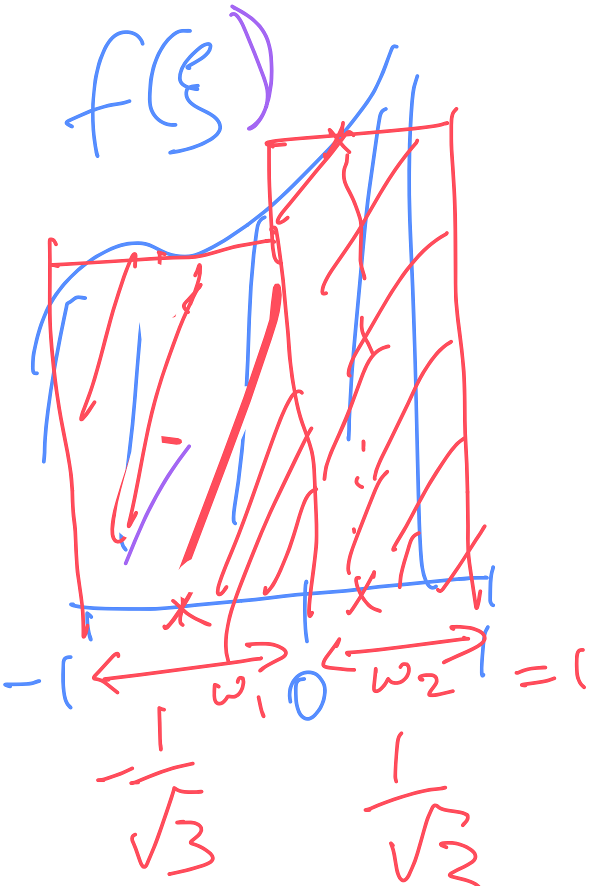
AE 510

Numerical integration



$$\int_{-1}^{1} f(\xi)\, d\xi =$$

cubic

$$f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

In code, store 10 decimal places

Table 7.1 in Belegundu text

n	w	ξ
$n=1$	$w_1 = 2$	$\xi = 0$
$n=2$	$w_1 = 1$ $w_2 = 1$	$\xi_1 = -\frac{1}{\sqrt{3}}$, $\xi_2 = \frac{1}{\sqrt{3}}$

