

# Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves

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**Introduction.** It is well known that an initial value problem for a nonlinear ordinary differential equation may very well fail to have a solution for all time; the solution may blow up after a finite time. The same is true for quasi-linear hyperbolic partial differential equations: solutions may break down after a finite time when their first derivatives blow up.

In these notes we study first order quasi-linear hyperbolic systems which come from conservation laws. Since a conservation law is an integral relation, it may be satisfied by functions which are not differentiable, not even continuous, merely measurable and bounded. We shall call these *generalized* solutions, in contrast to the regular, i.e., differentiable ones. The breakdown of a regular solution may merely mean that although a generalized solution exists for all time, it ceases to be differentiable after a finite time. All available evidence indicates that this is so. It turns out however that there are many generalized solutions with the same initial data, only one of which has physical significance; the task is to give a criterion for selecting the right one. A class of such criteria is described in these notes; they are called *entropy conditions*, for in the gas dynamical case they amount to requiring the increase of entropy of particles crossing a shock front.

These lectures deal with the mathematical side of the theory, i.e., with results that can be proved rigorously. We present whatever is known about existence and uniqueness of generalized solutions of the initial value problem subject to the entropy conditions. We also investigate the subtle dissipation introduced by the entropy condition and show that it causes a slow decay in signal strength.

As stated in the preface, no numerical results are presented; yet there is a very brief introduction to numerical methods in § 7 of the Notes. The approximate solutions that can be computed by these methods are not only enormously useful quantitatively, but there is hope that such methods can also be used to prove the existence of solutions and to study them qualitatively.

**1. Quasi-linear hyperbolic equations.** A first order system of quasi-linear equations in two independent variables is of the form

$$(1.1) \quad u_t + Au_x = 0,$$

where  $u$  is a vector function of  $x$  and  $t$ ,  $A$  a matrix function of  $u$  as well as of  $x$  and  $t$ . Such a system is called strictly hyperbolic if for each  $x$ ,  $t$  and  $u$  the matrix  $A = A(x, t, u)$  has real and distinct eigenvalues  $\tau_j = \tau_j(x, t, u)$ ,  $j = 1, \dots, n$ .

Similarly, a quasi-linear system in  $k + 1$  variables  $x^1, x^2, \dots, x^k, t$ ,

$$(1.2) \quad u_t + \sum_{i=1}^k A_i u_{x^i} = 0, \\ A_i = A_i(x, t, u),$$

is called hyperbolic if for each  $x$ ,  $t$ ,  $u$  and unit vector  $\omega$ , the matrix

$$\sum A_i \omega_i$$

has real and distinct eigenvalues  $\tau_j(x, t, u, \omega)$ ,  $j = 1, \dots, n$ .

The initial value problem for (1.1)—or (1.2)—is to find a solution  $u(x, t)$  with prescribed values at  $t = 0$ :

$$(1.3) \quad u(x, 0) = u_0(x).$$

We shall deduce now easily from the linear theory that this initial value problem has at most one solution in the class of  $C^1$  solutions. For let  $u$  and  $v$  both solve (1.1):

$$u_t + A(u)u_x = 0, \quad v_t + A(v)v_x = 0, \\ u(x, 0) = v(x, 0).$$

Subtracting the two equations we find that the difference  $d = u - v$  satisfies

$$(1.4) \quad d_t + A(u)d_x + [A(u) - A(v)]v_x = 0.$$

Assume that  $A$  is  $C^1$ ; then  $|A(u) - A(v)| \leq \text{const. } |d|$  for the quasi-linear equation (1.1), and similarly for (1.2). It follows that  $d = 0$ .

Does the initial value problem always have a solution? We shall sketch an argument, based on linear theory, that the answer is "yes" if the initial values are smooth enough. The solution is obtained by iterating the transformation  $u = \mathcal{T}v$  defined as follows:  $u$  is the solution of the linear initial value problem

$$u_t + \sum_{i=1}^k A_i(v)u_{x^i} = 0, \\ u(x, 0) = u_0(x).$$

Let us assume that the  $A_i$  are symmetric matrices. It is not hard to show, using the energy estimates for linear symmetric hyperbolic systems and the Sobolev inequalities, that the transformation  $\mathcal{T}$  has the following properties:

Suppose that  $u_0$  is of class  $C^N$ , where  $N > 1 + k/2$ . Define the norm  $\|u\|_{N,T}$  by

$$\|u\|_{N,T}^2 = \sup_{0 \leq t \leq T} \int \sum_{|\alpha| \leq N} |D_x^\alpha u|^2 dx.$$

(i) For  $R$  large enough and  $T$  small enough  $\mathcal{T}$  maps the ball  $B_{N,T}^R$  of radius  $R$ :

$$\|v\|_{N,T} \leq R$$

into itself.

(ii)  $\mathcal{T}$  is a contraction of  $B_{N,T}^R$  with respect to the  $\|\cdot\|_{0,T}$  norm:

$$\|\mathcal{T}v_1 - \mathcal{T}v_2\|_{0,T} \leq \frac{1}{2}\|v_1 - v_2\|_{0,T}.$$

Since  $B_{N,T}^R$  is closed in the  $\|\cdot\|_{0,T}$  norm, it follows that  $\mathcal{T}$  has a unique fixed point in  $B_{N,T}^R$  constructible by iteration. This fixed point solves the initial value problem for the quasi-linear equation (1.2).

This proof shows that (1.2), (1.3) has a solution in some finite time interval  $0 \leq t \leq T$ . Examples presented in § 3 and § 6 show that, in general, smooth solutions do not exist beyond some finite time interval. Since solutions are supposed to describe the state of a physical system, how is one to interpret the nonexistence of solutions in the large? We shall show in the next section that for quasi-linear equations which come from conservation laws there is a way of defining generalized solutions.

**2. Conservation laws.** A conservation law asserts that the rate of change of the *total amount* of substance contained in a fixed domain  $G$  is equal to the *flux* of that substance across the boundary of  $G$ . Denoting the *density* of that substance by  $u$ , and the flux by  $f$ , the conservation law is

$$(2.1) \quad \frac{d}{dt} \int_G u \, dx = - \int_{\partial G} f \cdot n \, dS;$$

here  $n$  denotes the outward normal to  $G$  and  $dS$  the surface element on  $\partial G$ , the boundary of  $G$ , so that the integral on the right in (2.1) measures outflow—hence the minus sign. Applying the divergence theorem and taking  $d/dt$  under the integral sign we obtain

$$\int_G (u_t + \operatorname{div} f) \, dx = 0.$$

Dividing by  $\operatorname{vol}(G)$  and shrinking  $G$  to a point where all partial derivatives of  $u$  and  $f$  are continuous we obtain the differential conservation law

$$(2.2) \quad u_t + \operatorname{div} f = 0.$$

We shall deal with systems of conservation laws

$$(2.3) \quad u_t^j + \operatorname{div} f^j = 0, \quad j = 1, \dots, n,$$

where each  $f^j$  is some nonlinear function of  $u^1, \dots, u^n$ . Carrying out the differentiations in (2.3) we get the first order quasi-linear system

$$(2.4) \quad u_t^j + \sum \frac{\partial f^j}{\partial u^k} u_{x^k}^i = 0.$$

Introducing the vector and matrix notation

$$(2.5) \quad u = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix}, \quad f_i = \begin{pmatrix} f_i^1 \\ \vdots \\ f_i^n \end{pmatrix},$$

$$A_i = \left( \frac{\partial f_i^j}{\partial u^k} \right) = \text{grad}_u f_i,$$

we can write (2.4) in the form

$$(2.6) \quad u_t + \sum A_i u_{x^i} = 0.$$

Since the  $f^j$  are nonlinear functions of  $u$ , the matrices  $A_i$  as defined by (2.5) are functions of  $u$ . We assume that the quasi-linear system (2.6) is strictly hyperbolic.

$u$  is called a *generalized solution* of the system of conservation laws (2.3) if it satisfies the integral form of these laws, i.e., if

$$(2.7) \quad \int_G u^j dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\partial G} f^j \cdot n dS dt = 0$$

holds for every smoothly bounded domain and for every time interval  $(t_1, t_2)$ . This is equivalent to requiring (2.3) to hold in the sense of distribution theory.

Let  $S(t)$  be a smooth surface moving with  $t$ ,  $u$  a continuously differentiable solution of (2.3) on either side of  $S$  which is discontinuous across  $S$ ; the condition which must be satisfied at each point of  $S$  if  $u$  is a generalized solution across  $S$  is

$$(2.8) \quad s[u^j] = [f^j] \cdot n.$$

Here  $[u]$  and  $[f]$  denote the difference between values of  $u$  and  $f$  respectively on the two sides of  $S$ ;  $n$  is the normal to  $S$  and  $s$  the speed with which  $S$  propagates in the direction  $n$ . Relation (2.8) is called the Rankine-Hugoniot jump condition; we shall prove it for the one-dimensional case in § 3.

We leave now these formal considerations and turn to solving the initial value problem within the class of these generalized solutions.

**3. Single conservation laws.** A single conservation law is an equation of the form

$$(3.1) \quad u_t + f_x = 0,$$

where  $f$  is some nonlinear function of  $u$ . Denoting

$$(3.2) \quad \frac{df}{du} = a(u)$$

we can write (3.1) in the form

$$(3.3) \quad u_t + a(u)u_x = 0,$$

which asserts that  $u$  is constant along trajectories  $x = x(t)$  which propagate with speed  $a$ :

$$(3.4) \quad \frac{dx}{dt} = a.$$

For this reason  $a$  is called the *signal speed*, the trajectories, satisfying (3.4), are called *characteristics*. Note that if  $f$  is a nonlinear function of  $u$ , both signal speed and characteristic depend on the solution  $u$ .

The constancy of  $u$  along characteristics combined with (3.4) shows that the characteristics propagate with constant speed; so they are straight lines. This leads to the following geometric solution of the initial value problem  $u(x, 0) = u_0(x)$ : Draw straight lines issuing from points  $y$  of the  $x$ -axis, with speed  $u_0(y)$  (see Fig. 1).

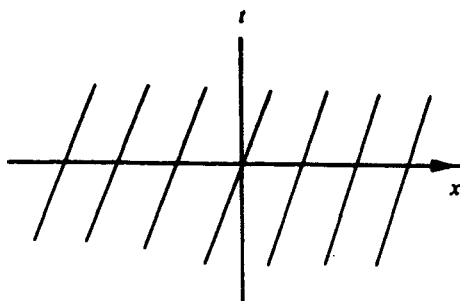


FIG. 1

As we shall show, if  $u_0$  is a  $C^1$  function, these straight lines simply cover a neighborhood of the  $x$ -axis. Since the value of  $u$  along the line issuing from the point  $y$  is  $u_0(y)$ ,  $u(x, t)$  is uniquely determined near the  $x$ -axis.

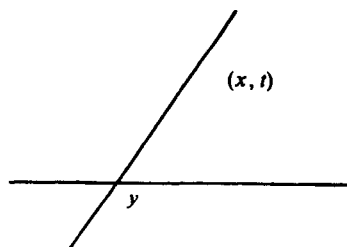


FIG. 2

An analytical form of this construction is shown in Fig. 2. Let  $(x, t)$  be any point,  $y$  the intersection of the characteristic through  $x, t$  with the  $x$ -axis. Then  $u = u(x, t)$  satisfies

$$(3.5) \quad u - u_0(x - ta(u)) = 0.$$

Assume  $u_0$  differentiable; then, according to the implicit function theorem, (3.5) can be solved for  $u$  as a differentiable function of  $x$  and  $t$  for  $t$  small enough, and

$$(3.6) \quad \begin{aligned} u_t &= -\frac{u'_0 a}{1 + u'_0 a_u t}, \\ u_x &= \frac{u'_0}{1 + u'_0 a_u t}. \end{aligned}$$

Substituting (3.6) into (3.3) we see immediately that  $u$  defined by (3.5) satisfies (3.3).

Let us assume that (3.3) is genuinely nonlinear, i.e., that  $a_u \neq 0$  for all  $u$ , say

$$(3.7) \quad a_u > 0.$$

Then if  $u'_0 \geq 0$  for all  $x$ ,  $u_t$  and  $u_x$  as given by formulas (3.6) remain bounded for all  $t > 0$ ; on the other hand, if  $u'_0$  is  $< 0$  at some point, both  $u_t$  and  $u_x$  tend to  $\infty$  as  $1 + u'_0 a_u t$  approaches zero. Both these facts can be deduced from the geometric form of the solution contained in Fig. 1 as follows.

In the first case, when  $u_0(x)$  is an increasing function of  $x$ , the characteristics issuing from the  $x$ -axis diverge in the positive  $t$  direction, so that the characteristics simply cover the whole half-plane  $t > 0$ . In the second case there are two points  $y_1$  and  $y_2$  such that  $y_1 < y_2$ , and  $u_1 = u_0(y_1) > u_0(y_2) = u_2$ ; then by (3.7) also  $a_1 = a(u_1) > a(u_2) = a_2$  so that the characteristics issuing from these points intersect at time  $t = (y_2 - y_1)/(a_1 - a_2)$  (see Fig. 3). At the point of intersection,  $u$  has to take on both values  $u_1$  and  $u_2$ , an impossibility.

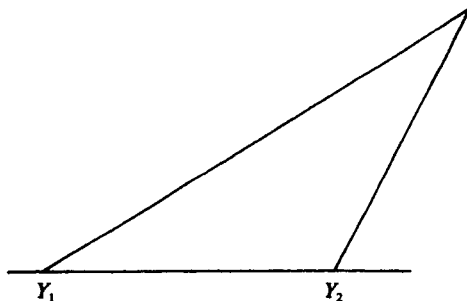


FIG. 3

Both the geometric and the analytic argument prove beyond the shadow of a doubt that if  $a(u_0(x))$  is not an increasing function of  $x$ , then *no function*  $u(x, t)$  exists for all  $t > 0$  with initial value  $u_0$  which solves (3.3) in the ordinary sense! We saw however in § 2 that bounded, measurable functions  $u$  which satisfy (3.1) in the sense of distributions can be regarded as satisfying the integral form of the conservation law of which (3.1) is the differential form. We turn now to the study of such distribution solutions, starting with the simplest kind—those satisfying

(3.1) in the ordinary sense on each side of a smooth curve  $x = y(t)$  across which  $u$  is discontinuous. We shall denote by  $u_l$  and  $u_r$  the values of  $u$  on the left and right sides, respectively, of  $x = y$ . Choose  $a$  and  $b$  so that the curve  $y$  intersects the

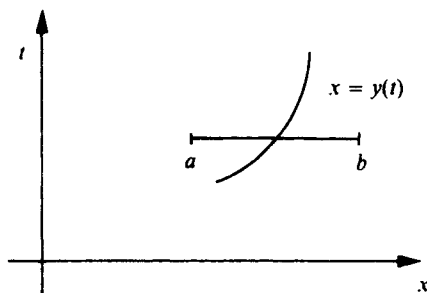


FIG 4

interval  $a \leq x \leq b$  at time  $t$  (see Fig. 4). Denoting by  $I(t)$  the quantity

$$I(t) = \int_a^b u(x, t) dx = \int_a^y + \int_y^b,$$

we have

$$(3.8) \quad \frac{dI}{dt} = \int_a^y u_t dx + u_t s + \int_y^b u_t dx - u_r s,$$

where we have used the abbreviation

$$(3.9) \quad s = \frac{dy}{dt}$$

for the speed with which the discontinuity propagates. Since on either side of the discontinuity (3.1) is satisfied, we may set  $u_t = -f_x$  in (3.8) for  $x < y$  and  $x > y$ , obtaining after carrying out the integration that

$$\frac{dI}{dt} = f_a - f_l + u_l s - f_b + f_r - u_r s.$$

Here we have used the handy abbreviations

$$\begin{aligned} f(u_l) &= f_l, & f(u_r) &= f_r, \\ f(u(a)) &= f_a, & f(u(b)) &= f_b. \end{aligned}$$

The conservation law asserts that

$$\frac{dI}{dt} = f_a - f_b.$$

Combining this with the above relation we deduce the *jump condition*

$$(3.10) \quad s[u] = [f],$$

where  $[u] = u_r - u_l$  and  $[f] = f_r - f_l$  denote the jump in  $u$  and in  $f$  across  $y$ .

