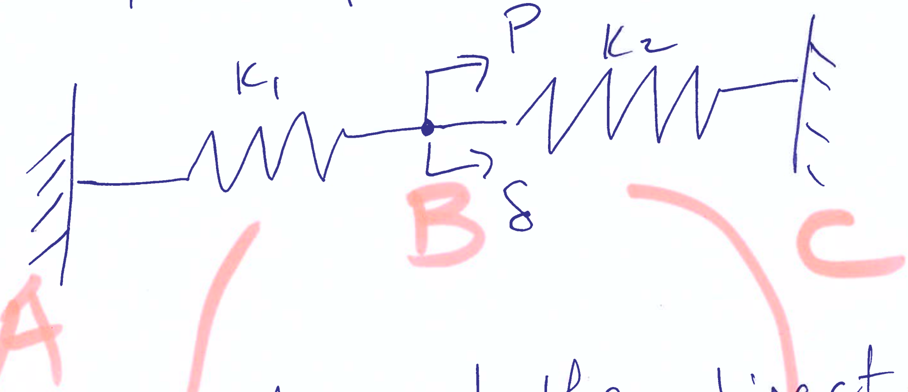
Potential Energy minimization

> ∘ At equilibrium
> ∘ (for elastic bodies) the potential energy is minimized.

P.E. Π

Process: Find displacements (or unknowns) by minimizing the potential energy

Note: FEM & RR are derived from P.E minim.



Find displacement δ

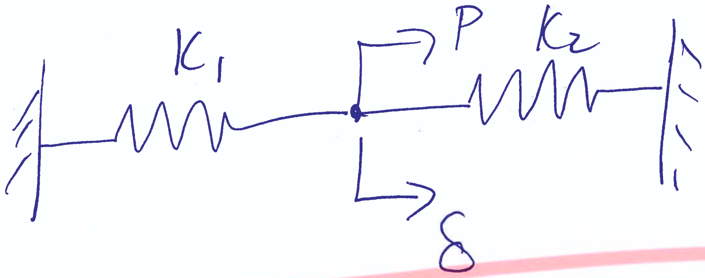
So far, we learned the direct method

$$\begin{array}{c|cc} & A & B \\ \hline A & K_1 & -K_1 \\ B & -K_1 & K_1 \end{array} \qquad \begin{array}{c|cc} & B & C \\ \hline B & K_2 & -K_2 \\ C & -K_2 & K_2 \end{array}$$

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & K_1 & -K_1 & 0 \\ B & -K_1 & K_1+K_2 & -K_2 \\ C & 0 & -K_2 & K_2 \end{array} \begin{bmatrix} 0 \\ \delta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ P \\ 0 \end{bmatrix}$$

$$\delta = \frac{P}{K_1+K_2}$$

New method: P.E. minimization



- no time involved

$$\boxed{\underset{\text{potential energy}}{\Pi} = \underset{\text{work}}{\text{Internal}} - \underset{\text{work}}{\text{External}}}$$

(related to body force, traction, point forces)

(related to stress, strain)

