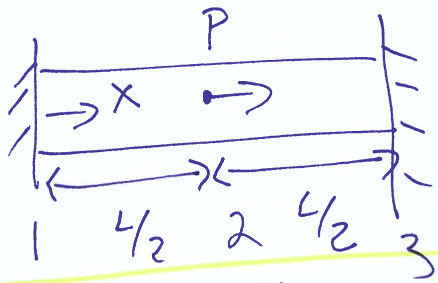
FEM 1D   Example: Recap



E, A, L given
Find displacement

$$\Pi = \frac{1}{2}\int_{x_1}^{x_2} E\left(\overset{(1)}{N}{}_1' u_1 + \overset{(1)}{N}{}_2' u_2\right)^2 A\,dx \; +$$

$$\frac{1}{2}\int_{x_2}^{x_3} E\left(\overset{(2)}{N}{}_1' u_2 + \overset{(2)}{N}{}_2' u_3\right)^2 A\,dx - P u_2 - R_1 u_1$$

$$- R_3 u_3$$

$$\Pi = \Pi^{(1)} + \Pi^{(2)}$$

Minimize P.E.

$$
\begin{bmatrix} \dfrac{\partial \Pi}{\partial u_1} \\[2ex] \dfrac{\partial \Pi}{\partial u_2} \\[2ex] \dfrac{\partial \Pi}{\partial u_3} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
=
\begin{array}{c}
\text{\# elements} \rightarrow \\
\begin{array}{c} \text{\# of unknowns} \\ \downarrow \end{array}
\begin{bmatrix}
\dfrac{\partial \Pi^{(1)}}{\partial u_1} + \boxed{\cancel{\dfrac{\partial \Pi^{(2)}}{\partial u_1}}}\,0 \\[2ex]
\dfrac{\partial \Pi^{(1)}}{\partial u_2} + \dfrac{\partial \Pi^{(2)}}{\partial u_2} \\[2ex]
\boxed{\cancel{\dfrac{\partial \Pi^{(1)}}{\partial u_3}}}\,0 + \dfrac{\partial \Pi^{(2)}}{\partial u_3}
\end{bmatrix}
\end{array}
$$

element (1) does not have node 3

Assembly process

F - element 1

$$
\frac{\partial \Pi^{(1)}}{\partial u_p} = \int_{x_1}^{x_2} EA\left(N_1^{(1)\prime} u_1 + N_2^{(1)\prime} u_2\right) N_1^{(1)\prime}\, dx - R_1
$$

$$
\frac{\partial \Pi^{(1)}}{\partial u_2} = \int_{x_1}^{x_2} EA\left(N_1^{(1)\prime} u_1 + N_2^{(1)\prime} u_2\right) N_2^{(1)\prime} - P
$$

$[K]$ local stiffness matrix   local (f) force vector

$$
\begin{bmatrix} \dfrac{\partial \Pi^{(1)}}{\partial u_1} \\[2ex] \dfrac{\partial \Pi^{(1)}}{\partial u_2} \end{bmatrix}
=
\left[\int_{x_1}^{x_2} EA\, dx
\begin{bmatrix} N_1^{(1)\prime 2} & N_2^{(1)\prime} N_1^{(1)\prime} \\ N_1^{(1)\prime} N_2^{(1)\prime} & N_2^{(1)\prime 2} \end{bmatrix}
\right]
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
-
\begin{bmatrix} R_1 \\ P \end{bmatrix}
$$

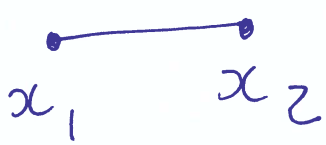
$$\boxed{K = \int_{x_1}^{x_2} EA \begin{bmatrix} N_1'^2 & N_1' N_2' \\ N_1' N_2' & N_2'^2 \end{bmatrix} dx}$$ for each element

$$N_1 = \frac{x_2 - x}{x_2 - x_1} \qquad N_1' = \frac{\partial N_1}{\partial x} = \frac{-1}{x_2 - x_1} = \frac{-1}{\ell}$$

$$N_2 = \frac{x - x_1}{x_2 - x_1} \qquad N_2' = \frac{+1}{x_2 - x_1} = \frac{1}{\ell}$$

for each element

$x_2 - x_1$ = length of the element



$$K = \int_{x_1}^{x_2} \left[EA \begin{bmatrix} \frac{1}{\ell^2} & -\frac{1}{\ell^2} \\ -\frac{1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} dx \right]$$

$$= \frac{EA}{\ell^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (x_2 - x_1) \rightarrow \ell$$

$$\boxed{K = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$ ⟸ used it in the first H.W.

↙ length of element j

$$\boxed{K_{11} = \frac{EA}{l} \cdot 1 \quad \text{code}}$$

Going back for element 2.

