$= 2\,h(1, 0)$

weight

$\xi = 1$ on 2-3

integration point

We still need to find B, det J, det J^*

You have to write a code that takes in the integration point (ξ, η) and element coordinates and RETURNS N matrix, 'dNdx' matrix, J matrix, det J

$$N = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(1-\xi)(1-\eta)}{4}, & \dots \end{bmatrix}$$

known (inputs) integration points

$$dNd\xi = \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix}}_{\text{input for } J} = \begin{bmatrix} -\frac{(1-\eta)}{4} & \dots \\ -(1-\xi)/4 & \dots \end{bmatrix}$$

ξ, η are known

$$J = dNd\xi \times \underbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}}_{\substack{\text{coord. matrix} \\ \text{(input)}}}$$

2 2
matrix

#5

$$\underset{\substack{2\times 4 \\ \text{matrix}}}{dNdx} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix} \; ?$$

$$= \underset{2\times 2}{\begin{bmatrix} J^{-1} \end{bmatrix}} \underset{2\times 4}{\begin{bmatrix} dNd\xi \end{bmatrix}}$$

Returns N, $dNdx$, J, $\det J$

Matlab code

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$ real element

function[N, dNdx, J, det J] = element(ξ, η, coord)

$$N = \begin{bmatrix} \frac{(1-\xi)(1-\eta)}{4} & \cdots \end{bmatrix}$$

$$dNd\xi = \begin{bmatrix} -\frac{(1-\eta)}{4} & \cdots \\ -\frac{(1-\xi)}{4} & \cdots \end{bmatrix}$$

$J = dNd\xi \times coord$

$dNdx = inv(J) \times dNd\xi$

$\rightarrow = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} \end{bmatrix}$

How to find the local stiffness matrix for element e

for $(\xi, \eta) \Rightarrow (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

- Call this function that returns

N, $dNdx$, J, $\det J$ at integration point (ξ, η)

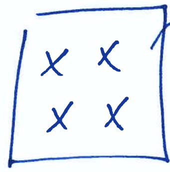
$$B = \begin{bmatrix} dNdx(1,1) & 0 & \\ 0 & dNdx(2,1) & \cdots \\ dNdx(2,1) & dNdx(1,1) & \end{bmatrix} \leftarrow \text{type}$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \\ 0 & \frac{\partial N_1}{\partial y} & \cdots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \end{bmatrix}$$

local stiffness = local stiffness +

$$B^T D B \det J \, t^e$$

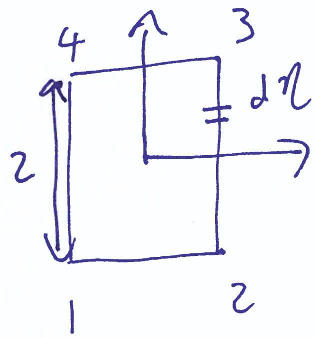
end



loop over the 4 integration points

Traction has this variable

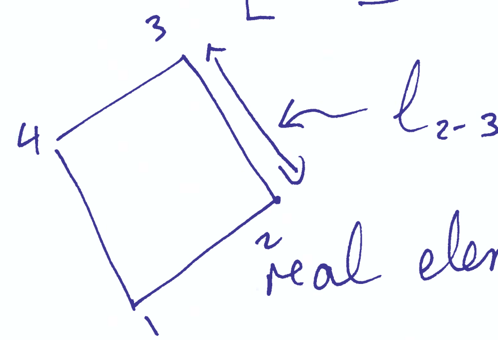
$\det J^*$ where $J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$



$$\int_{-1}^{1} h(\xi,\eta)\, d\eta$$

where $h(\xi,\eta) = N^T \begin{bmatrix} t_x \\ t_y \end{bmatrix} t^e \det J^*$

$\det J^*$ = ratio of line length



$$= \frac{l_{2-3}}{2}$$

$$\det J^* = \frac{\sqrt{dx^2 + dy^2}}{d\eta}$$

$$= \sqrt{\left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dy}{d\eta}\right)^2}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \boxed{\frac{\partial x}{\partial \eta} \quad \frac{\partial y}{\partial \eta}} \end{bmatrix}$$

