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CONVEX ENTROPIES AND HYPERBOLICITY FOR GENERAL EULER EQUATIONS*

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To the memory of Ami Harten

Abstract. The compressible Euler equations possess a family of generalized entropy densities of the form $\rho f(\sigma)$, where ρ is the mass density, σ is the specific entropy, and f is an arbitrary function. Entropy inequalities associated with convex entropy densities characterize physically admissible shocks. For polytropic gases, Harten has determined which $\rho f(\sigma)$ are strictly convex. In this paper we extend this determination to gases with an arbitrary equation of state. Moreover, we show that at every state where the sound speed is positive (i.e., where the Euler equations are hyperbolic) there exist $\rho f(\sigma)$ that are strictly convex, thereby establishing the converse of the general fact that the existence of a strictly convex entropy density implies hyperbolicity.

Key words. hyperbolic systems, viscosity solutions, entropy

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1. Introduction.

The general form of a first-order quasi-linear system of N partial differential equations over a D-dimensional spatial domain is

$$\partial_t \mathbf{u} + \sum_{i=1}^{D} \mathbf{A}^i(\mathbf{u}) \partial_{x^i} \mathbf{u} = 0 \,, \tag{1.1}$$

where $x = (x^1 \dots x^D)$, the vector of dependent variables $\mathbf{u} = \mathbf{u}(t, x)$ takes values in $\mathbb{R}^N$, and the coefficient matrices $\mathbf{A}^i = \mathbf{A}^i(\mathbf{u})$ take values in $\mathbb{R}^{N\times N}$. System (1.1) is said to be *hyperbolic at* $\mathbf{u}$, a given value in $\mathbb{R}^N$, if every real linear combination of the $\mathbf{A}^i(\mathbf{u})$ has real eigenvalues and a complete set of eigenvectors [9]—in other words, if every matrix of the form

$$\mathbf{A}(\mathbf{u}, k) \equiv \sum_{i=1}^{D} k_i \mathbf{A}^i(\mathbf{u}) \qquad \text{for some } k = (k_1 \dots k_D) \in \mathbb{R}^D \tag{1.2}$$

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is diagonalizable within the reals. System (1.1) is said to be *hyperbolic* if it is hyperbolic at every $\mathbf{u}$ in $\mathbb{R}^N$ where solutions of (1.1) might take on values. Hyperbolicity expresses the well posedness of the linearization of system (1.1) about the constant solution with value $\mathbf{u}$ and, hence, would seem to constitute a necessary condition for the well posedness of the full system for initial data with values near $\mathbf{u}$.

A property of hyperbolicity that aids in its verification is its invariance under changes of dependent variables. More specifically, given a diffeomorphism $\mathbf{v} \mapsto \mathbf{u} = \mathbf{u}(\mathbf{v})$ with Jacobian $\mathbf{J} \equiv \mathbf{u_v}$, the system (1.1) transforms to

$$\partial_t \mathbf{v} + \sum_{i=1}^{D} \mathbf{B}^i(\mathbf{v}) \partial_{x^i} \mathbf{v} = 0\,, \qquad \text{where} \quad \mathbf{B}^i(\mathbf{v}) = \mathbf{J}(\mathbf{v})^{-1} \mathbf{A}^i(\mathbf{u}(\mathbf{v}))\,\mathbf{J}(\mathbf{v})\,. \tag{1.3}$$

The hyperbolicity of (1.3) at $\mathbf{v}$ is then seen to be equivalent to that of (1.1) at $\mathbf{u}(\mathbf{v})$ by observing that every real linear combination of the $\mathbf{B}^i(\mathbf{v})$ is similar to the same linear combination of the $\mathbf{A}^i(\mathbf{v})$.

With hyperbolicity as the cornerstone, proofs of the local (in time) well posedness of (1.1) for classical (continuously differentiable) solutions can be built with the additional requirement of some mild regularity for the $\mathbf{A}^i$ [12]. However, due to the nonlinearities of (1.1), these classical solutions may develop singularities (usually shocks) at a finite time. The continuation of these solutions beyond this time requires that the notion of solution be enlarged. This can be done if the system can be formulated, possibly after a change of variables, as local conservation laws for a density $\mathbf{u}$ in the form

$$\partial_t \mathbf{u} + \nabla_{\!x} \cdot \mathbf{f}(\mathbf{u}) \equiv \partial_t \mathbf{u} + \sum_{i=1}^{D} \partial_{x^i} \mathbf{f}^i(\mathbf{u}) = 0\,, \tag{1.4}$$

where the flux $\mathbf{f} = (\mathbf{f}^1 \ldots \mathbf{f}^D)$ has components $\mathbf{f}^i = \mathbf{f}^i(\mathbf{u})$ that take values in $\mathbb{R}^N$. It will be assumed that $\mathbf{f}$ is at least once differentiable. When the differentiation is formally carried out, system (1.4) can be written in the matrix form (1.1) with $\mathbf{A}^i = \mathbf{f}^i_{\mathbf{u}}$, the gradient of $\mathbf{f}^i$ with respect to $\mathbf{u}$. By exploiting the divergence form of (1.4), notions of so-called *weak solutions* can be introduced. For example, $\mathbf{u}$ is said to solve (1.4) in the sense of distributions provided