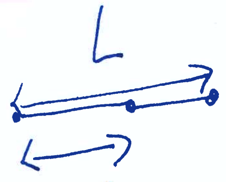
$$\begin{bmatrix} \frac{\partial \Pi^{(2)}}{\partial u_2} \\ \frac{\partial \Pi^{(2)}}{\partial u_3} \end{bmatrix} = \left[\int_{x_2}^{x_3} EA \begin{bmatrix} N_1'^2 & N_1'N_2' \\ N_1'N_2' & N_2'^2 \end{bmatrix} dx\right] \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} 0 \\ R_3 \end{bmatrix}$$

Plug these into $\begin{bmatrix} \frac{\partial \Pi}{\partial u_1} \\ \frac{\partial \Pi}{\partial u_2} \\ \frac{\partial \Pi}{\partial u_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- just provide local stiffness and local force to the code will do the assembly

only external forces go here

$$\begin{bmatrix} \frac{\partial \Pi}{\partial u_1} \\ \frac{\partial \Pi}{\partial u_2} \\ \frac{\partial \Pi}{\partial u_3} \end{bmatrix} \begin{matrix} \frac{\partial \Pi^{(1)}}{\partial u_1} \\ \frac{\partial \Pi^{(1)}}{\partial u_2} + \frac{\partial \Pi^{(2)}}{\partial u_2} \\ \frac{\partial \Pi^{(2)}}{\partial u_3} \end{matrix} \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} R_1 \\ P \\ R_3 \end{bmatrix}$$



$\ell = L/2$

$$= \frac{EA}{\ell}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} R_1 \\ P \\ R_3 \end{bmatrix}$$

Now you can delete rows/columns

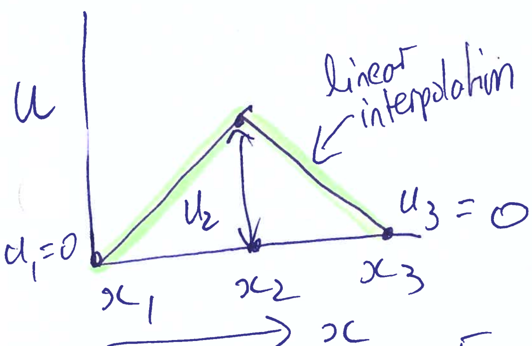
$$\frac{EA}{\ell}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} R_1 \\ P \\ R_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now you can apply B.C's and delete rows & columns - This is condensation

$$\frac{EA}{\ell} 2 u_2 - P = 0$$

$$u_2 = \frac{P\ell}{2EA} = \frac{PL}{4EA}$$

Interpolate: you can find solution anywhere
- This is an advantage over truss method, where you only know displacements at the ends



For 100 elements?

<u>Code</u>

$$\text{local stiffness} = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\text{local force} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leftarrow$$ always kept as zeros

except if you have body forces

$$+ \boxed{\text{thermal or body force}}$$

* Add external point forces at the end to the global force vector

What does your stiffness matrix look like for 100 elements?

$$\frac{EA}{l}\begin{bmatrix} 1 & -1 & & & & & \\ 1 & 2 & -1 & & & 0 & \\ & 1 & 2 & -1 & & & \\ & & 1 & 2 & & & \\ & 0 & & & \ddots & & \\ & & & & & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 = 0 \\ u_2 \\ \vdots \\ \vdots \\ u_N \neq 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ 0 \\ 0 \\ P \\ 0 \\ 0 \\ 0 \\ R_N \end{bmatrix}$$

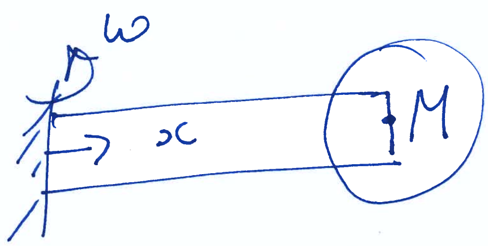
This is what your global stiffness looks like

Banded matrix
Tridiagonal matrix

Sparse matrix
(don't need to ~~sparse code~~ store the whole matrix)

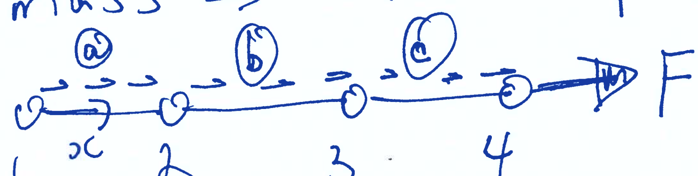
* The local stiffness remains the same. Don't need to write it over and over.

H.W.



Assume 1D

Point mass ⇒ external point force



Centrifugal force getting bigger

$$b = \frac{m}{V} x w^2 = \rho w^2 x$$

body force (N/m^3)

density

$$\int u^T b \, dV$$

The whole P.F.

$$\Pi = \frac{1}{2}\int_0^L EA u'^2 dx - \int \rho w^2 x u A dx$$

$$- (M w^2 L) u(L)$$

@

FEM code

local stiffness (same as previous)

$$= \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

length of element

The local force is different for this problem.

The point forces come at the end.

Let's consider a generic element (not the first or last)