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Weak Solutions of Nonlinear Hyperbolic Equations and Their Numerical Computation

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Introduction

This paper describes a finite difference scheme for the calculation of *time dependent* one-dimensional compressible fluid flows containing *strong shocks*. This method is closely related to one proposed by J. von Neumann (see [12]) and modified more recently by him and R. D. Richtmyer (see [13]), inasmuch as the path of the shock is *not* regarded as an interior boundary.* The novel feature of the method described here is the use of the *conservation form* of the hydrodynamic equations and, to a lesser extent, the particular way of differencing the equations.

Although the method was designed to deal with hydrodynamic problems, it can be used to construct solutions of discontinuous initial value problems for any hyperbolic system of first order nonlinear conservation laws (to be defined below) in any number of space variables. The evidence for the convergence of the method is a number of calculations carried out on high speed computing machines that show every sign of convergence. Although the flows calculated so far all belong to a somewhat special class, I fully believe that the method will reproduce the most general type of flow. The question of *accuracy* of the approximate solution with a given mesh-size, specifically the detrimental effect of *contact discontinuities* on accuracy, is discussed at the end of Section 1.

In addition to the numerical evidence, I succeeded in proving the convergence of the scheme for arbitrary bounded measurable initial data, for the single conservation law

$$u_t - [\log (a + be^{-u})]_x = 0,$$

a and b being arbitrary positive constants (or even functions depending on x and t). The proof, modeled after a procedure of E. Hopf, see [8], will be published in a separate note.

For the discussion of the difference scheme presented here I found it useful to develop the theory of weak solutions of nonlinear conservation laws a little more systematically than customary. The theory and the numerical evidence supporting it is presented in Section 1. Section 2 contains some remarks, plus

**Added in proof*: See also Ludford, Polachek and Seeger, [18], who employ a linear viscosity term but one which is artificially large. The difference scheme employed in [18] is implicit and is solved by iteration. The difference equations are centered in time.

one numerical example, about mixed initial and boundary value problems for conservation laws. Section 3 discusses the manner in which the approximate solutions computed by our difference scheme approach steady state solutions. Section 4 discusses irreversibility and Section 5 the manner of dependence of weak solutions on the initial data. Section 6 describes how the finite difference scheme would be applied to problems with more space variables and in particular to the equations of compressible flow in Eulerian variables.

1. *Theory of Weak Solutions*

We start with the definition of weak solutions applicable to a certain class of nonlinear hyperbolic systems. We take the number of space variables to be one and consider hyperbolic systems of first order equations, i.e. systems of the form

$$U_t + AU_x + B = 0. \tag{1}$$

U here is a column vector of unknown functions, A and B are matrix and vector coefficients depending on x, t and U. We shall consider systems in which the x-derivatives are perfect derivatives, i.e. AU_x is equal to F_x (plus possibly a vector function of x, t, U), F being a (in general nonlinear) vector function of x, t, U. Such equations,

$$U_t + F_x + B = 0, \tag{2}$$

are called *conservation laws.*

U is called a *weak solution* of equation (2) with initial value Φ if the integral relation

$$\iint \{W_t U + W_x F - WB\}\, dx\, dt + \int W(x, 0)\Phi(x)\, dx = 0, \tag{3}$$

obtained by multiplying (1) by W on the left, integrating the resulting equation and integrating by parts, holds for every *test vector* W which has continuous first derivatives and which vanishes outside of some bounded set.

Clearly, a genuine solution is a weak solution, and conversely: a weak solution with continuous first derivatives is a genuine solution. Weak solutions need not be differentiable; if U_1 and U_2 are two genuine solutions of (2) whose domains of definition in the x,t-plane are separated by a smooth curve, the two taken together will constitute a weak solution if and only if the slope τ of the separating curve and the value of U_1 and U_2 along the curve satisfy the condition

$$\frac{1}{\tau}(U_1 - U_2) = F(U_1) - F(U_2).$$

For the conservation laws of mass, momentum and energy, this relation embodies precisely the Rankine-Hugoniot shock conditions.

These facts about weak solutions are basic:

(a) *The class of weak solutions associated with a given system of equations depends on the form in which the equations are written.*

By this we mean: Suppose that new unknowns V can be introduced as certain nonlinear functions of the old:

$$V = H(U), \tag{4}$$

so that when the differential equation (2) is rewritten in terms of V and solved for V_t, each of the resulting equations is again a conservation law

$$V_t + G_x + C = 0. \tag{2'}$$

Weak solutions of (2′) can be defined the same way as before. But if U_0 is a weak solution of (2), $V_0 = H(U_0)$ is in general *not* a weak solution of (2′); in fact, it can be shown (at least in some special cases) that $V_0 = H(U_0)$ is a weak solution of (2′) if and only if U_0 is a genuine solution of (2) ("genuine" here means Lipschitz continuous); this result holds of course only if the transformation (4) is genuinely nonlinear.

This noninvariance is not paradoxical. Whenever the concept of solution is generalized one has to sacrifice some properties of the original notion for the sake of saving others.

This dependence on the form is especially relevant for the equations of unsteady compressible flow which contain *four* conservation laws: conservation of mass, momentum, energy and entropy. Any one of these equations can be deduced from the other three, that is, a genuine solution of any three is necessarily a genuine solution of the fourth. But a weak solution of any three is in general, not a weak solution of the fourth. In physical problems one is looking for weak solutions of the first three equations.

(b) *Weak solutions cannot be obtained as limits of genuine solutions.*

(c) *The initial values do not in general determine a unique weak solution.*

There are many known examples of different weak solutions (even infinitely many) of the same equation with the same initial data.[1] This shows that the initial value problem for weak solutions is not a meaningful one—unless some additional principle is given which selects a unique weak solution for each initial value problem. However, if we believe that our mathematical model does describe an aspect of the physical world, then there is indeed assigned to each initial function a unique weak solution, namely the one that occurs in nature. The problem is to characterize mathematically this physically relevant solution.

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[1]For example, the functions

$$u(x,t) = \begin{cases} 0 \text{ for } x < 0 \\ x/t \text{ for } 0 < x < t \\ 1 \text{ for } t < x \end{cases} \quad \text{and} \quad u(x,t) = \begin{cases} 0 \text{ for } 2x < t \\ 1 \text{ for } 2x > t \end{cases}$$

are both weak solution of the equation $u_t + (\frac{1}{2}u^2)_x = 0$ with initial value $\varphi(x) = 0$ for $x < 0$, $\varphi(x) = 1$ for $x > 0$.

First of all, we exclude all solutions where entropy of a particle has been decreased. It is not clear, however, whether this insures the uniqueness of the solution of the initial value problem especially if there are several space variables but even in the case of one space variable.** Some additional principle is needed to pick out a unique solution, such as:

(a) The weak solutions occurring in nature are limits of viscous flows.

(b) The weak solutions occurring in nature must be stable.

It is commonly believed that (a) does characterize uniquely the solutions occurring in nature. But whether the same is true of postulate (b) is seriously doubted by some.

We shall describe now in some detail the viscosity method, the notion of stability and the relation of the two.

Enlarge equation (2) by the additional term λU_{xx} on the right, obtaining a nonlinear parabolic system

$$U_t + F_x + B = \lambda U_{xx} . \tag{5}$$

The initial value problem $U(x, 0) = \Phi(x)$ can be solved for a fairly wide class of initial vectors Φ; it is commonly believed that the solution exists for a range of t which is independent of λ, and that if the initial vector is kept fixed and λ taken smaller and smaller, the corresponding solutions $U_\lambda(x, t)$ converge boundedly, almost everywhere in the strip $0 \leq t \leq T$ to a limit $U(x, t)$. This has been demonstrated so far only for a single equation,[2] $u_t + uu_x = 0$, by E. Hopf*** (see [8]), admitting all *bounded measurable* functions as initial values. For the hydrodynamic case only the convergence of *steady state* solutions[3] of the viscous flow equations to steady state weak solutions of the equations of ideal flow have been investigated.[4]

Granting the validity of the conjectures about the parabolic equation it is an easy matter to show, just as Hopf has shown for the equation considered by him, that the limit $U(x, t)$ of $U_\lambda(x, t)$ is a weak solution of equation (2). This follows from the fact that U, being a genuine solution of (5) is a weak solution of (5) as well; i.e. if we multiply (5) by any twice differentiable test vector W and integrate by parts we obtain the integral relation

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***Added in proof*: Recently, Germain and Bader, see [16], have found an analogue of the entropy condition for the equation $u_t + (u^2/2)_x = 0$ and have succeeded in showing that this and the jump condition characterizes a unique solution of any initial value problem.

[2]The nonlinear parabolic equation $u_t + uu_x = \lambda u_{xx}$ was first considered by Burgers, see [8].

***See also J. H. Cole, [17].

[3]A steady state solution is one that depends only on a particular linear combination of x and t.

[4]See Becker, [1], Thomas [14], Gilbarg [6], Grad [7], and Courant and Friedrichs [2], pp. 134-138.

$$(6) \qquad -\iint \{W_t U_\lambda + W_x F(U_\lambda) - WB\}\, dx\, dt - \int W(x, 0)\Phi(x)\, dx = \lambda \iint W_{xx} U_\lambda \, dx\, dt.$$

Keep Φ and W fixed and let λ tend to zero. The left side of (6) approaches the left side of (3) and the right side of (6) tends to zero. This proves that U satisfies (3) for all twice differentiable test vectors W (and therefore à fortiore for all once differentiable ones too).