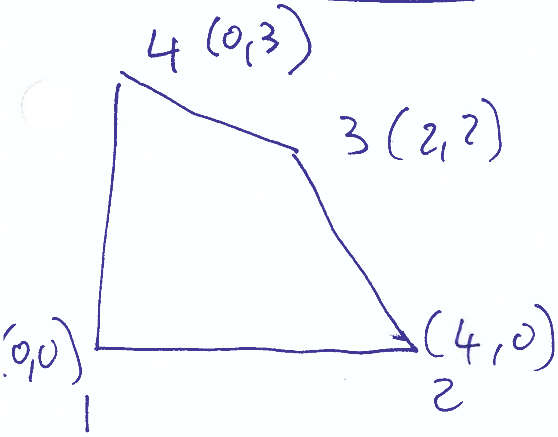
$\det J^* = \sqrt{J(2,1)^2 + J(2,2)^2}$

for 2-3 & 1-4 lines

$\det J^* = \sqrt{J(1,1)^2 + J(1,2)^2}$

for 1-2 & 3-4 lines

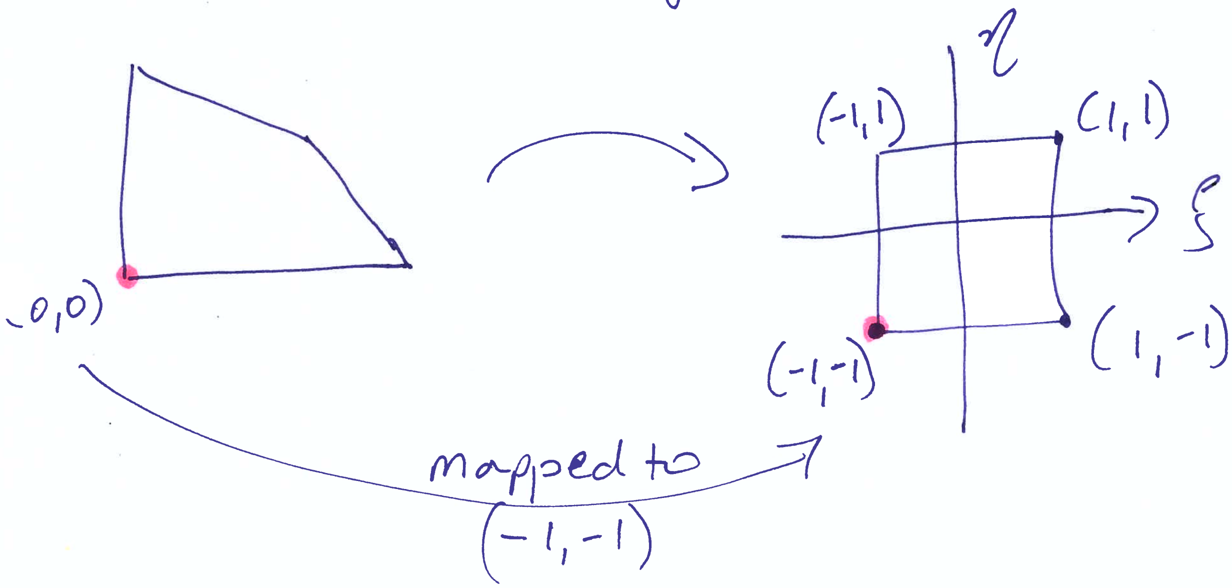
Quad element



a) Find $\frac{\partial N_1}{\partial y}$ @ node 1

b) Find $\int N_1 N_4 \, dA$ using one point integration

c) Find $\int_{l_{1-2}} N_1 \, dl$ along the line joining nodes 1 & 2.