$$\int_0^\infty \int_{\mathbb{R}^D} \left[(\partial_t \mathbf{w})^T \mathbf{u} + \sum_{i=1}^{D} (\partial_{x^i} \mathbf{w})^T \mathbf{f}^i(\mathbf{u}) \right] d\!\!\!^{D}_{\,} x\, dt = 0\,, \tag{1.5}$$

for every smooth vector-valued test function $\mathbf{w} = \mathbf{w}(t, x)$ with compact support within $(0, \infty) \times \mathbb{R}^D$.

The dilemma posed with the introduction of such weak solutions is that there are so many that classical solutions do not have a unique weak continuation. To be selected as physically meaningful, a weak solution must satisfy some additional conditions, such as being the limit as δ tends to zero of solutions $\mathbf{u}^\delta$ of the viscosity equation

$$\partial_t \mathbf{u}^\delta + \nabla_{\!x} \cdot \mathbf{f}(\mathbf{u}^\delta) = \delta \Delta_x \mathbf{u}^\delta\,. \tag{1.6}$$

Provided classical solutions of (1.6) can be shown to exist, it is not hard to show that if $\mathbf{u}^\delta \to \mathbf{u}$ strongly, then $\mathbf{u}$ will satisfy (1.5) and, hence, be a weak solution of (1.4). By such so-called viscosity methods, one may hope to establish the global (in time) well posedness of conservations laws. While there have been a number of successes of this approach, notably for the cases of $N = D = 1$ [8, 9] and $N = 2$, $D = 1$ [1, 2, 3, 15] and for some special systems [16], general success has been elusive.

In cases where viscosity methods have proven successful, the systems of conservations laws have formally admitted additional conservation laws whose densities play a role like entropy in physical systems. A differentiable scalar-valued function $s = s(\mathbf{u})$ is said to be an *entropy density* for the system (1.4) if classical solutions of (1.4) satisfy an additional local conservation law

$$\partial_t s(\mathbf{u}) + \nabla_x \cdot j(\mathbf{u}) = 0\,, \tag{1.7}$$

where the vector-valued function $j = (j^1 \dots j^D)$ is called the *entropy flux* for s. This will be the case for a given s if and only if for each $i = 1 \dots D$ there exists a real-valued function $j^i = j^i(\mathbf{u})$ such that

$$s_{\mathbf{u}}^T \mathbf{A}^i \equiv s_{\mathbf{u}}^T \mathbf{f}_{\mathbf{u}}^i = j_{\mathbf{u}}^{i\,T}\,. \tag{1.8}$$

Formally differentiating (1.8) with respect to $\mathbf{u}$ leads to the observation that

$$\mathbf{G}\mathbf{A}^i \quad \text{is symmetric for each } i = 1 \cdots D\,, \tag{1.9}$$

where $\mathbf{G} \equiv s_{\mathbf{uu}}$ is the Hessian matrix of s. Conversely, when $\mathbf{f}$ is twice differentiable, if a twice differentiable scalar-valued function $s = s(\mathbf{u})$ satisfies (1.9), then the Poincaré lemma ensures the existence of twice differentiable functions $j^i = j^i(\mathbf{u})$ such that (1.8) is satisfied. Even for certain cases when $\mathbf{f}$ is not twice differentiable, as for some physical systems, one can refine this result to show that if s satisfies (1.9), then there exist j^i that satisfy (1.8).

Suppose that s is an entropy density that is a convex function of $\mathbf{u}$, a so-called *convex entropy density.* It is not hard to show that if $\mathbf{u}^\delta \to \mathbf{u}$ strongly, where $\mathbf{u}^\delta$ satisfy the viscosity equation (1.6), then $\mathbf{u}$ will also satisfy in the sense of distributions the entropy inequality

$$\partial_t s(\mathbf{u}) + \nabla_x \cdot j(\mathbf{u}) \le 0\,. \tag{1.10}$$

In devising numerical methods for computing physically relevant weak solutions, we would like to make sure that a difference analog of this inequality is satisfied for some such entropy densities [5, 14, 19]. To do that, it is useful to know all the convex entropy densities.

The problem of finding nontrivial ($s_{\mathbf{uu}} \not\equiv 0$) entropy densities, convex or not, for a given system (1.4) of conservation laws reduces to solving (1.9). For a hyperbolic system of two conservation laws in one space dimension ($N = 2$, $D = 1$), condition (1.9) reduces to a single scalar equation which has many solutions, some of them convex. For any larger N or D, the system of equations that arises from (1.9) is grossly overdetermined and will generally have no solutions. However, for many equations of continuum mechanics, this overdetermined system has a solution, usually associated with either the physical entropy or the energy [6, 7]. This physical entropy density need not be convex, although that is the case for many simple models. Remarkably, the Euler equations of gas dynamics possess a whole family of entropy densities, even for arbitrary equations of state. In this paper we will characterize the convexity of members of this family. In doing so, we will extend to general gases Harten's result concerning gases with a γ-law (polytropic) equation of state [4].