We show now in an example that previously unsolvable initial value problems can be solved for all  $t$  with the aid of discontinuous solutions. Take

$$(3.11) \quad f(u) = \frac{1}{2}u^2,$$

$$u_0(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 1 - x & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } 1 \leq x. \end{cases}$$

The geometric solution (see Fig. 5) is single-valued for  $t \leq 1$  but double-valued

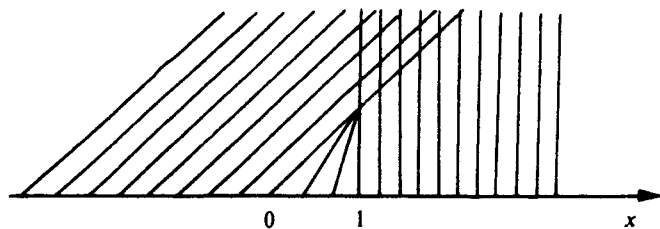


FIG. 5

thereafter. Now we define for  $t \geq 1$ ,

$$u(x, t) = \begin{cases} 1 & \text{for } x < \frac{1+t}{2}, \\ 0 & \text{for } \frac{1+t}{2} < x. \end{cases}$$

The discontinuity starts at  $(1, 1)$ ; it separates the state  $u_l = 1$  on the left from the state  $u_r = 0$  on the right; the speed of propagation was chosen according to the jump condition (3.10), with  $f(u) = \frac{1}{2}u^2$ :

$$s = \frac{0 - 1/2}{0 - 1} = \frac{1}{2}.$$

Introducing generalized solutions makes it possible to solve initial value problems which could not be solved within the class of genuine solutions. At the same time it threatens with the danger that the enlarged class of solutions is so large that there are several generalized solutions with the same initial data. The following example shows that this anxiety is well founded:

$$u_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } 0 < x. \end{cases}$$



The geometric solution (see Fig. 6) is single-valued for  $t > 0$  but does not deter-

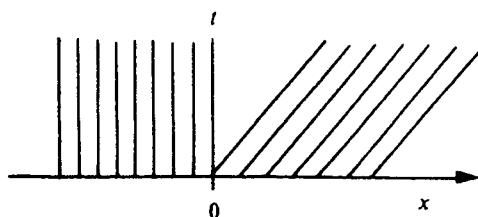


FIG. 6

mine the value of  $u$  in the wedge  $0 < x < t$ . We could fill this gap in the fashion of the previous example and set

$$(3.12) \quad u(x, t) = \begin{cases} 0 & \text{for } x < \frac{t}{2}, \\ 1 & \text{for } \frac{t}{2} < x. \end{cases}$$

The speed of propagation was so chosen that the jump condition (3.10) is satisfied. On the other hand, the function

$$(3.12') \quad u(x, t) = \frac{x}{t}, \quad 0 \leq x \leq t,$$

satisfies the differential equation (3.3) with  $a(u) = u$ , and joins continuously the rest of the solution determined geometrically. Clearly only one of these solutions can have physical meaning; the question is which?

We reject the discontinuous solution (3.12) for failure to satisfy the following criterion:

*The characteristics starting on either side of the discontinuity curve when continued in the direction of increasing  $t$  intersect the line of discontinuity. This will be the case if*

$$(3.13) \quad a(u_l) > s > a(u_r).$$

Clearly this condition is violated in the solution given by (3.12).

If all discontinuities of a generalized solution satisfy condition (3.13), no characteristic drawn in the direction of decreasing  $t$  intersects a line of discontinuity. This shows that for such solutions every point can be connected by a backward drawn characteristic to a point on the initial line; therein lies the significance of condition (3.13). When applied to the equations of compressible flow, this generalization amounts to requiring that material which crosses the discontinuity should suffer an increase of entropy. For this reason condition (3.13) will be called the *entropy condition*.

A discontinuity satisfying the jump relation (3.10) and the entropy condition (3.13) is called a *shock*. The task before us is to investigate whether every initial value problem for (3.1) has exactly one generalized solution, defined for all  $t \geq 0$ , which has only shocks as discontinuities.

We shall first treat the case when condition (3.7) is satisfied, i.e.,  $a(u)$  is an increasing function of  $u$ . Clearly this is so whenever  $f(u)$  is a *convex* function of  $u$

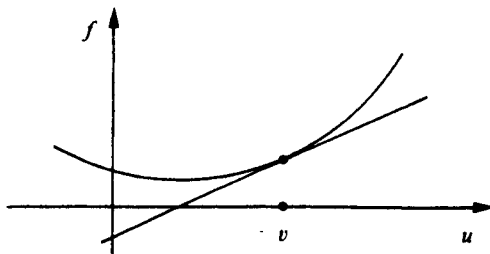


FIG. 7

(see Fig. 7). Such a function lies above all its tangents:

$$(3.14) \quad f(u) \geq f(v) + a(v)(u - v).$$

Let  $u$  be a genuine (i.e., continuous and differentiable) solution of (3.1), and suppose that  $u_0(x) = u(x, 0)$  is 0 for  $x$  large enough negative; then the same is true of  $u(x, t)$  for any  $t > 0$  for which  $u$  is defined. We introduce the integrated function  $U(x, t)$  defined as follows:

$$(3.15) \quad U(x, t) = \int_{-\infty}^x u(y, t) dy;$$

then

$$(3.16) \quad U_x = u.$$

Integrating (3.1) from  $-\infty$  to  $x$  and using (3.16) we obtain

$$(3.17) \quad U_t + f(U_x) = 0,$$

where we have adjusted  $f$  so that

$$(3.18) \quad f(0) = 0.$$

Applying inequality (3.14) with  $U_x = u$  and any number  $v$  we obtain that

$$(3.19) \quad U_t + a(v)U_x \leq a(v)v - f(v).$$

Denote by  $y$  the point where the line  $dx/dt = a(v)$  through  $x, t$  intersects the  $x$ -axis; clearly,

$$(3.20) \quad \frac{x - y}{t} = a(v).$$

Integrating (3.19) along this line from 0 to  $t$  we obtain, for  $t \geq 0$ ,

$$(3.21) \quad U(x, t) \leq U(y, 0) + t[a(v)v - f(v)].$$

Denote by  $b$  the inverse of the function  $a$ ; from (3.20) we obtain

$$(3.22) \quad b\left(\frac{x-y}{t}\right) = v.$$

Denote by  $g$  the function

$$(3.23) \quad g(z) = a(v)v - f(v), \quad v = b(z).$$

Clearly, since  $a$  and  $b$  are inverse functions,

$$\frac{dg}{dz} = v \frac{da}{dv} \frac{db}{dz} = b(z).$$

Denote  $a(0)$  by  $c$ ; then  $b(c) = 0$  and in view of the normalization (3.18), we have from (3.23) that  $g(c) = 0$ . Introducing the function  $g$  on the right side of (3.21) we obtain

$$(3.24) \quad U(x, t) \leq U(y, 0) + tg\left(\frac{x-y}{t}\right).$$

This inequality holds for all choices of  $y$ ; for that value of  $y$  for which  $v$ , given by (3.22), equals  $u(x, t)$ , the sign of equality holds in (3.19) along the whole characteristic  $dx/dt = u$  issuing from  $(x, t)$ ; therefore equality also holds in (3.24). We summarize this result as follows.

**THEOREM 3.1.** *Let  $u$  be a genuine solution of (3.1); then*

$$(3.25) \quad u(x, t) = b\left(\frac{x-y}{t}\right),$$

where  $y = y(x, t)$  is that value which minimizes

$$(3.26) \quad U_0(y) + tg\left(\frac{x-y}{t}\right) = G(x, y, t).$$

Here  $b$  is the inverse function of  $a$ , and  $g$  is defined by

$$(3.27) \quad \frac{dg}{dz} = b(z), \quad g(c) = 0, \quad \text{where } a(0) = c,$$

and

$$(3.28) \quad U_0(y) = \int_{-\infty}^y u_0(x) dx, \quad u_0(x) = u(x, 0).$$

Suppose  $u(x, t)$  is a generalized solution. Relation (3.17) is the integral form of the conservation law (3.1), so it follows that when  $f$  is convex, inequality (3.24) is valid for generalized solutions as well.

If all discontinuities of the generalized solution  $u$  are shocks, then every point  $(x, t)$  can be connected to a point  $y$  on the initial line by a backward characteristic.

For that value of  $y$  the sign of equality holds in (3.24); thus Theorem 3.1 applies also to generalized solutions of (3.1) whose discontinuities are shocks. We show that also the converse holds, thereby proving the existence of solutions, with shocks, with arbitrarily prescribed integrable initial data.

**THEOREM 3.2.** *Formula (3.25)–(3.26) defines a possibly discontinuous function  $u(x, t)$  for arbitrary integrable initial values  $u_0(x)$ ; the function  $u$  so defined satisfies (3.1) in the sense of distributions, and the discontinuities of  $u$  are shocks.*

*Proof.* (i) If  $u_0$  is integrable,  $U_0$  is bounded. As may be seen from (3.27),  $g$  is a convex function which achieves its minimum at  $z = c$ ; therefore, the function  $G$  defined by (3.26) achieves its minimum in  $y$  at some point or points.

**LEMMA 3.3.** *For fixed  $t$ , denote by  $y(x)$  any value of  $y$  where  $G(x, y)$  achieves its minimum. Then  $y(x)$  is a nondecreasing function of  $x$ .*

*Proof.* We have to show that for  $x_2 > x_1$ ,  $G(x_2, y)$  does not take on its minimum for  $y < y_1 = y(x_1)$ . In particular, we shall show that for  $y < y_1$ ,

$$(3.29) \quad G(x_2, y_1) < G(x_2, y).$$

For by the minimizing property of  $y_1$ ,

$$(3.30) \quad G(x_1, y_1) \leq G(x_1, y).$$

From Jensen's inequality we have that for  $x_1 < x_2$  and  $y < y_1$ ,

$$g\left(\frac{x_2 - y_1}{t}\right) + g\left(\frac{x_1 - y}{t}\right) < g\left(\frac{x_1 - y_1}{t}\right) + g\left(\frac{x_2 - y}{t}\right).$$

Multiplying this last inequality by  $t$  and adding to (3.30) we obtain (3.29); this completes the proof of the lemma.

It follows from the lemma that, for fixed  $t$ , for all but a denumerable set of  $x$  the minimum of  $G(x, y, t)$  is achieved at exactly one point  $y = y(x, t)$ , which is a monotonic increasing function of  $x$ . Thus  $u(x, t)$  is well defined by (3.25) at all but these points.

(ii) We can combine (3.25) and (3.26) into one formula

$$u = \lim u_N,$$

where

$$u_N(x, t) = \frac{\int b[(x - y)/t] \exp \{-NG\} dy}{\int \exp \{-NG\} dy}.$$

Similarly,

$$f(u) = \lim f_N,$$

where

$$f_N(x, t) = \frac{\int f(b[(x - y)/t]) \exp \{-NG\} dy}{\int \exp \{-NG\} dy}.$$

Denote by  $V_N$  the function

$$V_N = \log \int \exp \{ -NG \} dy.$$

Clearly,

$$u_N = -\frac{1}{N} \frac{\partial}{\partial x} V_N$$

and similarly we see that

$$f_N = \frac{1}{N} \frac{\partial}{\partial t} V_N$$

provided that we make use of the identity

$$zb(z) - g(z) = f(b(z)).$$

It follows from these relations that

$$u_{N_t} + f_{N_x} = 0;$$

letting  $N \rightarrow \infty$  we obtain in the limit relation (3.1) for  $u$ .

(iii) Since Lemma 3.3,  $y(x, t)$  is an increasing function of  $x$ , and since  $b$  is an increasing function, we have for  $x_1 < x_2$  that

$$\begin{aligned} u(x_1, t) &= b\left(\frac{x_1 - y_1}{t}\right) \geq b\left(\frac{x_1 - y_2}{t}\right) \\ &\geq b\left(\frac{x_2 - y_2}{t}\right) - k \frac{x_2 - x_1}{t} \\ &= u(x_2, t) - k \frac{x_2 - x_1}{t}; \end{aligned}$$

here  $k$  is a Lipschitz constant for  $b$ . The result is a one-sided Lipschitz condition for  $u$ :

$$(3.31) \quad \frac{u(x_2, t) - u(x_1, t)}{x_2 - x_1} \leq \frac{k}{t},$$

which shows that at points of discontinuity  $u_r < u_l$ , as required by the shock condition.

(iv) The solutions defined by (3.25), (3.26) have the semigroup property, i.e., that if  $u(x, t_1)$  is taken as new initial value, the corresponding solution at time  $t_2$  furnished by (3.25), (3.26) equals  $u(x, t_1 + t_2)$ . This is easily verified by a simple argument.

We turn now to the problem of uniqueness.

**THEOREM 3.4.** *Let  $u$  and  $v$  be two piecewise continuous generalized solutions of (3.1); assume that  $f$  is convex and that all discontinuities of both  $u$  and  $v$  are shocks.*

Then

$$\|u(t) - v(t)\|$$

is a decreasing function of  $t$ , where the norm is the  $L_1$  norm with respect to the  $x$  variable.

**COROLLARY.** If  $u = v$  at  $t = 0$ ,  $u = v$  for all  $t > 0$ . (This is the uniqueness theorem we were looking for.)

*Proof.* We can write the  $L_1$  norm of  $u - v$  as

$$(3.32) \quad \|u - v\| = \sum (-1)^n \int_{y_n}^{y_{n+1}} (u - v) dx,$$

where the points are so chosen that

$$u(x, t) - v(x, t) \text{ has the sign } (-1)^n$$

for  $y_n < x < y_{n+1}$ ; of course the  $y_n$  are functions of  $t$ .

