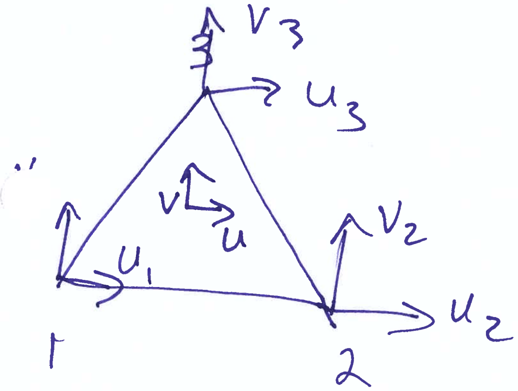


$$\underline{q} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = B\,\underline{q}$$

$$\tilde{\underline{\varepsilon}} = B\,\tilde{q}$$

At some point inside the triangle,

$$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix} = N q$$

shape function matrix

$$\underline{w} = N \tilde{q}$$

Principle of virtual work

$$\int_{\Omega^e} \underline{\sigma}\,\tilde{\underline{\varepsilon}}\, d\Omega = \int_{\Omega} \underline{w}^T \underline{f}\, d\Omega + \int_{\Gamma^e} \underline{w}^T \underline{t}\, d\Gamma$$

$$\tilde{q}^T \underbrace{\left[\int_{\Omega^e} B^T D B\, d\Omega\right]}_{\text{local stiffness matrix } 6\times 6} q = \tilde{q}^T \underbrace{\int_{\Omega^e} N^T \underline{f}\, d\Omega}_{\text{body force related 'local force' term } 6\times 1} + \tilde{q}^T \underbrace{\int_{\Gamma^e} N^T t\, d\Gamma}_{\text{traction related 'local force' term}}$$

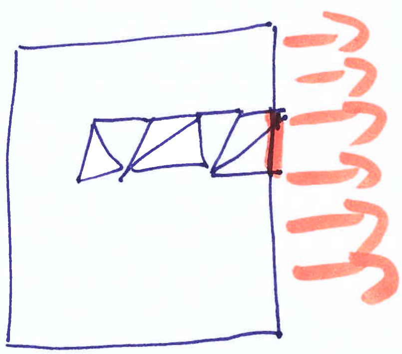
$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