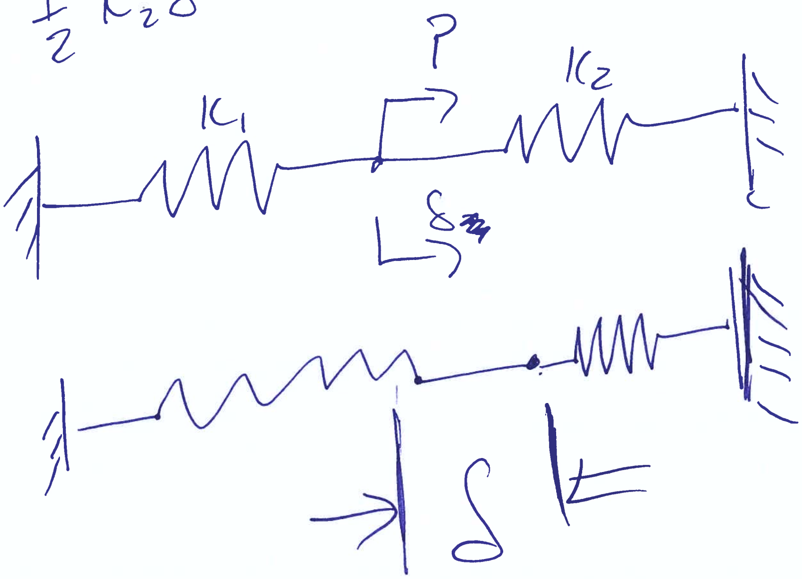
energy stored by the springs

$\frac{1}{2}K_1\delta^2 + \frac{1}{2}K_2\delta^2$

force vector · displacement vector

dot product

$P\delta$



before loading

after loading

$$\Pi = \left(\frac{1}{2}K_1\delta^2 + \frac{1}{2}K_2\delta^2\right) - (P\delta)$$

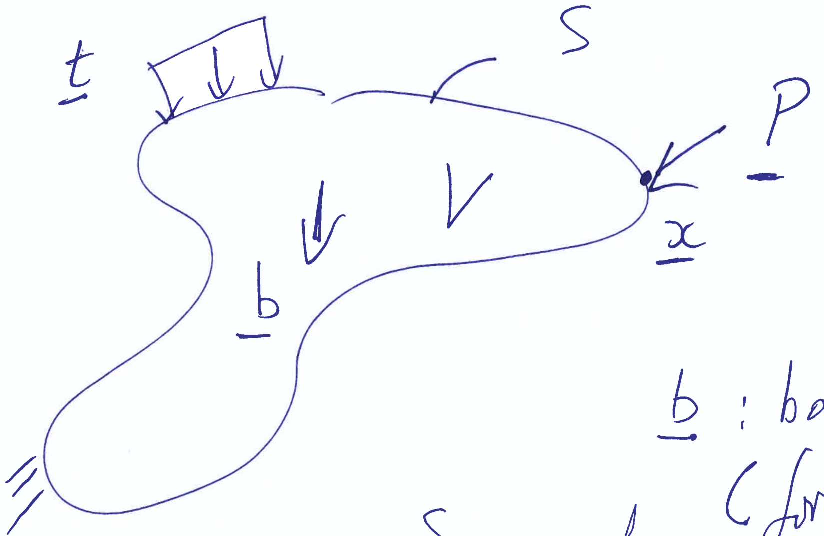
$$\frac{\partial \Pi}{\partial \delta} = 0 \text{ at minimum}$$

unknown

$$\frac{\partial \Pi}{\partial \delta} = k_1 \delta + k_2 \delta - P = 0$$

$$\delta = P/(k_1 + k_2)$$

For a continuum



$\underline{t}$: traction vector (force per unit area)

$\underline{b}$: body force vector (force per volume)

$\underline{S}$: surface area

V : volume

$\underline{P}$: point force at $\underline{x}$

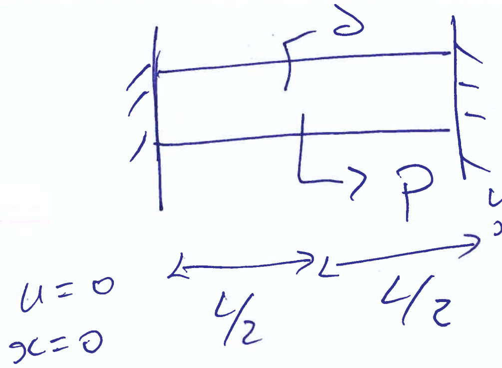
$$\Pi = \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{\sigma}\, dV - \int_S \underline{u}^T \underline{t}\, ds - \int_V \underline{u}^T \underline{b}\, dV$$

$$- \underline{u}(x)^T \underline{P}$$

3 stresses in 2D

$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \qquad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{normal strain} \\ \} \text{shear strain} \end{matrix}$$

$$\sigma^T \varepsilon = \varepsilon^T \sigma = \sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy}$$



- Elastic modulus given E
- area of cross-section A

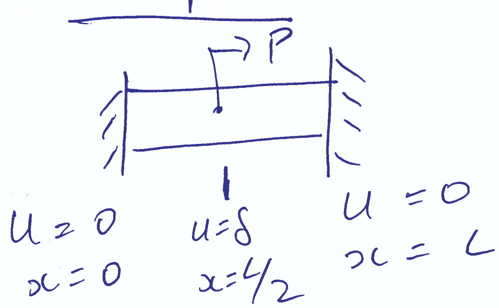
Find displacement.
For such problems we use the Rayleigh-Ritz method.