There are two cases:

(i)  $y_n$  is a point of continuity of both  $u$  and  $v$ ; in this case

$$(3.33) \quad u(y_n, t) = v(y_n, t).$$

Since values of solutions are constant along characteristics, it follows that in this case  $y_n$  is a linear function of  $t$ .

(ii)  $y_n$  is a point of discontinuity of one of the functions  $u$  or  $v$ ; in this case  $y_n$  is a shock curve.

Differentiate (3.32) with respect to  $t$ :

$$(3.34) \quad \begin{aligned} \frac{d}{dt} \|u - v\| = \sum (-1)^n & \left[ \int_{y_n}^{y_{n+1}} \frac{\partial}{\partial t} (u - v) dx \right. \\ & \left. + (u - v)(y_{n+1}) \frac{dy_{n+1}}{dt} - (u - v)(y_n) \frac{dy_n}{dt} \right]. \end{aligned}$$

On the interval  $y_n, y_{n+1}$  both  $u$  and  $v$  satisfy (3.1) in the sense of distributions. Setting  $u_t = -f(u)_x$ ,  $v_t = -f(v)_x$  in (3.34) we obtain, after carrying out the integration,

$$(3.35) \quad (-1)^n (f(v) - f(u))(y) + (u - v)(y) \frac{dy}{dt} \Big|_{y_n}^{y_{n+1}}.$$

In case (i)  $u(y) = v(y)$  which makes (3.35) equal 0. We turn now to case (ii). Suppose that  $u$  has a shock at  $y = y_{n+1}$  and that  $v = v(g)$  lies between  $u_l$  and  $u_r$ :

$$(3.36) \quad u_r < v < u_l.$$

Then  $u - v$  is positive in  $(y_n, y_{n+1})$ , so  $n$  is even. According to the jump relation (3.10),

$$\frac{dy}{dt} = s = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Substituting this into (3.35), we obtain at the endpoint  $y_{n+1}$  with  $u = u_l$ ,

$$(3.37) \quad \begin{aligned} & f(v) - f(u_l) + (u_l - v) \frac{f(u_l) - f(u_r)}{u_l - u_r} \\ &= f(v) - \left[ \frac{v - u_r}{u_l - u_r} f(u_l) + \frac{u_l - v}{u_l - u_r} f(u_r) \right]. \end{aligned}$$

Since  $f$  is a convex function and since by (3.36)  $v$  lies in the interval  $(u_r, u_l)$ , it follows from Jensen's inequality that the right side of (3.37) is negative. Similarly, also the contribution at the lower endpoint to (3.35) is negative. This shows that  $(d/dt)\|u - v\|$  is always  $\leq 0$ , so that as  $t$  increases  $\|u - v\|$  decreases. This completes the proof of the theorem.

The case when  $y$  is a discontinuity for both  $u$  and  $v$  can be treated similarly.

In the derivation of Theorem 3.4 we can omit the requirement that  $f$  be convex if we replace the entropy condition (3.13) by the following:

(i) If  $u_r < u_l$ , then the graph of  $f$  over  $[u_r, u_l]$  lies below the chord (see Fig. 8):

$$(3.38)_i \quad f(\alpha u_r + (1 - \alpha)u_l) \leq \alpha f(u_r) + (1 - \alpha)f(u_l) \quad \text{for } 0 \leq \alpha \leq 1.$$

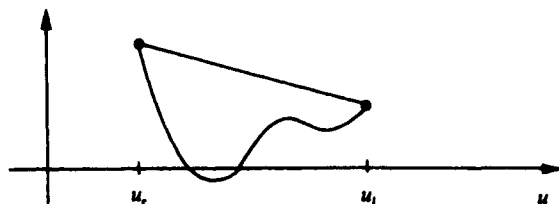


FIG. 8

(ii) If  $u_l < u_r$ , the graph of  $f$  over  $[u_l, u_r]$  lies above the chord (see Fig. 9):

$$(3.38)_{ii} \quad f(\alpha u_r + (1 - \alpha)u_l) \geq \alpha f(u_r) + (1 - \alpha)f(u_l) \quad \text{for } 0 \leq \alpha \leq 1.$$

Conditions (3.38) are called the *generalized entropy conditions*.

Let  $f(u)$  be an arbitrary  $C^1$  function of  $u$ ; let  $u(x, t)$  be a distribution solution of (3.1) which is a genuine solution outside of a finite number of discontinuities.

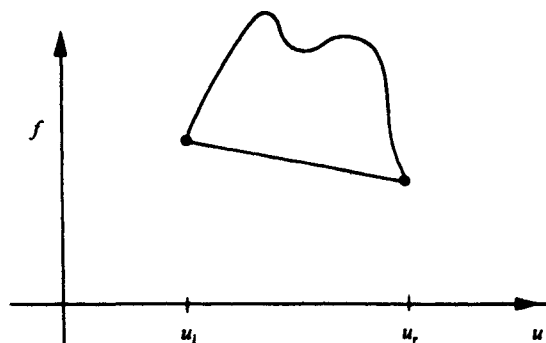


FIG. 9

A discontinuity in  $u$  is called a *shock* if  $u$ , and  $u_t$  satisfy one of the entropy conditions (3.38).

The proof of Theorem 3.4 yields the following more general result.

**THEOREM 3.5.** *Let  $f$  be any  $C^1$  function,  $u$  and  $v$  two distribution solutions of (3.1) of which all discontinuities are shocks. Then*

$$\|u(t) - v(t)\|$$

*is a decreasing function of  $t$ .*

It follows in particular that *two such solutions which are equal at  $t = 0$  are equal for all  $t$ .*

The proof of Theorem 3.5 also demonstrates the following converse.

**COROLLARY.** *If  $u$  is a distribution solution of (3.1) of which one of the discontinuities fails to satisfy the entropy condition (3.38), then there is a genuine solution  $v$  such that*

$$\|u(t) - v(t)\|$$

*is not a decreasing function of  $t$ .*

The uniqueness theorem stated above is not very interesting unless we can show that the generalized entropy condition is not too restrictive, i.e., that every initial value problem  $u(x, 0) = u_0(x)$  has a distribution solution  $u$  of which the discontinuities satisfy the generalized entropy condition. Such a solution  $u$  can be constructed, as the limit of solutions  $u_\lambda$  of the parabolic equation

$$(3.39) \quad \frac{\partial}{\partial t} u_\lambda + \partial_x f(u_\lambda) = \lambda \frac{\partial^2}{\partial x^2} u_\lambda, \quad \lambda > 0,$$

$$u_\lambda(x, 0) = u_0(x).$$

It follows easily from the maximum principle that (3.39) has at most one solution; it is also true that a solution  $u_\lambda$  exists for all  $t$ , and that as  $\lambda \rightarrow 0$  these solutions converge in the  $L_1$  sense to a limit  $u$ . We shall not present these proofs, but we shall show that this limit is a distribution solution of (3.1) which satisfies the generalized entropy condition. To show the truth of the first statement we multiply (3.39) by a  $C_0^\infty$  test function  $\phi$  and integrate by parts; we obtain

$$-\int \int [\phi_t u_\lambda + \phi_x f(u_\lambda)] dx dt = \lambda \int \phi_{xx} u_\lambda.$$

Let  $\lambda \rightarrow 0$ ; the left side converges to

$$-\int \int \phi_t u + \phi_x f(u) dx dt,$$

the right side converges to 0. This proves that  $u$  is a distribution solution of (3.1).

To prove that  $u$  satisfies the entropy condition, we show first that for any two solutions  $u_\lambda$  and  $v_\lambda$  of (3.39),

$$\|u_\lambda(t) - v_\lambda(t)\|$$



is a decreasing function of  $t$ . To see this we write

$$\|u_\lambda - v_\lambda\| = \sum (-1)^n \int_{y_n}^{y_{n+1}} (u_\lambda - v_\lambda) dx,$$

where  $u_\lambda - v_\lambda$  changes sign at the points  $y_n$ . Since solutions of (3.39) are continuous,

$$(3.40) \quad u_\lambda - v_\lambda = 0 \quad \text{at} \quad y_n.$$

Differentiate  $\|u_\lambda - v_\lambda\|$ :

$$\frac{d}{dt} \|u_\lambda - v_\lambda\| = \sum (-1)^n \int_{y_n}^{y_{n+1}} \frac{\partial}{\partial t} (u_\lambda - v_\lambda) dx + \sum (-1)^n (u_\lambda - v_\lambda) \frac{dy}{dt} \Big|_{y_n}^{y_{n+1}}.$$

On account of (3.40) the second term on the right is zero; substituting from (3.39) into the first term and carrying out the integration we obtain

$$\frac{d}{dt} \|u_\lambda - v_\lambda\| = \sum (-1)^n \lambda \frac{\partial}{\partial x} (u_\lambda - v_\lambda) \Big|_{y_n}^{y_{n+1}} - \sum (-1)^n (f(u_\lambda) - f(v_\lambda)) \Big|_{y_n}^{y_{n+1}}.$$

Again the second sum on the right is zero, on account of (3.40). We claim that each term in the first sum is nonpositive; for  $(-1)^n \lambda (u_\lambda - v_\lambda) = p$  is nonnegative in the interval and by (3.40) zero at the endpoints; so  $p_x$  is nonnegative at the left endpoint, nonpositive at the right endpoint. This completes the proof that

$$\|u_\lambda(t) - v_\lambda(t)\|$$

is a decreasing function of  $t$ . If  $u$  is the  $L_1$  limit of  $u_\lambda$  and  $v$  the  $L_1$  limit of  $v_\lambda$ , also  $\|u(t) - v(t)\|$  is decreasing.

It is not hard to show that every genuine solution  $v$  of (3.1) is the limit as  $\lambda \rightarrow 0$  of the solutions  $v_\lambda$  of the parabolic equation. Therefore it follows that if  $u$  is a distribution solution of (3.1) which is an  $L_1$  limit of solutions  $u_\lambda$  of (3.39) and if  $v$  is any genuine solution of (3.1), then  $\|u(t) - v(t)\|$  is a decreasing function of  $t$ . According to the corollary to Theorem 3.5 it follows then that all discontinuities of  $u$  satisfy the generalized entropy condition, as asserted.

Being rid of the convexity conditions makes it possible to extend these notions and the existence and uniqueness theorems to single conservation laws in any number of space variables.

**4. The decay of solutions as  $t$  tends to infinity.** Suppose  $f(u)$  is a convex function; then Theorem 3.2 gives an explicit expression for the solutions  $u$  of (3.1) in terms of their initial data:

$$(4.1) \quad u(x, t) = b \left( \frac{x - y}{t} \right),$$

where  $y$  minimizes

$$(4.2) \quad G(x, y, t) = U_0(y) + tg \left( \frac{x - y}{t} \right).$$

Let us see what we can deduce from (4.1) about the behavior of solutions for large  $t$ . We recall that  $g$  is a convex function and  $b$  a monotonically increasing one, and that  $g$  takes on its minimum at  $c = a(0)$  and

$$(4.3) \quad g(c) = 0.$$

We denote by  $k$  the quantity

$$(4.4) \quad k = \frac{1}{2}b'(c) = \frac{1}{2}g''(c).$$

Let us assume that  $f$  is strictly convex; then  $k > 0$ . We further assume that  $b'$  lies between two positive constants for all  $u$ :

$$(4.5) \quad k_- < \frac{1}{2}b' < k_+.$$

It follows from this and (4.3) that

$$(4.6) \quad g(z) \geq k_-(z - c)^2;$$

then

$$tg\left(\frac{x-y}{t}\right) \geq \frac{k_-}{t}(x-y-ct)^2.$$

Suppose the initial value  $u_0$  of  $u$  is in  $L_1$ ; then for every  $y$ ,  $U_0(y) = \int_{-\infty}^y u_0 dx$  is bounded in absolute value by  $\|u_0\| = M$ . Using (4.6) we see that for all  $y$ ,

$$(4.7) \quad -M + \frac{k}{t}(x-y-ct)^2 \leq G(x, y, t).$$

$G(x, x+ct, t) = U_0(x+ct)$  is  $\leq M$ ; this shows that  $G(x, y, t) \leq M$  at the minimizing point. Combining this with (4.7) we see that

$$(4.8) \quad \left| \frac{x-y}{t} - c \right| \leq \sqrt{\frac{2M}{kt}}.$$

It follows from (4.5) and  $b(c) = 0$  that

$$|b(z)| < 2k_+|z - c|;$$

combining this with (4.8) we obtain

$$\left| b\left(\frac{x-y}{t}\right) \right| \leq \frac{\text{const.}}{\sqrt{t}}, \quad \text{const.} = 2k_+ \sqrt{\frac{2M}{k_-}}.$$

Thus, by (4.1),

$$(4.9) \quad |u(x, t)| \leq \frac{\text{const.}}{\sqrt{t}}.$$

Suppose that the initial value  $u_0(x)$  is zero outside the interval  $(-A, A)$ ; then  $U_0(y)$  is zero for  $x < -A$ , some constant for  $A < x$ . According to (4.8), the minimum value of  $y$  lies in the interval

$$x - ct - \text{const.} \sqrt{t} \leq y \leq x - ct + \text{const.} \sqrt{t}.$$

If  $x < ct - \text{const.} \sqrt{t} - A$ ,  $y$  lies in the interval  $y < -A$  where the value of  $U_0(y)$  is independent of  $y$ ; therefore the minimum of  $G$  is taken on at the point which minimizes  $tg((x - y)/t)$ ; this value is  $y = x - ct$ . Similarly, for

$$x > ct + \text{const.} \sqrt{t} + A$$

the minimizing value of  $y$  is  $y = x - ct$ . Since  $b(c) = 0$ , we conclude from (4.1) that  $u(x, t) = 0$  for  $x$  outside of

$$(ct - \text{const.} \sqrt{t} - A, ct + \text{const.} \sqrt{t} + A).$$

This result, combined with (4.9), can be expressed thus: Every solution  $u$  whose initial value is zero outside a finite interval is, at time  $t$ , zero outside an interval whose length is  $O(\sqrt{t})$ ; inside that interval  $u$  is  $O(1/\sqrt{t})$ .

A more detailed analysis of the explicit formula yields the following more precise statement about the behavior of solutions for large  $t$ .

**THEOREM 4.1.** Define the 2-parameter family of functions  $v(p, q)$ ,  $p, q \geq 0$  as follows:

$$(4.10) \quad v(x, t; p, q) = \begin{cases} \frac{1}{h} \left( \frac{x}{t} - c \right) & \text{for } ct - \sqrt{pt} < x < ct + \sqrt{qt}, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $u(x, t)$  be any solutions with shocks of

$$(4.11) \quad u_t + f_x(u) = 0,$$

where  $f$  is convex,  $f'(0) = c$ ,  $f''(0) = h$ . Then

$$\|u(t) - v(t; p, q)\|$$

tends to 0 as  $t \rightarrow \infty$ , where  $\|\cdot\|$  is the  $L_1$  norm and

$$(4.12) \quad \begin{aligned} p &= -2h \min_y \int_{-\infty}^y u_0(x) dx, \\ q &= 2h \max_y \int_y^{\infty} u_0(x) dx. \end{aligned}$$

We shall not present a proof of this theorem but we shall present a verification of one of its consequences.

