B.C.'s Let's look at the first term

$$u'w\Big|_{x_1}^{x_2} = u'(x_2)w(x_2) - u'(x_1)w(x_1)$$

$$F(L) = P = \sigma(L)A = Eu'(L)A$$

$$u(0) = 0$$

0 L

Restriction

Constraint on weighting function
w has to be zero wherever
u is known

For one element, we'll apply shape functions

so far

$$\int_{x_1}^{x_2} (Eu'' + f)\, w\, dx = Eu'w\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} Eu'w'\, dx + \int_{x_1}^{x_2} fw\, dx = 0$$

$$u = N_1 u_1 + N_2 u_2$$

$$u' = \frac{\partial u}{\partial x} = N_1' u_1 + N_2' u_2$$

$$w = N_1 w_1 + N_2 w_2$$

$$w' = N_1' w_1 + N_2' w_2$$

Notation

$$w(x_1) = w_1$$

$$w(x_2) = w_2$$

$$u(x_1) = u_1$$

$$u(x_2) = u_2$$

First term

$$Eu'w\Big|_{x_1}^{x_2} = E\left[u'(x_2)w(x_2) - u'(x_1)w(x_1)\right]$$

$$= E\begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} -u_1' \\ u_2' \end{bmatrix}$$

Second term

$$\int_{x_1}^{x_2} Eu'w'\,dx = \int_{x_1}^{x_2} E\left(N_1'u_1 + N_2'u_2\right)\cdot\left(N_1'w_1 + N_2'w_2\right)dx$$

$$= \begin{bmatrix} w_1 & w_2 \end{bmatrix}\left[\int_{x_1}^{x_2}\begin{bmatrix} N_1' \\ N_2' \end{bmatrix} E \begin{bmatrix} N_1' & N_2' \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}dx\right]$$

Third term

$$\int_{x_1}^{x_2} fw\,dx = \int_{x_1}^{x_2} f\left(N_1w_1 + N_2w_2\right)dx$$

$$= \begin{bmatrix} w_1 & w_2 \end{bmatrix}\left[\int_{x_1}^{x_2}\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} f\,dx\right]$$

Summing the three terms

$$[w_1 \;\; w_2]\left[E\begin{bmatrix}-u_1'\\ u_2'\end{bmatrix} - \int_{x_1}^{x_2}\begin{bmatrix}N_1'\\ N_2'\end{bmatrix}E\begin{bmatrix}N_1' & N_2'\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}dx + \int_{x_1}^{x_2} f\begin{bmatrix}N_1\\ N_2\end{bmatrix}dx\right] = 0$$

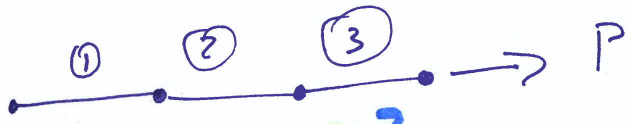
$$[w_1 \quad w_2]\begin{bmatrix}*\\ *\end{bmatrix} = 0$$

you can cancel w_1, w_2 because it's an arbitrary set of numbers.

We can simplify

$$\boxed{\underbrace{\int_{x_1}^{x_2}\begin{bmatrix}N_1'\\ N_2'\end{bmatrix}E\begin{bmatrix}N_1' & N_2'\end{bmatrix}dx}_{\text{local stiffness}}}\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \boxed{\underbrace{\int_{x_1}^{x_2} f\begin{bmatrix}N_1\\ N_2\end{bmatrix}dx + E\begin{bmatrix}-u_1'\\ u_2'\end{bmatrix}}_{\text{local force vector}}}$$

e.g.
H.W problem



$u=0$ 1 2

$$\begin{array}{c} \\ \\ \end{array}\int_{x_1}^{x_2} E \begin{array}{c}\begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} N_1^{(1)\prime 2} & N_1^{(1)\prime} N_2^{(1)\prime} \\ N_1^{(1)\prime} N_2^{(1)\prime} & N_2^{(1)\prime 2} \end{bmatrix}\end{array} dx \begin{array}{c} 1 \\ 2 \end{array} = K^{(1)}$$

$$\begin{array}{c} 1 \\ 2 \end{array} \int_{x_1}^{x_2} f \begin{bmatrix} N_1^{(1)} \\ N_2^{(1)} \end{bmatrix} dx + E \begin{bmatrix} -u_1^{(1)\prime} \\ u_2^{(1)\prime} \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array} = f^{(1)}$$