Finally, there is the fundamental connection between hyperbolicity and the strict convexity of entropy densities [6, 7, 13]. Specifically, an entropy density s is said to be *strictly convex* at $\mathbf{u}$ if the Hessian matrix $\mathbf{G} \equiv s_{\mathbf{uu}}(\mathbf{u})$ is positive definite. In that case (1.9) implies that every matrix $\mathbf{A} = \mathbf{A}(\mathbf{u}, k)$ given by (1.2) is symmetric with respect to the inner product associated to $\mathbf{G}$ and, hence, is diagonalizable within the reals. Alternatively, this fact could be deduced from (1.9) by arguing that $\mathbf{GA}$ is symmetric, so that

$$\mathbf{G}^{\frac{1}{2}}\,\mathbf{A}\,\mathbf{G}^{-\frac{1}{2}} = \mathbf{G}^{-\frac{1}{2}}\,\mathbf{GA}\,\mathbf{G}^{-\frac{1}{2}} \qquad \text{is symmetric}\,, \tag{1.11}$$

which shows that $\mathbf{A}$ is similar to a symmetric matrix and, hence, is diagonalizable within the reals. By either route, one concludes that if (1.4) possesses an entropy density that is strictly convex at $\mathbf{u}$ then (1.4) is hyperbolic at $\mathbf{u}$. While the converse of this fact does not generally hold, here we establish it for the Euler system of gas dynamics considered with an arbitrary equation of state.

2. The Euler equations of gas dynamics. The Euler system for a general gas in D dimensions written in conservation form is

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0\,, \tag{2.1a}$$

$$\partial_t (\rho u) + \nabla_x \cdot (\rho u \vee u) + \nabla_x p = 0\,, \tag{2.1b}$$

$$\partial_t \big(\tfrac{1}{2}\rho |u|^2 + \rho\varepsilon\big) + \nabla_x \cdot \big(\tfrac{1}{2}\rho|u|^2 u + \rho\varepsilon u + p u\big) = 0\,. \tag{2.1c}$$

These equations correspond to the local conservation of mass, momentum, and energy. Here $\rho(t,x) \geq 0$ is the mass density, $u(t,x) \in \mathbb{R}^D$ is the fluid velocity vector, $\varepsilon(t,x) \geq 0$ is the specific internal energy, and p is the pressure, which is given in terms of ρ and ε by an equation of state $p = p(\rho, \varepsilon)$. Here $\vee$ denotes the usual symmetric tensor product.

Thermodynamic equations of state are derived from the specific entropy σ which satisfies the differential identity

$$d\sigma = \frac{p}{\theta\rho^2}\, d\rho - \frac{1}{\theta}\, d\varepsilon\,, \tag{2.2}$$

where θ is the temperature, which is also given in terms of ρ and ε by an equation of state. Here we have chosen the sign convention of diminishing (dissipating) entropy, which, while more natural mathematically, differs from much of the physics literature. Physically, p and θ should have positive values when ρ and ε are positive. Hence, it follows from the thermodynamic identity (2.2) that in those cases σ must satisfy

$$\sigma_\rho = \frac{p}{\theta\rho^2} > 0\,, \qquad \sigma_\varepsilon = -\frac{1}{\theta} < 0\,, \tag{2.3}$$

where the subscripts denote partial derivatives. Given $\sigma = \sigma(\rho, \varepsilon)$, the equations of state for the pressure and temperature can be read from (2.3) as

$$p = -\rho^2 \frac{\sigma_\rho(\rho,\varepsilon)}{\sigma_\varepsilon(\rho,\varepsilon)}\,, \qquad \theta = -\frac{1}{\sigma_\varepsilon(\rho,\varepsilon)}\,. \tag{2.4}$$

For example, for a γ-law (polytropic) gas, one has

$$\sigma = \log\left(\frac{\rho}{\varepsilon^{\frac{1}{\gamma-1}}}\right)\,, \qquad p = (\gamma-1)\rho\varepsilon\,, \qquad \theta = (\gamma-1)\varepsilon\,, \tag{2.5}$$

where the constant $\gamma > 1$ is the adiabatic exponent. On the other hand, given equations of state for the pressure and temperature, $p = p(\rho, \varepsilon)$ and $\theta = \theta(\rho, \varepsilon)$, they must satisfy the cross-differentiation solvability condition for (2.2), namely, the Maxwell identity $\theta p_\varepsilon = p\,\theta_\varepsilon + \rho^2\theta_\rho$ of thermodynamics. Upon integrating (2.2), the specific entropy σ is then determined uniquely up to an additive constant which is usually normalized to a reference state, but the choice of which will not affect the equations of state (2.4).