We introduce the following abbreviations:

$$(4.13) \quad \begin{aligned} \min_y \int_{-\infty}^y u(x) dx &= I_-(u), \\ \max_y \int_y^{\infty} u(x) dx &= I_+(u). \end{aligned}$$

In terms of these (4.12) can be written as

$$\begin{aligned} p &= -2hI_-(u_0), \\ q &= 2hI_+(u_0). \end{aligned}$$

It follows easily from the definition of  $v$  in (4.10) that, for any  $T$ ,

$$\lim_{t \rightarrow \infty} \|u(t + T; p, q) - v(t; p, q)\| = 0.$$

It follows then from Theorem 4.1 that  $\|u(t + T; p, q) - v(t, p, q)\| \rightarrow 0$ ; applying (4.12) to  $u(x, t + T)$  in place of  $u(x, t)$  we conclude that for any  $T$

$$\begin{aligned} p &= -2hI_-(u(T)), \\ q &= 2hI_+(u(T)). \end{aligned}$$

In words:  $I_-$  and  $I_+$  are time invariant functionals of solutions.

We shall present now a direct proof for the invariance of  $I_-$ .

Let  $u$  be some solution of (4.11), possibly with shocks; denote by  $M(t)$ :

$$M(t) = I_-(u) = \min_y \int_{-\infty}^y u(x, t) dx$$

and by  $y(t)$  any of the values  $y$  where the indicated minimum is taken on. Our aim is to show that  $M$  is independent of  $t$ .

According to the integral form of the conservation law, for any  $t_1$  and  $t_2$  and any  $y$ ,

$$\int_{-\infty}^y u(x, t_2) dx = \int_{-\infty}^y u(x, t_1) dx + \int_{t_1}^{t_2} f(u(y, t)) dt;$$

here we have used the fact that  $u(-\infty, t) = 0$  and that  $f(0) = 0$ . Taking  $y$  to be  $y(t_1) = y_1$ ,  $y(t_2) = y_2$ , respectively, and using the definition of  $y$  as minimum, we obtain the inequalities

$$\begin{aligned} M(t_2) &\leq M(t_1) + \int_{t_1}^{t_2} f(y_1) dt, \\ M(t_1) &\leq M(t_2) - \int_{t_1}^{t_2} f(y_2) dt, \end{aligned}$$

which imply that

$$(4.14) \quad |M(t_2) - M(t_1)| \leq |t_1 - t_2|F,$$

where

$$(4.15) \quad F = \sup_{t_1 \leq t \leq t_2} |f(y_1, t)|, |f(y_2, t)|;$$

here  $f(y, t)$  abbreviates  $f(u(y, t))$ .

LEMMA.  $y(t)$ ,  $t$  is a point of continuity of  $u$ , and

$$(4.16) \quad u(y, t) = 0.$$

*Proof.* By the minimizing property of  $y$  we must have

$$(4.17) \quad u_l = u(y_-, t) \leq 0, \quad u_r = u(y_+, t) \geq 0.$$

Since  $f$  was assumed convex, the entropy condition is that  $u_l > u_r$ , so it follows from (4.17) that  $y(t)$  cannot be a point of discontinuity, and that  $u(y, t) = 0$ .

Since the set of minimizing points  $y(t)$ ,  $t$  is closed, it follows that in any compact portion of the  $(x, t)$ -plane,  $y(t)$ ,  $t$  has a positive distance from any shock of strength  $|u_r - u_l| = \varepsilon$ . From this it follows that as  $t_2 \rightarrow t_1$ , the oscillation of  $u(y_1, t)$  and of  $u(y_2, t)$  over  $(t_1, t_2)$  tends to 0. According to (4.16),  $u(y_1, t_1) = u(y_2, t_2) = 0$ , so  $u(y_1, t)$  and  $u(y_2, t)$  tend to 0 uniformly on  $(t_1, t_2)$  as  $t_2 \rightarrow t_1$ . Since  $f(0) = 0$ , it follows that likewise  $F$ , the maximum of  $f$  over this interval, tends to 0 as  $t_2 \rightarrow t_1$ . But then we deduce from (4.12) that

$$\frac{dM}{dt} = 0,$$

which proves the constancy of  $M$  and shows the invariance of the functional  $I_-$ . The invariance of  $I_+$  follows similarly; alternatively we observe that the minimum and maximum in (4.13) occur for the same value of  $y$ ; this implies that

$$I_-(u) + I_+(u) = \int_{-\infty}^{\infty} u(x) dx = I_0(u).$$

It is a consequence of the integral form of the conservation law that for solutions  $u$  which are zero at  $x = \pm \infty$ ,  $I_0(u)$  is independent of  $t$ ; thus the invariance of  $I_+$  follows from that of  $I_-$  and  $I_0$ .

The quantity  $I_0$  is a natural invariant built into the conservation law; it is remarkable that there exist other, "unnatural" invariants.

**THEOREM 4.2.** *Equation (3.1) has exactly 2 independent invariant functionals continuous in the  $L_1$  topology.*

*Proof.* Let  $I$  be any invariant functional continuous in the  $L_1$  topology; by Theorem 4.1,

$$I(u) = \lim_{t \rightarrow \infty} I(u(t, p, q)).$$

The value of the right side is determined by the values of  $p$  and  $q$ , which in turn are determined by  $I_-(u)$  and  $I_+(u)$ . Therefore, the value of  $I(u)$  is a function of the values of  $I_+(u)$  and  $I_-(u)$ , as asserted.

Using the explicit formula (4.1) we deduced, in (4.9) and Theorem 4.1, that solutions whose initial values lie in  $L_1$  decay to 0 as  $A \rightarrow \infty$ . We shall present now another method for studying the behavior of solutions as  $t \rightarrow \infty$ , one that does not rely on the explicit formula (4.1). With the aid of this method we can show that, as  $t \rightarrow \infty$ , solutions whose initial values are periodic tend uniformly to their mean value  $\bar{u}_0$ :

$$(4.18) \quad \bar{u}_0 = \frac{1}{p} \int_0^p u_0(x) dx.$$

As remarked in § 3, any differentiable solution  $u$  of (4.11) is constant along characteristics

$$(4.19) \quad \frac{dx}{dt} = a(u) = f'(u).$$

Let  $x_1(t)$  and  $x_2(t)$  be a pair of characteristics,  $0 \leq t \leq T$ . Then there is a whole one-parameter family of characteristics connecting the points of the interval  $[x_1(0), x_2(0)]$ ,  $t = 0$  with points of the interval  $[x_1(T), x_2(T)]$ ,  $t = T$ ; since  $u$  is constant along these characteristics,  $u(x, 0)$  on the first interval and  $u(x, T)$  on the second interval are equivariant. More generally, if  $\sigma$  and  $\tau$  are noncharacteristic curves each connecting  $x_1$  to  $x_2$ ,  $u$  along  $\sigma$  and  $\tau$  are equivariant. Since equivariant functions have the same total increasing and decreasing variations, we conclude that the total increasing and decreasing variations of a differentiable solution between any pair of characteristics are conserved.

Denote by  $D(t)$  the width of the strip bounded by  $x_1$  and  $x_2$ :

$$(4.20) \quad D(t) = x_2(t) - x_1(t).$$

Differentiating (4.20) with respect to  $t$  and using (4.19) we obtain

$$(4.21) \quad \frac{d}{dt}D(t) = \frac{dx_2}{dt} - \frac{dx_1}{dt} = a(u_2) - a(u_1).$$

Integrating with respect to  $t$  we obtain

$$(4.22) \quad D(T) = D(0) + [a(u_2) - a(u_1)]T.$$

Suppose there is a shock  $y$  present in  $u$  between the characteristics  $x_1$  and  $x_2$  (see Fig. 10). Since according to (3.13) characteristics on either side of a shock

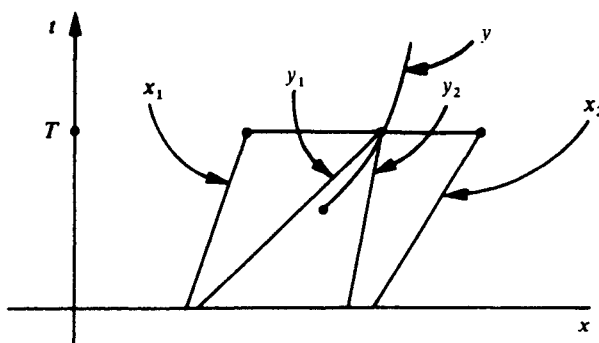


FIG. 10

run into the shock, there exist at any given time  $T$ , two characteristics  $y_1$  and  $y_2$  which intersect the shock  $y$  at exactly time  $T$ . Assuming that there are no other shocks present we conclude that the increasing variation of  $u$  on  $(x_1(t), y_1(t))$ , as well as on  $(x_2(t), y_2(t))$ , is independent of  $t$ . According to condition (3.13),  $a(u)$  decreases across shocks, so the increasing variation of  $a(u)$  along  $[x_1(T), x_2(T)]$  equals the sum of the increasing variations of  $a(u)$  along  $[x_1(0), y_1(0)]$  and along  $[y_2(0), x_2(0)]$ . This sum is in general less than the increasing variation of  $u$  along  $[x_1(0), x_2(0)]$ ; therefore we conclude that if shocks are present, the total increasing variation of  $a(u)$  between two characteristics decreases with time.

We give now a quantitative estimate of this decrease. Let  $I_0$  be any interval of the  $x$ -axis; we subdivide it into subintervals  $[y_{j-1}, y_j]$ ,  $j = 1, \dots, n$ , so that  $u(x, 0)$  is alternately increasing and decreasing on these intervals (we assume for simplicity that  $u_0$  is piecewise monotonic). We denote by  $y_j(t)$  the characteristic issuing from the  $j$ th point  $y_j$ , with the understanding that if  $y_j(t)$  runs into a shock,  $y_j(t)$  is continued as that shock.

It is easy to show that for any  $t > 0$ ,  $u(x, t)$  is alternately increasing and decreasing on the intervals  $(y_{j-1}(t), y_j(t))$ . Since across shocks  $u$  decreases, the total increasing variation  $A^+(T)$  of  $a(u)$  across the interval  $I(T) = [y_0(T), y_n(T)]$  is

$$(4.23) \quad \sum_{j \text{ odd}} a(u_j(T)) - a(u_{j-1}(T)) = A^+(T),$$

where  $u_{j-1}(T)$  denotes the value of  $u$  on the right edge of  $y_{j-1}(T)$ ,  $u_j(T)$  denotes the value of  $u$  on the left edge of  $y_j(T)$ ; in case that  $y_{j-1}(T)$  and  $y_j(T)$  are the same, the  $j$ th term in (4.23) is zero. Suppose,  $y_{j-1}(T) < y_j(T)$ ; then there exist characteristics  $x_{j-1}(t)$  and  $x_j(t)$  which start at  $t = 0$  inside  $(y_{j-1}, y_j)$  and which at  $t = T$  run into  $y_{j-1}(T)$ , and  $y_j(T)$  respectively. The value of  $u$  along  $x_j(t)$  is  $u_j(T)$ .

Denote  $x_j(t) - x_{j-1}(t)$  by  $D_j(t)$ ; according to (4.22)

$$D_j(T) = D_j(0) + [a(u_j) - a(u_{j-1})]T.$$

Summing over  $j$  odd and using (4.23) we obtain

$$\sum D_j(T) = \sum D_j(0) + A^+(T)T.$$

Since the intervals  $[x_{j-1}(T), x_j(T)] = [y_{j-1}(T), y_j(T)]$  are disjoint and lie in  $I(T)$ , the sum of their lengths cannot exceed the length  $L(T)$  of  $I(t)$ . So we deduce that

$$(4.24) \quad A^+(T) \leq \frac{L(T)}{T}.$$

Suppose the initial value of  $u$  is periodic, with period  $p$ . Then by uniqueness it follows that  $u$  is periodic for all  $p$ . Take  $L_0$  to be of length  $p$ ; then  $L(T)$  also has length  $p$  and we deduce from (4.24) that  $A^+(T) \leq p/T$ . Since the increasing variation of a periodic function per period is twice its total variation, we have proved the following theorem.

**THEOREM 4.3.** *For every space periodic solution  $u$  of (4.11),*

$$(4.25) \quad \frac{\text{total variation of } a(u(T)) \text{ per period}}{\text{period}} \leq \frac{2}{T}.$$

Relation (4.25) shows that the total variation of  $a(u)$  per period tends to zero as  $T$  tends to  $\infty$ . Since the mean value  $\bar{u}_0$  of any periodic solution of a conservation law is independent of  $t$ , it follows that  $u(x, T)$  tends to  $\bar{u}_0$  uniformly as  $T$  tends to  $\infty$ . Suppose  $|a'(\bar{u}_0)| = h \neq 0$ ; it follows from (4.25) that the total variation of  $u(T)$  per period is  $\leq 2p/hT$ ; so we conclude that for  $T$  large enough,

$$(4.26) \quad |u(x, t) - \bar{u}_0| < \frac{p}{ht}.$$

Comparison of (4.26) with (4.9) shows that periodic solutions decay *faster* than solutions whose initial values are integrable.

The estimate (4.26) is, in contrast to (4.9), absolute, inasmuch as the right side is independent of the amplitude of the solution.

**5. Hyperbolic systems of conservation laws.** In this section we shall study systems of conservation laws,

$$(5.1) \quad \frac{\partial}{\partial t} u_i + \frac{\partial}{\partial x} f_i = 0, \quad i = 1, \dots, n,$$

where each  $f_i$  is a function of  $u_1, \dots, u_n$ ; we shall denote the column vector formed by  $u_1, \dots, u_n$  by  $u$ . Carrying out the differentiation in (5.1) we obtain the quasi-linear system

$$(5.2) \quad u_t + A(u)u_x = 0,$$

where the  $i$ th row of the matrix  $A$  is the gradient of  $f_i$  with respect to  $u$ . We assume that the system (5.2) is hyperbolic, i.e., that for each value of  $u$  the matrix  $A$  has  $n$  real, distinct eigenvalues  $\lambda_1, \dots, \lambda_n$ , labeled in increasing order. Since  $A$  depends on  $u$ , so do the eigenvalues  $\lambda_k$  and the corresponding right and left eigenvectors  $r_k$  and  $l_k$ .

