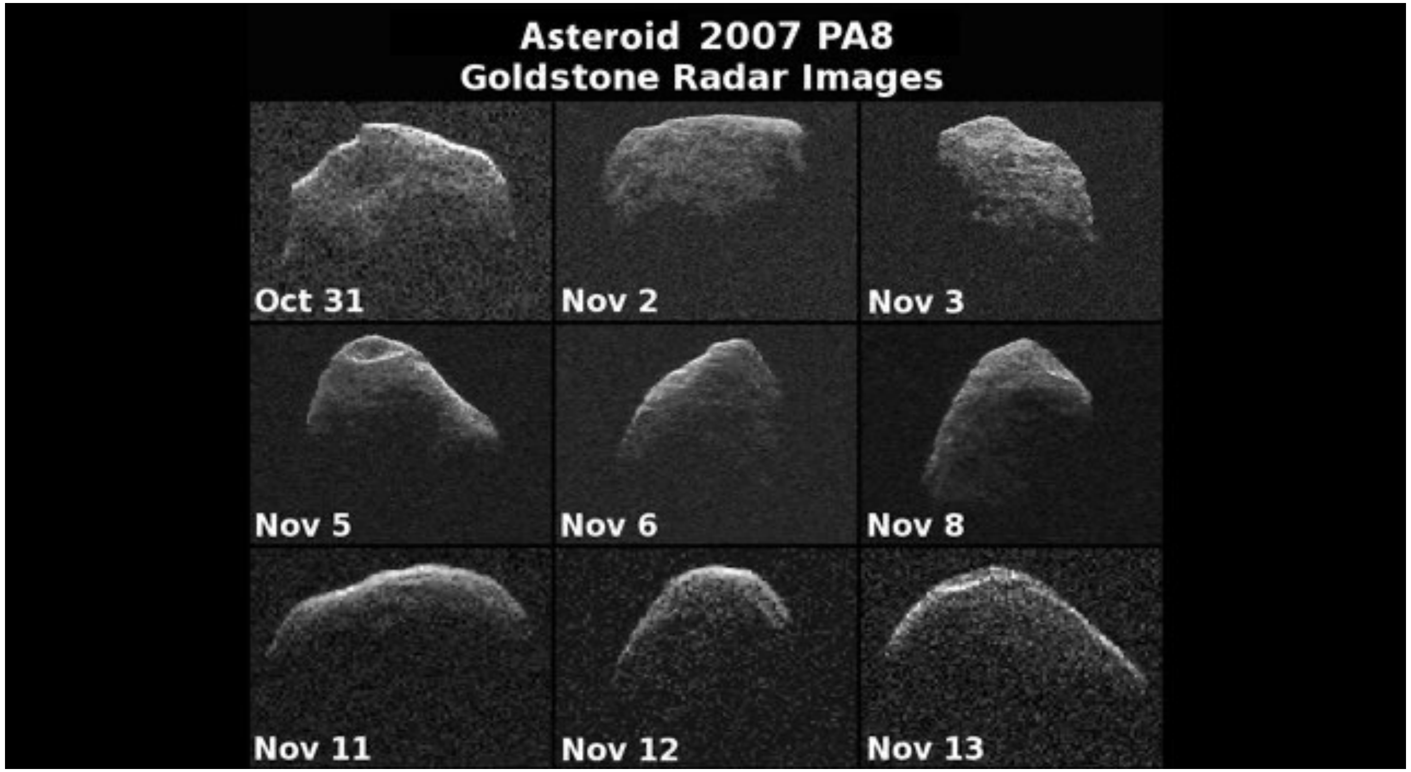
$$\gamma_{xy} = \lim_{\Delta x \to 0} \frac{u_y(x+\Delta x, y) - u_y(x, y)}{\Delta x} + \lim_{\Delta y \to 0} \frac{u_x(x, y+\Delta y) - u_x(x, y)}{\Delta y} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

In this course we use engineering strain.

Tensorial shear strain is $\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$

2D Strain: Rotation





In addition to axial elongations, the control volume can also undergo rotation. The rotation in 2D, $\omega_{xy}$is given as

$$\omega_{xy} \equiv \frac{1}{2}(\alpha_2 - \alpha_1) = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

For small deformations, ω_{xy} is small and does not affect the stress.

2D Strain: The Strain Matrix

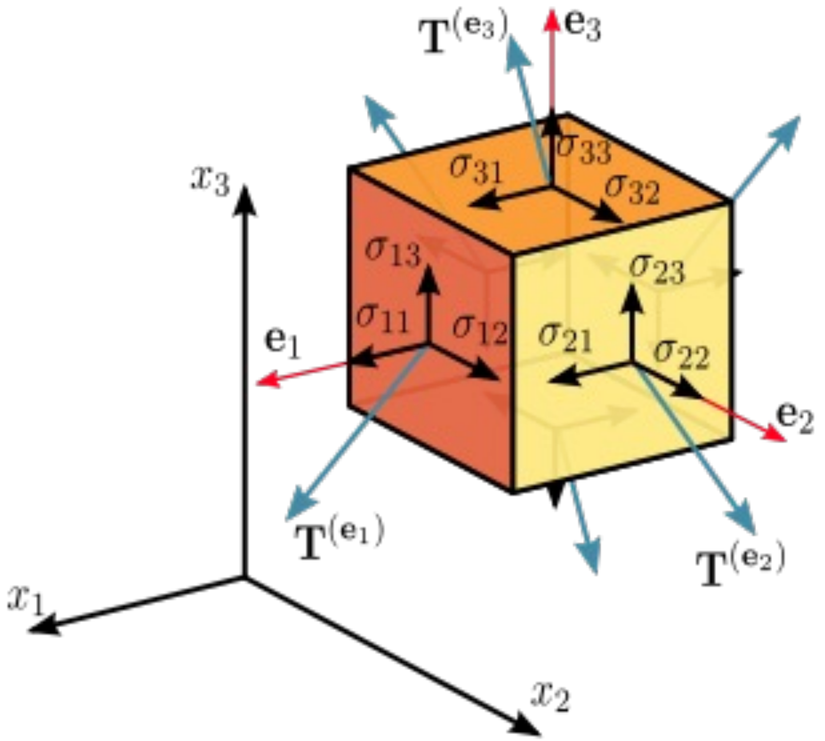
For our Finite Element Calculations, we will use the following notation for the strains

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}^T = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}}_{\nabla_s} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \nabla_S u$$

The gradient matrix ∇_S is defined as

$$\nabla_S = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

3D Traction vector and stress components



The traction is a force per unit area vector acting on a surface.

The orientation of the surface is denoted by unit normal $\vec{\mathbf{n}}$.