Rather than determine the hyperbolicity of the Euler system (2.1) in terms of its density $\mathbf{u} = (\rho, \rho u, \frac{1}{2}\rho|u|^2 + \rho\varepsilon)^T$, it is advantageous to transform it to a system for the variables $\mathbf{v} = (\rho, u, \varepsilon)^T$ as

$$\partial_t \rho + u \cdot \nabla_x \rho + \rho\, \nabla_x \cdot u = 0\,, \tag{2.6a}$$

$$\partial_t u + u \cdot \nabla_x u + \frac{1}{\rho}\, \nabla_x p \;= 0\,, \tag{2.6b}$$

$$\partial_t \varepsilon + u \cdot \nabla_x \varepsilon + \frac{p}{\rho} \nabla_x \cdot u = 0\,. \tag{2.6c}$$

This transformation is nonsingular provided $\rho > 0$. Assume p is differentiable. The hyperbolicity of the system is then determined by examining the diagonalizability of

$$\mathbf{B}(\mathbf{v}, k) = \begin{pmatrix} u \cdot k & \rho k^T & 0 \\ \dfrac{p_\rho}{\rho} k & u \cdot kI & \dfrac{p_\varepsilon}{\rho} k \\ 0 & \dfrac{p}{\rho} k^T & u \cdot k \end{pmatrix}, \tag{2.7}$$

where $\mathbf{B}(\mathbf{v}, k)$ is defined analogously to $\mathbf{A}(\mathbf{v}, k)$ in (1.2). This matrix has eigenvalues of $u \cdot k$ and $u \cdot k \pm c|k|$ where c^2 is defined by

$$c^2 = p_\rho + \frac{p}{\rho^2} p_\varepsilon\,. \tag{2.8}$$

That these eigenvalues be real clearly requires that $c^2 \geq 0$. When $c^2 = 0$, the matrix (2.7) does not have a complete set of eigenvectors, while when $c^2 > 0$, the eigenvalue $u \cdot k$ has multiplicity D and D independent eigenvectors. Therefore, for $\rho > 0$, the Euler system is hyperbolic at $\mathbf{v} = (\rho, u, \varepsilon)^T$ if and only if $c^2 > 0$. Notice that this characterization is independent of u, a property that reflects the Galilean invariance of the Euler system.

By applying the equations of state (2.4) to (2.8), the quantity c^2 may be expressed in terms of σ in the useful but unilluminating form

$$c^2 = -\frac{\rho^2}{\sigma_\varepsilon^3} \left(\left(\sigma_{\rho\rho} + \frac{2}{\rho}\sigma_\rho \right) \sigma_\varepsilon^2 - 2\sigma_{\varepsilon\rho}\sigma_\varepsilon\sigma_\rho + \sigma_{\varepsilon\varepsilon}\sigma_\rho^2 \right). \tag{2.9}$$

More insight can be gained into c^2 by reexpressing it using the equations of state (2.4) and a bit of calculus as

$$c^2 = p_\rho - \frac{\sigma_\rho}{\sigma_\varepsilon} p_\varepsilon = p_\rho + \left(\frac{\partial \varepsilon}{\partial \rho} \right)_\sigma p_\varepsilon = \left(\frac{\partial p}{\partial \rho} \right)_\sigma, \tag{2.10}$$

where the subscripted parentheses indicate that the enclosed derivative is to be taken while holding the subscript quantity fixed. Hence, from (2.10) it is seen that c^2 has the thermodynamic interpretation as the rate of change of pressure with respect to mass density while holding specific entropy constant. Of course, when the Euler system is hyperbolic, then $c > 0$ has an additional physical interpretation as the sound speed, as is easily seen through an analysis of the associated linearized problem.

Now let us turn to the investigation of entropy densities for the Euler system (2.1). Classical solutions of (2.1) will satisfy (2.6). By combining (2.2) with (2.6a) and (2.6c), the specific entropy σ is found to evolve according to

$$\partial_t \sigma + u \cdot \nabla_x \sigma = 0 \,, \tag{2.11}$$

which states that σ remains constant when moving with fluid parcels. It follows that any function of σ will also remain constant when moving with fluid parcels, which means that

$$\partial_t f(\sigma) + u \cdot \nabla_x f(\sigma) = 0 \,, \tag{2.12}$$

where $f = f(\sigma)$ is any differentiable function over $\mathbb{R}$. Combining (2.1a) with (2.12) shows that classical solutions of the Euler system (2.1) satisfy formal local conservation laws in the form

$$\partial_t \big(\rho f(\sigma)\big) + \nabla_x \cdot \big(\rho u f(\sigma)\big) = 0 \,. \tag{2.13}$$

The quantities $\rho f(\sigma)$ thereby constitute a large family of generalized entropy densities for the Euler system (2.1). While the existence of such a family of solutions to the overdetermined system (1.9) is remarkable, this phenomenon is not restricted to the Euler system [16]. Indeed, it was the existence of such a family for the Gaussian moment closure of the Boltzmann equation [10, 11] that in part motivated this investigation.