Observe that it was crucial for this argument that U be a *strong* limit of the sequence U_λ, i.e. that $\iint | U_\lambda - U |$ over any bounded set of the x,t-plane tend to zero. For if U_λ converges to U in the *weak* sense only, the sequence $F(U_\lambda)$ will converge in the weak sense but *not* to $F(U)$. This phenomenon can be expressed concisely:

A weak limit of weak solutions is not a weak solution unless it is also a strong limit.

A precise statement and proof of this, in the case of a single conservation law, is given in Section 5.

There are a number of different ways of introducing a viscosity term; the way equation (5) does it is perhaps the simplest, although when applied to the equations of compressible flow it does not exactly correspond to the action of viscosity or heat conduction. All these methods are expected to produce the same weak solution in the limit. The aim of this paper is to describe a different type of limiting process, a finite difference scheme and to show—by mathematical reasoning, plausibility arguments and numerical evidence—that this scheme furnishes the experimentally observed flows.

The difference scheme is as follows:

Replace the space derivatives F_x by the *symmetric* difference quotients $[F(U(x + \Delta x, t)) - F(U(x - \Delta x, t))]/2\Delta x$, the time derivative U_t by the *forward* difference quotient $[U(x, t + \Delta t) - \overline{U}(x, t)]/\Delta t$, where $\overline{U}(x, t)$ is an abbreviation for the average[5] of U at $(x + \Delta x, t)$ and $(x - \Delta x, t)$. If $U(x,0)$ is known, we can determine $U(x, t)$ for all values of t which are integer multiples of Δt; in particular the value of $U(x, 0)$ at the lattice point $x = m\Delta x$, $m = 0, \pm 2, \cdots$ determines $U(x, t)$ at all points of the staggered lattice $t = n\Delta t$, $n = 0, 1, 2 \cdots$; $x = m\Delta x$, $m + (-1)^n = 0, \pm 2, \pm 4, \cdots$.

Denote by Δ a particular choice of mesh width, $\Delta = (\Delta t, \Delta x)$ and by U_Δ the corresponding solution of the finite difference scheme with initial value Φ. $U_\Delta(x, t)$ is defined for all values of t which are integer multiples of Δt, and for sake of convenience we might as well put $U_\Delta(x, t)$ equal to $U_\Delta(x, \nu\Delta t)$ for $\nu\Delta t \leq t < (\nu + 1)\Delta t$.

I conjecture that if Δt and Δx tend to zero so that the classical Courant-Friedrichs-Lewy stability criterion is observed; i.e., the domain of dependence of

[5]If we replace $\overline{U}(x, t)$ by $U(x, t)$, the resulting finite difference scheme is unconditionally unstable.

a point with respect to the differential equation always stays within the domain of dependence of this point with respect to the difference equation, then the functions $U_\Delta(x, t)$ remain uniformly bounded and converge almost everywhere to a limit $U(x, t)$ for a wide class of initial vectors. Should this conjecture be true, it is an easy matter to show that the limit $U(x, t)$ is a weak solution of the original system of equations. The proof goes the same way as in the viscosity method, summation by parts replacing integration by parts. And, just as in that case, it is crucial for the argument that U be a *strong* limit of U_Δ ; if U were a weak limit only, it would in general not be a weak solution.

Very likely this conjecture holds only for systems of conservation laws which, in addition to being hyperbolic, satisfy some additional condition, possibly in the large, see e.g. Weyl's paper [15].

I succeeded in proving this conjecture for the single equation

$$u_t - [\log (a + be^{-u})]_x = 0.$$

Details of the proof will be published in a separate note; I would like to mention however that in the course of proving the convergence, I obtained a fairly explicit formula relating $u(x, t)$ to its initial values $\varphi(x)$. This formula is strikingly similar to the formula Hopf obtained for the solution of the equation $u_t + (u^2/2)_x = 0$ by the linear viscosity method and suggests this

CONJECTURE: Both the linear viscosity method and the finite difference scheme described above, when applied to any single homogeneous first order conservation law

$$u_t + [f(u)]_x = 0, \qquad f'' < 0,$$

and arbitrary bounded measurable initial data $u(x, 0) = \varphi(x)$, converge to the same limit $u(x, t)$ given by the explicit formula

$$u(x, t) = g\left(\frac{x - y_0}{t}\right)$$

where $y_0 = y_0(x, t)$ is that value of y which maximizes

$$\int_0^y \varphi(s)\, ds + tG\left(\frac{x - y}{t}\right).$$

The function $g(s)$ is defined as the inverse of $f'(u) = df(u)/du$, i.e. $f'(g(s)) = s$; $G(s)$ is defined as the integral of $g(s)$: $dG/ds = g$. The maximum problem defining y_0 has a unique solution for almost all x and t so that $u(x, t)$ is well defined for almost all x and t. A similar result holds if "f" is positive.

Next we turn to the principle of stability formulated as a

CONJECTURE: *Among all functions $U = S(\Phi)$, which assign to each vector Φ a weak solution U with initial value Φ there exists one which is continuous in some reasonable topology and it is the only one which is continuous in any reasonable topology.*[6]

[6] It is understood that whenever an initial value problem has a genuine solution, $S(\Phi)$ has to coincide with it and that the semigroup property is satisfied by the solutions furnished by S.

This conjecture is a bit vague since it does not define reasonable topology, nor does it specify the domain of initial vectors Φ. Maybe further investigations will indicate what the proper choice for these undefined objects is.

The first part of the conjecture, asserting the existence of a continuous assignment, is the classical principle that in a physical problem the solution must depend continuously on the data.

That there is only *one* way of assigning $U = S(\Phi)$ continuously was proposed merely as a possible explanation of why various types of apparently different limiting procedures, such as the various viscosity methods and finite difference schemes, pick out the same weak solution. Because of the systematic nature of these procedures one would expect that the solutions picked out by each of them do depend continuously on the initial data. (This can be proved rigorously in simple cases; see Section 5.) If, therefore, there is only one way of assigning a weak solution continuously to each initial vector, it would follow that these various limiting procedures do lead to the same weak solution. On the other hand, the reason why these various limiting procedures lead to the same result could easily be some type of dissipative mechanism common to all of them. In this connection it should be pointed out that if the parabolic equation (5) is approximated by the standard finite difference scheme (centered space derivatives, forward time derivative) and if Δx, Δt and λ are let to approach zero *simultaneously*, keeping $\Delta t/\Delta x$ constant and λ equal to $(\Delta x)^2/2\Delta t$, we arrive at the finite difference scheme proposed in this paper. The von Neumann-Richtmyer method is explicitly based on such a simultaneous carrying out of two limiting procedures.

We present now a partial list of experimental calculations carried out for the equation $u_t + (u^2/2)_x = 0$, $u_t + (u^3/3)_x = 0$, and the hydrodynamic equations both in Eulerian and Lagrangean coordinates.

In the first group of calculations the initial data were picked to have a constant value Φ_i for x negative and another constant value Φ_f for x positive. This choice of the initial data leads to homogeneous problems; i.e., the solution depends only on x/t. This has the advantage that advancing in time in the finite difference scheme has the same effect as refining the mesh. The computational plan was to keep on grinding out time cycles until it becomes evident whether the method converges, diverges to infinity or oscillates. All calculations performed so far converged and quite rapidly at that. Unless stated otherwise, the difference scheme used was the one described before.

(1) *Equation*: $u_t + (u^2/2)_x = 0.$

Initial function: $\varphi_i = 1, \quad \varphi_f = 0.$

$$\Delta t/\Delta x = 1.$$

Exact solution: $u(x, t) = \begin{cases} 1 \text{ for } x/t < 1/2 \\ 0 \text{ for } x/t > 1/2. \end{cases}$

TABLE I

$n = 44$		$n = 48$	
k	u	k	u
17	1.00000	19	1.00000
19	.99548	21	.99548
21	.76818	23	.76817
23	.21061	25	.21061
25	.02343	27	.02344
27	.00210	29	.00210
29	.00018	31	.00018

TABLE II
Rarefraction wave, $n = 48$, $\Delta t/\Delta x = 1$

k	u
47	.92695
45	.88187
43	.83994
41	.79948
39	.7599
37	.7209
35	.6825
33	.6444
31	.6066
29	.5692
27	.5321
25	.4954
23	.4590
21	.4229
19	.3873
17	.3523
15	.3177
13	.2839
11	.2509
9	.2189
7	.1881
5	.1587
3	.1310
1	.1055
−1	.0823
−3	.0619
−5	.0447
−7	.0306
−9	.0198
−11	.0120

Values of the solution after 44 and 48 time cycles are listed in Table I. The values of u for k less than 17 are equal to one within five figures, those for k greater than 31 are zero within five figures. Notice that there is a very rapid transition from $u = 1$ to $u = 0$ around $k = 22$ for $n = 44$, and $k = 24$ for $n = 48$; this corresponds closely to the exact solution which has a sharp discontinuity along the line $x = 2t$.

(2) *Equation*: same.