The components on the traction vector acting at a point on the plane with unit normal aligned in the x direction is $\vec{\boldsymbol{\sigma}}_x$ and its vector form is

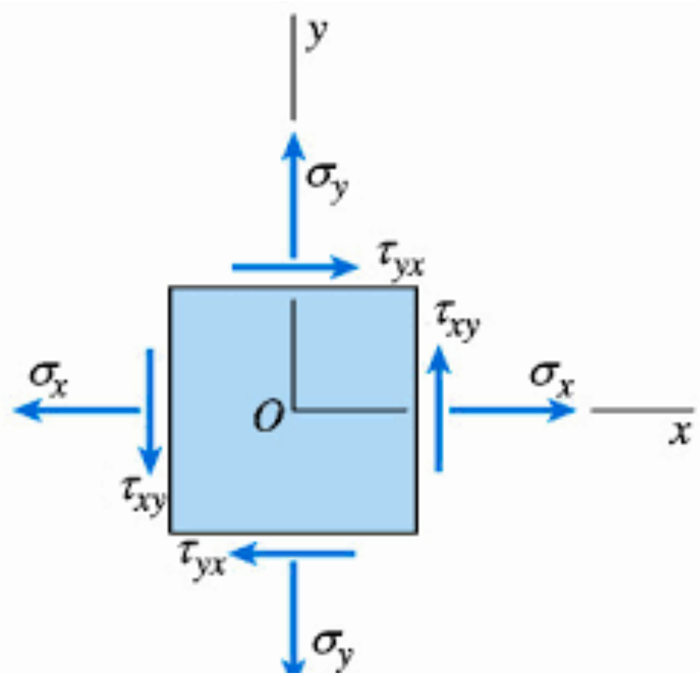
$$\vec{\boldsymbol{\sigma}}_x = \sigma_{xx}\,\vec{\mathbf{i}} + \sigma_{yy}\vec{\mathbf{j}}$$

1st component denotes normal to the plane

2nd component denotes direction of force

We usually drop the second subscript on the normal stresses.

2D Traction vector and stress components



The 2D stress state is described by 2 normal stresses σ_{xx}, σ_{yy} and shear stresses $\sigma_{xy} = \sigma_{yx}$.

The traction is a force per unit area vector acting on a surface. The orientation of the surface is denoted by unit normal $\vec{\mathbf{n}}$. The components on the traction vector acting at a point on the plane with unit normal aligned in the x direction is $\vec{\boldsymbol{\sigma}}_x$ and its vector form is

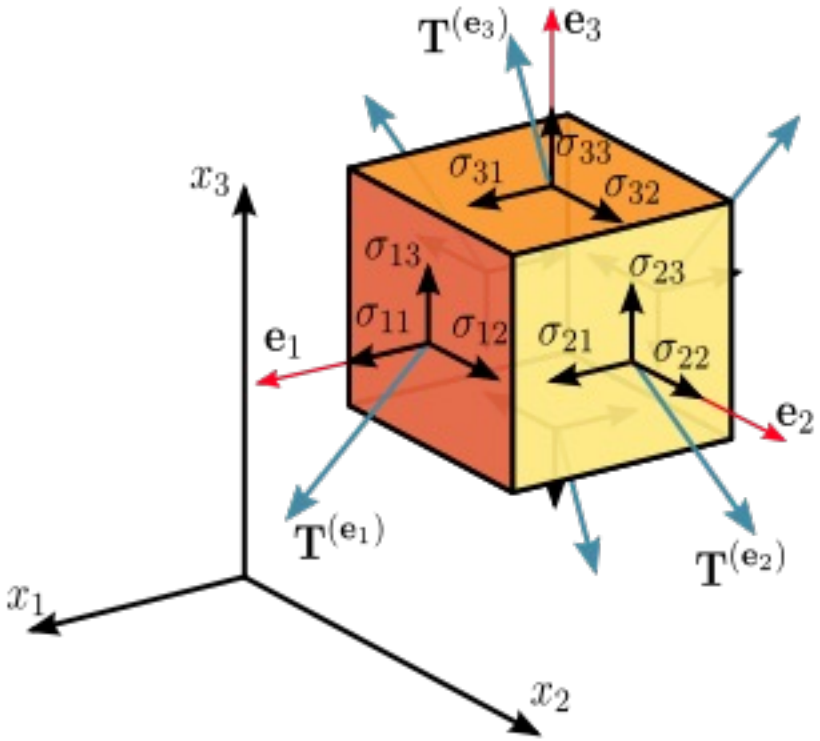
$$\vec{\boldsymbol{\sigma}}_x = \sigma_{xx}\,\vec{\mathbf{i}} + \sigma_{xy}\vec{\mathbf{j}}$$

The components of the traction vector acting on the plane with unit normal aligned in the y direction is $\vec{\boldsymbol{\sigma}}_y$ and its vector form is

$$\vec{\boldsymbol{\sigma}}_y = \sigma_{yx}\,\vec{\mathbf{i}} + \sigma_{yy}\vec{\mathbf{j}}$$

These are called stress vectors acting on the planes normal to x and y directions

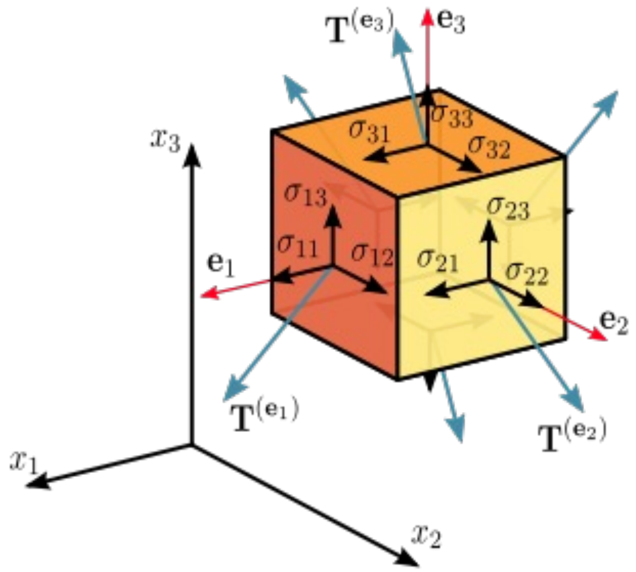
3D Traction vector and stress components

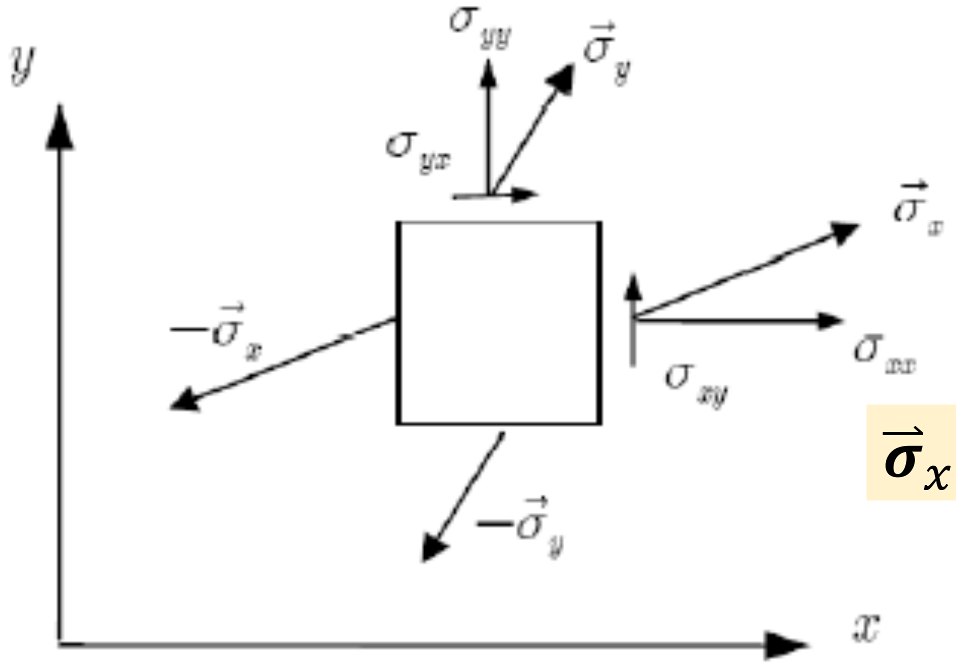


Positive stress components act of positive face in positive direction.

All stresses shown are positive.

