THE USE OF THE RIEMANN PROBLEM IN FINITE DIFFERENCE SCHEMES

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I INTRODUCTION

Given an arbitrary set of hyperbolic conservation laws for an unknown vector $\underline{u}$ (x,t)

$$\underline{u}_t + \underline{F}(\underline{u})_x = 0 \tag{1}$$

the Riemann problem is to solve (1) for the special case of initial data

$$\underline{u}\ (x,0) = \underline{u}_L\ (x < 0);\ \underline{u}\ (x,0) = \underline{u}_R\ (x > 0) \tag{2}$$

A general solution to the Riemann problem serves to encapsulate a great deal of information about whatever physical system is modelled by (1). For that reason, many authors have felt it to be an attractive "building block" for the construction of numerical methods, especially those intended to provide for the automatic handling of discontinuities. There have been two main lines of development. Godunov's (1959) method has recently been extended to second order by van Leer (1979), and Glimm (1965) devised a method with an element of random choice which makes shockwaves precise in their structure, but uncertain in their location. In comparison with more straightforward methods, both strategies prove expensive. The present work leads to methods which are cheaper, more flexible, accurate, and conservative. The penalty paid for these advantages is some weakening of the theoretical foundations. The individual Riemann problems which arise are solved by a direct, non-iterative, method which is not in general exact, but which does have certain special properties built into it.

II THE LINEAR SCALAR CASE

We begin by studying the simplest possible example of (1), namely the linear advection equation, $u_t + au_x = 0$, where u is a scalar unknown, and a is a constant. This has the analytic solution $u = f(x - at)$. We suppose that the solution is sought on a regularly spaced grid $x = i\Delta x$, $t = n\Delta t$. Any linear two-level difference scheme for this problem can be put into the form

$$u_i^{n+1} = \sum_k c_k\, u_{i+k}^n \tag{3}$$

Godunov showed such a scheme to be p^{th} order accurate if the coefficients c_k satisfy

$$\sum_k k^j c_k = (-\nu)^j \qquad j = 0,1,\ \ldots p \tag{4}$$

where ν is the Courant number, $\nu = a\Delta t/\Delta x$. It is very easy to put existing algorithms into the form (3), or to generate new ones by finding solutions of (4). The starting point for the present work is the observation that any scheme of the form (3) can be recast in an "increment form"

$$u_i^{n+1} - u_i^n = -\nu \sum_k \gamma_k\, (u_{i+k+1}^n - u_{i+k}^n) \tag{5}$$

Here the γ_k are new coefficients, related to the c_k, which must likewise satisfy a set of linear constraints so as to produce specified accuracy. Specifically,

$$\Sigma\gamma_k = 1 \; ; \; \Sigma k\gamma_k = (1+\nu)/2 \; ; \; \Sigma k^2\gamma_k = (1+\nu)(1+2\nu)/6 \text{ etc.} \qquad (6)$$

When put into the form (5), most well-known methods have a simple representation, for example

TABLE 1

	γ_0	γ_1	γ_2	γ_3
First order upwind	0	1	0	0
Lax-Wendroff (1960)	$(1-\nu)/2$	$(1+\nu)/2$	0	0
Fromm (1968)	$(1-\nu)/4$	1	$(\nu-1)/4$	0
Warming-Beam (1976)	0	$(3-\nu)/2$	$(\nu-1)/2$	0
Moretti (1979)	0	$(3-2\nu)/2$	$(3\nu-1)/2$	$-\nu/2$

What is more significant about (5), however, is that the method of implementing it gives rise to concepts which can be generalised fruitfully. For each interval (i, i+1) in turn we compute the quantity $-\nu(u^n_{i+1} - u^n_i)$, and then divide it into fractions proportional to γ_k (where $\Sigma\gamma_k = 1$). The k^{th} fraction is added as an increment to the mesh point (i+k). The process is illustrated in Fig 1. In moving from (3) to (5) there is a change in emphasis. Equation (3) answers the question "How does one find the solution at a given mesh point?". Equation (5) answers the question "What should one do with the data in a given mesh interval?". This change in emphasis foreshadows the subsequent emergence of the Riemann problem as crucial to the method.