① Guess a displacement fuction (e.g. $u = a_1 x^2 + a_2 x + a_3$)

② check if it satisfies displacement boundary conditions (force it to satisfy it)

③ Write potential energy & minimize to find the unknown constants (e.g. $a_1\, a_2\, a_3$)

Step 2



$x = 0 \Rightarrow$

$0\, u = a_1 \cancel{x^2} + a_2 \cancel{x} + a_3$

$a_3 = 0$

$x = L$

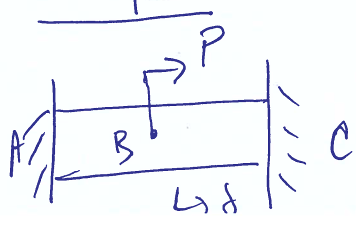
$u = 0 = a_1 L^2 + a_2 L$

$a_2 = - a_1 L$

$$u = a_1 x^2 + (-a_1 L) x + \cancel{a_3}$$

$$u = a_1 (x^2 - L x)$$

Step 3 write the potential energy



$$\Pi = \frac{1}{2} \int_V \sigma^T \varepsilon \, dV - P\delta - R_A(0) - R_c(0)$$

$$\sigma = E\,\varepsilon = E\frac{\partial u}{\partial x} = E\,a_1(2x-L)$$

$$\varepsilon = a_1(2x-L)$$

$$dV = A\,dx$$

$$\Pi = \frac{1}{2}\int_{x=0}^{x=L} E\left(a_1(2x-L)\right)^2 A\,dx - P\delta$$

$$\delta = u\left(x = L/2\right)$$
$$= a_1\left(\left(\frac{L}{2}\right)^2 - \frac{L^2}{2}\right)$$
$$= -a_1\frac{L^2}{4}$$

$$\Pi = \frac{1}{2}\int_0^L E\,a_1^2(2x-L)^2 A\,dx + \frac{P a_1 L^2}{4}$$

minimize w.r.t. a_1

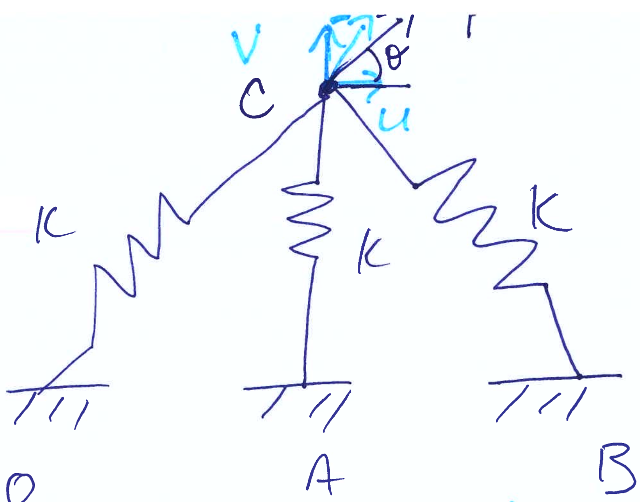
$$\frac{\partial\Pi}{\partial a_1} = 0 = \frac{1}{\cancel{2}}\int_0^L \cancel{2}\,E a_1(2x-L)^2 A\,dx + \frac{PL^2}{4}$$