Stress components





$$\vec{\boldsymbol{\sigma}}_x = \sigma_{xx}\,\vec{\mathbf{i}} + \sigma_{xy}\vec{\mathbf{j}}$$

Positive stress components act on the positive face in the positive direction

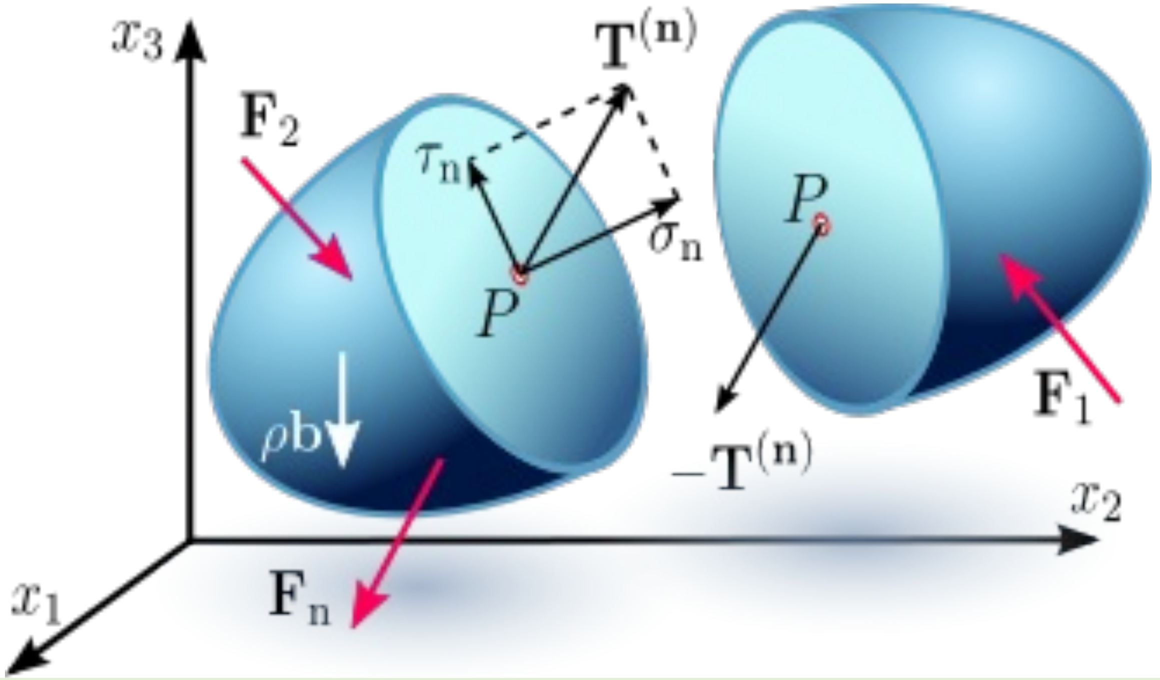
2D Stress Components in matrix or vector form

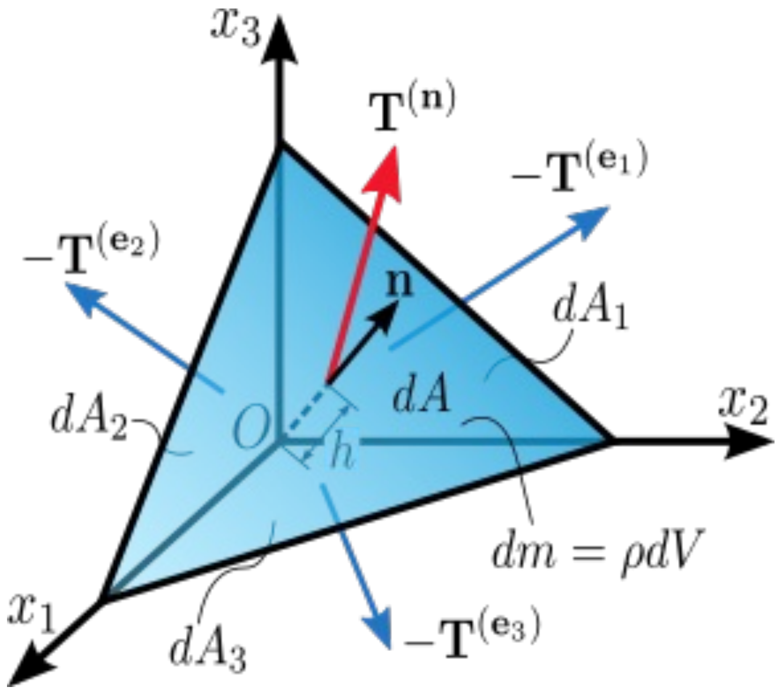
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}^T = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

or as

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

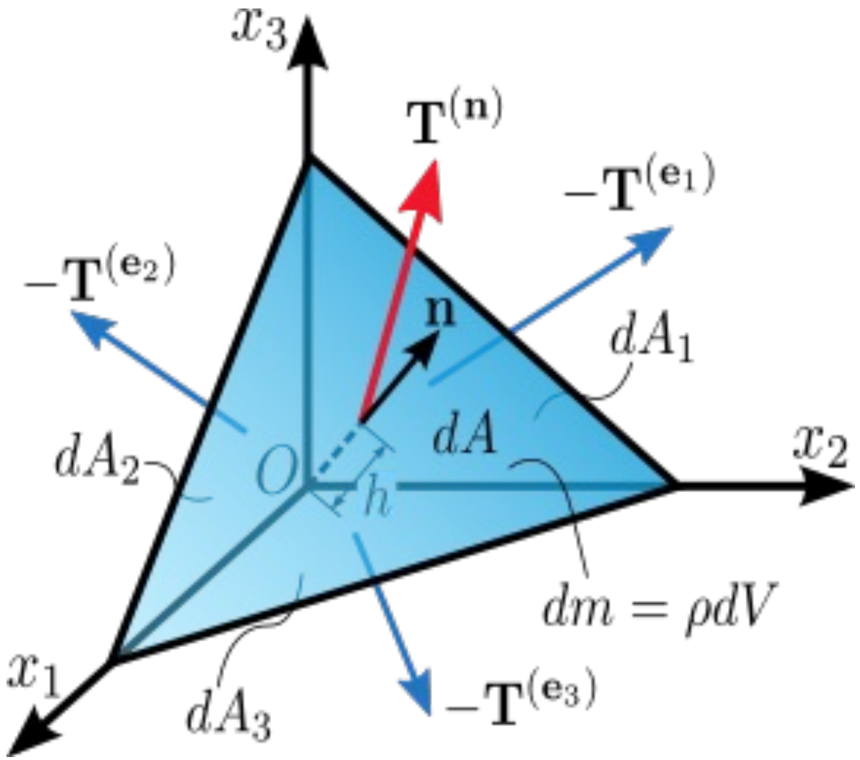
Traction on an arbitrary surface with unit normal **n**

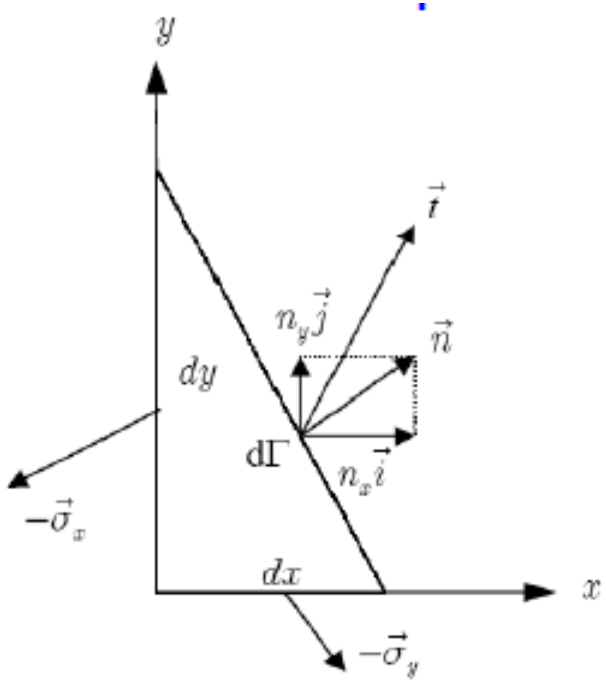




We can use the stresses to provide information about any arbitrary traction acting on any arbitrary surface with any orientation **n**

