Shock Waves and Entropy

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Introduction: We study systems of the first order partial differential equations in conservation form:

$$(1) \qquad \partial_t u^j + \partial_x f^j = 0 , \quad j = 1, \ldots, m, \ f^j = f^j(u^1, \ldots, u^m).$$

In many cases all smooth solutions of (1) satisfy an additional conservation law

$$(2) \qquad \partial_t U + \partial_x F = 0 , \quad F = F(u) ,$$

where U is a convex function of u. We study weak solutions of (1) which satisfy in addition the "entropy" inequality

$$(3) \qquad \partial_t U + \partial_x F \leq 0 .$$

We show that all weak solutions of (1) which are limits of solutions of modifications of (1) by the introduction of various kinds of dissipation satisfy the entropy inequality (3). We show that for weak solutions which contain discontinuities of moderate strength, (3) is equivalent to the usual shock condition involving the number of characteristics impinging on the shock. Finally we study all possible entropy conditions of form (3) which can be associated to a given hyperbolic system of two conservation laws.

1. We consider systems of first order nonlinear partial differential equations in conservation form:

$$\partial_t u^j + \partial_x f^j = 0 \,, \quad j\text{-}1, \ldots, m \,, \tag{1.1}$$

where ∂_t and ∂_x denote partial differentiation with respect to t and x, and where each f^j is a function of $u = \{u^1, \ldots, u^m\}$, in general nonlinear. For simplicity we take the number of space variables to be 1. Carrying out the differentiations in (1.1) we get the equations

$$\partial_t u^j + \sum f^j_\ell \, \partial_x u^\ell = 0 \tag{1.2}$$ $\frac{\partial f}{\partial u^\ell} \cdot \frac{\partial u}{\partial x}$

where the subscript ℓ denotes partial differentiation with respect to u^ℓ.

If u is a solution of (1.1) which is zero for $|x|$ large integrating (1.1) we obtain that

$$\int \partial_t u^j dx = 0$$ summation of $\frac{\partial u^j}{\partial t}$ along x is zero.

which means that the quantities

$$\int u^j dx \,, \quad j = 1, \ldots, m \tag{1.3}$$

are conserved, i.e. independent of t.

Let U be some function of $u^1, \ldots, u^m$; when does U satisfy a conservation law, i.e. an equation of form

$$\partial_t U + \partial_x F = 0 \quad ? \tag{1.4}$$

?

(1.4) can be written in the form

$$\sum U_j f^j_\ell = F_\ell \qquad \text{for} \quad \ell = 1, \ldots, m. \tag{1.5}$$

We suppose now that (1.5) has a solution, so that (1.1) implies an additional conservation law (1.4); and we suppose furthermore that the new conserved quantity U is a <u>convex</u> function of $u^1, \ldots, u^m$.

We observe that if the system (1.2) is symmetric, i.e.

$$f^j_\ell = f^\ell_j$$

then

$$U = \sum u_j^2$$

satisfies a conservation law, with

$$F = \sum u^j f^j - g \quad ,$$

where g is a function satisfying

$$g_\ell = f^\ell \ .$$

As observed in [3], if the convex function U satisfies a conservation law then multiplying (1.2) by $U_{j,\ell}$ puts the system into symmetric hyperbolic form. For the purposes of section 2 we shall require a little more, that (1.2) be strictly hyperbolic. This means that the matrix

$$f' = f^j_\ell \qquad (1.8)$$

flux tensor

has real and distinct eigenvalues. We denote these eigenvalues arranged in increasing order, by $c_1, c_2, \ldots, c_m$. They are called sound speeds, and are functions of u.

It is well known that the initial value problem is properly posed for symmetric hyperbolic systems, i.e. we may prescribe the values of u as arbitrary smooth functions at $t = 0$. Solutions are uniquely determined by their initial data but, since the governing equations are nonlinear, in general they exist only for a limited time range. For solutions which exist for all $t > 0$ we have to turn to weak solutions, which may be discontinuous solutions or slightly worse; these satisfy the conservation laws (1.1) only in the weak, or integral sense, i.e.

$$\iint_{t \geq 0} [u^j \partial_t \phi + f^j \partial_x \phi] dx dt + \int u(x,0)\, \phi(x,0) dx = 0 \qquad (1.9)$$

for all smooth test functions ϕ with bounded support in $t \geq 0$. As is well known, for piecewise continuous solutions (1.9) is equivalent to the Rankine-Hugoniot jump conditions

$$s[u^j] - [f^j] = 0 \tag{1.10}$$

where s is the speed with which the discontinuity is propagating, and $[u]$ denotes the difference between the values of u on the two sides of the discontinuity.

It is equally well known, see e.g. [8] for some simple examples, that weak solutions of conservation laws are not uniquely determined by their initial values. To pick out the physically relevant solutions among the many, some additional physical principle has to be introduced. This additional principle usually identifies the relevant solutions as limits of solutions of equations with some dissipation. Specifically, we consider the equations with artifical viscosity:

$$\partial u_t^j + \partial_x f^j = \varepsilon\, \partial_x^2 u^j, \quad \varepsilon > 0 . \tag{1.11}$$

Suppose that a sequence $u(x,t;\varepsilon)$ of solutions of (1.11) tends to a limit $u(x,t)$ boundedly, almost everywhere; then $\varepsilon\, \partial_x^2\, u(\varepsilon)$ tends to 0 in the topology for distributions, so that the limit u satisfies (1.1) in the weak sense.

