http://dx.doi.org/10.1090/trans2/026/05

DISCONTINUOUS SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS*

O. A. OLEĬNIK

CONTENTS

§1. Introduction

Many problems of mechanics, in particular problems of gas-dynamics, lead us to the study of discontinuous solutions of non-linear hyperbolic equations. Discontinuous solutions of non-linear hyperbolic equations have a series of properties which are not shared by discontinuous solutions of linear equations. The basic difference between linear and non-linear hyperbolic equations lies in the different nature of the physical processes which they describe. As we know, the propagation of strong perturbations in continuous media is described by non-linear differential equations, the propagation of weak perturbations by linear equations. We also know that in the case of a wave-motion described by a linear equation, the initial surface of a discontinuity is preserved as a discontinuity and it moves with the speed of sound. In the case of a wave-motion described by non-linear equations we have the following: either the initial discontinuity disappears immediately, or it spreads out in the form of shock waves, moving with a supersonic velocity [1].

These peculiarities of linear and non-linear hyperbolic equations can be

*The present article represents a detailed exposition of the contents of a lecture given by the author on June 27, 1956, at the Third All-Union Mathematical Conference under the title, "Certain problems of the theory of non-linear equations and an equation with a small parameter in the derivative of the highest order."

illustrated by the simplest equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{1.1}$$

and

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{1.2}$$

for example.

As an initial condition for the Cauchy problem, let us consider the discontinuous functions of the form

$$u_1(x) = 1, \text{ if } x \leq 0, \quad u_1(x) = -1, \text{ if } x > 0$$

and

$$u_2(x) = -1, \text{ if } x \leq 0, \quad u_2(x) = 1, \text{ if } x > 0.$$

For the linear equation (1.2) the solution of the Cauchy problem $u(t, x)$ with the initial condition

$$u(0, x) = u_1(x) \tag{1.3}$$

is defined uniquely in all points of the half-plane $t \geq 0$. This solution is given by the function

$$u(t, x) = \begin{cases} 1, & \text{if } x - t \leq 0, \\ -1, & \text{if } x - t > 0. \end{cases}$$

The function $-u(t, x)$ is a solution of equation (1.2) with the initial condition

$$u(0, x) = u_2(x). \tag{1.4}$$

At the points of the straight line $x - t = 0$ which is a characteristic of equation (1.2), these solutions have a discontinuity.

For the non-linear equation (1.1) solutions of the Cauchy problem with the initial conditions (1.3) and (1.4) are not uniquely defined even in an arbitrarily small neighborhood of the straight line $t = 0$, where the initial conditions are given. Indeed, let us construct in the space (t, x, u) the characteristics of equation (1.1) which pass through the points $(0, x, u(0, x))$, where $-\infty < x < +\infty$, and which are solutions of the system of equations (see [2])

$$\frac{du}{dt} = 0, \quad \frac{dx}{dt} = u.$$

The surface formed by these characteristics, in the case of a smooth function $u(0, x)$, defines a solution of the Cauchy problem. If $u(0, x) = u_1(x)$, then the projections of these characteristics on the plane (t, x) cover all points of the half-plane $t > 0$. The points lying between the straight lines $x - t = 0$ and $x + t = 0$ are covered by the projections of these characteristics twice and each time u assumes different values. It is evident that at the points lying between these straight lines for $t > 0$ a solution $u(t, x)$ cannot be defined uniquely by an initial condition alone.

If $u(0, x) = u_2(x)$, then the projections of the characteristics of equation (1.1), passing through the points $(0, x, u_2(x))$, cover in the half-plane $t > 0$ only those points which do not belong to the region enclosed within the straight lines $x - t = 0$ and $x + t = 0$ and, consequently, in this region a solution cannot be defined uniquely by an initial condition alone.

Thus, for a unique determination of a solution of the Cauchy problem for the non-linear equation (1.1) with discontinuous initial conditions (1.3) and (1.4), one needs additional conditions and a new definition of a solution of (1.1) in the class of discontinuous functions.

It is known that for non-linear hyperbolic equations a smooth solution of the Cauchy problem with smooth initial conditions given for $t = 0$ exists, in general, only for sufficiently small t. Thus, for instance, a smooth solution of the Cauchy problem (1.1) with the initial condition

$$u(0, x) = -\operatorname{th}\left[\frac{x}{2\epsilon}\right] \tag{1.5}$$

exists only for $t < 2\epsilon$. Indeed, the projections on the plane (t, x) of characteristics passing through the points $\left(0, x_0, -\operatorname{th}\left[\frac{x_0}{2\epsilon}\right]\right)$ and $\left(0, -x_0, -\operatorname{th}\left[-\frac{x_0}{2\epsilon}\right]\right)$, intersect in the point $P_{x_0} = \left(\frac{x_0}{\left[\operatorname{th}\frac{x_0}{2\epsilon}\right]}, 0\right)$ and yield different values of u at P_{x_0}. For $x_0 \to 0$ P_{x_0} tends to $(2\epsilon, 0)$.

However, for equations describing specific physical processes like, for instance, for the system of equations of gas-dynamics, a solution of the problem with initial conditions must exist for arbitrarily large values of the time t by virtue of the physical meaning of the problem. In this connection it is possible that a solution should be understood in a certain generalized sense.

Discontinuous solutions of systems of equations describing a motion of a gas were studied by Stokes, Riemann, Hugoniot and others. When considering the Cauchy problem for such a system describing a one-dimensional motion of a gas, one introduces additional relations among the functions to be found on the lines of a discontinuity. These relations express the fact that the laws of conservation of mass, energy, impulse are fulfilled and that entropy increases. The fulfillment of additional "conservation laws"-type relations on the lines of a discontinuity means that these functions satisfy a system of equations in a certain sense as integrals. The condition for the increase of entropy means that on the lines of a discontinuity of a solution certain inequalities, involving the limiting values of functions to be found, are fulfilled [1].

Let us explain what has been said above by the equation

$$\frac{\partial u}{\partial t} + \frac{\partial \phi(u)}{\partial x} = 0, \quad \phi'' \neq 0, \tag{1.6}$$

for example, which, as we shall show below, can be considered as the simplest model of equations of gas-dynamics.

We shall call piece-wise smooth and piece-wise continuous functions the functions of class K. Let a function $u(t, x)$ of class K satisfy equation (1.6) for $t > 0$ in the sense that for any smooth contour Γ, lying in the half-plane $t > 0$ and intersecting the discontinuity lines of the function $u(t, x)$ in a finite number of points, the equality

$$\int_{\Gamma} u\, dx - \phi(u)\, dt = 0 \tag{1.7}$$

is fulfilled. It is evident that for a continuously differentiable solution $u(t, x)$ of equation (1.6) the equality (1.7) always holds. Moreover, from relation

$$\iint_{D'} \left[\frac{\partial u}{\partial t} + \frac{\partial \phi(u)}{\partial x}\right] dx\, dt = \int_{\Gamma'} u\, dx - \phi(u)\, dt, \tag{1.8}$$

where Γ' is the boundary of region D', it follows that the function $u(t, x)$ for which equality (1.7) is fulfilled, satisfies equation (1.6) in the points of the region where $u(t, x)$ is continuously differentiable.

On the lines of a discontinuity $x = x(t)$ of the function $u(t, x)$ of class K, satisfying equation (1.7), the relation

$$\frac{dx}{dt} = \frac{\phi(u_1) - \phi(u_2)}{u_1 - u_2} \tag{1.9}$$

is fulfilled, where u_1 and u_2 are the limiting values of the function $u(t, x)$ in the point $(t, x(t))$ from right and left respectively, i.e., $u_1 = u(t, x(t) + 0)$, $u_2 = u(t, x(t) - 0)$. Indeed, let a closed contour Γ_δ be formed by the lines

$$t = t_1 + \delta, \quad t = t_1 - \delta, \quad x = x(t) + \alpha, \quad x = x(t) - \alpha, \tag{1.10}$$

where $\delta > 0$ and $\alpha > 0$ are certain sufficiently small constants.

Let us consider integral (1.7) taken around the contour Γ_δ:

$$\int_{t_1-\delta}^{t_1+\delta} \left[u(t, x(t)+\alpha)\frac{dx(t)}{dt} - \varphi(u(t, x(t)+\alpha))\right] dt - \int_{x(t_1+\delta)-\alpha}^{x(t_1+\delta)+\alpha} u(t_1+\delta, x)\, dx -$$
$$- \int_{t_1-\delta}^{t_1+\delta} \left[u(t, x(t)-\alpha)\frac{dx(t)}{dt} - \varphi(u(t, x(t)-\alpha))\right] dt + \int_{x(t_1-\delta)-\alpha}^{x(t_1-\delta)+\alpha} u(t_1-\delta, x)\, dx = 0.$$

Passing to the limit in this relation $\alpha \to 0$, we get

$$\int_{t_1-\delta}^{t_1+\delta}\left[u(t, x(t)+0)\frac{dx}{dt}-\varphi(u(t, x(t)+0))\right]dt=$$
$$=\int_{t_1-\delta}^{t_1+\delta}\left[u(t, x(t)-0)\frac{dx}{dt}-\varphi(u(t, x(t)-0))\right]dt.$$

Since $\delta > 0$ is an arbitrary small number, so the expressions under the integral sign in the right and the left parts of the last equality must coincide at each point, which means that at each point of a discontinuity line of $u(t, x)$ (1.9) is fulfilled.

Relation (1.9) is analogous to the "conservation laws"-type relations on the lines of a discontinuity for solutions of a system of a one-dimensional non-stationary motion of a gas.

For a function $u(t, x)$ of class K equality (1.7) is equivalent to the fact that for $u(t, x)$ for any continuously differentiable test-function $f(t, x)$ (i.e., equal to zero outside certain finite region D) the equality

$$\iint_D\left[\frac{\partial f}{\partial t}u(t, x)+\frac{\partial f}{\partial x}\phi(u(t, x))\right]dx\,dt=0 \tag{1.11}$$

holds. This integral equality is analogous to the relations with the help of which one introduces generalized solutions in the sense of S. L. Sobolev.

First, let us show that for a function $u(t, x)$ of class K satisfying equality (1.11), the relation (1.19) is fulfilled on the lines of a discontinuity. In those regions D' where $u(t, x)$ is continuously differentiable, from relation (1.11) by integration by parts we get

$$\iint_{D'} f\left(\frac{\partial u}{\partial t}+\frac{\partial \phi(u)}{\partial x}\right)dx\,dt=0$$

for any smooth function $f(t, x)$ vanishing on the boundary of D'. Hence, it follows that $u(t, x)$ satisfies in D' equation (1.6). Let us consider equality (1.11) for $u(t, x)$ in a region D_δ which is bounded by the contour Γ_δ, formed by the lines (1.10). Representing the left part of the considered equality as a sum of integrals, taken over the regions into which the region D is subdivided by a line of a discontinuity $x = x(t)$, and integrating by parts, we get

$$\int_{t_1-\delta}^{t_1+\delta}\left[u(t, x(t)+0)\frac{dx}{dt}-\varphi(u(t, x(t)+0))\right]f\,dt=$$
$$=\int_{t_1-\delta}^{t_1+\delta}\left[u(t, x(t)-0)\frac{dx}{dt}-\varphi(u(t, x(t)-0))\right]f\,dt, \tag{1.12}$$

since $f(t, x) = 0$ on the contour Γ_δ, and the function $u(t, x)$ satisfies equation (1.6) in each point of D_δ, not lying on the line of a discontinuity. Because of arbitrariness of $f(t, x)$ from (1.12) follows relation (1.9) in all points of the line of a discontinuity $x = x(t)$.

It is easy to show that for a function $u(t, x)$ of class K the equalities (1.7) and (1.11) are fulfilled, if $u(t, x)$ satisfies equation (1.6) in the points in which it has continuous derivatives, and satisfies relation (1.9) in the points of a discontinuity line. Thus, for instance, in order to prove that $u(t, x)$ satisfies (1.7), one has to consider relations (1.8) for each of the regions into which the region bounded by the contour Γ is subdivided by the lines of a discontinuity. The sum of these equalities gives (1.7), since the integrals taken along the discontinuity-lines of $u(t, x)$, cancel one another by virtue of relations (1.9).

Functions of class K, satisfying equality (1.7) or (1.11) can be considered as generalized solutions of equation (1.6). It is interesting to note that the set of the so defined generalized solutions of equation (1.6) does not coincide with the set of generalized solutions of equation

$$\alpha(u)\left[\frac{\partial u}{\partial t} + \frac{\partial \phi(u)}{\partial x}\right] = 0$$

where $\alpha(u) \neq 0$, since to them on the lines of a discontinuity there correspond different relations of the form (1.9).

Functions of class K, satisfying equation (1.6) in the sense of the integral equalities (1.7) or (1.11) form such a class of functions in which the Cauchy problem for equation (1.6) with one and the same initial condition has many solutions. This is what distinguishes the non-linear equation (1.6) from linear equations.

Thus, for instance, for any $a \geq 1$ the function

$$u_a(t, x) = \begin{cases} 1, & \text{if} \quad x \leqslant \frac{1-a}{2}t, \\ -a, & \text{if} \quad \frac{1-a}{2}t < x \leqslant 0, \\ a, & \text{if} \quad 0 < x \leqslant \frac{a-1}{2}t, \\ -1, & \text{if} \quad \frac{a-1}{2}t < x, \end{cases}$$

defined in the points of the half-plane $t \geq 0$, satisfies equation (1.1) for $t \geq 0$ in the sense of the integral equalities (1.7) and (1.11) and the initial condition (1.3). This can be easily verified using the fact that on the lines of a discontinuity of the function $u_a(t, x)$ the relation (1.9) is fulfilled.

Thus, in order to define a unique solution of the Cauchy problem in the generalized sense (1.7) or (1.11), one needs additional conditions. For equations of gas-dynamics such a condition is the condition of the increase of entropy. For the model equation (1.6) to this condition corresponds the fact that on the lines of a discontinuity the relation

$$u(t, x(t) - 0) > u(t, x(t) + 0) \tag{1.13}$$

is fulfilled if $\phi''(u) > 0$, and the relation

$$u(t, x(t) - 0) < u(t, x(t) + 0) \tag{1.14}$$

is fulfilled if $\phi''(u) < 0$.

Condition (1.13) is satisfied only by one of the functions $u_a(t, x)$, corresponding to $a = 1$. Consequently, a solution of the Cauchy problem (1.1) with condition (1.3) in the sense of the integral equalities (1.7) and (1.11), satisfying relation (1.13), is the function

$$u(t, x) = 1, \text{ if } x \leq 0 \text{ and } v(t, x) = -1, \text{ if } x > 0.$$

The straight line $x = 0$ is a discontinuity-line for this solution. These types of discontinuities of solutions of gas dynamics are called shock-waves. It can be shown that the solution of this problem is unique. It is interesting to note that a unique solution of the problem (1.1), (1.4) in the indicated sense is the function

$$\left.\begin{aligned} &u(t, x) = -1, \quad \text{if} \quad x \leqslant -t, \\ &u(t, x) = \frac{x}{t}, \quad \text{if} \quad -t \leqslant x \leqslant t, \\ &u(t, x) = 1, \quad \text{if} \quad x \geqslant t. \end{aligned}\right\} \tag{1.15}$$

This solution in its structure is analogous to those, which in gas-dynamics are called the rarefaction waves. Function (1.15) is continuous for $t > 0$. Thus, to a discontinuous initial condition (1.4) there corresponds a continuous solution of equation (1.1) in the sense of (1.7) or (1.11).

The construction of a generalized solution of the model equation (1.6) can be carried out also in a different way which has a deep analogy in equations of gas-dynamics.

In the study of a motion of a gas, taking into account viscosity and heat conduction corresponds to the introducing into a system of first order equations terms with derivatives of second order which contain small parameters as coefficients. It is natural to expect that when the coefficients of viscosity and heat conduction tend to zero, then the solutions of this system tend to solutions of equations of gas-dynamics. The limiting functions satisfy the required conditions on the lines of a discontinuity [4].

For the model equation (1.6) the introduction of terms corresponding to viscosity means the transition from equation (1.6) to the parabolic equation

$$\epsilon\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+\frac{\partial \phi(u)}{\partial x}, \tag{1.16}$$

where $\epsilon > 0$ is a small parameter. As it will be shown in §5, a smooth solution of the Cauchy problem for equation (1.16) for $t > 0$ exists for any discontinuous initial function from a certain class.

The definition of generalized solutions of equation (1.6) by means of introducing "viscocity" means, roughly speaking, the following: the function $u(t, x)$ is called a generalized solution of equation (1.6), if it is a limit in a certain norm of solutions $u_\epsilon(t, x)$ of equation (1.16) for $\epsilon \rightarrow 0$.

It is easy to see that if $u_\epsilon(t, x)$ tends to $u(t, x)$ in the L_1-norm, i.e., $\iint\limits_D |u_\epsilon - u|\,dx\,dt \rightarrow 0$ for $\epsilon \rightarrow 0$, then $u(t, x)$ is a solution of equation (1.6) in the sense of (1.11). Indeed, taking the limit for $\epsilon \rightarrow 0$ in the relation

$$\iint\limits_D \left[\varepsilon u_\varepsilon \frac{\partial^2 f}{\partial x^2} + u_\varepsilon \frac{\partial f}{\partial t} + \varphi(u_\varepsilon)\frac{\partial f}{\partial x}\right] dx\,dt = 0, \tag{1.17}$$

where $f(x, t)$ is a twice continuously differentiable function, vanishing outside the region D, we obtain relation (1.11). The equality (1.17) is obtained from equation (1.16) in an obvious way by multiplying it by $f(t, x)$, integrating over D and transforming the obtained integrals by integration by parts. When considering the Cauchy problem for equation (1.6) we evidently must require that the initial functions for $u_\epsilon(t, x)$ should tend for $\epsilon \rightarrow 0$ in a definite way to the initial function of the Cauchy problem for equation (1.6).

Introducing terms corresponding to "viscocity" into equation (1.6) leads to smoothing out of discontinuities of solutions of equation (1.6). Thus, to the initial condition (1.5) and equation (1.16) for $\phi(u) = u^2/2$ there corresponds a smooth solution $u(t, x) = -\operatorname{th}(x/2\epsilon)$ defined for all $t > 0$ which for $\epsilon \rightarrow 0$ tends to a generalized solution of problem (1.1), (1.3). As we have shown above, to the initial condition (1.5) there corresponds a smooth solution of equation (1.1) only for $t < 2\epsilon$.

Above we have described a series of arguments concerning possible approaches to the study of discontinuous solutions of non-linear equations. To the discussion of these questions are devoted also some sections of E. Hopf's work [5] and some of I. G. Petrovskiĭ's book [6] and the article [7] by P. Lax.

The present article is devoted to the study of discontinuous solutions of the quasi-linear equations

$$\frac{\partial u}{\partial t}+a(t, x, u)\frac{\partial u}{\partial x}+b(t, x, u)=0. \tag{1.18}$$

These equations we shall write in the form

$$\frac{\partial u}{\partial t}+\frac{\partial \phi(t, x, u)}{\partial x}+\psi(t, x, u)=0. \tag{1.19}$$

The generalized solutions of equation (1.19) for $\phi_{uu} \neq 0$, as it will be shown below, possess a series of properties resembling the properties of functions which describe the propagation of strong perturbations in a continuous medium. In this sense equation (1.19) is a model-equation for the system of equations of a one-dimensional non-stationary motion of a gas, as was first shown by Burgers [8]. Therefore, the study of equation (1.19) is of interest for applications.

At the present time the theory of generalized solutions of equation (1.19) is in a certain sense complete. Before turning to the discussion of the basic results related to the study of generalized solutions of equation (1.19), we give a brief review of the literature pertaining to the questions under consideration.

If we exclude works devoted to the physical studies connected with the consideration of discontinuous solutions, then the number of works related to the study of discontinuous solutions of non-linear equations is quite small.

The question concerning the construction of a solution of the Cauchy problem for equation (1.19) with discontinuous initial conditions, and the question concerning the relation of such solutions with the solutions of the parabolic equation

$$\epsilon \frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+\frac{\partial \phi(t, x, u)}{\partial x}+\psi(t, x, u) \tag{1.20}$$

for small ϵ, were considered by a series of authors.

In the work of E. Hopf [5] we have the construction of the limiting function for solutions $u_\epsilon(t, x)$ of the Cauchy problem to the equation

$$\epsilon \frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x} \tag{1.21}$$

for ϵ tending to zero with initial conditions given for $t=0$.

The solution of the Cauchy problem for equation (1.21) is reduced to the solution of the Cauchy problem for the heat equation and can be written in an explicit form. In this work the properties of the limiting function are studied in detail and the question is posed as to the determination of this function independently of the passage to the limit in equation (1.21), since it is natural to consider such a function as a generalized solution of equation (1.1).

In the articles by O. A. Oleĭnik [9]–[12] a definition is given for a generalized solution of the Cauchy problem with a discontinuous initial function for the equation

$$\frac{\partial \phi_1(t, x, u)}{\partial t}+\frac{\partial \phi_2(t, x, u)}{\partial x}=0, \quad \phi_{1u} \neq 0, \tag{1.22}$$

which is based on an integral equality of the form (1.7); its existence and uniqueness are proved; also are studied the properties of a generalized solution, its dependence on the initial function, and the connection with the limiting functions for $\epsilon \rightarrow 0$ of solutions of the Cauchy problem and the first boundary value problem for the equation

$$\epsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial \phi_1(t, x, u)}{\partial t} + \frac{\partial \phi_2(t, x, u)}{\partial x}. \tag{1.23}$$

In the work by P. Lax [7] a finite differences scheme for the construction of discontinuous solutions of equation (1.19) is proposed, and a hypothesis is stated to the effect that the solutions of the Cauchy problem for the finite differences equations with a refinement of the grid tend to the same limit as the solutions of the Cauchy problem for equation (1.20) when ϵ tends to zero. A series of interesting arguments concerning the construction of generalized solutions of quasi-linear equations and systems is stated in the paper [13] by P. Lax.

A different finite differences scheme for the construction of generalized solutions of a system of equations of gas-dynamics was proposed earlier in the work [14].

In the work by A. N. Tihonov and A. A. Samarskiĭ [15] a definition is given for a generalized solution of the Cauchy problem for equation (1.19) for a piece-wise smooth and piece-wise continuous initial function and the existence and uniqueness of such a solution for sufficiently small t are proved. The basis for this definition serves the integral relation of the form (1.7).

Piece-wise smooth solutions of equations of gas-dynamics and the question concerning the conditions for the uniqueness of a solution of the Cauchy problem for such equations were studied by S. K. Godunov in the work [16]. He proposed a finite differences scheme for the construction of discontinuous solutions of quasi-linear equations (without justifying its convergence).

In the work by O. A. Oleĭnik [17] a definition based on an integral equality of the form (1.11) is given for a generalized solution of the Cauchy problem for equation (1.19) and any bounded measurable initial function and the existence and uniqueness of the so-defined generalized solution of the Cauchy problem is proved. It is also shown that this definition of a generalized solution is equivalent to that which was given earlier in the papers [9]–[12] for $\psi(t, x, u) \equiv 0$.

N. D. Vvedenskaja [18], [19] has proved the existence of the generalized solution of the Cauchy problem for equation (1.19) defined in the work [17], using the finite differences method and has established a series of properties of this solution. She has proved the validity of the Lax's hypothesis which was mentioned above, and has shown that the generalized solution of equation (1.19) is the limiting function.

In the work by O. A. Ladyženskaja [20] for the initial functions $u_0(x)$ satisfying the condition $u_0'(x) < K$, a different proof is given for the convergence for $\epsilon \to 0$ of solutions of the Cauchy problem for equation (1.20) with $\psi \equiv 0$ to a generalized solution of equation (1.19) defined in [17].

In the work [21] the boundary value problem and the Cauchy problem for equation (1.19) is considered; it is proved that the Cauchy problem for equation (1.20) is solvable for any bounded measurable initial function and that for $\epsilon \to 0$ these solutions tend to a generalized solution of the Cauchy problem for equation (1.19).

The exposition which follows is based on works [17], [12], [18], [21]. Most of the results given here with detailed proofs are published for the first time.

In this article we shall not discuss the question pertaining to the systems of quasi-linear equations, although the methods employed in studying equation (1.19) are, in series of cases, in principle applicable also to certain systems of quasi-linear equations.

In the beginning of each section we indicate briefly the main results contained in it.

§2. Formulation of the Cauchy problem. Uniqueness theorem.

We shall consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial \phi(t, x, u)}{\partial x} + \psi(t, x, u) = 0 \tag{2.1}$$

under the assumption that the functions $\phi(t, x, u)$ and $\psi(t, x, u)$ are defined and continuous for all values of u and (t, x) in G, where G is the strip $\{-\infty < x < +\infty, 0 \le t \le T \le \infty\}$. For these values of variables, $\phi(t, x, u)$ has a continuous partial derivative with respect to u, which is bounded for $(t, x) \subset G$ and for bounded u, and it has continuous derivatives ϕ_{ux}, ϕ_{uu} $(\phi_{uu} \ge 0)$, and the function $\psi(t, x, u)$ has a continuous partial derivative with respect to u. In this section we shall give a definition of a generalized solution of the Cauchy problem for equation (2.1) and prove a uniqueness theorem for such a solution.

Let $u_0(x)$ be a measurable bounded function defined for all x.

Definition. A bounded measurable function $u(t, x)$ will be called a generalized solution of the Cauchy problem for equation (2.1) in the region G with the initial condition (2.2)

$$u(0, x) = u_0(x), \quad -\infty < x < +\infty, \tag{2.2}$$

if

1) for any continuously differentiable in G function $f(t, x)$ equal to zero outside some finite region and for $t = T$, we have the equality

$$\iint\limits_G \left[\frac{\partial f}{\partial t} u(t, x) + \frac{\partial f}{\partial x} \varphi(t, x, u(t, x)) - f\psi(t, x, u)\right] dx\,dt + \\ + \int\limits_{-\infty}^{+\infty} f(0, x) u_0(x)\,dx = 0, \qquad (2.3)$$

2) there exists a function $K(t, x_1, x_2)$, continuous for $0 < t < T$, $-\infty < x_1 < +\infty$, $-\infty < x_2 < +\infty$, such that for all points (t, x_1) and (t, x_2) of G we have

$$\frac{u(t, x_1) - u(t, x_2)}{x_1 - x_2} \leq K(t, x_1, x_2), \qquad (2.4)$$

if, when necessary, we redefine $u(t, x)$ on a set of measure zero.

We shall consider generalized solutions $u_1(t, x)$ and $u_2(t, x)$ as coinciding, if the functions $u_1(t, x)$ and $u_2(t, x)$ differ only on a set of measure zero.

Theorem 1. *A generalized solution of the Cauchy problem in a region G for the equation* (2.1) *with the initial condition* (2.2) *is unique.*

Proof. Let $u_1(t, x)$ and $u_2(t, x)$ be two solutions of the Cauchy problem (2.1), (2.2). We shall show that $u_1(t, x) = u_2(t, x)$ almost everywhere in G.

Let us construct the averaged functions u_1^h and u_2^h of u_1 and u_2, obtained by means of an infinitely differentiable averaging kernel $\omega(h, \overline{PP_1})$, depending only on the distance between the points P and P_1 and on the averaging radius.* In this connection we shall assume that for $t < 0$ and $t > T$ the function $u_1 = u_2 = 0$. As a particular averaging function $\omega(h, \overline{PP_1})$ we may take, for instance, the averaging kernel constructed in [3] (p. 18). Since u_1 and u_2 do not exceed in absolute value a certain number M, so the functions

$$u_1^h(P) = \iint \omega(h, \overline{PP_1})\, u_1(P_1)\, dP_1 \text{ and } u_2^h(P) = \iint \omega(h, \overline{PP_1})\, u_2(P_1)\, dP_1$$

are bounded in absolute value by the same constant M. It is known [3] that the averaged functions u_1^h, u_2^h are infinitely differentiable and converge in the mean for $h \to 0$ to the functions u_1, u_2 in any finite part of the region G. Since for u_1 and u_2 the condition 2) is fulfilled, so for sufficiently small h in any finite region D lying in the half-plane $t \geq \alpha > 0$

$$\frac{\partial u_1^h}{\partial x} < K \quad \text{and} \quad \frac{\partial u_2^h}{\partial x} < K$$

where the constant K depends only on α and R (R is the greatest distance of points of the region D from the origin), but does not depend on h. Indeed, from

*An averaging kernel is a smooth function $\omega(h, r)$, defined for $r \geq 0$, $h \geq 0$ and satisfying the conditions: $\omega(h, r) \geq 0$, $\omega(h, r) = 0$ for $r \geq h$, $\iint \omega(h, \overline{PP_1})\, dP_1 = 1$.

condition 2) it follows that there exists a constant K, depending only on α and R, such that the functions $u_1 - Kx$ and $u_2 - Kx$ are monotone in x. By virtue of the properties of the averaging kernel, the functions

$$(u_1 - Kx)^h \text{ and } (u_2 - Kx)^h$$

also are monotone functions, i.e., $\dfrac{\partial (u_i - Kx)^h}{\partial x} \leq 0$ $(i = 1, 2)$, and, consequently, $\dfrac{\partial u_1^h}{\partial x} \leq K$ and $\dfrac{\partial u_2^h}{\partial x} \leq K$, since $\dfrac{\partial (Kx)^h}{\partial x} = K$.

We shall show that

$$\iint_G F(t, x) [u_1 - u_2] \, dx \, dt = 0 \tag{2.5}$$

for any continuously differentiable function $F(t, x)$, equal to zero outside of a certain bounded region D_1, lying in the half-plane $t \geq \delta_1 > 0$, where δ_1 is an arbitrary small number. Evidently, from equality (2.5) it follows that $u_1 = u_2$ almost everywhere in G.

Let $M = \sup\{u_1, u_2\}$, $A = \max |\phi_u(t, x, u)|$ for $(t, x) \subset G$ and $|u| \leq M$. Let us consider in G a function $f(t, x)$, vanishing for $t = T$ and satisfying the equation

$$\frac{\partial f}{\partial t} + \Phi_h \frac{\partial f}{\partial x} - H_\rho f = F(t, x), \tag{2.6}$$

where

$$\Phi_h(t, x) = \frac{\varphi(t, x, u_1^h) - \varphi(t, x, u_2^h)}{u_1^h - u_2^h} = \int_0^1 \varphi_u(t, x, u_1^h + \tau(u_2^h - u_1^h)) \, d\tau.$$

and the functions H_ρ satisfy the following conditions: H_ρ are uniformly bounded for all x, t, $0 < \rho \leq 1$ and converge in the mean to the function

$$\Psi = \frac{\psi(t, x, u_1) - \psi(t, x, u_2)}{u_1 - u_2} = \int_0^1 \psi_u(t, x, u_1 + \tau(u_2 - u_1)) \, d\tau$$

for ρ tending to zero in any finite part of the region G; the function $H_\rho = 0$ for $t \leq \rho$ and has continuous derivatives of the second order. It is easy to construct such a sequence H_ρ, for example, by averaging Ψ and multiplying Ψ by a properly chosen bounded function vanishing for $t \leq \rho$.