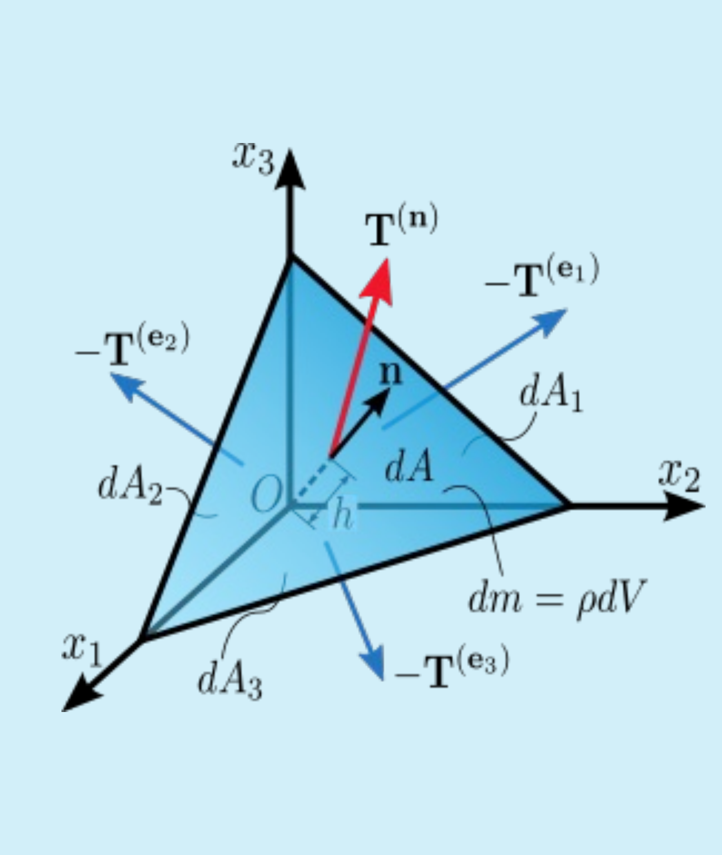
Traction on an arbitrary surface with unit normal **n**

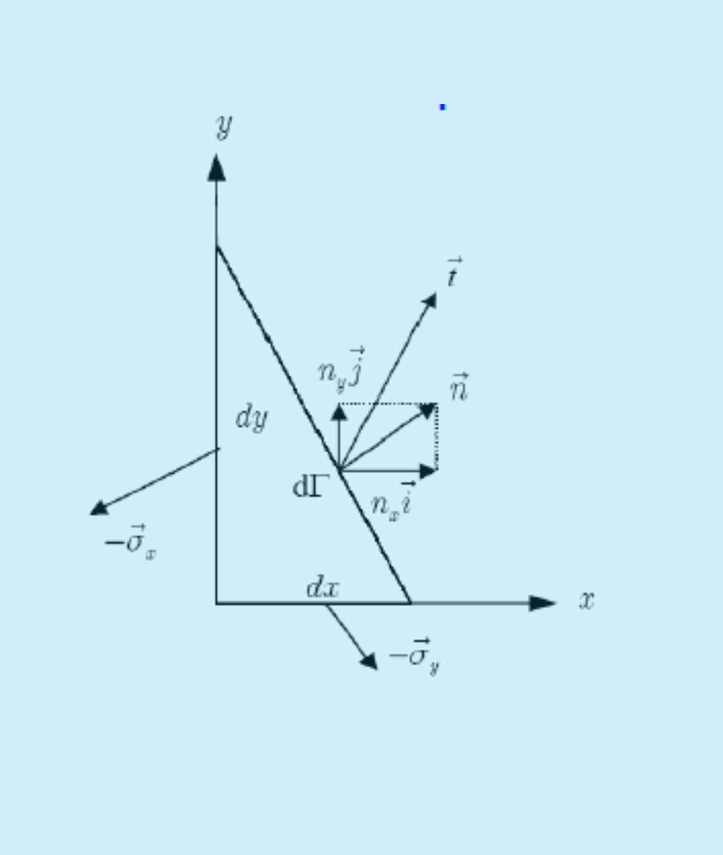




The stress vectors can be used to obtain the traction acting on any arbitrary surface with any orientation **n**

Traction on an arbitrary surface with unit normal **n**





The force equilibrium of the triangular body shown (unit thickness) requires that:

$$\vec{t}d\Gamma - \vec{\sigma}_x dy - \vec{\sigma}_y dx = 0 \Rightarrow$$

$$\vec{t}d\Gamma - \vec{\sigma}_x n_x d\Gamma - \vec{\sigma}_y n_y d\Gamma = 0 \Rightarrow$$

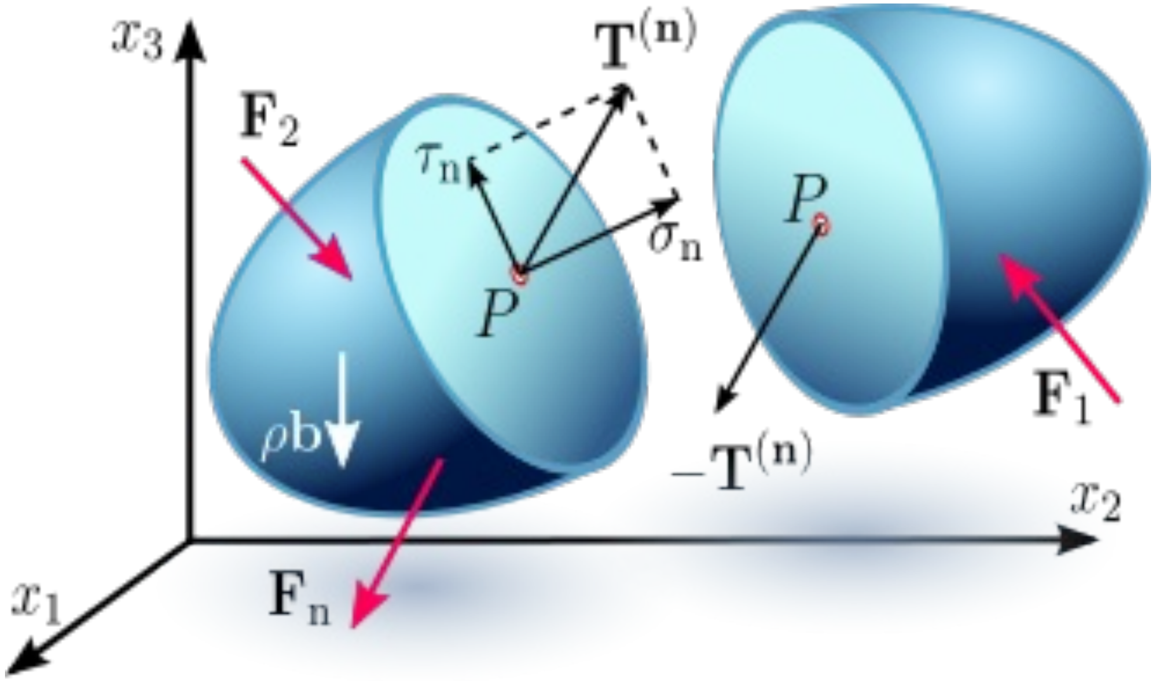
$$\vec{t} = \vec{\sigma}_x n_x + \vec{\sigma}_y n_y \Rightarrow$$

$$\vec{t} = \left(\sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}\right)n_x + \left(\sigma_{yx}\vec{i} + \sigma_{yy}\vec{j}\right)n_y \Rightarrow$$

$$\vec{t} = \underbrace{\left(\sigma_{xx}n_x + \sigma_{yx}n_y\right)}_{t_x}\vec{i} + \underbrace{\left(\sigma_{yx}n_x + \sigma_{yy}n_y\right)}_{t_y}\vec{j}$$