Example

$$I = \int_{-1}^{1} \left(3e^{x} + x^2 + \frac{1}{x+2}\, dx\right)$$

$$= 8.816 \text{ exact}$$

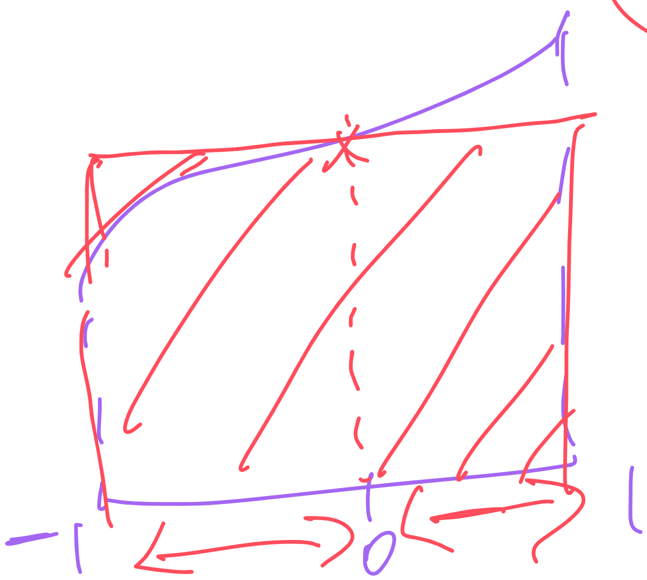
$n = 1$

$$I = \left(3e^{0} + 0^2 + \frac{1}{0+2}\right) \times 2$$

@ $\xi = 0$

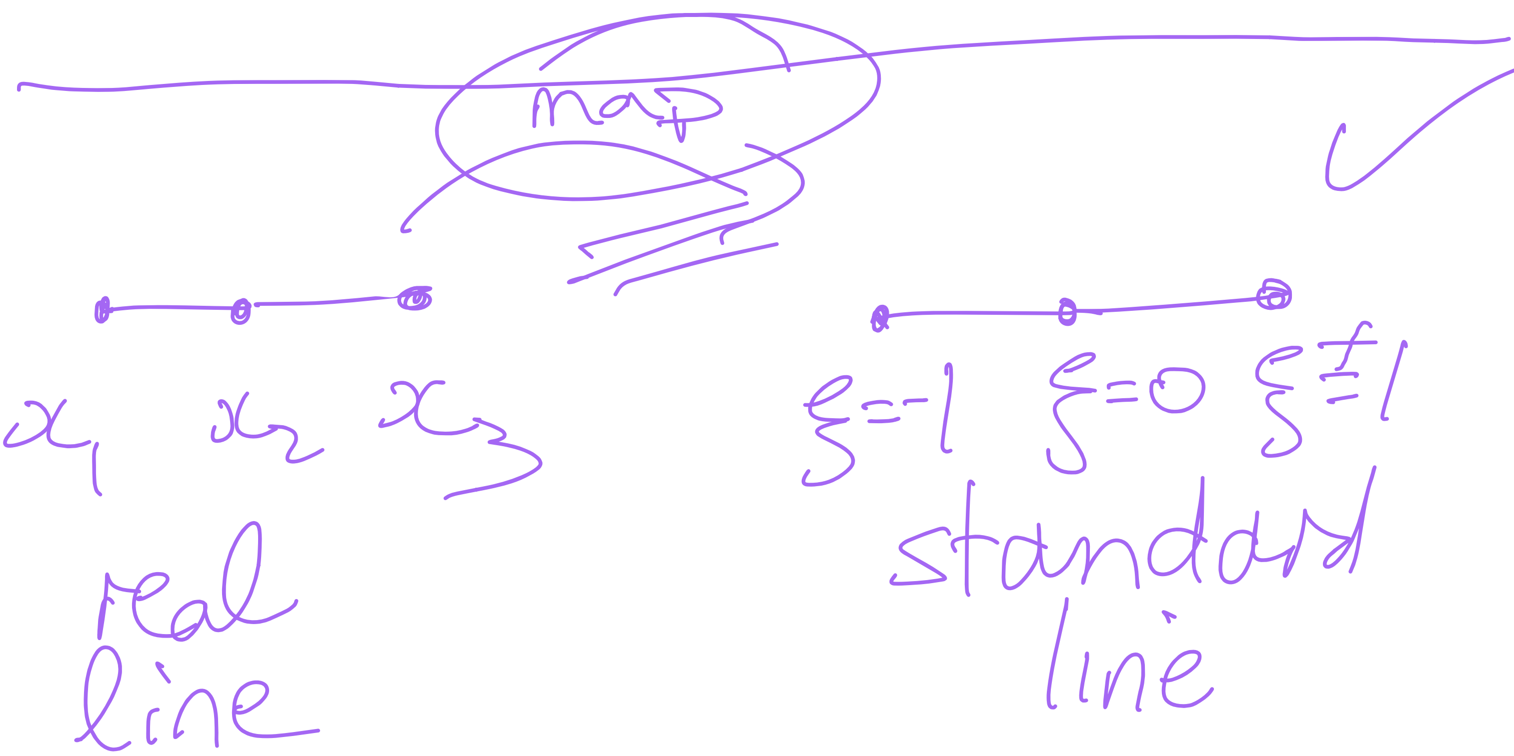
weight

$$= 7$$



$n = 2$, $I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$= \underline{\underline{8.786}}$



$$N = \left[\frac{\xi(\xi-1)}{2},\ 1-\xi^2,\ \frac{\xi(\xi+1)}{2}\right]$$

$$\text{coord} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \frac{dN}{d\xi} = \left[\frac{2\xi-1}{2},\ -2\xi,\ \frac{2\xi+1}{2}\right]$$

$$J = \frac{dN}{d\xi}(\text{coord}) = \begin{bmatrix} \frac{2\xi-1}{2}, & -2\xi, & \frac{2\xi+1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \left(\frac{2\xi-1}{2}\right)x_1 - 2\xi x_2 + \frac{2\xi+1}{2}x_3$$