Initial function: $\varphi_i = 0, \quad \varphi_f = 1.$

$$\Delta t/\Delta x = 1.$$

$$\textit{Exact solution: } u(x, t) = \begin{cases} 0 & \text{for } x < 0, \\ x/t & \text{for } 0 < x < t, \\ 1 & \text{for } t < x. \end{cases}$$

Values of the calculated solution after 48 time cycles are listed in Table II, and plotted in Figure 1. The dashed line in the graph is the exact solution.

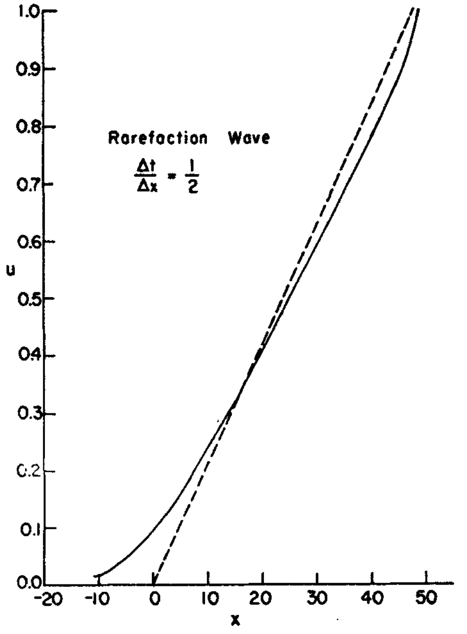


FIGURE 1

(3) *Equation*: same.

Initial function: same as in (2).

$$\Delta t/\Delta x = 1/2.$$

TABLE III

Rarefaction wave, $n = 63$, $\Delta t/\Delta x = 1/2$

k	u
64	.8553
60	.8170
56	.7758
52	.7322
48	.6869
44	.6405
40	.5933
36	.5457
32	.4980
28	.4506
24	.4039
20	.3580
16	.3134
12	.2704
8	.2295
4	.1911
0	.1555
−4	.1234
−8	.0949
−12	.0706
−16	.0505
−20	.0345
−24	.0225
−28	.0139

Values of calculated solution after 63 time cycle are listed in Table III and plotted in Figure 2. The dashed line in the graph is the exact solution.

(4) *Equation*: $u_t + (u^3/3)_x = 0$.

Initial function: $\varphi_i = 1, \quad \varphi_f = 0$.

$$\Delta t/\Delta x = 1.$$

Difference scheme: $u_k^{n+1} = u_k^n - 1/3[(u_k^n)^3 - (u_{k-1}^n)^3](\Delta t/\Delta x)$.

Exact solution: $u(x, t) = \begin{cases} 1 & \text{for } x/t < 1/3, \\ 0 & \text{for } x/t > 1/3. \end{cases}$

Values of the calculated solution after 25, 26 and 27 time cycles are listed in Table IV. The values of u, for x to the left of the range listed, are equal to one within five figures, to the right of the range listed, equal to zero within five

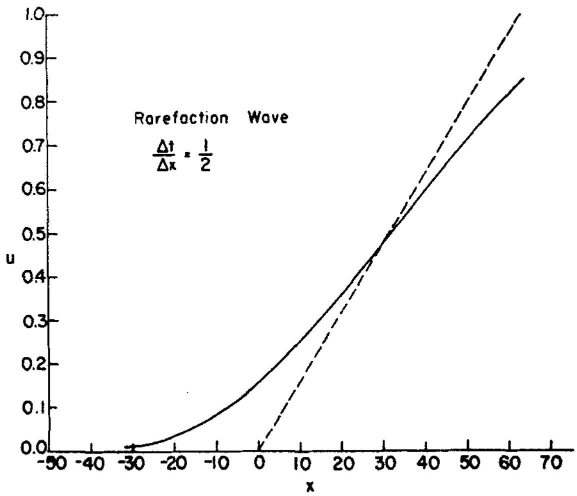


FIGURE 2

figures. The transition from $u = 1$ to $u = 0$ takes place very closely to where theory predicts it at $k = n/3$.

(5) *Equation*: Hydrodynamic equations in Eulerian form

$$\rho_t + (u\rho)_x = 0,$$

$$(u\rho)_t + (u^2\rho + p)_x = 0,$$

$$\left(\rho e + \frac{u^2\rho}{2}\right)_t + \left(\rho eu + \frac{u^3\rho}{2} + up\right)_x = 0.$$

Here ρ, u, p and e denote density, velocity, pressure and internal energy per unit mass. The equation of state expresses e as function of p and ρ; e.g. for an ideal gas e is $p/\rho(\gamma - 1)$.

In the computations the quantities ρ, $u\rho = m$ and $\rho(e + u^2/2) = E$, mass,

TABLE IV

$n = 25$		$n = 26$		$n = 27$	
k	u	k	u	k	u
6	1.00000				
7	.89330	7	.99989	7	1.00000
8	.10670	8	.65283	8	.98902
9	.00000	9	.01396	9	.34391
10		10	.00000	10	.00041

momentum and total energy per unit volume were used as dependent variables. In terms of these the equations are:

$$\rho_t + m_x = 0,$$

$$m_t + \left[(\gamma - 1)E + \frac{3 - \gamma}{2}\frac{m^2}{\rho}\right]_x = 0,$$

$$E_t + \left[\gamma \frac{m}{\rho} E - \frac{\gamma - 1}{2}\frac{m^3}{\rho^2}\right]_x = 0.$$

Initial Vector: $u_i = 2, \quad p_i = 50, \quad \rho_i = 50,$

$u_f = 0, \quad p_f = 0, \quad \rho_f = 10,$

$\gamma = 1.5,$

$\frac{\Delta t}{\Delta x} = 0.25.$

TABLE V

$n = 49$

ρ	$u/8$	p	k
4998	2500	4997	−6
4996	2501	4994	−4
4993	2501	4990	−2
4988	2503	4983	0
4981	2505	4972	2
4971	2508	4958	4
4959	2512	4941	6
4946	2515	4922	8
4931	2519	4904	10
4915	2522	4890	12
4898	2524	4882	14
4878	2524	4880	16
4852	2523	4887	18
4818	2520	4898	20
4777	2517	4913	22
4734	2513	4926	24
4695	2508	4928	26
4648	2492	4880	28
4497	2431	4620	30
3919	2196	3672	32
2622	1524	1753	34
1424	0490	0289	36
1047	0042	0012	38
1002	0001	0000	40
1000	0000	0000	42

$$\textit{Exact solution: } U = \begin{cases} U_i & \text{for } x/t > 2.5, \\ U_f & \text{for } x/t > 2.5. \end{cases}$$

Values of the calculated solution after 49 time cycles are given in Table V. There is a rapid transition from one state to the other around $k = 31$; this gives

TABLE VI

$n = 99$

ρ	$u/8$	p	k
4996	2501	4995	0
4994	2501	4992	2
4992	2502	4988	4
4988	2503	4983	6
4984	2504	4976	8
4979	2506	4969	10
4973	2508	4959	12
4966	2510	4949	14
4959	2512	4939	16
4952	2514	4929	18
4946	2516	4921	20
4942	2517	4914	22
4938	2518	4910	24
4937	2518	4909	26
4936	2518	4911	28
4937	2517	4915	30
4937	2515	4922	32
4935	2514	4931	34
4931	2512	4940	36
4923	2510	4949	38
4910	2508	4957	40
4893	2507	4965	42
4872	2505	4972	44
4850	2504	4977	46
4829	2503	4981	48
4814	2502	4984	50
4806	2502	4987	52
4809	2501	4987	54
4819	2499	4983	56
4824	2493	4956	58
4777	2469	4840	60
4508	2369	4388	62
3646	2027	3094	64
2208	1207	1150	66
1250	0278	0131	68
1024	0019	0005	70
1001	0000	0000	72
1000	0000	0000	74

124/49 = 2.48 for the speed of propagation of discontinuity, in pretty good agreement with the theoretically calculated value of 2.5.

Values of the calculated solution after 99 time cycles are given in Table VI. There is a rapid transition from one state to another around $k = 62$, giving as speed of propagation 248/99 = 2.50.

(6) *Equation*: Hydrodynamic equations in Eulerian form.