plane stress

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix}$$

plane strain

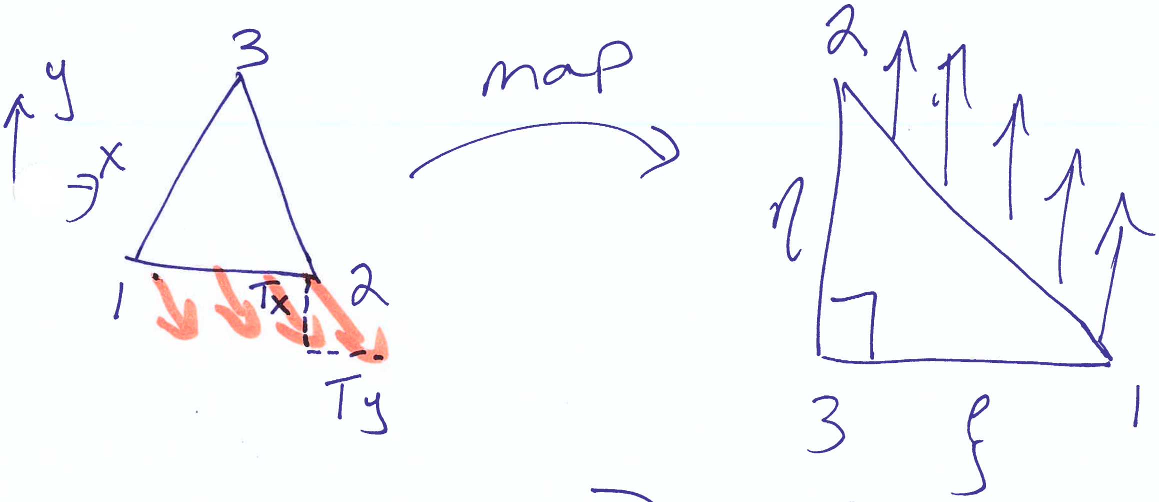
constitutive law $\sigma = D\, B\, q$



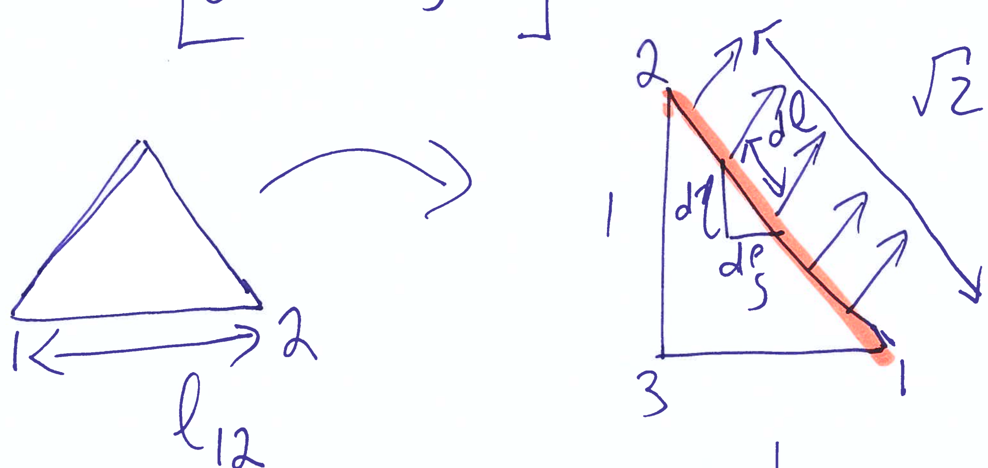
traction related 'local force' term

Traction related 'local force' term

$$\int_{\Gamma^e} N^T \underline{t}\, d\Gamma = \int_{\Gamma^e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} T_x \\ T_y \end{bmatrix} d\Gamma$$



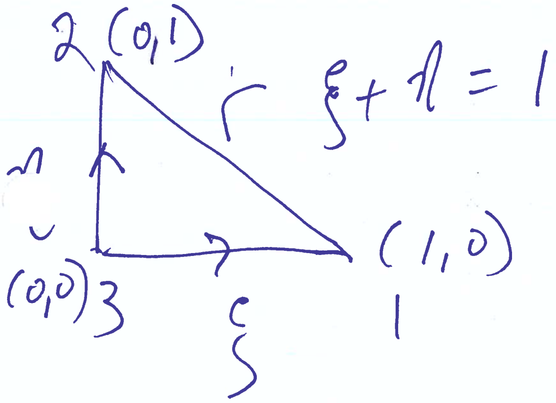
$$\int_{\Gamma^e} \begin{bmatrix} \xi & 0 \\ 0 & \xi \\ \eta & 0 \\ 0 & \eta \\ 1-\xi-\eta & 0 \\ 0 & 1-\xi-\eta \end{bmatrix} \begin{bmatrix} T_x \\ T_y \end{bmatrix} t^e \det J^* \sqrt{d\xi^2+d\eta^2}$$



$$dl = \sqrt{d\xi^2 + d\eta^2}$$

$\det J^*$ = ratio of line elements

$$= \frac{\text{length of real line}}{\text{length of corresponding line on the standard triangle}} = \frac{l_{12}}{\sqrt{2}}$$