When a generalized entropy density $\rho f(\sigma)$ is convex, the weak solutions obtained by the viscosity method will satisfy (in the sense of distributions) the inequality

$$\partial_t \big(\rho f(\sigma)\big) + \nabla_x \cdot \big(\rho u f(\sigma)\big) \le 0 \,. \tag{2.14}$$

Weak solutions of the Euler system (2.1) that also satisfy (2.14) for all convex generalized entropy densities $\rho f(\sigma)$ are called *entropy solutions*. In the context of γ-law gases, Tadmor [18] made use of this family to establish a minimum entropy principle for entropy solutions of the Euler system (2.1).

If any one of the generalized entropy densities $\rho f(\sigma)$ is strictly convex, then the general argument given in the introduction shows that the Euler system is hyperbolic. It is well known that the physical entropy density $\rho\sigma$ is strictly convex, for the case of a polytropic gas (2.5). However, this property is not shared by more general equations of state. Indeed, nonconvexity of the physical entropy density is associated with the existence of thermodynamic phase transitions and can lead to regimes in which the Euler system is not hyperbolic.

Our main result, the proof of which is deferred until the next section, is the following characterizations of those twice differentiable f for which $\rho f(\sigma)$ is strictly convex.

THEOREM 2.1. *Let σ be twice differentiable at $\mathbf{u} = (\rho, \rho u, \frac{1}{2}\rho|u|^2 + \rho\varepsilon)^T$, where ρ and ε are positive. Then for every twice differentiable f, the generalized entropy density $\rho f(\sigma)$ is strictly convex at $\mathbf{u}$ if and only if*

(i) *$f(\sigma)$ is strictly decreasing as a function of ε,*

(ii) $f(\sigma)$ *is strictly convex as a function of* $(1/\rho, \varepsilon)$.
This will be the case if and only if

$$c^2 > 0\,, \qquad f'(\sigma) > 0\,, \qquad f'(\sigma)\frac{1}{c_p} + f''(\sigma) > 0\,, \tag{2.15}$$

where $c_p = c_p(\rho, \varepsilon)$ *is the specific heat at constant pressure.*

Remark. The specific heat at constant pressure is defined by

$$c_p \equiv -\theta\left(\frac{\partial\sigma}{\partial\theta}\right)_p. \tag{2.16}$$

Equilibrium thermodynamics demands that $c_p \geq 0$. One can however have $c_p = \infty$ at a phase transition where σ is twice differentiable.

Remark. It was pointed out to us by a referee that characterization (2.15) was derived by Tartar [20] in the one-dimensional setting. In that setting, one can also find a version of it in the recent book by Serre [17]. His condition 2 is redundant.

Remark. Applying Theorem 2.1 to $f(\sigma) = \sigma$ shows that the physical entropy density is strictly convex if and only if $c^2 > 0$ and $c_p < \infty$, in which case $\rho\sigma$ lies in the interior of the convex cone of entropy densities $\rho f(\sigma)$ that satisfy (2.15).

Remark. For γ-law gases (2.5), it is readily computed from (2.8) that $c^2 = \gamma(\gamma-1)\varepsilon = \gamma\theta > 0$ and from (2.16) that $c_p = \gamma/(\gamma-1) < \infty$. By the previous remark, the physical entropy density is then seen to be strictly convex. Characterization (2.15) of Theorem 2.1 then reduces to

$$f'(\sigma) > 0\,, \qquad f'(\sigma)\frac{\gamma-1}{\gamma} + f''(\sigma) > 0\,, \tag{2.17}$$

which was proved by Harten in [4]. His analogue of (2.17) differs superficially from that here because his choice for the specific entropy (call it σ_H) is related to ours by $\sigma_H = -(\gamma-1)\sigma$. In extending his result to more general equations of state, we give a simpler proof. He worked in two dimensions for convenience only. Indeed, as remarked above, the result is essentially one-dimensional.

The condition $c^2 > 0$ in (2.15) reflects the general fact that the existence of an entropy density that is strictly convex at $\mathbf{u}$ implies the Euler system is hyperbolic at $\mathbf{u}$. However, the converse may be established by exploiting the freedom to choose an f so that (2.15) is satisfied. Indeed, provided $c^2 > 0$, if $c_p < \infty$, then (2.15) implies the physical entropy density $\rho\sigma$ is strictly convex, while if $c_p = \infty$ and one chooses any $f(\sigma)$ that satisfies $f'(\sigma) > 0$ and $f''(\sigma) > 0$, then the entropy density $\rho f(\sigma)$ is strictly convex. We have therefore proved the following.

THEOREM 2.2. *Let* σ *be twice differentiable at* $\mathbf{u} = (\rho, \rho u, \frac{1}{2}\rho|u|^2 + \rho\varepsilon)^T$ *where* ρ *and* ε *are positive. Then the Euler system* (2.1) *is hyperbolic at* $\mathbf{u}$ *if and only if there exists a generalized entropy density* $\rho f(\sigma)$ *that is strictly convex at* $\mathbf{u}$.