Initial vector: $u_i = 1, \quad p_i = 50, \quad \rho_i = 50,$

$$u_f = 0, \quad p_i = 0, \quad \rho_f = 10,$$

$$\gamma = 1.5,$$

$$\frac{\Delta t}{\Delta x} = 0.25.$$

TABLE VII

$n = 49$

ρ	$u/8$	p	k
4995	1251	4992	−20
4989	1253	4983	−18
4977	1256	4966	−16
4957	1263	4936	−14
4924	1273	4887	−12
4874	1288	4814	−10
4804	1309	4713	−8
4712	1338	4581	−6
4598	1373	4421	−4
4464	1416	4237	−2
4314	1464	4034	0
4154	1517	3822	2
3990	1572	3609	4
3831	1628	3405	6
3685	1682	3218	8
3561	1731	3055	10
3469	1772	2921	12
3415	1804	2816	14
3405	1824	2733	16
3431	1828	2655	18
3461	1806	2539	20
3412	1731	2302	22
3138	1554	1835	24
2541	1212	1127	26
1795	0704	0444	28
1270	0244	0095	30
1062	0047	0012	32
1010	0006	0001	34
1001	0000	0000	36
1000	0000	0000	38

TABLE VIII

$n = 99, \gamma = 1.5, \Delta t/\Delta x = .25$

ρ	$u/8$	p	k
4990	1252	4986	−28
4984	1254	4977	−26
4975	1257	4963	−24
4961	1261	4943	−22
4942	1267	4914	−20
4917	1275	4876	−18
4883	1285	4827	−16
4841	1298	4765	−14
4790	1314	4691	−12
4730	1333	4604	−10
4660	1355	4504	−8
4582	1380	4393	−6
4496	1408	4272	−4
4403	1438	4144	−2
4306	1470	4011	0
4205	1504	3874	2
4102	1539	3737	4
3999	1575	3602	6
3898	1610	3471	8
3800	1645	3346	10
3708	1679	3229	12
3623	1711	3122	14
3546	1740	3026	16
3480	1766	2943	18
3426	1789	2873	20
3384	1808	2817	22
3357	1822	2772	24
3346	1834	2739	26
3352	1841	2716	28
3378	1847	2700	30
3425	1849	2691	32
3495	1851	2685	34
3586	1850	2680	36
3695	1847	2671	38
3809	1838	2646	40
3898	1817	2583	42
3907	1769	2437	44
3735	1666	2134	46
3273	1461	1613	48
2525	1103	0932	50
1747	0618	0350	52
1253	0211	0075	54
1061	0043	0010	56
1012	0006	0001	58
1002	0000	0000	60
1000	0000	0000	62

$$\textit{Exact solution: } U = \begin{cases} U_i & \text{for } x/t < -0.225, \\ \text{rarefaction wave} & \text{for } -0.225 < x/t < 1.47, \\ U_s & \text{for } 1.47 < x/t < 1.84, \\ U_f & \text{for } 1.84 < x/t. \end{cases}$$

The value of U_s is: $u_s = 1.47$, $p_s = 27.1$, $\rho_s = 50$.

Results of the calculation after 49 time cycles are given in Table VII. There is a rapid transition around $k = 22$, corresponding to shock speed $88/49 = 1.79$, in fairly good agreement with the theoretically calculated shock speed of 1.84.

TABLE IX

$n = 49, \gamma = 2$

ρ	$u/8$	p	k
4996	2501	9986	−16
4992	2503	9970	−14
4985	2507	9942	−12
4973	2513	9895	−10
4955	2522	9823	−8
4929	2535	9720	−6
4894	2552	9584	−4
4850	2574	9415	−2
4798	2601	9216	0
4740	2630	8997	2
4678	2661	8768	4
4615	2693	8541	6
4556	2723	8327	8
4501	2751	8138	10
4453	2774	7981	12
4411	2792	7860	14
4372	2805	7773	16
4328	2814	7717	18
4272	2819	7683	20
4193	2822	7667	22
4082	2823	7660	24
3936	2823	7659	26
3762	2823	7657	28
3573	2824	7655	30
3393	2824	7652	32
3241	2824	7654	34
3130	2823	7656	36
3064	2824	7679	38
3004	2805	7573	40
3064	2882	7971	42
1741	1376	2197	44
1007	0007	0002	46
1000	0000	0000	48

Results of the calculation after 99 time cycles are given in Table VIII; the transition here occurs around $k = 46$ which gives a shock speed $184/99 = 1.86$, in good agreement with the theoretically calculated value.

In Table VIII, u and p appear to be fairly constant for awhile behind the shock, the value of u being $(0.184 \pm 0.001) \times 8 = 1.47 \pm 0.01$, the value of p around 27 ± 0.3. These are fairly close to $u_s = 1.47$ and $p_s = 27.1$, in spite of the fact that the value of ρ in this range differs considerably from ρ_s.

(7) *Equation*: Hydrodynamical equations in Eulerian form.

$$\textit{Initial vector}: \quad u_i = 2, \quad p_i = 100, \quad \rho_i = 50,$$

$$u_f = 0, \quad p_f = 0, \quad \rho_f = 10,$$

$$\gamma = 2,$$

$$\frac{\Delta t}{\Delta x} = 0.25.$$

$$\textit{Exact solution}: \; U = \begin{cases} U_i & \text{for } x < 0, \\ \text{rarefaction wave} & \text{for } 0 < x/t < 2.26, \\ U_s & \text{for } 2.26 < x/t < 3.40, \\ U_f & \text{for } 3.40 < x/t. \end{cases}$$

The value of U_s is: $u_s = 2.26$, $p_s = 76.5$, $\rho_s = 30$.

Results of the calculation after 49, respectively 99 steps are listed in Tables IX and X; u and p appear to be fairly constant behind the shock, the value of u being $8 \times 0.2824 = 2.26$, p equal to 76.5. This is very close to $u_s = 2.26$, $p_s = 76.5$.

Remark: The value of $\Delta t/\Delta x = 0.25$ is *larger* than its maximum permissible value according to the Courant-Fredrichs-Lewy theory at $p = 76.6$, $\rho = 30$. The indicated instability is indeed beginning to show around the shock front.

(8) *Equation*: Hydrodynamic equations in Lagrange mass variables

$$V_t - u_\xi = 0,$$

$$u_t + p_\xi = 0,$$

$$(e + \tfrac{1}{2}u^2)_t + (up)_\xi = 0.$$

V here denotes specific volume; if we introduce V, u and $E = e + \frac{1}{2}u^2$ as new unknowns (volume, momentum and energy per unit mass) the equations read

$$V_t - u_\xi = 0,$$

$$u_t + \left[(\gamma - 1)\frac{E - \frac{1}{2}u^2}{V}\right]_\xi = 0,$$

$$E_t + \left[(\gamma - 1)\frac{uE - \frac{1}{2}u^3}{V}\right]_\xi = 0.$$

TABLE X

$n = 99, \gamma = 2, \Delta t/\Delta x = .25$

ρ	$u/8$	p	k
4997	2501	9989	−24
4995	2502	9982	−22
4992	2503	9970	−20
4988	2505	9952	−18
4981	2509	9927	−16
4972	2513	9891	−14
4960	2519	9844	−12
4945	2527	9783	−10
4926	2536	9708	−8
4903	2548	9617	−6
4875	2562	9511	−4
4844	2577	9391	−2
4810	2595	9259	0
4772	2614	9117	2
4733	2634	8968	4
4692	2655	8816	6
4651	2676	8664	8
4611	2697	8515	10
4572	2717	8373	12
4535	2736	8242	14
4502	2753	8122	16
4472	2769	8017	18
4447	2782	7927	20
4425	2793	7853	22
4408	2802	7794	24
4395	2809	7749	26
4385	2814	7716	28
4377	2817	7692	30
4371	2820	7677	32
4365	2821	7667	34
4358	2822	7661	36
4349	2823	7657	38
4334	2823	7656	40
4312	2823	7655	42
4280	2823	7654	44
4234	2823	7654	46
4172	2823	7654	48
4091	2823	7654	50
3993	2823	7653	52
3878	2823	7653	54
3751	2824	7652	56
3619	2824	7652	58
3489	2824	7651	60
3368	2824	7651	62
3263	2824	7651	64
3178	2824	7650	66

TABLE X—*Continued*

$n = 99, \gamma = 2, \Delta t/\Delta x = .25$

ρ	$u/8$	p	k
3115	2825	7656	68
3066	2820	7628	70
3061	2841	7723	72
2981	2778	7342	74
3175	2949	8228	76
2734	2510	5891	78
3497	3137	9270	80
1993	1515	2139	82
3892	3514	2160	84
1853	1480	2426	86
1035	0045	0020	88
1000	0000	0000	90

Initial vector: $V_i = 1, \quad u_i = 4, \quad p_i = 8,$

$$V_f = 3, \quad u_f = 0, \quad p_f = 0,$$

$$\gamma = 2,$$

$$\frac{\Delta t}{\Delta x} = 0.25.$$

Exact solution: $U = \begin{cases} U_i & \text{for } \xi/t < 2, \\ U_f & \text{for } 2 < \xi/t. \end{cases}$

REMARK: The initial position $\xi = 0$ of the separation line of the two states was at $k = 100$. The tabulated results after 52, respectively 104 time cycles are listed in Tables XI and XII; the index k is listed in the leftmost column; the distance of the two consecutive lattice points is $2\Delta\xi$.

There is a rapid transition from the initial to the final state taking place around $k = 113$ for $n = 52$, and around $k = 126$ for $n = 104$, corresponding exactly to a shock speed of 2.

The experimental calculations presented have shown, I believe, the convergence of the method.