$$\sqrt{d\xi^2 + d\eta^2} = \sqrt{2\, d\xi^2}$$
$$= \sqrt{2}\, d\xi$$

$\xi + \eta = 1$

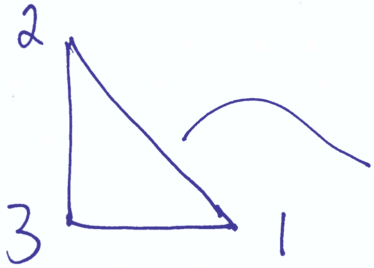
Take derivative

$\rightarrow d\xi + d\eta = 0$

$\rightarrow d\xi = -d\eta$

$$f^{traction} = t^e \int_{\xi=0}^{1} \begin{bmatrix} \xi & 0 \\ 0 & \xi \\ \eta & 0 \\ 0 & \eta \\ \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{0} \end{bmatrix} \frac{l_{12}}{\cancel{\sqrt{2}}} \cancel{\sqrt{2}}\, d\xi \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

Because



$\xi + \eta = 1$ on the line 1-2

Lets, assume that T_x & T_y are constants

$$\int_{\xi=0}^{1} \xi\, d\xi = \frac{1}{2} \qquad \int_0^1 \eta\, d\xi = \int_0^1 (1-\xi)\, d\xi = \frac{1}{2}$$

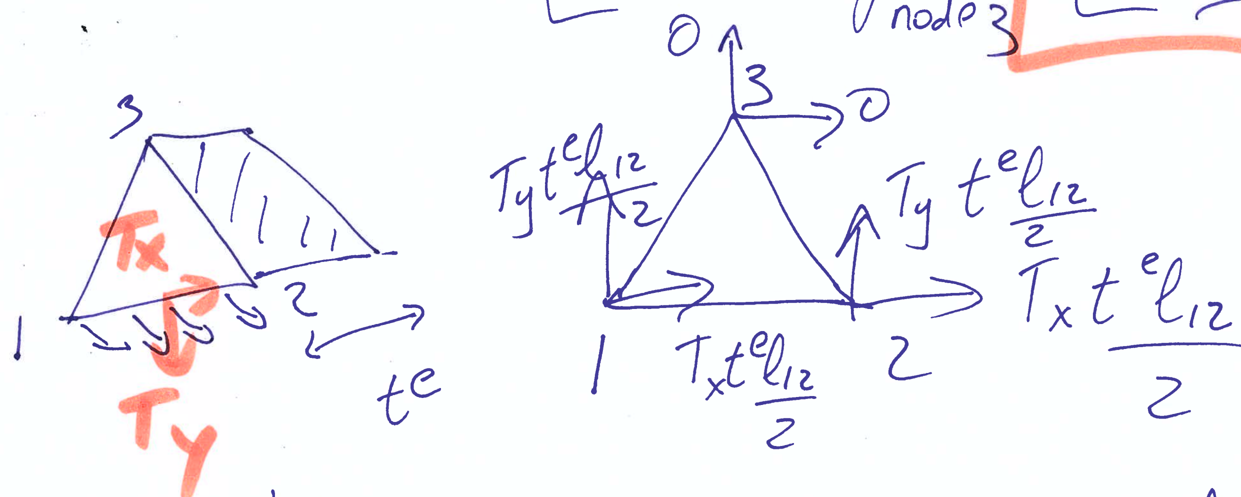
Traction

$$f^{\text{local force}} = \frac{t^e l_{12}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} T_x \\ T_y \end{matrix}} \\ \boxed{\begin{matrix} T_x \\ T_y \end{matrix}} \\ \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} \end{bmatrix} \frac{t^e l_{12}}{2}$$