We show next how to characterize such limit solutions directly, with the aid of the function U :

Multiplying (1.11) by U_j and summing we get

$$\partial_t U + \partial_x F = \varepsilon \sum U_j\, \partial_x^2 u^j . \tag{1.12}$$

Using the identity

$$\partial_x^2 U = \sum U_j \partial_x^2 u^j + \sum U_{jk} \partial_x u^j \partial_x u^k$$

and the convexity of U we deduce that

$$\partial_x^2 U \geq \sum U_j \partial_x^2 u^j .$$

Since ε is > 0 , using this to estimate the right side of 91.12) we get

$$\partial_t U + \partial_x F \leq \varepsilon\, \partial_x^2 F .$$

Letting $\varepsilon \to 0$ the right side tends to 0 in the topology of distributions and we deduce

Theorem 1.1: Let (1.1) be a system of conservation laws which implies an additional conservation law (1.4) where U is a strictly convex function. Then every weak solution of (1.1) which is the limit, boundedly a.e., of solutions of the viscous equation (1.11) satisfies the inequality

(1.13) $$\partial_t U + \partial_x F \leq 0 .$$

Remark A: Suppose u satisfies (1.13) and has compact support in x . Integrating (1.13) with respect to x gives

$$\int \partial_t U\, dx \leq 0$$

which implies that

(1.14) $$\int U\, dx$$

is a decreasing function of t .

Remark B: Suppose that u is a piecewise continuous weak solution of (1.1); then it is easy to deduce either from (1.13) or (1.14) that at a point of discontinuity

(1.15) $$s[U] - [F] \leq 0 ,$$

where s is the velocity with which the discontinuity propagates, and $[U]$, $[F]$ denote the jumps $U_{left} - U_{right}$ and $F_{left} - F_{right}$, respectively.

For compressible fluid flow (1.14) corresponds to the increase of total negative entropy, and (1.15) states that the classical entropy of particles upon crossing a shock increases. For this reason we shall call (1.13) and (1.15) entropy conditions.

The addition of a viscous term as in equation (1.11) is only one of many ways of introducing a slight amount of artificial dissipation into the system (1.1). Another way is to discretize the differential equations; one of the standard ways of doing this is to replace the operator ∂_t and ∂_x by the following difference operators. Denote by $T(h)$ translation in t by the amount h, and by $S(k)$ translation in x by the amount k. Define

$$D_t = \frac{1}{\Delta t}\left\{T(\Delta t) - \frac{S(\Delta x) + S(-\Delta x)}{2}\right\},$$

(1.16)

$$D_x = \frac{S(\Delta x) - S(-\Delta x)}{2\Delta x}.$$

We consider now the difference equation

$$D_t u_s + D_x f^j = 0 \tag{1.17}$$

and study limits of solutions of (1.17) as $\Delta t,\ \Delta x \to 0$ while the ratio $\frac{\Delta t}{\Delta x} = \lambda$ remains constant. We assume that $u(x,t)$ is the limit, boundedly and almost everywhere, of solutions $u(x,t,\Delta t,\Delta x) = u(\Delta)$ of (1.17); it follows that $D_t u(\Delta)$ and $D_x f(u(\Delta))$ tend, in the topology of distributions, to $\partial_t u$ and $\partial_x f(u)$, so that it follows that the limit function u satisfies the system of conservation laws (1.1). We shall show now, under an additional restriction on λ, that such a limit u satisfies the entropy inequality (1.13). It suffices to show that every solution of (1.17) satisfies the inequality

$$D_t U + D_x F \leq 0, \tag{1.18}$$

for the limit of (1.18) in the topology of distributions is (1.13). We introduce the following vector notation:

$$u(x,t+\Delta t) = u,\ u(x-\Delta x,t) = v,\ u(x+\Delta x,t) = w.$$

We regard u, v and w as column vectors. Then

$$D_t u = \frac{1}{\Delta t}\{u - \frac{v+w}{2}\}, \quad D_x f = \frac{1}{2\Delta x}\{f(w)-f(v)\};$$

substituting this into (1.17) and solving for u we get, with the notation

$$\frac{\Delta t}{\Delta x} = \lambda$$

(1.19) $$u = \frac{v+w}{2} + \frac{\lambda}{2}[f(v) - f(w)] .$$

Inequality (1.18) asserts that

(1.20) $$U(u) \leq \frac{U(v) + U(w)}{2} + \frac{\lambda}{2}[F(v) - F(w)] .$$

We deform v continuously into w; set

$$v(s) = sv + (1-s)w .$$

Since $v(0) = w$, both sides of (1.20) equal $U(w)$ for $s = 0$. So we can write the difference of the right and the left side in (1.20) as the integral of the difference of their derivatives with respect to s. This difference is

(1.21) $$\frac{1}{2}U'(v)(v-w) + \frac{\lambda}{2}F'(v)(v-w) - U'(u)\frac{du}{ds}$$

where U' and F' are the gradients of U and F, regarded as row vectors. From (1.19) we get

(1.22) $$\frac{du}{ds} = \frac{v-w}{2} + \frac{\lambda}{2}f'(v)(v-w);$$

f', the gradient of the vector quantity f, is a matrix.

In this notation we can write identity (1.6) as follows

(1.23) $$U'f' = F' .$$

Substituting (1.23) and (1.22) into (1.21) we get

$$\frac{1}{2}[U'(v) - U'(u)][I + \lambda f'(v)](v-w) \quad . \tag{1.24}$$

Next we set

$$w(r) = rv(s) + (1-r)w = rsv + (1-rs)w \ .$$

Since $w(1) = v(s)$, $w(0) = w$, we have

$$U'(v) - U'(u) \int_0^1 \frac{d}{dr} U'(u)dr \ .$$

We write

$$\frac{d}{dr} U'(u) = \frac{du^t}{dr} U'' \ ,$$

where U'' is the matrix of second derivatives of U and u^t the transpose of u. From (1.19) we get

$$\frac{du}{dr} = \frac{s}{2}[v-w-\lambda f'(w)(v-w)] \ .$$

Substituting these relations back into (1.25) we see that the difference between the right and left side of (1.20) is the double integral from 0 to 1 with respect to s and r of

$$\frac{s}{4}[[I-\lambda f'(w)](v-w]^t \cdot U'' [I+\lambda f'(v)](v-w) \ . \tag{1.25}$$