A large number of further shock calculations in Lagrange variables varying the initial state U_i and the value of $\Delta t/\Delta x$ were carried out by L. Baumhoff at Los Alamos. These calculations approximated the theoretically expected solutions very accurately with the exception noted further on and indicated that calculations carried out by this numerical scheme have these general features:

(i) *The width of the transition across a shock depends on the magnitude of $\Delta t/\Delta x$; it is narrowest if $\Delta t/\Delta x$ is taken as large as possible.*

TABLE XI

$n = 52$

k	V	E	u	p
076	1000	1600	4001	7993
077	1002	1602	4010	7959
078	1004	1603	4016	7935
079	1005	1604	4019	7920
080	1005	1604	4022	7911
081	1005	1604	4022	7908
082	1005	1604	4022	7909
083	1005	1604	4021	7913
084	1005	1604	4020	7918
085	1004	1603	4018	7925
086	1004	1603	4016	7932
087	1003	1603	4015	7939
088	1003	1602	4013	7946
089	1003	1602	4011	7953
090	1003	1602	4009	7960
091	1004	1603	4008	7966
092	1006	1605	4007	7971
093	1009	1607	4005	7976
094	1015	1612	4004	7980
095	1023	1618	4003	7984
096	1032	1625	4003	7987
097	1042	1634	4002	7989
098	1052	1642	4001	7992
099	1060	1648	4001	7994
100	1063	1650	4000	7995
101	1062	1650	4000	7996
102	1057	1645	4000	7997
103	1047	1638	4000	7998
104	1036	1629	3999	7998
105	1025	1620	3999	7999
106	1016	1612	3999	7999
107	1008	1606	3999	7999
108	1003	1602	3999	7999
109	1000	1600	3999	7999
110	0999	1599	3999	7999
111	1017	1556	3927	7717
112	1375	0931	2755	4011
113	2234	0190	0880	0680
114	2793	0020	0157	0068
115	2960	0001	0021	0006
116	2993	0000	0002	0000
117	3000	0000	0000	0000

TABLE XII

$n = 104$

k	V	E	u	p
47	1000	1600	4000	8000
56	1005	1604	4022	7912
77	1000	1600	4000	8000
100	1045	1636	4000	8000
114	1000	1600	4000	8000
123	1000	1600	4000	8000
124	1018	1557	3928	7716
125	1376	932	2755	4011
126	2235	191	881	680
127	2793	20	157	61
128	2961	2	21	5
129	2994	0	2	0
130	3000	0	0	0

The values of V, E, u, p vary monotonically between the lattice points $k = 47, 56, 77, 100, 114, 123$.

(ii) *The values of u and p converge faster than the values of ρ and V.*

The statement on the width of the shock is illustrated most strikingly in example 6, Table IX, where $\Delta t/\Delta x$ exceeds its maximum permissible value; here the shock transition takes place across one space interval. In fact, as J. Calkin pointed out, if U_i and U_f can be connected by a single shock, i.e. if $s[U_f - U_i]$ is equal to $F(U_f) - F(U_i)$, s being the shockspeed then, if we take $\Delta x/\Delta t$ to be exactly the shockspeed s, the numerically calculated solution is just $U_k^n = U_i$ for $k \leq sn$, $U_k^n = U_f$ for $k > sn$. Of course in hydrodynamics the character of the function F is such that the shock speed is always subsonic with respect to the state behind it, so that choosing $\Delta x/\Delta t$ to be s is necessarily an unstable mesh ratio. This is in sharp contrast to the linear case where s is the reciprocal of the slope of the characteristic curve involved and so $\Delta x/\Delta t = s$ is precisely the largest permissible mesh ratio.

Inaccuracies in the values of ρ and V are particularly noticeable if there are contact discontinuities present, i.e. lines across which u and p are continuous but not ρ; the reason for this is that contact discontinuities, in contrast to shocks, are very much like discontinuities of solutions of *linear*[7] hyperbolic equations and these, when calculated by the difference scheme described before, spread like $\sqrt{n}$. That is, after n time cycles the width of the contact discontinuity (measured, say, from the point where U is within 1% of U_i to the point where U is within 1% of U_f) is $O(\sqrt{n})$. One may check this by an explicit calculation

[7]In the same sense that solutions with contact discontinuities can be obtained as limits of continuous solutions; for proof of this fact, see a forthcoming paper of the author.

in a simple case, e.g. for the equation $u_t + u_x = 0$. In contrast, the width of shocks remains constant.

Of course whether or not there is any spreading of contact discontinuities depends on the difference scheme used and the magnitude of Δx and Δt. For instance, if we use the difference scheme described before for the equation $u_t + u_x = 0$ and choose $\Delta t/\Delta x = 1$, the solutions of the difference equations just propagate the initial data, unaltered in shape, at unit speed. There are other examples of equations and difference schemes where spreading does not occur. In all these examples the line of discontinuity passes through lattice points, and very likely this is a *necessary* condition on difference schemes for *linear* equations to avoid the spreading of initial discontinuities. This indicates that the spreading of contact discontinuities in difference calculations in *Eulerian* coordinates cannot be prevented unless one uses variable size space intervals which is tantamount to introducing Lagrange coordinates. In Lagrange coordinates a contact discontinuity is a straight line parallel to the t-axis so its path is known in advance. It is indeed possible to write down various schemes which, at least formally, maintain sharp contact discontinuities. The stability, effectiveness and accuracy of such schemes is being investigated by the author and L. Baumhoff, and by J. Calkin and N. Metropolis (who regard them from a somewhat different point of view, see a forthcoming report).

2. *Mixed Initial and Boundary Value Problems*

Let D be a domain bounded by an initial interval (say $-\infty < x \leq x_0, t = 0$) and a boundary curve (say $x = x_0, t \geq 0$) issuing from x_0 into the upper half-plane. In a mixed initial and boundary value problem for a system of equations we prescribe not only the values of all the unknowns (i.e. of all components of $\vec{U}$) on the initial interval, but we also prescribe a certain number (say r) of relations between the unknowns along the boundary curve. The problem is to determine the solution in the domain D; this problem has a unique solution if the number r of relations prescribed is equal to the number of those characteristics issuing from x_0 which enter the domain D.

The theory of mixed initial and boundary value problems for *genuine solutions* of linear and nonlinear equations is in a fairly satisfactory state; the corresponding theory for *weak solutions* of nonlinear conservation equations is less well known. Here again one would expect to find, just as in the case of the pure initial value problem, new features which are not present in either the linear theory or the theory of genuine solutions.

Since most hydrodynamic problems arising in physics are mixed problems, there is a great need for effective finite difference schemes to compute their solutions. In this paper I would like to report on a method for handling a particularly simple case: reflection of a gas from a rigid wall (i.e. $u(x_0, t) = 0$). This boundary condition can be taken into account in the finite difference scheme simply by putting the value of u at the first lattice point *beyond* the wall equal to the negative of its value at the last lattice point before the wall, while the values of p, V and E (Lagrange) or p, ρ and E (Euler) are transferred unchanged. I hope to be able to report in the near future on numerical schemes for handling more general boundary conditions.

Calculations were carried out in Lagrange coordinates using the following initial data: $U = U_i$ for all lattice points to the left of $k = 155$, $U = U_f$ for all lattice points between $k = 155$ and $k = 199$; the rigid wall is located at $k = 199$.

U_i and U_f were chosen as in example 8: $u_i = 4$, $V_i = 1$, $p_i = 8$, $u_f = 0$, $V_f = 3$, $p_f = 0$; γ was chosen as 2.

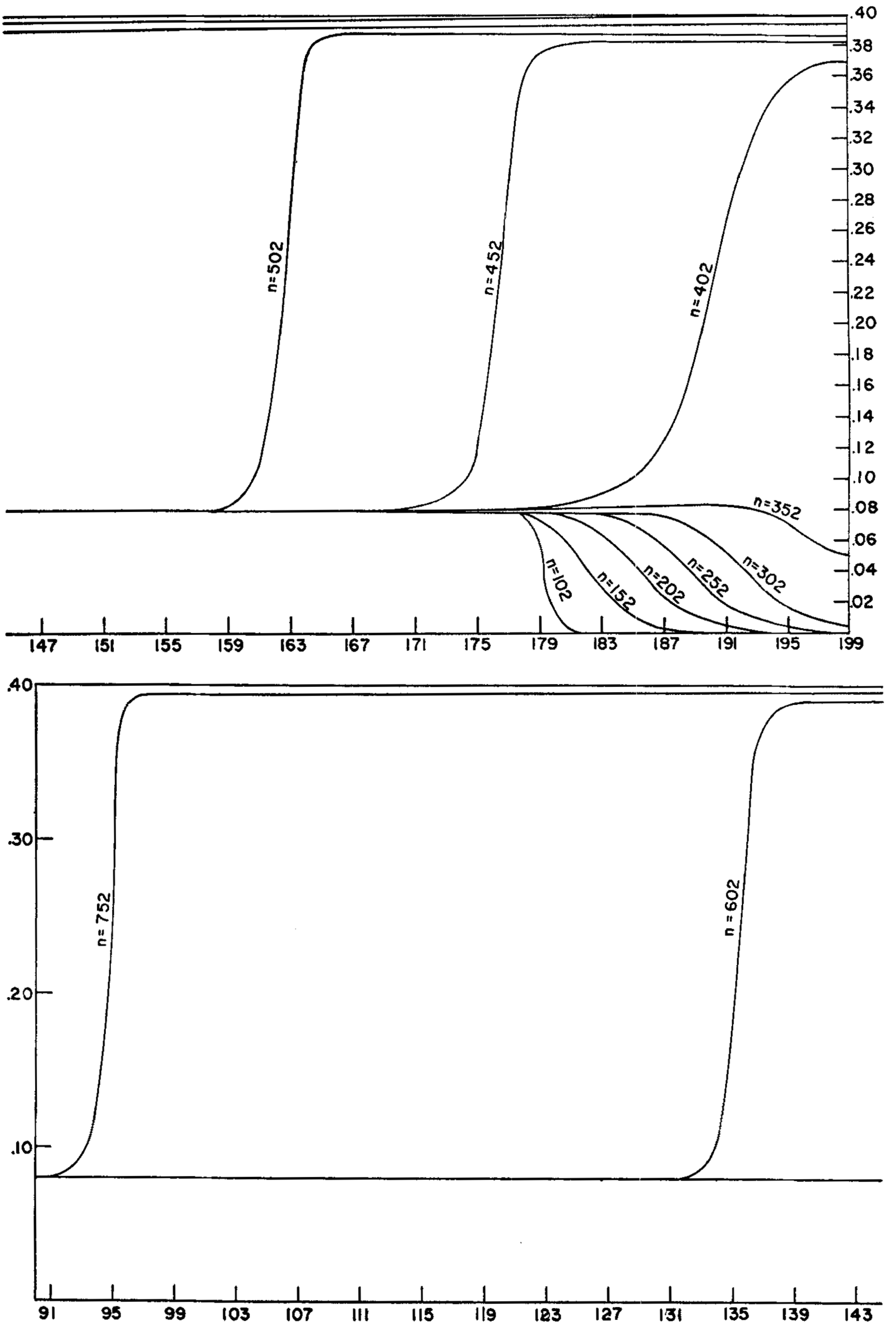
The exact solution is as follows: the shock propagates at the speed 2 until it reaches the wall, then it is reflected and propagated back at the speed of 8; the state of the gas U_r behind the reflected shock is $u_r = 0$, $V_r = 0.5$, $p_r = 40$. The original choice of $\Delta t/\Delta x$ was 0.25, which is the maximum permissible value for $p = 8$ and $V = 1$, and which is much too high at $p = 40$ and $V = 0.5$. Accordingly, $\Delta t/\Delta x$ was changed to 0.07 after 102 time cycles: this ratio is pretty close to the maximum permissible value of $\Delta t/\Delta x$ in the compressed state U_r. Calculations were carried on for an additional 650 time cycles. Taking Δx to be one, this corresponds to a total passage of time of $102 \times 0.25 + 65 \times 0.07 = 71$ units. During this time the shock will have reached the wall and will be reflected 216 units. Since meshpoints are $2\Delta x = 2$ units apart, this would locate the reflected shock at $k = 91$.

