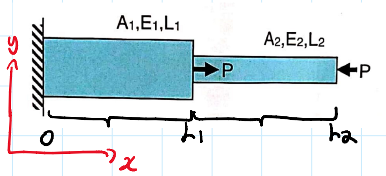
Exam 1

Tuesday, February 14, 2023 6:24 PM

a)



a) The potential energy formula is:

$$\Pi = \frac{1}{2}\int_V \varepsilon^T \sigma \, dV - \underbrace{\cancel{\int_S u^T t \, dS} - \cancel{\int_V u^T b \, dV}}_{\text{since we have no traction or body force.}} - u^T P$$

Π = Internal Work − External Work

$$\boxed{\Pi = \frac{1}{2}\int E_1 \frac{\partial u}{\partial x} A_1 dx + \frac{1}{2}\int E \frac{\partial u}{\partial x} A_2 \, dx - L_1 \cdot P - L_2 \cdot (-P)}$$

b) For element ①

$$K_1 = \begin{array}{cc} ① & ② \\ \end{array}$$
$$K_1 = \begin{bmatrix} E_1A_1/L_1 & -EA_1/L_1 \\ -EA_1/L_1 & E_1A_1/L_1 \end{bmatrix} \begin{array}{c} ① \\ ② \end{array} \qquad F_1 = \begin{bmatrix} R_{①}^{0} \\ F_{①}^{1} \end{bmatrix}$$

For element ②

$$K_2 = \begin{bmatrix} E_2A_2/L_2 & -E_2A_2/L_2 \\ -E_2A_2/L_2 & E_2A_2/L_2 \end{bmatrix} \begin{array}{c} ② \\ ③ \end{array} \qquad F_2 = \begin{bmatrix} F_{②}^{1} \\ -P_{②}^{2} \end{bmatrix}$$

We assemble the system:

$$\begin{array}{ccc} ① & ② & ③ \end{array}$$
$$\begin{bmatrix} E_1A_1/L_1 & -E_1A_1/L_1 & 0 \\ -E_1A_1/L_1 & E_1A_1/L_1 + E_2A_2/L_2 & -E_2A_2/L_2 \\ 0 & -E_2A_2/L_2 & E_2A_2/L_2 \end{bmatrix} \begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} R_0 \\ P+0 \\ -P \end{bmatrix}$$

$$\begin{bmatrix} E_1A_1/L_1 + E_2A_2/L_2 & -E_2A_2/L_2 \\ \end{bmatrix} \begin{bmatrix} u_2 \\ \end{bmatrix} = \begin{bmatrix} P \\ \end{bmatrix}$$

$$\begin{bmatrix} E_1A_1/L_1+E_2A_2/L_2 & -E_2A_2/L_2 \\ -E_2A_2/L_2 & E_2A_2/L_2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} P \\ -P \end{bmatrix}$$

Solving by MATLAB, we obtain:

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{L_2 P}{A_2 E_2} \end{bmatrix}$$

We obtain stress by using the formula:

$$\tau^{①} = E\left(\frac{\delta_1 - \delta_0}{L_1}\right)$$

$$\left[\tau^{①} = E\left(\frac{0-0}{L_1}\right) = 0\right]$$

$$\tau^{②} = E\left(\frac{\delta_2 - \delta_1}{L_2}\right)$$

$$= E\left(\frac{\left(\frac{-L_2 P}{A_2 E_2}\right)}{L_2}\right)$$

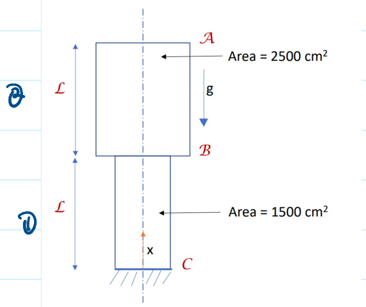
$$= E\left(\frac{-\cancel{L_2} P}{A_2 E_2 \cancel{L_2}}\right)$$

$$\left[\tau^{②} = -\left(\frac{EP}{A_2 E_2}\right)\right]$$

c)

i) T
ii) T
iii) F
iv) T
v) T

3)



For linear interpolants:

$$\begin{bmatrix} N_1^{(2)} = \dfrac{x_{i+1} - x}{x_{i+1} - x_i} \\ N_2^{(2)} = \dfrac{x - x_i}{x_{i+1} - x_i} \end{bmatrix}$$

$$\Pi^{(1)} = \frac{1}{2}\int_0^L EA_1\left(N_1^{(1)'}u_1 + N_2^{(1)'}u_2\right)^2 dx - \int_V b\left(N_1^{(1)}u_1 + N_2^{(1)}u_2\right)A_1\,dx - R_1u_1 - Fu_2$$

$$= \frac{1}{2}\int_0^L EA_1\left(N_1^{(1)'}u_1 + N_2^{(1)'}u_2\right)^2 dx - \int_0^L \rho g\left(N_1^{(1)}u_1 + N_2^{(1)}u_2\right)A_1\,dx - R_1u_1 - \underbrace{F}_{0}u_2$$

$$\Pi^{(2)} = \frac{1}{2}\int_L^{2L} EA_2\left(N_1^{(2)'}u_2 + N_2^{(2)}u_3\right)^2 dx - \int_L^{2L} \rho g\left(N_1^{(2)}u_2 + N_2^{(2)}u_3\right)A_2\,dx$$