By assumption U'' is positive definite; this implies that <u>for λ small enough</u> (1.25) <u>is positive</u>. The precise restriction on λ is as follows:

Denote by m and M the minimum and maximum eigenvalue of U'' in that portion of u-space in which we are operating, and denote by c the norm of f' there. Denote $v - w$ by z. The the following is a lower bound for the expression to the right of $s/4$ in (1.25):

$$m\|z\|^2 - M(2c\lambda+c^2\lambda^2)\|z\|^2 \ .$$

Clearly this is positive for $z \neq 0$ if

(1.26) $$c\lambda \leq \sqrt{1+m/M}-1 .$$

Thus we have proved

Theorem 1.2: Suppose all differentiable solutions of the system of conservation laws (1.1) satisfy an additional conservation law (1.4), where U is a strictly convex function of u. Then all weak solutions of (1.1) which are the limits, boundedly a.e., of solutions of the difference equation (1.17) satisfy the entropy inequality (1.13), provided that condition (1.26) is fulfilled.

Remark: In the case of f' symmetric, the norm c of f' equals the absolute value c_{max} of the largest eigenvalue of f'. In this case $U = \sum u_j^2$ is a conserved quantity; for this U, $U'' = 2I$, so $m/M = 1$ and condition (1.26) becomes

$$c_{max} \frac{\Delta t}{\Delta x} \leq \sqrt{2} - 1 = .414 .$$

This is slightly more stringent than the Courant-Friedrichs-Lewy necessary condition for convergence,

$$c_{max} \frac{\Delta t}{\Delta x} \leq 1 .$$

In the nonsymmetric case (1.26) is a still more stringent version of the C-F-L criterion, since then the norm c of f' is $> c_{max}$, and the condition number m/M of U'' is < 1. We remark that Theorem 1.1 was also proved by Kružkov at the end of [14], and he has suggested condition (1.13) as a generalized entropy condition.

2. In this section we assume that (1.1) is strictly hyperbolic, i.e. that the matrix $f' = (f^j)$ has distinct real eigenvalues $c_1, \ldots, c_m$, indexed in increasing order. The c_j are the characteristic speeds; they are functions of u. We also assume that the system is genuinely nonlinear in the sense of [8].

At a point of discontinuity of a solution we shall denote the value of u on the left, respectively right side of the discontinuity as follows:

$$u_{\text{left}} = v \,, \quad u_{\text{right}} = w \,. \tag{2.1}$$

A point of discontinuity is called a <u>k-shock</u> if

a) The Rankine-Hugoniot relation

$$s[v-w] = f(v) - f(w) \tag{2.2}$$

holds

b) There are exactly $k-1$ of the characteristic speeds $c_j(v) < s$ and exactly $(m-k)$ speeds $c_j(w) > s$:

$$\begin{aligned} &c_{k-1}(v) < s < c_k(v) \,, \\ &c_k(w) < s < c_{k+1}(w) \,. \end{aligned} \tag{2.3}$$

We shall call (2.3) the <u>shock condition.</u>

<u>Theorem 2.1:</u> Suppose that the system of conservation laws (1.1) is strictly hyperbolic, and that there is a strictly convex function U of u which satisfies the additional conservation law (1.4). Let u be a weak solution of (1.1) which has a discontinuity propagating with speed s, and suppose that the values of v and w on the left and right sides of the discontinuity are close. Then the shock condition (2.3) is satisfied if and only if the strict entropy condition (1.15):

$$s[U(v) - U(w)] - F(v) + F(w) < 0 \tag{2.4}$$

is satisfies.

<u>Proof:</u> It was shown in [8] that all states w near v which satisfy the R-H condition (2.2) form m one-parameter families $w_k(r)$, $s_k(r)$ where $w_k(0) = v$. If the parametrization is so taken that

$$\left.\frac{ds_k}{dr}\right|_{r=0} > 0 \tag{2.5}$$

then those w_k which correspond to $r < 0$ satisfy the shock condition (2.3).

To prove (2.4) we shall substitute for w one of these families $w_j(r)$ and expand the left side of (2.4) in powers of r; the crux of the argument is to show that the lowest power r^p which is different from 0 is odd, and that the coefficient of r^p is positive.

Let's denote differentiation with respect to r by a dot $\cdot$ and, as before, the gradient with respect to u by prine $'$. The crucial exponent p turns out to be 3, so we have to calculate the first 3 derivatives of the left side of (2.4) at $r = 0$. Differentiating (2.2) we get

$$\dot{s}[v-w] - s\dot{w} = -\dot{f}(w). \tag{2.6}$$

The derivative of the left side of (2.4) in r is

$$\dot{s}\,[U(v) - U(w)] - s\dot{U} + \dot{F}\ . \tag{2.7}$$

Using relation (1.6), $F' = U'f'$, we can write

$$\dot{F} = F'\dot{w} = U'f'\dot{w} = U'\dot{f}\ .$$

Substituting for $\dot{f}$ from (2.6) we get

$$\dot{F} = \dot{s}U'[w-v] + sU'\dot{w}\ .$$

Substituting this into (2.7) and noting that $\dot{U} = U'\dot{w}$ we get the following expression:

$$\dot{s}\,[U(v) - U(w)] + \dot{s}U'[w-v]\ .$$

Differentiating once more we get

$$\ddot{s}\,[U(v)-U(w)] + \ddot{s}\ U'[w-v] + \dot{s}\ \dot{U}'[w-v]\ .$$

Since $w(0) = v$, this is clearly zero at $r = 0$.

Differentiating once more, and setting $r = 0$ we get, after eliminating, those terms which are zero when $w = v$,

$$\dot{s}\,\dot{U}'\dot{w} .$$

The remaining term can be written as

$$\dot{s}\,\dot{w} + U''\dot{w} ; \tag{2.8}$$

since according to (2.5) the parametrization is so chosen that $\dot{s}$ is positive, and since U'' is positive because of the strict convexity of U , it follows that (2.8) is positive. This proves inequality (2.4) of theorem 2.1.