TABLE XIII

k	V	u	ϕ
88	0.9989	3.995	8.017
89	.9980	3.992	8.031
90	.9942	3.976	8.094
91	.9792	3.911	8.365
92	.9272	3.653	9.591
93	.7967	2.836	13.96
94	.6187	1.334	25.48
95	.5221	.292	36.41
96	.5039	.071	39.09
97	.5022	.050	39.36
98	.5019	.047	39.40

Table XIII lists the data around $k = 91$ after 752 time cycles. These calculations indicate that the reflected shock is located between $k = 91$ and 96, in pretty good agreement with theory.

The values of u beyond $k = 98$ decrease monotonically to $u = 0$ at $k = 199$; those of p increase monotonically to $p = 39.86$ at $k = 199$. The value of V decreases monotonically to $V = 0.5005$ which is reached at $k = 118$; from then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n=502
n=452
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n=352
n=302
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n=202
n=152
n=102
147
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155
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167
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.20
.10
n=752
n=602
91
95
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123
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131
135
139
143

FIGURE 3

on it increases monotonically to $V = 0.5109$ which is reached at $k = 174$, and then again decreases monotonically until it reaches $V = 0.4950$ at $k = 199$.

Curves depicting the pressure distribution at various times are given in Figure 3.

REMARK: The time cycle $n = 102$ was a bit too early to switch from $\Delta t/\Delta x = 0.25$ to $\Delta t/\Delta x = 0.07$ since at that time the shock was still twenty meshpoints away from the wall. This resulted in a rather smeared out shock for a considerable portion of the calculations, as may be seen from the pressure curves. Nevertheless this did not affect the situation at the 752-nd time cycle.

An analogous calculation, starting with the same initial data, but keeping $\Delta t/\Delta x$ equal to 0.07 throughout the calculation led to substantially the same result.

3. *Finite Difference Approximations to Steady State Solutions*

The exact solutions in examples 1, 4, 5 and 8 were steady state solutions, i.e. they depended only on a linear combination of x and t. In this section we would like to analyze the manner in which the approximate solutions computed by our finite difference scheme approach steady state solutions.

The numerical evidence indicates fairly clearly that the width of the shock is constant; furthermore there is a strong indication, especially in examples 1, 4 and 8 that the shapes of the transition curves tend to a definite limiting shape. For instance in example 1 the profile of the transition curve changes only by one figure in the fifth place between the 44-th and the 48-th time cycle. The limiting shape can be characterized as the *steady state* solution of the finite difference equation. That is, into the difference equation

$$u_k^{n+1} = \tfrac{1}{2}(u_{k+1}^n + u_{k-1}^n) + \tfrac{1}{4}(u_{k-1}^n)^2 - (u_{k+1}^n)^2$$

we put $u_k^n = f(k + n/2)$, so that $f(x)$ satisfies the nonlinear difference equation

$$\frac{f(x-1) + f(x+1)}{2} + \frac{f^2(x-1) - f^2(x+1)}{4} = f(x + \tfrac{1}{2}). \tag{7}$$

and the boundary conditions

$$f(-\infty) = 1, \qquad f(\infty) = 0. \tag{8}$$

CONJECTURE: *The difference equation* (7) *has a continuous, monotonic solution* $f(x)$ *as function of the real variable* x, *subject to the boundary condition* (8); *this solution is unique except for an arbitrary phase shift.*

Furthermore: *Iterates of the transformation* $Tg = g'$ *defined by*

$$g'\left(x + \frac{1}{2}\right) = \frac{g(x-1) + g(x+1)}{2} + \frac{g^2(x-1) - g^2(x+1)}{4}$$

converge to a solution of the steady state equation.

TABLE XIV

$n = 44$		$n = 48$	
k	u	k	u
17	1.00000	19	1.00000
19	.99195	21	.99195
21	.71566	23	.71566
23	.17449	25	.17449
25	.01858	27	.01859
27	.00165	29	.00165
29	.00014	31	.00014

This last statement means: If $g_0(x)$ is any function defined over the odd integers and equal to 1 for x large enough negative, 0 for x large enough positive, and if we denote $T^n g_0$ by $g_n(x)$, then $g_n(x)$ tends uniformly to $f(x + \alpha)$, where $f(x)$ is the steady state solution of (7), (8) (made unique, say, by fixing $f(0)$ to be $\frac{1}{2}$). The phase shift α depends of course on the initial distribution $g_0(x)$.

Observe that $g_n(x)$ is defined only at points congruent $n/2$ modulo 2; consequently the only values of $f(x + \alpha)$ that enter this limiting statement are at points congruent 0 or $\frac{1}{2}$ modulo one. The somewhat exceptional situation in this example (and in examples 4 and 8 as well) arose because $\Delta t/\Delta x$ was chosen commensurable with the speed of propagation of the discontinuity.

The calculations in example 1 verify the conjecture. As further check,

TABLE XV

x	g	x	g
−3.0	0.995	−3.5	0.998
−2.5	.982	−3.0	.992
−2.0	.947	−2.5	.972
−1.5	.877	−2.0	.925
−1.0	.768	−1.5	.840
−0.5	.626	−1.0	.716
0.0	.471	−0.5	.566
0.5	.327	0.0	.412
1.0	.211	0.5	.277
1.5	.128	1.0	.174
2.0	.074	1.5	.104
2.5	.042	2.0	.060
3.0	.023	2.5	.034
3.5	.013	3.0	.019
4.0	.007	3.5	.010
		4.0	.006

calculations were carried out for this initial function: $g_0(x) = 1$ for odd negative x less than minus 1, $g_0(-1) = 0.9$, $g_0(x) = 0$ for positive odd x. The results after 44, respectively 48 time cycles, listed in Table XIV, support the conjecture.

Table XV lists g_{45}, g_{46}, g_{47} and g_{48}, both for original and the second choice of g_0.

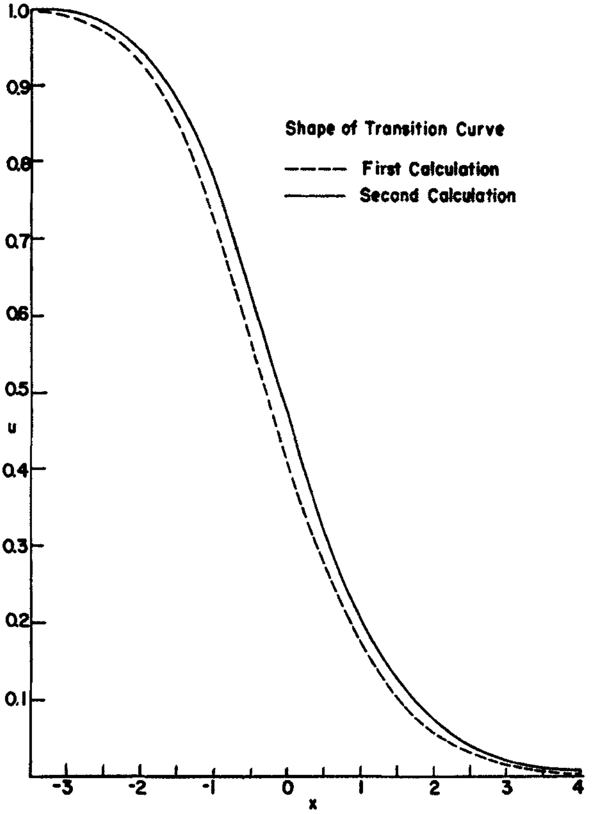

FIGURE 4

Figure 4 shows plots of these values; the two curves do indeed appear to be translates of each other, the phase shift being approximately 0.2.