We combine the two equations & obtain:

$$\Pi = \left(\frac{1}{2}\int_0^L EA_1\left(N_1^{(1)'}u_1 + N_2^{(1)'}u_2\right)^2 dx - \int_0^L -\rho g\left(N_1^{(1)}u_1 + N_2^{(1)}u_2\right)A_1\,dx - R_1u_1 - \underbrace{F}_{0}u_2\right)$$

$$+\left(\frac{1}{2}\int_L^{2L} EA_2\left(N_1^{(2)'}u_2 + N_2^{(2)}u_3\right)^2 dx - \int_L^{2L} -\rho g\left(N_1^{(2)}u_2 + N_2^{(2)}u_3\right)A_2\,dx\right)$$

For each element, we take derivative with respect to deformations

$$\frac{\partial \Pi^{(1)}}{\partial u_1} = \int_0^L EA_1\left(N_1^{(1)'}u_1 + N_2^{(1)'}u_2\right)N_1^{(1)} - \int_0^L -\rho g\,N_1^{(1)}A_1\,dx - R_1$$

$$\frac{\partial \Pi^{(1)}}{\partial u_2} = \int_0^L EA_1\left(N_1^{(1)'}u_1 + N_2^{(1)'}u_2\right)N_1^{(1)} - \int_0^L -\rho g\,N_2^{(1)}A_1\,dx - F$$

$$
\begin{bmatrix} \dfrac{\partial \pi^{(1)}}{\partial u_1} \\ \dfrac{\partial \pi^{(1)}}{\partial u_2} \end{bmatrix} = \int_0^L EA_1 \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dx - \begin{bmatrix} \int_0^L \rho g N_1^{(1)} A_1 \, dx - R_1 \\ \int_0^L \rho g N_2^{(1)} A_1 \, dx - F \end{bmatrix}
$$

$$
= \frac{EA_1}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} -\int_0^L \rho g N_1^{(1)} A_1 \, dx - R_1 \\ -\int_0^L \rho g N_2^{(1)} A_1 \, dx - F \end{bmatrix}
$$

For element (2), we have very similar structure:

$$
\begin{bmatrix} \dfrac{\partial \pi^{(2)}}{\partial u_2} \\ \dfrac{\partial \pi^{(2)}}{\partial u_3} \end{bmatrix} = \int_L^{2L} EA_2 \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} dx - \begin{bmatrix} -\int_L^{2L} \rho g N_1^{(2)} A_2 \, dx \\ -\int_L^{2L} \rho g N_2^{(2)} A_2 \, dx \end{bmatrix}
$$

$$
= L\,EA_2 \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} -\int_L^{2L} \rho g N_1^{(2)} A_2 \, dx \\ -\int_L^{2L} \rho g N_2^{(2)} A_2 \, dx \end{bmatrix}
$$

$$
= \frac{EA_2}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} -\int_L^{2L} \rho g N_1^{(2)} A_2 \, dx \\ -\int_L^{2L} \rho g N_2^{(2)} A_2 \, dx \end{bmatrix}
$$

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{EA_1}{L} & -\frac{EA_1}{L} & 0 \\ -\frac{EA_1}{L} & \frac{EA_1}{L} + \frac{EA_2}{L} & -\frac{EA_2}{L} \\ 0 & -\frac{EA_2}{L} & \frac{EA_2}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} -\int_0^L \rho g N_1^{(1)} A_1 \, dx - R_1 \\ -\int_0^L \rho g N_2^{(1)} A_1 \, dx - \int_L^{2L} \rho g N_1^{(2)} A_2 \\ -\int_L \rho g N_2^{(2)} A_2 \, dx \end{bmatrix}
$$