III THE NON-LINEAR SCALAR CASE

The same strategy can be readily applied to solve the non-linear scalar equation $u_t + F_x = 0$ where F is an arbitrary non-linear function of u. One begins by defining, for each interval, a local Courant number $\nu_{i+\frac{1}{2}} = (\Delta F\Delta t)/(\Delta u\Delta x)$. One selects an algorithm, perhaps from Table 1, and computes local weights, $\gamma_{k,i+\frac{1}{2}} = \gamma_k(\nu_{i+\frac{1}{2}})$. Then the quantity $-\nu_{i+\frac{1}{2}}\,\Delta u$ is split into fractions proportional to $\gamma_{k,i+\frac{1}{2}}$, and the k^{th} fraction is added as an increment to the mesh point (i+k). It can easily be checked that the total of all increments added at all mesh points from all sources is exactly that required to maintain conservation. It is less easy to verify that the accuracy of the non-linear algorithm is the same as that of the linear version, but in all cases checked so far this has proved to be the case.

IV SETS OF LINEAR EQUATIONS

We now extend the method to deal with sets of linear equations $\underline{u}_t + A\,\underline{u}_x = 0$ where $\underline{u}$ is now a vector of m unknowns, and A is a constant m x m matrix with a complete set

of real eigenvectors, $\underline{e}_1$, $\underline{e}_2$, ... $\underline{e}_m$. The general analytic solution of the initial-value problem is well known, and consists of projecting $\underline{u}$ (x,0) onto the eigenvectors thus

$$\underline{u}(x,0) = \sum_{j=1}^{m} f_j(x)\ \underline{e}_j \tag{7}$$

then

$$\underline{u}(x,t) = \sum_{j=1}^{m} f_j(x-\lambda_j t)\underline{e}_j \tag{8}$$

where λ_j is the eigenvalue of A corresponding to $\underline{e}_j$. Thus, once the decomposition of the data onto eigenvectors has been accomplished, the solution, whether analytical or numerical, can proceed exactly as in the scalar case. A way of implementing the method is illustrated in Fig 2. In each interval, the measured vector difference $\Delta\underline{u}$ is split up into components associated with each wave system, ie $\Delta\underline{u} = \Sigma a_i \underline{e}_i = \Sigma(\Delta\underline{u})_i$. (In making this decomposition, we are of course solving the linear Riemann problem.) Weights $\gamma_{k,j}$ are computed from $\gamma_{k,j} = \gamma_k(\nu_j)$ and increments $\Delta\underline{u}_{k,j} = \gamma_{k,j}\ (\Delta\underline{u})_j$ are added at the mesh point (i+k). In this linear problem, all the wave systems are independent, and all properties of accuracy, stability, etc, are therefore carried over unchanged from the scalar case. It is worth mentioning, however, that there is no need to use the same algorithm for each wave system.

V SETS OF NON-LINEAR EQUATIONS

We now combine the two preceding generalisations and study the non-linear system $\underline{u}_t + \underline{F}_x = 0$ where $\underline{u}$ is again a vector of m unknowns, but $\underline{F}$ is now an arbitrary vector function of $\underline{u}$. It is also convenient to write

$$\underline{u}_t + A\underline{u}_x = 0 \tag{9}$$

where the m x m matrix A is the Jacobian $\partial F/\partial u$, and its elements are no longer constants, but functions of $\underline{u}$. We still assume, of course, that A always has a complete set of real eigenvectors.

The general strategy will be to combine the previous lines of attack. Thus, in each interval we find components of $\Delta\underline{u}$, $\Delta\underline{F}$ due to each wave system

$$\Delta\underline{u} = \Sigma a_i \underline{e}_i \ ; \ \Delta\underline{F} = \Sigma \lambda_i a_i \underline{e}_i \tag{10a; b}$$

where, although the notation is as in Section IV, the eigenvalues λ and eigenvectors $\underline{e}$ are no longer constants but locally defined in each interval. Multiplication of each λ_j by $\Delta t/\Delta x$ produces a local Courant number ν_j appropriate to the j^{th} wave. Reference to Table 1 produces weights $\gamma_{k,j}$ which can be used to distribute the increment $\Delta t\ \Delta\underline{F}_j/\Delta x$ over the mesh.