Genuine nonlinearity played an important role for single conservation laws; this was the requirement that  $\lambda$  be a nonconstant function of  $u$ , i.e., that  $\lambda_u \neq 0$ . The analogous condition for system is not merely that  $\text{grad}_u \lambda_k$  be nonzero, but that it be not orthogonal to  $r_k$ , the corresponding eigenvector. If this is so, we call the  $k$ th field *genuinely nonlinear*, and normalize  $r_k$  so that

$$(5.3) \quad r_k \cdot \text{grad } \lambda_k = 1.$$

If on the other hand  $r_k \cdot \text{grad } \lambda_k \equiv 0$ , we call the  $k$ th characteristic field *linearly degenerate*.

We turn now to the study of piecewise continuous solutions of (5.1) in the integral sense; each of the  $n$  conservation laws must satisfy the Rankine-Hugoniot jump condition, i.e.,

$$(5.4) \quad s[u_k] = [f_k], \quad k = 1, \dots, n,$$

must hold across every discontinuity, where  $s$  is the speed of propagation of the discontinuity.

Next we formulate an *entropy condition* for systems. For single convex (or concave) equations this condition requires that the characteristics on either side of a discontinuity run into the line of discontinuity, which is the case if the characteristic speed on the left is greater, on the right less, than  $s$ :

$$\lambda(u_l) > s > \lambda(u_r).$$

For systems we require that for some index  $k$ ,  $1 \leq k \leq n$ ,

$$(5.5') \quad \lambda_k(u_l) > s > \lambda_k(u_r)$$



while

$$(5.5'') \quad \lambda_{k-1}(u_l) < s < \lambda_{k+1}(u_r).$$

These inequalities assert that  $k$  characteristics impinge on the line of discontinuity from the left and  $n - k + 1$  from the right, a total of  $n + 1$ . This information carried by these characteristics plus the  $n - 1$  relations obtained from (5.4) after eliminating  $s$  are sufficient to determine the  $2n$  values which  $u$  takes on on both sides of the line of discontinuity.

A discontinuity across which (5.4) and (5.5) are satisfied is called a  $k$ -shock.

We give now a description of all weak  $k$ -shocks, i.e., those where  $u_r$  and  $u_l$  differ little; it is understood that  $u_l$  is to the left of  $u_r$ .

**THEOREM 5.1.** *The set of states  $u_r$  near  $u_l$  which are connected to some given state  $u_l$  through a  $k$ -shock form a smooth one-parameter family  $u_r = u(\varepsilon)$ ,  $-\varepsilon_0 < \varepsilon \leq 0$ ,  $u(0) = u_l$ ; the shock speed  $s$  also is a smooth function of  $\varepsilon$ .*

We shall omit the proof of this result but shall calculate the first two derivatives of  $u(\varepsilon)$  at  $\varepsilon = 0$ . Differentiating the jump relation

$$s[u - u_l] = f(u) - f(u_l)$$

we obtain, using the symbol  $\dot{\phantom{x}} = d/d\varepsilon$ ,

$$(5.6) \quad \dot{s}[u] + s\dot{u} = \dot{f} = A\dot{u}.$$

At  $\varepsilon = 0$  we have  $[u] = 0$ , so there

$$s(0)\dot{u}(0) = A(u_l)\dot{u}(0),$$

which can be satisfied with  $\dot{u} \neq 0$  only if  $s(0)$  is an eigenvalue of  $A$ :

$$(5.7) \quad s(0) = \lambda_k(u_l),$$

and  $\dot{u}(0)$  an eigenvector:

$$(5.8) \quad \dot{u}(0) = \alpha r_k(u_l).$$

By reparametrizing we can make  $\alpha = 1$ . Differentiating (5.6) once more and setting  $\varepsilon = 0$  we obtain (omitting the subscript  $k$ )

$$s\ddot{u} + 2\dot{s}\dot{u} = A\ddot{u} + \dot{A}\dot{u}.$$

Substituting (5.7) and (5.8), with  $\alpha = 1$  we obtain

$$(5.9) \quad \lambda\ddot{u} + 2\dot{s}r = A\ddot{u} + \dot{A}r.$$

To determine  $\dot{s}$  and  $\ddot{u}$  we turn to the eigenvalue relation

$$\lambda r = Ar,$$

restricted to  $u = u(\varepsilon)$  and differentiate with respect to  $\varepsilon$ :

$$\lambda\dot{r} + \dot{\lambda}r = A\dot{r} + \dot{A}r.$$

Subtract this from (5.9):

$$(5.10) \quad \lambda(\ddot{u} - \dot{r}) + (2\dot{s} - \dot{\lambda})r = A(\ddot{u} - \dot{r}).$$

Taking the scalar product with the left eigenvector  $l$  belonging to  $\lambda$  we obtain

$$2\dot{s} - \dot{\lambda} = 0.$$

Since by (5.3),  $\dot{\lambda} = \dot{u} \operatorname{grad} \lambda = r \operatorname{grad} \lambda = 1$ , we obtain

$$(5.11) \quad 2\dot{s} = \dot{\lambda} = 1.$$

Equation (5.10) has the solution  $\ddot{u} - \dot{r} = \beta r$ ; by a change of parameter  $\varepsilon$  we can accomplish that  $\beta = 0$ ; so

$$(5.12) \quad \ddot{u}(0) = \dot{r}(0).$$

It follows from (5.11) that, modulo terms  $O(\varepsilon^2)$ ,

$$\lambda(\varepsilon) = \lambda(0) + \varepsilon,$$

$$s(\varepsilon) = \lambda(0) + \varepsilon/2.$$

Thus the entropy condition (5.5)

$$\lambda(u_l) = \lambda(0) > s(\varepsilon) > \lambda(\varepsilon) = \lambda(u_r)$$

is satisfied for  $\varepsilon < 0$  and not for  $\varepsilon > 0$ ; that is why in Theorem 5.1 the parameter  $\varepsilon$  is restricted to  $\varepsilon \leq 0$ .

Next we turn to an important class of continuous solutions, centered rarefaction waves; these are solutions which depend only on the ratio  $(x - x_0)/(t - t_0)$ ,  $x_0, t_0$  being the center of the wave.

Let  $u$  be a rarefaction wave centered at the origin:

$$(5.13) \quad u(x, t) = h(x/t).$$

Let us denote  $x/t$  by  $\xi$  and differentiation with respect to  $\xi$  by  $'$ ; substituting (5.13) into (5.2) we obtain

$$-\frac{x}{t^2}h' + \frac{1}{t}Ah' = 0,$$

which is the same as

$$[A(h) - \xi]h'(\xi) = 0.$$

This is satisfied by

$$(5.14) \quad \xi = \lambda(h(\xi)), \quad h' = \alpha r(h).$$

Using relation (5.3) we obtain, after differentiating the first relation in (5.14) and using the second, that

$$1 = \lambda(h(\xi))' = h' \operatorname{grad} \lambda = \alpha r \operatorname{grad} \lambda = \alpha,$$

where  $\lambda = \lambda_k$  is one of the eigenvalues of  $A$ ;  $h$  is called a *k-rarefaction wave*. Setting  $\alpha = 1$  in (5.14) we obtain

$$(5.14') \quad h' = r(h).$$

Abbreviate  $\lambda(u_i)$  by  $\lambda$ ; the differential equation (5.14') has a unique solution satisfying the initial condition

$$(5.15) \quad h'(\lambda) = u_i;$$

$h$  is defined for all  $\xi$  close enough to  $\lambda$ .

Let  $\varepsilon$  be a number  $\geq 0$ , so small that  $h$  is defined at  $\lambda + \varepsilon$ ; denote by  $u_r$  the value

$$(5.16) \quad u_r = h(\lambda + \varepsilon).$$

We construct now the following piecewise smooth function  $u(x, t)$ , defined for  $t \geq 0$  (see Fig. 11):

$$(5.17) \quad u(x, t) = \begin{cases} u_i & \text{for } x < \lambda t, \\ h(x/t) & \text{for } \lambda t \leq x \leq (\lambda + \varepsilon)t, \\ u_r & \text{for } (\lambda + \varepsilon)t < x. \end{cases}$$

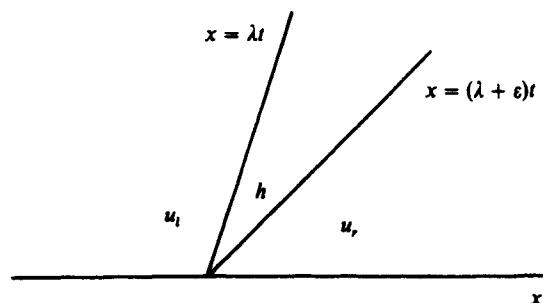


FIG. 11

This function  $u$  satisfies the differential equation (5.2) in each of the three regions, and is continuous across the lines separating the regions. We shall say that in  $u$  the states  $u_i$  and  $u_r$  are connected by a centered  $k$ -rarefaction wave.

**THEOREM 5.2.** *Given a state  $u_i$ , there is a one-parameter family of states  $u_r = u(\varepsilon)$ ,  $0 \leq \varepsilon \leq \varepsilon_0$ , which can be connected to  $u_i$  through a centered  $k$ -rarefaction wave.*

Theorems 5.1 and 5.2 can be combined.

**THEOREM 5.3.** *Given a state  $u_i$ , it can be connected to a one-parameter family of states  $u_r = u(\varepsilon)$ ,  $-\varepsilon_0 < \varepsilon < \varepsilon_0$ , on the right of  $u_i$  through a centered  $k$ -wave, i.e., either a  $k$ -shock or a  $k$ -rarefaction wave;  $u(\varepsilon)$  is twice continuously differentiable with respect to  $\varepsilon$ .*

The only part that needs proof is the continuity of  $du/d\varepsilon$  and  $d^2u/d\varepsilon^2$  at  $\varepsilon = 0$ . From (5.8) and (5.14) we see, since  $\alpha = 1$  in both cases, that  $du/d\varepsilon = r(u_i)$  for  $\varepsilon = \pm 0$ ; to show that  $d^2u/d\varepsilon^2$  is continuous at  $\varepsilon = 0$  we differentiate (5.14') with respect to  $\varepsilon$  to obtain

$$(5.18) \quad \frac{d^2}{d\varepsilon^2}u(+0) = h''(\lambda) = r'.$$

Since  $u'(0) = \tilde{u}(0)$ , we see, comparing (5.18) and (5.12), that  $u'' = \tilde{u}''$  at  $\varepsilon = 0$ . This completes the proof of Theorem 5.3.

If the  $k$ th characteristic field of (5.1) is degenerate, then (5.1) has discontinuous solutions whose speed of propagation is

$$s = \lambda_k(u_l) = \lambda_k(u_r).$$

These are called contact discontinuities.

We turn now to the so-called *Riemann initial value problem*, where the initial function  $u_0$  is

$$(5.19) \quad u_0(x) = \begin{cases} u_0 & \text{for } x < 0, \\ u_n & \text{for } 0 < x, \end{cases}$$

where  $u_0$  and  $u_n$  are two vectors.

**THEOREM 5.4.** *If the states  $u_0$  and  $u_n$  are sufficiently close, the initial value problem (5.19) has a solution. This solution consists of  $n + 1$  constant states  $u_0, u_1, \dots, u_n$ , separated by centered rarefaction or shock waves, one of each family (see Fig. 12).*

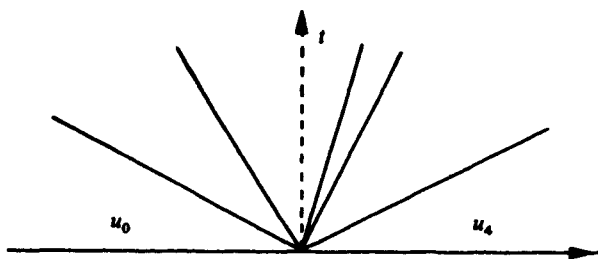


FIG. 12

*Proof.* The state  $u_0$  can be connected through a 1-wave to a one-parameter family of states  $u_1(\varepsilon_1)$  to the right of  $u_0$ ;  $u_1$  in turn can be connected through a 2-wave to a one-parameter family  $u_2(\varepsilon_1, \varepsilon_2)$  of states to the right of  $u_1$ , etc. Thus  $u_0$  can be connected through a succession of  $n$  waves to an  $n$ -parameter family of states  $u_n(\varepsilon_1, \dots, \varepsilon_n)$ . By (5.8)–(5.14),

$$\frac{\partial u}{\partial \varepsilon_j} = r_j;$$

since the  $r_j$  are linearly independent, it follows from the implicit function theorem that a small  $\varepsilon$ -ball is mapped one-to-one onto a neighborhood of  $u_0$ ; this completes the proof of Theorem 5.4.

We describe now a method developed by James Glimm for solving any initial value problem  $u(x, 0) = u_0(x)$  where the oscillation of  $u_0$  is small. The solution  $u$  is obtained as the limit as  $h \rightarrow 0$  of approximate solutions  $u_h$  constructed as follows:

(I)  $u_h(x, 0)$  is a piecewise constant approximation to  $u_0(x)$ :

$$(5.20) \quad u_h(x, 0) = m_j \quad \text{for } jh < x < (j+1)h, \quad j = 0, \pm 1, \dots,$$

where  $m_j$  is some kind of mean value of  $u_0(x)$  over the interval  $(jh, (j+1)h)$ .

(II) For  $0 \leq t < h/\lambda$ ,  $u_h(x, t)$  is the exact solution of (5.1) with initial values  $u_h(x, 0)$  given by (5.20); here  $\lambda$  is an upper bound for  $2|\lambda_k(u)|$ . This exact solution is constructed by solving the Riemann initial value problems

$$(5.21_j) \quad u(x, 0) = \begin{cases} m_{j-1} & \text{for } x < jh, \\ m_j & \text{for } jh < x, \end{cases}$$

$j = 0, \pm 1, \dots$ . Since the oscillation of  $u_0$  is small,  $m_{j-1}$  and  $m_j$  are close and so according to Theorem 5.4 this initial value problem has a solution consisting of constant states separated by shocks or centered rarefaction waves issuing from the points  $x = jh, t = 0$  (see Fig. 13). As long as

$$(5.22) \quad t < \frac{h}{\lambda}$$

these waves do not intersect each other and so the solutions of the initial value problem (5.21<sub>j</sub>) can be combined into a single exact solution  $u_h$ .