In matrix form $\underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{t} = \underbrace{\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}}_{\tau} \underbrace{\begin{bmatrix} n_x \\ n_y \end{bmatrix}}_{n}, \quad t = \tau n$

Types of forces

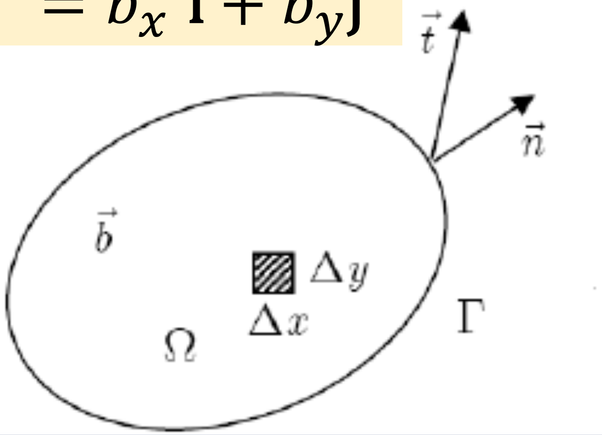


In 2 Dimensions

In 2D, the forces acting on the body are tractions $\vec{\boldsymbol{t}}$ along the boundary Γ and body forces $\vec{\boldsymbol{b}}$ per unit volume

$$\vec{\boldsymbol{t}} = t_x \,\vec{\mathbf{i}} + t_y \vec{\mathbf{j}}$$

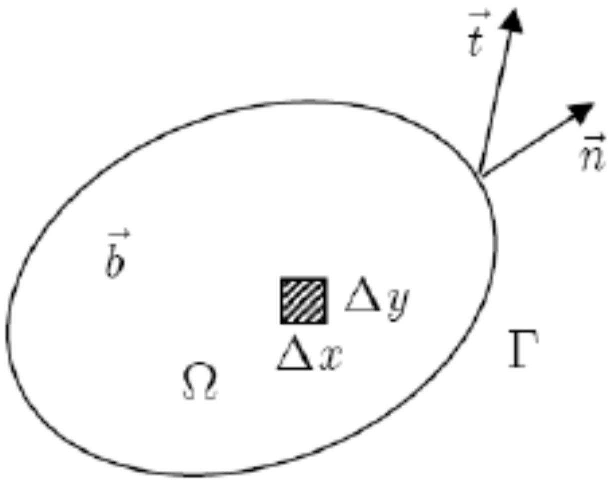
$$\vec{\boldsymbol{b}} = b_x \,\vec{\mathbf{i}} + b_y \vec{\mathbf{j}}$$

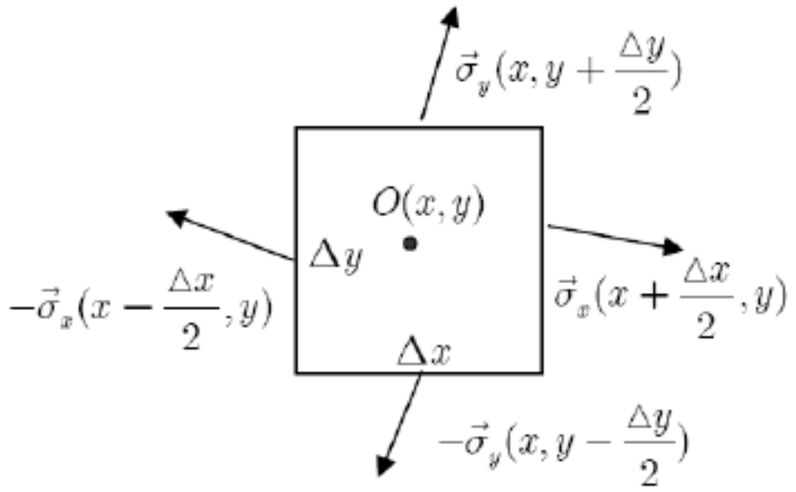


Examples of body forces: magnetic forces, gravitational forces. Thermal forces can also be interpreted as body forces

Stress equilibrium

In 2 Dimensions





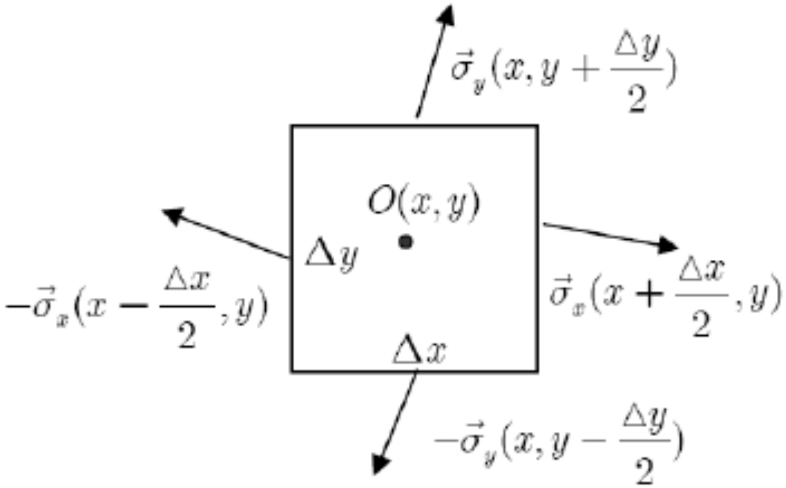
Consider equilibrium of the 2D infinitesimal element of unit thickness on the plane:

$$-\bar{\boldsymbol{\sigma}}_x(x-\frac{\Delta x}{2},y)\Delta y+\bar{\boldsymbol{\sigma}}_x(x+\frac{\Delta x}{2},y)\Delta y-\bar{\boldsymbol{\sigma}}_y(x,y-\frac{\Delta y}{2})\Delta x+\bar{\boldsymbol{\sigma}}_y(x,y+\frac{\Delta y}{2})\Delta x+\bar{b}(x,y)\Delta x\Delta y=0\Rightarrow$$

$$\frac{\bar{\boldsymbol{\sigma}}_x(x+\frac{\Delta x}{2},y)-\bar{\boldsymbol{\sigma}}_x(x-\frac{\Delta x}{2},y)}{\Delta x}+\frac{\bar{\boldsymbol{\sigma}}_y(x,y+\frac{\Delta y}{2})-\bar{\boldsymbol{\sigma}}_y(x,y-\frac{\Delta y}{2})}{\Delta y}+\bar{b}(x,y)=0 \underset{\Delta x,\Delta y\to 0}{\Rightarrow}$$

Stress equilibrium

In 2 Dimensions



$$\frac{\partial \vec{\sigma}_x}{\partial x} + \frac{\partial \vec{\sigma}_y}{\partial y} + \vec{b}(x,y) = 0$$