$$= \cancel{\xi(x_1 - 2x_2 + x_3)} + \frac{L^e}{2}$$

where $L^e = x_3 - x_1$

If $x_2 = \dfrac{x_3 + x_1}{2}$

$$J = \frac{L^e}{2}$$

Recall that

$$B = \frac{du}{dx} = J^{-1}\frac{du}{d\xi}$$

$$= \frac{2}{L^e}\left[\frac{2\xi-1}{2},\ -2\xi,\ \frac{2\xi+1}{2}\right]$$

$$K = \int_{x_1}^{x_3} B^T D B A\, dx =$$

$$D \to E$$

$$= \int_{-1}^{1} EA\left(\frac{2}{L^e}\right)^2 \underbrace{\begin{bmatrix} \left(\xi-\frac{1}{2}\right)^2 & -2\xi\left(\xi-\frac{1}{2}\right) & \xi^2-\frac{1}{4} \\ & 4\xi^2 & -2\xi\left(\xi+\frac{1}{2}\right) \\ & & \left(\xi+\frac{1}{2}\right)^2 \end{bmatrix}}_{\text{quadratic in } \xi}$$

$$\underbrace{\left(\frac{L^e\, d\xi}{2}\right)}_{\det J\, d\xi \;=\; dx}$$

We can use 2 point integration

$$n = 2, \quad \xi_1 = -\frac{1}{\sqrt{3}}, \quad \xi_2 = \frac{1}{\sqrt{3}}$$

$$w_1 = w_2 = 1$$

$$K = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= B^T E B A \det J \Big|_{\xi = -\frac{1}{\sqrt{3}}} + B^T E B A \det J \Big|_{\xi = \frac{1}{\sqrt{3}}}$$

$$= \frac{EA}{3L^e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

In code

for i = 1 to nel

local stiffness = zero matrix

for j = 1 to nint

# of integration points

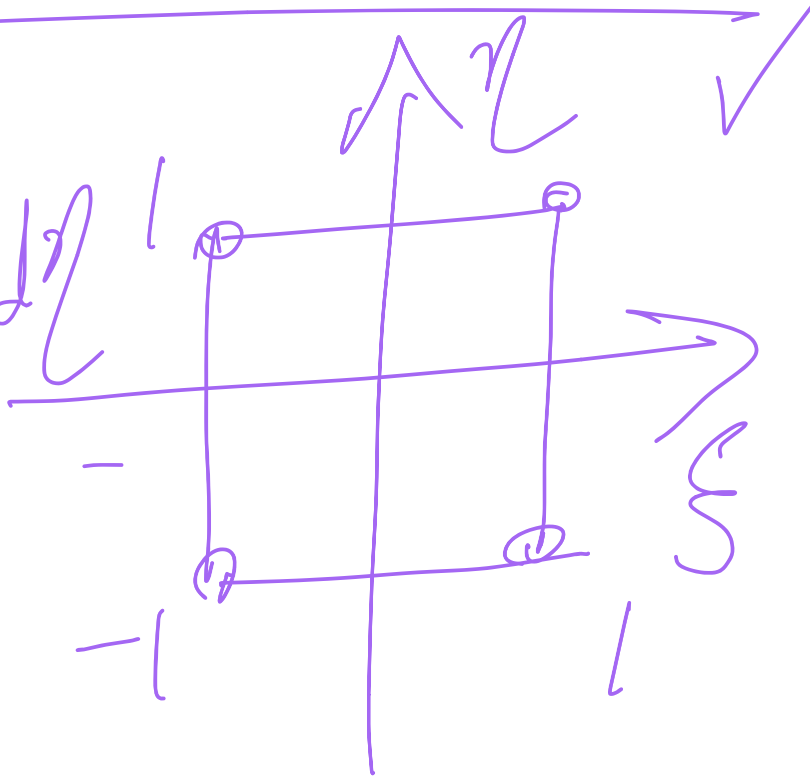
→ B, det J etc.

localstiffness = local stiffness + $(EA) B^T B \det J$;

Assembly

end

$$I = \int_{\eta=-1}^{1}\int_{\xi=-1}^{1} f(\xi,\eta)\, d\xi\, d\eta$$



standard element

$$= \int_{-1}^{1} \underbrace{\left[\int_{-1}^{1} f(\xi, \eta)\, d\xi \right]}_{= w_1 f(\xi_1) + w_2 f(\xi_2) + \dots} d\eta$$

weights

functions

$$= \int_{-1}^{1} \underbrace{\left(w_1 f(\xi_1, \eta) + w_2 f(\xi_2, \eta) + \dots \right)}_{g(\eta)} d\eta$$

$$= \int_{-1}^{1} g(\eta)\, d\eta = w_1 g(\eta_1) + w_2 g(\eta_2) + \dots$$

Substitute

$$g(\eta) = w_1 f(\xi_1, \eta) + w_2 f(\xi_2, \eta) + \dots$$

$$= w_1 \left(w_1 f(\xi_1, \eta_1) + w_2 f(\xi_2, \eta_1) \right) + w_2 \left(w_1 f(\xi_1, \eta_2) + w_2 f(\xi_2, \eta_2) \right) + \dots$$