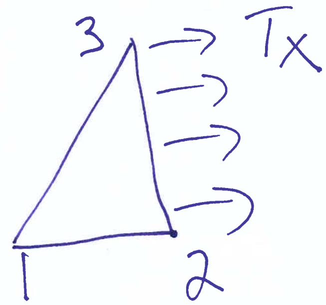
force on node 1

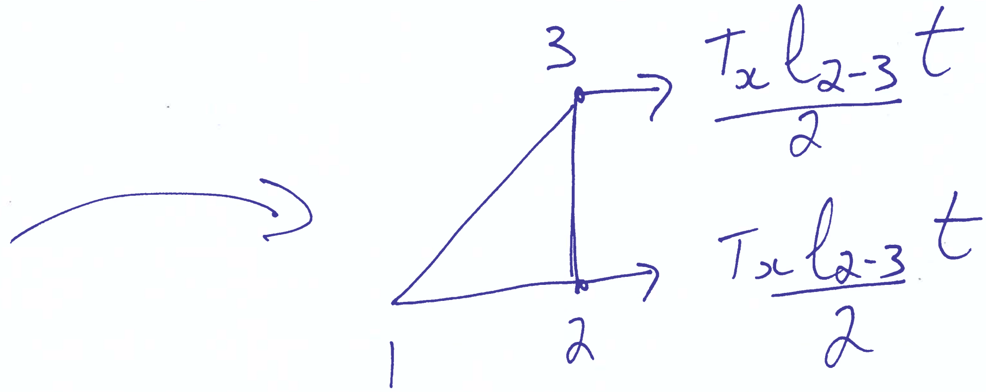
force on node 2

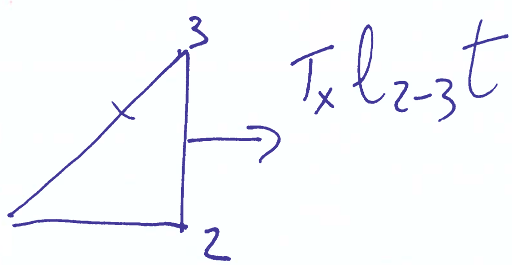
force on node 3



Example







$$f^{\text{local}} = \begin{bmatrix} 0 \\ 0 \\ T_x \\ 0 \\ T_x \\ 0 \end{bmatrix} \frac{l_{2-3}\, t}{2}$$

Similarly if the body force were constants

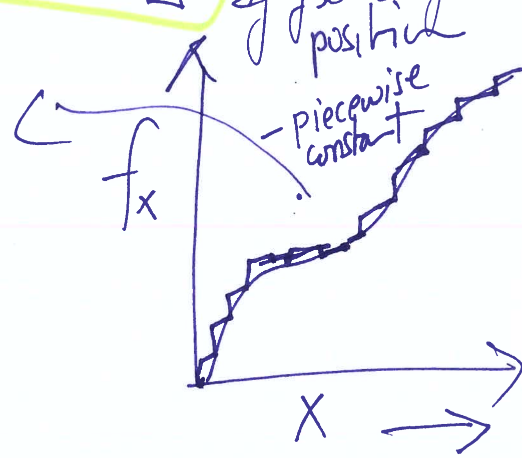
$$f^{local}_{b.f} = \int_{\xi=0}^{1}\int_{\eta=0}^{1-\xi} \begin{bmatrix} \xi & 0 \\ 0 & \xi \\ \eta & 0 \\ 0 & \eta \\ 1-\xi-\eta & 1-\xi-\eta \end{bmatrix} t^e \overbrace{\det J}^{2A^e} \, d\xi \, d\eta \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$$\int_0^1\int_0^{1-\xi} \xi \, d\xi \, d\eta = \int_0^1 \xi(1-\xi)\, d\xi = \frac{\xi^2}{2} - \frac{\xi^3}{3}\Bigg|_0^1 = \frac{1}{6}$$

$$f^{local}_{b.f.} = 2t^eA^e \left(\int_{\xi=0}^{1}\int_{\eta=0}^{1-\xi} \begin{bmatrix} \xi & 0 \\ 0 & \xi \\ \eta & 0 \\ 0 & \eta \\ 1-\xi-\eta & 0 \\ 0 & 1-\xi-\eta \end{bmatrix} d\xi \, d\eta \right) \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$$f^{local}_{b.f.} = \frac{t^eA^e}{3} \begin{bmatrix} f_x \\ f_y \\ f_x \\ f_y \\ f_x \\ f_y \end{bmatrix}$$