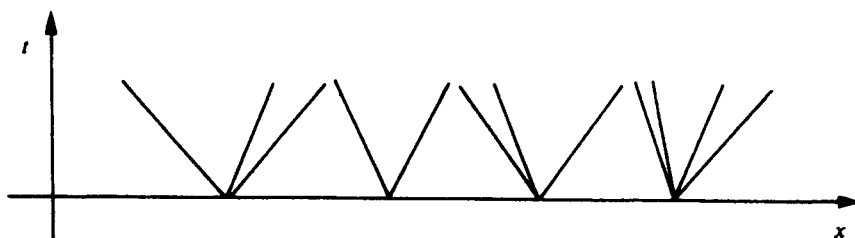


FIG. 13

(III) We repeat the process, with  $t = h/\lambda$  as new initial time in place of  $t = 0$ . It is not at all obvious that this process yields an approximate solution  $u_h$  which is defined for all  $t$ ; to prove this one must show that the oscillation of  $u(x, nh)$  remains small, uniformly for  $n = 1, 2, \dots$  so that the Riemann problems (5.21<sub>j</sub>) can be solved, and so that  $\lambda$  does not tend to  $\infty$ . This estimate turns out to depend very sensitively on the kind of averaging used to compute the mean values  $m_j$ . In the scheme introduced by Glimm the quantities  $m_j$  are computed as follows: A sequence of random numbers  $\alpha_1, \alpha_2, \dots$ , uniformly distributed in  $[0, 1]$ , is chosen;  $m_j^n$ , the mean value of  $u(x, nh/\lambda)$  over the interval  $(jh, (j+1)h)$  is taken to be

$$(5.23) \quad m_j^n = u(jh + \alpha_n h, nh/\lambda).$$

Glimm shows:

(A) Given any  $\varepsilon$ , we can choose  $\eta$  so small that if the oscillation and total variation of  $u_0$  are  $< \eta$  then for any  $t$ , the oscillation and total variation of  $u(x, t)$  along any space-like line is  $< \varepsilon$ .

(B) A subsequence of  $u_h$  converges in the  $L_1$  norm with respect to  $x$ , uniformly in  $t$ , to a limit  $u$ .

(C) For almost all choices of the random sequence  $\{\alpha_n\}$ , this limit  $u$  is a solution in the integral sense of the conservation law (5.1).

For proof we refer to Glimm's paper; here we merely point out how Glimm's scheme treats a particularly simple initial value problem, a Riemann initial value problem:

$$u_0(x) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } 0 < x, \end{cases}$$

where  $u_l$  and  $u_r$  are so chosen that the exact solution  $u$  consists of the two states  $u_l, u_r$  separated by a single shock

$$(5.24) \quad u(x, t) = \begin{cases} u_l & \text{for } x < st, \\ u_r & \text{for } st < x, \end{cases}$$

where  $s$  is the speed with which the shock separating the two states propagates. By assumption,  $\lambda > |s|$ . Let us assume that  $0 < s$ ; then Glimm's recipe (5.23) gives

$$u_n(x, h/\lambda) = \begin{cases} u_l & \text{for } x < J_1 h, \\ u_r & \text{for } J_1 h < x, \end{cases}$$

where

$$J_1 = \begin{cases} 1 & \text{if } \alpha_1 < s/\lambda, \\ 0 & \text{if } s/\lambda < \alpha_1. \end{cases}$$

Repeating this analysis  $n$  times we obtain

$$(5.25) \quad u_n(x, nh/\lambda) = \begin{cases} u_l & \text{for } x < J_n h, \\ u_r & \text{for } J_n h < x, \end{cases}$$

where  $J_n$  = number of  $\alpha_j, j = 1, \dots, n$ , which satisfy

$$(5.26) \quad \alpha_j < \frac{s}{\lambda}.$$

Since  $\{\alpha_j\}$  is a uniformly distributed random sequence,

$$(5.27) \quad \frac{J_n}{n} \rightarrow \frac{s}{\lambda}$$

with probability 1; this shows that the approximate solution given by (5.25) tends almost certainly to the exact solution given by (5.24).

Note that if, instead of using random sequences we use a *single sequence of equidistributed numbers*  $\{\alpha_j\}$ , i.e., numbers for which (5.27) holds, we conclude that  $u_n$  tends to  $u$  as  $h \rightarrow 0$ .

We conclude by stating precisely the existence theorem whose proof was outlined above, and by stating some open problems.

**THEOREM 5.5.** *The initial value problem for the system of conservation laws (5.1) has a solution for all  $t$  provided that the initial function  $u_0$  has sufficiently small oscillation and total variation.*

What is lacking at present is a proof that the solution  $u$  constructed by Glimm's scheme satisfies the entropy condition, and that it is uniquely determined by  $u_0$ . Some remarks about both points will be made in § 6.

Another outstanding problem is to remove the requirement that  $u_0$  have small oscillation.

Inequalities (3.14), (3.38) and (5.5) are criteria which reject certain discontinuities as physically unrealizable even though the conservation laws are satisfied across them; we designated these criteria as *entropy conditions*. We shall introduce now a notion of *entropy* which can be related to these criteria.

We start with a system of conservation laws (5.1). Let  $U$  be some function of  $u_1, \dots, u_n$ . When does  $U$  satisfy a conservation law, i.e., a law of the form

$$(5.28) \quad U_t + F_x = 0,$$

where  $F$  is some function of  $u_1, \dots, u_n$ ? Carrying out the differentiation in (5.28) we obtain

$$\text{grad } U \cdot u_t + \text{grad } F \cdot u_x = 0,$$

where the gradient is with respect to  $u$ . To deduce this from (5.2),

$$(5.2) \quad u_t + Au_x = 0,$$

we multiply (5.2) on the left with  $\text{grad } U$ ; (5.28) results if and only if the relation

$$(5.29) \quad \text{grad } U \cdot A = \text{grad } F$$

holds. This is a system of  $n$  partial differential equations for  $U$  and  $F$ ; for  $n \geq 2$  it is overdetermined and has no solution in general; there are, however, special cases of some importance with a nontrivial solution, for example, in gas dynamics. A general class of equations where a solution exists are the symmetric ones, i.e., when  $A$  is a symmetric matrix. In this case,

$$(5.30) \quad \frac{\partial f_j}{\partial u_i} = \frac{\partial f_i}{\partial u_j}.$$

Relation (5.30) is the compatibility relation for the existence of a function  $g(u)$  satisfying

$$\frac{\partial g}{\partial u_i} = f_i.$$

It is then easy to verify that

$$(5.31) \quad U = \sum u_j^2 \quad \text{and} \quad F = \sum u_j f_j - g$$

satisfy (5.29).

The role of entropy conditions is to distinguish those discontinuous solutions which are physically realizable from those which are not. Another way to characterize the physically realizable solutions is to identify them as limits of solutions of equations in which a small dissipative mechanism has been added to

the laws already embodied in (5.1). A particular example of such a dissipative mechanism is *artificial viscosity*; here the augmented equation is

$$(5.32) \quad u_t + Au_x = \lambda u_{xx}, \quad \lambda > 0.$$

Multiply this by  $\text{grad } U$ ; if (5.29) is satisfied we obtain

$$(5.33) \quad U_t + F_x = \lambda \text{grad } U \cdot u_{xx}.$$

We have the identity

$$U_{xx} = \text{grad } U \cdot u_{xx} + U_{ij} u_x^i u_x^j,$$

where

$$U_{ij} = \frac{\partial^2 U}{\partial u^i \partial u^j}.$$

Suppose  $U$  is *convex*, i.e., the matrix of its second derivatives is positive definite; then we deduce that

$$U_{xx} \geq \text{grad } U \cdot u_{xx};$$

substituting this into (5.33) we obtain, since  $\lambda > 0$ , that

$$U_t + F_x \leq \lambda U_{xx}.$$

Let  $\lambda \rightarrow 0$ ; the right side tends to zero in the sense of distributions and we deduce the following theorem.

**THEOREM 5.6.** *Let (5.1) be a system of conservation laws which implies an additional conservation law (5.28); suppose that  $U$  is strictly convex. Let  $u(x, t)$  be a distribution solution of (5.1) which is the limit, boundedly, a.e., of solutions of (5.32) containing the artificial viscous term. Then  $u$  satisfies the inequality*

$$(5.34) \quad U(u)_t + F(u)_x \leq 0.$$

The following are immediate consequences of (5.34):

(a)

$$(5.35) \quad \int U(t) dx,$$

if finite, is a decreasing function of  $t$ .

(b) Suppose  $u$  is piecewise continuous; then across a discontinuity

$$(5.36) \quad s[U_l - U_r] - [F_l - F_r] \leq 0.$$

We shall call conditions (5.34) and (5.36) *entropy conditions*; to justify the name we have to show compatibility with the previous designations.

**THEOREM 5.7.** *Suppose that the system of conservation laws (5.1) is hyperbolic and genuinely nonlinear in the sense of (5.3). Suppose there is a strictly convex function  $U$  of  $u$  which satisfies the additional conservation law (5.28). Let  $u$  be a solution of (5.1) in the integral sense which has a discontinuity propagating with*



speed  $s$ . Suppose that the values on the two sides of the discontinuity are close; then the entropy condition (5.5) holds if and only if the entropy condition (5.36) holds.

*Proof.* According to Theorem 5.1, the states  $u_r$  which can be connected to a given state  $u_l$  through a  $k$ -shock, form a one-parameter family of states  $u_r = u(\varepsilon)$ ,  $-\varepsilon_0 < \varepsilon < 0$ . Denote by  $E(\varepsilon)$  the quantity on the left of (5.36):

$$E(\varepsilon) = s[U(u_r) - U(u_l)] - F(u_r) + F(u_l).$$

A brief calculation using (5.8) and (5.11), and which we omit, shows that the values of the first two derivatives with respect to  $\varepsilon$  of  $E(\varepsilon)$  are zero at  $\varepsilon = 0$ , and the value of the third derivative at  $\varepsilon = 0$  is

$$\ddot{E} = \frac{1}{2} {}^t r U'' r.$$

Here  ${}^t r$  is the transpose of the right eigenvector  $r$  and  $U''$  is the matrix of second derivatives of  $U$ . Since  $U$  is convex,  $\ddot{E}$  is positive; this shows that  $E(\varepsilon)$  is an increasing function of  $\varepsilon$  near  $\varepsilon = 0$ . But then  $E(\varepsilon)$  is negative for  $\varepsilon$  negative; since by Theorem 5.1 the shock condition (5.5) restricts  $u_r(\varepsilon)$  to negative values of  $\varepsilon$ , Theorem 5.7 follows.

Next we show that the entropy condition (5.34) is equivalent to the entropy condition in the large (3.38) imposed on solutions of single equations. We note that in this case  $U$  can be taken to be any function of  $u$ ;  $F$  can be determined from (5.29) by integration.

**THEOREM 5.8.** *The entropy condition (3.38) is satisfied if and only if (5.36) is satisfied for all  $U, F$  which satisfy (5.29) and where  $U$  is convex.*

*Proof.* Suppose  $u_l < u_r$ ; let  $z$  be any number between  $u_l$  and  $u_r$ , and set

$$U(u) = \begin{cases} 0 & \text{for } u < z, \\ u - z & \text{for } z \leq u. \end{cases}$$

It follows from (5.29) that

$$F(u) = \begin{cases} 0 & \text{for } u < z, \\ f(u) - f(z) & \text{for } z \leq u. \end{cases}$$

Substituting these into (5.36) and using the jump relation

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

we obtain, after rearrangement,

$$f(z) \geq \frac{u_r - z}{u_r - u_l} f(u_l) + \frac{z - u_l}{u_r - u_l} f(u_r),$$

which is precisely condition (3.38<sub>ii</sub>); the other case can be handled similarly.

**6. Pairs of conservation laws.** More is known about systems of two conservation laws than systems consisting of more than two. What is special about systems of only two equations is the existence of Riemann invariants, which we now proceed to describe.

We write the system as

$$(6.1) \quad \begin{aligned} u_t + f_x &= 0, \\ v_t + g_x &= 0, \end{aligned}$$

where  $f$  and  $g$  are functions of  $u$  and  $v$ . Carrying out the differentiation in (6.1) we obtain

$$(6.2) \quad \begin{pmatrix} u \\ v \end{pmatrix}_t + A \begin{pmatrix} u \\ v \end{pmatrix}_x = 0,$$

where

$$A = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}.$$

We assume that (6.2) is hyperbolic, which means that the matrix  $A$  has real and distinct eigenvalues; we denote them by  $\lambda$  and  $\rho$ , so arranged that  $\lambda < \rho$ ; of course  $\lambda$  and  $\rho$  both are functions of  $u$  and  $v$ . We denote the corresponding left and right eigenvector by  $l$  and  $r$ ; that is

$$l_\lambda A = \lambda l_\lambda, \quad A r_\lambda = \lambda r_\lambda$$

and

$$l_\rho A = \rho l_\rho, \quad A r_\rho = \rho r_\rho.$$

The eigenvectors, too, depend on  $u$  and  $v$ . We shall consider functions  $w$  of  $u$  and  $v$  which satisfy the first order equation

$$(6.3_\lambda) \quad \text{grad } w \cdot r_\lambda = 0.$$

This equation asserts that  $w$  is constant along the trajectories of the vector field  $r_\lambda$ . We can construct solutions of (6.3) by taking a curve  $C$  which is not tangent to  $r_\lambda$  at any point, and assigning arbitrary values for  $w$  along it. We shall choose  $w$  to be *strictly increasing* along  $C$ . The value of  $u$  is then determined along every trajectory intersecting  $C$ , and  $w$  has distinct values along distinct trajectories.

The function  $z$  of  $u, v$  is defined analogously as solution of

$$(6.3_\rho) \quad \text{grad } z \cdot r_\rho = 0.$$

Since  $w$  has distinct values along distinct  $r_\lambda$ -trajectories, and  $z$  has distinct values along  $r_\rho$ -trajectories, and since in a simply connected domain of  $u, v$  space an  $r_\rho$ -trajectory intersects an  $r_\lambda$ -trajectory in at most one point, it follows that the mapping

$$u, v \rightarrow w, z$$

is one-to-one over any simply connected domain.

It is well known that the left and right eigenvectors of a matrix with distinct eigenvalues are biorthogonal. It is here that we exploit that  $n = 2$ : since by (6.3)  $\text{grad } w$  is orthogonal to  $r_\lambda$ , it follows that  $\text{grad } w$  is a left eigenvector with eigenvalue  $\rho$ :

$$(6.4_\rho) \quad \text{grad } w A = \rho \text{ grad } w.$$

Similarly,

$$(6.4_\lambda) \quad \text{grad } z A = \lambda \text{ grad } z.$$

Multiply (6.2) by  $\text{grad } w$ ; using the above relation, and the chain rule, we obtain

$$(6.5) \quad w' = w_t + \rho w_x = 0,$$

where  $'$  denotes differentiation in the direction

$$(6.6) \quad \frac{dx}{dt} = \rho.$$

Similarly,

$$(6.7) \quad z' = z_t + \lambda z_x = 0,$$

where  $\cdot$  is differentiation in the direction

$$(6.8) \quad \frac{dx}{dt} = \lambda.$$

Curves satisfying (6.6) and (6.8) are called  $\lambda$  and  $\rho$  characteristics respectively. Relations (6.5) and (6.7) can be stated in words as follows.

As functions of  $x$  and  $t$ ,  $w$  is constant along  $\rho$ -characteristics,  $z$  is constant along  $\lambda$ -characteristics. After their discoverer,  $w$  and  $z$  are called *Riemann invariants*.

