AEROSP 590: Conservation and Stability

Andi Zhou

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1 Introduction

The following report presents a discussion regarding conservation properties and their corresponding stability of the a typical conservation law in multiple dimensions. We first discuss conservation and stability regarding a 1-D conservation, and we then move to 2-D and eventually numerical studies and confirmations.

2 1-Dimension Conservation Law

We first look at the following basic conservation equation:

$$0 = \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} \tag{1}$$

In this case, $\mathbf{u}$ represents a conservative state (such as air density ρ and temperature T) and $\mathbf{f}$ is the flux of that particular state.

2.1 1st Order Conservation

We first prove that the above equation is conserved. In finite volume method, the first step towards proving conservation starts by taking the integral of the above equation to obtain the **weak form**. The biggest difference between finite volume method and finite difference method lies in the fact that finite volume method starts with the integral form of the conservation, which generally allows for *more information*. Finite difference method involves directly approximating the above equation with finite difference stencil, which **does not account for discontinuities** in the domain. We are proving conservation in a *semi-discrete fashion*, meaning that we only investigate the law spatially.

We first define a finite domain made of N finite cells, illustrated as follow:

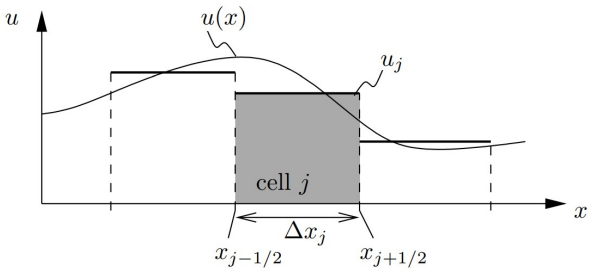

Figure 1: Caption

In finite volume methods, we store the state as a cell centered average and we use half indices $j \pm 1/2$ to represent edges.

We first begin by taking the integral of the state across one cell. Supposing we are centered on

an arbitrary cell j:

$$\begin{aligned}
\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx + \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{f}}{\partial x} dx = 0 \\
\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx + \mathbf{f}|_{x_{j-1/2}}^{x_{j+1/2}} = 0 \\
\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx = -\mathbf{f}|_{x_{j-1/2}}^{x_{j+1/2}} \\
\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx = -(\mathbf{f}_{x_{j+1/2}} - \mathbf{f}_{x_{j-1/2}})
\end{aligned} \tag{2}$$

We then arrive at the final semi-discrete equation for a single cell:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx = \mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}} \tag{3}$$

We then refer to our grid presented in Figure 12. We assume a grid of N elements, and sum over each element with respect to Equation 14 in order to examine the flux conservation. We obtain the following formula:

$$\sum_{j=1}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx = \sum_{j=1}^{N} (\mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}}) \tag{4}$$

We expand the summation on the right hand side, which we then obtain the following:

$$\begin{aligned}
\sum_{j=1}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx = \sum_{j=1}^{N} (\mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}}) \\
= (\mathbf{f}_{x_{1-1/2}} - \mathbf{f}_{x_{1+1/2}}) + (\mathbf{f}_{x_{2-1/2}} - \mathbf{f}_{x_{2+1/2}}) + (\mathbf{f}_{x_{3-1/2}} - \mathbf{f}_{x_{3+1/2}}) \dots (\mathbf{f}_{x_{N-1/2}} - \mathbf{f}_{x_{N+1/2}})
\end{aligned} \tag{5}$$

We could observe that all interior fluxes cancel out, because they all share the common edge $j \pm 1/2$. Therefore, we are left with the following:

$$\sum_{j=1}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx = \mathbf{f}_{1/2} - \mathbf{f}_{N+1/2} \tag{6}$$

The only flux contributions left are the boundary fluxes, since all interior fluxes cancelled out. This is called the *telescoping* property. In other words, this equation speaks that what comes through the boundaries determines the time variation of state throughout all cells. This is consistent of the general statement of finite volume conservation: *what comes in equals to what comes out.*