$$[N] = \left[\frac{1}{4}(1-\xi)(1-\eta), \frac{1}{4}(1+\xi)(1-\eta), \frac{1}{4}(1+\xi)(1+\eta), \frac{1}{4}(1-\xi)(1+\eta)\right]$$

Short hand

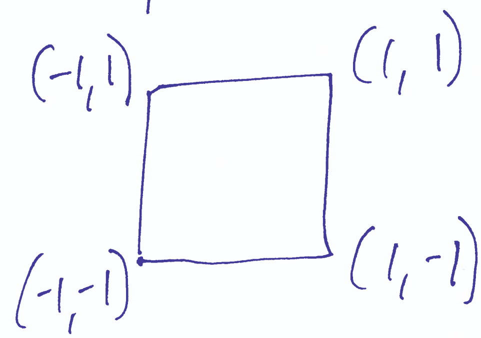
$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$$

For node 1, $\xi_i = -1$, $\eta_i = -1$

$$N_1 = \frac{1}{4}(1+\xi(-1))(1+\eta(-1))$$

$$dNd\xi = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$$

$$\frac{dN}{d\xi} = \begin{bmatrix} -\frac{(1-\eta)}{4} ; & \frac{(1-\eta)}{4} ; & \frac{(1+\eta)}{4} ; & -1\frac{(1+\eta)}{4} \\ -\frac{(1-\xi)}{4} & -\frac{(1+\xi)}{4} & \frac{(1+\xi)}{4} & \frac{(1-\xi)}{4} \end{bmatrix}$$



Shorthand

$$\frac{\partial N_i}{\partial \xi} = \frac{1}{4}\xi_i(1+\eta\eta_i)$$

$$\frac{\partial N_i}{\partial \eta} = \frac{1}{4}\eta_i(1+\xi\xi_i)$$

node	ξ_i	η_i
1	-1	-1
2	1	-1
3	1	1
4	-1	1

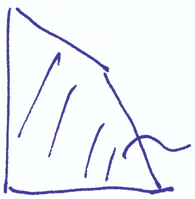
Coord (real element) $= \begin{bmatrix} 0 & 0 \\ 4 & 0 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$

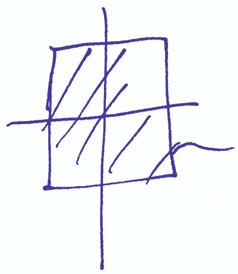
$$\boxed{J = dNd\xi \times coord}$$

$$J = \begin{bmatrix} \frac{3-\eta}{2} & -\left(\frac{1+\eta}{4}\right) \\ -\left(\frac{1+\xi}{2}\right) & \frac{5-\xi}{4} \end{bmatrix} \Longrightarrow \boxed{\det(J) = \frac{7-2\xi-3\eta}{4}}$$

not just ratio of volumes, it depends on position.

$= \frac{7}{4}$ @ $\xi = 0, \eta = 0$


$A = 7$


$A = 4$

a)

$$dNdx = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix}$$

asked to find

Plug in $\xi = -1$, $\eta = -1$ to get value @ node 1

J becomes $\begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix}$ at $\xi = -1, \eta = -1$ (node 1)

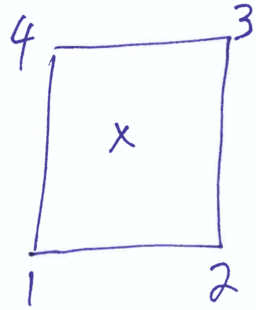
$$dNdx = J^{-1} * dNd\xi$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2/3 \end{bmatrix} * \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & +\frac{1}{2} \end{bmatrix}$$

$$\frac{dN_1}{dy} = 0 \times -\frac{1}{2} + \frac{2}{3} \times -\frac{1}{2} = \boxed{-\frac{1}{3}} \text{ at node 1}$$

b) Find $\iint_A N_1 N_4 \, dA$ — use one point integration

real element $\rightarrow A$; $N_1 N_4$ in (ξ, η)