Remark. It is clear from the above argument that in fact there are many generalized entropy densities that are strictly convex at every such point in the domain of hyperbolicity for the Euler system.

3. The characterization of strict convexity. The main task here is to prove Theorem 2.1. Specifically, we will determine those twice differentiable functions f for which the generalized entropy density $s = \rho f(\sigma)$ has a positive definite Hessian as a function of the mass, momentum, and energy densities of the Euler system (2.1). Denote the vector of these densities by $\mathbf{u}$, so that

$$\mathbf{u} \equiv \begin{pmatrix}\rho & m & e\end{pmatrix}^T \equiv \begin{pmatrix}\rho & \rho u & \frac{1}{2}\rho|u|^2 + \rho\varepsilon\end{pmatrix}^T, \tag{3.1}$$

where m and e are the momentum and energy densities. By using the fact that $\rho_{\mathbf{uu}} = 0$, the Hessian of $s = \rho f(\sigma)$ at $\mathbf{u}$, which we denote $\mathbf{G}$, is seen to have the form

$$\mathbf{G} \equiv s_{\mathbf{uu}} = f'(\sigma)\,\mathbf{H} + f''(\sigma)\,\rho\,\mathbf{g}\,\mathbf{g}^T\,, \tag{3.2}$$

where

$$\mathbf{g} \equiv \sigma_{\mathbf{u}}\,, \qquad \mathbf{H} \equiv \rho\,\sigma_{\mathbf{uu}} + \rho_{\mathbf{u}}\,\sigma_{\mathbf{u}}^T + \sigma_{\mathbf{u}}\,\rho_{\mathbf{u}}^T\,. \tag{3.3}$$

The vector $\mathbf{g}$ is the gradient of the specific entropy and the matrix $\mathbf{H}$ is the Hessian of the physical entropy density $\rho\sigma$, as can be seen by noticing that (3.2) reduces to $\mathbf{G} = \mathbf{H}$ upon setting $f(\sigma) = \sigma$.

Carrying out the differentiations of σ indicated in (3.3), $\mathbf{g}$ and $\mathbf{H}$ take the form

$$\begin{aligned} \mathbf{g} &= \sigma_\rho \mathbf{a} + \sigma_\varepsilon \mathbf{b}\,, \\ \mathbf{H} &= \left(\rho\sigma_{\rho\rho} + 2\sigma_\rho\right)\mathbf{a}\,\mathbf{a}^T + \rho\sigma_{\varepsilon\rho}\left(\mathbf{a}\,\mathbf{b}^T + \mathbf{b}\,\mathbf{a}^T\right) + \rho\sigma_{\varepsilon\varepsilon}\mathbf{b}\,\mathbf{b}^T - \sigma_\varepsilon \mathbf{C}\,, \end{aligned} \tag{3.4}$$

where the vectors $\mathbf{a}$ and $\mathbf{b}$ and the matrix $\mathbf{C}$ are defined by

$$\mathbf{a} \equiv \rho_{\mathbf{u}}\,, \qquad \mathbf{b} \equiv \varepsilon_{\mathbf{u}}\,, \qquad \mathbf{C} \equiv -\left(\rho\,\varepsilon_{\mathbf{uu}} + \mathbf{a}\,\mathbf{b}^T + \mathbf{b}\,\mathbf{a}^T\right)\,. \tag{3.5}$$

The explicit dependence of u and ε on $\mathbf{u}$ is seen from (3.1) to be

$$u = \frac{m}{\rho}\,, \qquad \varepsilon = \frac{e}{\rho} - \frac{|m|^2}{2\rho^2}\,. \tag{3.6}$$

Using these relations, the derivatives appearing in (3.5) are found to be

$$\begin{gathered} \rho_{\mathbf{u}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \varepsilon_{\mathbf{u}} = \begin{pmatrix} -\dfrac{e}{\rho^2} + \dfrac{|m|^2}{\rho^3} \\ -\dfrac{m}{\rho^2} \\ \dfrac{1}{\rho} \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} \frac12|u|^2 - \varepsilon \\ -u \\ 1 \end{pmatrix}\,, \\ \varepsilon_{\mathbf{uu}} = \begin{pmatrix} 2\dfrac{e}{\rho^3} - 3\dfrac{|m|^2}{\rho^4} & 2\dfrac{m^T}{\rho^3} & -\dfrac{1}{\rho^2} \\ 2\dfrac{m}{\rho^3} & -\dfrac{1}{\rho^2}I & 0 \\ -\dfrac{1}{\rho^2} & 0 & 0 \end{pmatrix} = \frac{1}{\rho^2}\begin{pmatrix} 2\varepsilon - 2|u|^2 & 2u^T & -1 \\ 2u & -I & 0 \\ -1 & 0 & 0 \end{pmatrix}\,. \end{gathered} \tag{3.7}$$

Using (3.7) to evaluate (3.5) shows that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{C}$ are simply

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \mathbf{b} = \frac{1}{\rho}\begin{pmatrix} \frac12|u|^2 - \varepsilon \\ -u \\ 1 \end{pmatrix}\,, \qquad \mathbf{C} = \frac{1}{\rho}\begin{pmatrix} |u|^2 & -u^T & 0 \\ -u & I & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{3.8}$$

Substituting (3.8) into (3.4) then expresses $\mathbf{g}$ and $\mathbf{H}$ explicitly in terms of the partial derivatives of the specific entropy σ.