Similar checks were carried out for examples 4 and 8; in each case the conjecture was verified.

It should be remarked that the nonlinear difference equation (7) has no solution if the boundary values in (8) are switched; i.e., $f(-\infty)$ is required to be zero, $f(\infty)$ one. This expresses the fact that the finite difference method furnishes compression shocks but no rarefaction shocks; I have no rigorous proof of this nonexistence at present. The result, if true, would be an analogue of a known result on viscous flows in steady state (See footnote 4). For completeness I shall present this result for the simplified equation $u_t + uu_x = \lambda u_{xx}$.

Let $u_0(x, t)$ be a steady state solution of this equation; that is, u_0 is a function of $x - ct$ only: $u_0(x, t) = f(x - ct)$. $f(\xi)$ satisfies the nonlinear ordinary differential equation

$$cf' + ff' = \lambda f''.$$

Integrating both sides with respect to ξ gives

$$\lambda f' = cf + \tfrac{1}{2}f^2 + k, \tag{9}$$

k being some constant.

We are interested in those solutions $f(\xi)$ of (9) which exist for all ξ and which approach constant values u_i and u_f as ξ tends to $-\infty$ and $+\infty$ respectively. Clearly if f approaches constant values, there must be two sequences of ξ tending to $+\infty$ and $-\infty$ respectively for which $f'(\xi)$ tends to zero, and so $u_i = f$ and $uf = f$ are zeros of the quadratic function on the right of (9). But a quadratic function with two real roots is *negative* between its two roots, and so by (9) f' is negative for all ξ; this shows that u_f cannot exceed u_i. Conversely, if u_f is less than u_i, the two states can be connected by a solution of (9); the formula

$$\xi = \frac{2\lambda}{u_i - u_f} \log \frac{u^i - u}{u - u_f}$$

gives the shape of the connecting curve.

We emphasized at the beginning of this paper that the class of weak solutions depends on the form in which the equation is written. I would like to present here an example which shows that if in a conservation equation the exact space derivative is *not* replaced by an exact difference, the limit of the solutions of the difference equations is *not* a weak solution of the original conservation law.

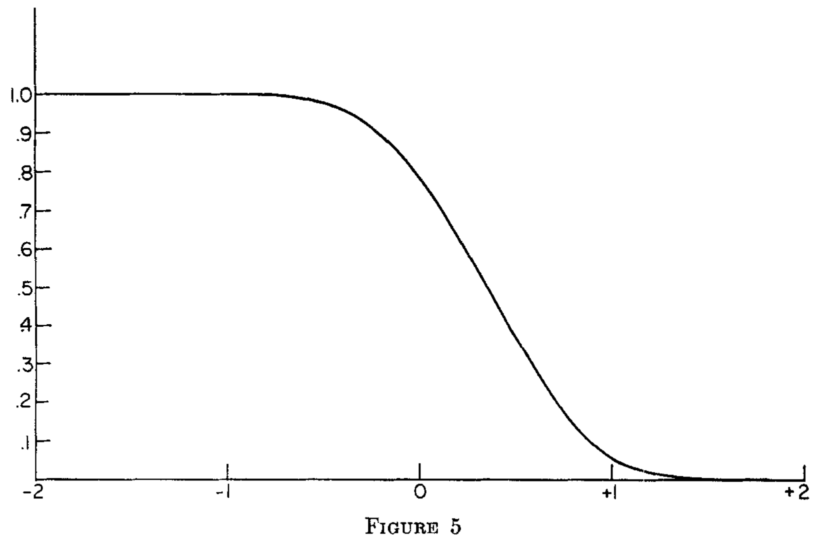

FIGURE 5

The equation in this example is the same as in example 4, $u_t + u^2 u_x = 0$; the following difference scheme was used:

$$u_k^{n+1} = u_k^n - \frac{(u_k^n)^2 + (u_{k-1}^n)^2}{2} (u_k^n - u_{k-1}^n) \frac{\Delta t}{\Delta x} ;$$

that is, instead of regarding the term $u^2 u_x$ as a perfect derivative, u^2 was regarded as coefficient of u_x. u_k^0 was chosen as 1 for $k < 0$, zero for $k > 0$; $\Delta t/\Delta x$ was chosen as one, and calculations were carried out for 48 time cycles. At all cycles the solution differed from 1 or zero only at two or three space points; the position of the interval of transition seemed to propagate at about the speed of 0.388; this differs appreciably from 0.333, the propagation speed of the exact weak solution. The shape of the transition curves seemed to settle down to a steady state shown in Figure 5.

The calculations were repeated with slightly altered initial data, taking u_1^0 to be 0.5 and u_k^0 for $k \neq 1$ the same as before. The solution again seemed to converge to a steady profile propagating at the speed of 0.388. The shape of

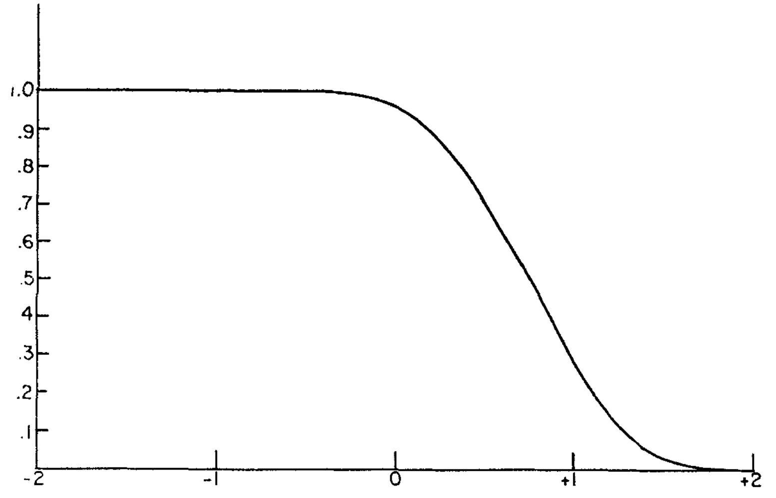

FIGURE 6

the profile, shown in Figure 6, seemed to be the same—except for a phase shift—as for the other set of initial values.

4. *Irreversibility*

The class of *genuine solutions* of differential equations is *reversible* in time; that is, if $U(x, t)$ is a genuine solution of the equation

$$U_t + AU_x + B = 0 \tag{1}$$

(in which we exclude, for sake of simplicity, explicit dependence of the coefficients on time) in the strip $0 < t < T$, then $U'(x, t) = U(x, T - t)$, is, clearly, a genuine solution of

$$U'_t - AU'_x - B = 0.$$

Likewise, if $U(x, t)$ is a *weak solution* of the equation

$$U_t + F_x + B = 0, \tag{2}$$

$U'(x, t) = U(x, T - t)$ is a weak solution of

$$U'_t - F(U')_x - B = 0. \tag{2'}$$

But if $U(x, t)$ is a *physically relevant* solution of (2), U' need not be a physically relevant solution of (2'). This can be summarized thus: *the class of physically relevant weak solutions is irreversible.*

There is nothing paradoxical in this loss of reversibility; it is just another instance of that universal experience that when concepts are generalized, not all properties of the original can be retained. In our case we wanted to generalize the notion of solution of the initial value problem and retain the property of uniqueness and continuous dependence; this turned out to be at the expense of reversibility.

Note that both the viscosity method and the finite difference schemes described in this paper, as well as the von Neumann-Richtmyer scheme, described in [13], distinguish the positive t direstion from the negative one; thus it is not surprising that the class of solutions produced by these methods is irreversible. Conversely, any limiting procedure that can be expected to furnish the physically relevant solutions must be *unsymmetric* in time.

5. *Continuous Dependence of the Solution on the Initial Data*

In the section where the viscosity method and the finite difference scheme were described I asserted that on account of the systematic nature of these processes one would expect the solutions furnished by these methods to depend continuously on the initial data. This is of course merely a plausibility argument, no proof. However, if the number of unknowns is one and the equation is homogeneous in the first derivatives, it is possible to give a rigorous proof of this fact. The most convenient topology for this purpose is a sort of a *weak* topology. More precisely, we introduce the following norm for functions $\varphi(x)$:

$$|\varphi| = \max_{a,b} \left| \int_a^b \varphi(\xi)\, d\xi \right|.$$

We shall refer to this norm as the *weak norm*.

Let u and u' be two solutions of the equation $u_t + f(u)_x = \lambda u_{xx}$ with initial values φ and φ'. We shall show that their deviation, in the sense of the norm introduced, at any positive time t does not exceed twice their deviation at

$t = 0$. This would prove the continuous dependence of solutions of our parabolic equation on the initial data *uniformly* with respect to λ, and thus in the limit $\lambda \to 0$ as well.

