Kinetic Energy

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$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \text{ (mass)}$$
$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho u v) = 0, \text{ (x-momentum)}$$
$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v^2 + p) = 0, \text{ (y-momentum)}$$
$$\frac{\partial \rho e^t}{\partial t} + \frac{\partial}{\partial x}(u(\rho e^t + p)) + \frac{\partial}{\partial y}(v(\rho e^t + p)) = 0, \text{ (Energy)}$$

$$\frac{\partial(\rho k)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{(\rho u)^2}{\rho} + \frac{(\rho v)^2}{\rho}\right) = u\frac{\partial}{\partial t}(\rho u) + v\frac{\partial}{\partial t}(\rho v) - \frac{1}{2}(u^2 + v^2)\frac{\partial \rho}{\partial t}$$

From momentum conservation, we know that:

$$\frac{\partial \rho u}{\partial t} = -\frac{\partial}{\partial x}(\rho u^2 + p) - \frac{\partial}{\partial y}(\rho u v)$$
$$\frac{\partial \rho v}{\partial t} = -\frac{\partial}{\partial x}(\rho u v) - \frac{\partial}{\partial y}(\rho v^2 + p)$$

From conservation of mass:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v)$$
$$= -\left(\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right)$$

We substitute & obtain:

$$\frac{\partial(\rho k)}{\partial t} = -u\left(\frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho u v)\right) - v\left(\frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v^2 + p)\right) + \frac{1}{2}(u^2 + v^2)\left(\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right)$$

$$= -u\frac{\partial}{\partial x}(\rho u^2 + p) - u\frac{\partial}{\partial y}(\rho u v) - v\frac{\partial}{\partial x}(\rho u v) - v\frac{\partial}{\partial y}(\rho v^2) + \frac{1}{2}(u^2 + v^2)\left(\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right)$$

$$= -u\frac{\partial}{\partial x}(\rho u^2) - u\frac{\partial}{\partial x}(p) - u\frac{\partial}{\partial y}(\rho u v) - v\frac{\partial}{\partial x}(\rho u v) - v\frac{\partial}{\partial y}(\rho v^2) - v\frac{\partial}{\partial y}(p) + \frac{1}{2}(u^2 + v^2)\left(\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right) \checkmark$$

$$= -u\frac{\partial}{\partial x}(\rho u^2) - u\frac{\partial}{\partial x}(p) - u\frac{\partial}{\partial y}(\rho u v) - v\frac{\partial}{\partial x}(\rho u v) - v\frac{\partial}{\partial y}(\rho v^2) - v\frac{\partial}{\partial y}(p) + \frac{1}{2}\left(u^2\frac{\partial}{\partial x}(\rho u) + u^2\frac{\partial}{\partial y}(\rho v) + v^2\frac{\partial}{\partial x}(\rho u) + v^2\frac{\partial}{\partial y}(\rho v)\right) \checkmark$$

$$= -u\frac{\partial}{\partial x}(u(\rho u)) - u\frac{\partial}{\partial x}(p) - u\frac{\partial}{\partial y}(\rho u v) - v\frac{\partial}{\partial x}(\rho u v) - v\frac{\partial}{\partial y}(v(\rho v)) - v\frac{\partial}{\partial y}(p) + \frac{1}{2}u^2\frac{\partial}{\partial x}(\rho u) + \frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v) + \frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u) + \frac{1}{2}v^2\frac{\partial}{\partial y}(\rho v) \checkmark$$

$$= -\rho u^2\frac{\partial}{\partial x}(u) - u^2\frac{\partial}{\partial x}(\rho u) - u\frac{\partial}{\partial x}(p) - u\frac{\partial}{\partial y}(\rho u v) - v\frac{\partial}{\partial x}(\rho u v) - \rho v^2\frac{\partial}{\partial y}(v) - v^2\frac{\partial}{\partial y}(\rho v) - v\frac{\partial}{\partial y}(p) + \frac{1}{2}u^2\frac{\partial}{\partial x}(\rho u) + \frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v) + \frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u) + \frac{1}{2}v^2\frac{\partial}{\partial y}(\rho v) \checkmark$$

$$= -\rho u^2\frac{\partial}{\partial x}(u) - \frac{1}{2}u^2\frac{\partial}{\partial x}(\rho u) - u\frac{\partial}{\partial x}(p) - \frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v) - \rho u v\frac{\partial}{\partial y}(u) - \frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u) - \rho u v\frac{\partial}{\partial x}(v) - \rho v^2\frac{\partial}{\partial y}(v) - \frac{1}{2}v^2\frac{\partial}{\partial y}(\rho v) - v\frac{\partial}{\partial y}(p) + \frac{1}{2}u^2\frac{\partial}{\partial x}(\rho u) + \frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v) + \frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u) + \frac{1}{2}v^2\frac{\partial}{\partial y}(\rho v)$$