The main simplification in our analysis is achieved through the introduction of the invertible matrix $\mathbf{D}$, defined by

$$\mathbf{D} \equiv \begin{pmatrix} 1 & u^T & \frac12|u|^2 + \varepsilon \\ 0 & \rho I & \rho u \\ 0 & 0 & \rho \end{pmatrix}\,. \tag{3.9}$$

It is readily checked from (3.8) that

$$\mathbf{Da} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{Db} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{DCD}^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

Using these relations in conjunction with (3.4) leads to

$$\mathbf{Dg} = \begin{pmatrix} \sigma_\rho \\ 0 \\ \sigma_\varepsilon \end{pmatrix}, \qquad \mathbf{DHD}^T = \rho \begin{pmatrix} \sigma_{\rho\rho} + \dfrac{2}{\rho}\sigma_\rho & 0 & \sigma_{\varepsilon\rho} \\ 0 & -\sigma_\varepsilon I & 0 \\ \sigma_{\varepsilon\rho} & 0 & \sigma_{\varepsilon\varepsilon} \end{pmatrix}. \tag{3.11}$$

Substituting these expressions into (3.2) then yields

$$\frac{1}{\rho}\mathbf{D}\,\mathbf{G}\,\mathbf{D}^T = f'(\sigma) \begin{pmatrix} \sigma_{\rho\rho} + \dfrac{2}{\rho}\sigma_\rho & 0 & \sigma_{\varepsilon\rho} \\ 0 & -\sigma_\varepsilon I & 0 \\ \sigma_{\varepsilon\rho} & 0 & \sigma_{\varepsilon\varepsilon} \end{pmatrix} + f''(\sigma) \begin{pmatrix} \sigma_\rho \\ 0 \\ \sigma_\varepsilon \end{pmatrix} \begin{pmatrix} \sigma_\rho & 0 & \sigma_\varepsilon \end{pmatrix}. \tag{3.12}$$

In light of the invertibility of $\mathbf{D}$, the question of the positive definiteness of $\mathbf{G}$, the Hessian of the generalized entropy density $\rho f(\sigma)$ at $\mathbf{u}$, is thereby equivalent to the question of the positive definiteness of the matrix on the right side of (3.12) at $\mathbf{u}$.

The matrix on the right side of (3.12) clearly has an eigenvalue of $-f'(\sigma)\,\sigma_\varepsilon$ with eigenspace $\mathbb{E}$ of dimension D given by $\mathbb{E} = \{(\,0\ v\ 0\,)^T : v \in \mathbb{R}^D\}$. By (2.3), this eigenvalue will be positive if and only if $f'(\sigma) > 0$. The orthogonal complement of $\mathbb{E}$, denoted $\mathbb{E}^\perp$, is a two-dimensional invariant subspace spanned by $(\,1\ 0\ 0\,)^T$ and $(\,0\ 0\ 1\,)^T$. In terms of this basis, the matrix acts on $\mathbb{E}^\perp$ as the 2×2 matrix

$$M \equiv f'(\sigma) \begin{pmatrix} \sigma_{\rho\rho} + \dfrac{2}{\rho}\sigma_\rho & \sigma_{\varepsilon\rho} \\ \sigma_{\varepsilon\rho} & \sigma_{\varepsilon\varepsilon} \end{pmatrix} + f''(\sigma) \begin{pmatrix} \sigma_\rho \\ \sigma_\varepsilon \end{pmatrix} \begin{pmatrix} \sigma_\rho & \sigma_\varepsilon \end{pmatrix}. \tag{3.13}$$

Hence, the matrix on the right side of (3.12) is positive definite if and only if $f'(\sigma) > 0$ and the 2×2 matrix M is positive definite. The first of these conditions is clearly equivalent to (i) of Theorem 2.1 by (2.3), while the second is seen to be equivalent to (ii) upon noticing that the Hessian of $f(\sigma)$ with respect to $(1/\rho, \varepsilon)$ is just M multiplied on the left and right by the diagonal matrix $\mathrm{diag}(-\rho^2, 1)$.

