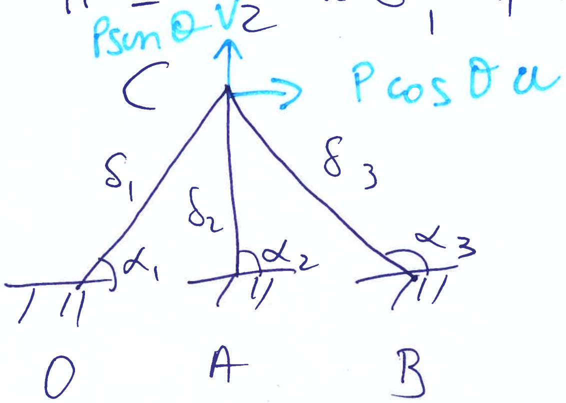
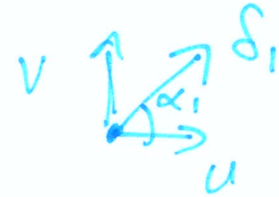
$$\Rightarrow K\begin{bmatrix} C_1^2+C_2^2+C_3^2 & C_1S_1+C_2S_2+C_3S_3 \\ C_1S_1+C_2S_2+C_3S_3 & S_1^2+S_2^2+S_3^2\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}P\cos\theta\\ P\sin\theta\end{bmatrix}$$

New method P.E. minimization

$$\Pi = \frac{1}{2}K\delta_1^2 + \frac{1}{2}K\delta_2^2 + \frac{1}{2}K\delta_3^2 - P\cos\theta u - P\sin\theta v$$





$$\delta_1 = u\cos\alpha_1 + v\sin\alpha_1$$
$$= uC_1 + vS_1$$

$$\Pi = \frac{1}{2}K(uC_1+vS_1)^2 + \frac{1}{2}K(uC_2+vS_2)^2 \text{ etc.}$$
$$+\frac{1}{2}K(uC_3+vS_3)^2 - P\cos\theta u - P\sin\theta v$$

Minimize wrt unknowns u, v

$$\begin{bmatrix}\dfrac{\partial\Pi}{\partial u}\\ \dfrac{\partial\Pi}{\partial v}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}K\overbrace{\delta_1}^{uC_1+vS_1}C_1 + K\delta_2C_2 + K\delta_3C_3 - P\cos\theta\\ K\delta_1S_1 + K\delta_2S_2 + K\delta_3S_3 - P\sin\theta\end{bmatrix}$$

Multiply out first row to set Eq.

$$K(uC_1+vS_1)C_1 + K(uC_2+vS_2)C_2 + K(uC_3+vS_3)C_3$$
$$-P\cos\theta = 0$$

$$(KC_1^2 + KC_2^2 + KC_3^2)u + (KS_1C_1 + KS_2C_2 + KS_3C_3)V = P\cos\theta$$

$$(KS_1^2 + KS_2^2 + KS_3^2)V + (KS_1C_1 + KS_2C_2 + KS_3C_3)u = P\sin\theta$$

Put in matrix form

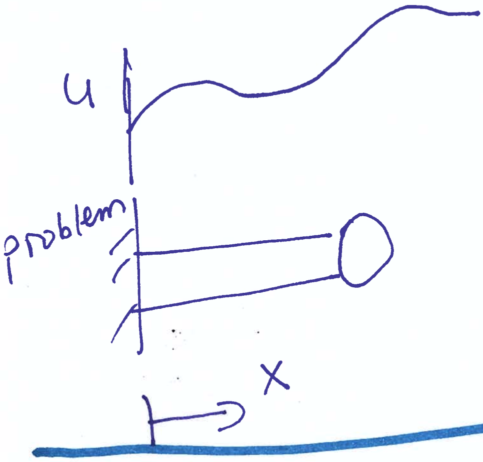
$$\begin{bmatrix} KC_1^2 + KC_2^2 + KC_3^2 & KS_1C_1 + KS_2C_2 + KS_3C_3 \\ KS_1C_1 + KS_2C_2 + KS_3C_3 & KS_1^2 + KS_2^2 + KS_3^2 \end{bmatrix} \begin{bmatrix} u \\ V \end{bmatrix} = \begin{bmatrix} P\cos\theta \\ P\sin\theta \end{bmatrix}$$

same as direct method

Finite Element Method (1D)