$$a_1 = \frac{-\frac{PL^2}{4}}{\int_0^L E(2x-L)^2 A\,dx} = -\frac{3P}{4LEA}$$

$$u = a_1 x^2 + a_2 x$$

$$a_1 = -\frac{3P}{4LEA}$$

$$a_2 = \frac{3P}{4EA}$$



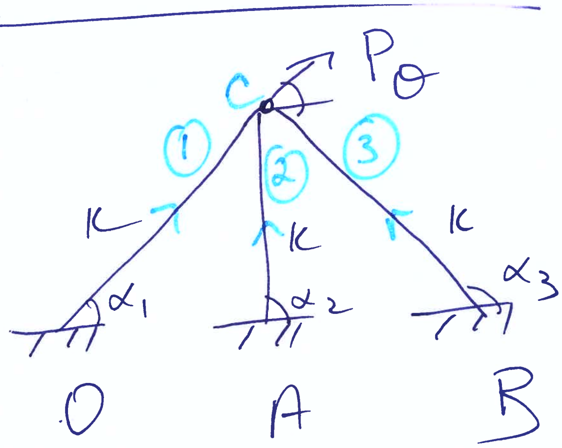
u, v are displacements of C

O, A, B fixed

Force P at angle θ

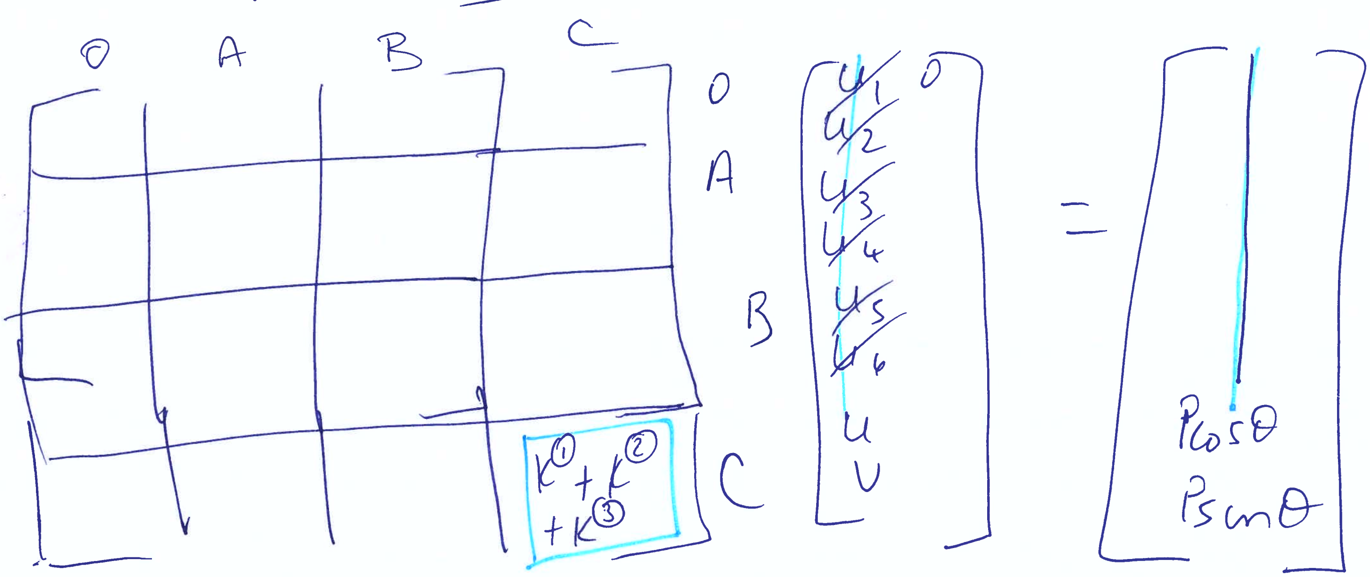
Find the displacements u, v.

Direct Method



$$\begin{array}{ccc} O & C & \\ \begin{bmatrix} K^{(1)} & -K^{(1)} \\ -K^{(1)} & K^{(1)} \end{bmatrix} & & \begin{array}{c} O \\ C \end{array} \end{array} \qquad \begin{array}{ccc} B & C & \\ \begin{bmatrix} K^{(3)} & -K^{(3)} \\ -K^{(3)} & K^{(3)} \end{bmatrix} & & \begin{array}{c} B \\ C \end{array} \end{array}$$

$$\begin{array}{ccc} A & C & \\ \begin{bmatrix} K^{(2)} & -K^{(2)} \\ -K^{(2)} & K^{(2)} \end{bmatrix} & & \begin{array}{c} A \\ C \end{array} \end{array}$$



$$\left(K^{(1)} + K^{(2)} + K^{(3)}\right) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} P\cos \\ P\sin\theta \end{bmatrix}$$

$$K^{(1)} = K \begin{bmatrix} C_1^2 & C_1 S_1 \\ C_1 S_1 & S_1^2 \end{bmatrix} \quad \text{etc.}$$