Lecture 6

Wednesday, January 25, 2023 9:15 AM

Homogeneous & Isotropic Turbulence

$$R_{ij}(\vec{r}) = F(r)\, r_i r_j + G(r)\, \delta_{ij}$$

valid only in HIT

[happens in freestream turbulence]

$$R_{11}(r\vec{e}_1) = F(r) r_1^2 + G(r)$$
$$R_{22}(r\vec{e}_2) = F(r)$$

$\vec{r} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$

$$R_{11}(r\vec{e}_1) = u^2 f(r) \quad ②$$
$$R_{22}(r\vec{e}_2) = u^2 g(r) \quad ③$$

$$F(r) r^2 + G(r) = u^2 f(r)$$
$$\left[G(r) = u^2 g(r)\right] \quad ⑥$$

$$\Rightarrow F(r) r^2 + u^2 g(r) = u^2 f(r)$$
$$\Rightarrow \left[F(r) = \frac{u^2 (f(r) - g(r))}{r^2}\right] \quad ⑦$$

⑥ & ⑦ $\Rightarrow$ ①

$$\Rightarrow R_{ij}(\vec{r}) = u^2 \left[\left(\frac{f(r) - g(r)}{r^2} \right) r_i r_j + g(r)\, \delta_{ij} \right]$$

$$\Downarrow$$

$$R_{ij}(\vec{r}) = u^2 \left[f(r)\, \delta_{ij} + \frac{r}{2} \frac{df}{dr} \left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \right]$$

$$R_{ij}(\vec{r}) = u'^2\left[f(r)\,\delta_{ij} + \frac{r}{2}\frac{df}{dr}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right]$$

$$E_{ij}(\vec{k}) = \frac{1}{(2\pi)^3}\iiint_{r_3\, r_2\, r_1} R_{ij}(\vec{r})\, e^{-i\vec{k}\vec{r}}\, d\vec{r}$$

Energy Spectrum Tensor

Velocity Correlation Tensor

$$\boxed{E_{ij} = F.T.(R_{ij})}$$

General form (for HIT)

$$\boxed{E_{ij}(\vec{k}) = A(k)\,k_i k_j + B(k)\,\delta_{ij}}$$

Incompressibility $\Rightarrow k_i E_{ij} = 0$

$$k_1 E_{1j} + k_2 E_{2j} + k_3 E_{3j} = 0$$

$$k_i A(k) k_i k_j + k_i B(k)\,\delta_{ij} = 0$$

$$k_i A(k) k_i k_i + k_i B(k) = 0$$

$$A(k) k^2 k_i + B(k) k_i = 0$$

$$\Rightarrow \boxed{B(k) = -A(k) k^2} \quad (10)$$

$$\Rightarrow \boxed{E_{ij}(\vec{k}) = k^2 A(k)\left(\frac{k_i k_j}{k^2} - \delta_{ij}\right)} \quad (10) \to (8)$$

(11)

$$E_{ii}(k) = E_{11}(r) + E_{22}(k) + E_{33}(k)$$

$$= k^2 A(k)\,(1-3)$$

$$k^2 = k_1^2 + k_2^2 + k_3^2$$

$$E_{11} = k^2 A(k)\left(\frac{k_1 k_1}{k^2} - \delta_{11} + \frac{k_2 k_2}{k^2} - \delta_{22} + \frac{k_3 k_3}{k^2} - \delta_{33}\right)$$

$\Rightarrow E_{ii}(k) = 2k^2 A(k)$

$$k^2 A(k) = -\frac{1}{2} E_{ii}(k) -$$

$$\Rightarrow \left[E_{ij}(\vec{k}) = -\frac{1}{2} E_{ii}(k) \left(\frac{k_i k_j}{k^2} - \delta_{ij} \right) \right]$$

$$\Downarrow$$

$\frac{1}{2}\langle \vec{u} \cdot \vec{u} \rangle$ = Turbulent Kinetic Energy (T.K.E)

$$\left[\frac{1}{2} \langle \vec{u} \cdot \vec{u} \rangle = \int_0^\infty \int_{S(k)} \frac{1}{2} E_{ii}(k) \, dS(k) \, dk \right]$$

Our attempt @ relating $E_{ii}(k)$ to something physical