Waves

(Your name here)

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Consider the linear advection equation with constant coefficient $a \in \mathbb{R}$:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,\ u = u(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t > 0. \tag{1}$$

1. Prove, mathematically, that solutions to this equation at time t are obtained by propagating the initial profile $u(x, 0)$ by a distance at.
2. Prove the same result in two spatial dimensions (denote (a_x, a_y) the advection coefficients).
3. What happens if we make a a function of space?
4. How would a source term effect the solution?

Consider the 1D linear system:

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = 0,\ \mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^{3\times 1}, \tag{2}$$

where $A \in \mathbb{R}^{3\times 3}$ is diagonalizable.

1. Describe how to get exact solutions to this problem from the initial data.
2. Consider the same system extended to two dimensions (let B be the 3×3 matrix multiplying the spatial derivative of $\mathbf{u}$ with respect to y). Assume that B is diagonalizable as well. Under what conditions can we repeat the procedure of the previous question?

Consider the 1D compressible Euler system:

1. Express the system in quasi-linear form, i.e. like equation (2). Explain how you derived $A = A(\mathbf{u})$.
2. For smooth initial conditions, and (at least) for some finite time, the solution of this system can be described analytically in a way similar to solutions of the 1D linear system. Derive the three nonlinear wave equations.
3. Explain how to prepare initial data to generate
   (a) an entropy wave.
   (b) a right-going acoustic wave.
   (c) a left-going acoustic
4. Can we prepare initial data to generate two waves (a discarding the third one). For example, can we setup a problem where an acoustic wave propagates to the right and an entropy wave propagates to the left?
5. For certain initial conditions, discontinuous solutions can develop after some time. With what you've figured in the preceding questions, setup a problem where a discontinuity forms.
6. The entropy condition states that there should be an entropy jump across the discontinuous solution. How is this jump counted (is entropy jumping from left to right, or from right to left) and how does this jump evolve over time (does it just show up and stay fixed?) ? Feel free to illustrate with numerical examples (smooth initial profile).

Consider the 2D compressible Euler system:

1. Express the system in quasi-linear form.
2. Under what conditions can we describe solutions in terms of waves? For each scenario, describe a procedure to generate the exact solution.

References

[1] Author1, Author 2 : Paper title, *Journal of Scientific Computing*, volume #, year #, pages #-#.