A noncalculational proof can be given using the following result of Foy, [2]:

If two nearby states v and w can be connected through a shock, then they can be connected through a viscous profile, i.e. a steady progressing solution of (1.11) of the form

$$u(x,t,\varepsilon) = w\left(\frac{x-st}{\varepsilon}\right), \quad w(-\infty) = v, \quad w(\infty) = w \ . \tag{2.9}$$

Substituting this form of u into (1.11) gives for the function w the ordinary differential equation

$$-s\dot{w} + \dot{f} = \dot{w} . \tag{2.10}$$

Clearly, the discontinuous solution

$$u(x,t) = \begin{cases} v & \text{for } x < st \\ w & \text{for } st < x \end{cases} \tag{2.11}$$

is the weak limit of $w(\frac{x-st}{\varepsilon})$ as $\varepsilon \to 0$. Therefore according to theorem 1.1, the solution (2.11) satisfies the entropy condition.

Recently Conley and Smoller, [1], have shown that, for a fairly general class of systems of two conservation laws, any two states v and w which can be connected through a shock can also be connected through a viscous profine. It follows from the above argument that for such systems the restriction that v and w be close can be removed from theorem 2.1.

In his important paper [4] Glimm constructs solutions of systems of conservation laws as the limit of approximate solutions. These are piecewise continuous weak solutions in each strip $k\Delta t < t < (k+1)\Delta t$, and all their discontinuities are shocks; in addition the oscillation of these solutions is small. If there is a convex function which satisfies an additional conservation law, it follows from theorem 2.1 that the entropy condition

$$\partial_t U + \partial_x F \leq 0 \tag{2.12}$$

is satisfied by each approximate solution in each strip. Let ϕ be a smooth, positive test function with compact support. Multiply (2.12) by ϕ, integrate over each strip; integrating by parts with respect to t over each strip and summing over all strips we get

$$\sum_{k=1}^{\infty} \int \phi(x, k\Delta)[U(x, k\Delta+) - U(x, k\Delta-)]dx + \iint (-(\partial_t \phi)U + \phi\partial_x F)dxdt \leq 0 . \tag{2.13}$$

Lemma (5.1) in Glimm's paper shows that the sum in (2.13) tends to zero for a suitably selected subsequence. This leaves us in the limit with

$$\iint (-\phi_t U + \phi F_x)dxdt \leq 0$$

for all positive test function ϕ supported in $t > 0$. Integrating by parts with respect to t we get that

$$\iint \phi[\partial_t U + \partial_x F]dxdt \leq 0$$

for all such ϕ. Clearly this implies (2.12). Thus we have proved

Theorem 2.2: Suppose that the system of conservation laws (1.1) is strictly hyperbolic, and that there is a convex function U of u which satisfies the additional conservation law (1.4). Then all weak solutions of (1.1) constructed by Glimm's method satisfy the entropy inequality (2.12).

3. What systems admit an additional conservation law where the additional conserved quantity U is a convex function of the original ones? We saw that symmetric systems do, and so does the system consisting of the laws of conservation of mass, momentum and energy for a compressible gas. A systematic search for additional conservation laws was carried out by Rozdestvenskii, [12]; in this section we record some observations on the existence and utility of additional convex conservation laws.

We start with a single conservation law:

$$\partial_t u + \partial_x f = 0 . \tag{3.1}$$

In this case we may choose for U any convex function; F is then determined by integrating the compatibility relation (1.6):

$$U'f' = F' . \tag{3.2}$$

The entropy condition,

$$\partial_t U + \partial_x F \leq 0 , \tag{3.3}$$

was derived for smooth U only; by passing to the limit in the topology of distributions we deduce (3.3) for any convex U, smooth or not.

Every convex function lies in the convex cone generated by the functions $U(u) = |u-z|$, z some constant, and by the linear functions. In [7], Krushkov takes (3.3) for all U of this form to be the definition of the relevant class of weak solutions of the analogue of (3.1) for n space variables; he proves existence of such solutions with arbitrary initial data, and announces their uniqueness.

We present now some known consequences of the entropy inequality (3.3); the first was found independently by Krushkov and by Hopf, [6]:

Suppose that u is a piecewise continuous weak solution of (3.1) which satisfies (3.3); let's denote the speed of propagation of a discontinuity by s, and denote by v and w the values of u on the left, respectively right side of the discontinuity. According to (1.15), (3.3) implies that

$$s[U(v) - U(w)] - F(v) + F(w) < 0 . \tag{3.4}$$

Suppose that $w < v$; let z be any number between w and v, and set

$$U(u) = \begin{cases} 0 & \text{for } u < z \\ u - z & \text{for } z < u \end{cases} \tag{3.5}$$

It follows from (3.2) that then

$$F(u) = \begin{cases} 0 & \text{for } u < z \\ f(u) - f(z) & \text{for } z < u . \end{cases} \tag{3.6}$$

Substituting these into (3.4), and using the jump relation

$$s = \frac{f(v) - f(w)}{v - w} \tag{3.7}$$

we get a relation which, after rearrangement, becomes

$$f(z) \leq \frac{v-z}{v-w} f(v) + \frac{z-w}{v-w} f(v) \quad \text{for } w \leq z \leq v . \tag{3.8$_+$}$$

The geometrical meaning of (3.8) is that the graph of f over the interval $[w, z]$ lies below the chord connecting $(w, f(w))$ with $(v, f(v))$; for $w > v$ the opposite inequality $(3.8)_-$ obtains; these inequalities are Oleinik's celebrated condition (E), see [10].

We remark that in [11] B. Quinn has shown that $(3.8)_+$ is necessary and sufficient for L_1 contraction, more precisely:

If a pair of piecewise continuous solutions u_1 and u_2 both satisfy $(3.8)_\pm$, then

$$\int |u_1(x,t) - u_2(x,t)|\, dx \tag{3.9}$$

is a decreasing function of t; conversely, if (3.9) is a decreasing function of t for a certain piecewise continuous u_1 and every continuous u_2, then u_1 satisfies $(3.8)_\pm$.

