Kinetic energy

Ayoub Gouasmi

March 13th

Let's start with the 1D compressible Euler equations. We have:

$$\begin{aligned} \frac{\partial \rho}{\partial t} \;+\; \frac{\partial}{\partial x}\big(\rho u\big) \;&=0, && (1)\\ \frac{\partial \rho u}{\partial t} \;+\; \frac{\partial}{\partial x}\big(\rho u^2 \;+\; p\big) \;&=0, && (2)\\ \frac{\partial \rho e^t}{\partial t} \;+\; \frac{\partial}{\partial x}\big(u(\rho e^t \;+\; p)\big) \;&=0. && (3) \end{aligned}$$

where $e^t := e + k,\ k := \frac{u^2}{2}$. We assume a calorically perfect gas, so pressure and internal energy are related through:

$$p \;:=\; (\gamma - 1)\rho e.$$

Let's express the time derivative of ρk (the quantity of interest) in terms of the time derivatives of mass, momentum and total energy (the quantities whose equations we have). Using the chain-rule, we obtain:

$$\frac{\partial (\rho k)}{\partial t} \;=\; \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{(\rho u)^2}{\rho}\right) \;=\; u\frac{\partial}{\partial t}\big(\rho u\big) \;-\; \frac{1}{2}u^2\frac{\partial \rho}{\partial t}.$$

We can substitute the time derivatives on the right-hand side with spatial derivatives through equations (1) and (2). This gives:

$$\begin{aligned} \frac{\partial (\rho k)}{\partial t} \;&=\; -\,u\frac{\partial}{\partial x}\big(\rho u^2 \;+\; p\big) \;+\; \frac{1}{2}u^2\frac{\partial}{\partial x}\big(\rho u\big)\\ &=\; -\,u\frac{\partial}{\partial x}\big(\rho u^2\big) \;+\; \frac{1}{2}u^2\frac{\partial}{\partial x}\big(\rho u\big) - u\frac{\partial p}{\partial x}\\ &=\; -\,u^2\frac{\partial}{\partial x}\big(\rho u\big) \;-\; \rho u^2\frac{\partial u}{\partial x} \;+\; \frac{1}{2}u^2\frac{\partial}{\partial x}\big(\rho u\big) - u\frac{\partial p}{\partial x}\\ &=\; -\,\frac{1}{2}u^2\frac{\partial}{\partial x}\big(\rho u\big) \;-\; \rho u^2\frac{\partial u}{\partial x} \;-\; u\frac{\partial p}{\partial x}\\ &=\; -\,\frac{\partial}{\partial x}\left(\frac{1}{2}u^2(\rho u)\right) \;-\; u\frac{\partial p}{\partial x}\\ &=\; -\,\frac{\partial}{\partial x}\left(\frac{1}{2}u^2(\rho u) \;+\; pu\right) \;+\; p\frac{\partial u}{\partial x}\\ &=\; -\,\frac{\partial}{\partial x}\big(u(\rho k + p)\big) \;+\; p\frac{\partial u}{\partial x} \end{aligned}$$

This gives the final result:

$$\frac{\partial (\rho k)}{\partial t} \;+\; \frac{\partial}{\partial x}\big(u(\rho k + p)\big) \;=\; p\frac{\partial u}{\partial x}. \qquad (4)$$

This equation does not express conservation unless the right-hand side term of equation (4) is zero. This is the case for incompressible flow since for such flows:

$$\frac{\partial u}{\partial x} \;=\; 0.$$

Let's do it in 2D now. We have:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{5}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho u v) = 0, \tag{6}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v^2 + p) = 0, \tag{7}$$

$$\frac{\partial \rho e^t}{\partial t} + \frac{\partial}{\partial x}\big(u(\rho e^t + p)\big) + \frac{\partial}{\partial y}\big(v(\rho e^t + p)\big) = 0. \tag{8}$$

where $e^t := e + k,\ k := \frac{u^2+v^2}{2}$. Just as before, express the time derivative of ρk (the quantity of interest) in terms of the time derivatives of mass, momentum and total energy (the quantities whose equations we have). Using the chain-rule, we obtain:

$$\frac{\partial(\rho k)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{(\rho u)^2}{\rho} + \frac{(\rho v)^2}{\rho}\right) = u\frac{\partial}{\partial t}(\rho u) + v\frac{\partial}{\partial t}(\rho v) - \frac{1}{2}(u^2 + v^2)\frac{\partial \rho}{\partial t}.$$

We can substitute the time derivatives on the right-hand side with spatial derivatives through equations (5), (6) and (7).

Question 1: This gives:

$$\begin{aligned}\frac{\partial(\rho k)}{\partial t} &= \text{Figure}\\ &= \text{it}\\ &= \text{out}\\ &= \text{with}\\ &= \text{chain}\\ &= \text{rules}\end{aligned}$$

This gives the final result:

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial}{\partial x}\big(u(\rho k + p)\big) + \frac{\partial}{\partial y}\big(v(\rho k + p)\big) = p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{9}$$

For incompressible flows the right-hand side term is zero (divergence free).

Question 2: Find a problem that illustrates that kinetic energy is not conserved for compressible flows. Let's stick to the 1D system (1) - (3) and set periodic boundary conditions. Run it with a decent numerical scheme on a decently fine grid (so that your numerical solution can be trusted) and show a time history of:

$$\int_\Omega (\rho k)\ dV$$

Question 3: Consider the two-dimensional compressible Navier-Stokes equation (just add a viscous diffusion term). Prove that viscous diffusion dissipates kinetic energy.

References