However:

$$\vec{\sigma}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}$$

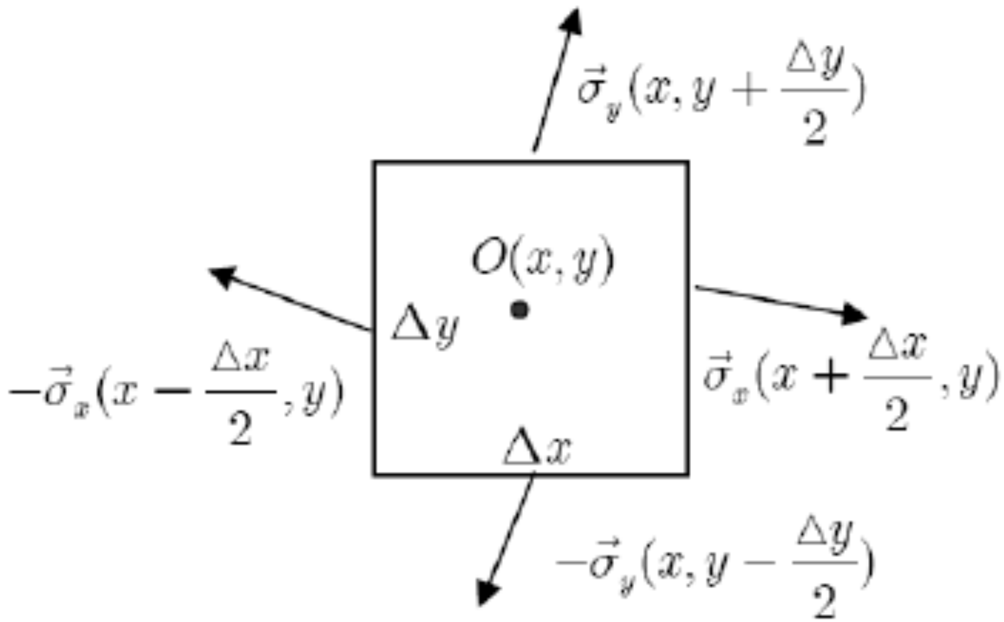
$$\vec{\sigma}_y = \sigma_{yx}\vec{i} + \sigma_{yy}\vec{j}$$

Combining these two equations yields the equilibrium equations in the x and y directions:

$$\frac{\partial\left(\sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}\right)}{\partial x} + \frac{\partial\left(\sigma_{yx}\vec{i} + \sigma_{yy}\vec{j}\right)}{\partial y} + \vec{b}(x,y) = 0 \Rightarrow \begin{array}{l} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{array} \quad \text{or} \quad \begin{array}{l} \nabla \bullet \vec{\sigma}_x + b_x = 0 \\ \nabla \bullet \vec{\sigma}_y + b_y = 0 \end{array}$$

Stress equilibrium

In 2 Dimensions



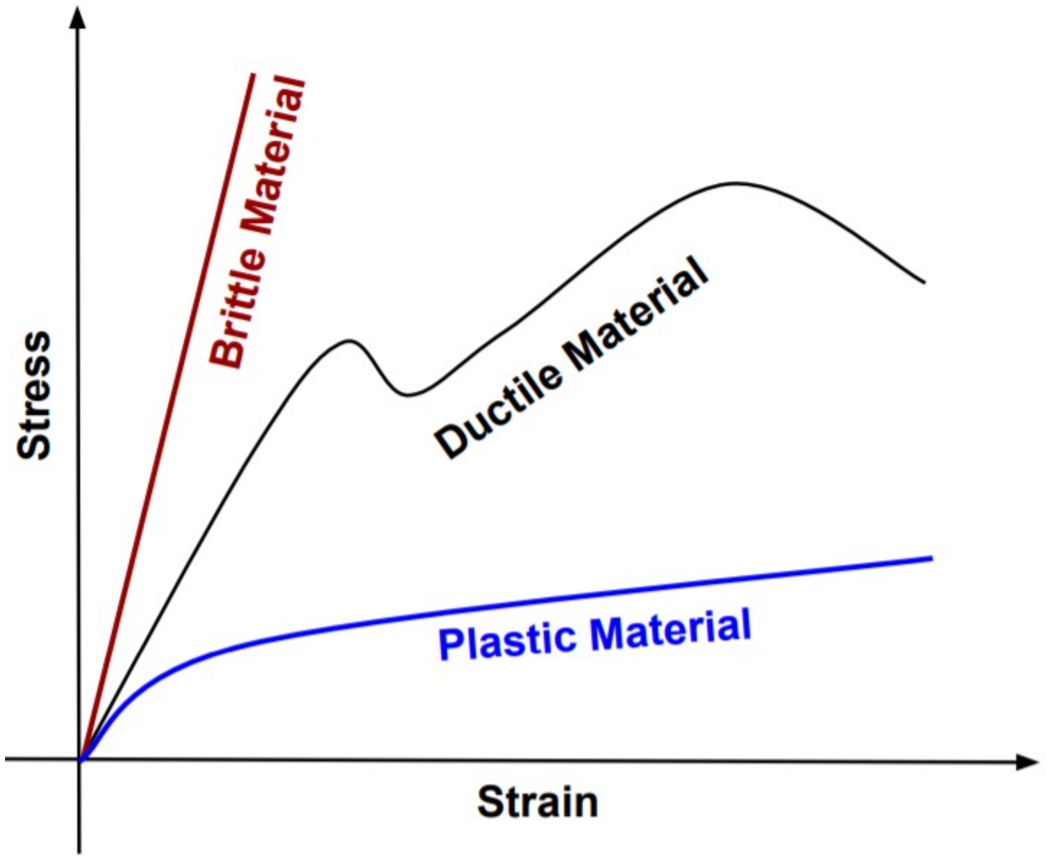
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0$$

We can rewrite the equilibrium equations as matrix equations:

$$\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\nabla_s^T} \underbrace{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}}_{\sigma} + \underbrace{\begin{bmatrix} b_x \\ b_y \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0} \Rightarrow \qquad \nabla_s^T \sigma + b = 0$$

Constitutive Equations



- A Constitutive Equation is the relationship between stress and strain. Material behavior characterized as: elastic, viscoelastic, plastic, etc.
- Here, we discuss linear elastic isotropic behavior
- In 1D, a linear elastic material is described using Hooke's law, $\sigma = E\varepsilon$ where E is Young's modulus
- In 2D, the linear elastic behavior (relationship between stress and strain) is written as