The equation for the characteristics of equation (2.6) has the form

$$\frac{dx}{dt} = \Phi_h(t, x). \tag{2.7}$$

Let $x(t) = x_h(t, t_1, x_1)$ be a characteristic, passing through the point (t_1, x_1). The function $f(t, x)$ satisfying equation (2.6) and equal to zero for $t = T$, is representable in the point (t_1, x_1) of G in the form

$$f(t_1, x_1) = \int_T^{t_1} F(s, x_h(s, t_1, x_1)) \exp\left[\int_{t_1}^{s} H_\rho(\tau, x_h(\tau, t_1, x_1))\, d\tau\right] ds. \qquad (2.8)$$

Since $|\Phi_h(t, x)| \leq A$, so the function $f(t, x)$ is equal to zero for $|x| > R_1 + AT$, where R_1 is the maximal distance of points of D_1 from the origin. From (2.8) it follows also that the functions $f(t, x)$ are uniformly bounded in G for all h and ρ and have continuous derivatives of the first order.

We shall show that for a fixed ρ and $t \geq \alpha$ the derivatives $\frac{\partial f}{\partial x}$ are uniformly bounded in G for all sufficiently small h. To this end we observe that $\left|\frac{\partial x_h}{\partial x_1}\right| < C$ for $t \geq \alpha > 0$, $|x_h| \leq R_1 + AT$ and sufficiently small h, where C does not depend on h. Indeed, let $\frac{\partial x_h}{\partial x_1} = z_h$. Differentiating (2.7) with respect to x_1, we obtain:

$$\frac{\partial z_h}{\partial t} = \frac{\partial \Phi_h}{\partial x} z_h \quad \text{and} \quad z_h = e^{\int_{t_1}^{t} \frac{\partial \Phi_h}{\partial x} dt}, \qquad (2.9)$$

where

$$\frac{\partial \Phi_h}{\partial x} = \int_0^1 \left\{ \varphi_{ux} + \varphi_{uu}(t, x, u_1^h + \tau(u_2^h - u_1^h)) \left[\frac{\partial u_1^h}{\partial x}(1-\tau) + \frac{\partial u_2^h}{\partial x}\tau\right]\right\} d\tau.$$

Since ϕ_{ux} are uniformly bounded for $|u| < M$ and $|x| \leq R_1 + AT$, $\phi_{uu} \geq 0$, $\frac{\partial u_1^h}{\partial x} < K_1$, $\frac{\partial u_2^h}{\partial x} < K_1$, if $|x| \leq R_1 + AT$, $t \geq \alpha$, so $\frac{\partial \Phi_h}{\partial x} < K_2$, where K_2 depends on α but does not depend on h. Therefore

$$|z_h| = \left|\frac{\partial x_h}{\partial x_1}\right| \leq e^{K_2(T-\alpha)} \qquad (2.10)$$

for $t \geq \alpha$, $|x| \leq R_1 + AT$. From (2.8) we have:

$$\frac{\partial f}{\partial x_1} = \int_T^{t_1} \frac{\partial F}{\partial x} \frac{\partial x_h}{\partial x_1} \exp\left[\int_{t_1}^{s} H_\rho(\tau, x_h)\, d\tau\right] ds + $$
$$+ \int_T^{t_1} \left\{F \cdot \exp\left[\int_{t_1}^{s} H_\rho(\tau, x_h)\, d\tau\right] \cdot \int_{t_1}^{s} \frac{\partial H_\rho}{\partial x} \frac{\partial x_h}{\partial x_1} d\tau\right\} ds. \qquad (2.11)$$

Hence, using inequality (2.10), we find that $\partial f/\partial x$ are uniformly bounded in absolute value with respect to h for $t \geq \alpha$ and a fixed ρ.

We shall show that for a fixed ρ the variation of the function $f(t, x)$ as a function of x is uniformly bounded with respect to h and t. Let $r = \min\{\delta_1, \rho\}$.

For $t < r$ equation (2.6) has the form

$$\frac{\partial f}{\partial t} + \Phi_h \frac{\partial f}{\partial x} = 0 \quad \text{or} \quad \frac{df}{dt} = 0,$$

where df/dt means the derivative of f in the direction of the characteristic. Hence, it follows that the function $f(t, x)$ is constant on characteristics (2.7) for $t \leq r$. For $t \geq r$ the derivatives $\partial f/\partial x$ are uniformly bounded for all sufficiently small h and therefore for $t \geq r$

$$\operatorname*{Var}_x f(t, x) = \int_{-\infty}^{+\infty} \left| \frac{\partial f}{\partial x} \right| dx \leq V_\rho,$$

where V_ρ is a certain constant. Consequently, for all $t \leq T$ and sufficiently small h

$$\operatorname*{Var}_x f(t, x) \leq V_\rho.$$

Using relations (2.3) for the functions u_1 and u_2, we obtain:

$$\iint_G F(u_1 - u_2)\, dx\, dt = \iint_G (u_1 - u_2)\left[\frac{\partial f}{\partial t} + \Phi_h \frac{\partial f}{\partial x} - H_\rho f \right] dx\, dt =$$
$$= \iint_G \left[(\Phi_h - \Phi) \frac{\partial f}{\partial x} + (\Psi - H_\rho) f \right] dx\, dt, \qquad (2.12)$$

where $\Phi = \dfrac{\phi(t, x, u_1) - \phi(t, x, u_2)}{u_1 - u_2}$. We shall show that the right side of (2.12) is arbitrarily small for sufficiently small ρ and h, and since the left side of this equality does not depend on ρ and h, so (2.5) is valid.

Let $\epsilon > 0$ be an arbitrary given number. We choose ρ so small so that

$$\left| \iint_G (\Psi - H_\rho)\, f\, dx\, dt \right|$$

is less than $\epsilon/3$ for all h. This is possible since f is uniformly bounded with respect to h and ρ, $f = 0$ for $|x| > R_1 + AT$ and H_ρ converge in the mean to Ψ for $\rho \to 0$ in any finite part of G. Fixing ρ in the indicated way, we choose $\alpha > 0$ so small that

$$\left| \iint_{0 \leq t \leq \alpha} (\Phi - \Phi_h) \frac{\partial f}{\partial x}\, dx\, dt \right| < \frac{\epsilon}{3} \qquad (2.13)$$

for all h. This is possible, since the left side of (2.13) does not exceed

$$\alpha [\max_G |\Phi| + \max_G |\Phi_h|] \cdot \int_{-\infty}^{+\infty} \left| \frac{\partial f}{\partial x} \right| dx \leq \alpha M_1 V_\rho$$

and, consequently, is arbitrarily small for small α. For α and ρ chosen in this way

$$\left| \iint_{T \geq t \geq \alpha} (\Phi - \Phi_h) \frac{\partial f}{\partial x}\, dx\, dt \right| \leq \frac{\epsilon}{3},$$

if h is sufficiently small since, for $t \geq \alpha$, $|\partial f/\partial x|$ are uniformly bounded with respect to h for a fixed ρ and equal to zero for $|x| > R_1 + AT$, and for, $h \to 0$, Φ_h converge uniformly in the mean to Φ in any finite part of G.

Consequently,

$$\left|\iint_G F(u_1 - u_2)\,dx\,dt\right| < \epsilon,$$

and since $\epsilon > 0$ is an arbitrary number, so (2.5) is valid and $u_1 = u_2$ almost everywhere in G. The theorem is proved.

From the proof of Theorem 1 we get the following stronger assertion.

Let u_1 and u_2 be generalized solutions of the Cauchy problem of equation (2.1) *with conditions*

$$u_1(0, x) = u_0^1(x) \quad \text{and} \quad u_2(0, x) = u_0^2(x)$$

and let $|u_1(t, x)| \leq M$, $|u_2(t, x)| \leq M$ in G and $u_0^1(x) = u_0^2(x)$ for $a \leq x \leq b$. Then $u_1(t, x) = u_2(t, x)$ almost everywhere in the region bounded by the straight lines $t = 0$, $x - a = At$, $x - b = -At$, where $A = \max|\phi_u(t, x, u)|$ for $|u| \leq M$ and $(t, x) \subset G$.

A generalized solution of equation (2.1) *assuming on the segment $[a, b]$ of the straight line $t = 0$ the values of a given bounded measurable function $u_0(x)$,* we shall call a bounded measurable function $u(t, x)$, defined in a region $D(t > 0)$, whose boundary contains the segment $[a, b]$, if 1) for $u(t, x)$ the equality (2.3) is fulfilled for any continuously differentiable function $f(t, x)$ equal to zero outside a certain finite region whose boundary lies inside D for $t > 0$, 2) for the points (t, x_1) and (t, x_2) in D the relation (2.4) is fulfilled.

In a way similar to the proof of Theorem 1, one can prove the following theorem.

Theorem 2. *Two generalized solutions $u_1(t, x)$, $u_2(t, x)$ of equation* (2.1), *defined in the regions D_1 and D_2 respectively, and assuming on the segment $[a, b]$ of the straight line $t = 0$ the same values $u_0(x)$, coincide in any region $D_{\tau\alpha}$ bounded by the straight lines*

$$t = 0, \; x - a - \alpha = At, \; x - b + \alpha = -At, \; t = \tau,$$

where $A = \max|\phi_u(t, x, u)|$ for $|u| \leq M$ and $(t, x) \subset D_1 + D_2$, $M = \sup_{D_1 + D_2}\{|u_1|, |u_2|\}$, $\alpha \geq 0$, $\tau > 0$, provided the region $D_{\tau\alpha}$ is contained in the intersection of the regions D_1 and D_2.

Remark. Theorem 1 on the uniqueness of a generalized solution of equation (2.1) with condition (2.2) can be proved under the assumption that the function $K(t, x_1, x_2)$ in relation (2.4) is infinite for a certain set of values t whose closure has measure zero. However, in §4 generalized solutions of the Cauchy problem (2.1), (2.2) will be constructed with the function $K(t, x_1, x_2)$ continuous for $t > 0$.

This shows that a weaker assumption concerning the function $K(t, x_1, x_2)$ does not lead to an extension of the class of generalized solutions of the Cauchy problem and that for $t > 0$ among generalized solutions of equation (2.1) the "rarefaction waves" cannot occur (see §1).

§3. Finite differences schemes.*

The existence of a generalized solution of the Cauchy problem of equation (2.1) with condition (2.2) will be proved by means of a finite differences method. Moreover, using finite differences schemes, we shall indicate a way for an approximate computation of generalized solutions and establish a series of properties of these solutions. To this end, in the present section we shall study a finite differences scheme proposed by Lax [7].

Let the half-plane $t \geq 0$ be covered by a net by means of the straight lines

$$t = kh, \quad x = nl,$$

where h and l are certain positive numbers, n runs through all the integral values, and k assumes the values of all non-negative integers. According to the scheme of Lax, equation (2.1) is replaced in the half-plane $t > 0$ by the following system of finite differences equations

$$\frac{u_n^{k+1} - \frac{u_{n-1}^k + u_{n+1}^k}{2}}{h} + \frac{\varphi(kh, (n+1)l, u_{n+1}^k) - \varphi(kh, (n-1)l, u_{n-1}^k)}{2l} + \\ + \psi(kh, (n+1)l, u_{n+1}^k) = 0, \tag{3.1}$$

where for abbreviation the notation

$$u_i^j = u(jh, il), \quad k = 1, 2, \cdots; \quad n = 0, 1, 2, \cdots, -1, -2, \cdots$$

is introduced.

These schemes belong to the so-called solved schemes, since the value of u_n^{k+1} at an intersection point of the net $((k+1)h, nl)$ is computed directly from the values of this function at the intersection points lying on the straight line $t = kh$.

We shall assume that for all u and (t, x) of G there exist continuous partial derivatives $\phi_x, \phi_u, \phi_{xx}, \phi_{xu}, \phi_{uu}, \psi_x, \psi_u$, which are bounded for bounded u and (t, x) in G, and that $\phi_{uu} \geq 0$ for all u and (t, x) of G; in this connection $\phi_{uu} \geq \mu > 0$ for bounded u and $0 \leq t \leq \tau$, where τ and μ are certain positive numbers.

Let us add to the finite differences equations (3.1) the initial condition

*The bibliography, devoted to the application of the finite differences method to the study of partial differential equations, is given, for instance, in [22].

$$u_n^0 = u_0(nl) \tag{3.2}$$

and we shall assume that $|u_0(nl)| \leq m$.

In all that follows we shall assume that there exists a constant M and a continuously differentiable function $V(v)$ such that

$$\max_{\substack{(t,x)\subset G,\\ |u|\leqslant v}} |\varphi_x + \psi| \leqslant V(v), \quad V'(v) \geqslant 0 \text{ and } \int_m^M \frac{dv}{V(v)+\alpha} \geqslant T$$

for a certain $\alpha > 0$. Let us denote by Ω the region $\{|u| \leq M,\ (t, x) \subset G\}$ in the space (t, x, u) and let $A = \max_\Omega |\phi_u|$.

Lemma 1. *If* $|u_n^0| \leq m$, *h and l are such that* $Ah/l < 1$, *then for sufficiently small l the solutions u_n^k of the finite differences equations* (3.1) *satisfying condition* (3.2) *do not exceed M in absolute value for all n and* $k \leq T/h$.

Proof. The finite differences equations (3.1) can be written in the form

$$\begin{aligned} u_n^{k+1} = u_{n-1}^k \left[\frac{1}{2} + \varphi_u(kh, (n+1)\,l, \theta_n^k)\,\frac{h}{2l}\right] + \\ + u_{n+1}^k \left[\frac{1}{2} - \varphi_u(kh, (n+1)\,l, \theta_n^k)\,\frac{h}{2l}\right] - \varphi_x(kh, (n+1)\,l, u_{n+1}^k)\,h - \\ - \psi(kh, (n+1)\,l, u_{n+1}^k)\,h - \varphi_{xx}(kh, \gamma_n^k, u_{n+1}^k)\,hl, \end{aligned} \tag{3.3}$$

where θ_n^k is a certain intermediate value between u_{n+1}^k and u_{n-1}^k, and γ_n^k is an intermediate value between $(n-1)\,l$ and $(n+1)\,l$.

Let $M^k = \max_n |u_n^k|$ and $M^k \leq M$. Then by virtue of the condition $Ah/l \leq 1$ the coefficients of u_{n-1}^k and u_{n+1}^k in formula (3.3) are non-negative and their sum is equal to one. Therefore, from (3.3) we have:

$$M^{k+1} \leq M^k + hV(M^k) + \max_\Omega |\phi_{xx}| \cdot hl.$$

Let l be so small that $\max_\Omega |\phi_{xx}|\, l < \alpha$. Then

$$\frac{M^{k+1} - M^k}{h} \leq V(M^k) + \alpha. \tag{3.4}$$

Let us consider a solution of the differential equation

$$\frac{dz}{dt} = V(z) + \alpha, \tag{3.5}$$

satisfying the condition $z(0) = m$. Since $d^2z/dt^2 = V'(z)(V(z) + \alpha) \geq 0$ and all solutions of equation (3.5) have their convex side turned downward, so from (3.4) it follows that $M^k \leq z(kh)$ for $k \leq T/h$. The function $z(t)$ is defined by the equation

$$\int_m^{z(t)} \frac{dz}{V(z)+\alpha} = t$$

and by virtue of our hypotheses $z(t) \le M$ for $t \le T$. Consequently, $M^k \le M$ for $k \le T/h$, which was to be proved.

In all that follows we shall assume that $Ah/l < 1$ and u_n^k are solutions of the finite differences equations (3.1) with condition (3.2), where $|u_0(nl)| \le m$.

Lemma 2. *There exists a constant E such that for all n and for $k \le T/h$*

$$\frac{u_n^k - u_{n-2}^k}{2l} < \frac{E}{kh}, \tag{3.6}$$

where E does not depend on h and l.

Proof. Let us denote $z_n^k = \dfrac{u_n^k - u_{n-2}^k}{2l}$. From equations (3.1) we get the following equation for z_n^k:

$$z_n^{k+1} = \frac{z_{n-1}^k + z_{n+1}^k}{2} + \frac{2\varphi\left(kh,(n-1)l,u_{n-1}^k\right)h}{4l^2} - \\ - \frac{\varphi\left(kh,(n-3)l,u_{n-3}^k\right) + \varphi\left(kh,(n+1)l,u_{n+1}^k\right)}{4l^2}\cdot h - \\ - \frac{\psi\left(kh,(n+1)l,u_{n+1}^k\right) - \psi\left(kh,(n-1)l,u_{n-1}^k\right)}{2l}\,h.$$

Using Taylor's formula, we get

$$z_n^{k+1} = z_{n-1}^k\left[\frac{1}{2} + \varphi_u\left(kh,(n-1)l,u_{n-1}^k\right)\frac{h}{2l} - \varphi_{xu}\left(kh,(n-1)l,\bar{\theta}_n^k\right)h\right] + \\ + z_{n+1}^k\left[\frac{1}{2} - \varphi_u\left(kh,(n-1)l,u_{n-1}^k\right)\frac{h}{2l} - \varphi_{xu}\left(kh,(n-1)l,\tilde{\theta}_n^k\right)h - \\ - \psi_u\left(kh,(n-1)l,\hat{\theta}_n^k\right)h\right] - \left[\varphi_{xx}\left(kh,\gamma_1,u_{n+1}^k\right) + \varphi_{xx}\left(kh,\gamma_2,u_{n-3}^k\right) + \\ + 2\psi_x\left(kh,\gamma_3,u_{n+1}^k\right)\right]\frac{h}{2} - \left(z_{n+1}^k\right)^2\cdot\varphi_{uu}\left(kh,(n-1)l,\theta_1\right)\frac{h}{2} - \\ - \left(z_{n-1}^k\right)^2\varphi_{uu}\left(kh,(n-1)l,\theta_2\right)\frac{h}{2}. \tag{3.7}$$

First let us carry out the estimate of z_n^k for values $k \le \tau/h$, where $\tau > 0$ is defined so that for $|u| \le M$ and $0 \le t \le \tau$, $\phi_{uu} \ge \mu > 0$. Let us introduce the notation

$$a = \max_{\Omega} 2|\varphi_{xu}| + \max_{\Omega} |\psi_u|,$$
$$b = \max_{\Omega} |\varphi_{xx}| + \max_{\Omega} |\psi_x|,$$
$$\widetilde{a} = a + b,$$
$$c = \min\left(\frac{\mu}{2}, A\frac{(1+\widetilde{a}h)}{4M}\right).$$

Let $\widetilde{z}_n^k = \max\{z_{n-1}^k, z_{n+1}^k, 0\}$. We shall assume h to be so small that the coefficients for z_{n-1}^k and z_{n+1}^k in the right side of (3.7) are positive. From (3.7) it follows that

$$z_n^{k+1} \leqslant \widetilde{z}_n^k(1 + ah) + bh - c(\widetilde{z}_n^k)^2 h$$

and

$$z_n^{k+1} \leqslant \widetilde{z}_n^k(1 + \widetilde{a}h) + bh - c(\widetilde{z}_n^k)^2 h. \tag{3.8}$$

Since $|u_n^k| \leq M$, so $z_n^k \leq M/l$. From the conditions $Ah/l < 1$ and $c < A(1 + \widetilde{a}h)/4M$ it follows that $z_n^k < M/Ah$ and

$$z_n^k < \frac{1 + \widetilde{a}h}{4hc}. \tag{3.9}$$

Let us consider the polynomial $H(y) = y(1 + \widetilde{a}h) + bh - chy^2$. Evidently $H'(y) = 1 + \widetilde{a}h - 2ych \geq 0$ for $y < (1 + \widetilde{a}h)/2ch$. Let $M^k = \max[\max_n\{z_n^k\}, 0]$. From estimate (3.9) for z_n^k and (3.8) it follows that

$$z_n^{k+1} \leq M^k(1 + \widetilde{a}h) + bh - c(M^k)^2 h$$

and that the right part in the last inequality is positive. Therefore

$$M^{k+1} \leq M^k(1 + \widetilde{a}h) + bh - c(M^k)^2 h. \tag{3.10}$$

For the quantities $V^k = M^k + 1$ from (3.10) we obtain the relation

$$V^{k+1} \leqslant V^k(1 + \widetilde{a}h + 2ch) + bh - ah - c(V^k)^2 h \leqslant V^k(1 + \widetilde{c}h) - ch(V^k)^2.$$

Since for sufficiently small h the inequality $0 < 1 - \widetilde{c}h < 1$ holds, so

$$V^{k+1} \leq V^k + \frac{\widetilde{c}h}{1 - \widetilde{c}h} V^k - ch(V^k)^2. \tag{3.11}$$

Let $W^k = (1 - \widetilde{c}h)^k V^k$, multiplying inequality (3.11) by $(1 - \widetilde{c}h)^{k+1}$, we get:

$$W^{k+1} \leq W^k - ch(W^k)^2 \cdot (1 - \widetilde{c}h)^{-k+1}.$$

Since $(1 - \widetilde{c}h)^{-k+1} \geq 1$ for $k > 0$, so

$$W^{k+1} \leq W^k - ch(W^k)^2. \tag{3.12}$$

From inequality (3.12) it is easy to get an estimate for W^k, and, consequently, also for M^k. To this end we consider the differential equation

$$\frac{dw}{dt} = -cw^2. \tag{3.13}$$

All positive solutions of this equation have their convex side turned downward since $d^2w/dt^2 = 2cw^3$. The solution $w(t)$ of equation (3.13) satisfying the condition $w(0) = W^0$, has the form

$$w(t) = \frac{1}{ct + \frac{1}{W^0}}.$$

It is evident that $W^k \leq w(kh) \leq 1/ckh$. Hence

$$(1 - \tilde{c}h)^k (M^k + 1) \leq \frac{1}{ckh} \text{ and } M^k \leq \frac{E}{kh},$$

where the constant E does not depend on h and k.

Thus we proved the lemma for $kh \leq r$. For $kh > r$ from (3.7) it follows that

$$z_n^{k+1} \leq \tilde{z}_n^k (1 + ah) + bh, \quad M^{k+1} \leq M^k (1 + ah) + bh$$

and $M^k \leq B_1 + B_2 M^{\left[\frac{r}{h}\right]}$, where B_1, B_2 do not depend on h and k. Consequently, estimate (3.6) is valid for all $k \leq T/h$ and n.

In what follows we shall consider u_n^k only in such points (kh, nl) for which the number $(k - n)$ is even.

These intersection points we shall denote by S_1. It is easy to see that the values u_n^k are defined uniquely in all points (kh, nl) of S_1 by the initial values u_n^0 for even n since by virtue of obvious properties of the finite differences scheme (3.1), the values u_n^k in the points (kh, nl) for $(k - n)$ even and $(k - n)$ odd are computed independently.

Lemma 3. *For $k > \alpha/h$ and any $X > 0$*

$$\Sigma \left| u_{n+2}^k - u_n^k \right| < C, \tag{3.14}$$

where the summation is taken over all indices n, satisfying the conditions: $|n| \leq X/l$ and $(n - k)$ is even. In this connection the constant C depends on α and X and does not depend on h and l.

Proof. Consider the function $v_n^k = u_n^k - C_1 nl$ and let us choose a constant C_1 so large that $v_{n+2}^k - v_n^k \leq 0$ for $kh > \alpha$. This is possible since by virtue of Lemma 2

$$v_{n+2}^k - v_n^k = u_{n+2}^k - u_n^k - C_1 \cdot 2l \leq \frac{E}{kh} \cdot 2l - C_1 \cdot 2l.$$

Evidently

$$\sum \left| v_{n+2}^k - v_n^k \right| = -\sum \left(v_{n+2}^k - v_n^k \right) \leqslant 2 \max_{|n| \leqslant \frac{X}{l}} \left| v_n^k \right| \leqslant 2M + 2C_1 X, \tag{3.15}$$

if the summation is taken over n for $|n| \leq X/l$ and $(n-k)$ even. Let us consider the left side of (3.14). Using estimate (3.15), we obtain:

$$\sum |u_{n+2}^k - u_n^k| \leqslant \sum |u_{n+2}^k - C_1(n+2)l - (u_n^k - C_1 nl)| + \sum C_1 \cdot 2l \leqslant$$
$$\leqslant \sum |v_{n+2}^k - v_n^k| + 2C_1 X \leqslant 2M + 4C_1 X < C,$$

which was to be shown.

Lemma 4. *If* $l^2/h <$ const, *then for an even* $k-p$

$$\Sigma \left| u_n^k - u_n^p \right| \cdot 2l \leq L[(k-p)h]^\beta, \tag{3.16}$$

and for an odd $k-p$

$$\Sigma \left| u_n^k - u_{n+1}^p \right| \cdot 2l \leq L[(k-p)h]^\beta, \tag{3.17}$$

where the summation is taken over n *corresponding to the points of* S_1 *and such that* $|n| \leq X/l$; kh *and* ph *are greater* $\alpha > 0$, *the constant* L *depends on* α *and* X, *but does not depend on* h *and* l; $\beta = 1/3$. *If* $h/l > \delta > 0$, *then* $\beta = 1$.

Proof. First, we consider the case when $h/l > \delta > 0$. This condition means that u_n^k is determined in some intersection point by u_n^0 on a finite segment of the straight line $t = 0$ whose length is bounded for all h and l, i.e., for each point there exists a finite region of dependence. Let $k > p$ and let $k - p$ be even. The case when $(k-p)$ is odd is treated similarly. Let us express the values u_n^k in terms of u_n^p. From formula (3.3) we have:

$$u_n^k = a_{n,\,n-1}^{k,\,k-1} u_{n-1}^{k-1} + a_{n,\,n+1}^{k,\,k-1} u_{n+1}^{k-1} + \eta_{k-1}, \tag{3.18}$$

where $a_{n,n-1}^{k,k-1} + a_{n,n+1}^{k,k-1} = 1$, $|\eta_{k-1}| \leq \eta h$, η is a certain constant, independent of k and n. Applying successively formula (3.18) for the corresponding indices n and k, we obtain:

$$u_n^k = \sum_{j=n-(k-p)}^{n+k-p} a_{n,\,j}^{k,\,p} u_j^p + \eta_p, \tag{3.19}$$

where

$$\left.\begin{aligned} &\sum_j a_{n,\,j}^{k,\,p} = 1, \quad a_{n,\,j}^{k,\,p} \geqslant 0, \quad |\eta_p| \leqslant \eta(k-p)h, \\ &a_{n,\,j}^{k,\,p} = a_{n,\,j+1}^{k,\,p+1} \left[\frac{1}{2} + \varphi_u\left(ph,\,(j+2)l,\,\theta_{j+1}^p\right)\frac{h}{2l}\right] + \\ &\qquad + a_{n,\,j-1}^{k,\,p+1}\left[\frac{1}{2} - \varphi_u\left(ph,\,jl,\,\theta_{j-1}^p\right)\frac{h}{2l}\right]. \end{aligned}\right\} \tag{3.20}$$

Using formula (3.19) and relations (3.20) we get:

$$\sum_{|n|\leq \frac{X}{l}} \left| u_n^k - u_n^p \right| \cdot 2l \leqslant \sum_{|n|\leq \frac{X}{l}} \left| \sum_{j=n-(k-p)}^{n+(k-p)} a_{n,j}^{k,p} \left(u_j^p - u_n^p\right) \right| 2l + 2X\eta\,(k-p)\,h \leqslant$$

$$\leqslant \sum_{|n|\leq \frac{X}{l}} \sum_{j=n-(k-p)}^{n+(k-p)} a_{n,j}^{k,p} \left| u_j^p - u_n^p \right| 2l + 2X\eta\,(k-p)\,h \leqslant$$

$$\leqslant \sum_{|n|\leq \frac{X}{l}} \sum_{j=n-(k-p)}^{n+(k-p)} a_{n,j}^{k,p} \sum_{r=j}^{n} \left| u_r^p - u_{r-2}^p \right| 2l + 2X\eta\,(k-p)\,h \leqslant$$

$$\leqslant \sum_{|n|\leq \frac{X}{l}} \sum_{j=n-(k-p)}^{n+k-p} a_{n,j}^{k,p} \sum_{r=n-(k-p)}^{n+(k-p)} \left| u_r^p - u_{r-2}^p \right| 2l + 2X\eta\,(k-p)\,h.$$

Since $\sum_j a_{n,j}^{k,p} = 1$, so the right side of the last inequality does not exceed

$$\sum_{|n|\leq \frac{X}{l}} \sum_{r=n-(k-p)}^{n+(k-p)} \left| u_r^p - u_{r-2}^p \right| 2l + 2X\eta\,(k-p)\,h \leqslant$$

$$\leqslant 2l\,(k-p) \sum_{|r|\leq \frac{X}{l}+k-p} \left| u_r^p - u_{r-2}^p \right| + 2X\eta\,(k-p).$$

According to Lemma 3 and the hypothesis $h/l > \delta > 0$, the last expression does not exceed

$$2\,(k-p)\,lC_2 + 2\eta\,(k-p)\,hX \leq L\,(k-p)\,h.$$

Thus, the lemma is proved under the assumption $h/l > \delta > 0$.