However, we have not taken *first order* into account here yet. First order FVM could essentially be considered as a flux approximation. In the beginning of this section, we introduced the concept that finite volume method stores state value as cell averages. However, this means that we could not obtain the flux values at the edges (locations of half indices). Therefore, we need some form of approximations. In first order methods, we approximate the flux on the cell edges as a function of the average state value $\mathbf{u}$. Then, Equation 5 becomes the following:

$$\begin{aligned}
\sum_{j=1}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx = (\mathbf{f}(u_1) - \mathbf{f}(u_2)) + (\mathbf{f}(u_2) - \mathbf{f}(u_3)) + (\mathbf{f}(u_3) - \mathbf{f}(u_4)) \dots (\mathbf{f}(u_{N-1}) - \mathbf{f}(u_N)) \\
\sum_{j=1}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx = \mathbf{f}(u_1) - \mathbf{f}(u_N)
\end{aligned} \tag{7}$$

We see that the telescoping property still exist, and the overall solution conservation is not affected. This brings up an important statement that the *order of the approximation in FVM does not affect the general conservation statement.* In addition, as we will see in the last section, *having a conservative scheme does not guarantee solution stability.*

2.2 Diffusion

Now we look at the conservation law with the diffusion term on the right hand side. Typically, conservation with diffusion is represented as follow:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \frac{\partial^2 \mathbf{u}}{\partial^2 x} \tag{8}$$

To determine whether this equation is conserved, we determine whether the diffusion term could be casted as a **flux**, which in this case, it can.

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} &= \frac{\partial^2 \mathbf{u}}{\partial^2 x} \\ \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} &= \frac{\partial}{\partial x}(\frac{\partial \mathbf{u}}{\partial x}) \end{aligned} \tag{9}$$

In this case, $\mathbf{G} = \frac{\partial \mathbf{u}}{\partial x}$ is the **diffusive flux**, and we could write $\frac{\partial^2 \mathbf{u}}{\partial^2 x} = \frac{\partial \mathbf{G}}{\partial x}$, which bears structural similarity to the flux term $\frac{\partial \mathbf{f}}{\partial x}$ We could write out its contributions again for each element $\frac{\partial \mathbf{u}}{\partial x}|_{x_{j-1/2}}^{x_{j+1/2}} = G|_{x_{j-1/2}}^{x_{j+1/2}}$ and sum up over the whole domain. We would again have the telescoping effect and every flux contribution would cancel out except for the boundary fluxes, and the entire equation could be considered as non-conservative.

Appendix A discusses specifically what constitutes as an conservative equation, and performs a derivation of the 2D kinetic energy transport equation from the Euler's equations. In general, a state is conservative when all states, (and terms with dependency on states) could be casted as a **flux**. In the Euler's equation, we will soon see that there exists some terms that could not be casted within the derivative.

2.3 Source Terms

Source terms follows the same rule as that of diffusion. If the source term could be casted in general as a flux term, then it remains conservative. Mathematically, we could write the following:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial x} = \mathbf{q} \tag{10}$$

where $\mathbf{Q}$ in this case represents the source term. If we could format $\mathbf{Q}$ as a flux, that is:

$$\mathbf{q} = \frac{\partial \mathbf{Q}}{\partial x} \tag{11}$$

then we could again integrate over the domain and achieve the telescoping effect.

2.4 2nd Order Conservation

We then look at the second order finite volume method and its relationship with conservation. We recall that first order FVM stores the state as cell averages per each cell, and we use the cell averages to approximate the flux on the edges. In 2nd order, we make use of neighboring cells to calculate the gradient within each cell, and use the gradient information to approximate the fluxes on the edge. In short, we have a linear representation of the state within each cell in 2nd order FVM, and the flux function now depend on cell-edge values.