Simplify, 2 point integration

$$= w_1^2 f(\xi_1, \eta_1) + w_1 w_2 f(\xi_2, \eta_1) +$$

$$w_1 w_2 f(\xi_1, \eta_2) + w_2^2 f(\xi_2, \eta_2)$$

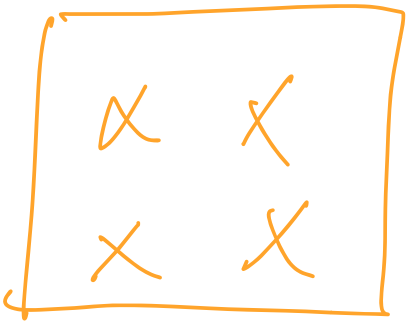
$$= \sum_{i=1}^{4} W_i f(\underline{P}^i)$$

$$W_1 = w_1^2 \qquad \underline{P}^1 = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}$$

$$W_2 = w_1 w_2 \qquad \underline{P}^2 = \begin{pmatrix} \xi_2 \\ \eta_1 \end{pmatrix}$$

$$W_3 = w_1 w_2 \qquad \underline{P}^3 = \begin{pmatrix} \xi_1 \\ \eta_2 \end{pmatrix}$$

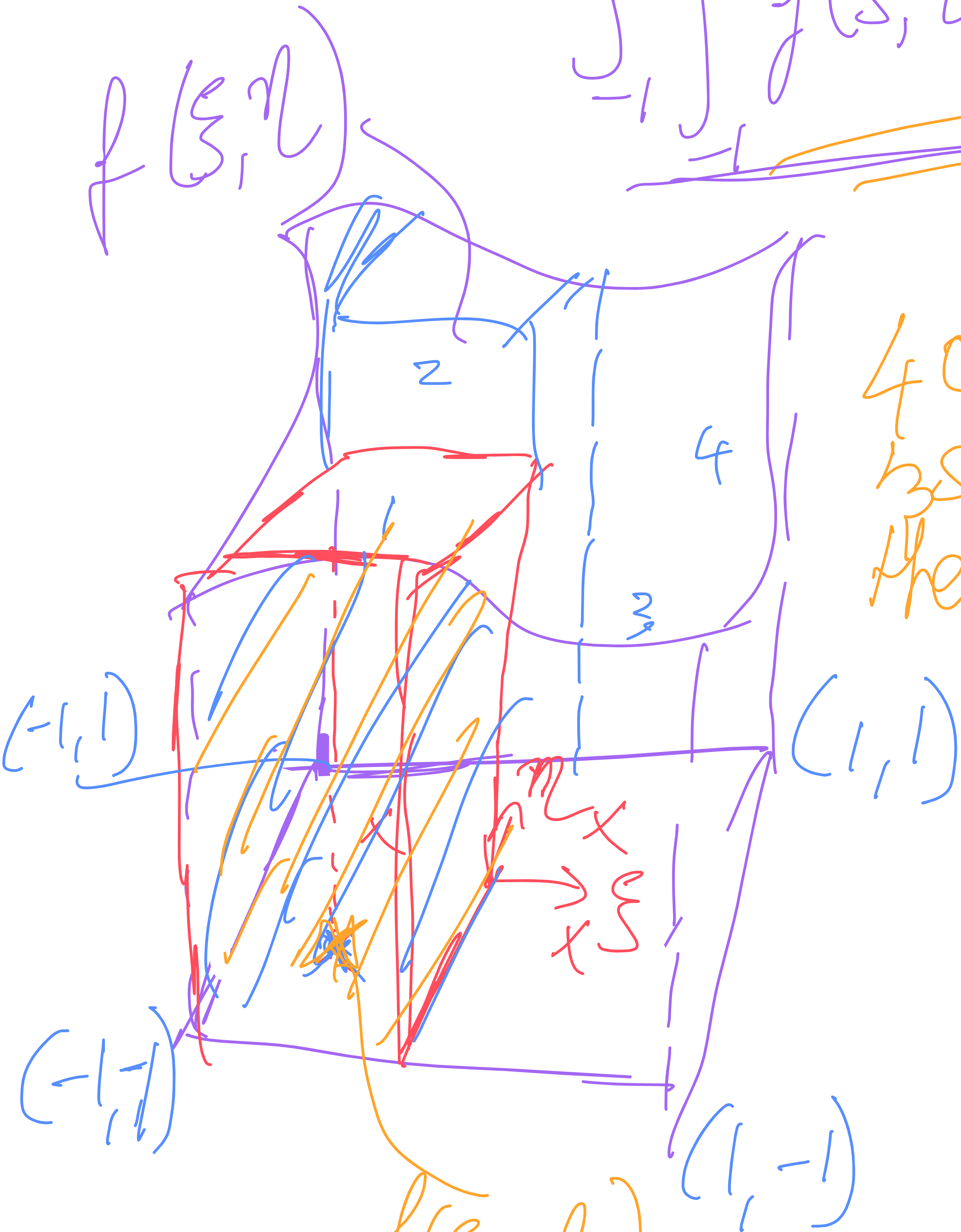
$$W_4 = w_2^2 \qquad \underline{P}^4 = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$