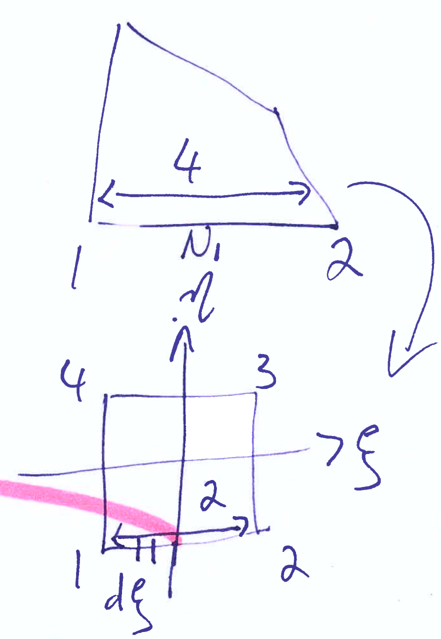
$$\int_{-1}^{1}\int_{-1}^{1} N_1 N_4 \det J \, d\xi \, d\eta = 4 \left(N_1 N_4 \det J \right)\Big|_{@(0,0)}$$

$$N_1 = \frac{(1-\xi)(1-\eta)}{4} \qquad N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

Plug in $\xi = 0$, $\eta = 0$

$$\Rightarrow 4 \times \frac{1}{4} \times \frac{1}{4} \times \frac{7}{4} = \boxed{\frac{7}{16}}$$

c) $$\int_{l_{1-2}} N_1 \, dl = \int_{\xi=-1}^{1} N_1 \det J^* \, d\xi$$



$\eta = -1$

$$J = \begin{bmatrix} \frac{3-\eta}{2} & -\left(\frac{1+\eta}{4}\right) \\ -\left(\frac{1+\xi}{2}\right) & \frac{5-\xi}{4} \end{bmatrix}$$

$$\det J^* = \sqrt{\left(\frac{3-\eta}{2}\right)^2 + \left(\frac{1+\eta}{4}\right)^2}$$

On line 1-2, $\eta = -1$

$$= \sqrt{\left(\frac{3+1}{2}\right)^2 + \cancel{\left(\frac{1-1}{4}\right)^2}}$$

$$= 2$$

For 4 noded quad

only $\rightarrow$ $\det J^* = \frac{l_{1-2}}{2}$

$\eta = -1$

$$\int_{\xi=-1}^{1} \frac{(1-\xi)(1-\eta)}{4} \, 2 \, d\xi = \int_{-1}^{1} \underbrace{\frac{(1-\xi)\cancel{4}}{\cancel{4}}}_{f(\xi)} d\xi$$

$$= 2$$

Because

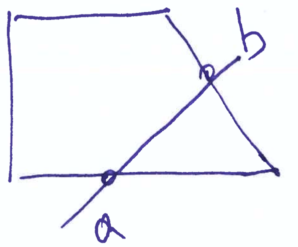
1 point integration

$\hookrightarrow 2 \times f(0) = 2$

2 point integration $\Rightarrow f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$$= 1 + \frac{1}{\sqrt{3}} + 1 - \frac{1}{\sqrt{3}} = 2$$

$\int N_1\, dl$ on line a-b



$$\int_{l_{a-b}} N_1\, dl = \int_{\xi_a}^{\xi_b} N_1 \frac{dl}{d\xi}\, d\xi$$

$\rightarrow$? $\det J^*$

$$\frac{dl}{d\xi} = \sqrt{\left(\frac{\partial x}{\partial \xi} + \frac{\partial x}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right)^2}$$

Find equation of line a-b, & differentiate

$$a\xi + b\eta = c$$

$$\frac{\partial \eta}{\partial \xi} = -\frac{a}{b}$$

Calculation of stresses

Plane stress: $$D = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Plane strain

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Calculate stress

$$\underset{3\times 1}{\sigma} = \underset{3\times 3}{D}\,\underset{3\times 8}{B}\,\underset{8\times 1}{q}$$