In order to prove Lemma 4 in the general case, i.e., without the hypothesis on the finiteness of the dependence region, let us study the behavior of the coefficients $a_{n,j}^{k,p}$. A similar study for linear equations was carried out in the work [23]. Let us compare the coefficients $a_{n,j}^{k,p}$ in formula (3.19) with the coefficients $\alpha_j^{k,p}$ and $A_j^{k,p}$ which are defined by the following relations:

$$\left.\begin{aligned} \alpha_j^{k,p} &= \alpha_{j+1}^{k,p+1}\,\bar{q} + \alpha_{j-1}^{k,p+1}\,(1-\bar{q}), \\ A_j^{k,p} &= A_{j+1}^{k,p+1}\,\tilde{q} + A_{j-1}^{k,p+1}\,(1-\tilde{q}). \end{aligned}\right\} \tag{3.21}$$

In this connection $A_n^{k,k} = \alpha_n^{k,k} = 1$, $\alpha_r^{k,p} = A_r^{k,p} = 0$ for $|n-r| > k-p$,

$$\bar{q} = \min_{\Omega} \left[\frac{1}{2} + \varphi_u(t,\,x,\,u)\,\frac{h}{2l} \right],\quad \tilde{q} = \max_{\Omega} \left[\frac{1}{2} + \varphi_u(t,\,x,\,u)\,\frac{h}{2l} \right].$$

We shall show that for any m

$$\overline{P}_m^{k,\,p}=\sum_{j=n-k+p}^{m} a_j^{k,\,p}\leqslant P_{n,\,m}^{k,\,p}=\sum_{j=n-k+p}^{m} a_{n,\,j}^{k,\,p}\leqslant\sum_{j=n-k+p}^{m} A_j^{k,\,p}=\widetilde{P}_m^{k,\,p}. \quad (3.22)$$

Here the summation extends over the indices j for which $(j-p)$ is even. Using relations (3.20), we get:

$$P_{n,\,m}^{k,\,p}=\sum_{s=n-k+p}^{m}\left\{a_{n,\,s+1}^{k,\,p+1}\left[\frac{1}{2}+\varphi_u\left(ph,\ (s+2)\,l,\ \theta_{s+1}^p\right)\frac{h}{2l}\right]+\right.$$

$$\left.+a_{n,\,s-1}^{k,\,p+1}\left[\frac{1}{2}-\varphi_u\left(ph,\ sl,\ \theta_{s-1}^p\right)\frac{h}{2l}\right]\right\}=$$

$$=\sum_{s=n-k+(p+1)}^{m-1}\left\{a_{n,\,s}^{k,\,p+1}+a_{n,\,m+1}^{k,\,p+1}\left[\frac{1}{2}+\varphi_u\left(ph,\ (m+2)\,l,\ \theta_{m+1}^p\right)\frac{h}{2l}\right]\right\}=$$

$$=P_{n,\,m-1}^{k,\,p+1}+\left(P_{n,\,m+1}^{k,\,p+1}-P_{n,\,m-1}^{k,\,p+1}\right)\left[\frac{1}{2}+\varphi_u\left(ph,\ (m+2)\,l,\ \theta_{m+1}^p\right)\frac{h}{2l}\right]=$$

$$=P_{n,\,m+1}^{k,\,p+1}\left[\frac{1}{2}+\varphi_u\left(ph,\ (m+2)\,l,\ \theta_{m+1}^p\right)\frac{h}{2l}\right]+$$

$$+P_{n,\,m-1}^{k,\,p+1}\left[\frac{1}{2}-\varphi_u\left(ph,\ (m+2)\,l,\ \theta_{m+1}^p\right)\frac{h}{2l}\right]. \quad (3.23)$$

The inequalities

$$\left.\begin{aligned}\overline{P}_m^{k,\,p}&=\overline{P}_{m+1}^{k,\,p+1}\,\overline{q}+\overline{P}_{m-1}^{k,\,p+1}\,(1-q),\\ \widetilde{P}_m^{k,\,p}&=\widetilde{P}_{m+1}^{k,\,p+1}\,\widetilde{q}+\widetilde{P}_{m-1}^{k,\,p+1}\,(1-\widetilde{q}).\end{aligned}\right\} \quad (3.24)$$

are proved similarly. We prove relations (3.22) by induction on $k-p$. For $k-p=0$ relations (3.22) are clearly fulfilled. We shall show that if (3.22) is fulfilled for $k-(p+1)$, then it is also fulfilled for $k-p$. Indeed, from equalities (3.23) and (3.24) it follows that

$$\overline{P}_m^{k,\,p}=\overline{P}_{m+1}^{k,\,p+1}\,\overline{q}+\overline{P}_{m-1}^{k,\,p+1}(1-q)\leqslant$$

$$\leqslant P_{n,\,m+1}^{k,\,p+1}\,\overline{q}+P_{n,\,m-1}^{k,\,p+1}\,(1-\overline{q})=P_{n,\,m-1}^{k,\,p+1}+\overline{q}\left(P_{n,\,m+1}^{k,\,p+1}-P_{n,\,m-1}^{k,\,p+1}\right)\leqslant$$

$$\leqslant P_{n,\,m-1}^{k,\,p+1}+\left[\frac{1}{2}+\varphi_u\left(ph,\ (m+2)\,l,\ \theta_{m+1}^p\right)\frac{h}{2l}\right]\left(P_{n,\,m+1}^{k,\,p+1}-P_{n,\,m-1}^{k,\,p+1}\right)=P_{n,\,m}^{k,\,p}.$$

In exactly the same way we get

$$P_{n,\,m}^{k,\,p}=P_{n,\,m+1}^{k,\,p+1}\left[\frac{1}{2}+\varphi_u\left(ph,\ (m+2)\,l,\ \theta_{m+1}^p\right)\frac{h}{2l}\right]+$$

$$+P_{n,\,m-1}^{k,\,p+1}\left[\frac{1}{2}-\varphi_u\left(ph,\ (m+2)\,l,\ \theta_{m+1}^p\right)\frac{h}{2l}\right]\leqslant$$

$$\leqslant\widetilde{P}_{m-1}^{k,\,p+1}+\widetilde{q}\left(\widetilde{P}_{m+1}^{k,\,p+1}-\widetilde{P}_{m-1}^{k,\,p+1}\right)=\widetilde{P}_m^{k,\,p}.$$

Hence it follows that for $0\leq p\leq k$

$$\overline{P}_m^{k,\,p}\leqslant P_{n,\,m}^{k,\,p}\leqslant\widetilde{P}_m^{k,\,p}.$$

We estimate the sums $\overline{P}_m^{k,p}$ and $\widetilde{P}_m^{k,p}$ using the Chebyshev inequality [24].

Let us consider a motion of a particle on a straight line such that the particle is at the origin of the coordinates at the initial moment, and it moves to the left in a unit of time by l with probability $\overline{q}$, and to the right by l with probability $1-\overline{q}$. It is easy to see that the coefficient $a_j^{k,p}$ is equal to the probability of the event that in $k-p$ time-units the particle returns to the point with the coordinate $(j-n)l$. Let us compute the mathematical expectation and the variance for the random variable $\overline{\xi}_{kp}$, equal to the coordinate of the particle after $k-p$ time-units. The mathematical expectation is

$$M(\overline{\xi}_{kp}) = (k-p)\,l\,(1-2\overline{q}) = -(k-p)\cdot\min_{\Omega}\phi_u\cdot h.$$

The variance is $D(\overline{\xi}_{kp}) = 4\overline{q}\,(1-\overline{q})\,l^2\,(k-p)$.

The quantity $\overline{P}_m^{k,p}$ is the probability of the event that after $k-p$ time-units the coordinate $\overline{\xi}_{kp}$ of the particle does not exceed $(m-n)l$. According to the Chebyshev inequality

$$P\,(\overline{\xi}_{kp} < M\,(\overline{\xi}_{kp}) - \varepsilon) \leqslant \frac{D(\overline{\xi}_{kp})}{\varepsilon^2} \text{ and } P\,(\overline{\xi}_{kp} > M\,(\overline{\xi}_{kp}) + \varepsilon) \leqslant \frac{D(\overline{\xi}_{kp})}{\varepsilon^2},$$

where ϵ is any positive number, or with our notation

$$\overline{P}_m^{k,p} \leqslant \frac{4\overline{q}\,(1-\overline{q})\,l^2\,(k-p)}{\varepsilon^2}, \quad \text{if} \quad (m-n)l < -\min_{\Omega}\varphi_u\cdot(k-p)\,h-\varepsilon$$

and

$$1-\overline{P}_m^{k,p} \leqslant \frac{4\overline{q}\,(1-\overline{q})\,l^2\,(k-p)}{\varepsilon^2}, \quad \text{if} \quad (m-n)l > -\min_{\Omega}\varphi_u\cdot(k-p)\,h+\varepsilon. \tag{3.25}$$

Similarly we get

$$\widetilde{P}_m^{k,p} \leq \frac{4\widetilde{q}\,(1-\widetilde{q})\,l^2\,(k-p)}{\epsilon^2}, \tag{3.26}$$

if $(m-n)\,l < -\max_{\Omega}\phi_u\cdot(k-p)\,h-\epsilon$, and

$$1-\widetilde{P}_m^{k,p} \leq \frac{4\widetilde{q}\,(1-\widetilde{q})\,l^2\,(k-p)}{\epsilon^2},$$

if $(m-n)\,l > -\max_{\Omega}\phi_u\,(k-p)\,h+\epsilon$.

From (3.25), (3.26) and (3.22) we get the estimates

$$\sum_{s=n-k+p}^{m} a_{n,s}^{k,p} < \frac{D_1\,(k-p)\,hl^2}{h\varepsilon^2}, \quad \text{if} \quad (m-n)l < -\max_{\Omega}\varphi_u\cdot(k-p)\,h-\varepsilon$$

$$\sum_{s=m}^{n+(k-p)} a_{n,s}^{k,p} < \frac{D_2\,(k-p)\,hl^2}{h\varepsilon^2}, \quad \text{if} \quad (m-n)l > -\min_{\Omega}\varphi_u\cdot(k-p)\,h+\varepsilon,$$

where D_1 and D_2 are constants independent of h and l. Let us take $\epsilon^2 = [(k-p)h]^{2/3}$. We get

$$\left.\begin{aligned} &\sum_{s=n-k+p}^{m_1} a_{n,s}^{k,p} < D_1 \frac{l^2}{h}[(k-p)h]^{\frac{1}{3}}, \\ &\text{where} \quad lm_1 \leqslant -\max_{\Omega} \varphi_u (k-p)h - [(k-p)h]^{\frac{1}{3}} + nl = \bar{R}_n l, \\ &\sum_{s=m_2}^{n+(k-p)} a_{n,s}^{k,p} < D_2 \frac{l^2}{h}[(k-p)h]^{\frac{1}{3}}, \\ &\text{where} \quad lm_2 \geqslant -\min_{\Omega} \varphi_u (k-p)h + [(k-p)h]^{\frac{1}{3}} + nl = \tilde{R}_n l. \end{aligned}\right\} \quad (3.27)$$

In a way similar to that by which we proved the lemma under the assumption $h/l > \delta > 0$, using estimates (3.27), we obtain

$$\sum_{|n| \leqslant \frac{X}{l}} |u_n^k - u_n^p| 2l \leqslant \sum_{|n| \leqslant \frac{X}{l}} \sum_{\bar{R}_n < s < \tilde{R}_n} a_{n,s}^{k,p} |u_s^p - u_n^p| 2l +$$
$$+ \sum_{|n| \leqslant \frac{X}{l}} \sum_{\substack{s \geqslant \tilde{R}_n \\ s \leqslant \bar{R}_n}} a_{n,s}^{k,p} |u_s^p - u_n^p| 2l + 2\gamma_1 X(k-p)h \leqslant$$
$$\leqslant \sum_{|n| \leqslant \frac{X}{l}} \sum_{\bar{R}_n < s < \tilde{R}_n} a_{n,s}^{k,p} \sum_{\bar{R}_n < r < \tilde{R}_n} |u_r^p - u_{r-2}^p| 2l +$$
$$+ 2X \cdot 2MD_3 \frac{l^2}{h}[(k-p)h]^{\frac{1}{3}} + 2\gamma_1 X(k-p)h \leqslant$$
$$\leqslant 2l|\tilde{R}_n - \bar{R}_n| \sum_{|r| \leqslant \frac{X}{l} + |\tilde{R}_n - \bar{R}_n|} |u_r^p - u_{r-2}^p| + 2X \cdot 2MD_3 \frac{l^2}{h}[(k-p)h]^{\frac{1}{3}} +$$
$$+ 2\gamma_1 X(k-p)h.$$

Using Lemma 3 and the expressions for $l\bar{R}_n$ and $l\tilde{R}_n$, we find that the left part of the last inequality does not exceed $L[(k-p)h]^{1/3}$, which was to be shown.

Lemma 5. *Let u_n^k and v_n^k be solutions on S_1 of the finite differences equation* (3.1) *with the condition $|u_n^0| \leq m$ and $|v_n^0| \leq m$. Then*

$$\sum_{|n| \leqslant N} |u_n^k - v_n^k| 2l \leqslant \delta(1+Bh)^k,$$

where $B = \max_{\Omega} |\psi_u|$, $k \leq T/h$, if $\sum_{|n| \leq N+k} |u_n^0 - v_n^0| 2l \leq \delta$.

Proof. Let us denote $W_n^k = u_n^k - v_n^k$. From equations (3.1) it follows:

$$W_n^k = W_{n+1}^{k-1}\left[\frac{1}{2} - \varphi_u((k-1)h,\ (n+1)l,\ \tilde{\theta}_{n+1}^{k-1})\frac{h}{2l} - \right.$$
$$\left. - \psi_u((k-1)h,\ (n+1)l,\ \bar{\theta}_{n+1}^{k-1})h\right] +$$
$$+ W_{n-1}^{k-1}\left[\frac{1}{2} + \varphi_u\left((k-1)h,\ (n-1)l,\ \tilde{\theta}_{n-1}^{k-1}\right)\frac{h}{2l}\right],$$

where $\tilde{\theta}_n^k$, $\bar{\theta}_n^k$ are certain intermediate values for u_n^k and v_n^k.

For sufficiently small h the coefficients of W_{n+1}^k and W_{n-1}^k are non-negative by virtue of the condition $Ah/l < 1$. Therefore

$$\sum_{|n|\leqslant N} |W_n^k| \leqslant \sum_{|n|\leqslant N}\left\{|W_{n+1}^{k-1}|\left[\frac{1}{2} - \varphi_u\left((k-1)h,\ (n+1)l,\ \tilde{\theta}_{n+1}^{k-1}\right)\frac{h}{2l} - \right.\right.$$
$$\left. - \psi_u\left((k-1)h,\ (n+1)l,\ \bar{\theta}_{n+1}^{k-1}\right)h\right] +$$
$$\left. + |W_{n-1}^{k-1}|\left[\frac{1}{2} + \varphi_u\left((k-1)h,\ (n-1)l,\ \tilde{\theta}_{n-1}^{k-1}\right)\frac{h}{2l}\right]\right\} \leqslant$$
$$\leqslant \sum_{|n|\leqslant N+1} |W_n^{k-1}|\left[1 - \psi_u\left((k-1)k,\ nl,\ \bar{\theta}_n^{k-1}h\right)\right] \leqslant \sum_{|n|\leqslant N+1} |W_n^{k-1}|(1+Bh).$$

From this follows easily the inequality

$$\sum_{|n|\leqslant N} |W_n^k|\, 2l \leqslant \sum_{|n|\leqslant N+k} |W_n^0|\,(1+Bh)^k \cdot 2l,$$

which was to be shown.

Let us consider in the points of the half-plane $t \geq 0$ a family of functions U_{hl}, constructed from u_n^k in the following fashion. In the points (t, x), satisfying the condition $\{kh \leq t < (k+1)h,\ nl \leq x < (n+2)l\}$ for $(k-n)$ even, the function U_{hl} is equal to u_n^k. Thus the function U_{hl} is defined in all points of the half-plane $t \geq 0$ and in the intersection points of S_1 coincides with u_n^k.

For the functions U_{hl} we shall prove the following theorem.

Theorem 3. *If* $l^2/h <$ const, $Ah/l < 1$, *then it is possible to select from the family of functions* $\{U_{hl}\}$ *an infinite sequence* $\{U_{hl}^i\}$ *such that for* h *and* $l \to 0$ *for* $i \to \infty$ *and for any* $X > 0$

$$\int_{-X}^{X} |U_{hl}^i(t,\ x) - u(t,\ x)|\, dx \to 0 \quad \textit{for} \quad i \to \infty. \tag{3.28}$$

The limit function $u(t, x)$ *is measurable,* $|u(t, x)| \leq M$ *in* G *and*

$$\int_0^T \int_{-X}^{X} |U_{hl}^i(t,\ x) - u(t,\ x)|\, dx\, dt \to 0 \quad \textit{for} \quad i \to \infty. \tag{3.29}$$

Proof.* From Lemma 3 it follows that the functions of the family $\{U_{hl}\}$ as functions of x have a variation uniformly bounded with respect to h and l on every finite segment of the straight line $t = \text{const} > 0$. Let $t = t_m$ $(m = 1, 2, \cdots)$ be a countable everywhere dense set on the segment $[0, T]$. By Helly's theorem [25], on any straight line $t = \text{const} > 0$, from the family of functions $\{U_{hl}\}$ one can select a sequence, converging in every point of this straight line for $h, l \to 0$. Consequently, by means of the diagonal process we can select from $\{U_{hl}\}$ a sequence $\{U^i_{hl}\}$ ($i \to \infty$ for $h \to 0$ and $l \to 0$), converging in every point of the family of straight lines $t = t_m$ $(m = 1, 2, \cdots)$ for $i \to \infty$.

Let us show that for any $t > 0$

$$\int_{-X}^{X} |U^i_{hl}(t, x) - U^j_{hl}(t, x)|\, dx \to 0, \quad \text{if} \quad i \to \infty,\ j \to \infty. \tag{3.30}$$

By definition of $U_{hl}(t, x) = U_{hl}([t/h]h, x)$, where $[\nu]$ is the integral part of ν. Let $\{t_{m_s}\}$ be a sequence converging to t for $m_s \to \infty$. We have:

$$\begin{aligned}\int_{-X}^{X} |U^i_{hl}(t, x) - U^j_{hl}(t, x)|\, dx \leqslant \int_{-X}^{X} \left|U^i_{hl}\left(\left[\frac{t}{h}\right]h, x\right) - U^i_{hl}\left(\left[\frac{t_{m_s}}{h}\right]h, x\right)\right| dx + \\ + \int_{-X}^{X} \left|U^j_{hl}\left(\left[\frac{t}{h}\right]h, x\right) - U^j_{hl}\left(\left[\frac{t_{m_s}}{h}\right]h, x\right)\right| dx + \\ + \int_{-X}^{X} |U^i_{hl}(t_{m_s}, x) - U^j_{hl}(t_{m_s}, x)|\, dx.\end{aligned} \tag{3.31}$$

Let us consider separately the cases when $\left(\left[\frac{t}{h}\right] - \left[\frac{t_{m_s}}{h}\right]\right)$ is even and odd. If $\left(\left[\frac{t}{h}\right] - \left[\frac{t_{m_s}}{h}\right]\right)$ is even, then

$$I \equiv \int_{-X}^{X} \left|U^i_{hl}\left(\left[\frac{t}{h}\right]h, x\right) - U^i_{hl}\left(\left[\frac{t_{m_s}}{h}\right]h, x\right)\right| dx \leqslant \sum_{|n| \leqslant \frac{X}{l}+1} \left|u_n^{\left[\frac{t}{h}\right]} - u_n^{\left[\frac{t_{m_s}}{h}\right]}\right| 2l,$$

and according to Lemma 4 this quantity does not exceed

$$L_1\left[\left(\left[\frac{t}{h}\right] - \left[\frac{t_{m_s}}{h}\right]\right)h\right]^{\frac{1}{3}} \leqslant L_1[|t - t_{m_s}| + h]^{\frac{1}{3}} \tag{3.32}$$

*We remark that Theorem 3 is valid also without the hypothesis $\phi_{uu} > 0$, if $u_0(x)$ is a function of bounded variation on the entire straight line. In this case Lemma 3 can be proved in a way similar to the proof of Lemma 14 in §7, and in the proof of Lemma 4 the hypothesis $\phi_{uu} > 0$ is not used. The proof of Theorem 3 is based only on Lemmas 3 and 4.

and, consequently, tends to zero if $t_{m_s} \to t$, and l and $h \to 0$. If $\left(\left[\frac{t}{h}\right] - \left[\frac{t_{m_s}}{h}\right]\right)$ is odd, then

$$I \leqslant \sum_{|n| \leqslant \frac{X}{l}+1} \left| u_n^{\left[\frac{t}{h}\right]} - u_{n+1}^{\left[\frac{t_{m_s}}{h}\right]} \right| 2l + \sum_{|n| \leqslant \frac{X}{l}+1} \left| u_n^{\left[\frac{t}{h}\right]} - u_{n-2}^{\left[\frac{t}{h}\right]} \right| l$$

and according to Lemmas 3 and 4 this quantity does not exceed

$$L_1 [\,|t - t_{m_s}| + h]^{\frac{1}{3}} + C_1 l \tag{3.33}$$

and, consequently, tends to zero for $t_{m_s} \to t$, $h \to 0$, $l \to 0$. The last integral in the right side of (3.31) tends to zero for $h \to 0$ and $l \to 0$ according to the choice of the sequence $\{t_m\}$. Consequently, the left side of (3.31) is arbitrarily small for sufficiently small h and l, which was to be shown.

Let us denote by $u(t, x)$ the limit function for the sequence $\{U^i_{hl}\}$. From (3.14) and Helly's theorem [25] it follows that a certain subsequence of $\{U^i_{hl}\}$ tends for a fixed t to $u(t, x)$ everywhere on the straight line under consideration. Since all functions of the family $\{U_{hl}\}$ are bounded by the constant M, so $|u(t, x)| \leq M$ almost everywhere in G.

Moreover, it can be shown that integral (3.30) tends to zero for $i \to \infty$, $j \to \infty$ uniformly in t, if $t \geq \tau > 0$. Indeed, for a given $\delta > 0$ one can select from the sequence $\{t_m\}$ a finite set, having the following property: for any $t \geq \tau$ a t_m can be found from this set such that for all sufficiently small h and l the quantities (3.32) and (3.33) are smaller than the given δ. If h and l are sufficiently small, then for this finite set of the straight lines $t = t_m$ the last integral in relation (3.31) does not exceed δ.

From the uniform convergence with respect to t of integral (3.30) for $i \to \infty$, $j \to \infty$ it follows that

$$\int_{\tau}^{T} \int_{-X}^{X} |U^i_{hl}(t, x) - U^j_{hl}(t, x)|\, dx\, dt \to 0$$

and the limit function $u(t, x)$ as a function of the variables x and t, is measurable. It is also evident that

$$\int_{0}^{T} \int_{-X}^{X} |U^i_{hl}(t, x) - u(t, x)|\, dx\, dt \to 0 \quad \text{for} \quad i \to \infty. \tag{3.34}$$

Lemma 6. *For* $|x_1 - x_2| > 2l$, $t > h$

$$\frac{U_{hl}(t, x_1) - U_{hl}(t, x_2)}{x_1 - x_2} < \frac{2E}{(t-h)}, \tag{3.35}$$

where the constant E *is as defined in Lemma* 2.

Proof. Let $x_1 > x_2$. According to the definition of $U_{hl}(t, x)$ we have the equalities

$$U_{hl}(t, x_2) = U_{hl}\left(\left[\frac{t}{h}\right]h,\ x_2 - \xi_2\right), \quad U_{hl}(t, x_1) = U_{hl}\left(\left[\frac{t}{h}\right]h,\ x_1 - \xi_1\right),$$

where $0 \leq \xi_1 < 2l$, $0 \leq \xi_2 < 2l$ and the points $\left(\left[\frac{t}{h}\right]h,\ x_1 - \xi_1\right)$, $\left(\left[\frac{t}{h}\right]h,\ x_2 - \xi_2\right)$ belong to the intersection points S_1. Hence it follows that

$$\frac{U_{hl}(t, x_1) - U_{hl}(t, x_2)}{x_1 - x_2} = \frac{\sum (u_n^k - u_{n-2}^k)}{x_1 - x_2}, \tag{3.36}$$

where the summation is taken over n from $n_2 = x_2 - \xi_2$ to $n_1 = x_1 - \xi_1$ and such that $(n - k)$ is even, $k = \left[\frac{t}{h}\right]$. Using Lemma 2 and equality (3.36) we find that the left side of this equality does not exceed

$$\frac{E \cdot (x_1 - \xi_1 - x_2 + \xi_2)}{(t-h)(x_1 - x_2)} \leqslant \frac{E}{t-h} + \frac{\xi_2 E}{(t-h)(x_1 - x_2)} \leqslant \frac{2E}{t-h},$$

which was to be shown.

Above we have studied in detail the solutions of a system of finite differences equations (3.1), set up for differential equation (2.1). We shall not pause here on other finite differences schemes for equation (2.1) which can be studied in a similar fashion. This question will be considered briefly at the end of the next section.

§4. Construction of generalized solutions of the Cauchy problem.

We shall prove the existence of a generalized solution of the Cauchy problem in the region G for equation (2.1) with condition (2.2). In this connection we shall use the finite differences scheme (3.1) which we have studied in detail in the preceding section.

Lemma 7. *The sequence $\{U_{hl}^i\}$, constructed in Theorem 3, converges for $i \to \infty$ in the sense of (3.28) and (3.29) to a function $u(t, x)$, satisfying the relation*

$$\iint_G \left[\frac{\partial f}{\partial t} u(t, x) + \frac{\partial f}{\partial x} \varphi(t, x, u(t, x)) - f\psi(t, x, u(t, x))\right] dx\,dt + \\ + \int_{-\infty}^{+\infty} f(0, x)\, u_0(x)\, dx = 0 \tag{4.1}$$

for any thrice continuously differentiable in G function $f(t, x)$, equal to zero outside a finite region, also for $t = T$, if $l^2/h \to 0$ and

$$\int_{-\infty}^{+\infty} f(0, x)\left[u_0(x) - U_{hl}^i(0, x)\right] dx \to 0 \tag{4.2}$$

for $i \to \infty$.*

If $l^2/h \to 2\epsilon > 0$ *and condition* (4.2) *is fulfilled for* $i \to \infty$, *then the limit function* $u_\epsilon(t, x)$ *satisfies the relation*

$$\iint_G \left[\frac{\partial f}{\partial t} u_\varepsilon + \frac{\partial f}{\partial x} \varphi(t, x, u_\varepsilon) - f\psi(t, x, u_\varepsilon) + \varepsilon \frac{\partial^2 f}{\partial x^2} u_\varepsilon \right] dx\, dt + \\ + \int_{-\infty}^{+\infty} f(0, x)\, u_0(x)\, dx = 0. \qquad (4.3)$$

Proof. The functions u_n^k are defined in every intersection point of S_1, i.e., in the points (kh, nl) if $(k-n)$ is even, and satisfy the finite differences equations (3.1). If $(k-n)$ is odd, let u_n^k in the point (kh, nl) be equal to u_{n-1}^k. Evidently, we may suppose that in such points u_n^k are solutions of the following system of finite differences equations

$$\frac{u_n^k - \dfrac{u_{n-1}^k + u_{n+1}^k}{2}}{h} + \frac{\varphi(kh, nl, u_{n+1}^k) - \varphi(kh, (n-2)l, u_{n-1}^k)}{2l} + \\ + \psi(kh, nl, u_{n+1}^k) = 0 \qquad (4.4)$$

with the initial condition $u_n^0 = u_{n-1}^0$.

Let us write equations (3.1) in the points of S_1 in the form

$$\frac{u_n^{k+1} - u_n^k}{h} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2l^2} \cdot \frac{l^2}{h} + \frac{\varphi(kh, (n+1)l, u_{n+1}^k)}{2l} - \\ - \frac{\varphi(kh, (n-1)l, u_{n-1}^k)}{2l} + \psi(kh, (n+1)l, u_{n+1}^k) = 0. \qquad (4.5)$$

In exactly the same way we transform equations (4.4) in the intersection points not belonging to S_1. We shall denote them by S_2.

Let us multiply each of the equation (4.5) by $f_n^k = f(kh, nl)$ and write the obtained expressions as follows:

$$\frac{f_n^{k+1} u_n^{k+1} - f_n^k u_n^k}{h} - u_n^{k+1} \frac{f_n^{k+1} - f_n^k}{h} + \frac{l^2}{2h} \frac{2f_n^k - f_{n+1}^k - f_{n-1}^k}{l^2} u_n^k + \\ + \frac{f_{n+1}^k u_n^k - f_n^k u_{n-1}^k}{2h} + \frac{f_{n-1}^k u_n^k - f_n^k u_{n+1}^k}{2h} + f_n^k \psi(kh, (n+1)l, u_{n+1}^k) + \\ + \frac{f_{n+1}^k \varphi(kh, (n+1)l, u_{n+1}^k) - f_{n-1}^k \varphi(kh, (n-1)l, u_{n-1}^k)}{2l} - \\ - \varphi(kh, (n+1)l, u_{n+1}^k) \frac{f_{n+1}^k - f_n^k}{2l} - \varphi(kh, (n-1)l, u_{n-1}^k) \frac{f_n^k - f_{n-1}^k}{2l} = 0. \qquad (4.6)$$

---

*Condition (4.2) is evidently valid if $\int_{-X}^{X} |u_0(x) - U_{hl}^i(0, x)|\, dx \to 0$ for $i \to \infty$ and any $X > 0$.