- This is an extension of the Rayleigh-Ritz method using local basis functions

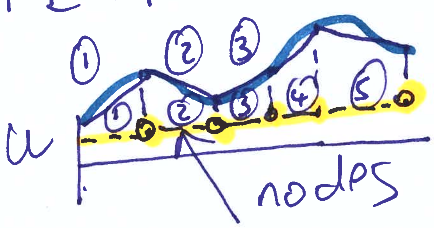
R.R. - uses global basis function



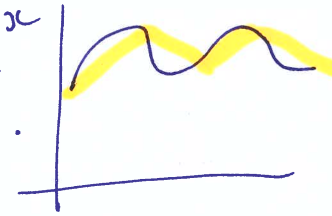
$$u = ax^3 + bx^2 + cx + d$$

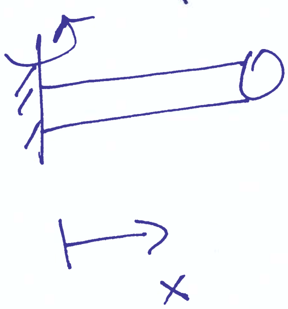
$$a, b, c, d = ?$$

FEM → local basis function



$$u = A\sin x + Be^{2x} + \dots$$





$$u^{(5)} = a + bx$$

We have 2 constants per element

If we have 5 elements, we need to solve for 10 constants

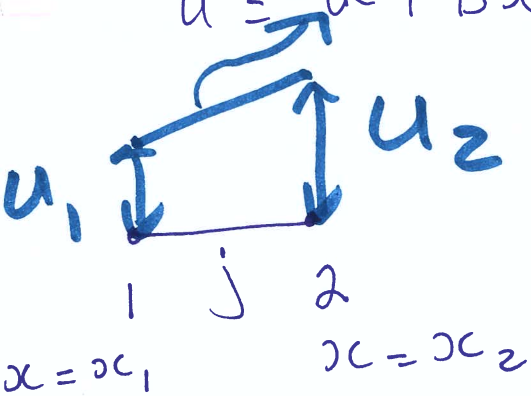
Why FEM? - easy to code

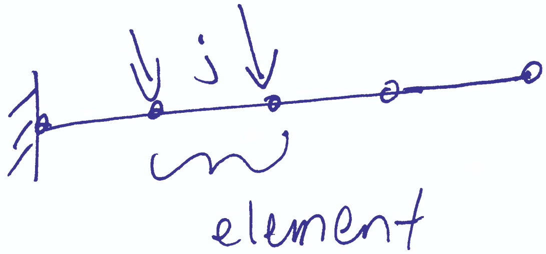
- can represent any function

FEM

Interpolation for 1 element

$u = a + bx$





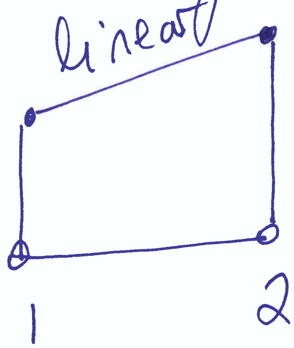
u_1, u_2 displacements on node 1 & 2 of element j