We shall derive now another consequence of (3.3):

Let u be a weak solution of (3.1) which satisfies (3.3), and which is 0 at $x = \pm\infty$. Then as observed in (1.14), for U convex, and $U(0) = 0$,

$$\int U(u(x,t_2))dx \le \int U(u(x,t_1))dx \quad \text{for } t_1 < t_2 . \tag{3.10}$$

Choose for U the convex function

$$U(u) = |M - u| - M \tag{3.11}$$

where

$$M = \text{ess.}\sup_x u(x,t_1) . \tag{3.12}$$

Since $u(x,t_1) \le M$, a.e., it follows that

$$U(u(x,t_1)) = -u(x,t_1) \text{ a.e.;} \tag{3.13}$$

on the other hand, since $|M - u| \ge M - u$,

$$U(u(x,t_2)) \ge -u(x,t_2) . \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.10) we get that

$$-\int u(x,t_2)dx \le -\int u(x,t_1)dx . \tag{3.15}$$

On the other hand it follows from (3.1) that

$$\int u(x,t)dt$$

is independent of t; therefore in (3.15) the sign of equality holds. But this can be if and only if in (3.14) equality holds a.e.; this is the case if and only if

$$u(x,t_2) \le M \quad \text{for almost all} \quad x .$$

In view of the definition of (3.12) of M this result can be expressed as follows:

$$\operatorname*{Ess\,sup}_x u(x,t)$$

is a decreasing function of t for every weak solution u which satisfies (3.3) for all convex U. Similarly,

$$\operatorname*{Ess\,inf}_x u(x,t)$$

is an increasing function of t.

We turn now to pairs of conservation laws:

$$\begin{aligned} &\partial_t u^1 + \partial_x f^1 = 0 \\ (3.16) \qquad & \\ &\partial_t u^2 + \partial_x f^2 = 0 , \end{aligned}$$

these can be written in the form

$$\partial_t u + f' \partial_x u = 0 . \tag{3.17}$$

The compatibility relation (1.6) is

$$F' = U' f' , \tag{3.18}$$

a pair of first order equations for the two functions F and U. These equations are linear; it is easy to show that if the nonlinear system (3.17) is strictly hyperbolic, so is† (3.18). For suppose that the matrix f' has 2 distinct real eigenvalues; let r be a right eigenvector of f':

$$f'r = cr . \tag{3.19}$$

Multiplying (3.18) by r on the right gives

$$F'r = U'f'r = cU'r , \tag{3.20}$$

a linear combination in which both F and U are differentiated in the direction r. The existence of two such directions shows that (3.18) is hyperbolic, and its characteristic directions are those of the right eigenvectors of f'.

We can eliminate F from (3.18) by differentiating the first equation with respect to u^2, the second with respect to u^1, and subtracting one from the other. We get a homogeneous 2^{nd} order equation for U of the form

$$SU = a_{11}U_{11} + a_{12}U_{12} + a_{22}U_{22} = 0 \tag{3.21}$$

where S is the second order operator with coefficients

$$a_{11} = -f_2^1 , \; a_{12} = f_1^1 - f_2^2 , \; a_{22} = f_1^2 . \tag{3.22}$$

Equation (3.21), being derived from a hyperbolic first order system, is itself hyperbolic, which means that the quadratic form

$$a_{11}\xi^2 + a_{12}\xi\eta + a_{22}\eta^2 \tag{3.23}$$

is indefinite.

We turn now to the question: does equation (3.21) have convex solutions? It is pretty easy to see that it does in the small, on the basis of this

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†and similarly, as shown by Loewner, if (3.17) is elliptic, so is (3.18).

Lemma 3.1: Let a_{ij} be a symmetric $n \times n$ matrix which is indefinite; then there exists a positive definite matrix U_{ij} such that

$$\sum a_{ij} U_{ij} = 0 . \tag{3.24}$$

Proof: Since a_{ij} is indefinite, there exist vectors $\{\xi_i\} = \xi$ such that

$$\sum a_{ij}\, \xi_i\, \xi_j = 0 . \tag{3.25}$$

The set of these separates the set of those vectors where the quadratic form $\xi\, a\, \xi$ is positive from the set where the form is negative. It follows that the set of ξ satisfying (3.25) spans the whole space; denote by $\xi^1, \ldots, \xi^n$ a spanning set. Now define U_{ij} by

$$U_{ij} = \sum_k \xi_i^k \xi_j^k ;$$

since the ξ^k span the whole space, U_{ij} is positive definite; on the other hand it follows from (3.25) that condition (3.24) holds; this completes the proof of the lemma.

Applying the lemma to the quadratic form (3.23) at some point v we conclude that there exists a positive definite U_{ij} which satisfies (3.21) at v. By solving an appropriate Cauchy problem we can construct a solution U whose second derivatives at v equal U_{ij}; this solution will be convex near v. Thus we have proved

Theorem 3.2: A homogeneous second order hyperbolic equation has a convex solution in the neighborhood of every point.

We show now that the compatibility equation (3.18) has solutions with U convex in any domain G where a certain inequality, see (3.39), is satisfied. We do not claim that this condition is necessary.

We shall construct a one-parameter family of such solutions; we start with approximate solutions of the form

(3.26) $$U_{\text{approx}} = e^{k\phi}V, \quad F_{\text{approx}} = e^{k\phi}H ,$$

where

(3.27) $$V = \sum_0^N V^j/k^j , \quad H = \sum_0^N H^j/k^j ;$$

ϕ, V^j and H^j are independent of k. Solutions of this sort, with $i\phi$ in place of ϕ, were constructed in [9]. For this reason we only sketch the details.