Multiplying (4.4) by f_n^k, we get:

$$\frac{f_n^{k+1}u_n^{k+1}-f_n^k u_n^k}{h}-u_n^{k+1}\frac{f_n^{k+1}-f_n^k}{h}+\frac{l^2}{2h}\frac{2f_n^k-f_{n+1}^k-f_{n-1}^k}{l^2}u_n^k+$$
$$+\frac{f_{n+1}^k u_n^k-f_n^k u_{n-1}^k}{2h}+\frac{f_{n-1}^k u_n^k-f_n^k u_{n+1}^k}{2h}+f_n^k\psi(kh,\,nl,\,u_{n+1}^k)+$$
$$+\frac{f_{n+1}^k\varphi(kh,\,nl,\,u_{n+1}^k)-f_{n-1}^k\varphi(kh,\,(n-2)\,l,\,u_{n-1}^k)}{2l}-$$
$$-\varphi(kh,\,nl,\,u_{n+1}^k)\frac{f_{n+1}^k-f_n^k}{2l}-\varphi(kh,\,(n-2)\,l,\,u_{n-1}^k)\frac{f_n^k-f_{n-1}^k}{2l}=0. \quad (4.7)$$

Let us sum up equalities (4.6) over all n and k $(k>0)$ such that $k+1-n$ is even, and equalities (4.7) over all n and k $(k>0)$ such that $k+1-n$ is odd. We may suppose that f_n^k is equal to zero for $k=\left[\frac{T}{h}\right]$ for all n. After multiplying by hl and obvious reductions we get:

$$hl\Big\{\sum_{n,\,k>0}\Big[u_n^{k+1}\frac{f_n^k-f_n^{k+1}}{h}-\frac{l^2}{2h}\frac{u_n^k(f_{n+1}^k-2f_n^k+f_{n-1}^k)}{l^2}\Big]+$$
$$+\sum_{S_1}f_n^k\psi(kh,\,(n+1)\,l,\,u_{n+1}^k)+\sum_{S_2}f_n^k\psi(kh,\,nl,\,u_{n+1}^k)-$$
$$-\sum_{S_1}\varphi(kh,\,(n+1)\,l,\,u_{n+1}^k)\frac{f_{n+1}^k-f_n^k}{2l}-\sum_{S_1}\varphi(kh,\,(n-1)\,l,\,u_{n-1}^k)\frac{f_n^k-f_{n-1}^k}{2l}-$$
$$-\sum_{S_2}\varphi(kh,\,nl,\,u_{n+1}^k)\frac{f_{n+1}^k-f_n^k}{2l}-\sum_{S_2}\varphi(kh,\,(n-2)\,l,\,u_{n-1}^k)\frac{f_n^k-f_{n-1}^k}{2l}\Big\}-$$
$$-\sum_n u_n^0 f_n^0 l=0. \qquad (4.8)$$

According to the definition of $U_{hl}(t,\,x)$ and continuity of $\frac{\partial f}{\partial x}$

$$\sum_n u_n^0 f_n^0 l=\sum_n u_0(2nl)\frac{f_{2n}^0+f_{2n+1}^0}{2}2l=\int_{-\infty}^{+\infty}U_{hl}(0,\,x)\,f(0,\,x)\,dx+O(l).$$

Since the function $U_{hl}(t,\,x)$ is piece-wise constant and $f(t,\,x)$ is thrice continuously differentiable and equal to zero for $t=T$ and sufficiently large x, so

$$\sum_{n,\,k>0}u_n^{k+1}\frac{f_n^{k+1}-f_n^k}{h}hl=\iint_G U_{hl}\frac{\partial f}{\partial t}\,dx\,dt+\delta_1,$$

$$\sum_{n,\,k>0}\frac{l^2}{2h}u_n^k\frac{f_{n+1}^k-2f_n^k+f_{n-1}^k}{l^2}hl=\frac{l^2}{2h}\iint_G U_{hl}\frac{\partial^2 f}{\partial x^2}\,dx\,dt+\delta_2,$$

$$hl\Big[\sum_{S_1} f_n^k\psi(kh,\,(n+1)\,l,\,u_{n+1}^k)\,hl+\sum_{S_2} f_n^k\psi(kh,\,nl,\,u_{n+1}^k)\Big]=$$

$$=\iint_G f\psi(t,\,x,\,U_{hl})\,dx\,dt+\delta_3,$$

$$hl\Big[\sum_{S_1}\varphi(kh,\,(n+1)\,l,\,u_{n+1}^k)\frac{f_{n+1}^k-f_n^k}{2l}+\sum_{S_1}\varphi(kh,\,(n-1)\,l,\,u_{n-1}^k)\frac{f_n^k-f_{n-1}^k}{2l}+$$

$$+\sum_{S_2}\varphi(kh,\,nl,\,u_{n+1}^k)\frac{f_{n+1}^k-f_n^k}{2l}+\sum_{S_2}\varphi(kh,\,(n-2)\,l,\,u_{n-1}^k)\frac{f_n^k-f_{n-1}^k}{2l}\Big]=$$

$$=\iint_G\varphi(t,\,x,\,U_{hl})\frac{\partial f}{\partial x}\,dx\,dt+\delta_4,$$

where δ_1, δ_2, δ_3, δ_4 tend uniformly to zero for $h\to 0$ and $l\to 0$.

Using this, equality (4.8) for the functions U_{hl}^i can be written in the form

$$-\iint_G U_{hl}^i\frac{\partial f}{\partial t}\,dx\,dt-\frac{l^2}{2h}\iint_G\frac{\partial^2 f}{\partial x^2}U_{hl}^i\,dx\,dt+\iint_G f\psi(t,\,x,\,U_{hl})\,dx\,dt-$$

$$-\iint_G\varphi(t,\,x,\,U_{hl}^i)\frac{\partial f}{\partial x}\,dx\,dt-\int_{-\infty}^{+\infty}U_{hl}^i(0,\,x)\,f(0,\,x)\,dx=\delta(h,\,l),\qquad(4.9)$$

where $\delta(h,\,l)\to 0$ for $h\to 0$ and $l\to 0$. Let i tend to ∞ and also $l^2/2h\to 0$. From equality (4.9) and condition (4.2) it follows that the limit function $u(t,\,x)$ satisfies the relation

$$\iint_G\Big[u\,\frac{\partial f}{\partial t}+\varphi(t,\,x,\,u)\frac{\partial f}{\partial x}-\psi(t,\,x,\,u)\,f\Big]\,dx\,dt+\int_{-\infty}^{+\infty}u_0(x)\,f(0,\,x)\,dx=0.$$

If $l^2/2h\to\epsilon>0$ for $i\to\infty$, so passing to the limit in equality (4.9), we obtain equality (4.3). Thus the lemma is proved.

Theorem 4. *A generalized solution of the Cauchy problem in the region G for equation* (2.1) *with condition* (2.2) *exists.*

Proof. Let l and k tend to zero in such a way that besides the condition $Ah/l<1$ which by hypothesis is always fulfilled, the condition $l^2/2h\to 0$ is fulfilled. By Theorem 3 from the sequence of functions $\{U_{hl}\}$ one can select a subsequence $\{U_{hl}^i\}$ converging in the sense of (3.28) and (3.29) to a bounded in G measurable function $u(t,\,x)$. By Lemma 7 the function $u(t,\,x)$ satisfies relation (2.3) for any thrice continuously differentiable function $f(t,\,x)$, equal to zero outside a finite region, also for $t=T$, if condition (4.2) is fulfilled which, evidently, can be easily satisfied by an appropriate choice of the initial values U_n^0.

Relation (2.3) is also fulfilled for any continuously differentiable in G function $f(t, x)$, equal to zero outside a finite region, also for $t = T$. This follows from the fact that the sequence of the averaged functions f^h converges uniformly in G to the function f, the sequences of the derivatives $\frac{\partial f^h}{\partial x}, \frac{\partial f^h}{\partial t}$ converge in the mean to the derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}$, and in relation (2.3) we may pass to the limit for $h \to 0$ for the sequence of the averaged functions f^h.

We shall show now that $u(t, x)$ satisfies condition 2), mentioned in the definition of a generalized solution of the Cauchy problem. According to Lemma 3 the functions U_{hl}^i as functions of x, have a variation uniformly bounded on each finite segment of the straight line $t = \text{const} > 0$ for all h and l. Therefore, by Helly's theorem [25] a certain subsequence of this sequence converges in each point of such a segment. It is evident that the limit function must coincide with $u(t, x)$ almost everywhere. For the functions U_{hl} the inequality (3.35) is valid. Passing to the limit for $h \to 0$ and $t \to 0$ in this inequality in the indicated subsequence of $\{U_{hl}^i\}$, we find that for $u(t, x)$ the condition 2) of the definition of a generalized solution is fulfilled in the form $\frac{u(t, x_1) - u(t, x_2)}{x_1 - x_2} \leq \frac{2E}{t}$, where the constant E does not depend on t, and x_1, x_2. In this connection $u(t, x)$ possibly would have to be changed on a set of measure zero on each straight line $t = \text{const}$. Thus, the function $u(t, x)$ is a generalized solution of the Cauchy problem (2.1), (2.2).

Remark. For the proof of the existence of a generalized solution of the Cauchy problem (2.1), (2.2) we could use such a net of straight lines $t = kh$, $x = nl$, so that $Ah/l < 1$ and $h/l > \delta > 0$ by setting, for example, $Ah/l = 1/2$. In this connection the condition $l^2/2h \to 0$ for $h \to 0$ and $l \to 0$ will be evidently fulfilled. Theorem 4 can be proved under weaker assumptions as to the functions $\phi(t, x, u)$ and $\psi(t, x, u)$, since one may drop the requirement of boundedness of the derivatives of ϕ and ψ, mentioned in the beginning of §2, for $(t, x) \subset G$ and for bounded u, and require only their continuity for $(t, x) \subset G$ and all u. Under such assumptions condition 2) will be fulfilled with the function $K(t, x_1, x_2)$, depending, generally speaking, on t, x_1, x_2.

Theorem 5. *Let the sequence* $\{U_{hl}\}$ *be such that* $Ah/l < 1$, $l^2/2h \to 0$ *for* $h \to 0$ *and* $l \to 0$, *and condition* (4.2) *of Lemma* 7 *is fulfilled. Then the sequence* $\{U_{hl}\}$ *converges for* $h, l \to 0$ *to a generalized solution* $u(t, x)$ *of the Cauchy problem* (2.1), (2.2) *in the sense that for any* $X > 0$, $t > 0$

$$\int_{-X}^{X} |U_{hl}(t, x) - u(t, x)|\, dx \to 0 \quad \textit{for} \quad h \to 0 \textit{ and } l \to 0 \qquad (4.10)$$

and, moreover,

$$\int_0^T \int_{-X}^{X} |U_{hl}(t, x) - u(t, x)|\, dx\, dt \to 0 \quad \text{for} \quad h \to 0 \text{ and } l \to 0. \tag{4.11}$$

Proof. From Theorems 3 and 4 it follows that from the sequence $\{U_{hl}\}$ one can select a subsequence, converging in the sense of (4.10) and (4.11) to a generalized solution $u(t, x)$ of the Cauchy problem (2.1), (2.2).

We shall show now that the entire sequence $\{U_{hl}\}$ converges to $u(t, x)$ for $h, l \to 0$. Suppose to the contrary. Then there exists such a subsequence of the sequence $\{U_{hl}\}$, so that integral (4.10) or (4.11) is greater than a certain $\delta > 0$ for each of its terms. But by virtue of Theorems 3, 4 and Theorem 1 on the uniqueness of a solution of the Cauchy problem, from this subsequence one can single out another subsequence converging to $u(t, x)$ in the sense of (4.10) and (4.11), i.e., for its terms these integrals must tend to zero. The obtained contradiction proves Theorem 5.

We remark that on the basis of Theorem 5, the finite differences scheme (3.1) can be used for the construction of an approximate solution of the Cauchy problem (2.1), (2.2) in the region G.

In the case when the coefficients of equation (2.1) are given for all u and (t, x) belonging to a certain finite region D in the half-plane $t > 0$ whose boundary contains the segment $[a, b]$ of the axis x, one can construct a generalized solution of equation (2.1), assuming on the segment $[a, b]$ values of a given bounded measurable function $u_0(x)$ in a certain region adjacent to $[a, b]$. In this connection one must impose the same conditions on the functions $\phi(t, x, u)$ and $\psi(t, x, u)$ for $(t, x) \subset \overline{D}$ and all u, as one has imposed on these functions when considering the Cauchy problem in the region G. The solution which is to be found will be defined in a region bounded by the straight lines

$$t = 0, \quad x - a - \alpha = Bt, \quad x - b + \alpha = -Bt, \quad t = \tau,$$

where $B > A$ and $A = \max |\phi_u|$ for $|u| \le M$ and $(t, x) \subset D$, if this region is contained in D; α and τ are arbitrary positive numbers. For the construction of such a solution one can use the finite differences scheme (3.1) under the condition $Ah/l = A/B$.

For the construction of a generalized solution of the Cauchy problem (2.1), (2.2) one can use also other finite differences schemes.

Thus, for instance, in the case when for $|u| \le M$ and for $(t, x) \subset G$ the condition $\phi_u(t, x, u) \ge 0$ is fulfilled, for the construction of a generalized solution of the Cauchy problem (2.1), (2.2) one can use a finite differences scheme in which equation (2.1) with condition (2.2) is replaced by the following system of the finite differences equations

$$\frac{u_n^{k+1}-u_n^k}{h}+\frac{\varphi(kh,\, nl,\, u_n^k)-\varphi(kh,\, (n-1)\,l,\, u_{n-1}^k)}{l}+$$
$$+\psi(kh,\, nl,\, u_n^k)=0, \qquad u_n^0=u_0(nl). \qquad (4.12)$$

If for $|u| \leq M$ and $(t, x) \subset G$ the condition $\phi_u(t, x, u) \leq 0$ is fulfilled, then the construction of a generalized solution of problem (2.1), (2.2) can be carried out using the finite differences equations

$$\frac{u_n^{k+1}-u_n^k}{h}+\frac{\varphi(kh,\, (n+1)\,l,\, u_{n+1}^k)-\varphi(kh,\, nl,\, u_n^k)}{l}+$$
$$+\psi(kh,\, nl,\, u_n^k)=0, \qquad u_n^0=u_0(nl). \qquad (4.13)$$

The finite differences schemes (4.12) and (4.13) are a special case of a finite differences scheme proposed by S. K. Godunov. According to this scheme the values u_n^k are determined by the following equations:

$$u_n^{k+1}=u_n^k+\frac{\left[\varphi\left(kh,\left(n-\frac{1}{2}\right)l,\, u_{n-\frac{1}{2}}^k\right)-\varphi\left(kh,\left(n+\frac{1}{2}\right)l,\, u_{n+\frac{1}{2}}^k\right)\right]h}{l}-$$
$$-\psi(kh,\, nl,\, u_n^k)\,h,$$

where $k = 0, 1, 2, \cdots, n = 0, 1, 2, \cdots, -1, -2, \cdots,$ $u_{j+\frac{1}{2}}^k = u_{j+1}^k$, if

$$\varphi_u\left(kh,\ \left(j+\frac{1}{2}\right)l,\, u_{j+1}^k\right)<0 \text{ and}$$
$$\Phi_{j+\frac{1}{2}}^k=\frac{\varphi\left(kh,\left(j+\frac{1}{2}\right)l,\, u_{j+1}^k\right)-\varphi\left(kh,\left(j+\frac{1}{2}\right)l,\, u_j^k\right)}{u_{j+1}^k-u_j^k}<0;$$
$$u_{j+\frac{1}{2}}^k=u_j^k, \quad \text{if} \quad \varphi_u\left(kh,\left(j+\frac{1}{2}\right)l,\, u_j^k\right)>0 \text{ and } \Phi_{j+\frac{1}{2}}^k>0,$$

and in the remaining cases $u_{j+\frac{1}{2}}^k$ is found from the equation

$$\varphi_u\left(kh,\ \left(j+\frac{1}{2}\right)l,\, u_{j+\frac{1}{2}}^k\right)=0.$$

We shall not consider the questions of convergence for these finite differences schemes. Their study is carried out in a similar fashion in the work [19].

§5. Solution of the Cauchy problem for the non-linear parabolic equations. Generalized solutions of non-linear equations of the first order as a limit of solutions of parabolic equations.

In §1 we said that the introduction into equation (2.1) of a term of the form $\epsilon \dfrac{\partial^2 u}{\partial x^2}$ with a small parameter $\epsilon > 0$, corresponds to taking into account the viscosity in the model problem of gas dynamics and it leads to smoothing out of the solution.

In the present section we shall show that the Cauchy problem in a region G for the equation

$$\varepsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \frac{\partial \varphi(t, x, u)}{\partial x} + \psi(t, x, u) \tag{5.1}$$

with the initial condition

$$u(0, x) = u_0(x) \tag{5.2}$$

is solvable for any bounded measurable initial function $u_0(x)$ and a generalized solution of the Cauchy problem for equation (2.1) with condition (2.2) can be obtained as a limit of solutions of the Cauchy problem of equation (5.1) with the same initial condition, when the parameter ϵ tends to zero.

We shall suppose that $\phi(t, x, u)$ and $\psi(t, x, u)$ satisfy conditions formulated in the beginning of §3, and prove the following theorem on the existence of a solution of the Cauchy problem (5.1), (5.2).*

Theorem 6. *There exists a function $u_\epsilon(t, x)$, bounded in G, which for $t > 0$ has all continuous derivatives occurring in equation* (5.1), *satisfies this equation and assumes initial conditions* (5.2) *in the following sense: for any continuous function $f(t, x)$ equal to zero for a sufficiently large in absolute value x,*

$$\int_{-\infty}^{+\infty} [f(t, x) u_\epsilon(t, x) - f(0, x) u_0(x)]\, dx \to 0 \text{ for } t \to 0. \tag{5.3}$$

If $u_0(x)$ is continuous for $x = x_1$, then $\lim\limits_{t \to 0, x \to x_1} u_\epsilon(t, x) = u_0(x_1)$.

Proof.** For the construction of the solution $u_\epsilon(t, x)$ we shall use the finite difference scheme (3.1); in this connection we shall assume that $l^2/2h = \epsilon$. From Theorem 3 and Lemma 7 it follows that there exists a sequence of functions $\{U^i_{hl}\}$ which for $i \to \infty$ converges in the sense (3.28), (3.29) to a function $u_\epsilon(t, x)$, satisfying relation (4.3) and $|u_\epsilon(t, x)| < M$.*** In the same way as in the proof of Theorem 4, we show that the limit function $u_\epsilon(t, x)$ satisfies the condition

$$\frac{u_\varepsilon(t, x_1) - u_\varepsilon(t, x_2)}{x_1 - x_2} < \frac{2E}{t}. \tag{5.4}$$

---

*For smooth initial functions $u_0(x)$ the Cauchy problem for equation (5.1) was solved in a different way in work [26].

**Theorem 6 can be proved in a different way, using results of work [26], without the hypothesis $\phi_{uu} > 0$. Let $u^n_0(x)$ be a sequence of uniformly bounded smooth functions converging almost everywhere to $u_0(x)$. Then the solutions $u_n(t, x)$ of the Cauchy problem for equation (5.1) with the conditions $u_n(0, x) = u^n_0(x)$, constructed in [26], for $t > 0$ and $n \to \infty$ converge to the solution of the Cauchy problem (5.1), (5.2), having the properties indicated in Theorem 6. For the proof of this fact one uses estimates of derivatives of $u_n(t, x)$ similar to the estimates of Bernštein.

***We shall call a bounded measurable function $u_\epsilon(t, x)$ satisfying relation (4.3), a generalized solution of the Cauchy problem (5.1), (5.2).

Let us show now that the function $u_\epsilon(t, x)$ for $t > 0$ has continuous derivatives occurring in equation (5.1).

If in relation (4.3) one takes for $f(t, x)$ the averaging kernel $\omega(h, \overline{PP_1})$, then for sufficiently small h in a neighborhood of the point (t_0, x_0) for $t_0 > 0$ we get

$$\epsilon \frac{\partial^2 u^h}{\partial x^2} - \frac{\partial u^h}{\partial t} = \frac{\partial \phi^h}{\partial x} + \psi^h, \tag{5.5}$$

where u^h is the average of the function $u_\epsilon(t, x)$ corresponding to the averaging kernel $\omega(h, \overline{PP_1})$, ϕ^h and ψ^h are the averages of the functions $\phi(t, x, u_\epsilon(t, x))$, $\psi(t, x, u_\epsilon(t, x))$, respectively. Let $0 < \rho < t_0$ and $F_\rho(t, x)$ be an infinitely differentiable function equal to zero outside of the circle K_ρ with the radius ρ and the center in the point (t_0, x_0), and equal to one in the circle $K_{\frac{\rho}{2}}$ with the radius $\frac{\rho}{2}$ and the center in the same point. Let

$$w_h^\rho = u^h \cdot F_\rho .$$

Then from equation (5.5) we get:

$$\varepsilon \frac{\partial^2 w_h^\rho}{\partial x^2} - \frac{\partial w_h^\rho}{\partial t} = F_\rho \frac{\partial \varphi^h}{\partial x} + F_\rho \psi^h + \\ + 2\varepsilon \frac{\partial u^h}{\partial x} \frac{\partial F_\rho}{\partial x} + \varepsilon \frac{\partial^2 F_\rho}{\partial x^2} u^h - \frac{\partial F_\rho}{\partial t} u^h \equiv \Phi_h^\rho . \tag{5.6}$$

We shall show that the integrals

$$\iint_G |\Phi_h^\rho| \, dx\, dt \tag{5.7}$$

are uniformly bounded with respect to h. From condition (5.4) it follows that the function $u_\epsilon(t, x) - (2E/t)\, x$ is monotone in x, and since the averaging kernel $\omega(h, \overline{PP_1}) \geq 0$ and depends only on the distance between P and P_1, so $u^h - ((2E/t)\, x)^h$ is also monotone in x for $t > 0$. Therefore, u^h, as a function of x, has a bounded variation for $t > q$ on any finite segment and, consequently, (see [25])

$$\iint_{K_\rho} \left| \frac{\partial u^h}{\partial x} \right| dx\, dt \tag{5.8}$$

does not exceed a constant independent of h. It is easy to show that $\phi(t, x, u_\epsilon(t, x))$, as a function of x, has a bounded variation for $t > \rho$ on any finite segment of the variable x. Indeed,

$$\sum_j |\varphi(t, x_j, u_\varepsilon(t, x_j)) - \varphi(t, x_{j+1}, u_\varepsilon(t, x_{j+1}))| \leqslant \max_\Omega |\varphi_x| \cdot L + \\ + \max_\Omega |\varphi_u| \cdot \sum_j |u_\varepsilon(t, x_j) - u_\varepsilon(t, x_{j+1})|, \tag{5.9}$$

where x_j are the points of an arbitrary subdivision of a segment of length L. Evidently,

$$\varphi^h(t, x) = \iint_G \omega(h, [(\xi - x)^2 + (t-\eta)^2]^{\frac{1}{2}}) \varphi(\eta, \xi, u_\varepsilon(\eta, \xi))\, d\xi\, d\eta =$$

$$= \iint_G \omega(h, [\beta^2 + (t-\eta)^2]^{\frac{1}{2}}) \varphi(\eta, \beta + x, u_\varepsilon(\eta, \beta + x))\, d\beta\, d\eta$$

and

$$\sum_j |\varphi^h(t, x_j) - \varphi^h(t, x_{j+1})| =$$

$$= \sum_j \left| \iint_G \omega(h, [\beta^2 + (t-\eta)^2]^{\frac{1}{2}}) [\varphi(\eta, \beta + x_j, u_\varepsilon(\eta, \beta + x_j)) - \right.$$

$$\left. - \varphi(\eta, \beta + x_{j+1}, u_\varepsilon(\eta, \beta + x_{j+1}))]\, d\beta\, d\eta \right|.$$

Hence, using inequality (5.9), we easily find that ϕ^h, as a function of x, has a bounded variation for $t > \rho$ and, consequently,

$$\iint_{K_\rho} \left| \frac{\partial \phi^h}{\partial x} \right| dx\, dt \tag{5.10}$$

does not exceed a certain constant independent of h. Since in the right side of (5.6) all functions, except $\partial \phi^h/\partial x$ and $\partial u^h/\partial x$, are uniformly bounded with respect to h, so from (5.8) and (5.10) it follows that integrals (5.7) are uniformly bounded with respect to h.

We shall show that $\dfrac{\partial w_h^\rho}{\partial x}$ are bounded in $L_2(K_\rho)$ for $\rho = \rho_0 < t_0$, i.e.,

$$\iint_{K_\rho} \left(\frac{\partial w_h^\rho}{\partial x^2} \right)^2 dx\, dt < C_1 \tag{5.11}$$

for all h. To this end we multiply (5.6) by w_h^ρ and integrate its right and left sides over K_ρ. Transforming the terms of the obtained equality by integrating by parts and taking into account that on the boundary of K_ρ the function w_h^ρ is equal to zero, we get

$$\iint_{K_\rho} \varepsilon \left(\frac{\partial w_h^\rho}{\partial x} \right)^2 dx\, dt = \left| \iint_{K_\rho} w_h^\rho \Phi_h^\rho\, dx\, dt \right| \leqslant \max_{K_\rho} |w_h^\rho| \iint_{K_\rho} |\Phi_h^\rho|\, dx\, dt.$$

Since in the circle $K_{\frac{\rho}{2}}$ the equality $w_h^\rho = u^h$ is fulfilled, so from (5.11) it follows that

$$\iint_{K_{\frac{\rho}{2}}} \left(\frac{\partial u^h}{\partial x} \right)^2 dx\, dt \leqslant C_1. \tag{5.12}$$

According to a theorem of S. L. Sobolev (see [3], p. 42) from (5.12) it follows that $u_\epsilon(t, x)$ has a generalized derivative $\partial u_\epsilon/\partial x$ which is square summable in $K_{\frac{\rho}{2}}$.

Now it is not difficult to show that the right side of (5.6) is uniformly bounded in $L_2(K_\rho)$ for $\rho = \rho_0/4$, i.e.,

$$\iint_{K_\rho} (\Phi_h^\rho)^2\, dx\, dt < C_2$$

for all h. To this end, it is sufficient, evidently, to establish that $\partial\phi^h/\partial x$ are bounded in $L_2(K_\rho)$ uniformly with respect to h for $\rho = \rho_0/4$, since for $\partial u^h/\partial x$ we have established inequality (5.12), and the remaining functions occurring in Φ_h^ρ are bounded in absolute value uniformly with respect to h. The uniform with respect to h boundedness of $\partial\phi^h/\partial x$ in $L_2(K_\rho)$ for $\rho = \rho_0/4$ follows from the fact that u_ϵ and $\phi(t, x, u_\epsilon(t, x))$ have in $K_{\frac{\rho_0}{2}}$ generalized derivatives with respect to x which are square-summable, $\partial\phi^h/\partial x = (\partial\phi/\partial x)^h$, and, as it is known (see [27], p. 242),

$$\iint_{K_{\frac{\rho_0}{4}}} \left[\left(\frac{\partial\varphi}{\partial x}\right)^h\right]^2 dx\, dt \leqslant \iint_{K_{\frac{\rho_0}{2}}} \left(\frac{\partial\varphi}{\partial x}\right)^2 dx\, dt.$$

Now we shall undertake the estimating of the integrals

$$\int_{-\infty}^{+\infty} [\Phi_h^\rho(t, x)]^2\, dx. \tag{5.13}$$

Multiplying the right and the left parts of (5.6) by $\partial w_h^\rho/\partial t$ and integrating over the circle K_ρ for $\rho = \rho_0/4$, we get

$$-\iint_{K_\rho} \left[\left(\frac{\partial w_h^\rho}{\partial t}\right)^2 - \varepsilon\frac{\partial}{\partial x}\left(\frac{\partial w_h^\rho}{\partial x}\cdot\frac{\partial w_h^\rho}{\partial t}\right) + \varepsilon\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial w_h^\rho}{\partial x}\right)^2\right] dx\, dt =$$

$$= \iint_{K_\rho} \Phi_h^\rho \frac{\partial w_h^\rho}{\partial t}\, dx\, dt. \tag{5.14}$$

Since w_h^ρ vanishes on the boundary of K_ρ, so from (5.14) it follows that

$$\iint_{K_\rho} \left(\frac{\partial w_h^\rho}{\partial t}\right)^2 dx\, dt \leqslant \sqrt{\iint_{K_\rho} (\Phi_h^\rho)^2\, dx\, dt}\sqrt{\iint_{K_\rho} \left(\frac{\partial w_h^\rho}{\partial t}\right)^2 dx\, dt}$$

and

$$\iint_{K_\rho} \left(\frac{\partial w_h^\rho}{\partial t}\right)^2 dx\, dt < C_2 \tag{5.15}$$

for $\rho = \rho_0/4$, where the constant C_2 does not depend on h.

From equation (5.6) it follows that $\partial^2 w_h^\rho / \partial x^2$ are bounded uniformly with respect to h in $L_2(K_{\rho_0/4})$, since $\partial w_h^\rho/\partial t$ and Φ_h^ρ have this property, i.e., for $\rho = \rho_0/4$

$$\iint_{K_\rho} \left(\frac{\partial^2 w_h^\rho}{\partial x^2}\right)^2 dx\,dt < C_3 \text{ for all } h. \tag{5.16}$$

Multiplying (5.6) by $\partial w_h^\rho/\partial t$, integrating over the strip $0 \le t \le t_1$, and taking into account that outside K_ρ the right and the left parts of (5.6) are equal to zero, we get

$$\int_{-\infty}^{+\infty} \left(\frac{\partial w_h^\rho(t_1, x)}{\partial x}\right)^2 dx < C_4, \tag{5.17}$$

where C_4 does not depend on h and t_1, and $\rho = \rho_0/4$. Since in the circle $K_{\rho/2}$, $w_h^\rho = u^h$, so from (5.17) it follows that for $\rho = \rho_0/8$

$$\int_{L(\rho,\, t_1)} \left(\frac{\partial u^h(t_1, x)}{\partial x}\right)^2 dx < C_4, \tag{5.18}$$

where $L(\rho, t_1)$ is a segment of the straight line $t = t_1$ belonging to K_ρ.

Since u^h, as functions of x, have a variation uniformly bounded with respect to h, so using Helly's theorem [25] and the theorem on the existence of a generalized derivative ([3], p. 42), from (5.18) we find that on the segment $L(\rho_0/8, t)$ $u_\epsilon(t, x)$ has a generalized derivative $\partial u_\epsilon/\partial x$, square-summable on this segment, and that

$$\int_{L\left(\frac{\rho_0}{8},\, t\right)} \left(\frac{\partial u_\varepsilon(t, x)}{\partial x}\right)^2 dx < C_4. \tag{5.19}$$

From (5.19) it follows that for all t

$$\int_{L\left(\frac{\rho_0}{8},\, t\right)} \left[\frac{\partial \varphi(t, x, u_\varepsilon(t, x))}{\partial x}\right]^2 dx < C_5, \tag{5.20}$$

where the constant C_5 does not depend on t.

For the proof of the uniform boundedness with respect to h and t of integrals (5.13) for $\rho = \rho_0/16$ by virtue of inequality (5.18) it suffices to show that

$$\int_{L(\rho,\, t)} \left[\frac{\partial \varphi^h(t, x)}{\partial x}\right]^2 dx < C_5$$

for all sufficiently small h and t. Using the definition of ϕ^h and the Bouniakowski inequality, we get

$$\int\limits_{L(\rho,\, t)} \left(\frac{\partial \varphi^h}{\partial x}\right)^2 dx = \int\limits_{L(\rho,\, t)} \left(\iint\limits_G \omega \frac{\partial \varphi(\eta,\, \xi,\, u_\varepsilon(\eta,\, \xi))}{\partial \xi} d\xi\, d\eta\right)^2 dx =$$
$$= \int\limits_{L(\rho,\, t)} \left(\iint\limits_G \sqrt{\omega}\cdot\sqrt{\omega}\frac{\partial \varphi}{\partial \xi} d\xi\, d\eta\right)^2 dx \leqslant$$
$$\leqslant \int\limits_{L(\rho,\, t)} \left(\iint\limits_G \omega\, d\xi\, d\eta \cdot \iint\limits_G \omega \left(\frac{\partial \varphi}{\partial \xi}\right)^2 d\xi\, d\eta\right) dx. \qquad (5.21)$$

Since the integral $\iint\limits_G \omega\, d\xi\, d\eta = 1$, so making a change of variables and changing the order of integration, we find that the right part of (5.21) does not exceed

$$\int\limits_{L(\rho,\, t)} \left(\iint\limits_G \omega\left(h,\, (\beta^2+\gamma^2)^{\frac{1}{2}}\right)\left[\frac{\partial \varphi(t+\gamma,\, x+\beta,\, u_\varepsilon(t+\gamma,\, x+\beta))}{\partial x}\right]^2 d\beta\, d\gamma\right) dx =$$
$$= \iint\limits_G \omega \left(\int\limits_{L(\rho,\, t)} \left(\frac{\partial \varphi}{\partial x}\right)^2 dx\right) d\beta\, d\gamma,$$

and these integrals do not exceed C_5 by virtue of (5.20).