Mathematically, Equation 5 becomes the following:

$$\begin{aligned} \sum_{j=1}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx &= \sum_{j=1}^{N} (\mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}}) \\ &= (\mathbf{f}_{x_{1-1/2}} - \mathbf{f}_{x_{1+1/2}}) + (\mathbf{f}_{x_{2-1/2}} - \mathbf{f}_{x_{2+1/2}}) + (\mathbf{f}_{x_{3-1/2}} - \mathbf{f}_{x_{3+1/2}}) \dots (\mathbf{f}_{x_{N-1/2}} - \mathbf{f}_{x_{N+1/2}}) \\ &= (\mathbf{f}(\mathbf{u}_{1-1/2}) - \mathbf{f}(\mathbf{u}_{1+1/2})) + (\mathbf{f}(\mathbf{u}_{2-1/2}) - \mathbf{f}(\mathbf{u}_{2+1/2})) + (\mathbf{f}(\mathbf{u}_{3-1/2}) - \mathbf{f}(\mathbf{u}_{3+1/2})) \dots (\mathbf{f}(\mathbf{u}_{N-1/2}) - \mathbf{f}(\mathbf{u}_{N+1/2})) \end{aligned} \tag{12}$$

Again with telescoping, we obtain the following equation of conservation:

$$\sum_{j=1}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx = \mathbf{f}(\mathbf{u}_{1/2}) - \mathbf{f}(\mathbf{u}_{N+1/2}) \tag{13}$$

In the above derivation, notice we replaced cell averaged value from Equation 7 $\mathbf{u}_1, \mathbf{u}_2 \dots \mathbf{u}_N$ with cell edge values $\mathbf{u}_{1\pm1/2}, \mathbf{u}_{2\pm1/2} \dots \mathbf{u}_{N\pm1/2}$. However, the conservation of the equation is not affected as all interior fluxes are cancelled out. Therefore, solver order does not impact the conservation law.

2.5 Temporal Discretization

In finite volume method, each cell holds one constant average of the state. This means we could write the temporal derivative as follow:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{d\mathbf{u}}{dt} dx = \Delta x_i \frac{du_i}{dt} \tag{14}$$

We now write the fully discrete equation as:

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \mathbf{u}}{\partial t} dx &= \mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}} \\ \Delta x_j \frac{d\mathbf{u}_j}{dt} &= \mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}} \\ \frac{d\mathbf{u}_j}{dt} &= \frac{1}{\Delta x_j}(\mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}}) \end{aligned} \tag{15}$$

We select three different rules for time integration: Forward Euler, Backward Euler, and the Midpoint Rule.

2.5.1 Forward Euler

Forward Euler follows as:

$$\frac{d\mathbf{u}}{dt} \approx \frac{\mathbf{u}_j^{n+1} - \mathbf{u}_j^n}{\Delta t} \tag{16}$$

In our case, we have:

$$\begin{aligned} \frac{\mathbf{u}_j^{n+1} - \mathbf{u}_j^n}{\Delta t} &= \frac{1}{\Delta x_j}(\mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}}) \\ \mathbf{u}_j^{n+1} - \mathbf{u}_j^n &= \frac{\Delta t}{\Delta x_j}(\mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}}) \\ \mathbf{u}_j^{n+1} &= \mathbf{u}_j^n + \frac{\Delta t}{\Delta x_j}(\mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}}) \end{aligned} \tag{17}$$

2.5.2 Backward Euler

Backward Euler is an implicit method, meaning the updated states are on the right hand side of the time integration equation:

$$\frac{u_{n+1} - u_n}{\Delta t} = f(t_{n+1}, u_{n+1}) \tag{18}$$

where $f(t_{n+1}, y_{n+1})$ is an arbitrary function. In our case, it is the residual calculation $\frac{1}{\Delta x_j}(\mathbf{f}_{x_{j-1/2}} - \mathbf{f}_{x_{j+1/2}})$ but with an updated state. We organize the equation into the following form:

$$\begin{aligned} u_{n+1} - \Delta t f(t_{n+1}, u_{n+1}) - u_n &= 0 \\ u_{n+1} + \frac{\Delta t}{\Delta x_j}(\mathbf{f}^{n+1}_{x_{j-1/2}} - \mathbf{f}^{n+1}_{x_{j+1/2}}) - u_n &= 0 \end{aligned} \tag{19}$$

Using u_0 as an initial guess, we could solve this equation for the new state u^{n+1} using Newton-Raphson method.