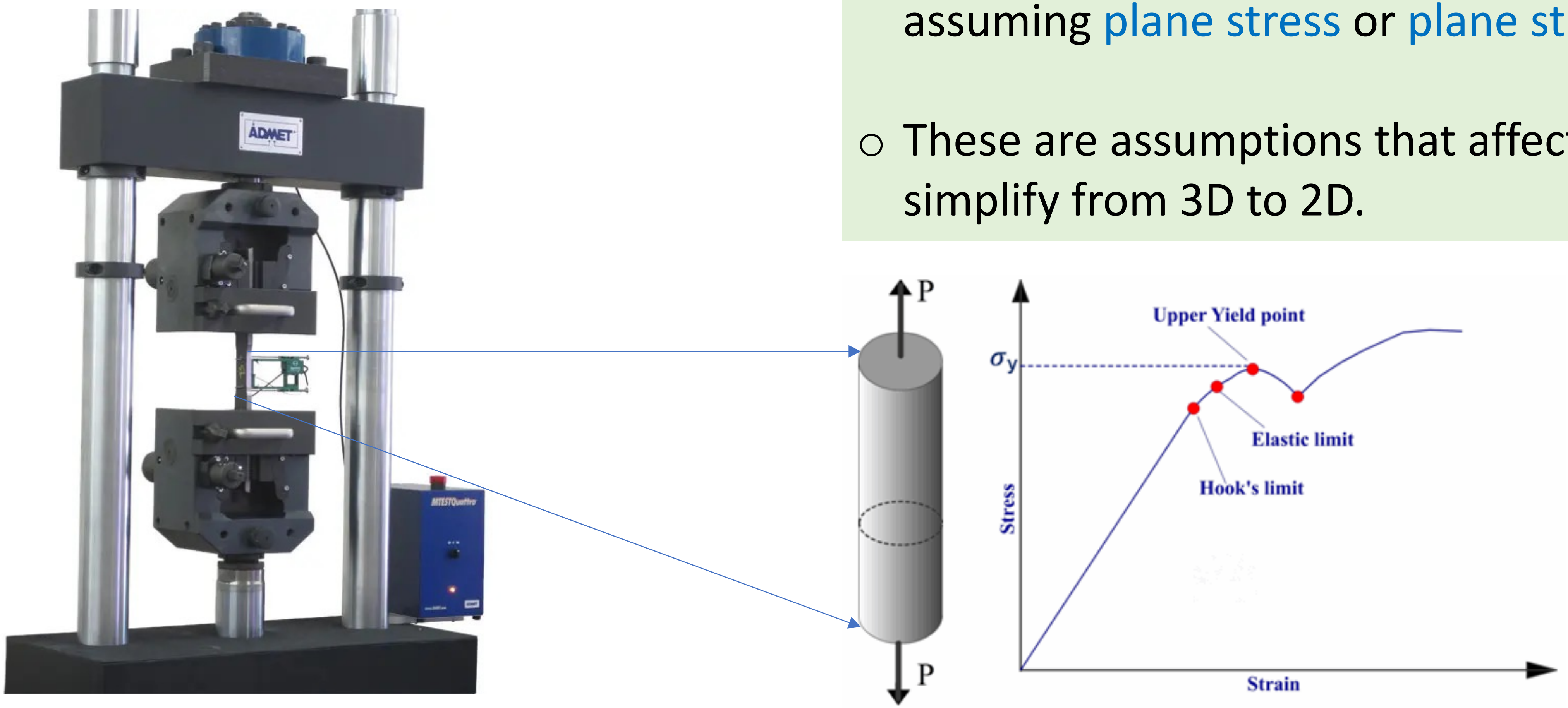
$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$$

where $\boldsymbol{D}$ is a 3 x 3 matrix.

Constitutive Equations

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$$

- This is the generalized version of Hooke's law
- ***D*** is a symmetric positive definite matrix
- The form of ***D*** depends on whether you are assuming plane stress or plane strain
- These are assumptions that affect how we simplify from 3D to 2D.



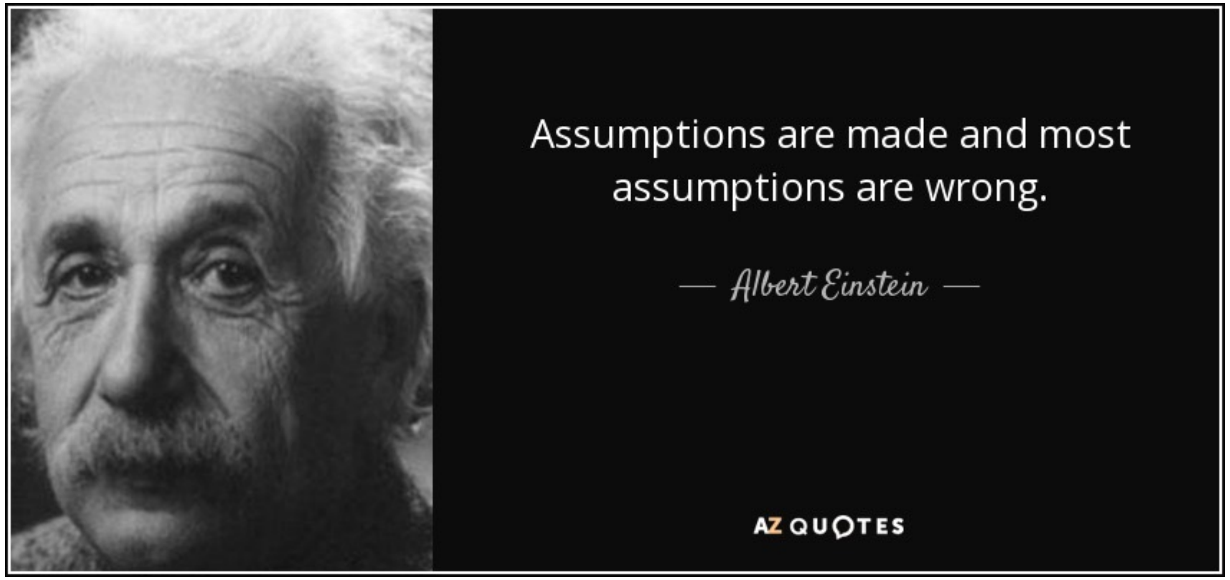
Plane strain

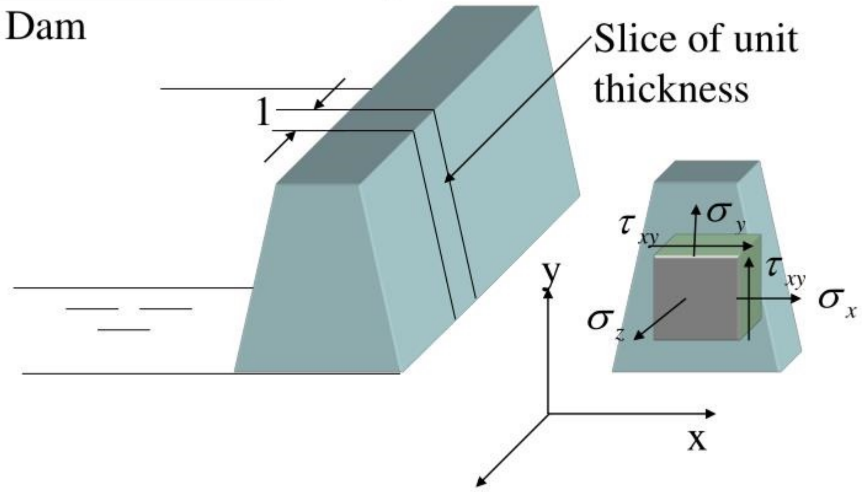
Plane strain assumes the body is thick relative to the xy- plane in which the model is constructed

The strain normal to the plane ε_z is assumed to be zero along with the shear strains involving z - $\gamma_{xz}$and γ_{yz}

This does not mean you neglect the stresses in the z-direction

When a body is thick, significant stresses can develop on the z- faces, in particular the normal stresses σ_{zz} can be quite large