If f_x is a function of position



Generally, for constant body force

$$f^{\substack{\text{local}\\ \text{bodyforce}}} = t^e \left(\int_0^1 \int_0^{1-\xi} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} d\eta \, d\xi \right) \begin{bmatrix} b_x \\ b_y \end{bmatrix} 2A$$

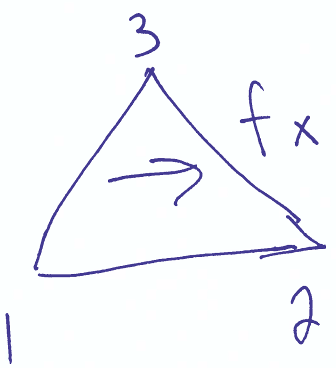
We need to find

$$\int_0^1 \int_0^{1-\xi} N_1 \, d\eta \, d\xi = \frac{1}{6}$$

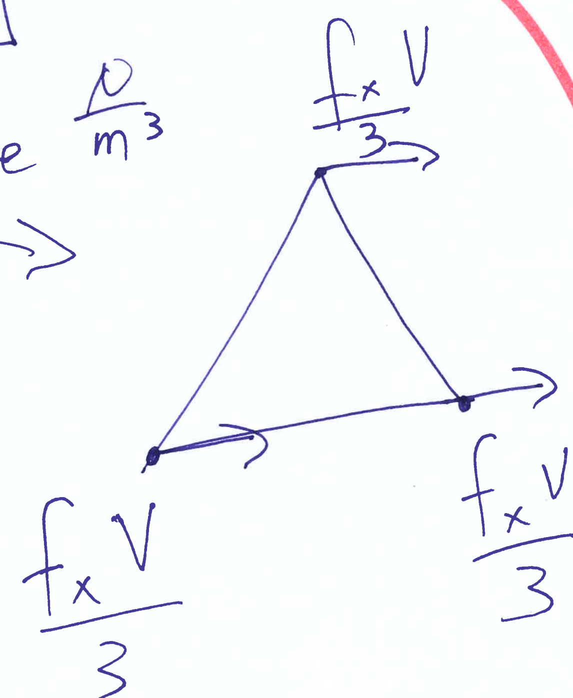
$$\int_0^1 \int_0^{1-\xi} N_2 \, d\eta \, d\xi = \int_0^1 \frac{(1-\xi)^2}{2} d\xi = \frac{(1-\xi)^3}{6} = \frac{1}{6}$$

$$\int_0^1 \int_0^{1-\xi} N_3 \, d\eta \, d\xi = \frac{1}{6}$$

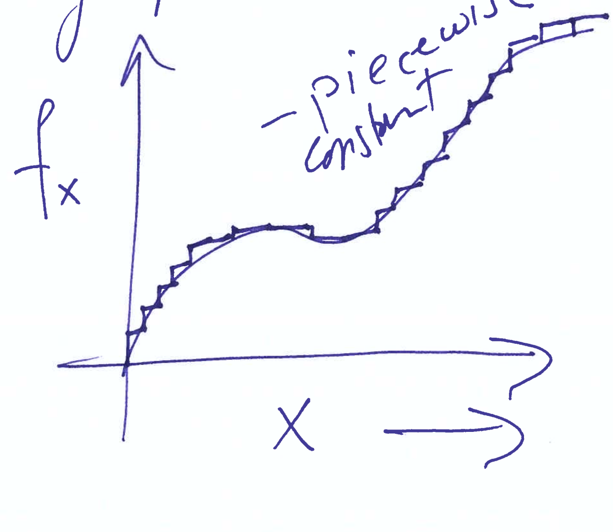
$$f^{\text{local}}_{\text{b.f.}} = \frac{t^e A^e}{3} \begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix}$$



units $\frac{\text{force}}{\text{volume}}$ $\frac{N}{m^3}$



If f_x is a function of position



If $b_x = y^2$

$y = N_1 y_1 + N_2 y_2 + N_3 y_3$

$:= (N_1 y_1 + N_2 y_2 + N_3 y_3)^2$