Substituting (3.26) in (3.18) gives, after division by ek^{ϕ}, the equation

$$k\phi'Vf' + V'f' = k\phi'H + H' .$$

We substitute (3.27) into the above equation; equating coefficients of various powers of k we get

(3.28) $$V^o\phi'f' = H^o\phi' ,$$

and

(3.29) $$V^j\phi'f' + V^{(j-1)'} = H^h\phi' + H^{(j-1)'} .$$

Equation (3.28) asserts that ϕ' is a left eigenvector of f':

(3.30) $$\phi'f' = c\phi'$$

with

(3.31) $$cV^0 = H^0 .$$

Such a function ϕ, called a <u>phase function,</u> is easily constructed since a left eigenvector is characterized by orthogonality to the right eigenvector r corresponding to the other eigenvalue:

(3.32) $$\phi'r = 0$$

where

(3. 33) $$f'r = sr\,, \quad s \neq c\,.$$

Substituting (3. 30) into (3. 29) gives

$(3.34)^j$ $$(cV^j - H^j)\phi' = H^{(j-1)'} - V^{(j-1)'}f'.$$

The first step in solving equations (3. 34) is to multiply $(3.34)^j$ by r; using (3. 33) we get

$$0 = (H^{j-1} - sV^{j-1})'r\,.$$

Using (3. 31) we get

$$(c-s)\,V^{o}{}'r + c'rV^{o} = 0\,.$$

This is a first order equation for V^o which can be solved once we prescribe the value of V^o on a noncharacteristic initial curve. Notice that if we prescribe positive values for V^o initially, V^o is positive everywhere.

Having determined V^o and H^o, equation $(3.34)^j$ gives one linear relation between V^1 and H^1; proceeding recursively we can determine all V^j and H^j.

The functions U_{approx} and F_{approx} of form (3. 26), (3. 27) constructed in this fashion satisfy the approximate equation

$$U'_{approx}f' = F'_{approx} + e^{k\phi}R_n/k^N\,.$$

We shall now construct another solution

(3. 35) $$U'_N f' = F'_N + e^{k\phi}R_N/k^N$$

of this same equation such that

(3. 36) $$U_N\,, F_N = e^{k\phi}0(1/k^N).$$

Then

$$U = U_{approx} - U_{N}, \; F = F_{approx} - F_{N}$$

are exact solutions of $U'f' = F'$, of which the leading term is (3.26).

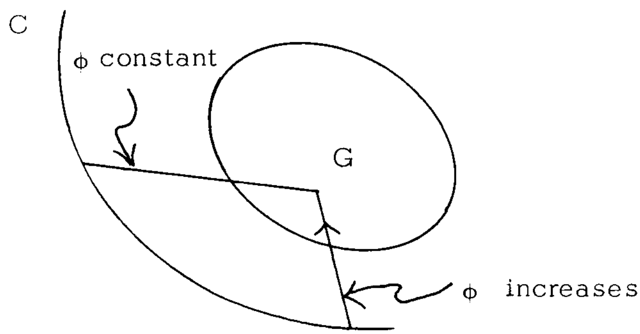
To construct a solution of (3.35) which satisfies (3.36) in some domain G of the u-plane we assign initial values zero for U_N, F_N along a non-characteristic curve C with these properties:

i) C doesn't intersect G .

ii) G is contained in the domain of determinacy of C .

In what follows we assume that ϕ has no ciritcal points, i.e. $\phi' \neq 0$ everywhere. Then ϕ is monotonic along every curve whose tangent is not parallel to r appearing in (3.32); in particular ϕ is monotonic along characteristics corresponding to the other eigenvector of f'. Our last condition is

iii) Along these other characteristics ϕ increases in the direction from C toward G .



Condition iii) determines the side of G on which C lies.

It is easy to show, using standard estimates in the maximum norm for the hyperbolic equation (3.35), that if the initial values of U_N and F_N are chosen to be zero on C , then $U_N \leq 0(e^{k\phi}/k^N)$ in G as $k \to +\infty$. This completes the construction of exact solutions.

When is the function U just constructed convex? The answer can be read off from the first 2 leading terms in the asymptotic expression for the quadratic form of U'':

$$
\begin{aligned}
e^{-k\phi}\{U_{11}\xi^2+2U_{12}\xi\eta+U_{22}\eta^2\} &= k^2\{\phi_1^2\xi^2+2\phi_1\phi_2\xi\eta+\phi_2^2\eta^2\}V^o + \\
&+ k\{\phi_1^2\xi^2+2\phi_1\phi_2\xi\eta+\phi_2^2\eta^2\}V^1 \\
&+ 2k\{\phi_1V_1^o\xi^2+(\phi_2V_1^o+\phi_1V_2^o\xi\eta+\phi_2V_2^o\eta^2\} \\
&+ k\{\phi_{11}\xi^2+2\phi_{12}\xi\eta+\phi_{22}\eta^2\}V^o.
\end{aligned} \tag{3.37}
$$

The coefficient of k^2 is

$$(\phi_1\xi + \phi_2\eta)^2 V^o ,$$

a positive quantity except along

$$\xi = -\phi_2 , \quad \eta = \phi_1 . \tag{3.38}$$

The coefficient of k consists of 3 terms, of which the first 2 are zero along the line (3.38). We impose now the condition that the third term be positive along this line:

$$\phi_{11}\phi_2^2 - 2\phi_{12}\phi_1\phi_2 + \phi_{22}\phi_1^2 > 0 . \tag{3.39}$$

It follows then that U is convex for k large enough, positive.

Let u be any differentiable solution of (3.17):

$$\partial_t u + f'\partial_x u = 0 .$$

Multiplying by ϕ' and using (3.30) we get

$$\partial_t\phi + c\partial_x\phi = 0 ;$$

this equation asserts that ϕ is constant along one of the characteristics of the nonlinear equation (3.17). Such a function is called a Riemann invariant; thus the phase functions of the linear compatibility equation (3.18) are the Rieman invariants of the nonlinear equation (3.17).