Thus, we have estimated integrals (5.13) independently of h and t for $\rho = \rho_0/16$.

Now we shall show that $\partial u_\epsilon/\partial x$ is bounded and continuous in the circle K_ρ with the radius $\rho = \rho_0/64$. To this end we make use of the preceding estimates and of the fundamental solution of the heat conduction equation.

We shall consider w_h^ρ in the circle K_ρ as a solution of the heat conduction equation

$$\epsilon \frac{\partial^2 w_h^\rho}{\partial x^2} - \frac{\partial w_h^\rho}{\partial t} = \Phi_h^\rho$$

with the initial condition $\omega_h^\rho(0, x) = 0$. Therefore, for $t \geq 0$ w_h^ρ can be represented by means of a known formula (see [28]) as

$$w_h^\rho(t,\, x) = \int\limits_0^t \int\limits_{-\infty}^{+\infty} G(t,\, x,\, t_1,\, x_1)\, \Phi_h^\rho(t_1,\, x_1)\, dx_1\, dt_1, \qquad (5.22)$$

where

$$G(t,\, x,\, t_1,\, x_1) = -\frac{1}{2\sqrt{\pi}\sqrt{(t-t_1)\,\varepsilon}}\, e^{-\frac{(x-x_1)^2}{4(t-t_1)\,\varepsilon}}$$

It is evident that

$$\frac{\partial w_h^\rho}{\partial x} = \int\limits_0^t \int\limits_{-\infty}^{+\infty} \frac{\partial G}{\partial x}\, \Phi_h^\rho\, dx_1\, dt_1$$

and

$$\left|\frac{\partial w_h^\rho}{\partial x}\right| \leqslant \int_0^t \left(\sqrt{\int_{-\infty}^{+\infty}\left(\frac{\partial G}{\partial x}\right)^2 dx_1}\cdot\sqrt{\int_{-\infty}^{+\infty}(\Phi_h^\rho)^2\,dx_1}\right)dt_1.$$

It is easy to calculate that

$$\int_{-\infty}^{+\infty}\left(\frac{\partial G}{\partial x}\right)^2 dx_1 = M_1\frac{1}{|t-t_1|^{\frac{3}{2}}},$$

where M_1 is a certain constant. Using the obtained estimate for integrals (5.13) for $\rho = \rho_0/16$, we get

$$\left|\frac{\partial w_h^\rho}{\partial x}\right| \leqslant M_2\int_0^t \frac{1}{(t-t_1)^{\frac{3}{4}}}\,dt_1 \leqslant M_3 \quad \text{for} \quad \rho=\frac{\rho_0}{16}. \tag{5.23}$$

Since inside of the circle $K_{\rho/2}$, $w_h^\rho = u^h$, so from (5.23) it follows that in the circle K_ρ for $\rho = \rho_0/32$ the derivatives $\partial u^h/\partial x$ are uniformly bounded in absolute value with respect to h and therefore

$$\left|\frac{\partial u_\epsilon}{\partial x}\right| < C_6 \tag{5.24}$$

in the circle K_ρ for $\rho = \rho_0/32$, where C_6 is a certain constant.

Passing to the limit for $h \to 0$ in equality (5.22) for $\rho = \rho_0/32$, we find that for (t, x) belonging to $K_{\rho/2}$,

$$u_\varepsilon(t, x) =$$

$$= \int_0^t\int_{-\infty}^{+\infty} G(t, x, t_1, x_1)\left[F_\rho\frac{\partial\varphi}{\partial x} + F_\rho\psi + 2\varepsilon\frac{\partial u_\varepsilon}{\partial x}\frac{\partial F_\rho}{\partial x} + \varepsilon\frac{\partial^2 F_\rho}{\partial x^2}u_\varepsilon - \frac{\partial F_\rho}{\partial t}u_\varepsilon\right]dx_1\,dt_1 \equiv$$

$$\equiv \int_0^t\int_{-\infty}^{+\infty} G(t, x, t_1, x_1)\,\Phi^\rho(t_1, x_1)\,dx_1\,dt_1. \tag{5.25}$$

Since by virtue of (5.24) Φ^ρ are bounded in absolute value for $\rho = \rho_0/32$, so from (5.25) it follows that $u_\epsilon(t, x)$ is continuous in the circle K_{ρ_1} $(\rho_1 = \rho_0/64)$ and the derivative

$$\frac{\partial u_\varepsilon}{\partial x} = \int_0^t\int_{-\infty}^{+\infty}\frac{\partial G}{\partial x}\,\Phi^\rho\,dx_1\,dt_1$$

is also continuous in this circle.

Next, we shall consider the proof of the fact that $u_\epsilon(t, x)$ has a derivative

$\partial^2 u_\epsilon/\partial x^2$ which is continuous in the circle K_ρ for $\rho = \rho_1/16$.

First we show that for $\rho = \rho_1$

$$\iint_G \left[\frac{\partial \Phi_h^\rho}{\partial x}\right]^2 dx\,dt < C_7 \tag{5.26}$$

for sufficiently small h. Since the estimate (5.16) is valid, so for the proof of (5.26) it suffices, evidently, to show that the integrals

$$\iint_{K_\rho} \left(\frac{\partial^2 \varphi^h}{\partial x^2}\right)^2 dx\,dt \quad \text{and} \quad \iint_{K_\rho} \left(\frac{\partial \psi^h}{\partial x}\right)^2 dx\,dt$$

are bounded uniformly with respect to h for $\rho = \rho_1$. By virtue of a known theorem (see [27])

$$\iint_{K_\rho} \left(\frac{\partial^2 \varphi^h}{\partial x^2}\right)^2 dx\,dt = \iint_{K_\rho} \left[\left(\frac{\partial^2 \varphi}{\partial x^2}\right)^h\right]^2 dx\,dt \leqslant \iint_{K_{2\rho}} \left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2 dx\,dt,$$

and the existence of a generalized derivative $\partial^2\phi/\partial x^2$, square-summable in K K_ρ $(\rho = 2\rho_1)$, follows from the fact that for $h \to 0$ $\partial\phi(t, x, u^h)/\partial x$ converge in the mean to $\partial\phi(t, x, u_\epsilon)/\partial x$ in the circle K_ρ for $\rho = 2\rho_1$ and the derivatives $\partial^2\phi(t, x, u^h)/\partial x^2$, by virtue of (5.23) and (5.16), are bounded uniformly with respect to h in $L_2(K_\rho)$ for $\rho = 2\rho_1$ (see [3], the theorem on p. 42). The uniform with re-pect to h boundedness in $L_2(K_\rho)$ of $\partial\psi^h/\partial x$ is proved in exactly the same way as was proved the boundedness in $L_2(K_\rho)$ of the function $\partial\phi^h/\partial x$.

Consider equation (5.6) for $\rho = \rho_1$ Differentiating the right and the left sides of (5.6) with respect to x, we obtain for $\partial w_h^\rho/\partial x = w_{hx}^\rho$ an equation of the form

$$\varepsilon \frac{\partial^2 w_{hx}^\rho}{\partial x^2} - \frac{\partial w_{hx}^\rho}{\partial t} = \frac{\partial \Phi_h^\rho}{\partial x}. \tag{5.27}$$

Using the fact that $\partial\Phi_h^\rho/\partial x$ are bounded uniformly with respect to h in $L_2(K_\rho)$ for $\rho = \rho_1$, multiplying equation (5.27) by $\partial w_{hx}^\rho/\partial t$, integrating the right and the left parts of this equation over K_ρ $(\rho = \rho_1)$, we get

$$\iint_{K_\rho} \left(\frac{\partial^2 w_h^\rho}{\partial t\,\partial x}\right)^2 dx\,dt < C_8, \tag{5.28}$$

where the constant C_8 does not depend on h. In a similar way we find that for $\rho = \rho_1$

$$\int_{L(\rho,\,t)} \left(\frac{\partial^2 w_h^\rho(t,\,x)}{\partial x^2}\right)^2 dx < C_9 \tag{5.29}$$

for all t and h.

Since Φ_h^ρ vanish outside the circle K_ρ, so integrating by parts we get from the formula (5.22)

$$\frac{\partial w_h^\rho}{\partial x} = \int_0^t \int_{-\infty}^{+\infty} G(t, x, t_1, x_1) \frac{\partial \Phi_h^\rho}{\partial x} dx_1 dt_1$$

and

$$\frac{\partial^2 w_h^\rho}{\partial x^2} = \int_0^t \int_{-\infty}^{+\infty} \frac{\partial G(t, x, t_1, x_1)}{\partial x} \frac{\partial \Phi_h^\rho}{\partial x} dx_1 dt_1. \tag{5.30}$$

Using estimate (5.29), we obtain the uniform boundedness with respect to h and t for $\rho = \rho_1/4$ of the integrals

$$\int_{L(\rho,\, t)} \left(\frac{\partial^2 u^h}{\partial x^2}\right)^2 dx \quad \text{and} \quad \int_{L(\rho,\, t)} \left(\frac{\partial^2 \varphi^h}{\partial x^2}\right)^2 dx,$$

and, consequently, also of the integrals

$$\int_{L(\rho,\, t)} \left(\frac{\partial \Phi_h^\rho}{\partial x}\right)^2 dx \tag{5.31}$$

in a similar way, as we obtained the estimates of integrals (5.13).

With the help of (5.30) for $\rho = \rho_1/4$ and of the obtained estimate of integrals (5.31), we establish the uniform boundedness of $\partial^2 u^h/\partial x^2$ in the circle K_ρ for $\rho = \rho_1/8$, and also the boundedness and continuity of $\partial^2 u_\epsilon/\partial x^2$ in the circle K_ρ for $\rho = \rho_1/16$ in a similar way as we proved the continuity of $\partial u_\epsilon/\partial x$.

Thus, it is proved that in a circle with a sufficiently small radius with the center in the point (t_0, x_0) there exist continuous derivatives $\partial u_\epsilon/\partial x$, $\partial^2 u_\epsilon/\partial x^2$.

The existence of a generalized derivative $\partial u_\epsilon/\partial t$ in this circle follows from the inequality (5.15). Therefore, passing to the limit for $h \to 0$ in equation (5.5), we find that for $t > 0$ u_ϵ satisfies equation (5.1) almost everywhere. From the continuity of $u_\epsilon(t, x)$, $\partial u_\epsilon/\partial x$, $\partial^2 u_\epsilon/\partial x^2$ for $t > 0$ and (5.1) follows the continuity of $\partial u_\epsilon/\partial t$ and that equation (5.1) is fulfilled in all points of G for $t > 0$.

Now we shall show that $u_\epsilon(t, x)$ satisfies condition (5.3). Let $f(t, x)$ be a twice-continuously differentiable function equal to zero for $t = T$ and for sufficiently large in absolute value x. Since $u_\epsilon(t, x)$ satisfies equation (5.1) for $t > 0$, so multiplying this equation by $f(t, x)$, integrating over the region $0 < t_0 \leq t \leq T$ and transforming the obtained equality by integration by parts, we obtain:

$$\iint\limits_{t_0 \leq t \leq T} \left[\frac{\partial f}{\partial t} u_\varepsilon + \varepsilon \frac{\partial^2 f}{\partial x^2} u_\varepsilon + \frac{\partial f}{\partial x} \varphi(t, x, u_\varepsilon) - f\psi(t, x, u_\varepsilon)\right] dx\, dt +$$

$$+ \int\limits_{-\infty}^{+\infty} f(t_0, x)\, u_\varepsilon(t_0, x)\, dx = 0 \quad (5.32)$$

for any $t_0 > 0$. For the function $u_\epsilon(t, x)$, as it was proved, the equality (4.3) is fulfilled. Subtracting from equality (5.32) equality (4.3), we obtain:

$$-\iint\limits_{0 \leq t \leq t_0} \left[\frac{\partial f}{\partial t} u_\varepsilon + \varepsilon \frac{\partial^2 f}{\partial x^2} u_\varepsilon + \frac{\partial f}{\partial x} \varphi(t, x, u_\varepsilon) - f\psi(t, x, u_\varepsilon)\right] dx\, dt +$$

$$+ \int\limits_{-\infty}^{+\infty} [f(t_0, x)\, u_\varepsilon(t_0, x) - f(0, x)\, u_0(x)]\, dx = 0. \qquad (5.33)$$

Since $u_\epsilon(t, x)$ is bounded in G, so the integral over the region $0 \leq t \leq t_0$ in (5.33) tends to zero for $t_0 \to 0$ and, consequently, (5.3) is fulfilled for twice-differentiable functions equal to zero for sufficiently large in absolute value x. It is easy to see that condition (5.3) must be fulfilled also for continuous test-functions, since they can be approximated uniformly by functions as smooth as we please.

If $u_0(x)$ for $x = x_0$ is continuous, so $\lim\limits_{t \to 0, x \to x_0} u_\epsilon(t, x) = u_0(x_0)$. This follows easily from the fact that $u_\epsilon(t, x)$ can be represented for $t > 0$ in the form:

$$u_\varepsilon(t, x) = -\int\limits_{-\infty}^{+\infty} G(t, x, 0, x_1)\, u_0(x_1)\, dx_1 +$$

$$+ \int\limits_0^t \int\limits_{-\infty}^{+\infty} \left[-\frac{\partial G}{\partial x_1} \varphi(t_1, x_1, u_\varepsilon) + G\psi(t_1, x, u_\varepsilon)\right] dx_1\, dt_1. \qquad (5.34)$$

We shall obtain formula (5.34) in the proof of the following theorem.

Theorem 7. *A bounded solution $u_\epsilon(t, x)$ of the Cauchy problem* (5.1), (5.2) *is unique, i.e., there exists a unique bounded in G function $u_\epsilon(t, x)$ such that for $t > 0$ it has continuous derivatives occuring in equation* (5.1), *satisfies this equation and assumes for $t = 0$ the initial values in the sense of* (5.3).

Proof.* Let $F_n(x)$ be a family of twice continuously differentiable functions,

*Similarly, one can prove the uniqueness of a generalized solution of the Cauchy problem (5.1), (5.2). From the uniqueness of a generalized solution and Theorem 6 it follows that any generalized solution of the Cauchy problem (5.1), (5.2) for $t > 0$ has continuous derivatives, appearing in equation (5.1), and satisfies this equation.

satisfying the following conditions: $F_n(x) = 0$ for $|x| > n+1$, $F_n(x) = 1$ for $|x| < n$, $|F_n(x)| < N$, $|F'_n(x)| < N$, $|F''_n(x)| < N$, where the constant N does not depend on n.

The function $v^n_\epsilon = u_\epsilon \cdot F_n$ for $t > t_0 > 0$ satisfies the equation

$$\varepsilon \frac{\partial^2 v^n_\varepsilon}{\partial x^2} - \frac{\partial v^n_\varepsilon}{\partial t} =$$
$$= F_n \frac{\partial \varphi(t, x, u_\varepsilon)}{\partial x} + F_n \psi(t, x, u_\varepsilon) + 2\varepsilon \frac{\partial u_\varepsilon}{\partial x} F'_n + \varepsilon F''_n u_\varepsilon = \Phi_n(t, x),$$

whose right side is bounded and equal to zero for sufficiently large in absolute value x. Therefore, for $t > t_0$ it can be represented in the form

$$v^n_\varepsilon(t, x) = -\int_{-\infty}^{+\infty} G(t, x, t_0, x_1)\, v^n_\varepsilon(t_0, x_1)\, dx_1 +$$
$$+ \int_{t_0}^{t} \int_{-\infty}^{+\infty} G(t, x, t_1, x_1)\, \Phi_n(t_1, x_1)\, dx_1\, dt_1. \tag{5.35}$$

Let us transform the last integral in equality (5.35) by integrating by parts. We obtain:

$$v^n_\varepsilon(t, x) = -\int_{-\infty}^{+\infty} G(t, x, t_0, x_1)\, v^n_\varepsilon(t_0, x_1)\, dx_1 +$$
$$+ \int_{t_0}^{t} \int_{-\infty}^{+\infty} \left[-\frac{\partial (GF_n)}{\partial x_1} \varphi + F_n G\psi + \varepsilon F''_n u_\varepsilon \cdot G - 2\varepsilon \frac{\partial (GF'_n)}{\partial x_1} u_\varepsilon \right] dx_1\, dt_1.$$

Now let t_0 tend to zero. By virtue of condition (5.3), we get for $t > 0$

$$v^n_\varepsilon(t, x) = -\int_{-\infty}^{+\infty} G(t, x, 0, x_1)\, u_0(x_1)\, F_n(x_1)\, dx_1 + \int_0^t \int_{-\infty}^{+\infty} \left[-\frac{\partial G}{\partial x_1} F_n \varphi - GF'_n \varphi + \right.$$
$$\left. + F_n \psi G - 2\varepsilon \frac{\partial G}{\partial x_1} F'_n u_\varepsilon - \varepsilon G F''_n u_\varepsilon \right] dx_1\, dt_1. \tag{5.36}$$

In equality (5.36) we pass to the limit for $n \to \infty$. By virtue of conditions imposed on the functions $F_n(x)$ and of the boundedness of $u_\epsilon(t, x)$ in G, we get for $t > 0$

$$u_\varepsilon(t, x) = -\int_{-\infty}^{+\infty} G(t, x, 0, x_1)\, u_0(x_1)\, dx_1 -$$
$$- \int_0^t \int_{-\infty}^{+\infty} \left[\frac{\partial G}{\partial x_1} \varphi(t_1, x_1, u_\varepsilon) - G\psi(t_1, x_1, u_\varepsilon) \right] dx_1\, dt_1. \tag{5.37}$$

Suppose that there exist two solutions u_ϵ and $\tilde{u}_\epsilon$ for the Cauchy problem (5.1), (5.2). Let us write equality (5.37) for each of them and subtract one from the other. We get:

$$u_\varepsilon(t, x) - \widetilde{u}_\varepsilon(t, x) =$$
$$= \int_0^t \int_{-\infty}^{+\infty} \left\{ \frac{\partial G}{\partial x_1} \varphi_u(t_1, x_1, \theta)(\widetilde{u}_\varepsilon - u_\varepsilon) + G\phi_u(t_1, x_1, \theta)(u_\varepsilon - \widetilde{u}_\varepsilon) \right\} dx_1 \, dt_1. \quad (5.38)$$

Let $\sup_{0<t\leq\tau} |u_\epsilon - \widetilde{u}_\epsilon| = \gamma$ and let (t_s, x_s) be a sequence of points such that $|u_\epsilon(t_s, x_s) - \widetilde{u}_\epsilon(t_s, x_s)| \to \gamma$ for $s \to \infty$ and $t_s > 0$. Since the functions G and $\partial G/\partial x_1$ are summable in any strip $\{0 \leq t \leq \tau\}$, so from (5.38) we find that

$$|u_\epsilon(t_s, x_s) - \widetilde{u}_\epsilon(t_s, x_s)| \leq \gamma C, \quad (5.39)$$

where C is a certain constant. Passing to the limit for $s \to \infty$ in (5.39), we get:

$$\gamma \leq \gamma C. \quad (5.40)$$

Evidently, if $\tau > 0$ is sufficiently small, so $C < 1$, and from relation (5.40) it follows that $\gamma = 0$. Thus we have proved that $u_\epsilon(t, x) = \widetilde{u}_\epsilon(t, x)$ for $0 < t \leq \tau$.

Further, we prove in a similar way that these functions coincide in the strip $\tau \leq t \leq 2\tau, \cdots, N\tau \leq t \leq T$, where N is such that $(N+1)\tau > T > NT$. This proves the theorem on the uniqueness of the solution of the Cauchy problem (5.1), (5.2).

Theorem 8. *For $\epsilon \to 0$ the solutions $u_\epsilon(t, x)$ of the Cauchy problem* (5.1), (5.2) *converge to a generalized solution $u(t, x)$ of the Cauchy problem* (2.1), (2.2), *obtained in Theorem* 4, *in the sense that for any $t (0 < t \leq T)$ and $X > 0$*

$$\int_{-X}^{X} |u_\epsilon(t, x) - u(t, x)| \, dx \to 0 \text{ for } \epsilon \to 0. \quad (5.41)$$

Proof. It is evident that

$$\int_{-X}^{X} |u_\varepsilon(t, x) - u(t, x)| \, dx \leqslant \int_{-X}^{X} |u_\varepsilon(t, x) - U_{hl}(t, x)| \, dx +$$
$$+ \int_{-X}^{X} |u(t, x) - U_{hl}(t, x)| \, dx, \quad (5.42)$$

where U_{hl} are functions constructed in §3.

Let $\{\epsilon_i\}$ be a certain sequence, converging to zero and let condition (4.2) be valid for the sequence U_{hl}. Let us choose U^i_{hl} so that $l^2/2h = \epsilon_i$ and h are so small that

$$\int_{-X}^{X} |u_{\epsilon_i}(t, x) - U^i_{hl}(t, x)| \, dx < \frac{\delta}{2}.$$

The latter is possible by virtue of Theorems 6, 7, and Lemma 7.

Let us consider the sequence $\{U^i_{hl}\}$ constructed in this way. Since $h \to 0$ and $l^2/2h = \epsilon_i \to 0$ for $i \to \infty$, so by Theorem 5

$$\int_{-X}^{X} |u(t, x) - U_{hl}^{i}(t, x)|\, dx < \frac{\delta}{2},$$

if i is sufficiently large. Thus, the left part of (5.42) is arbitrarily small for sufficiently small ϵ. The theorem is proved.

Theorem 9. *Let $u(t, x)$ be the generalized solution of the Cauchy problem* (2.1), (2.2), *constructed in Theorem* 5. *Let (t_0, x_0) be a point in G such that by changing $u(t, x)$ on the straight line $t = t_0 > 0$ on a set of measure zero one can make $u(t, x)$ continuous in x at the point (t_0, x_0). Then*

$$u_\epsilon(t_0, x_0) \to u(t_0, x_0) \;\; \textit{for} \;\; \epsilon \to 0. \tag{5.43}$$

Proof. As it was shown in the proof of Theorem 6, the functions $u_\epsilon(t, x)$ satisfy condition (5.4), where E does not depend on t and ϵ. Therefore, for $t_0 > 0$ there exists a constant H, independent of ϵ, and such that the functions $v_\epsilon = u_\epsilon(t_0, x) - Hx$ are monotone decreasing functions of x.

Now let us change $u(t, x)$ on a set of measure zero so that it is continuous at the point (t_0, x_0). Let $v = u - Hx$. Suppose that (5.43) is not fulfilled. Then there exist $\delta > 0$ and a sequence of $\epsilon \to 0$ such that

$$|v_\epsilon(t_0, x_0) - v(t_0, x_0)| > \delta.$$

Suppose that for the infinite sequence of ϵ we have $v_\epsilon(t_0, x_0) > v(t_0, x_0) + \delta$. (The case $v_\epsilon(t_0, x_0) < v(t_0, x_0) - \delta$ is treated similarly.) Let us choose $\alpha > 0$ so that for all x in the interval $x_0 - \alpha < x < x_0$ the inequality $|v(t_0, x) - v(t_0, x_0)| \leq \frac{\delta}{2}$. Then

$$\int_{x_0-\alpha}^{x_0} |u_\varepsilon(t_0, x) - u(t_0, x)|\, dx = \int_{x_0-\alpha}^{x_0} |v_\varepsilon(t_0, x) - v_\varepsilon(t_0, x_0) +$$
$$+ [v_\varepsilon(t_0, x_0) - v(t_0, x_0)] + [v(t_0, x_0) - v(t_0, x)]|\, dx. \tag{5.44}$$

Since for $x_0 - \alpha < x < x_0$ the inequalities $v_\epsilon(t_0, x) - v_\epsilon(t_0, x_0) \geq 0$,

$$v_\epsilon(t_0, x_0) - v(t_0, x_0) > \delta \text{ and } |v(t_0, x) - v(t_0, x_0)| \leq \frac{\delta}{2}$$

are fulfilled, so the left part of equality (5.44) is greater than $\delta/2$ for arbitrarily small ϵ, which contradicts Theorem 8.

§6. Properties of generalized solutions of non-linear equations.

In the present section we shall study properties of generalized solutions of the Cauchy problem (2.1), (2.2) and properties of solutions of equation (2.1), assuming prescribed values on a finite interval $[a, b]$.

First of all, we prove a theorem on the character of the dependence of a solution of the Cauchy problem (2.1), (2.2) in the region G on the initial function $u_0(x)$.

Theorem 10. *Let $u(t, x)$ and $v(t, x)$ be the generalized solutions of the Cauchy problem for equation* (2.1) *in the region G, constructed in Theorem* 5 *and corresponding to the initial conditions*

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad -\infty < x < +\infty,$$

where $u_0(x)$ and $v_0(x)$ are measurable functions $|u_0(x)| < m$, $|v_0(x)| < m$. Then

$$\int_{a_1}^{a_2} |u(t, x) - v(t, x)|\, dx < \delta e^{Bt}, \tag{6.1}$$

if

$$\int_{a_1 - At}^{a_2 + At} |u_0(x) - v_0(x)|\, dx < \delta, \tag{6.2}$$

where $a_1, a_2, \delta > 0$ $(a_1 < a_2)$ are arbitrary numbers, $A = \max\limits_{\Omega} |\phi_u|$, $B = \max\limits_{\Omega} |\psi_u|$ and the region Ω is defined by the conditions:

$$\{(t, x) \subset G, \ |u| \le M\} \text{ and } M = \sup_G \{|u(t, x)|, \ |v(t, x)|\}.$$

Proof. Let U_{hl} and V_{hl} be the sequences constructed in Theorem 5 for the initial functions $u_0(x)$ and $v_0(x)$, respectively, which converge in the sense of (4.10) to the solutions $u(t, x)$ and $v(t, x)$. Moreover, we shall assume that $Ah/l = 1 - \beta$, where $\beta > 0$ is an arbitrary small number, and the functions $U_{hl}(0, x)$ and $V_{hl}(0, x)$ converge to $u_0(x)$ and $v_0(x)$, respectively, in the sense that

$$\int_{-X}^{X} |u_0(x) - U_{hl}(0, x)|\, dx \to 0, \quad \int_{-X}^{X} |v_0(x) - V_{hl}(0, x)|\, dx \to 0 \quad \text{for } h, l \to 0$$

for any $X > 0$.

From Lemma 5 it follows that

$$\int_{a_1}^{a_2} |U_{hl}(t, x) - V_{hl}(t, x)|\, dx < \delta (1 + hB)^{\left[\frac{t}{h}\right]}, \tag{6.3}$$

if

$$\int_{a_1 - t \cdot \frac{A}{1-\beta}}^{a_2 + t \cdot \frac{A}{1-\beta}} |U_{hl}(0, x) - V_{hl}(0, x)|\, dx < \delta. \tag{6.4}$$

Passing to the limit in inequalities (6.3) and (6.4) for $h, l \to 0$ and then for $\beta \to 0$, by virtue of Theorem 5 and the assumptions we made, we obtain (6.1) under condition (6.2).

Remark. It is evident that Theorem 10 is valid for a solution of equation (2.1), assuming initial values $u_0(x)$ on a finite segment $[a, b]$ in the region indicated in Theorem 2, and for such a_1, a_2 so that the segment $[a_1 - At, a_2 + At]$ belongs to $[a, b]$.

We shall use the following Lemmas 8 and 9 in the proof of another theorem on the dependence of a solution of the Cauchy problem (2.1), (2.2) on the initial functions.

Lemma 8. *A generalized solution $u(t, x)$ of the Cauchy problem* (2.1), (2.2) *in the region G constructed in Theorem* 5 *satisfies the relation*

$$\int_{a_1}^{a_2} |u(t_1, x) - u(t_2, x)|\, dx \le L_1(t_1 - t_2) \tag{6.5}$$

where $t_1 > t_2 \ge \alpha > 0$, a_1 and a_2 are arbitrary numbers $(a_1 < a_2)$, the constant L_1 depends only on α, a_1, a_2 and it can be chosen to be one and the same for all solutions $u(t, x)$ of problem (2.1), (2.2) *for which $|u_0(x)| \le m$.*

Proof. The assertion of Lemma 8 follows easily from Lemma 4 and Theorem 5. Indeed, as in the proof of Theorem 3, we shall show that

$$\int_{a_1}^{a_2} |U_{hl}(t_1, x) - U_{hl}(t_2, x)|\, dx \le L_1(t_1 - t_2 + h + l) \tag{6.6}$$

for all $U_{hl}(t, x)$, satisfying the conditions of Theorem 5, and for $Ah/l = 1/2$. Passing to the limit in (6.6) for $h \to 0$, $l \to 0$, by virtue of Theorem 5 we get inequality (6.5).

Lemma 9. *A generalized solution $u(t, x)$ of the Cauchy problem* (2.1), (2.2) *in the region G can be changed on a set of measure zero so that the obtained function $u(t, x)$ has the following properties: for $t > 0$ the function $u(t, x)$, as a function of x, has not more than countably many discontinuity points, on every finite segment of the straight line $t = \text{const} > 0$ the variation of $u(t, x)$, as a function of x, is bounded, and the relation*

$$\frac{u(t, x_1) - u(t, x_2)}{x_1 - x_2} < \frac{2E}{t} \tag{6.7}$$

is fulfilled, where the constant E is one and the same for all $u(t, x)$ for which $u(0, x) = u_0(x)$ does not exceed m in absolute value.

Proof. Relation (6.7) was obtained in the proof of Theorem 4, where E was the constant indicated in Lemma 2. From the proof of Lemma 2 one sees that the constant E is one and the same for all u_n^k, for which $|u_n^0| \le m$. From (6.7) it follows that $u(t, x) - \frac{2E}{t}x$ is a monotone decreasing function of x, and therefore $u(t, x)$, as a function of x, on every straight line $t = \text{const} > 0$ has not more than countably many discontinuity points and has a bounded variation on every finite segment of this straight line.