In § 3 about single conservation laws we gave a geometric argument for the nonexistence of continuous solutions beyond a certain time. One ingredient of that proof was the constancy of  $u$  along characteristics; the other was the fact that characteristics are straight lines. The first ingredient is present here:  $w$  and  $z$  are constant along characteristics, but it is no longer true that characteristics are straight lines; so the simple geometric reasoning given in § 3 cannot be extended to systems. We present now a different nonexistence proof for a single equation which is capable of generalization.

The equation (3.3) satisfied by  $u$  is

$$u_t + a(u)u_x = 0;$$

differentiate this with respect to  $x$ :

$$u_{tx} + a u_{xx} + a_u u_x u_x = 0.$$

Abbreviate  $u_x$  by  $q$ ; the above can be written then as

$$(6.9) \quad q' + a_u q^2 = 0,$$

where  $q'$  abbreviates the directional derivative

$$q' = q_t + a q_x.$$

Equation (6.9) is an ordinary differential equation for  $q$  along the characteristic and can be integrated explicitly:

$$(6.10) \quad q(t) = \frac{q_0}{1 + q_0 k t},$$

where  $q_0 = q(0)$  and  $k = a'(u)$  constant along the characteristic. This formula shows that if  $q_0 k > 0$ ,  $q(t)$  is bounded for all  $t > 0$ , while if  $q_0 k < 0$ ,  $q(t)$  blows up at  $t = -1/q_0 k$ .

We imitate the above proof for the system (6.5), (6.7) as follows. Differentiate (6.5) with respect to  $x$ :

$$w_{tx} + \rho w_{xx} + \rho_w w_x^2 + \rho_z w_x z_x = 0.$$

Abbreviating  $w_x$  by  $p$  we can put this as

$$(6.11) \quad p' + \rho_w p^2 + \rho_z p z_x = 0.$$

From (6.7) we deduce that

$$(6.12) \quad z_x = \frac{z'}{\rho - \lambda};$$

substituting this into (6.11) we obtain

$$(6.13) \quad p' + \rho_w p^2 + \frac{\rho_z}{\rho - \lambda} z' p = 0.$$

Let  $h$  denote a function of  $w$  and  $z$  which satisfies

$$h_z = \frac{\rho_z}{\rho - \lambda}.$$

Since according to (6.5),  $w' = 0$ ,

$$h' = h_w w' + h_z z' = \frac{\rho_z}{\rho - \lambda} z'.$$

Substituting this into (6.13) we obtain

$$p' + \rho_w p^2 + h' p = 0.$$

Multiply this by  $e^h$ ; using the abbreviations

$$(6.14) \quad e^h p = q, \quad e^{-h} \rho_w = k,$$

the resulting equation can be rewritten as

$$(6.15) \quad q' + k q^2 = 0.$$

This is an ordinary differential equation for  $q$  along each  $\rho$ -characteristic, similar to (6.9) except that the coefficient  $k$  of  $q^2$  is not constant. Nevertheless an explicit formula for  $q$  can be written:

$$(6.16) \quad q(t) = \frac{q_0}{1 + q_0 K(t)},$$

where  $q_0 = q(0)$  and

$$K(t) = \int_0^t k \, dt,$$

the integration along the  $\rho$ -characteristic.

Clearly, the boundedness or not of  $q(t)$  hinges on whether  $q_0 K(t)$  ever takes on the value  $-1$ . This is easily analyzed if  $\rho_w \neq 0$  anywhere; since the sign of  $w$  is arbitrary, we might as well assume that

$$(6.17) \quad \rho_w > 0.$$

Suppose the initial values of  $w, z$  are bounded:  $|w|, |z| < M$ . The same inequalities hold for all  $t > 0$ , since the value of  $w$  or  $z$  at any point equals the value of  $w$  or  $z$  at that point on the initial line to which  $P$  can be connected by a  $\rho$ , or  $\lambda$  characteristic. Once we know that  $(w, z)$  stays for all time in a bounded set, we can conclude from (6.17) that the function  $k$  defined as  $e^{-h} \rho_w$  in (6.14) is bounded from below for all  $t$  and  $x$  by a positive constant  $k_0$ . Its integral  $K$  then satisfies

$$K(t) \geq k_0 t \quad \text{for all } t \geq 0.$$

Substituting this into (6.16) we conclude that if  $q_0 > 0$ ,  $q(t)$  stays bounded, if  $q_0 < 0$ ,  $q(t)$  becomes unbounded after a finite time. We see from (6.14) that the sign of  $q_0$  is the same as that of  $p_0$ , the initial value of  $w_x$ . So we can summarize what we have proved as follows.

**THEOREM 6.1.** *Suppose condition (6.17) is satisfied for a system of equations (6.2). Let  $u, v$  be a solution whose initial values are bounded; then if  $w_x(x, 0)$  is  $< 0$  at any point, the derivatives of the solution become unbounded after a finite time.*

A similar result holds with respect to the other variable  $z$ , and the following converse holds:

*Suppose that for a system (6.2),  $\rho_w > 0$  and  $\lambda_z > 0$ . Suppose  $u, v$  is a solution of (6.2) whose initial values are bounded, and suppose that both  $w(x, 0)$  and  $z(x, 0)$  are increasing functions of  $x$ . Then the first derivatives of the solution remain uniformly bounded, and the solution exists and is differentiable for all  $t > 0$ .*

**Remark.** It is easy to verify that the condition  $\rho_w \neq 0$  is the same as the genuine nonlinearity condition (5.3).

We turn now to solutions with shocks. Since there are two families of characteristics, there are two families of discontinuities; we shall refer to them as  $\rho$ -shocks and  $\lambda$ -shocks.

How does the Riemann invariant  $w$  change across a  $\lambda$ -shock? According to Theorem 5.1, the states  $u$ , which can be connected to  $u_l$  across a  $\lambda$ -shock form on one-parameter family  $u(\varepsilon)$ ,  $\varepsilon < 0$ ; the first derivative of  $u$  with respect to  $\varepsilon$  is given by (5.8):

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}(0) = r_\lambda.$$

Let us calculate  $\dot{w}$ :

$$\dot{w} = \text{grad } w \cdot \begin{pmatrix} \dot{u} \\ v \end{pmatrix} = \text{const. grad } w \cdot r_\lambda,$$

which according to (6.4<sub>1</sub>) is zero.

A similar calculation, based on (5.12), shows that also

$$\dot{w} = 0.$$

However  $\ddot{w}$  is in general not zero; if we impose the requirement that  $\ddot{w} \neq 0$ , say

$$(6.18) \quad \ddot{w} > 0,$$

we conclude that, at least for weak shocks,

$$(6.19) \quad w(u_l) > w(u_r).$$

Consider a solution containing a finite number of weak shocks. Let  $(x, t)$  be any point,  $t > 0$ ; draw a backward  $\rho$ -characteristic  $C$  through this point. According to the shock condition (5.5),  $C$  cannot run into a  $\rho$ -shock; so  $C$  can be continued all the way down to the initial line.  $C$  will intersect a finite number of  $\lambda$ -shocks; between two points of intersection  $w$  is constant. Since  $\lambda < \rho$ , it follows from (6.19) that  $w$  increases along  $C$  as  $t$  decreases. So we conclude:

If (6.18) holds, then

$$(6.20) \quad w(x, t) < w(y, 0),$$

where  $y$  is the point where the  $\rho$ -characteristic through  $(x, t)$  intersects the line  $t = 0$ .

We turn now to the asymptotic behavior of solutions for large  $t$ . This problem was studied for single conservation laws in §4; the main tool there was the conservation of increasing and decreasing variation of continuous solutions between two characteristics. Thanks to the existence of Riemann invariants, we have a conservation of variation of  $w$  between  $\rho$ -characteristics and of  $z$  between  $\lambda$ -characteristics, valid for continuous solutions. We saw that for solutions of a single conservation law with shocks the variation between two characteristics decreases as  $t$  increases; the same argument applied to the Riemann invariants shows that the presence of  $\rho$ -shocks causes the variation of  $w$  between characteristics to decrease with increasing time, and similarly the presence of  $\lambda$ -shocks causes the variation of  $z$  to decrease. We have however the additional task of assessing the effect of  $\lambda$ -shocks on the variation of  $w$  and of  $\rho$ -shocks on  $z$ . This has been carried out in Glimm-Lax for solutions whose oscillation is small. The precise result proved there is the following.

**THEOREM 6.2.** *Suppose condition (6.18), and an analogous condition for  $z$ , is satisfied for a system of conservation laws (6.1). Then the initial value problem for (6.1) has a solution for all bounded, measurable initial data whose oscillation is small enough. The total variation of this solution on an interval of length  $t$  at time  $t$  is bounded by a constant. For periodic solutions, the total variation of  $u$  and  $v$  per period decays as*

$$\frac{\text{const}}{t}.$$

Not a great deal is known about uniqueness; Oleinik has studied solutions of systems of the special form

$$u_t - v_x = 0, \quad v_t + f(u)_x = 0,$$

where  $f$  is an increasing function of  $u$ . She has shown that solutions which contain a finite number of shocks and centered rarefaction waves are uniquely determined by their initial data.

We turn now to the entropy condition (5.36) described in the last section:

$$U_t + F_x \leq 0$$

for all  $U(u, r)$  and  $F(u, r)$  which satisfy (5.29):

$$(6.21) \quad \text{grad } U \cdot A = \text{grad } F.$$

We can eliminate  $F$  from this system of equations by differentiation; we get a homogeneous second order equation for  $U$ :

$$(6.22) \quad aU_{uu} + bU_{uv} + cU_{vv} = 0,$$

where

$$(6.23) \quad a = -f_v, \quad b = f_u - g_v, \quad c = g_u.$$

It is easy to verify that if the original nonlinear system (6.1) is hyperbolic, so is the linear system (6.21), and the derived equation (6.22). The question is: does a second order hyperbolic equation (6.22) have convex solutions? It is fairly easy to show that in the small the answer is "yes," on the basis of this observation: If the real symmetric matrix  $a_{ij}$  is indefinite, there exists a positive definite matrix  $U_{ij}$  such that

$$\sum a_{ij}U_{ij} = 0.$$

We show now that the compatibility equation (6.21) has solutions where  $U$  is convex, provided that a certain condition, see (6.31) below, is satisfied; we do not claim that this condition is necessary.

We shall construct families of solutions depending on a parameter  $k$  in this fashion:

$$(6.24) \quad U = e^{k\varphi}V + U_N, \quad F = e^{k\varphi}H + F_N,$$

where

$$(6.25) \quad V = \sum_0^N V_j/k^j, \quad H = \sum_0^N H_j/k^j.$$

$\varphi$ ,  $V_j$  and  $H_j$  are functions of  $u$  and  $v$ .  $U_N$  and  $F_N$  are functions of  $u$ ,  $v$  and  $k$  as well, of order  $k^{-N} e^{k\varphi}$ , i.e.,

$$(6.26) \quad |D^j(F_N + U_N)| \leq O(e^{k\varphi}/k^{N-j}).$$

Expansions of this sort, with  $i\varphi$  in place of  $\varphi$ , are customary in geometrical optics.

The formal construction in the real case is the same as in the imaginary case: We substitute (6.24), (6.25) into (6.21), divide by  $e^{k\varphi}$  and equate like powers of  $k$  to obtain

$$(6.27) \quad V_0(\text{grad } \varphi)A = H_0 \text{ grad } \varphi,$$

$$(6.28) \quad V_j(\text{grad } \varphi)A + V_{j-1} = H_j \text{ grad } \varphi + H_{j-1},$$

$j = 1, 2, \dots, N$ . We solve these recursively. Equation (6.27) asserts that  $\text{grad } \varphi$  is a left eigenvector of  $A$ , with eigenvalue  $H_0/V_0$ . Such a function is called a *phase function* in geometrical optics; according to (6.4) this condition means that  $\varphi$  is a *Riemann invariant*.

Having found  $\varphi$ , we substitute it into (6.28) which we solve recursively, assigning arbitrary initial values for  $U_j$  on a noncharacteristic curve. If the initial value for  $U_0$  is chosen to be positive,  $U_0$  will be positive everywhere.

We have to solve a first order system to determine  $U_N$  and  $F_N$ ; the inhomogeneous term in this system is  $O(e^{k\varphi}/k^N)$  and, with the proper choice for an initial curve,  $U_N$  and  $V_N$  will satisfy (6.26).

When is the function  $U$  defined by (6.24), (6.25) convex for  $k$  large enough, i.e., when is  $Q = \xi^2 U_{uu} + \xi\eta U_{uv} + \eta^2 U_{vv}$  positive for all  $\xi, \eta$ ? The answer can be read off from the first two leading terms of  $Q$  in its asymptotic expansion in powers of  $k$ . The coefficient of  $k^2 e^{k\varphi}$  is

$$(\varphi_u^2 \xi^2 + 2\varphi_u \varphi_v \xi\eta + \varphi_v^2 \eta^2)U_0,$$

which is equal to

$$(6.29) \quad (\varphi_u \xi + \varphi_v \eta)^2 U_0.$$

As we have remarked before,  $U_0$  can be chosen to be positive throughout; therefore, the above form is positive except along

$$(6.30) \quad (\xi, \eta) = (-\varphi_v, \varphi_u).$$

The coefficient of  $ke^{k\varphi}$  consists of three terms; two of them are zero along (6.30); the remaining one is

$$(6.31) \quad \varphi_{uu}\varphi_v^2 - 2\varphi_{uv}\varphi_u\varphi_v + \varphi_{vv}\varphi_u^2.$$

We make the assumption that there exists a Riemann invariant  $\varphi$  for which (6.31) is positive; if this is so,  $U$  given by (6.24) is convex for  $k$  large enough.

If (6.31) is positive, the function

$$\psi(u, v) = e^{k\varphi}$$

is convex; since  $\psi$  is a function of the Riemann invariant  $\varphi$  it is itself another Riemann invariant, and so our result can be formulated thus: *If there exists a convex Riemann invariant  $\psi$  in a domain of the  $(u, v)$ -plane, there exist functions of the form*

$$(6.32) \quad \begin{aligned} U &= e^{k\psi} \{U_0 + O(1/K)\}, \\ F &= e^{k\psi} \{F_0 + O(1/k)\}, \end{aligned}$$

which satisfy (6.21); furthermore  $U$  is convex for  $k$  large enough.



What can we deduce from the entropy condition

$$(6.33) \quad U_t + F_x \leq 0$$

for functions of the form (6.32)?