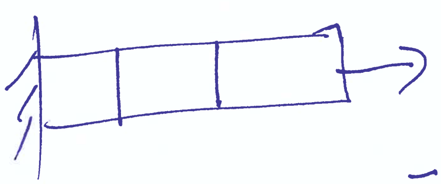
Assemble

$$E \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \multicolumn{4}{c}{\left[\begin{array}{cccc} \int_{x_1}^{x_2} N_1^{(1)\prime 2} dx & \int_{x_1}^{x_2} N_1^{(1)\prime} N_2^{(1)\prime} dx & 0 & 0 \\ \int_{x_1}^{x_2} N_1^{(1)\prime} N_2^{(1)\prime} dx & \int_{x_1}^{x_2} N_2^{(1)\prime 2} dx + \int_{x_2}^{x_3} N_1^{(2)\prime 2} dx & \int_{x_2}^{x_3} N_1^{(2)\prime} N_2^{(2)\prime} dx & 0 \\ 0 & \int_{x_2}^{x_3} N_1^{(2)\prime} N_2^{(2)\prime} dx & \int_{x_2}^{x_3} N_2^{(2)\prime 2} dx + \int_{x_3}^{x_4} N_1^{(3)\prime 2} dx & \int_{x_3}^{x_4} N_1^{(3)\prime} N_2^{(3)\prime} dx \\ 0 & 0 & \int_{x_3}^{x_4} N_1^{(3)\prime} N_2^{(3)\prime} dx & \int_{x_3}^{x_4} N_2^{(3)\prime 2} dx \end{array}\right]} \end{array} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$= \begin{bmatrix} \int_{x_1}^{x_2} f N_1 \, dx \quad (1) \\ \int_{x_1}^{x_2} f N_2 \, dx \quad (1) + \int_{x_2}^{x_3} f N_1 \, dx \quad (2) \\ \int_{x_2}^{x_3} f N_2 \, dx \quad (2) + \int_{x_3}^{x_4} f N_1 \, dx \quad (3) \\ \int_{x_3}^{4} f N_2 \, dx \quad (3) \end{bmatrix}$$

$$+ \; E \begin{bmatrix} -u_1' \\ \cancel{u_2' - u_2'} \\ \cancel{u_3' - u_3'} \\ u_4' \end{bmatrix}$$

with $E u_4' = \dfrac{F(L)}{A} = \dfrac{P}{A}$



P

$$-E u_1' = \frac{R_1}{A}$$

HW 3 $\quad u'' + x = 0$

$0 < x < 1 \quad u(0) = 0$

$u(1) = 0$

- solve using Galerkin FE with 2 noded 1D element

This is the same as the previous problem where I solved $E \frac{\partial^2 u}{\partial x^2} + f = 0$

with $E = 1$, $f = x$

* no code

2D finite element method

- Deriving stiffness matrix and force vector for deformation (elastic) problems

Indicial notation (3D)

$$u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$$
$$= \underline{u} \cdot \underline{v} \text{ (dot product)}$$

(repeated index i)

$$a_{ij} v_j = a_{i1} v_1 + a_{i2} v_2 + a_{i3} v_3$$

$$= \begin{bmatrix} a_{11} v_1 + a_{12} v_2 + a_{13} v_3 \\ a_{21} v_1 + a_{22} v_2 + a_{23} v_3 \\ a_{31} v_1 + a_{32} v_2 + a_{33} v_3 \end{bmatrix} = [A] \underline{v}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$\underline{\underline{A}} : a_{ij}$ ⟸ vector of a vector which is a matrix