Now turn to establishing the characterization (2.15). The positive definiteness of M clearly implies and, as we will see, is characterized by the positivity of both $\det(M)$ and the quantity

$$\begin{aligned} \begin{pmatrix} -\sigma_\varepsilon & \sigma_\rho \end{pmatrix} M \begin{pmatrix} -\sigma_\varepsilon \\ \sigma_\rho \end{pmatrix} &= f'(\sigma) \begin{pmatrix} -\sigma_\varepsilon & \sigma_\rho \end{pmatrix} \begin{pmatrix} \sigma_{\rho\rho} + \dfrac{2}{\rho}\sigma_\rho & \sigma_{\varepsilon\rho} \\ \sigma_{\varepsilon\rho} & \sigma_{\varepsilon\varepsilon} \end{pmatrix} \begin{pmatrix} -\sigma_\varepsilon \\ \sigma_\rho \end{pmatrix} \\ &= f'(\sigma) \frac{c^2}{\rho^2\theta^3}, \end{aligned} \tag{3.14}$$

where expression (2.9) was used for c^2. Indeed, the positivity of this quantity implies that M has at least one positive eigenvalue, while the positivity of $\det(M)$ implies that both eigenvalues have the same sign; hence, both eigenvalues must be positive, whereby M must be positive definite. Therefore, given that $f'(\sigma) > 0$, it follows from (3.14) that M will be positive definite if and only if $c^2 > 0$ and $\det(M) > 0$.

By combining the facts put forth in the previous two paragraphs, we conclude that the matrix on the right side of (3.12) is positive definite at $\mathbf{u}$ if and only if

$$c^2 > 0\,, \qquad f'(\sigma) > 0\,, \qquad \det(M) > 0\,. \tag{3.15}$$

A direct calculation then shows that

$$\begin{aligned}\det(M) &= \big(f'(\sigma)\big)^2 \det\begin{pmatrix} \sigma_{\rho\rho} + \dfrac{2}{\rho}\sigma_\rho & \sigma_{\varepsilon\rho} \\ \sigma_{\varepsilon\rho} & \sigma_{\varepsilon\varepsilon}\end{pmatrix} \\ &\qquad + f'(\sigma)\, f''(\sigma) \begin{pmatrix} -\sigma_\varepsilon & \sigma_\rho \end{pmatrix} \begin{pmatrix} \sigma_{\rho\rho} + \dfrac{2}{\rho}\sigma_\rho & \sigma_{\varepsilon\rho} \\ \sigma_{\varepsilon\rho} & \sigma_{\varepsilon\varepsilon}\end{pmatrix} \begin{pmatrix} -\sigma_\varepsilon \\ \sigma_\rho \end{pmatrix} \\ &= \Big(f'(\sigma)\beta + f''(\sigma)\Big)\, f'(\sigma)\, \frac{c^2}{\rho^2\theta^3}\,,\end{aligned} \tag{3.16}$$

where β is given by

$$\beta \equiv \frac{\rho^2\theta^3}{c^2} \det\begin{pmatrix} \sigma_{\rho\rho} + \dfrac{2}{\rho}\sigma_\rho & \sigma_{\varepsilon\rho} \\ \sigma_{\varepsilon\rho} & \sigma_{\varepsilon\varepsilon}\end{pmatrix}\,. \tag{3.17}$$

One can then conclude that (3.15) is equivalent to (2.15) provided $\beta = 1/c_p$.

To evaluate β, observe that differentiation of the equations of state (2.4) gives the relation

$$\begin{pmatrix} p_\rho & \theta_\rho \\ p_\varepsilon & \theta_\varepsilon \end{pmatrix} = \frac{\rho^2}{{\sigma_\varepsilon}^2} \begin{pmatrix} \sigma_{\rho\rho} + \dfrac{2}{\rho}\sigma_\rho & \sigma_{\varepsilon\rho} \\ \sigma_{\varepsilon\rho} & \sigma_{\varepsilon\varepsilon}\end{pmatrix} \begin{pmatrix} -\sigma_\varepsilon & 0 \\ \sigma_\rho & \dfrac{1}{\rho^2} \end{pmatrix}\,. \tag{3.18}$$

By taking determinants and again using (2.4), one obtains

$$\det\begin{pmatrix} p_\rho & \theta_\rho \\ p_\varepsilon & \theta_\varepsilon \end{pmatrix} = \rho^2\theta^3 \det\begin{pmatrix} \sigma_{\rho\rho} + \dfrac{2}{\rho}\sigma_\rho & \sigma_{\varepsilon\rho} \\ \sigma_{\varepsilon\rho} & \sigma_{\varepsilon\varepsilon}\end{pmatrix}\,, \tag{3.19}$$

whereby it follows from (3.17) that β can be expressed as

$$\beta = \frac{1}{c^2} \det\begin{pmatrix} p_\rho & \theta_\rho \\ p_\varepsilon & \theta_\varepsilon \end{pmatrix} = \sigma_\varepsilon \frac{\theta_\varepsilon p_\rho - \theta_\rho p_\varepsilon}{\sigma_\varepsilon p_\rho - \sigma_\rho p_\varepsilon} = -\frac{1}{\theta}\left(\frac{\partial\theta}{\partial\sigma}\right)_p = \frac{1}{c_p}\,, \tag{3.20}$$

where c^2 is evaluated using the first equality of (2.10), and c_p enters through its definition (2.16). This proves Theorem 2.1.

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