4 points on square

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta$$

4 cubes & sum them up



$f(\xi_1,\eta_1)$

Area of base = W_1

height = $f(\xi_1, \eta_1)$

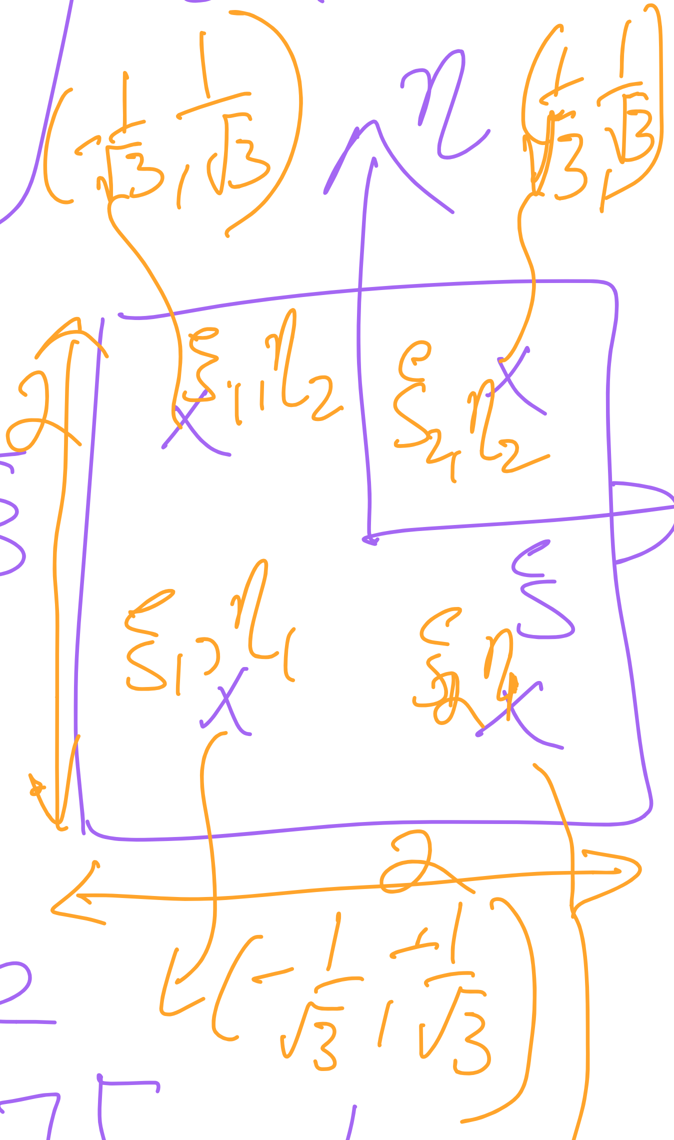
$\boxed{\text{Volume} = W_1 f(\xi_1, \eta_1)}$