Theorem 11. *Let* $u^n(t, x)$ *be a generalized solution of the Cauchy problem of equation* (2.1) *in the region* G *with the initial condition* $u^n(0, x) = u_0^n(x)$, *and* $|u_0^n(x)| \leq m$ $(n = 1, 2, \cdots)$. *Let*

$$\int_{-\infty}^{+\infty} f(x)\,[u_0^n(x) - u_0(x)]\,dx \longrightarrow 0$$

for $n \longrightarrow \infty$ *for any continuous function* $f(x)$, *equal to zero for sufficiently large in absolute value* x. *Then the sequence* $u^n(t, x)$ *converges in* G *for* $n \longrightarrow \infty$ *to the generalized solution* $u(t, x)$ *of the Cauchy problem for equation* (2.1) *with the initial condition* $u(0, x) = u_0(x)$ *in the sense that for any finite part* G_1 *of the region* G

$$\iint_{G_1} |u^n(t, x) - u(t, x)|\,dx\,dt \longrightarrow 0 \text{ for } n \longrightarrow \infty. \tag{6.8}$$

Proof. Using the assertions of Lemmas 8 and 9, in a way similar to that as in the proof of Theorem 3, it is easy to show that from the sequence $\{u^n(t, x)\}$ one can select such a subsequence $\{u^{n'}(t, x)\}$ so that

$$\iint_{G_1} |u^{n'}(t, x) - \tilde{u}(t, x)|\,dx\,dt \longrightarrow 0 \text{ for } n' \longrightarrow \infty \tag{6.9}$$

for any finite part G_1 of the region G; in this connection $\tilde{u}(t, x)$ is a measurable function in the region G and $|\tilde{u}(t, x)| < M$.

Passing to the limit for $n' \longrightarrow \infty$ in relation (2.3) for functions of the sequence $\{u^{n'}(t, x)\}$, we find that relation (2.3) is fulfilled for $\tilde{u}(t, x)$ with the initial function $u_0(x)$. From (6.7) and Helly's theorem [25] it follows that a certain subsequence of $\{u^{n'}\}$ converges to $\tilde{u}(t, x)$ almost everywhere on each straight line $t = \text{const} > 0$. Since for all functions $u^n(t, x)$ the condition (6.7) is fulfilled, so also the limit function $\tilde{u}(t, x)$ satisfies condition 2) of the definition of a generalized solution. Thus, we have proved that $\tilde{u}(t, x)$ is a generalized solution of problem (2.1), (2.2), i.e., $\tilde{u}(t, x) = u(t, x)$. From the uniqueness of the solution of the Cauchy problem (2.1), (2.2), it follows that the entire sequence $\{u^n(t, x)\}$ converges to $u(t, x)$ in the sense of (6.8) for $n \longrightarrow \infty$.

Theorem 12. *A generalized solution* $u(t, x)$ *of the Cauchy problem* (2.1), (2.2) *in the region* G, *constructed in Theorem* 4, *satisfies the initial condition* (2.2) *in the sense that for any continuous function* $f(x)$ *equal to zero for sufficiently large in absolute value* x

$$\int_{-\infty}^{+\infty} f(x)\,[u(t, x) - u_0(x)]\,dx \longrightarrow 0 \text{ for } t \longrightarrow 0. \tag{6.10}$$

Proof. In a way similar to the proof of Lemma 7, it is easy to show that for any t_1 $(0 < t_1 \leq T)$ and any continuously differentiable function $f(t, x)$, equal to zero for sufficiently large in absolute value x, the relation

$$\iint\limits_{0\leqslant t\leqslant t_1}\left[\frac{\partial f}{\partial t}u+\frac{\partial f}{\partial x}\varphi(t,x,u)-f\psi(t,x,u)\right]dx\,dt+\int_{-\infty}^{+\infty}u_0(x)f(0,x)\,dx-$$
$$-\int_{-\infty}^{+\infty}u(t_1\,x)f(t_1,x)\,dx=0 \qquad (6.11)$$

holds. Let $f(t, x)$ depend only on x. Then

$$\iint\limits_{0\leqslant t\leqslant t_1}\left[\frac{df}{dx}\varphi(t,x,u)-f\psi(t,x,u)\right]dx\,dt+\int_{-\infty}^{+\infty}f(x)\left[u_0(x)-u(t_1,x)\right]dx=0. \qquad (6.12)$$

Since $f(x)$ is equal to zero for sufficiently large in absolute value x, and $u(t, x)$ and df/dx are bounded in G, so from (6.12) it follows that (6.10) is fulfilled for a continuously differentiable test-function $f(x)$. That (6.10) is fulfilled for continuous test-function $f(x)$, follows from the fact that such a function can be approximated uniformly by continuously differentiable test-functions for which (6.10) has been proved already.

Further, in this section we shall undertake a more detailed study of the structure of generalized solutions of equation (2.1) in the case when $\psi(t, x, u) \equiv 0$. The basic results pertaining to this question were published with detailed proofs in the author's work [12], and therefore we shall not reproduce the proof contained in the indicated work.

We shall consider the equation

$$\frac{\partial u}{\partial t}+\frac{\partial\phi(t,x,u)}{\partial x}=0 \qquad (6.13)$$

and we shall construct a generalized solution of this equation, assuming on a given segment $[a, b]$ of the straight line $t = 0$ values of a bounded measurable function $u_0(x)$ in a certain region adjacent to $[a, b]$. Using Theorem 2, we shall show that this solution coincides with a generalized solution of the Cauchy problem (2.1), (2.2) in any finite part $\{|x| < X,\ 0 < t \leq T\}$ of the region G, if the segment $[a, b]$ is chosen sufficiently large.

We shall subject the function $\phi(t, x, u)$ to an additional restriction. Let $\phi_{uu} > 0$ and suppose that any two points of G, not lying on the same straight line parallel to the axis x, can be connected in a unique way by the projection of a characteristic of equation (6.13) on the plane (t, x). The characteristics of equation (6.13) are solutions of the system of equations

$$\frac{dx}{dt}=\phi_u(t,x,u),\quad \frac{du}{dt}=-\phi_x(t,x,u). \qquad (6.14)$$

Thus, we assume that the boundary value problem for the equation of the second order which we obtain from system (6.14) by eliminating u, is solvable. Sufficient conditions for this are given, for instance, in [29]. It is also evident that our assumption is fulfilled when, for instance, ϕ depends only on u and ϕ_u assumes all values when $-\infty < u < +\infty$.

We shall construct a generalized solution of equation (6.13) assuming the values $u_0(x)$ on the segment $[a, b]$, in a region G_1 lying in the half-plane $t > 0$ and bounded by the straight lines $t = 0$, $t = T$ and the projections on the plane (t, x) of the characteristics of (6.14), passing through the points $(0, a, \overline{\lim\limits_{x\to a}}\, u_0(x))$ and $(0, b, \underline{\lim\limits_{x\to b}}\, u_0(x))$. It is easy to show that G_1 contains in its interior a region bounded by the straight lines $t = 0$, $t = T$, $x - a = At$, $x - b = -At$, where $A = \max\limits_{\Omega} |\phi_u|$.

Let us construct a function $U(t, t_1, x_1, \xi)$ as follows: we join the points (t_1, x_1) and $(0, \xi)$, belonging to G_1, by the projection $x = X(t, t_1, x_1, \xi)$ of a characteristic of equation (6.13) and from the values dX/dt along this curve we determine $U(t, t_1, x_1, \xi)$ from the equality $dX/dt = \phi_u(t, X, U)$. Thus, $u = U(t, t_1, x_1, \xi)$ and $x = X(t, t_1, x_1, \xi)$ is a solution of system (6.14), whose projection on the plane (t, x) connects the points (t_1, x_1) and $(0, \xi)$. As it is shown in the work [12], the function $U(t, t_1, x_1, \xi)$ is a continuous function of its arguments.

Consider the function

$$J(t, x, s) = \int_a^s [u_0(\xi) - U(0, t, x, \xi)]\, d\xi, \tag{6.15}$$

where (t, x) is any point of the region G, $a \le s \le b$. The function $J(t, x, s)$ has the following properties: for each point (t, x) of G the function $J(t, x, s)$, as a function of s, assumes the least value in the interior points of the segment $[a, b]$ (see [12], Lemma 2). Denote by $s_+(t, x)$ the least upper bound and by $s_-(t, x)$ the greatest lower bound of the set on which $J(t, x, s)$, as a function of s, assumes the least value.

It turns out that the generalized solution of equation (6.13) in the region G_1, assuming on $[a, b]$ the values $u_0(x)$, is the function

$$u(t, x) = U(t, t, x, s_+(t, x)).$$

The proof of this assertion is based on the following auxiliary propositions, proved in [12].

1. If $x < x_1$, so $s_+(t, x) \le s_-(t, x_1)$.
2. In each point (t_1, x_1) of the region G_1 one has the relations

$$\underline{\lim}_{(t,\,x)\to(t_1,\,x_1)} s_-(t, x) = s_-(t_1, x_1), \quad \overline{\lim}_{(t,\,x)\to(t_1,\,x_1)} s_+(t, x) = s_+(t_1, x_1),$$
$$s_-(t_1, x_1 - 0) = s_-(t_1, x_1), \quad s_+(t_1, x_1 + 0) = s_+(t_1, x_1).$$

3. If $J(t_0, x_0, s_0) = J(t_0, x_0, s_+(t_0, x_0))$, then in all points (t_1, x_1) the projections on the plane (t, x) of characteristic (6.14), connecting the points (t_0, x_0) and $(0, s_0)$, the equality

$$s_-(t_1, x_1) = s_+(t_1, x_1) = s_0$$

holds.

4. For all points (t_1, x_1) of the region G_1 and $a \leq s \leq b$, the equality

$$J(t_1, x_1, s) =$$
$$= \int_a^s u_0(\xi)\, d\xi + \int_s^{x_1} U(t, t_1, x_1, s)\, dx - \int_0^{t_1} \varphi(t, x, U(t, t_1, x_1, s))\, dt + F_1(t_1, x_1)$$

is true, where $F_1(t_1, x_1)$ is a certain function, depending only on x_1 and t_1, $x = X(t, t_1, x_1, s)$. This equality can also be written in the form

$$J(t_1, x_1, s) = \int u\, dx - \phi(t, x, u)\, dt + F_1(t_1, x_1),$$

where the line-integral is taken over the segment $[a, s]$ of the straight line $t = 0$ and the arc of the curve $x = X(t, t_1, x_1, s)$ from the point $(0, s)$ to (t_1, x_1), $u = u_0(t)$ for $t = 0$ and $u = U(t, t_1, x_1, s)$ on the remaining part of the contour.

With the help of these assertions one establishes in [12] the following properties of the function $u(t, x) = U(t, t, x, s_+(t, x))$.

Theorem 13. *The discontinuity points of the function $u(t, x) = U(t, t, x, s_+(t, x))$ are situated on lines; the x-coordinate of such a line is a unique and continuous function of t. The set of the discontinuity lines of the function $u(t, x)$ of which none is a part of another, is at most countable.*

Theorem 14. *In each discontinuity point of the function $u(t, x) = U(t, t, x, s_+(t, x))$ the limits $u(t, x+0)$ and $u(t, x-0)$ exist and the inequality $u(t, x-0) > u(t, x+0)$ holds.*

Theorem 15. *Let $x = x(t)$ be the equation of some discontinuity line of the function $u(t, x) = U(t, t, x, s_+(t, x))$. For each point $(t_1, x(t_1))$ on the discontinuity line we have the relation*

$$\frac{dx(t_1+0)}{dt} = \lim_{\substack{t_2 \to t_1 \\ t_2 > t_1}} \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\varphi(t_1, x(t_1), u(t_1, x(t_1) - 0)) - \varphi(t_1, x(t_1), u(t_1, x(t_1) + 0))}{u(t_1, x(t_1) - 0) - u(t_1, x(t_1) + 0)}$$

Theorem 16. *Let a certain function $u(t, x)$ have the properties:*

1. *If a point (t_1, x_1), of the region G_1 is 0-point of continuity of the function $u(t, x)$, then the characteristic, passing through the point $(t_1, x_1, u(t_1, x_1))$, belongs to $u(t, x)$ for $0 < t \leq t_1$. We say that a point (t^*, x^*, u^*) belongs to $u(t, x)$ if $u^* = u(t^*, x^*)$ and $u(t, x)$ is continuous in the point (t^*, x^*).*

2. *If a point (t_1, x_1) is a discontinuity point of the function $u(t, x)$, then at*

least two characteristics of (6.14) *can be found such that all their points for* $0 < t < t_1$ *belong to* $u(t, x)$, *and the projections of these characteristics pass through the point* (t_1, x_1).

3. *The equality*

$$\int \phi(t, x, u(t, x))\, dt - u(t, x)\, dx = 0$$

is fulfilled, where the integral is taken over the contour formed by the projections of any two characteristics, mentioned in Property 2, *and the straight line* $t = t_0$, *where* $0 \le t_0 < t_1$.

4. *The function* $u(t, x)$ *is bounded and measurable in* G_1 *and*

$$\lim_{\delta\to 0} \int_{x_1}^{x_2} [u_0(x) - u(\delta, x)]\, dx = 0$$

for any segment $[x_1, x_2]$, *belonging to* $[a, b]$.

Then $u(t, x) = U(t, t, x, s_+(t, x))$ *in the region* G_1.

Since the conditions, indicated in Theorem 16, define the function $U(t, t, x, s_+(t, x))$ **uniquely, so the generalized solution of equation (6.13) in the region** G_1, **assuming the values** $u_0(x)$ **on the segment** $[a, b]$, **can be defined as the function** $u(t, x)$, **satisfying the conditions of Theorem 16. Such a definition of a generalized solution of equation (6.13) is adapted in work[12].**

We cite, in addition, the following interesting properties of the function

$$u(t, x) = U(t, t, x, s_+(t, x)).$$

Theorem 17. *The function* $u(t, x)$ *satisfies the relation*

$$\int u\, dx - \phi(t, x, u(t, x))\, dt = 0, \tag{6.16}$$

where the integral is taken over the contour formed by the straight lines $t = t_0$, $t = 0$, *and the projections of the two characteristics, passing through the points* $(t_0, x_1, u(t_0, x_1))$ *and* $(t_0, x_2, u(t_0, x_2))$. *In this connection* (t_0, x_1) *and* (t_0, x_2) *are points of continuity of* $u(t, x)$.

Theorem 18. *Let* $f(t, x)$ *be a continuously differentiable function equal to zero outside a region* D, *whose boundary for* $t > 0$ *lies inside* G_1. *Then for the function* $u(t, x)$ *we have the relation*

$$\iint_{G_1} \left[\frac{\partial f}{\partial t} u + \frac{\partial f}{\partial x} \varphi(t, x, u)\right] dx\, dt + \int_a^b f(0, x)\, u_0(x)\, dx = 0.$$

This theorem is proved in exactly the same way as Theorem 8 in work [12]. It means that $u(t, x) = U(t, t, x, s_+(t, x))$ **satisfies condition 1) of the definition of the generalized solution of equation (6.13), mentioned in §2.**

Theorem 19. *The function* $u(t, x) = U(t, t, x, s_+(t, x))$ *in* G_1 *is a generalized solution of equation* (6.13), *assuming on the segment* $[a, b]$ *values of the function* $u_0(x)$. *If* $|u_0(x)| \le m$, *then in the region* G_1 $|u(t, x)| \le M$. *In the region bounded by the straight lines* $t = 0$, $t = T$, $x - a = At$, $x - b = -At$, $u(t, x)$ *coincides with the generalized solution of the Cauchy problem* (2.1), (2.2), *constructed in* §4. *Here* $A = \max|\phi_u|$ *for* $(t, x) \subset G$ *and* $|u| \le M$.

Proof. First we show that $|u(t, x)| \le M$. It is easy to prove that $|U(0, t, x, s_+(t, x))| \le m$. Indeed, the function $J(t, x, s) = \int_a^s [u_0(\xi) - U(0, t, x, \xi)]d\xi$ assumes the least value for $s = s_+(t, x)$. Therefore, for a positive and sufficiently small δ

$$\frac{1}{\delta} \int\limits_{s_+(t, x)}^{s_+(t, x)+\delta} [u_0(\xi) - U(0, t, x, \xi)]\, d\xi =$$

$$= \frac{1}{\delta} \int\limits_{s_+(t, x)}^{s_+(t, x)+\delta} u_0(\xi)\, d\xi - \frac{1}{\delta} \int\limits_{s_+(t, x)}^{s_+(t, x)+\delta} U(0, t, x, \xi)\, d\xi \geqslant 0. \qquad (6.17)$$

Since for $\delta \to 0$ the last integral tends to $U(0, t, x, s_+(t, x))$ by virtue of the continuity of this function, and $|u_0(\xi)| \le m$, so from (6.17) it follows that $U(0, t, x, s_+(t, x)) \le m$. Similarly one proves

$$U(0, t, x, s_+(t, x)) \ge -m.$$

Concerning the function ϕ_x, as it was pointed out in §3, we assume that $\max|\phi_x(t, x, u)| \le V(v)$ for $(t, x) \subset G$, $|u| \le v$ and

$$\int_m^M \frac{dv}{V(v) + \alpha} \ge T. \qquad (6.18)$$

For $t_1 = t$ the function $U(t, t, x, s_+(t, x))$ coincides with the solution of the equation

$$\frac{du}{dt_1} = -\phi_x(t_1, X(t_1, t, x, s_+(t, x)), u),$$

which for $t_1 = 0$ is equal to $U(0, t, x, s_+(t, x))$. From condition (6.18) it follows easily that

$$|U(t, t, x, s_+(t, x))| \le M.$$

As we already remarked, Theorem 18 means that for $u(t, x)$ condition 1) of the definition of a generalized solution, assuming given values $u_0(x)$ on the segment $[a, b]$, is fulfilled. We shall show now that condition 2) is fulfilled. We observe that for $x_1 > x_2$

$$\frac{U(t, t, x_1, s_+(t, x_1)) - U(t, t, x_2, s_+(t, x_2))}{x_1 - x_2} < \frac{U(t, t, x_1, s_+(t, x_2)) - U(t, t, x_2, s_+(t, x_2))}{x_1 - x_2}, \qquad (6.19)$$

since $U(t, t, x_1, s_+(t, x_2)) > U(t, t, x_1, s_+(t, x_1))$ by virtue of the monotoneity of the function ϕ_u. Consider the function

$$U(t, t, x, \eta),$$

when $(t, x) \subset G_1$ and $\eta \subset [a, b]$. It is evident that the right part of (6.19) does not exceed $\max \left| \frac{\partial U(t, t, x, \eta)}{\partial x} \right|$, when $\eta \subset [a, b]$ and the point (t, x) lies on a segment of the straight line $t = \text{const}$, belonging to G_1. The existence and continuity with respect to t, x, η for $t > 0$ of the derivative $\frac{\partial U(t, t, x, \eta)}{\partial x}$ can be easily proved in a way similar to that as one proved the differentiability with respect to s of the function $U(t, t_1, x_1, s)$ in the proof of Lemma 6 in work [12].

The function $U(t, t, x, s_+(t, x))$ coincides with the solution $u(t, x)$ of the Cauchy problem for equation (6.13) with condition (2.2), constructed in §4, in the region D_{ab} bounded by the straight lines $t = 0$, $t = T$, $x - a = At$, $x - b = -At$ on the basis of Theorem 2. Theorem 19 is proved.

Thus, we have shown that in the region D_{ab} the solution $u(t, x)$ of the Cauchy problem for equation (6.13) with condition (2.2), constructed in §4, can be defined in each point with the help of the integral $J(t, x, s)$ as the function $U(t, t, x, s_+(t, x))$. It is evident that the region D_{ab} contains any finite part of the region G, if the segment $[a, b]$ is chosen sufficiently large. Consequently, for a solution $u(t, x)$ of the Cauchy problem for equation (2.1) with condition (2.2) for $\psi(t, x, u) \equiv 0$, Theorems 13, 14, 15, pertaining to the properties of discontinuity points of the function $u(t, x)$, are true. The solution $u(t, x)$ is composed of characteristics (6.14), whose projections on the plane (t, x) intersect the axis x according to Theorem 16 and for $u(t, x)$ the integral relation (6.16) is fulfilled.

Many of these properties resemble the properties of functions describing the propagation of strong perturbations in a continuous medium, in particular, the properties of discontinuous solutions of equations of gas-dynamics.

§7. On the limit functions of the solutions of the boundary value problems for a parabolic equation when the coefficient ϵ in the highest order derivative tends to zero.

We consider the question of the behavior for $\epsilon \to 0$ of the solutions $u_\epsilon(t, x)$ of the equation

$$\varepsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \frac{\partial \varphi(t, x, u)}{\partial x} + \psi(t, x, u), \quad \varepsilon > 0, \tag{7.1}$$

defined in the rectangle $R\{0 \le t \le T,\ 0 \le x \le 1\}$ and satisfying the conditions

$$u_\epsilon(0, x) = u_0(x), \quad u_\epsilon(0, t) = u_1(t), \quad u_\epsilon(1, t) = u_2(t). \tag{7.2}$$

We shall assume that for the functions $\phi(t, x, u)$, $\psi(t, x, u)$, $u_0(x)$, $u_1(t)$, $u_2(t)$ the conditions of Theorem 1, proved in work [26], are fulfilled and, consequently, in the rectangle R there exists a solution $u_\epsilon(t, x)$ of equation (7.1), continuous in R together with the derivatives occurring in this equation, and satisfying conditions (7.2). Moreover, we shall assume that $\phi_{uu} \geq \mu > 0$.

First, we prove several lemmas, clarifying properties of the family $u_\epsilon(t, x)$ in the rectangle R.

Lemma 10. *In the rectangle* R

$$|u_\epsilon(t, x)| < M_1, \tag{7.3}$$

where the constant M_1 *does not depend on* ϵ.

Proof. Inequality (7.3) can be easily obtained using the generalized maximum principle for equation (7.1) and taking into account the condition of Theorem 1 of [26] which asserts that for all u and (t, x) in R

$$\frac{\partial}{\partial u}(\phi_x + \psi) \geq c$$

where c is a certain constant.

Lemma 11. *Let* $\phi_u(t, 0, u_1(t)) \geq \alpha_1 > 0$ *and* $\phi_u(t, 1, u_2(t)) \leq \alpha_2 < 0$ *for* $0 \leq t \leq T$. *Then*

$$\frac{\partial u_\epsilon}{\partial x} \leq E_1 \tag{7.4}$$

in R, *where the constant* E_1 *does not depend on* ϵ.

Proof. Let $\partial u_\epsilon/\partial x = v_\epsilon$. Differentiating equation (7.1) with respect to x, we obtain an equation for v_ϵ of the form

$$\varepsilon \frac{\partial^2 v_\varepsilon}{\partial x^2} = \frac{\partial v_\varepsilon}{\partial t} + \varphi_u(t, x, u_\varepsilon)\frac{\partial v_\varepsilon}{\partial x} + \varphi_{uu}(t, x, v_\varepsilon)v_\varepsilon^2 + A_\varepsilon(t, x)v_\varepsilon + B_\varepsilon(t, x), \tag{7.5}$$

where $A_\epsilon(t, x)$ and $B_\epsilon(t, x)$ are certain functions bounded for all (t, x) in R and all ϵ.

Let P be a point at which v_ϵ assumes the greatest value. If P lies inside R or on the straight line $t = T$, so from (7.5) it follows that

$$\phi_{uu}v_\epsilon^2 + A_\epsilon v_\epsilon + V_\epsilon \leq 0 \tag{7.6}$$

and since $\phi_{uu} \geq \mu > 0$, so

$$v_\epsilon < K_0, \tag{7.7}$$

where K_0 does not depend on ϵ. If P lies on the straight line $t = 0$, so

$$v_\varepsilon(P) = \frac{\partial u_\varepsilon}{\partial x} \leqslant \max_{0 \leqslant x \leqslant 1}\left|\frac{du_0(x)}{dx}\right|. \tag{7.8}$$

If P lies on the straight line $x = 0$, then in this point $\dfrac{\partial v_\epsilon}{\partial x} = \dfrac{\partial^2 u_\epsilon}{\partial x^2} \leq 0$ and from equation (7.1) it follows that in the point $P(t, 0)$

$$\frac{du_1(t)}{dt} + \varphi_x(t, 0, u_1(t)) + \varphi_u(t, 0, u_1(t))\frac{\partial u_\varepsilon}{\partial x} + \psi(t, 0, u_1(t)) \leqslant 0.$$

Since $\phi_u(t, 0, u_1(t)) \geq \alpha_1 > 0$, so in the point P

$$\frac{\partial u_\varepsilon}{\partial x} \leqslant \frac{-\psi(t, 0, u_1(t)) - \varphi_x(t, 0, u_1(t)) - \dfrac{du_1(t)}{dt}}{\alpha_1} \leqslant K_1, \tag{7.9}$$

where K_1 does not depend on ϵ. If P lies on the straight line $x = 1$, then in this point $\dfrac{\partial v_\epsilon}{\partial x} = \dfrac{\partial^2 u_\epsilon}{\partial x^2} \geq 0$ and

$$\frac{\partial u_\varepsilon}{\partial x} \leqslant \frac{\psi(t, 1, u_2(t)) + \varphi_x(t, 1, u_2(t)) + \dfrac{du_2(t)}{dt}}{-\alpha_2} \leqslant K_2, \tag{7.10}$$

where the constant K_2 also does not depend on ϵ.

From estimates (7.7), (7.8), (7.9), (7.10) it follows that

$$v_\epsilon = \frac{\partial u_\epsilon}{\partial x} < E_1$$

for all points (t, x) in R and all ϵ.

If one does not impose any conditions on $\phi_u(t, 0, u_1(t))$ and $\phi_u(t, x, u_2(t))$, the following weaker assertion is true.

Lemma 12. *In the rectangle* $R_\delta\{0 \leq t \leq T,\ 0 < \delta \leq x \leq 1 - \delta\}$

$$\frac{\partial u_\epsilon}{\partial x} < E_\delta, \tag{7.11}$$

where the constant E_δ *depends on* δ *and does not depend on* ϵ.

Proof. In equation (7.5) we introduce a new unknown function $\tilde{v}_\epsilon = v_\epsilon \cdot F_\delta(x)$, where $F_\delta(x)$ is a twice continuously differentiable function, satisfying the conditions: $F_\delta(x) = 1$ for $\delta \leq x \leq 1 - \delta$, $F_\delta(x) = 0$ for $0 \leq x \leq \delta/2$ and for $1 - \delta/2 \leq x \leq 1$, $0 \leq F_\delta(x) \leq 1$ for all x and $\dfrac{(F'_\delta(x))^2}{F_\delta(x)} < C_\delta$.

For $\tilde{v}_\epsilon(t, x)$ we obtain the equation

$$F_\delta(x)\varepsilon\frac{\partial^2\tilde{v}_\varepsilon}{\partial x^2} - F_\delta(x)\frac{\partial\tilde{v}_\varepsilon}{\partial t} - \varphi_u(t, x, u_\varepsilon)F_\delta(x)\frac{\partial\tilde{v}_\varepsilon}{\partial x} - 2\varepsilon F'_\delta(x)\frac{\partial\tilde{v}_\varepsilon}{\partial x} -$$
$$-\varphi_{uu}(t, x, u_\varepsilon)\tilde{v}_\varepsilon^2 + \left[2\varepsilon\frac{(F'_\delta(x))^2}{F_\delta(x)} - \varepsilon F''_\delta(x) + \varphi_u(t, x, u_\varepsilon)F'_\delta - A_\varepsilon(t, x)F_\delta(x)\right]\tilde{v}_\varepsilon -$$
$$-B_\varepsilon(t, x)(F_\delta(x))^2 = 0. \tag{7.12}$$

If $\tilde{v}_\epsilon$ assumes the greatest value on the sides Γ of the rectangle R, i.e., on the sides $x = 0$, $t = 0$, $x = 1$, then

$$\widetilde{v}_\varepsilon = F_\delta(x)\frac{\partial u_\varepsilon}{\partial x} \leqslant \max_{0\leqslant x\leqslant 1}\frac{\partial u_0(x)}{\partial x}\,. \tag{7.13}$$

If $\widetilde{v}_\epsilon$ assumes the greatest value inside R or for $t = T$, then in this point, as it follows from equation (7.12),

$$\varphi_{uu}\widetilde{v}_\varepsilon^2 - \left[2\varepsilon\frac{(F'_\delta)^2}{F_\delta} - \varepsilon F''_\delta + \varphi_u F'_\delta - A_\varepsilon F_\delta\right]\widetilde{v}_\varepsilon + B_\varepsilon(F_\delta)^2 \leqslant 0.$$

Since $\phi_{uu} \geq \mu > 0$, then the last inequality can be satisfied only for

$$\widetilde{v}_\epsilon \leq \widetilde{E}_\delta, \tag{7.14}$$

where $\widetilde{E}_\delta$ is a certain positive constant not depending on ϵ. From (7.13) and (7.14) it follows that inside of the rectangle R

$$\frac{\partial u_\varepsilon}{\partial x} \leqslant \max\left\{\widetilde{E}_\delta,\ \max_{0\leqslant x\leqslant 1}\frac{du_0(x)}{dx}\right\} = E_\delta,$$

which was to be proved.

For equation (7.1) with condition (7.2) we consider the following finite differences scheme. Let the straight lines $t = kh$, $x = nl$ $(k = 0, 1, \cdots, T/h;$ $n = 0, 1, 2, \cdots, 1/l)$ form a net of straight lines covering the rectangle R; $h > 0$ and $l > 0$ are such that T/h and $1/l$ are integers, $1/l$ is even and $\epsilon = l^2/2h$. For the points of intersection (kh, nl) $(0 < k \leq T/h;$ $0 < n < 1/l)$ of the net we consider the finite differences equations

$$\frac{l^2}{2h}\,\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{l^2} =$$
$$= \frac{u_n^{k+1} - u_n^k}{h} + \frac{\varphi(kh,\ (n+1)\,l,\ u_{n+1}^k) - \varphi(kh,\ (n-1)\,l,\ u_{n-1}^k)}{2l} + \psi(kh,\ (n+1)\,l,\ u_{n+1}^k) \tag{7.15}$$

with conditions

$$u_n^0 = u_0(nl), \quad u_0^k = u_1(kh), \quad u_{\frac{1}{l}}^k = u_2(kh). \tag{7.16}$$

It is evident that the finite differences equations (7.15) coincide with the finite differences equations (3.1). In exactly the same way as one has proved Lemma 1, one could show that for sufficiently small l

$$\left|u_n^k\right| < M_2 \tag{7.17}$$

in R, assuming that condition (3.3) is fulfilled. Everywhere in the following we shall suppose that $Ah/l < 1$, where $A = \max|\phi_u|$ for $(t, u) \subset R$ and $|u| \leq M_1 + M_2$.