we then rearrange the system & obtain

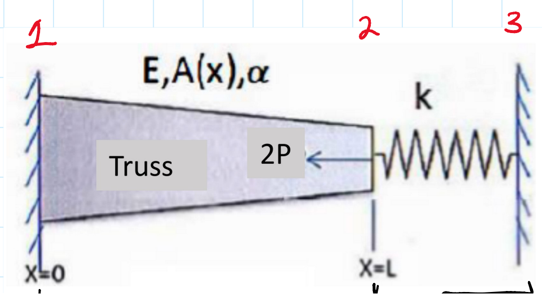
$$\begin{bmatrix} -\int_0^L \rho g N_1^{①} A_1\,dx - R_1 \\ -\int_0^L \rho g N_2^{①} A_1\,dx \quad -\int_L^{2L} \rho g N_1^{②} A_2 \\ -\int_L \rho g N_2^{②} A_2\,dx \end{bmatrix} = \begin{bmatrix} \frac{EA_1}{L} & -\frac{EA_1}{L} & 0 \\ -\frac{EA_1}{L} & \frac{EA_1}{L}+\frac{EA_2}{L} & -\frac{EA_2}{L} \\ 0 & -\frac{EA_2}{L} & \frac{EA_2}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

We now subsitute in our units, where we watch out for unit conversion. In particular $\rho g = 0.06 N/cm^3$
$= 6E\text{-}8/m^3$

$$\begin{bmatrix} -0.12\cdot 10^{-7} \\ -0.0750\cdot 10^{-7} \end{bmatrix} = \begin{bmatrix} \frac{EA_1}{L}+\frac{EA_2}{L} & -\frac{EA_2}{L} \\ -\frac{EA_2}{L} & \frac{EA_2}{L} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -0.13 \\ -0.16 \end{bmatrix} \cdot 10^{-14}$$

2)



Element ① Element ②

We take an average of the bar to approximate its stiffness:

$$A_{average} = \frac{A_1 + A_2}{2}$$

$$k_{avg} = E A_{avg} / L$$

At the ends, we have that

$$k_1 = \frac{EA_1}{L}, \quad k_2 = \frac{EA_2}{L}$$

We take an weighted average such that:

$$k_{bar} = \frac{1}{3}\left(\frac{EA_{avg}}{L}\right) + \frac{1}{3}\left(\frac{EA_1}{L}\right) + \frac{1}{3}\left(\frac{EA_2}{L}\right) \quad (1)$$

The stiffness matrix of the bar then becomes:

$$k_{bar} = \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} k_{bar} & -k_{bar} \\ -k_{bar} & k_{bar} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

We then have enough information to assemble a stiffness matrix:

$$\underbrace{\begin{bmatrix} k_{bar} & -k_{bar} & 0 \\ -k_{bar} & k_{bar} + k & -k \\ 0 & -k & k \end{bmatrix}}_{\text{stiffness matrix } K} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1 \\ -2P \\ R_2 \end{bmatrix}}_{\text{external force}}$$

a)

$$\begin{bmatrix} k_{bar} & -k_{bar} & 0 \end{bmatrix} \begin{bmatrix} u_1 \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix}$$

a)

$$\underbrace{\begin{bmatrix} k_{bar} & -k_{bar} & 0 \\ -k_{bar} & k_{bar}+k & -k \\ 0 & -k & k \end{bmatrix}}_{\text{stiffness matrix } K} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1 \\ -2P \\ R_2 \end{bmatrix}}_{\text{external force}}$$

$$(k_{bar}+k)\,u_2 = -2P$$

$$u_2 = -2P/(k_{bar}+k)$$

$$\boxed{u_2 = \frac{-2P}{\left(k+\frac{E(A_1+A_2)}{2L}\right)}}$$

b) If we have thermal forces, we modify the force vector such that:

$$\begin{bmatrix} k_{bar} & -k_{bar} & 0 \\ -k_{bar} & k_{bar}+k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} R_1 - EA\alpha(20) \\ -2P + EA\alpha(20) \\ R_2 \end{bmatrix}$$

since we approximated stiffness, $EA\alpha(20)$ becomes

$EA\alpha(20) = K_{bar} \cdot L\,\alpha(20)$ according to equation ②

we again obtain:

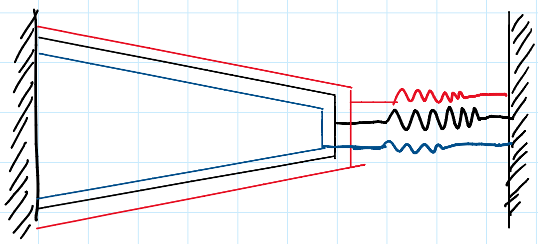
$$(k_{bar}+k)\,u_2 = -2P + K_{bar} L\,\alpha(20)$$

$$u_2 = \frac{-2P + K_{bar} L\,\alpha(20)}{K_{bar}+K}$$

$$\left[u_2 = \frac{-2P + K_{bar} h \alpha (20)}{\left(k + \frac{E(A_1 + A_2)}{2L}\right)} \right]$$

c) If we decrease temperature, the thermal induced load ΔT then switches and become negative, which would induce a negative deformation.

Drawing each scencerio, we have:



Red represents thermal expansion while blue represents thermal contraction.