**THEOREM 6.3.** *Let  $u, v$  be a solution in the integral sense of the conservation laws (6.1), which satisfies the entropy condition (6.33) for all  $U, F$  of the form (6.32),  $k$  large enough. Then*

$$\max_x \psi(u(x, t), v(x, t))$$

*is a decreasing function of  $t$ .*

*Sketch of proof.* Integrate (6.33) over a lens-shaped region contained between  $S_1$  and  $S_2$  (see Fig. 14). We obtain

$$(6.34) \quad \int_{S_2} (Un_t + Fn_x) ds \leq \int_{S_1} U dx.$$

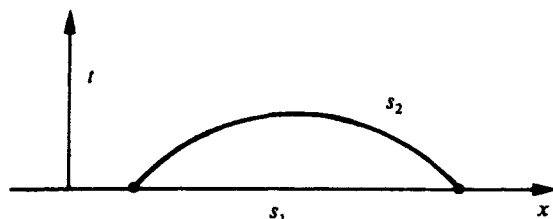


FIG. 14

If  $S_2$  is so chosen that

$$n_t U^0 + n_x F^0 > 0,$$

the  $k$ th root of the left side of (6.34) tends as  $k \rightarrow \infty$  to the maximum of  $\psi$  on  $S_2$ , while the  $k$ th root of the right side tends to the maximum of  $\psi$  on  $S_1$ . The resulting inequality proves Theorem 6.3, and even a little more.

*Remark.* The conclusion of Theorem 6.3 agrees with the statement (6.20), deduced under the assumption (6.18). It turns out that inequality (6.18) is equivalent with the positivity of (6.31).

#### Notes.

1. *Quasilinear equations.* The energy inequality for symmetric hyperbolic systems is due to Friedrichs and Lewy, for nonsymmetric hyperbolic systems, see Leray and Gårding, and Calderon. The existence theorem using the contractive character of  $\mathcal{T}$  is due to Schauder. For a more detailed discussion of these approaches, see Chapter VI of Courant and Hilbert.

For the case of functions of one space variable one can employ estimates in the maximum norm instead of the energy norm. This is done as follows: differentiate

(1.1) with respect to  $x$ ; denoting  $u_x$  by  $p$  we obtain a relation of the following sort:

$$(1) \quad p_t + Ap_x + Q(p) = 0,$$

where  $Q$  is a quadratic function of  $p$ . Denote by  $l_j$  the left eigenvector of  $A$ :

$$(2) \quad l_j A = \lambda_j l_j, \quad j = 1, \dots, n.$$

Multiplying (1) by  $l_j$  we obtain the following relation for  $p_j = l_j p$ :

$$(3) \quad \partial_t p_j + \lambda_j \partial_x p_j + q_j(p) = 0,$$

where  $q_j$  is some quadratic function of the vector  $p$ . The first two terms in (3) are a directional derivative of  $p_j$  in a so-called characteristic direction, given by

$$\frac{dx}{dt} = \lambda_j.$$

Integrating (3) along the  $j$ th characteristic connecting the point  $(x, t)$  to some point on the initial line  $t = 0$  we obtain

$$(4) \quad p_j(x, t) = p_j(0) + \int_0^t q_j(p) dt.$$

Denote by  $M(T)$  the maximum of all  $p_j(x, t)$ ,  $j = 1, \dots, n$ ,  $-\infty < x < \infty$ ,  $0 \leq t \leq T$ ; we obtain from (4) that

$$(5) \quad M \leq M(0) + cTM^2.$$

From this we deduce easily that

$$(6) \quad M(T) \leq \frac{1}{2cT} \{1 - \sqrt{1 - 4cTM(0)}\},$$

valid as long as

$$(7) \quad T \leq \frac{1}{4cM(0)}.$$

Using this estimate for the first derivatives of solutions one can show that the initial value problem (1.1), (1.3) has a solution in the time interval (7), where  $M(0)$  is defined as

$$\max |l_j \partial_x u_0(x)|.$$

The nonexistence Theorem 6.1 shows that the restriction (7) is, roughly, necessary unless one puts conditions on the initial values as in Theorem 6.2.

**2. Conservation laws.** An important class of hyperbolic systems of conservation laws are the ones governing the flow of compressible, nonviscous non-heat conductive fluids. There are five conserved quantities: mass, momenta and energy per unit volume:

$\rho$  = mass per unit volume = density,

$M$  = momenta per unit volume =  $\rho V$ , where  $(u, v, w) = V$  is flow velocity,

$E$  = energy per unit volume = internal + kinetic energy =  $\rho e + \frac{1}{2} \rho V^2$ ,  
where  $e$  = internal energy per unit mass and  $V^2 = u^2 + v^2 + w^2$ .

The fluxes are partly due to material being transported across the boundary with the velocity of the flow; for the momenta there is an additional flux due to the momentum imparted by the pressure force at the boundary, and there is an energy flux due to the work done by the pressure force at the boundary. If there is no heat conduction this accounts for all energy changes. For a nonviscous fluid the pressure is a scalar  $p$ , exerted equally in all directions. The formulas for the fluxes are:

$$\begin{aligned}\text{mass flux} &= \rho V = M, \\ \text{momentum flux} &= \begin{cases} \rho u V + (p, 0, 0), \\ \rho v V + (0, p, 0), \\ \rho w V + (0, 0, p), \end{cases} \\ \text{energy flux} &= (E + p)V.\end{aligned}$$

Internal energy  $e$ , pressure  $p$  and density  $\rho$  are related by an equation of state:

$$p = p(e, \rho).$$

Using this the fluxes can be expressed in terms of  $\rho$ ,  $M$  and  $E$ .

The jump relations for shocks in gas dynamics were first stated by Riemann, incorrectly, for he conserved entropy instead of energy. The correct relations were found by Rankine and by Hugoniot.

Other important nonlinear hyperbolic systems of conservation laws are the equations governing the motion of a shallow layer of water, and the equations of hydrodynamic flows (see Courant and Hilbert, Chapter VI).

3. *Single conservation laws.* The starting point of the rigorous theory of single conservation laws has been a paper by Hopf in 1950, where the explicit formula stated in Theorem 3.1 was given in the special case of the quadratic conservation law  $f(u) = \frac{1}{2}u^2$ . The formula for arbitrary convex  $f$  is stated in Lax (1954), and analyzed in Lax (1957).

The revealing Theorems 3.4 and 3.5 about the decrease of the  $L_1$  norm of the difference of two solutions are due to Barbara Quinn;  $L_1$  contraction also plays a role in the work of Oleinik (1957). Condition (3.38) is due to Oleinik (1959); she showed that solutions satisfying (3.38) are uniquely determined by their initial data, and Kalashnikov proved that solutions of (3.39) converge as  $\lambda \rightarrow 0$  to a solution which satisfies (3.38).

There is a parallel theory of single conservation laws in  $n$  space variables. Existence theorems are contained in Conway and Smoller, Volpert, Krushkov (1969), and Kotlow. A uniqueness theorem for piecewise continuous solutions has been given by Douglis and by Quinn; a more general uniqueness theorem has been given by Krushkov.

4. *The decay of solutions.* In his 1950 paper Hopf studied the large time behavior of solutions of quadratic conservation laws; the extension to any convex  $p$  is given in Lax (1957). The more refined Theorem 4.1, and Theorem 4.2 about the two and

only conserved quantities is given in Lax (1970). The law of decrease of increasing variation, and the method for proving it is taken from Glimm and Lax (see also Lax (1972)). For the nonconvex case, see Dafermos (1972).

5. *Hyperbolic systems of conservation laws.* The shock condition (5.5) and Theorems 5.1, 5.2, 5.3, 5.4 are given in Lax (1957). In case the  $k$ th field is linearly degenerate, i.e.,  $\text{grad } \lambda_k \cdot r_k \equiv 0$ , there exist discontinuous solutions where the discontinuity is a  $k$ -characteristic with respect to either side. Such discontinuities are called *contact discontinuities*. It can be shown that solutions with contact discontinuities only are the limits of continuous solutions.

Foy has shown that if  $u_l$  and  $u_r$  can be connected by a weak shock, then they can be connected by a viscous profile, i.e., a plane wave solution of the equation

$$u_t + f_x = \lambda u_{xx}$$

with artificial viscous term. By *plane wave* we mean a solution of the form

$$u(x, t) = \frac{v(x - st)}{\lambda},$$

where  $v$  is independent of  $\lambda$  and satisfies the ordinary differential equation

$$-sv' + f(v)' = v''.$$

Conley and Smoller have studied viscous profiles for strong shocks.

The main tool in Glimm's existence theorem, beside the difference scheme, is a functional which measures the potential interaction contained in the Cauchy data along any space-like curve. Glimm shows that this functional decreases with time.

The notion of entropy discussed here has been proposed by Lax (1971), and Krushkov (1970). The theory of the symmetric case is due to Godunov and was applied by him to the compressible flow equations.

6. *Hyperbolic systems of two conservation laws.* The nonexistence Theorem 6.1 and Theorem 6.2 are from Lax (1964); another version has been given by Zabusky (1962).

Johnson and Smoller have shown that under assumption (6.18) the Riemann initial value problem can be solved uniquely for two arbitrary initial states, not necessarily close. They have shown how to solve the initial value problem for such systems under a monotonicity assumption for the initial values. Nishida has shown that for the system

$$\begin{aligned} u_t - v_x &= 0, \\ v_t + \left(\frac{1}{w}\right)_x &= 0, \end{aligned}$$

the initial value problem can be solved for arbitrary initial values  $u(x), v_0(x)$ ,  $u_0 \geq 0$ . Nishida's work has been extended by Bakhvalov, DiPerna, Greenberg and Nishida and Smoller.

The Riemann initial value problem in gas dynamics for a broad class of equations of state has been studied by Wendroff.

Theorem 6.2 is due to Glimm and Lax. Further uniqueness theorems, in the absence of rarefaction waves have been given by Rozhdestvenskii and by Hurd. The construction of the entropy function (6.24) is carried out in greater detail in Lax (1971).

7. *Difference schemes.* No set of lectures on hyperbolic conservation laws should end without mention of the various effective difference schemes for computing solutions of conservation laws. These are used to compute solutions of specific initial value problems which come up in scientific and technological problems; problems involving two space dimensions can be handled as well. In addition to providing numerical answers to specific questions, one hopes that numerical calculations will reveal patterns which play a role in the theory to be developed about solutions of these equations.

If it were possible to prove rigorously that solutions of finite difference equations converge, this would provide a proof of the existence of solutions with arbitrarily prescribed data. So far this has been accomplished only for single conservation laws, and for a very crude difference scheme proposed by Lax (1954):

$$(8) \quad u_k^{n+1} = \frac{u_{k+1}^n + u_{k-1}^n}{2} + \frac{\Delta t}{2\Delta x} \{f_{k-1}^n - f_{k+1}^n\}.$$

Here  $u_k^n$  abbreviates an approximation to  $u$  at  $t = n\Delta t$ ,  $x = k\Delta x$ , and  $f_k^n$  abbreviates  $f(u_k^n)$ . The convergence of this scheme for a special case was verified by Lax (1957); convergence for any convex  $f$  was proved by Vvedenskaya. Convergence for any number of space variables was shown by Conway and Smoller, and also by Kotlow.

The approximation (8) is in conservation form: that is, if we think of  $u_k^n$  as an approximation to the average value of  $u$  over the cell  $[(k - \frac{1}{2})\Delta x, (k + \frac{1}{2})\Delta x]$  at time  $t = n\Delta t$ , (8) is of the generic form

$$(9) \quad u_k^{n+1} = u_k^n + \frac{\Delta t}{\Delta x} \{\bar{f}_{k-1/2} - \bar{f}_{k+1/2}\},$$

i.e., where the average value of  $u$  in the  $k$ th cell at time  $t = (n + 1)\Delta t$  differs from the average at time  $n\Delta t$  by the average of the amount that has entered and left at the endpoints during the time elapsed. The conservation character of the approximate equation (9) is expressed by the fact that the amount that enters the  $k$ th cell during the time interval  $\{n\Delta t, (n + 1)\Delta t\}$  through the left endpoint is exactly equal to the amount which leaves the  $(k - 1)$ st cell through its right endpoint during the same time interval.

In (8),  $\bar{f}_{k+1/2}$  was taken to be

$$(10) \quad \bar{f}_{k+1/2} = \frac{f_{k+1}^n + f_k^n}{2} + \frac{\Delta x}{\Delta t} \frac{u_k^n - u_{k+1}^n}{2};$$

this is a rather poor approximation to the average flux at  $k + \frac{1}{2}$  during  $\{n, n + 1\}$ , since the flux instead of being an average is evaluated at the earliest time  $n$ . In addition, for  $\Delta t$  small, the presence of a rather large amount of artificial viscosity proportional to  $(\Delta x)^2 u_{xx}/\Delta t$  causes additional errors. This second term has to be included to stabilize (8); to ensure stability one has to impose in addition the CFL condition (see Courant, Friedrichs and Lewy):

$$(11) \quad \frac{\Delta t}{\Delta x} \leq \frac{1}{\lambda_{\max}}.$$

A more accurate choice of  $\tilde{f}_{k+1/2}$  has been proposed by Lax and Wendroff (1960), (1964). Starting with the Taylor series

$$u^{n+1} = u + \Delta t u_t + \frac{(\Delta t)^2}{2} u_{tt}$$

and the formulas

$$u_t = -f_x = -Au_x, \quad u_{tt} = -f_{xt} = -f_{tx} = (-Au_t)_x = (A^2 u_x)_x = (Af_x)_x,$$

set in (9)

$$(12) \quad \tilde{f}_{k+1/2} = \frac{f_{k+1}^n + f_k^n}{2} - \frac{\Delta t}{2\Delta x} A_{k+1/2} (f_{k+1} - f_k).$$

Since this formula centers the flux properly at time  $t = (n + \frac{1}{2})\Delta$ , it is more accurate than (8); it can be shown that this formula is stable if the CFL condition (11) is satisfied.

The following modification of (12), proposed by Richtmyer (see Richtmyer and Morton), turns out to be more practical:

$$(13) \quad \tilde{f}_{k+1/2} = f(\bar{u}_{k+1/2}^{n+1/2}),$$

where

$$(14) \quad \bar{u}_{k+1/2}^{n+1/2} = \frac{u_k^n + u_{k+1}^n}{2} + \frac{\Delta t}{2\Delta x} \{f_k^n - f_{k+1}^n\}.$$

A further interesting modification which has been introduced by R. W. MacCormack has been especially efficient in the case of several space variables.

Recently, still more accurate schemes have been devised by Rusanov and by Burstein and Mirin.

Another type of difference scheme has been introduced by Godunov; his starting point is the same as in Glimm's scheme, but the approximate average value at time  $(n + 1)\Delta t$  over the  $k$ th cell is defined to be the average of the exact solution computed there. This average value is computed from the flux relation, i.e., (9) is used, with  $\tilde{f}_{k+1/2}$  taken as the exact value of  $f$  at the interface between the  $k$ th and  $(k + 1)$ st cell.

Calculations performed with the methods described above produce approximate solutions in which a shock is spread over a finite number—usually two to four—of

meshpoints. This is in sharp contrast to numerical solutions of linear equations, where discontinuities are spread over regions which are proportional to some power of the number of time steps taken.

A different computing method, called particle-in-cell, has been developed by Harlow. This method is particularly effective in several space dimensions.

A study of the formation of steady state profiles for solutions of difference equations was begun by Jennings (1973).

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