Lemma 13. *If $l^2/2h = \epsilon > 0$ and $h \to 0$, then*

$$\Sigma \left|u_\epsilon(kh, nl) - u_n^k\right| 2l \to 0,$$

where the summation is taken over all n $(0 \le n \le 1/l)$ *for which* $k - n$ *is even.*

Proof. The function $u_\epsilon(t, x)$ has in R continuous derivatives occurring in equation (7.1). Therefore in the points (kh, nl) the function $u_\epsilon(kh, nl) = v_n^k$ satisfies the system of finite differences equation

$$\frac{l^2}{2h}\frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{l^2} = \frac{v_n^{k+1} - v_n^k}{h} + \frac{\varphi(kh, (n+1)l, v_{n+1}^k) - \varphi(kh, (n-1)l, v_{n-1}^k)}{2l} + \\ + \psi(kh, (n+1)l, v_{n+1}^k) + \sigma(t, x, h, l) \quad (7.18)$$

and the conditions: $v_n^0 = u_0(nl)$, $v_0^k = u_1(kh)$, $v_{\frac{1}{l}}^k = u_2(kh)$, where $\sigma(t, x, h, l)$ tends uniformly in R to zero for $h \to 0$.

Let us rewrite equations (7.15) and (7.18) in the form

$$u_n^{k+1} = \frac{1}{2}u_{n-1}^k + \frac{1}{2}u_{n+1}^k - \frac{h}{2l}\varphi(kh, (n+1)l, u_{n+1}^k) + \\ + \frac{h}{2l}\varphi(kh, (n-1)l, u_{n-1}^k) - \psi(kh, (n+1)l, u_{n+1}^k)h$$

and

$$v_n^{k+1} = \frac{1}{2}v_{n-1}^k + \frac{1}{2}v_{n+1}^k - \frac{h}{2l}\varphi(kh, (n+1)l, v_{n+1}^k) + \\ + \frac{h}{2l}\varphi(kh, (n-1)l, v_{n-1}^k) - \psi(kh, (n+1)l, v_{n+1}^k)h - \sigma\cdot h.$$

Let $w_n^k = v_n^k - u_n^k$. The function w_n^k satisfies the equation

$$w_n^{k+1} = \frac{1}{2}w_{n-1}^k + \frac{1}{2}w_{n+1}^k - \frac{h}{2l}\varphi_u(kh, (n+1)l, \theta_{n+1}^k)w_{n+1}^k + \\ + \frac{h}{2l}\varphi_u(kh, (n-1)l, \theta_{n-1}^k)w_{n-1}^k + \psi_u(kh, (n+1)l, \tilde{\theta}_{n+1}^k)hw_{n+1}^k - \sigma\cdot h \quad (7.19)$$

and the conditions

$$w_0^k = 0, \quad w_{\frac{1}{l}}^k = 0, \quad w_n^0 = 0,$$

where θ_n^k and $\tilde{\theta}_n^k$ are certain intermediate values between u_n^k and v_n^k.

We sum $\left|w_n^{k+1}\right|$ over all n such that $0 \le n \le 1/l$ and $(k + 1 - n)$ is even. Using (7.19), we get:

$$\sum_n |w_n^{k+1}|\, 2l \leqslant \sum_n |w_n^k|\, 2l + Bh\sum_n |w_n^k|\, 2l + h\tilde{\sigma}(h, l), \quad (7.20)$$

where $\tilde{\sigma} = \max_R(\sigma(t, x, h, l))$, $B = \max|\psi_u(t, x, u)|$ for $(t, x) \subset R$, $|u| \le M_1 + M_2$.

Since $\sum_n |w_n^0| 2l = 0$, so from (7.20) it follows that for $0 \le k \le T/h$

$$\sum_n |w_n^{k+1}| 2l \leqslant -\frac{\tilde{\sigma}}{B} + \frac{\tilde{\sigma}}{B} e^{B(k+1)h}, \quad \tilde{\sigma} \to 0 \quad \text{for} \quad h,\ l \to 0$$

and, consequently, $\sum_n |w_n^k| 2l \to 0$ uniformly in R for $h,\ l \to 0$ which was to be proved.

Let

$$V^k = \sum_n |u_{n+2}^k - u_n^k|$$

for even k and

$$V^k = \sum_n |u_{n+2}^k - u_n^k| + |u_1(kh) - u_1^k| + |u_2(kh) - u_{\frac{1}{l}-1}^k|$$

for k odd, where the summation is taken over all n $(0 \le n \le 1/l)$ for which $k - n$ is even. The following assertion is true.

Lemma 14. *For all $0 \le k \le T/h$*

$$V^k < \tilde{E}$$

where the constant $\tilde{E}$ does not depend on ϵ and h.

Proof. **Rewrite equations (7.15) in the form**

$$u_n^{k+1} = \left[\frac{1}{2} + \varphi_u(kh,\ (n+1)l,\ \bar{\theta}_n^k)\cdot\frac{h}{2l}\right] u_{n-1}^k +$$

$$+ \left[\frac{1}{2} - \varphi_u(kh,\ (n+1)l,\ \bar{\theta}_n^k\right] u_{n+1}^k - \psi(kh,\ (n+1)l,\ u_{n+1}^k)h - \varphi_x(kh,\ \zeta_n,\ u_{n-1}^k)h, \tag{7.21}$$

where $(n-1)l \le \zeta_n \le (n+1)l$, $\bar{\theta}_n^k$ are certain intermediate values between u_{n-1}^k and u_{n+1}^k. We introduce the notation

$$\tilde{u}_n^k = \left[\frac{1}{2} + \varphi_u(kh,\ (n+1)l,\ \bar{\theta}_n^k)\frac{h}{2l}\right] u_{n-1}^k + \left[\frac{1}{2} - \varphi_u(kh,\ (n+1)l,\ \bar{\theta}_n^k)\frac{h}{2l}\right] u_{n+1}^k. \tag{7.22}$$

It is evident that $\tilde{u}_k^n \subset [u_{n-1}^k,\ u_{n+1}^k]$. From formula (7.21) it follows that

$$u_n^{k+1} = \tilde{u}_n^k - \psi(kh,\ (n+1)l,\ u_{n+1}^k)h - \varphi_x(kh,\ \zeta_n,\ u_{n-1}^k)h. \tag{7.23}$$

Let $k+1$ be even. In this case, using (7.23), we get:

$$\sum_{n=0}^{\frac{1}{l}-2} |u_{n+2}^{k+1} - u_n^{k+1}| \leqslant \sum_{n=2}^{\frac{1}{l}-4} |\tilde{u}_{n+2}^k - \tilde{u}_n^k| + |\tilde{u}_2^k - u_1((k+1)h)| +$$

$$+|u_2((k+1)h)-\widetilde{u}^k_{\frac{1}{l}-2}|+\max|\psi_u|\cdot h\cdot\sum_{n=2}^{\frac{1}{l}-4}|u^k_{n+3}-u^k_{n+1}|+\max|\psi_x|h+$$

$$+\max|\varphi_{xx}|h+\max|\varphi_{xu}|\cdot h\sum_{n=2}^{\frac{1}{l}-4}|u^k_{n+1}-u^k_{n-1}|+|\psi(kh,\ 3l,\ u^k_3)|h+$$

$$+|\varphi_x(kh,\ \zeta_2,\ u^k_1)|h+\left|\psi\left(kh,\ \ 1-l,\ \ u^k_{\frac{1}{l}-1}\right)\right|h+|\varphi_x(kh,\ \zeta_{\frac{1}{l}-2},\ u^k_{\frac{1}{l}-3})|\,h.$$

Since $\widetilde{u}^k_n$ is a certain intermediate value between u^k_{n+1} and u^k_{n-1}, so from the last relation it follows that

$$V^{k+1}\leqslant V^k+|u_1((k+1)h)-u_1(kh)|+|u_2((k+1)h)-$$
$$-u_2(kh)|+A_1hV^k+A_2h, \qquad (7.24)$$

where the constants A_1 and A_2 do not depend on h and ϵ.

The functions $u_1(t)$ and $u_2(t)$ have continuous derivatives and therefore from (7.24) it follows that

$$V^{k+1}\leq V^k+A_1hV^k+A_2h. \qquad (7.25)$$

Now let $k+1$ be odd. In this case, using (7.23), we get:

$$\sum_{n=1}^{\frac{1}{l}-3}|u^{k+1}_{n+2}-u^{k+1}_n|+|u_1((k+1)h)-u^{k+1}_1|+|u_2((k+1)h)-u^{k+1}_{\frac{1}{l}-1}|\leqslant$$

$$\leqslant\sum_{n=1}^{\frac{1}{l}-3}|\widetilde{u}^k_{n+2}-\widetilde{u}^k_n|+\sum_{n=1}^{\frac{1}{l}-3}|u^k_{n+3}-u^k_{n+1}|\cdot\max|\psi_u|\cdot h+\max|\psi_x|h+\max|\varphi_{xx}|h+$$

$$+\sum_{n=1}^{\frac{1}{l}-3}|u^k_{n+1}-u^k_{n-1}|\cdot\max|\varphi_{xu}|\cdot h+|\widetilde{u}^k_1-u_1((k+1)h)|+|\widetilde{u}^k_{\frac{1}{l}-1}-u_2((k+1)h)|+$$

$$+|\psi(kh,\ 2l,\ u^k_2)|h+|\varphi_x(kh,\ \xi_1,\ u^k_0)|h+|\psi(kh,\ 1,\ u^k_{\frac{1}{l}})|h+|\varphi_x(kh,\ \xi_{\frac{1}{l}-1},\ u^k_{\frac{1}{l}-2})|\,h.$$

Hence, as in the case of $k+1$ even, it follows that

$$V^{k+1}\leq V^k+A_1hV^k+A_2h,$$

where A_1, A_2 are certain constants not depending on ϵ and h. Therefore for all k $(0\leq k\leq T/h)$

$$V^k\leq\left[V^0+\frac{A_2}{A_1}\right]e^{A_1kh}-\frac{A_2}{A_1}\leq\widetilde{E},$$

where $\widetilde{E}$ does not depend on ϵ and h.

Lemma 15. *For* $0 < \delta \leq x_1 < x_2 \leq 1 - \delta$, $\epsilon < \epsilon_0$ *and sufficiently small* $|t_1 - t_2|$

$$\int_{x_1}^{x_2} |u_\epsilon(t_1, x) - u_\epsilon(t_2, x)|\, dx \leq L_\delta |t_1 - t_2|^{\frac{1}{3}},$$

where L_δ *does not depend on* ϵ.

Proof. First we show that

$$\sum_{n=n_1}^{n_2} |u_n^k - u_n^p| \cdot 2l \leq L_\delta |kh - ph|^{\frac{1}{3}} \tag{7.26}$$

for $\delta \leq n_1 l < n_2 l \leq 1 - \delta$, sufficiently small $|kh - ph|$ and $k - p$ even $(k > p)$; the summation is carried out over such n so that $n - k$ is even.

The proof of relation (7.26) will be conducted in a way similar to the proof of Lemma 4.

Using formula (7.21) and conditions (7.16), we get for an even k

$$u_n^k = \sum_{j=2}^{n-2} a_{n,\,j}^{k,\,p}\, u_j^p + \sum_{r=p}^{k} a_{n,\,0}^{k,\,r}\, u_0^r + \sum_{r=p}^{k} a_{n,\,\frac{1}{l}}^{k,\,r}\, u_{\frac{1}{l}}^r + \gamma_{k,\,p}, \tag{7.27}$$

where

$$\gamma_{kp} \leqslant \gamma \cdot (k-p)h, \quad a_{n,\,j}^{k,\,p} = 0 \quad \text{for} \quad j > n + (k-p) \text{ and } j < n - (k-p), \quad a_{n,\,0}^{k,\,r} = 0$$

for $r > k - n$ and $a_{n,\,\frac{1}{l}}^{k,\,r} = 0$ for $r > k - \left(\frac{1}{l} - n\right)$, $\sum_{j=2}^{n-2} a_{n,\,j}^{k,\,p} + \sum_{r=p}^{k} a_{n,\,0}^{k,\,r} + \sum_{r=p}^{k} a_{n,\,\frac{1}{l}}^{k,\,r} = 1$;

the summation is carried out over j for $j - p$ even and over r for r even. Clearly,

$$a_{n,\,j}^{k,\,p} = a_{n,\,j+1}^{k,\,p+1}\left[\frac{1}{2} + \varphi_u(ph,\ (j+2)l,\ \bar{\theta}_{j+1}^p)\frac{h}{2l}\right] + \\ + a_{n,\,j-1}^{k,\,p+1}\left[\frac{1}{2} - \varphi_u(ph,\ jl,\ \bar{\theta}_{j-1}^p)\frac{h}{2l}\right],$$

if p is even, and $2 \leq j \leq 1/l - 2$, and if p is odd and $3 \leq j \leq 1/l - 3$. For an odd p

$$a_{n,\,1}^{k,\,p} = a_{n,\,2}^{k,\,p+1}\left[\frac{1}{2} + \varphi_u(ph,\ 3l,\ \bar{\theta}_2^p)\frac{h}{2l}\right],$$

$$a_{n,\,\frac{1}{l}-1}^{k,\,p+1} = a_{n,\,\frac{1}{l}-2}^{k,\,p+1}\left[\frac{1}{2} - \varphi_u\left(ph,\,(1-l),\,\bar{\theta}_{\frac{1}{l}-2}^p\right)\frac{h}{2l}\right].$$

For an even p

$$a_{n,\,0}^{k,\,p} = a_{n,\,1}^{k,\,p+1}\left[\frac{1}{2} + \varphi_u(ph,\ 2l,\ \bar{\theta}_1^p)\frac{h}{2l}\right],$$

$$a_{n,\,\frac{1}{l}}^{k,\,p} = a_{n,\,\frac{1}{l}-1}^{k,\,p+1}\left[\frac{1}{2} - \varphi_u(ph,\ 1,\ \bar{\theta}_{\frac{1}{l}-1}^p)\frac{h}{2l}\right].$$

Let

$$\bar{q}=\min_{\Omega}\left[\frac{1}{2}+\varphi_u(t,\ x,\ u)\frac{h}{2l}\right],\quad \tilde{q}=\max_{\Omega}\left[\frac{1}{2}+\varphi_u(t,\ x,\ u)\frac{h}{2l}\right],$$

where Ω are defined by the conditions: $\{|u|\leq M_2,\ (t,\ x)\subset R\}$.

We shall show that for any m

$$\begin{aligned}\bar{Q}^{k,\ p}_{n,\ m}&\equiv\sum_{j=0}^{m}\bar{a}^{k,\ p}_{n,\ j}+\sum_{r=p+1}^{k}\bar{a}^{k,\ r}_{n,\ 0}\leqslant Q^{k,\ p}_{n,\ m}\equiv\\&\equiv\sum_{j=0}^{m}a^{k,\ p}_{n,\ j}+\sum_{r=p+1}^{k}a^{k,\ r}_{n,\ 0}\leqslant\sum_{j=0}^{m}\tilde{a}^{k,\ p}_{n,\ j}+\sum_{r=p+1}^{k}\tilde{a}^{k,\ r}_{n,\ 0}\equiv\tilde{Q}^{k,\ p}_{n,\ m},\end{aligned}\tag{7.28}$$

where $\bar{a}^{k,p}_{n,j}\ [\tilde{a}^{k,p}_{n,j}]$ are coefficients which we obtain in formula (7.27) using finite differences scheme (7.21) and boundary conditions (7.16), if instead of $1/2+\phi_u(kh,\ (n+1)\,l,\ \bar{\theta}^k_n)\ h/2l$ and $1/2-\phi_u(kh,\ (n+1)l,\ \bar{\theta}^k_n)\ h/2l$ we take $\bar{q}$ and $1-\bar{q}$ [$\tilde{q}$ and $1-\tilde{q}$], respectively. Relations (7.28) can be easily proved by induction in a way similar to the proof of relation (3.22) in Lemma 4.

Indeed, summing over j for $p-j$ even and even r, we get, if p is even,

$$\begin{aligned}Q^{k,\ p}_{n,\ m}&=\sum_{j=0}^{m}a^{k,\ p}_{n,\ j}+\sum_{r=p+1}^{k}a^{k,\ r}_{n,\ 0}=\sum_{j=2}^{m}a^{k,\ p+1}_{n,\ j+1}\left[\frac{1}{2}+\varphi_u(ph,\ (j+2)\,l,\ \bar{\theta}^p_{j+1})\frac{h}{2l}\right]+\\&+\sum_{j=2}^{m}a^{k,\ p+1}_{n,\ j-1}\left[\frac{1}{2}-\varphi_u(ph,\ jl,\ \bar{\theta}^p_{j-1})\frac{h}{2l}\right]+\\&+\sum_{r=p+1}^{k}a^{k,\ r}_{n,\ 0}+a^{k,\ p+1}_{n,\ 1}\left[\frac{1}{2}+\varphi_u(ph,\ 2l,\ \bar{\theta}^p_1)\frac{h}{2l}\right]=\\&=\sum_{j=1}^{m-1}a^{k,\ p+1}_{n,\ j}+\sum_{r=p+1}^{k}a^{k,\ r}_{n,\ 0}+a^{k,\ p+1}_{n,\ m+1}\left[\frac{1}{2}+\varphi_u(ph,\ (m+2)\,l,\ \bar{\theta}^p_{m+1})\frac{h}{2l}\right]=\\&=Q^{k,\ p+1}_{n,\ m-1}+(Q^{k,\ p+1}_{n,\ m+1}-Q^{k,\ p+1}_{n,\ m-1})\left[\frac{1}{2}+\varphi_u(ph,\ (m+2)\,l,\ \bar{\theta}^p_{m+1})\frac{h}{2l}\right]=\\&=Q^{k,\ p+1}_{n,\ m-1}\left[\frac{1}{2}-\varphi_u(ph,\ (m+2)\,l,\ \bar{\theta}^p_{m+1})\frac{h}{2l}\right]+\\&\qquad+Q^{k,\ p+1}_{n,\ m-1}\left[\frac{1}{2}+\varphi_u(ph,\ (m+2)\,l,\ \bar{\theta}^p_{m+1})\frac{h}{2l}\right].\end{aligned}$$

The same equality is valid also for odd p.

Assuming that (7.28) is true for $p+1$, from the last relation we find that

$$\begin{aligned}Q^{k,\ p}_{n,\ m}&\leqslant\tilde{Q}^{k,\ p+1}_{n,\ m-1}\left[\frac{1}{2}-\varphi_u(ph,\ (m+2)\,l,\ \bar{\theta}^p_{m+1})\frac{h}{2l}\right]+\\&+\tilde{Q}^{k,\ p+1}_{n,\ m+1}\left[\frac{1}{2}+\varphi_u(ph,\ (m+2)\,l,\ \bar{\theta}^p_{m+1})\frac{h}{2l}\right]\leqslant\tilde{Q}^{k,\ p+1}_{n,\ m-1}+\end{aligned}$$

$$+(\widetilde{Q}^{k,\,p+1}_{n,\,m+1}-\widetilde{Q}^{k,\,p+1}_{n,\,m-1})\left[\frac{1}{2}+\varphi_u(ph,\ (m+2)\,l,\ \theta^p_{m+1})\frac{h}{2l}\right]\leqslant$$
$$\leqslant\widetilde{Q}^{k,\,p+1}_{n,\,m-1}+(\widetilde{Q}^{k,\,p+1}_{n,\,m+1}-\widetilde{Q}^{k,\,p+1}_{n,\,m-1})\,\widetilde{q}=\widetilde{Q}^{k,\,p}_{n,\,m}.$$

In exactly the same way we find that $Q^{k,p}_{n,m} \geq \bar{Q}^{k,p}_{n,m}$ and we get relation (7.28) for an odd p.

Now we estimate $\sum_{j=m_1}^{m_2} a^{k,p}_{n,j}$, where $0 < m_1 < m_2 < 1/l$. Clearly

$$\sum_{j=m_1}^{m_2} a^{k,\,p}_{n,\,j}=Q^{k,\,p}_{n,\,m_2}-Q^{k,\,p}_{n,\,m_1}\geqslant\bar{Q}^{k,\,p}_{n,\,m_2}-\widetilde{Q}^{k,\,p}_{n,\,m_1}. \tag{7.29}$$

Therefore it suffices to estimate $\widetilde{Q}^{k,p}_{n,m}$ and $\bar{Q}^{k,p}_{n,m}$.

Consider a motion of a particle $\bar{S}\,[\widetilde{S}]$ on the segment $[0, 1]$ for which the particle, which is in the point with the coordinate $x = nl$ at the initial moment, moves to the left by l in one time-unit with the probability $\bar{q}[\widetilde{q}]$ and to the right by l with the probability $1-\bar{q}\,[1-\widetilde{q}\,]$ under the condition that the particle is absorbed, if it falls into the point with the coordinate $x = 0$ or $x = 1$. Evidently, $\bar{a}^{k,p}_{n,j}[\widetilde{a}^{k,p}_{n,j}]$ is the probability of the event that after $k-p$ units of time the coordinate of the particle is equal to jl, and $\bar{Q}^{k,p}_{n,m}\,[\,\widetilde{Q}^{k,p}_{n,m}\,]$ is the probability of the event that the coordinate of the particle does not exceed ml.

At the same time we consider the motion of a particle $\bar{R}[\widetilde{R}]$ on a straight line for which a particle, being at the initial moment in the point with the coordinate nl, moves in a unit of time to the left by l with probability $\bar{q}[\widetilde{q}]$ and to the right by l with probability $1-\bar{q}[1-\widetilde{q}\,]$. Let $\bar{P}^{k,p}_{n,m}\,[\widetilde{P}^{k,p}_{n,m}]$ be the probability of the event that after $k-p$ units of time the coordinate of the particle $\bar{R}[\widetilde{R}]$ does not exceed ml. When proving Lemma 4 we have shown that

$$\left.\begin{aligned}
\widetilde{P}^{k,\,p}_{n,\,m}&<\widetilde{D}\,\frac{l^2}{h}\,[(k-p)\,h]^{\frac{1}{3}}, \quad \text{if} \quad lm<ln-\max_{\Omega}\varphi_u\,(k-p)\,h-[(k-p)\,h]^{\frac{1}{3}},\\
1-\widetilde{P}^{k,\,p}_{m,\,n}&<\widetilde{D}\,\frac{l^2}{h}\,[(k-p)\,h]^{\frac{1}{3}}, \quad \text{if} \quad lm>ln-\max_{\Omega}\varphi_u\,(k-p)\,h+[(k-p)\,h]^{\frac{1}{3}},
\end{aligned}\right\} \tag{7.30}$$

where $\widetilde{D}$ is a certain constant. Similar estimates are valid also for $\bar{P}^{k,p}_{n,m}$. We shall use these inequalities for estimating $\bar{Q}^{k,p}_{n,m}$ and $\widetilde{Q}^{k,p}_{n,m}$. Evidently

$$\widetilde{P}^{k,\,p}_{n,\,m_2}-\widetilde{P}^{k,\,p}_{n,\,m_1}=\widetilde{Q}^{k,\,p}_{n,\,m_2}-\widetilde{Q}^{k,\,p}_{n,\,m_1}+\sum_{i=1}^{k-p}\widetilde{P}^n_i\widetilde{W}, \tag{7.31}$$

where $\widetilde{P}^n_i$ is the probability of the fact that the particle $\widetilde{R}$, moving on the straight line as we indicated above, and which is at the initial moment in the point with the coordinate nl, after i units of time will return for the first time into the point with the coordinate $x = 0$ or $x = 1$, and $\widetilde{W}_i$ is the probability of the event that this particle, being in the point with the coordinate $x = 0$ or $x = 1$, after $k - p - i$ units of time will fall into the segment $[lm_1, lm_2]$.

From (7.31) we have

$$\widetilde{Q}^{k,p}_{n,m_2} - \widetilde{Q}^{k,p}_{n,m_1} \geqslant \widetilde{P}^{k,p}_{n,m_2} - \widetilde{P}^{k,p}_{n,m_1} - \max_{1 \leqslant i \leqslant k-p} \{\widetilde{W}_i\}, \tag{7.32}$$

since $\sum_{i=1}^{k-p} \widetilde{P}^n_i \leq 1$.

We observe that we consider such n so that $\delta \leq nl \leq 1 - \delta$. We shall assume that $(k - p)h$ is so small that

$$h(k - p) \cdot \max_{\Omega} |\phi_u| + [(k - p)h]^{\frac{1}{3}} < \frac{\delta}{2}. \tag{7.33}$$

Now we estimate $\max_i \{\widetilde{W}_i\}$. If the particle $\widetilde{R}$ after i units of time has returned into the point $x = 0$, then the probability of the fact that this particle $\widetilde{R}$ after $k - p - i$ units of time would fall into the segment $[m_1 l, m_2 l]$ is less than $1 - \widetilde{P}^{k,p+i}_{0,m_1}$. If the condition $lm_1 > -h(k - p) \cdot \max_{\Omega} \phi_u + [(k - p)h]^{1/3}$ is fulfilled, then from (7.30) it follows that

$$1 - \widetilde{P}^{k,p+i}_{0,m_1} < \widetilde{D}\frac{l^2}{h}[(k-(p+i))h]^{\frac{1}{3}} \leqslant \widetilde{D}\frac{l^2}{h}[(k-p)h]^{\frac{1}{3}}. \tag{7.34}$$

If the particle $\widetilde{R}$ after i units of time has returned into the point $x = 1$, then the probability of the fact that this particle $\widetilde{R}$ after $k - p - i$ units of time would fall into the segment $[m_1 l, m_2 l]$ is less than $\widetilde{P}^{k,p+i}_{1/l,m_2}$. If $lm_2 < 1 - h(k - p) \cdot \max_{\Omega} \phi_u - [(k - p)h]^{1/3}$, then from (7.30) it follows that

$$\widetilde{P}^{k,p+i}_{\frac{1}{l},m_2} < \widetilde{D}\frac{l^2}{h}[k-(p+i)h]^{\frac{1}{3}} \leqslant \widetilde{D}\frac{l^2}{h}[(k-p)h]^{\frac{1}{3}}.$$

Thus, $\max_i \{\widetilde{W}_i\} \leq 2\widetilde{D}\frac{l^2}{h}[(k - p)h]^{1/3}$ if

$$lm_1 > -h(k-p)\max_{\Omega}\varphi_u + [(k-p)h]^{\frac{1}{3}}$$

and

$$lm_2 < 1 - h(k-p)\max_{\Omega}\varphi_u - [(k-p)h]^{\frac{1}{3}}.$$

Let

$$\frac{\delta}{2} \leqslant lr_1 \leqslant ln - h(k-p)\max_{\Omega}\varphi_u - [(k-p)h]^{\frac{1}{3}} = lR^n_1,$$

$$1 - \frac{\delta}{2} \geqslant lr_2 \geqslant ln - h(k-p)\max_{\Omega}\varphi_u + [(k-p)h]^{\frac{1}{3}} = lR^n_2.$$

Then from inequalities (7.32) and (7.30) it follows that

$$\widetilde{Q}^{k,\,p}_{n,\,r_2} - \widetilde{Q}^{k,\,p}_{n,\,r_1} \geqslant \widetilde{P}^{k,\,p}_{n,\,r_2} - \widetilde{P}^{k,\,p}_{n,\,r_1} - 2\widetilde{D}\frac{l^2}{h}[(k-p)\,h]^{\frac{1}{3}} =$$

$$= 1 - (1 - \widetilde{P}^{k,\,p}_{n,\,r_2}) - \widetilde{P}^{k,\,p}_{n,\,r_1} - 2\widetilde{D}\frac{l^2}{h}[(k-p)\,h]^{\frac{1}{3}} \geqslant 1 - 4\widetilde{D}\frac{l^2}{h}[(k-p)\,h]^{\frac{1}{3}}$$

and

$$1 - (\widetilde{Q}^{k,\,p}_{n,\,r_2} - \widetilde{Q}^{k,\,p}_{n,\,r_1}) \leqslant 4\widetilde{D}\frac{l^2}{h}[(k-p)\,h]^{\frac{1}{3}}.$$

Analogous estimates hold for $\bar{Q}^{k,p}_{n,m}$. Therefore, using (7.29) and the last inequalities, we get:

$$\sum_{R_1^n < j < R_2^n} a^{k,\,p}_{n,\,j} \geqslant \bar{Q}^{k,\,p}_{n,\,R_2^n} - \widetilde{Q}^{k,\,p}_{n,\,R_1^n} = \widetilde{Q}^{k,\,p}_{n,\,R_2^n} - \widetilde{Q}^{k,\,p}_{n,\,R_1^n} +$$

$$+ (1 - \widetilde{Q}^{k,\,p}_{n,\,R_2^n}) - (1 - \bar{Q}^{k,\,p}_{n,\,R_2^n}) > 1 - (8\widetilde{D} + 4\bar{D})\frac{l^2}{h}[(k-p)\,h]^{\frac{1}{3}}. \qquad (7.35)$$

With the help of estimate (7.35) the proof of inequality (7.26) is carried out in exactly the same way as one has proved Lemma 4 with the help of estimates (3.27). In this connection the use of the assertion of Lemma 14 is essential.

In order to obtain the assertion of Lemma 15, we observe that

$$\int_{x_1}^{x_2} |u_\varepsilon(t_1,\,x) - u_\varepsilon(t_2,\,x)|\,dx \leqslant \sum_{n=n_1}^{n_2} |u_n^k - u_n^p|\cdot 2l + \int_{x_1}^{x_2} |u_\varepsilon(t_1, x) - U^k(x)|\,dx +$$

$$+ \int_{x_1}^{x_2} |u_\varepsilon(t_2,\,x) - U^p(x)|\,dx,$$

where $hk = t_1$, $hp = t_2$, $U^k(x)$ is the function equal u_n^k for $nl \le x < (n+2)\,l$. According to Lemma 13 the last two integrals tend to zero for $h \to 0$. From this and from (7.36) follows Lemma 15.