2.5.3 Midpoint Rule

The midpoint rule is another implicit method that follows the format as:

$$u_{n+1} = u_n + \Delta t f(t_n + \frac{\Delta t}{2}, \frac{1}{2}(u_n + u_{n+1})) \tag{20}$$

In our case, this could be written as:

$$\begin{gathered} u_{n+1} - \Delta t f(t_{n+1}, u_{n+1}) - u_n = 0 \\ u_{n+1} + \frac{\Delta t}{\Delta x_j}(\frac{1}{2}((\mathbf{f}^{n+1}_{x_{j-1/2}} - \mathbf{f}^{n+1}_{x_{j+1/2}}) + (\mathbf{f}^{n}_{x_{j-1/2}} - \mathbf{f}^{n}_{x_{j+1/2}}))) - u_n = 0 \end{gathered} \tag{21}$$

Again, a nonlinear solver such as Newton-Raphson could again be used in this case.

2.5.4 Temporal Discretization on Conservation

Let us then discuss the effects on conservation from different temporal discretization. As we have mentioned in the earlier sections, conservation is determined via the balance of fluxes that both enter and leave a specific cell. If the flux on the edge does not equal based on calculations from the neighboring two cells, then conservation is not maintained. In terms of temporal discretization, how the equation is integrated with respect to time DOES NOT affect how the system is conserved. As we will see in Section 4 where we physically model a linear advection equation, even a simulation that is unstable can still be conserved.

3 2-Dimension Conservation Law

We now turn our attention to 2-D conservation law:

$$\frac{d\mathbf{u}}{dt} + \nabla \cdot \mathbf{f} \tag{22}$$

where the divergence operator is restricted in $x - y$. We first investigate conservation for an arbitrary 2D mesh.

3.1 2D Unstructured Mesh

We prove that in an unstructured mesh, the conservation law in 2-Dimension still holds.

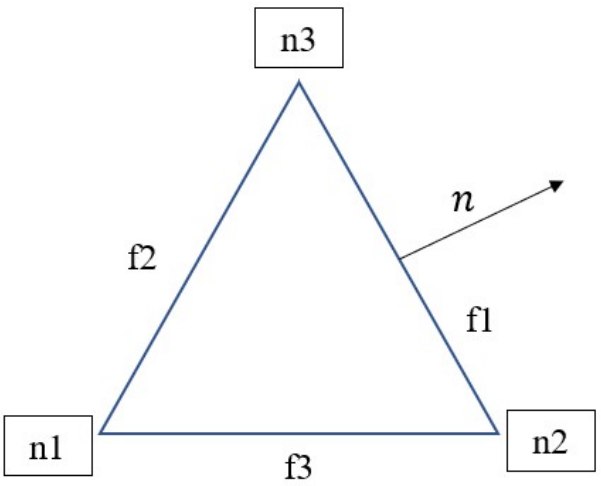

Figure 2: Illustration of a single triangle cell

In the above triangle cell, each vertex shares the same numbering as the face directly opposite from it. In addition, the normal on each face *always* points outward into the neighboring cell, unless it is a boundary element.

We again start by integrating Equation 22 according to each cell's area. This time, however, the integral becomes an area integral due to 2-D.

$$\iint_{\Omega} \frac{d\mathbf{u}}{dt} d\Omega + \iint_{\Omega} \nabla \cdot \mathbf{f} d\Omega = 0 \tag{23}$$

We use divergence theorem, which effectively turns the area integral into a line integral which we could integrate over numerically with ease:

$$\oint_{\partial\Omega} F \cdot \hat{n} ds = \iint_{\Omega} \text{div} F dA \tag{24}$$

The flux term then becomes that of the following:

$$\begin{aligned} \iint_{\Omega} \frac{d\mathbf{u}}{dt} d\Omega + \oint_{\partial\Omega} \mathbf{f} \cdot \hat{n} ds = 0 \\ \iint_{\Omega} \frac{d\mathbf{u}}{dt} d\Omega + \sum_{i=1}^{3} (F_i \cdot n_i) s_i = 0 \end{aligned} \tag{25}$$

where we replaced the line integral with a numerical summation over each edge since we have a fixed normal due to our triangular cell. Note that although triangle is used as an example in this case, technically any *closed* element could be used.