Any function $p(\phi)$ of a Riemann invariant is another Riemann invariant. Denoting $p(\phi)$ by ψ and differentiating with respect to ϕ by a dot we have

$$\begin{aligned}(3.40)\qquad &\psi_{11}\xi^2 + 2\psi_{12}\xi\eta + \psi_{22}\eta^2 \\ &= \ddot{p}\,\{\phi_1^2\xi^2 + 2\phi_1\phi_2\xi\eta + \phi_2^2\eta^2\} \\ &+ \dot{p}\,\{\phi_{11}\xi^2 + 2\phi_{12}\xi\eta + \phi_{22}\eta^2\}\,.\end{aligned}$$

We deduce from this that if ϕ satisfies (3.38), so does every increasing function ψ of ϕ. This shows that property (3.39), except for the sign, does not depend on the particular choice for ϕ.

It is easy to decide when (3.39) can be satisfied. Denote by dot differentiation in the direction

$$(3.41)\qquad \dot{u} = r\,,$$

where r is the right eigenvector appearing in (3.32). Differentiating (3.30) in the above direction and multiplying by r we get, using (3.32) and (3.33) that

$$(c-s)\,\dot{\phi}'\,r = \phi'\dot{f}'r\,.$$

In view of (3.41), $\dot{\phi}'r$ is the left side of (3.39); therefore (3.39) can be satisfied if and only if

$$(3.42)\qquad \phi'\dot{f}'r \neq 0\,,$$

a condition on the derivatives and right and left eigenvectors of f'.

We shall now use the solutions (3.26) constructed to prove

Theorem 3.3: Let $u(x,t)$ be a weak solution of the conservation laws (3.17), defined for $t \geq 0$, which satisfies the entropy condition

$$U_t + F_x \leq 0 \tag{3.43}$$

for all convex solutions U of (3.18). Let ϕ be a Riemann invariant which satisfies (3.39) in a domain G which contains all values of $u(x,t)$. Then

$$\operatorname*{Max}_x \ \phi(u(x,t)) \tag{3.44}$$

is a decreasing function of t.

Actually we shall prove a sharper theorem of which Theorem 3.3 is a corollary:

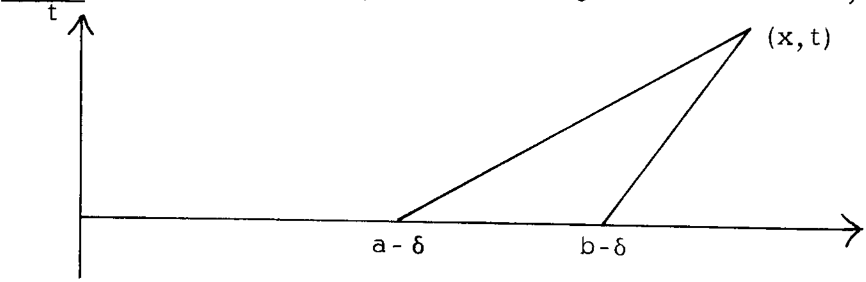
Theorem 3.4: Denote by c_{min} and c_{max} the minimum and maximum of $c(u)$ in G, where c is the eigenvalue in (3.30). Then at any point (x,t), $t > 0$,

$$\varphi(u(x,t)) \leq \sup_{a \leq y \leq b} \phi(u(y,0)) \tag{3.45}$$

where

$$a = x - tc_{max}, \ b = x - tc_{min} . \tag{3.46}$$

Proof: Integrate (3.43) over the triangle T shown below,



where δ is some positive quantity. Using the divergence theorem we get

$$(3.47) \qquad \int_{a-\delta}^{(x,t)} + \int_{(x,t)}^{b+\delta} [Un_t + Fn_x]ds \le \int_{a-\delta}^{b+\delta} U(y,0)dy ,$$

where (n_x, n_t) is the outward normal to T. Substituting the special solutions with leading term (3.26), the leading term on the left in (3.47), can be written, using (3.31), as

$$\text{const} \int e^{k\phi}[c_{max} - c + \frac{\delta}{t}] V^o ds$$

$$+ \text{const} \int e^{k\phi}[c - c_{min} + \frac{\delta}{t}] V^o ds ;$$

since V^o and δ are >0, this is bounded from below by

$$\text{const} \int_{a-\delta}^{(x,t)} \int_{(x,t)}^{b+\delta} e^{k\phi} ds .$$

On the other hand the right side of (3.47) is bounded from above by

$$\text{const} \int_{a-\delta}^{b+\delta} e^{k\phi} dy .$$

The k^{th} root of the former tends, as $k \to \infty$ to $\text{Max}\ \phi(u)$ on the segments connecting $a - \delta$, (x,t) and $b + \delta$, while the k^{th} root of the latter tends to $\text{Max}\ \phi(u)$ along $(a-\delta, b+\delta)$. Therefore inequality (3.47) implies, if we take the k^{th} root, let $k \to \infty$ and $\delta \to 0$, that (3.45) holds.

We have shown in Theorems 1.1 and 2.2 that weak solutions which are the limits of solutions of the viscous equation, or of Glimm's scheme, satisfy all entropy inequalities (3.43). It follows therefore that the estimates for the Riemann invariants asserted in Theorems 3.3 and 3.4 hold for such weak solutions. This conclusion is not new; we indicate the relation of condition (3.39) to known results.

We start with the observation that if ϕ satisfies (3.39) and if the function p is chosen so that $\ddot{p}$ is very much larger than $\dot{p}$, then the first term on the right in (3.40) is larger than the second except near these values where $\phi_1\xi + \phi_2\eta = 0$. For such values the second term is, by (3.39) positive, so that the right side of (3.40) is positive. But that means that $\delta = p(\phi)$ is convex. Thus (3.39) implies the existence of a convex† Riemann invariant ψ.