$$= \frac{1}{2}\frac{\partial}{\partial x}(-\rho u^3) - u\frac{\partial}{\partial x}(p) \quad -\frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v) - \rho v\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) - \frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u) - \rho u\frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) \quad -\frac{1}{2}\frac{\partial}{\partial y}(-\rho v^3) - v\frac{\partial}{\partial y}(p)$$
$$= \frac{1}{2}\frac{\partial}{\partial x}(-\rho u \cdot u^2) - u\frac{\partial}{\partial x}(p) \quad -\frac{1}{2}\frac{\partial}{\partial y}(\rho v)u^2) - \frac{1}{2}\frac{\partial}{\partial x}((\rho u)v^2) \quad -\frac{1}{2}\frac{\partial}{\partial y}(-\rho v \cdot v^2) - v\frac{\partial}{\partial y}(p)$$

$$= \frac{1}{2}\frac{\partial}{\partial x}(-\rho u \cdot u^2) - u\frac{\partial}{\partial x}(p) - \frac{1}{2}\frac{\partial}{\partial x}((\rho u)v^2) - \frac{1}{2}\frac{\partial}{\partial y}(\rho v \cdot v^2) - v\frac{\partial}{\partial y}(p) - \frac{1}{2}\frac{\partial}{\partial y}((\rho v)u^2)$$

$$= \frac{-1}{2}\frac{\partial}{\partial x}\left(\rho u \cdot u^2 - \rho u \cdot v^2\right) - u\frac{\partial}{\partial x}(p)$$
$$= -\frac{\partial}{\partial x}\left(\rho u\left(\frac{u^2+v^2}{2}\right)\right) - u\frac{\partial}{\partial x}(p) \quad -\frac{1}{2}\frac{\partial}{\partial y}(-\rho v \cdot v^2 - \rho v u^2) - v\frac{\partial}{\partial y}(p)$$

$$\frac{\partial(\rho k)}{\partial t} = -\frac{\partial}{\partial x}(\rho u k + p u) + p\frac{\partial}{\partial x}(u) - \frac{\partial}{\partial y}(\rho v k + p v) + p\frac{\partial}{\partial y}(v)$$

$$\boxed{\frac{\partial(\rho k)}{\partial t} + \frac{\partial}{\partial x}(\rho u k + p u) + \frac{\partial}{\partial y}(\rho u k + p v) = p\frac{\partial}{\partial x}(u) + p\frac{\partial}{\partial y}(v)}$$

The 2-D Navier-Stokes is described as follow:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad \text{(mass)}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} - \frac{1}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) = 0 \quad \text{(x-momentum)}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v^2 + p)}{\partial y} - \frac{1}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right) = 0 \quad \text{(y-momentum)}$$

Again we express the kinetic energy, ρk, in terms of time derivative of mass and momentum:

$$\frac{\partial(\rho k)}{\partial t} = \frac{\partial\left(\rho\left(\frac{u^2+v^2}{2}\right)\right)}{\partial t}$$

$$= u\frac{\partial}{\partial t}(\rho u) + v\frac{\partial}{\partial t}(\rho v) - \frac{1}{2}(u^2+v^2)\frac{\partial \rho}{\partial t}$$

we express in terms of conservation of mass & momentum from N-S:

$$\frac{\partial(\rho k)}{\partial t} = u\frac{\partial}{\partial t}(\rho u) + v\frac{\partial}{\partial t}(\rho v) - \frac{1}{2}(u^2+v^2)\frac{\partial \rho}{\partial t}$$

$$= u\left(-\frac{\partial(\rho u^2+p)}{\partial x} - \frac{\partial(\rho uv)}{\partial y} + \frac{1}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)\right) \quad \text{I}$$

$$+$$

$$v\left(-\frac{\partial(\rho uv)}{\partial x} - \frac{\partial(\rho v^2+p)}{\partial y} + \frac{1}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)\right) \quad \text{II}$$

$$+$$

$$+\frac{1}{2}(u^2+v^2)\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right) \quad \text{III}$$

We divide & conquer:

I: $$-u\frac{\partial(\rho u^2+p)}{\partial x} - u\frac{\partial(\rho uv)}{\partial y} + \frac{u}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)$$

$$=$$

$$-u\frac{\partial(\rho u^2)}{\partial x} - u\frac{\partial(p)}{\partial x} - u\frac{\partial(\rho uv)}{\partial y} + \frac{u}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)$$

$$-u^2\frac{\partial(\rho u)}{\partial x} - \rho u^2\frac{\partial(u)}{\partial x} - u\frac{\partial(p)}{\partial x} - u^2\frac{\partial(\rho v)}{\partial y} - \rho uv\frac{\partial u}{\partial y} + \frac{u}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)$$

$$-u^2\frac{\partial(\rho u)}{\partial x} - \rho u^2\frac{\partial u}{\partial x} - u\frac{\partial(p)}{\partial x} - u^2\frac{\partial(\rho v)}{\partial y} - \frac{1}{2}\rho v\frac{\partial u^2}{\partial y} + \frac{u}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)$$

II: $$v\left(-\frac{\partial(\rho uv)}{\partial x} - \frac{\partial(\rho v^2+p)}{\partial y} + \frac{v}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)\right)$$

$$=$$

$$-v\frac{\partial(\rho uv)}{\partial x} - v\frac{\partial(\rho v^2+p)}{\partial y} + \frac{v}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)$$

$$-v\frac{\partial(\rho uv)}{\partial x} - v\frac{\partial(\rho v^2)}{\partial y} - v\frac{\partial p}{\partial y} + \frac{v}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)$$

$$-v^2\frac{\partial(\rho u)}{\partial x} - \frac{1}{2}\rho u\frac{\partial v^2}{\partial x} - v^2\frac{\partial(\rho v)}{\partial y} - \rho v^2\frac{\partial(v)}{\partial y} - v\frac{\partial p}{\partial y} + \frac{v}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)$$

III: $$\frac{1}{2}(u^2+v^2)\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right)$$

$$=$$

$$\frac{1}{2}\left(u^2\frac{\partial}{\partial x}(\rho u) + u^2\frac{\partial}{\partial y}(\rho v) + v^2\frac{\partial}{\partial x}(\rho u) + v^2\frac{\partial}{\partial y}(\rho v)\right)$$

$$=$$

$$+\frac{1}{2}u^2\frac{\partial}{\partial x}(\rho u) + \frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v) + \frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u) + \frac{1}{2}v^2\frac{\partial}{\partial y}(\rho v)$$

$$-\frac{1}{2}u^2\frac{\partial(\rho u)}{\partial x} - \rho u^2\frac{\partial u}{\partial x} - u\frac{\partial(p)}{\partial x} - \frac{1}{2}u^2\frac{\partial(\rho v)}{\partial y} - \frac{1}{2}\rho v\frac{\partial u^2}{\partial y} + \frac{u}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) - \frac{1}{2}v^2\frac{\partial(\rho u)}{\partial x} - \frac{1}{2}\rho u\frac{\partial v^2}{\partial x} - \frac{1}{2}v^2\frac{\partial(\rho v)}{\partial y} - \rho v^2\frac{\partial(v)}{\partial y} - v\frac{\partial p}{\partial y} + \frac{v}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)$$

$$-\frac{1}{2}\frac{\partial}{\partial x}(\rho u\cdot u^2) - u\frac{\partial p}{\partial x} - \frac{1}{2}\frac{\partial}{\partial y}(\rho v\cdot u^2) + u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)$$

$$+\frac{1}{2}u\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}u\frac{\partial}{\partial y}(\rho v)+\frac{1}{2}v\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}v\frac{\partial}{\partial y}(\rho v)$$

$$-\frac{1}{2}\frac{\partial}{\partial x}(\rho u\cdot u^2) - u\frac{\partial p}{\partial x} - \frac{1}{2}\frac{\partial(\rho v\cdot u^2)}{\partial y} + \frac{u}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial(\rho u\cdot v^2)}{\partial x}\right) - \frac{1}{2}\frac{\partial(\rho v^3)}{\partial y} - v\frac{\partial p}{\partial y} + \frac{v}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)$$

$$-\left(\frac{1}{2}\frac{\partial}{\partial x}(\rho u\cdot u^2 + \rho u\cdot v^2) + pu\right) + p\frac{\partial u}{\partial x} - \left(\frac{1}{2}\frac{\partial}{\partial y}\left(\rho v^3 + \rho v u^2\right) + pv\right) + p\frac{\partial v}{\partial x} + \frac{u}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) + \frac{v}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)$$

$$= -\frac{\partial}{\partial x}(\rho uk + pu) + p\frac{\partial u}{\partial x} - \frac{\partial}{\partial y}(\rho vk + pv) + p\frac{\partial v}{\partial x} + \frac{u}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) + \frac{v}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho uk + pu) + \frac{\partial}{\partial y}(\rho vk + pv) - p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) - \frac{u}{Re_r}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) - \frac{v}{Re_r}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right) = 0$$

The addition of viscous forces eats away kinetic energy.