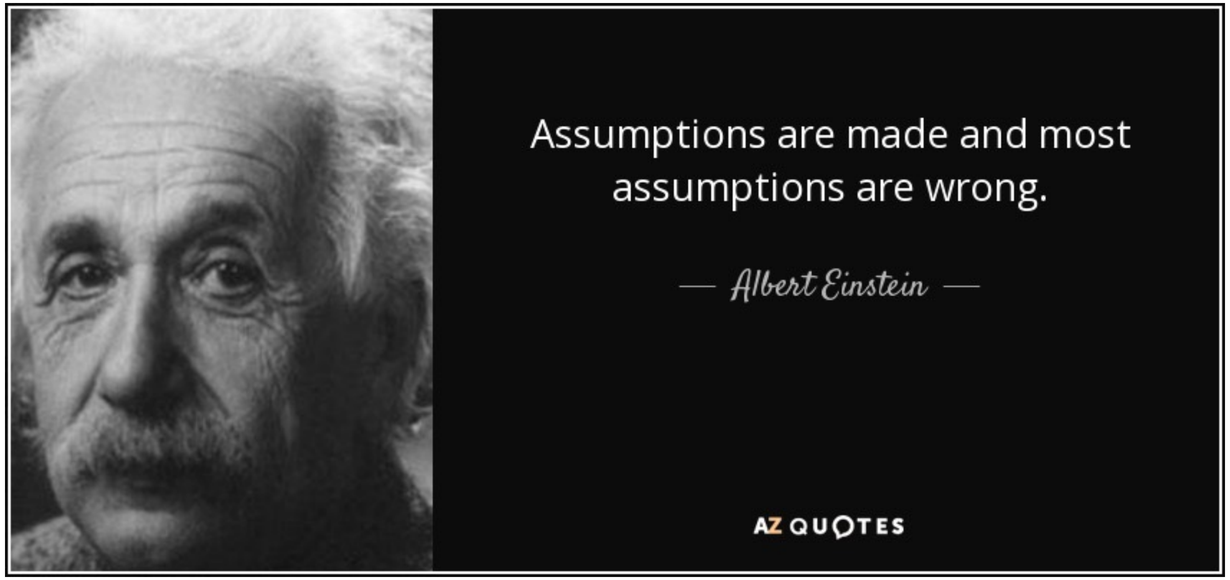
Plane stress

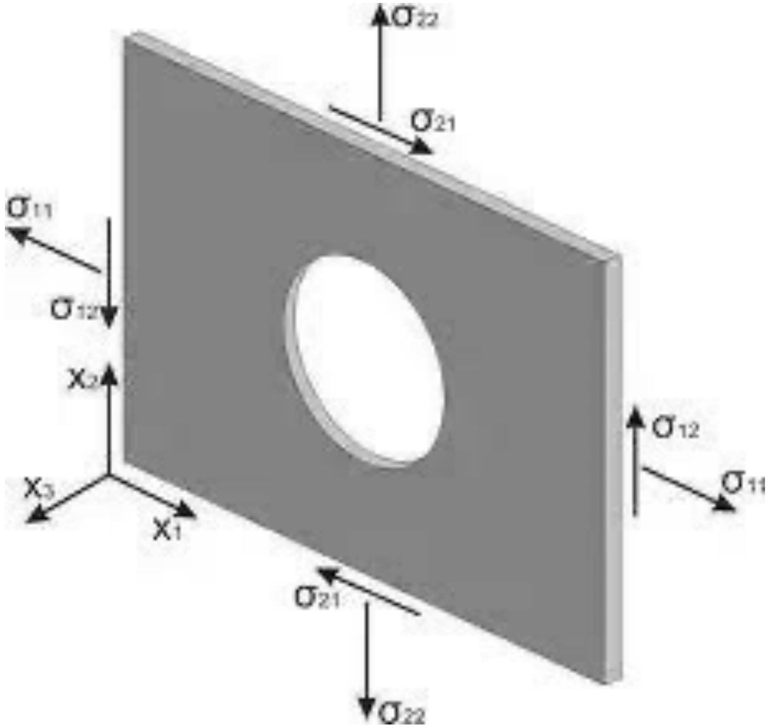
Plane stress assumes the body is thin relative to the dimensions of the xy- plane

We assume no loads are applied on the z faces and the stress normal to the xy- plane σ_{zz} is zero

This does not mean you neglect the strains in the z-direction

For a thin body, since the stresses σ_{zz} must vanish on the outer surfaces, there is no mechanism for significant normal stresses σ_{zz} to develop in the body





Constitutive Equations

- We assume an isotropic material (i.e. properties independent of coordinate system)

- For an isotropic material, ***D*** is the same regardless of the coordinate system. (This is not true for all materials.)

$$\boldsymbol{\sigma} = \boldsymbol{D\varepsilon}$$

  where ***D*** is a 3 x 3 matrix.

- To write the components of ***D***, recall *E* (Young's modulus) and we introduce ν Poisson's ratio

Plane Stress

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Plane Strain

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Only independent properties for linear elastic isotropic material

Constitutive Equations: Some important observations

- For an isotropic elastic material we have only two independent material properties: *E* and ν.

- We can also write ***D*** in terms of other quantities such as the shear modulus *G* and the bulk modulus *K*.

$$G = \frac{E}{2(1+\nu)}$$

$$K = \frac{E}{3(1-\nu)}$$

- Note that for plane strain, as $\nu \rightarrow 0.5$, ***D*** goes to infinity

Plane Stress

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Plane Strain

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

For incompressible materials, $\nu \rightarrow 0.5$

Modeling incompressible materials (in plane strain and 3D) requires special attention in FEM (and special elements).

Strong form of an elasticity problem

Equilibrium equation (Force balance)

$$\nabla_s^T \boldsymbol{\sigma} + b = 0 \text{ or equivalently: } \quad \vec{\nabla}\cdot\vec{\boldsymbol{\sigma}}_x + b_x = 0$$

$$\vec{\nabla}\cdot\vec{\boldsymbol{\sigma}}_y + b_y = 0$$

Kinematics equation (strain-displacement)

$$\boldsymbol{\varepsilon} = \nabla_s u$$

Constitutive equation (stress-strain)

$$\boldsymbol{\sigma} = \boldsymbol{D\varepsilon}$$

Boundary conditions: The portion of the boundary where the traction is prescribed is called Γ_t and the portion where the displacement is prescribed is called Γ_u

- ✓ The traction boundary condition is written as $\boldsymbol{\tau} n = \bar{t} \;\; on \;\; \Gamma_t$ or equivalently:

$$\vec{\boldsymbol{\sigma}}_x \cdot \vec{n} = \bar{t}_x, \vec{\boldsymbol{\sigma}}_y \cdot \vec{n} = \bar{t}_y \;\; on \;\; \Gamma_t$$

- ✓ The displacement boundary condition is written as $\vec{u} = \bar{\vec{u}} \;\; on \;\; \Gamma_u$

Strong form of an elasticity problem: Boundary Conditions

Essential Boundary Conditions: The displacement boundary condition is the essential boundary condition satisfied by the displacement field.

Natural Boundary Conditions: The traction boundary condition is a natural boundary condition.

The displacement and traction cannot both be prescribed on the same part of the boundary, thus

$$\Gamma_u \cap \Gamma_t = 0$$

However, on any portion of the boundary, either the traction or the displacement must be prescribed

$$\Gamma_u \cup \Gamma_t = \Gamma$$

Strong form isotropic linear elasticity

Find the displacement field $\vec{\mathbf{u}}$ on Ω such that

$$\vec{\nabla} \bullet \vec{\boldsymbol{\sigma}}_x + b_x = 0, \vec{\nabla} \bullet \vec{\boldsymbol{\sigma}}_y + b_y = 0 \quad on \;\; \Omega$$

$$where \;\; \boldsymbol{\sigma} = D \nabla_s u$$

$$with$$

$$\vec{\boldsymbol{\sigma}}_x \bullet \vec{n} = \bar{t}_x, \vec{\boldsymbol{\sigma}}_y \bullet \vec{n} = \bar{t}_y \;\; on \;\; \Gamma_t$$

$$\vec{u} = \bar{\vec{u}} \;\; on \;\; \Gamma_u$$