$\underline{v} : v_i$ ⟸ single free subscript means vector

Last time we did divergence theorem

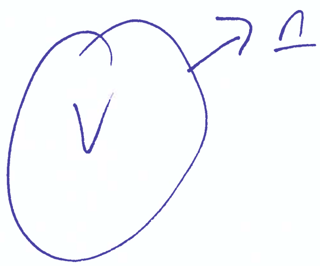
$$\int_V (a_{ij})_{,j} \, dV = \int_V (a_{i1,1} + a_{i2,2} + a_{i3,3}) \, dV$$

repeated

$$= \int_V \begin{array}{c} i=1 \\ i=2 \\ i=3 \end{array} \begin{bmatrix} a_{11,1} + a_{12,2} + a_{13,3} \\ a_{21,1} + a_{22,2} + a_{23,3} \\ a_{31,1} + a_{32,2} + a_{33,3} \end{bmatrix} dV$$

Divergence theorem

$$\int_V (a_{ij})_{,j} \, dV = \int_S a_{ij} n_j \, ds$$



$\underline{n} = n_j =$ unit normal vector to the surface

Galerkin method in 2D

$\rightarrow$ derive local stiffness matrix

Equilibrium equation using indicial notation

$$\sigma_{ij,j} + f_i = \underline{0}$$

in 1D $i=1, j=1$

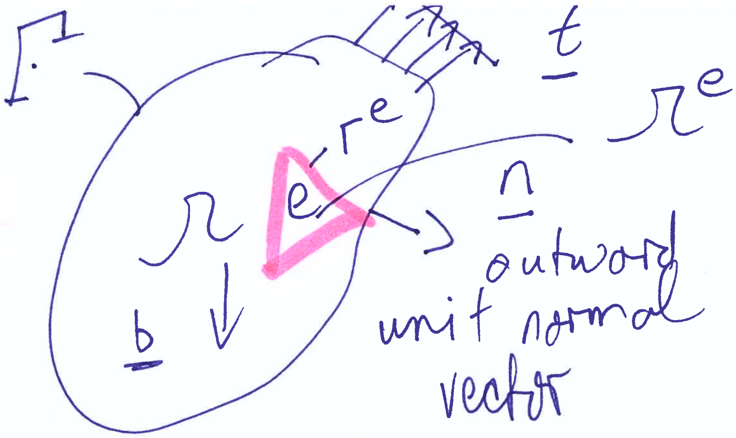
$$\sigma_{11,1} + f_1 = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + f_x = 0$$

In 2D $i,j = \{1,2\}$

$$\sigma_{i1,1} + \sigma_{i2,2} + f_i = \underline{0}$$

$$\begin{array}{c} i=1 \\ (x) \\ i=2 \\ (y) \end{array} \begin{bmatrix} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + f_x \\ \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + f_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



e: finite element

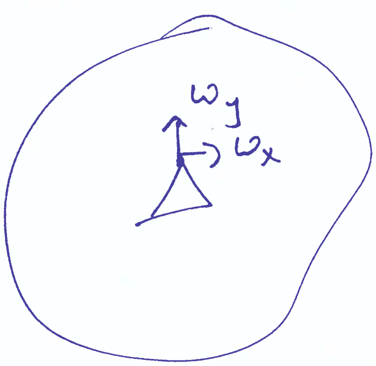
Γ^e - surface area of element

Ω^e - volume of the element

Strong form of that one element

$$\int_{\Omega^e} \left(\sigma_{ij,j} + f_i\right) w_i \; d\Omega = 0$$

w_i = weighting function which is a vector field



$$\underline{w} = \begin{bmatrix} w_x \\ w_y \end{bmatrix} = w_i \qquad \underline{2D}$$

Step 1: u-v rule of differentiation

$$\sigma_{ij,j} w_i = \left(\sigma_{ij} w_i\right)_{,j} - \left(\sigma_{ij} w_{i,j}\right)$$