Proof: Denote by U and U' integrals of u and u' with respect to x; that is, $U(x, t)$ and $U'(x, t)$ are two functions for which $\partial U/\partial x$ is u, $\partial U'/\partial x$ is u'. Clearly U and U' are determined only modulo a function of t. U and U' satisfy, modulo a function of t, the nonlinear parabolic equation

$$U_t + f(U_x) = \lambda U_{xx} \; ; \tag{10}$$

by adding to U and U' a suitable function of t we can achieve that these modified functions satisfy the equation (10) exactly. Imagine such terms already added on; there is still room for an arbitrary additive constant in U and U' which we choose so that $U(0, 0)$ and $U'(0, 0)$ are zero.

Subtract the equations satisfied by U and U' and apply the mean value theorem to the second term on the left; we obtain

$$(U - U')_t + f_u(\bar{u}) \cdot (U - U')_x = \lambda(U - U')_{xx} \, ,$$

where $\bar{u}$ denotes a value between u and u'. The resulting equation is a linear parabolic equation for $U - U'$; for this the *maximum principle* holds, therefore $| U(x, t) - U'(x, t) |$ is, for t positive,

$$\leq \max_x | U(x, 0) - U' \ (x, 0) | = \max_x \left| \int_0^x u(\xi, 0) - u'(\xi, 0) \, d\xi \right| .$$

This last quantity is clearly $\leq | u(x, 0) - u'(x, 0) | = | \varphi - \varphi' |$. Since $\int_a^b (u(x, t) - u'(x, t)) \, dx$ is equal to $U(b, t) - U'(b, t) - [U(a, t) - U'(a, t)]$, the maximum of this quantity with respect to a and b, $| u(t) - u'(t) |$, does not exceed $2 | \varphi - \varphi' |$.

The above result on solutions of the nonlinear parabolic equation has an analogue for the solutions of the finite difference scheme

$$u_k^{n+1} = u_{k+1}^n + \tfrac{1}{2} u_{k-1}^n - [f(u_{k+1}^n) - f(u_{k-1}^n)] \frac{\Delta t}{2\Delta x} \, . \tag{11}$$

If we introduce as new unknown

$$U_k^n = \sum_{j=0}^{k} u_k^n + c^n,$$

c^n being a function of n but not of k, U_k^n will satisfy the equation

$$U_k^{n+1} = U_{k+1}^n + \tfrac{1}{2} U_{k-1}^n - f(U_{k+1}^n - U_{k-1}^n) \frac{\Delta t}{2\Delta t} \tag{12}$$

modulo a function n; and if the constants c^n are chosen suitably (12) will be satisfied exactly.

Let u_k^n and $\tilde{u}_k^n$ be two solutions of (11) and U_k^n and $\tilde{U}_k^n$ the corresponding solutions of (12). Subtract (12) and ($\tilde{12}$) and apply the mean value theorem to the second term on the right:

$$D_k^{n+1} = D_{k+1}^n + \tfrac{1}{2}D_{k-1}^n - f_u(\bar{u})[D_{k+1}^n - D_{k-1}^n]\frac{\Delta t}{2\Delta x}\,; \tag{13}$$

D_k^n is an abbreviation for the difference of U_k^n and $\tilde{U}_k^n$. The stability criterion demands that $f_u\ \Delta t/\Delta x$ should not exceed one; assume that this condition is satisfied. Then (13) is a recursion relation for D_k^n with *positive* coefficients whose sum is one, and for these the *maximum principle* applies: D_k^n never exceeds its largest value for $n = 0$. From this result one can deduce that the solutions constructed by the finite difference scheme are continuously dependent on the initial data in the sense of the weak norm.

The result we have just proved about the continuous dependence of the solutions constructed by the viscosity method on the initial data in the *weak* topology has this rather remarkable consequence:

The dependence of u on φ is completely continuous.

More precisely: Denote by $\{\varphi\}$ the collection of measurable functions bounded by some constants and vanishing outside some fixed interval[8], and denote by $\{u\}$ the corresponding weak solutions constructed by the viscosity method. We claim that the set $\{u\}$ is *compact in the L_1 sense.*

REMARK: The same result holds for the solutions constructed by the finite difference scheme.

Proof: Let $[u]$ be an infinite subset of the set of weak solutions considered; we wish to show that $[u]$ has a point of accumulation in the L_1 sense. Let $[\varphi]$ be the set of their initial data. The original collection $\{\varphi\}$ of initial functions is *compact* in the sense of the *weak* norm introduced before; therefore it is possible to select an infinite subsequence φ_i in $[\varphi]$ which converges to a limit φ in the sense of the weak norm. According to the lemma proved before, the sequence of corresponding solutions u_i converges in the sense of the weak norm to the solution u with initial value φ. However, according to a principle stated in one of the earlier sections, a weak limit of weak solutions is a weak solution if and only if it is a strong (i.e. L_1) limit. Therefore we can conclude that the sequence u_i tends to u in the L_1 sense, and this proves the compactness of the set $\{u\}$ in the L_1 set.

For sake of completeness we shall state and prove rigorously the principle about weak limits of weak solutions. We have to assume to start with that our equation $u_t + f(u)_x = 0$ is genuinely nonlinear, meaning that f_u, the coefficient of u_x, genuinely depends on u, i.e. that its derivative with respect to u doesn't vanish: $f_{uu} \neq 0$, say f_{uu} is greater in the relevant range of u than some positive quantity.

LEMMA: If the sequence of functions u_n converges[9] in the *weak sense* to a

[8]These conditions imposed on the set of initial functions can be replaced by others.

[9]The range of the functions $u_n(x, t)$ is assumed to lie in a finite range: $a \leq u \leq b$.

limit u, then $f(u_n)$ converges in the weak sense to $f(u)$, if and only if u_n tends to u in the L_1 sense.

Proof: If $f(u_n)$ converges weakly to $f(u)$, $\int vf(u_n)$ must approach $\int vf(u)$ for every square integrable function v. Take for v any positive function and consider the integral

$$I(u) = \int vf(u)$$

as functional of u. It is an easy consequence of the assumption that f_{uu} is positive, that the second variation of $I(u)$ is positive definite, i.e.

$$I[u_n] - I[u] = \int vf_u(u) \cdot (u_n - u) + R,$$

where R is greater than const. $\int v(u_n - u)^2$. This implies the lower semicontinuity of the functional $I[u]$ in the *strict* sense:

$$\liminf I[u_n] \geq I[u]$$

for any sequence u_n weakly convergent to u; the sign of equality holding *if and only if* u is a strong limit of the sequence u_n.

The results derived can be summarized thus:

If the sequence of initial vectors φ_i converges in the weak norm, the corresponding solutions u_i converge in the L_1 sense in every bounded subset of the halfplane $t > 0$.

It follows from this result that for almost all t the functions $u_i(x, t)$ converge in the L_1 sense on every bounded interval of the x-axis. Most likely this is true for all values of t without *any* exception; this fact is expressed concisely by this statement:

The transformation mapping the initial state into the state at time t is completely continuous.

This is a particularly striking manifestation of irreversibility, in sharp contrast with the state of affairs in the theory of linear equations. There the transformations mapping the initial data into their value at time t is invertible.

The completely continuous dependence of the solution on the initial data can also be read off the explicit expression for the solution given in section 1. It would be interesting to know whether something like this is true for systems with more unknowns, in particular the equations of compressible flow.

6. *The Case of More Space Variables*

Much of what was said for the case of one space variable can be generalized if there are more space variables. In particular the same staggered difference scheme can be employed; the space derivatives are still replaced by centered difference quotients, and the time derivative U_t by a forward difference quotient

$$U_k^{n+1} - \overline{U}_k^n/\Delta t,$$

where $\overline{U}_k^n$ is the average of the value of U at the $2m$ neighbors of the lattice point k. The convergence of this scheme under a certain restriction on the ratio of Δt to the space increments has been proved for *linear* symmetric hyperbolic equations by K. O. Friedrichs.

The method can be applied to the Euler form of the flow equations in two (and three) dimensions. The conservation equations are:

$$\rho_t + (u\rho)_x + (v\rho)_y = 0,$$

$$(u\rho)_t + (u^2\rho + p)_x + (uv\rho)_y = 0,$$

$$(v\rho)_t + (uv\rho)_x + (v^2\rho + p)_y = 0,$$

$$E_t + (uE + up)_x + (vE + vp)_y = 0.$$

One would operate with ρ, $u\rho$, $v\rho$, and E as unknown functions.

A large portion of the work reported here was performed for the Los Alamos Scientific Laboratory. The calculations for examples 1, 2, 3, 4 were carried out on IBM calculators The calculations for examples 5, 6 and 7, as well as the calculations of Lester Baumhoff, were carried out on the Los Alamos MANIAC. The calculations for example 8 and for the reflection of a shock from a rigid wall were carried out on the UNIVAC of the Eckert-Mauchly Corporation. My thanks are due to Stewart Schlesinger for coding problems 1, 2, 4 and for programming problems 5, 6, 7, to Paul Stein and Paul Rosenthal for programming and coding the problems for the UNIVAC, to Lois Cook for coding problems 5, 6, 7, and to George Pimbley for coding problem 3. My thanks are also due to members of the staff of the Theoretical Physics Division of the Los Alamos Laboratory for their interest in this work; I had many fruitful conversations with them on finite difference methods.

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