Lemma 16.* *The functions $u_\epsilon(t, x)$ form in R a compact family, i.e., from any sequence $u_\epsilon(t, x)$ $(\epsilon \to 0)$ one can select a subsequence $\{u_\epsilon(t, x)\}$ such that*

$$\int_0^1 |u_\varepsilon(t,\,x) - u(t,\,x)|\,dx \to 0 \quad \textit{for} \quad \varepsilon \to 0 \qquad (7.36)$$

and

$$\iint_R |u_\varepsilon(t, x) - u(t,\,x)|\,dx\,dt \to 0 \quad \textit{for} \quad \varepsilon \to 0, \qquad (7.37)$$

where $u(t, x)$ is a certain measurable in R function and $|u(t, x)| < M_1$.

---

*Lemma 16, as well as Theorem 3, is valid also without the hypothesis $\phi_{uu} > 0$.

Proof. The assertion of Lemma 16 is proved with the help of Lemmas 12 and 15 in a way similar to that of the proof of Theorem 3 in §3.

Lemma 17. *The function* $u(t, x)$, *obtained in Lemma* 16, *satisfies the relation*

$$\iint\limits_R \left[\frac{\partial f}{\partial t} u + \frac{\partial f}{\partial x} \varphi(t, x, u) - f\psi(t, x, u)\right] dx\, dt + \int_0^1 f(0, x) u_0(x)\, dx = 0 \quad (7.38)$$

for any continuously differentiable function $f(t, x)$ *equal to zero in a neighborhood of the straight lines* $t = T$, $x = 0$, $x = 1$.

Proof. Let us multiply equation (7.1) by $f(t, x)$ and integrate over R. Integrating by parts, we find that

$$\iint\limits_R \left[\varepsilon \frac{\partial^2 f}{\partial x^2} u_\varepsilon + \frac{\partial f}{\partial t} u_\varepsilon + \frac{\partial f}{\partial x} \varphi(t, x, u_\varepsilon) - f\psi(t, x, u_\varepsilon)\right] dx\, dt +$$

$$+ \int_0^1 f(0, x) u_0(x)\, dx = 0. \quad (7.39)$$

Passing to the limit for $\epsilon \to 0$ in relation (7.39), on the basis of Lemma 16, we get (7.38).

Lemma 18. *Let* $\phi_u(t, 0, u_1(t)) \geq \alpha_1 > 0$, $\phi_u(t, 1, u_2(t)) \leq \alpha_2 < 0$, $\phi_u(t, x, u_\epsilon(t, x)) \geq 0$ *for* $0 \leq x \leq \delta$ *and* $\phi_u(t, x, u_\epsilon(t, x)) \leq 0$ *for* $1 - \delta \leq x \leq 1$, *if* ϵ *is sufficiently small and* $\delta > 0$ *is a certain number. Then the function* $u(t, x)$, *obtained in Lemma* 16, *satisfies the relation*

$$= \iint\limits_R \left[\frac{\partial f}{\partial t} u + \frac{\partial f}{\partial x} \varphi(t, x, u) - f\psi(t, x, u)\right] dx\, dt +$$

$$+ \int_0^1 f(0, x) u_0(x)\, dx + \int_0^T f(t, 0) \varphi(t, 0, u_1(t))\, dt - \int_0^T f(t, 1) \varphi(t, 1, u_2(t))\, dt \quad (7.40)$$

for any continuously differentiable function $f(t, x)$ *equal to zero for* $t = T$.

Proof. We multiply (7.1) by $f(t, x)$, integrate over R and transform by integrating by parts. We get:

$$\iint\limits_R \left[\varepsilon \frac{\partial^2 f}{\partial x^2} u_\varepsilon + \frac{\partial f}{\partial t} u_\varepsilon + \frac{\partial f}{\partial x} \varphi(t, x, u_\varepsilon) - f\psi(t, x, u_\varepsilon)\right] dx\, dt + \int_0^1 f(0, x) u_0(x)\, dx +$$

$$+ \int_0^T f(t, 0) \varphi(t, 0, u_1(t))\, dt - \int f(t, 1) \varphi(t, 1, u_2(t))\, dt + \varepsilon \int_0^T \frac{\partial u_\varepsilon(t, 1)}{\partial x} f\, dt -$$

$$-\varepsilon\int_0^T \frac{\partial u_\varepsilon(t,0)}{\partial x} f\,dt + \varepsilon\int_0^T u_\varepsilon \frac{\partial f(t,0)}{\partial x}\,dt - \varepsilon\int_0^T u_\varepsilon \frac{\partial f(t,1)}{\partial x}\,dt = 0.$$

Since $|u_\epsilon(t,x)| < M_1$, so to prove Lemma 18 it suffices to show that the integrals

$$\varepsilon\int_0^T \frac{\partial u_\varepsilon(t,0)}{\partial x} f(t,0)\,dt \quad\text{and}\quad \varepsilon\int_0^T \frac{\partial u_\varepsilon(t,1)}{\partial x} f(t,1)\,dt \tag{7.41}$$

tend to zero for $\epsilon \to 0$. In Lemma 11 it was shown that with our assumptions concerning $u_1(t)$ and $u_2(t)$ the inequality

$$\frac{\partial u_\epsilon}{\partial x} < E_1 \text{ in } R \tag{7.42}$$

holds.

Let the function $f(t,0) \geq 0$ on the segment $0 \leq t_1 \leq t \leq t_2 \leq T$ and let $0 < x' < \delta_1 < \delta$. We multiply equation (7.1) by $f(t,x)$ and integrate over the rectangle $Q'\{t_1 \leq t \leq t_2,\ 0 \leq x \leq x'\}$. We have:

$$\int_{t_1}^{t_2} \varepsilon f(t,x') \frac{\partial u_\varepsilon(t,x')}{\partial x}\,dt - \int_{t_1}^{t_2} \varepsilon f(t,0) \frac{\partial u_\varepsilon(t,0)}{\partial x}\,dt =$$

$$= \int_{t_1}^{t_2} f(t,0)\,[\varphi(t,x',u_\varepsilon(t,x')) - \varphi(t,0,u_1(t))]\,dt + O(\varepsilon,\delta_1).$$

Hence it follows that

$$\int_{t_1}^{t_2} \varepsilon f(t,x') \frac{\partial u_\varepsilon(t,x')}{\partial x}\,dt - \int_{t_1}^{t_2} \varepsilon f(t,0) \frac{\partial u_\varepsilon(t,0)}{\partial x}\,dt \leqslant c_1(\varepsilon+\delta_1) +$$

$$+ \int_{t_1}^{t_2} f(t,0)\,\varphi_u(t,x',\tilde{u}_\varepsilon)\,[u_\varepsilon(t,x') - u_1(t)]\,dt, \tag{7.43}$$

where $\tilde{u}_\epsilon$ is a certain intermediate value between $u_\epsilon(t,x')$ and $u_1(t)$, c_1 is a certain constant. For sufficiently small x' the function $\phi_u(t,x',\tilde{u}_\epsilon) \geq 0$, since $\phi_u(t,x',u_1(t)) \geq \alpha_1/2$ by virtue of the continuity of the function ϕ_u and by the hypothesis that $\phi_u(t,x',u_\epsilon(t,x')) \geq 0$ and $\phi_{uu} > 0$. Therefore, using (7.42) to estimate the last integral in (7.43), we get

$$\int_{t_1}^{t_2} \varepsilon f(t,0) \frac{\partial u_\varepsilon(t,0)}{\partial x}\,dt \geqslant \int_{t_1}^{t_2} \varepsilon f(t,x') \frac{\partial u_\varepsilon(t,x')}{\partial x}\,dt - c_2(\varepsilon+\delta_1).$$

Integrating this inequality with respect to x' from zero to δ_1 and transforming the double integral by integration by parts, we get

$$\delta_1 \int_{t_1}^{t_2} \varepsilon f(t,\ 0) \frac{\partial u_\varepsilon(t,\ 0)}{\partial x} dt \geqslant - \int_0^{\delta_1} \int_{t_1}^{t_2} \varepsilon \frac{\partial f}{\partial x} u_\varepsilon \, dx \, dt +$$

$$+ \int_{t_1}^{t_2} \varepsilon [f(t,\ \delta_1) u_\varepsilon(t,\ \delta_1) - f(t,0) u_1(t)] \, dt - c_2 \delta_1 (\varepsilon + \delta_1) \geqslant -c_2 \delta_1 (\varepsilon + \delta_1) + c_3 \varepsilon. \quad (7.44)$$

From inequality (7.42) it follows that

$$\int_{t_1}^{t_2} \varepsilon f(t,\ 0) \frac{\partial u_\varepsilon(t,\ 0)}{\partial x} dt < \varepsilon \widetilde{M} E_1 (t_2 - t_1). \quad (7.45)$$

Since δ_1 is arbitrary, so from (7.44) and (7.45) it follows that

$$\int_{t_1}^{t_2} \epsilon f(t,\ 0) \frac{\partial u_\epsilon(t,\ 0)}{\partial x} dt$$

tends to zero for $\epsilon \longrightarrow 0$.

Thus, if $f(t,\ 0)$ and $f(t,\ 1)$ change the sign on the segment $[0,\ T]$ a finite number of times, so integrals (7.41) tend to zero for $\epsilon \longrightarrow 0$, and for such twice continuously differentiable functions $f(t,\ x)$ the equality (7.40) holds, if $f(T,\ x) = 0$. Since any continuously differentiable in R function, together with the first derivatives, can be approximated uniformly by functions of this class, so by passing to the limit in (7.40), we find that this equality holds for all required functions. Lemma 18 is proved.

Lemma 19. *The function $u(t,\ x)$, obtained in Lemma 16, (if, possibly, we change it on a set of measure zero), satisfies the relation*

$$\frac{u(t,\ x_1) - u(t,\ x_2)}{x_1 - x_2} < E_1, \quad (7.46)$$

if the assumptions of Lemma 11 are fulfilled, and the relation

$$\frac{u(t,\ x_1) - u(t,\ x_2)}{x_1 - x_2} < K(t,\ x_1,\ x_2) \quad (7.47)$$

in the general case, where the function $K(t,\ x_1,\ x_2)$ is continuous, when $0 < x_1 < 1$, $0 < x_2 < 1$, $t \geq 0$.

Proof. From (7.11) and Helly's theorem [25] it follows that a certain subsequence $u_\epsilon(t,\ x)$ converges in each point of R on the straight line $t = \text{const}$ to a certain function which coincides with $u(t,\ x)$ almost everywhere. Therefore, relations (7.46) and (7.47) are consequences of inequalities (7.4) and (7.11).

Lemmas 17 and 19 show that the function $u(t,\ x)$, obtained in Lemma 16, is a generalized solution of equation (2.1) in R assuming on $[0,\ 1]$ values of the function $u_0(x)$ (see §2). We give now conditions which determine such a solution uniquely.

Lemma 20. *A bounded and measurable in R function $U(t, x)$, satisfying relations* (7.38), (7.47) *and the conditions*

$$\varphi_u(t, x, U(t, x)) \leqslant \beta_1 < 0, \quad \text{if} \quad 0 \leqslant x \leqslant \delta,$$
$$\varphi_u(t, x, U(t, x)) \geqslant \beta_2 > 0, \quad \text{if} \quad 1-\delta \leqslant x \leqslant 1,$$

for certain β_1, β_2 and $\delta > 0$, is unique.

Proof. Lemma 20 can be proved in a way similar to that of the proof of Theorem 1 on the uniqueness of a solution of the Cauchy problem (2.1), (2.2).

Suppose that there exist in R two functions $U_1(t, x)$ and $U_2(t, x)$ satisfying the conditions of Lemma 20. We let U_1 and U_2 be equal to zero outside R and consider the averaged functions U_1^h and U_2^h. It is evident that in the rectangle $R_\eta\{0 < \eta \leq x \leq 1-\eta,$ $0 \leq t \leq T\}$ (η is an arbitrary number)

$$\frac{\partial U_1^h}{\partial x} < K_\eta \text{ and } \frac{\partial U_2^h}{\partial x} < K_\eta. \tag{7.48}$$

As in the proof of Theorem 1, we consider in R equation (2.6). The equation of the characteristics of this equation has the form

$$\frac{dx}{dt} = \frac{\varphi(t, x, U_1^h) - \varphi(t, x, U_2^h)}{U_1^h - U_2^h} = \int_0^1 \varphi_u(t, x, U_1^h + \tau(U_2^h - U_1^h))\, d\tau. \tag{7.49}$$

We shall show that for $\eta \leq x \leq \delta/2$, sufficiently small h and $0 \leq \tau \leq 1$

$$\varphi_u(t, x, U_1^h + \tau(U_2^h - U_1^h)) \leqslant \frac{\beta_1}{2} < 0, \tag{7.50}$$

and for $1 - \delta/2 \leq x \leq 1 - \eta$, sufficiently small h and $0 \leq \tau \leq 1$

$$\varphi_u(t, x, U_1^h + \tau(U_2^h - U_1^h)) \geqslant \frac{\beta_2}{2} > 0. \tag{7.51}$$

Indeed, since $\phi_u(t, x, u)$ is continuous, one can find a number $\rho > 0$ so small that $\phi_u(t', x', U_1(t, x)) \leq \beta_1/2$ and $\phi_u(t', x', U_2(t, x)) \leq \beta_1/2$, if $\eta \leq x \leq \delta/2$ and $|t - t'| < \rho$, $|x - x'| < \rho$. Therefore, $\phi_u(t', x', u) \leq \beta_1/2$ for all u, equal $U_1(t, x)$ and $U_2(t, x)$ when (t, x) varies in the ρ-neighborhood of the point (t', x'). Consequently, $\phi_u(t, x, U_1^h(t, x)) \leq \beta_1/2$ and $\phi_u(t, x, U_2^h(t, x)) \leq \beta_1/2$ for sufficiently small h and $\eta \leq x \leq \delta/2$, and condition (7.50) is fulfilled by virtue of the monotoneity of ϕ_u. Similarly one proves (7.51).

Let $F(t, x)$ be equal to zero in a neighborhood of the boundary of R. Then there exists a solution $f(t, x)$ of equation (2.6), equal to zero in a certain neighborhood of the straight lines $t = T$, $x = \eta$, $x = 1 - \eta$ (η is sufficiently small). This follows from the fact that the characteristics of (7.49), by virtue of conditions (7.50) and (7.51), enter into the rectangle R_η through these straight lines and come out through the side $t = 0$. From conditions (7.48) it follows that in R_η for all h and a fixed ρ $|\partial f/\partial x| < R_\eta$. With the help of the function $f(t, x)$ constructed in this way, the proof of Lemma 20 is carried out in exactly the same fashion as that of Theorem 1.

Lemma 21. *A bounded and measurable in R function $V(t, x)$, satisfying conditions* (7.40) *and* (7.46), *is unique.*

Proof. Suppose that there exist two functions V_1 and V_2 satisfying (7.40) and (7.46). We extend these functions to all (t, x) by setting $V_i(t, x) = V_i(t, 0)$ for $x \leq 0$ and $V_i(t, x) = V_i(t, 1)$ for $x \geq 1$ $(i = 1, 2)$. Then by virtue of condition (7.46)

$$\frac{\partial V_1^h}{\partial x} < K \text{ and } \frac{\partial V_2^h}{\partial x} < K.$$

Using this, we can show in exactly the same way as in the proof of Theorem 1 that

$$\iint_R (V_1 - V_2) F(t, x)\, dx\, dt = 0$$

for any continuously differentiable function $F(t, x)$, from which it follows that $V_1 = V_2$ in R.

Theorem 20. *The solutions $u_\epsilon(t, x)$ of equation* (7.1), *satisfying conditions* (7.2), *converge in R for $\epsilon \to 0$ to the function $U(t, x)$ in the sense that*

$$\iint_R |u_\varepsilon(t, x) - U(t, x)|\, dx\, dt \to 0 \textit{ and } \int_0^1 |u_\varepsilon(t, x) - U(t, x)|\, dx\, dt \to 0 \textit{ for } \varepsilon \to 0, \tag{7.52}$$

if for sufficiently small ϵ, for a certain $\delta > 0$ and an arbitrarily small $\eta > 0$,

$$\left.\begin{aligned} &\varphi_u(t, x, u_\varepsilon(t, x)) \leqslant \beta_1(\eta) < 0 \quad \textit{for} \quad \eta \leqslant x \leqslant \delta, \\ &\varphi_u(t, x, u_\varepsilon(t, x)) \geqslant \beta_2(\eta) > 0 \quad \textit{for} \quad 1 - \delta \leqslant x \leqslant 1 - \eta. \end{aligned}\right\} \tag{7.53}$$

The function $U(t, x)$ satisfies conditions of Lemma 20 *in any*

$$R_\eta\{\eta \leq x \leq 1 - \eta,\ 0 \leq t \leq T\}.$$

Proof. According to Lemma 16, from the family of functions $u_\epsilon(t, x)$ one can select a subsequence convergent in the sense of (7.36) and (7.37) for $\epsilon \to 0$. From Lemmas 17, 19 and condition (7.53) it follows that the limiting function satisfies in R all conditions of Lemma 20 and therefore is defined uniquely. From the uniqueness of the limiting function it follows that any sequence $u_\epsilon(t, x)$ converges to this function for $\epsilon \to 0$ in the sense of (7.52).

Thus, if $u_\epsilon(t, x)$ satisfies the conditions mentioned in Theorem 20, then the limiting function for $u_\epsilon(t, x)$ for $\epsilon \to 0$ does not depend on the functions $u_1(t)$ and $u_2(t)$ and is determined uniquely by $u_0(x)$.

Theorem 21. *The solutions* $u_\epsilon(t, x)$ *of equation* (7.1) *satisfying conditions* (7.2), *converge in* R *for* $\epsilon \to 0$ *to a function* $V(t, x)$, *defined by the conditions of Lemma* 21 *in the sense that*

$$\left.\begin{aligned} &\iint_R |u_\varepsilon(t, x) - V(t, x)|\, dx\, dt \to 0 \\ \text{and} \quad &\int_0^1 |u_\varepsilon(t, x) - V(t, x)|\, dx \to 0 \end{aligned}\right\} \text{ for } \varepsilon \to 0, \tag{7.54}$$

if for sufficiently small ϵ *and for a certain* $\delta > 0$

$$\left.\begin{aligned} &\varphi_u(t, x, u_\varepsilon(t, x)) \geqslant \alpha_1 > 0 \quad \text{for} \quad 0 \leqslant x \leqslant \delta, \\ &\varphi_u(t, x, u_\varepsilon(t, x)) \leqslant \alpha_2 < 0 \quad \text{for} \quad 1 - \delta \leqslant x \leqslant 1. \end{aligned}\right\} \tag{7.55}$$

Proof. By virtue of condition (7.55) and Lemmas 18 and 19, the limiting function $u(t, x)$ constructed in Lemma 16, satisfies all conditions of Lemma 21 and, consequently, coincides with $V(t, x)$. From the uniqueness of the limiting function follows the convergence of the entire sequence $u_\epsilon(t, x)$ for $\epsilon \to 0$ in the sense of (7.54).

We give some sufficient conditions under which the hypotheses of Theorems 20 and 21 are fulfilled. Let Γ be the sides $x = 0$, $x = 1$, $t = 0$ of the rectangle R.

Lemma 22. *For* $u_\epsilon(t, x)$ *for sufficiently small* ϵ *and a certain* $\delta > 0$ *the conditions* (7.53) *are fulfilled, if there exist in* R *twice continuously differentiable solutions* $u_1(t, x)$ *and* $u_2(t, x)$ *of equation* (2.1) *such that*

$$u_1(t, x) \geqslant u_\varepsilon(t, x) \text{ on } \Gamma \text{ and } \phi_u(t, x, u_1(t, x)) \leqslant 2\beta_1 < 0 \quad \text{for} \quad 0 \leqslant x \leqslant \delta,$$

and

$$u_2(t, x) \leqslant u_\varepsilon(t, x) \text{ on } \Gamma \text{ and } \phi_u(t, x, u_2(t, x)) \geqslant 2\beta_2 > 0 \quad \text{for} \quad 1 - \delta \leqslant x \leqslant 1.$$

Proof. Denote $w_1 = u_1 - u_\epsilon$. The function w_1 satisfies the equation

$$\varepsilon \frac{\partial^2 w_1}{\partial x^2} = \frac{\partial w_1}{\partial t} + \varphi_u(t, x, u_\varepsilon) \frac{\partial w_1}{\partial x} + \left(\frac{\partial u_1}{\partial x} \varphi_{uu} + \varphi_{xu} + \psi_u \right) w_1 + \varepsilon \frac{\partial^2 u_1}{\partial x^2}. \tag{7.56}$$

In equation (7.56) we make the transformation $w_1 = e^{\alpha t} W_1$ and choose $\alpha > 0$ so large that in the obtained equation for W_1 the coefficient of W_1 would be greater than one. By hypothesis $W_1 \geq 0$ on Γ. Therefore, if W_1 assumes negative values, then the negative minimum is assumed in the interior of R. In this point

$$\left(\alpha + \frac{\partial u_1}{\partial x} \varphi_{uu} + \varphi_{xu} + \psi_u \right) e^{\alpha t} W_1 + \varepsilon \frac{\partial^2 u_1}{\partial x^2} \geqslant 0$$

and

$$W_1 \geqslant \frac{-\varepsilon \dfrac{\partial^2 u_1}{\partial x^2} e^{-\alpha t}}{\left(\alpha + \dfrac{\partial u_1}{\partial x} \varphi_{uu} + \varphi_{xu} + \psi_u \right)} \geqslant -\varepsilon \mu_1, \qquad w_1 \geqslant -\varepsilon \mu_2.$$

Therefore everywhere in R we have $u_1 + \epsilon\mu_2 \geq u_\epsilon$. Since $\phi_u(t, x, u_1) \leq 2\beta_1 < 0$ for $0 \leq x \leq \delta$, so for sufficiently small ϵ the function $\phi_u(t, x, u_1 + \mu_2\epsilon) \leq \beta_1$ and by virtue of the condition $\phi_{uu} > 0$, the function $\phi_u(t, x, u_\epsilon(t, x)) \leq \beta_1$ for $0 \leq x \leq \delta$.

With the help of the function $u_2(t, x)$, one shows in a similar fashion that for $u_\epsilon(t, x)$ the second condition of (7.53) is fulfilled.

Lemma 23. *For $u_\epsilon(t, x)$, for sufficiently small ϵ and a certain $\delta > 0$ the conditions* (7.55) *are fulfilled, if there exist in R twice continuously differentiable solutions $v_1(t, x)$ and $v_2(t, x)$ of equation* (2.1) *such that*

$$v_1(t, x) \leqslant u_\varepsilon(t, x) \text{ on } \Gamma \text{ and } \phi_u(t, x, v_1(t, x)) \geqslant 2\alpha_1 > 0 \quad \text{for} \quad 0 \leqslant x \leqslant \delta,$$

and

$$v_2(t, x) \geqslant u_\varepsilon(t, x) \text{ on } \Gamma \text{ and } \phi_u(t, x, v_2(t, x)) \leqslant 2\alpha_2 < 0 \quad \text{for} \quad 1-\delta \leqslant x \leqslant 1.$$

The proof of Lemma 23 is carried out in exactly the same fashion as the proof of Lemma 22.

Theorem 22. *If the functions $u_1(t)$ and $u_2(t)$ are such that $\phi_u(t, 0, u_1(t)) \leq \beta_1 < 0$ and $\phi_u(t, 1, u_2(t)) \geq \beta_2 > 0$, then the solutions $u_\epsilon(t, x)$ of equation* (7.1) *with conditions* (7.2) *converge for $\epsilon \to 0$ in the sense of* (7.52) *to the function $U(t, x)$ defined in Lemma* 20.

We shall not give here the detailed proof of Theorem 22. We point out only that under the conditions of this theorem, in any rectangle

$$Q_\sigma\{0 \leq x \leq 1, \quad t_0 \leq t \leq t_0 + \sigma\},$$

if σ is sufficiently large, one can construct functions $u_1(t, x)$ and $u_2(t, x)$ satisfying in Q_σ the conditions of Lemma 22 and in this way prove that conditions (7.53) of Theorem 20 are fulfilled. We remark that if $\phi(t, x, u) \equiv \phi(u)$, $\psi(t, x, u) \equiv 0$, $\phi_u \to \pm\infty$ for $u \to \pm\infty$, then in the rectangle Q_σ one can set $u_1(t, x) = \Phi\left[\frac{x - x_1}{t - t_1}\right]$ and $u_2(t, x) = \Phi\left[\frac{x - x_2}{t - t_2}\right]$, where $\Phi(u)$ is the inverse function of ϕ_u, and x_1, $t_0 - t_1$, $1 - x_2$, $t_0 - t_2$ are positive and sufficiently small. Analogously one can construct the functions u_1 and u_2 also in the general case.

With the help of Lemma 23 one can give a series of sufficient conditions for $u_0(x)$, $u_1(t)$, $u_2(t)$ in order that the functions $u_\epsilon(t, x)$ converge for $\epsilon \to 0$ in R to $V(t, x)$.

Similarly one can consider the case when for $u_\epsilon(t, x)$ in a neighborhood of the straight line $x = 0$ a condition of the form (7.53), and in a neighborhood of the straight line $x = 0$, a condition of the form (7.55) is fulfilled.

The question on the behavior of $u_\epsilon(t, x)$ is studied in greater detail in a different way in the work [11]. The limiting functions $U(t, x)$ and $V(t, x)$ have properties similar to those which we established for the solution of the Cauchy problem (2.1), (2.2). If $\psi(t, x, u) \equiv 0$, so these functions can be defined in each point by means of finding the minima of certain integrals in a way similar to that as one has constructed the solutions of the Cauchy problem of equation (6.13) in §6 (see [11] and [21]).

BIBLIOGRAPHY

[1] L. D. Landau and E. M. Lifšic, *Mechanics of continuous media*, 2nd ed., GITTL, Moscow, 1953. (Russian)

[2] I. G. Petrovskiĭ, *Lectures on the theory of ordinary differential equations*, GITTL, Moscow, 1952. (Russian)

[3] S. L. Sobolev, *Some applications of functional analysis in mathematical physics*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950. (Russian)

[4] R. Courant and K. Friedrichs, *Supersonic flow and shock waves*, Interscience, New York, 1948.

[5] E. Hopf, *The partial differential equation* $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. 3 (1950), 201–230.

[6] I. G. Petrovskiĭ, *Lectures on partial differential equations*, 2nd ed., GITTL, Moscow, 1953; German transl., Teubner, Leipzig, 1955.

[7] P. Lax, *Weak solutions of nonlinear hyperbolic equations and their numerical computation*, Comm. Pure Appl. Math. 7 (1954), 159–193.

[8] J. M. Burgers, *Application of a model system to illustrate some points of the statistical theory of free turbulence*, Nederl. Akad. Wetensch. Proc. 43 (1940), 2–12.

[9] O. A. Oleĭnik, *On the Cauchy problem for nonlinear equations in a class of discontinuous functions*, Dokl. Akad. Nauk SSSR 95 (1954), 451–455. (Russian)

[10] ———, *On the Cauchy problem for nonlinear equations in a class of discontinuous functions*, Uspehi Mat. Nauk 9 (1954), no. 3 (61), 231–233. (Russian)

[11] ———, *Boundary value problems for partial differential equations involving a small parameter in the highest order derivatives and the Cauchy problem for nonlinear equations in the large*, Ibid. 10 (1955), no. 3 (65), 229–234. (Russian)

[12] ———, *The Cauchy problem for nonlinear differential equations of the first order with discontinuous initial conditions*, Trudy Moskov. Mat. Obšč. 5 (1956), 433–454. (Russian)

[13] P. Lax, *The initial value problem for nonlinear hyperbolic equations in two independent variables*, Contributions to the theory of partial differential equations, pp. 211–229, Princeton Univ. Press, Princeton, N. J., 1954.

[14] J. von Neumann and R. D. Richtmyer, *A method for numerical calculation of hydrodynamic shocks*, J. Appl. Phys. 21 (1950), 232–237.

[15] A. N. Tihonov and A. A. Samarskiĭ, *On discontinuous solutions of a quasi-linear equation of the first order,* Dokl. Akad. Nauk SSSR 99 (1954), 27–30. (Russian)

[16] S. K. Godunov, *On the uniqueness of a solution of hydrodynamic equations,* Mat. Sb. (N.S) 40 (82) (1956), 467–478. (Russian)

[17] O. A. Oleĭnik, *On discontinuous solutions of non-linear differential equations,* Dokl. Akad. Nauk SSSR 109 (1956), 1098–1101. (Russian)

[18] N. D. Vvedenskaja, *Solution of the Cauchy problem for a non-linear equation with discontinuous initial conditions by the method of finite differences,* Dokl. Akad. Nauk SSSR 111 (1956), 517–520. (Russian)

[19] ———, *Application of the finite differences method to the construction of generalized solutions of non-linear equations,* Dissertation, Math. Inst., Academy of Sciences of the USSR, Moscow, 1957. (Russian)

[20] O. A. Ladyženskaja, *On construction of discontinuous solutions of quasi-linear hyperbolic equations as a limit of solutions of the corresponding parabolic equations when the "viscosity coefficient" tends to zero,* Dokl. Akad. Nauk SSSR 111 (1956), 291–294. (Russian)

[21] O. A. Oleĭnik and N. D. Vvedenskaja, *Solution of the Cauchy problem and the boundary value problem for the quasi-linear equations in a class of discontinuous solutions,* Ibid. 113 (1957), 503–506. (Russian)

[22] A. F. Filippov and V. S. Rjaben'skiĭ, *On stability of difference equations,* GITTL, Moscow, 1956. (Russian)

[23] N. S. Bahvalov, *Conditions for convergence and the order of error in solving the Cauchy problem for a linear equation of the first order by the finite differences method,* Prikl. Mat. Meh. 20 (1956), 279–283. (Russian)

[24] B. V. Gnedenko, *Course in theory of probabilities,* GITTL, Moscow, 1950.

[25] I. P. Natanson, *Theory of functions of a real variable,* GITTL, Moscow, 1950.

[26] O. A. Oleĭnik and T. D. Ventcel', *The first boundary value problem and the Cauchy problem for quasi-linear equations of the parabolic type,* Mat. Sb. (N.S.) 41 (83) (1957), 105–128. (Russian)

[27] V. I. Smirnov, *Course of higher mathematics,* Vol. V, OGIZ, Moscow, 1947.

[28] S. L. Sobolev, *Equations of mathematical physics,* GITTL, Moscow, 1954.

[29] Giovanni Sansone, *Equazioni differenziali nel campo reale,* Vol. 2, 2nd ed., Zanichelli, Bologna, 1949.

Translated by:

George Biriuk