In 1D, we had

$-\frac{1}{\sqrt{3}}$ $\quad$ $\frac{1}{\sqrt{3}}$

-1 $\quad$ 1

In 2D



$\xi_1 = -\frac{1}{\sqrt{3}}$ $\quad$ $\eta_1 = -\frac{1}{\sqrt{3}}$

$\xi_2 = \frac{1}{\sqrt{3}}$ $\quad$ $\eta_2 = \frac{1}{\sqrt{3}}$

The weights are

$W_1 = W_2 = W_3 = W_4 = 1$

$$\int_{\eta=-1}^{1}\int_{\xi=-1}^{1}$$

$$\left(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right)$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta = f\left(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right) + f\left(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right)$$

$$f(\xi,\eta) = \xi^3\eta^3$$

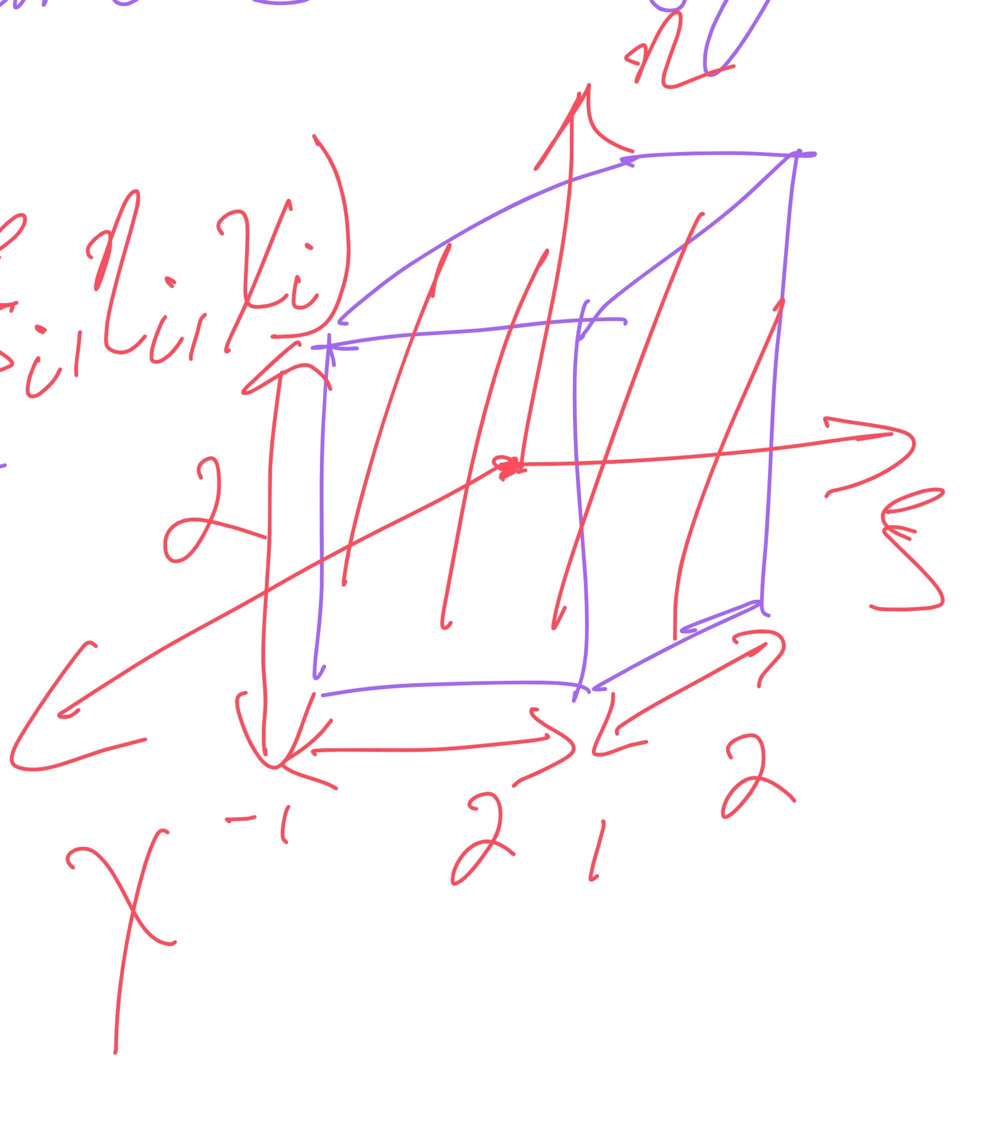
$$= \xi^4\eta$$

3D integrals

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi, \eta, \chi)\, d\xi\, d\eta\, d\chi$$

Basic elements will go from -1 to 1

$$= \sum_{i=1}^{8} W_i f(\xi_i, \eta_i, \chi_i)$$



3D table

n	$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\chi)\,d\xi\,d\eta\,d\chi$
1	$8 f(0,0,0)$
2	$f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ $+ \dots + 8$ terms
⋮	