What we have so far left unspecified is the precise means by which the eigenvalues and eigenvectors are to be estimated. To provide such estimates we need to solve, at least approximately, the non-linear Riemann problem. We could of course use the

exact solution but that would impose extra computing costs which do not seem to be justified by the results. Instead, we seek an approximation having the following properties

i As $\underline{u}_{i+1} \to \underline{u}_i$, then (for all j) λ_j and $\underline{e}_j$ tend to their linearised values. This is an obvious necessity for accuracy in smooth parts of the flow.

ii We require that λ_j, $\underline{e}_j$, a_j are such that both (10a) and (10b) are satisfied exactly for arbitrary Δu. This is rather a loose requirement - it gives 2 m equations in $m^2 + m$ unknowns, hence m(m - 1) degrees of freedom. However, two rather nice properties flow from it.

The first is that since (10b) is satisfied exactly then the total sum of all increments will be such as to ensure conservation, as in Section III. The second property arises whenever the states $\underline{u}_i$, $\underline{u}_{i+1}$ can be connected by a single shock or contact discontinuity, ie whenever $\Delta\underline{F} = S\Delta\underline{u}$ for some scalar S. In that case, provided the $\underline{e}_j$ are linearly independent, we must have, for some s, $S = \lambda_s$, and $\Delta\underline{u} = \underline{e}_s$. Thus, in this case, the method will return the exact solution.

In combination, I call properties (i) and (ii) Property U. It ensures accuracy in the two situations most commonly encountered; that is to say the flow is either smooth, or close to a single discontinuity. Where several discontinuities intersect, the present approximation has no special virtue (other than conservation) but in practice the error seems highly localised. Working out an approximation having Property U for a given set of equations is not difficult in principle, but may involve tedious algebra. A technique which partly overcomes this has been devised, and will be published elsewhere. Here we merely note the results for the unsteady Euler equations in one space dimension. The linearised eigenvectors and eigenvalues for that problem are

$$\underline{e}_1 = \begin{pmatrix} 1 \\ u-a \\ H-ua \end{pmatrix}, \quad \underline{e}_2 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix}, \quad \underline{e}_3 = \begin{pmatrix} 1 \\ u+a \\ H+ua \end{pmatrix} \tag{11}$$

$$\lambda_1 = u-a \;, \quad \lambda_2 = u \;, \quad \lambda_3 = u+a \tag{12}$$

where $H = \frac{\gamma p}{(\gamma-1)\rho} + \frac{1}{2}u^2$ and $a^2 = (\gamma-1)\left[H-\frac{1}{2}u^2\right]$ (13a; b)

Evidently the first part of Property U is ensured by taking u, a, H to be any valid averages over the interval. The second part can be achieved by defining those averages more particularly. The simplest so far discovered are

$$u_{i+\frac{1}{2}} = \frac{(\rho^{\frac{1}{2}}u)_i + (\rho^{\frac{1}{2}}u)_{i+1}}{\rho_i^{\frac{1}{2}} + \rho_{i+1}^{\frac{1}{2}}}; \quad H_{i+\frac{1}{2}} = \frac{(\rho^{\frac{1}{2}}H)_i + (\rho^{\frac{1}{2}}H)_{i+1}}{\rho_i^{\frac{1}{2}} + \rho_{i+1}^{\frac{1}{2}}} \tag{14a; b}$$

and $a_{i+\frac{1}{2}}$ is obtained by substituting (14a) and (14b) into (13b).