Let ψ be a convex Riemann invariant; let $u(x,t,\varepsilon)$ be solutions of the viscous equation

$$(3.47) \qquad \partial_t u + f'\partial_x u = \varepsilon\partial_x^2 u, \quad \varepsilon > 0 .$$

Multiply this equation by ψ'; using relation (3.30): $\psi'f'=c\psi'$ we get

$$\partial_t\psi + c\partial_x\psi = \varepsilon\psi'\partial_x^2 u .$$

Using the identity

$$\partial_x^2\psi = \psi'\partial_x^2 + \partial_x u^t\psi''\partial_x u$$

and the convexity of ψ we conclude that

$$\partial_t\psi + c\partial_x\psi \le \varepsilon\partial_x^2\psi .$$

The maximum principle holds for solutions of such a differential inequality and tells us that

$$\operatorname*{Max}_x \psi(u(x,t;\varepsilon))$$

is a decreasing function of t. But then the same is true of their a.e. limits as $\varepsilon \to 0$; this proves Theorem 3.3 for this class of solutions.

We turn now to Glimm's scheme; in Theorem 2.1 we have shown that if U is a convex function which satisfies an additional conservation law, then U decreases across

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† Smoller and Johnson in [13] have shown that condition (3.42) implies that the curves ϕ = const. are convex. This implies the existence of a convex $\psi = p(\phi)$.

shocks; a similar result holds for convex Riemann invariants:

Lemma 3.5: If the Riemann invariant ϕ satisfies (3.39), then ϕ decreases across a shock of the family opposite to ϕ.

Remark: This decrease was stipulated in Glimm-Lax, [5], precisely for the purpose of proving that the Riemann invariant is a decreasing function.

Sketch of proof: Consider all states w close to a given state v which can be connected on the right to v through a shock of a fixed kind, i.e. which satisfy the jump relations

$$s[w-v] + f(v) - f(w) = 0 \tag{3.48}$$

and the shock inequality (2.3). We saw in Section 2 that these states w form a one parameter family $w(p)$, $p \leq 0$, under the normalization $w(0) = v$, $\dot{s}(0) > 0$, where dot denotes differentiation with respect to the parameter.

Differentiating (3.48) gives

$$\dot{s}(w-v) + s\dot{w} - f'\dot{w} = 0\,. \tag{3.49}$$

Multiply (3.49) by ϕ' on the left; using (3.30) and that $\phi'\dot{w} = \dot{\phi}$ we get

$$\phi'\dot{s}(w-v) = (c-s)\dot{\phi}\,. \tag{3.50}$$

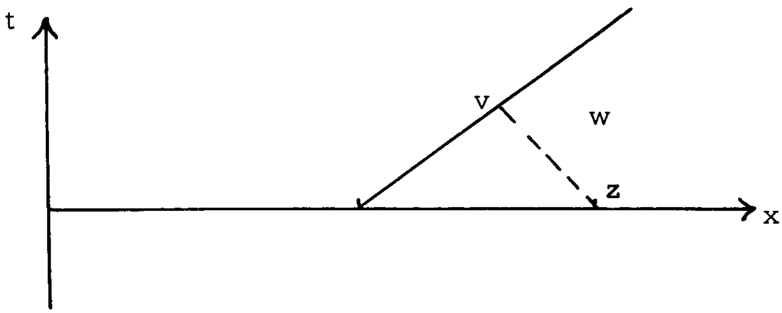
Since $w(0) = v$, (3.50) implies that $\dot{\phi}(0) = 0$. Differentiate (3.50) with respect to p; using $\dot{\phi}(0) = 0$ we deduce $\ddot{\phi}(0) = 0$; differentiating once more we get

$$\dddot{\phi}(0) = \frac{\dot{s}}{c-s}\,\dot{\phi}'\dot{w}\,. \tag{3.51}$$

By (3.39), $\dot{\phi}'\dot{w} > 0$ and by choice of normalization $\dot{s} > 0$; furthermore we are restricted to negative values of the parameter. So it follows from (3.51) that for w near v,

$$\operatorname{sgn}(\phi(v)-\phi(w)) = \operatorname{sgn}(c-s). \tag{3.52}$$

Consider now a flow with a single shock going with speed s greater than the sound speed c of the opposite family:



It follows from (3.52) that

$$\phi(v) < \phi(w) ;$$

on the other hand $\phi(w) = \phi(z)$, where z is the value of u at that point on the initial line which can be connected to the shock by a characteristic of the opposite family. So the value v of ϕ along the other side of the shock is $<$ the value of ϕ at some point of the initial line. The same conclusion holds for shocks of the other family; this proves Lemma 3.5.

In Glimm's difference scheme the initial interval is divided into subintervals and the initial function is approximated by one which is constant in each subinterval; this problem is solved exactly in a time interval taken so short that the waves issuing from the point of discontinuity do not interact. At the end of that time interval the solution is approximated by a piecewise constant function obtained by setting u in each subinterval equal to its value at a randomly chosen point in that subinterval. It follows from Lemma 3.5 that for any Riemann invariant which satisfies (3.39) each approximate solution u_Δ satisfies the conclusion of

Theorem 3. 3. But then so does their limit.

It is not hard to show that for almost all choices of the random points, the approximate solutions u_Δ satisfy the conclusions of Theorem 3. 4. Therefore so do their limits, for almost all random choices.

Another difference scheme, devised by Godunov, starts similarly by solving exactly a piecewise constant initial value problem for a short time, but the conversion into piecewise constant data at the end of that time interval is accomplished differently: in each subinterval u is set equal to its average over that subinterval. If ψ is a convex Riemann invariant, this process decreases the maximum value of $\psi(u)$; so these approximate solutions satisfy the conclusion of Theorem 3. 3. But then so does their limit.

It would be useful to determine all convex solutions of the second order equation (3. 21), so that one can study weak solutions which satisfy the entropy condition with respect to all additional convex conservation laws. The most important question is: are such solutions uniquely determined by their initial data? Another interesting task is to derive from these entropy conditions an analogue for systems of Oleinik's condition E .

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