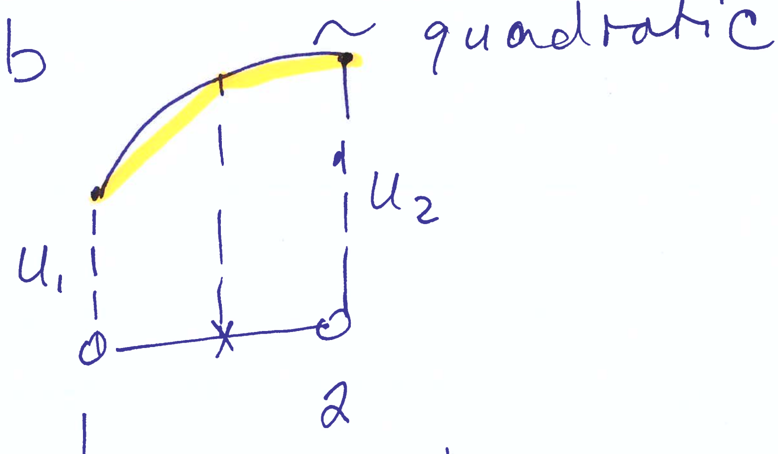
$u = u_1$ @ $x = x_1$

$u = u_2$ @ $x = x_2$

$$\boxed{\begin{aligned} u_1 &= a + bx_1 \\ u_2 &= a + bx_2 \end{aligned}}$$

solve for, a, b





Interpolation for 1D elements

$$\boxed{u = \frac{x_2 - x}{x_2 - x_1} u_1 + \frac{x - x_1}{x_2 - x_1} u_2}$$

Check if $u = u_1$ @ $x = x_1$ ✓

$u = u_2$ @ $x = x_2$ ✓

Shorthand notation

$$u = N_1 u_1 + N_2 u_2$$

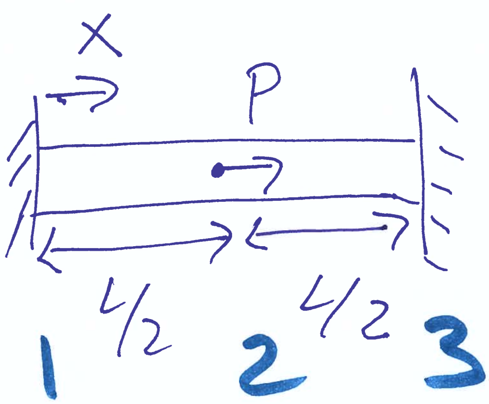
unknown displacements

$$N_1 = \frac{x_2 - x}{x_2 - x_1} \qquad N_2 = \frac{x - x_1}{x_2 - x_1}$$

functions of x

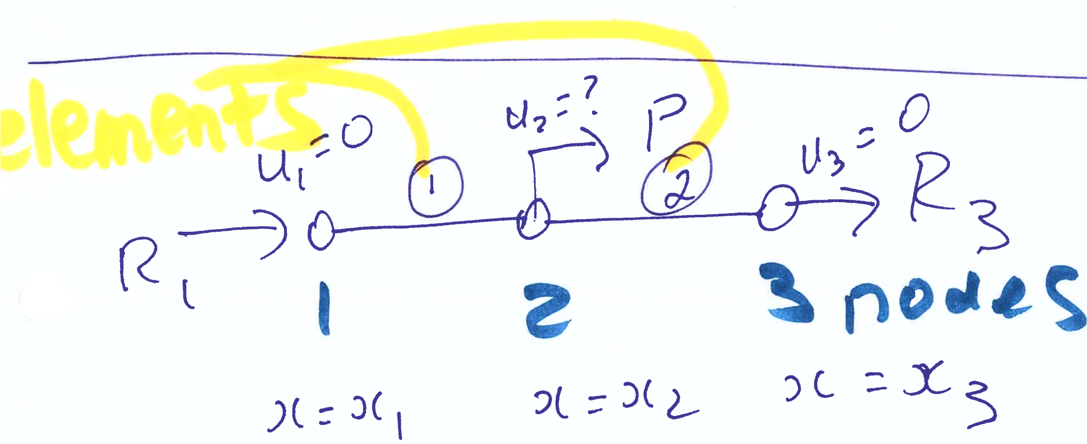
N_1, N_2 are called shape functions

Example



E, A, L given

Find displacement



$x_1 = 0$

$x_2 = L/2$

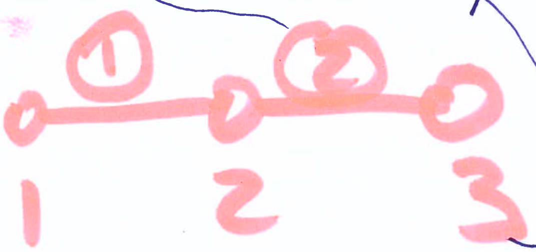
$x_3 = L$

$$\Pi = \underbrace{\frac{1}{2}\int_{x_1}^{x_2} E\,\varepsilon^{(1)2} A\,dx}_{\text{internal energy in 1D}} + \frac{1}{2}\int_{x_2}^{x_3} E\,\varepsilon^{(2)2} A\,dx - P u_2 - \underbrace{R_1 u_2 - R_3 u_3}_{\substack{\text{need to be included}\\ \text{although we know that}\\ u_1 = 0\\ u_3 = 0}}$$

$$\varepsilon^{(1)} = \frac{\partial u^{(1)}}{\partial x} = \frac{\partial N_1^{(1)}}{\partial x} \underbrace{u_1}_{\text{global}} + \frac{\partial N_2^{(1)}}{\partial x} \underbrace{u_2}_{\text{global}}$$

$$= N_1^{(1)\prime} u_1 + N_2^{(1)\prime} u_2$$

$$\varepsilon^{(2)} = N_1^{(2)\prime} u_2 + N_2^{(2)\prime} u_3$$



global

Plug $\varepsilon^{(1)}$ & $\varepsilon^{(2)}$ back into Π

$$\Pi = \frac{1}{2}\int_{x_1}^{x_2} E \underbrace{\left(N_1^{(1)\prime} u_1 + N_2^{(1)\prime} u_2\right)}_{\varepsilon^{(1)}}{}^2 A\,dx + \ldots$$