VI THE PROBLEM OF SPURIOUS OSCILLATIONS

It was first proved by Godunov that any numerical method of the form (3), having second-order accuracy or better, would produce an oscillating response to step data. Since (5) is merely (3) rewritten, nothing so far said contributes directly to the removal of these "wiggles". However, the techniques which we have described allow us to identify wiggles as originating with a particular wave system. This seems to permit control mechanisms rather more delicate than those in currect use. Mechanisms such as "artificial viscosity" or "flux limitation" act to suppress wiggles without acknowledging the information which they carry. Although space does not allow it to be described here, a new mechanism for the control of wiggles has been developed, based on non-linear combinations of the algorithms in Table 1. Details will be published elsewhere, but the results produced on a standard test problem are shown in the next section.

Alternatively, one accepts wiggles but seeks to reduce them. Almost invariably this approach relies on directionally biased formulae which maintain the physically correct flow of information, eg Fromm (1968), Steger and Warming (1978), Moretti (1979), Leonard (1979). The present work is very compatible with these ideas.

VII NUMERICAL RESULTS

Results have been obtained for the shock tube problem used as a test case by Sod (1978). When the upwind algorithm from Table 1 was used in conjunction with the approximate Riemann solution, the results agreed to 4 or 5 significant figures with those from the standard Godunov method employing the exact Riemann solution. This confirms the opinion that the exact solution is not needed. The results in Fig 3 were obtained by choosing weights corresponding to Fromm's algorithm. They compare well with most of the results surveyed by Sod, especially when it is realised that they are obtained without any recourse to artificial viscosity. Fig 4 shows results obtained from the control mechanism, mentioned in Section VI. These results show no overshoots, and are third-order accurate in smooth regions.

VIII FURTHER DEVELOPMENTS

The material above has concentrated on explicit two-level algorithms for one-dimensional problems. However, the approach of analysing the data as a set of approximate Riemann problems appears to be very widely applicable. Both multilevel schemes, and implicit schemes, have been developed in the same way. In these cases also, it is possible to begin with the linear advection equation and go through a similar process of generalisation to treat non-linear systems. The extension to multidimensional problems is possible via operator splitting, and has been applied to practical transonic airfoil calculations by Sells (1980).

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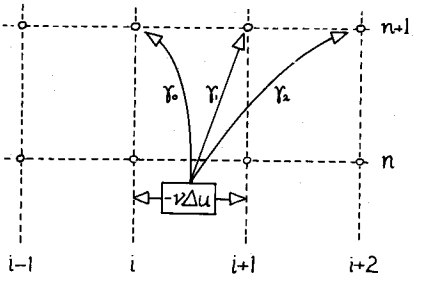

Fig 1 Dispersal of scalar increments

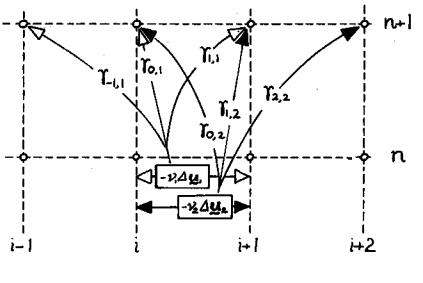

Fig 2 Dispersal of vector increments

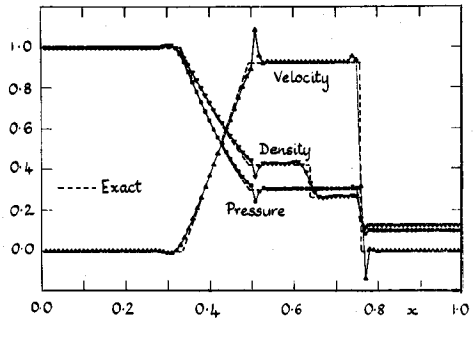

Fig 3 Results from Fromm's algorithm

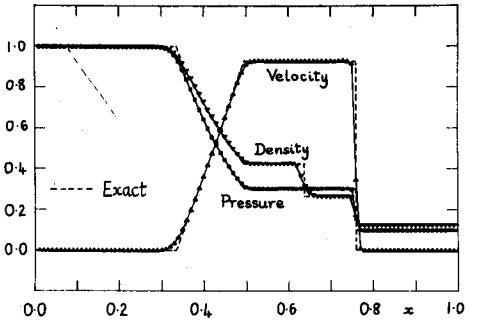

Fig 4 Results from optimised algorithm