Conservation and Stability

(Your name here)

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Consider a 1D system of conservation laws:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = 0, \ \mathbf{u} \in \mathbb{R}^K, \ \mathbf{f} \in \mathbb{R}^K. \tag{1}$$

1. Consider a 1^{st} order finite volume discretization of system (1). Prove that $\mathbf{u}$ is conserved at the semi-discrete level (hint: telescoping). Does it amount to a stability statement (consider boundary conditions)?
2. What if there is a diffusion term ?
3. What if we add a source term ?
4. What about higher-order finite-volume schemes? Does conservation still hold? Why? You can pick an example finite-volume scheme from the literature.
5. Let's now consider time-integration. Choose three different time-integration schemes and prove that conservation of $\mathbf{u}$ at the semi-discrete level implies conservation at the fully discrete level.

Now consider the 2D system:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}_x}{\partial x} + \frac{\partial \mathbf{f}_y}{\partial y} = 0, \ \mathbf{u} \in \mathbb{R}^K, \ \mathbf{f}_x \in \mathbb{R}^K, \mathbf{f}_y \in \mathbb{R}^K. \tag{2}$$

1. Consider a finite-volume discretization. Prove that $\mathbf{u}$ is conserved as well. You can start by assuming the mesh to be structured. Then prove the result for a general unstructured mesh.
2. Find a 2nd order finite-volume scheme for a triangular unstructured mesh in the literature, and detail its steps (how are fluxes computed at edges?).
3. Consider a discontinuous Galerkin (DG) scheme in space. DG schemes are known to be conservative (i.e. conserve $\mathbf{u}$). What is implicitly assumed in that statement?

Consider the 1D advection equation ($K = 1, \mathbf{u} = u, \mathbf{f} = au, a = 1$) on a domain $[0, 1]$. Periodic boundary conditions. Initial condition: Gaussian pulse centered at $x = 0.5$.

1. Run it with a first-order finite-volume scheme until $t = 0.5s$ with the six schemes listed below and discuss your results (what time step/ mesh size values to use is up to you):
   - In space: central flux, upwind flux.
   - In time: Forward Euler, Backward Euler, Midpoint rule.

   Are you surprised by certain results?
2. Do all these schemes conserve u ? Confirm your statement numerically.
3. Prove that u^2 also satisfies a conservation equation. Do any of the schemes conserve u^2 at the semi-discrete level and/or fully-discrete level? Numerical and/or theoretical proof expected.

Try answering all the questions. Do cite the papers you learned [1] (including the page/section that contains the relevant information). You are encouraged to introduce your own notation. Last but not least, <u>carry out the proofs yourself</u>, do not defer to some paper.

References

[1] Author1, Author 2 : Paper title, *Journal of Scientific Computing*, volume #, year #, pages #-#.