$$\Pi = \frac{1}{2}\int_{x_1}^{x_2} E\Big(\underbrace{N_1^{(1)\prime}u_1 + N_2^{(1)\prime}u_2}_{\varepsilon^{(1)}}\Big)^2 A\,dx \; + \qquad \Pi^{(1)}$$

$$\frac{1}{2}\int_{x_2}^{x_3} E\left(N_1^{(2)\prime}u_2 + N_2^{(2)\prime}u_3\right)^2 A\,dx \qquad \Pi^{(2)}$$

$$- P u_2 - R_1 u_1 - R_3 u_3$$

$$\Pi = \Pi^{(1)} + \Pi^{(2)}$$

$$\Pi^{(1)} = \frac{1}{2}\int_{x_1}^{x_2} E\left(N_1^{(1)\prime}u_1 + N_2^{(1)\prime}u_2\right)^2 A\,dx - P u_2$$

$$- R_1 u_1$$

Do not double count P

$$\Pi^{(2)} = \frac{1}{2}\int_{x_2}^{x_3} E\left(N_1^{(2)\prime}u_2 + N_2^{(2)\prime}u_3\right)^2 A\,dx - R_3 u_3$$

Next step: P. E. minimization

$$\begin{bmatrix} \dfrac{\partial \Pi}{\partial u_1} \\ \dfrac{\partial \Pi}{\partial u_2} \\ \dfrac{\partial \Pi}{\partial u_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \Pi^{(1)}}{\partial u_1} + \cancel{\boxed{\dfrac{\partial \Pi^{(2)}}{\partial u_1}}}^{\,0} \\ \dfrac{\partial \Pi^{(1)}}{\partial u_2} + \dfrac{\partial \Pi^{(2)}}{\partial u_2} \\ \cancel{\boxed{\dfrac{\partial \Pi^{(1)}}{\partial u_3}}}_{\,0} + \dfrac{\partial \Pi^{(2)}}{\partial u_3} \end{bmatrix}$$

Assembly process

element (1) does not have node 3

F - element 1

$$\frac{\partial \Pi^{(1)}}{\partial u_p} = \int_{x_1}^{x_2} EA\left(N_1^{(1)\prime} u_1 + N_2^{(1)\prime} u_2\right) N_1^{(1)\prime}\, dx - R_1$$

$$\frac{\partial \Pi^{(1)}}{\partial u_2} = \int_{x_1}^{x_2} EA\left(N_1^{(1)\prime} u_1 + N_2^{(1)\prime} u_2\right) N_2^{(1)\prime} - P$$

[K] local stiffness matrix — local (f) force vector

$$\begin{bmatrix} \dfrac{\partial \Pi^{(1)}}{\partial u_1} \\ \dfrac{\partial \Pi^{(1)}}{\partial u_2} \end{bmatrix} = \left[\int_{x_1}^{x_2} EA\, dx \begin{bmatrix} N_1^{(1)\prime 2} & N_2^{(1)\prime} N_1^{(1)\prime} \\ N_1^{(1)\prime} N_2^{(1)\prime} & N_2^{(1)\prime 2} \end{bmatrix}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} R_1 \\ P \end{bmatrix}$$

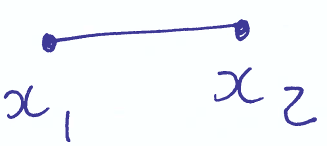
$$K = \boxed{\int_{x_1}^{x_2} EA \begin{bmatrix} N_1'^2 & N_1'N_2' \\ N_1'N_2' & N_2'^2 \end{bmatrix} dx}$$ for each element

$$\left.\begin{aligned} N_1 &= \frac{x_2 - x}{x_2 - x_1} \\ N_2 &= \frac{x - x_1}{x_2 - x_1} \end{aligned}\right\}$$ for each element

$$N_1' = \frac{\partial N_1}{\partial x} = \frac{-1}{x_2 - x_1} = \frac{-1}{\ell}$$

$$N_2' = \frac{+1}{x_2 - x_1} = \frac{1}{\ell}$$

$x_2 - x_1$ = length of the element



$$K = \int_{x_1}^{x_2} \left[EA \begin{bmatrix} \frac{1}{\ell^2} & -\frac{1}{\ell^2} \\ -\frac{1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} dx \right]$$

$$= \frac{EA}{\ell^{\cancel{2}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (x_2 - x_1) \to \cancel{\ell}\ \ell$$

$$\boxed{K = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

used it in the first H.W.

length of element j

$K_{11} = \frac{EA}{l} \cdot 1$

code