We could already observe telescoping effect from this equation. If we were to sum over every element of the mesh with Equation 25, the ideal algorithm would walk along each cell (typically in counter-clockwise fashion) and add the contribution of the flux to its state. Imagine we are walking on an interior edge, as demonstrated in Figure 3

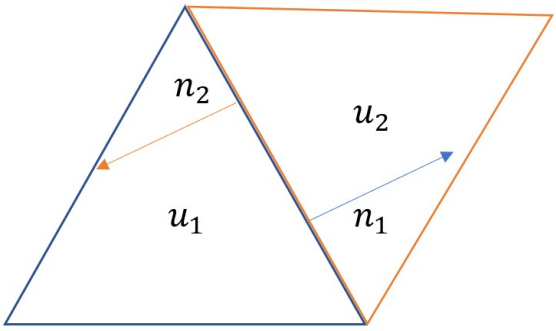

Figure 3: 2 connected elements

Assuming we are looping through on a per element basis, we would visit this edge twice. Once when we walk around element u_1, and a second time when we loop around element u_2. Since the edge is connected, we would obtain the same $\mathbf{f}$ magnitude each time. This condition is guaranteed mathematically by Equation 25 and numerically via the usage of different numerical fluxes. However, the normals are defined as pointing out of each individual element. Therefore, the two $\mathbf{f}$ quantities on each side would become an equal but opposite *vector*. This would effectively cancel out all interior fluxes within a mesh, and leaving out only the boundary contributions, analogous to Equation 14 in 1D.

3.2 2nd Order Finite Volume Scheme

In this section we introduce the Monotonic Upstream-centered Scheme for Conservation Laws (MUSCL), a second order finite volume method that is extremely popular among today's commercial solvers. We discuss the algorithm in a pseudo-code fashion and explain the gradient calculations.

Algorithm 1 MUSCL Scheme for Steady Calculations

```
Compute/store edge normals, cell areas, etc
Initialize the state from first-order solve (Warm-Start)
while |R| > tolerance do
    Initialize the gradient, ∇u|_i, to zero on each element
    for f = 1:N_face do
        Let L/R be the elements adjacent to face f
        Let n→ be the normal pointing out of L
        Let Δl be the length of face f
        set u to the average of the L and R cell averages
        add u n→ Δl to ∇u|_i on L and subtract it from R
    end for
    Divide each ∇u|_i by its element area, A_i
    Initialize residual and wave speed on each cell to zero
    for f = 1 : N_face do
        Let L/R be the elements adjacent to face f
        Compute the L/R states on the face midpoint
        Compute the flux on face f using the edge states
        Increment residual on L, decrement it on R
        Add wave speed to tallies on L and R cells
    end for
    Compute the time step on each cell
    Update the state: use RK2 or higher-order method
end while
```

Note that the above scheme pseudo-code is only for steady-state simulations. For transient simulation, all cells would take the lowest time-step calculated within the entire mesh.

3.3 Conservation on Discontinuous Galerkin Method

Discontinous Galerkin (DG) differs from Finite Volume Method in that instead of storing each state as averages in each cell, the states can now be interpolated by using different basis functions. For triangular elements, we typically use Full-Order basis. Consider the 1-D linear advection equation again:

$$\frac{d\mathbf{u}}{dt} + \frac{\partial \mathbf{F}}{\partial x} = 0 \tag{26}$$

In DG, states are approximated using basis functions as the following:

$$u(x,t) \approx \sum_{m=1}^{Ne} \sum_{j=1}^{p+1} U_{m,j}(t)\phi_{m,j}(x) \tag{27}$$

To obtain the weak form in DG, we multiply the advection law by the basis functions ϕ_i, which we obtain:

$$\begin{gathered} \int_{\Omega_k} \phi_{k,j} \left[\frac{d\mathbf{u}}{dt} + \frac{\partial F}{\partial x} \right] dx = 0 \\ \int_{\Omega_k} \phi_{k,i} \frac{d\mathbf{u}}{dt} dx + \int_{\Omega_k} \phi_{k,j} \frac{\partial F}{\partial x} dx = 0 \\ \int_{\Omega_k} \phi_{k,i} \frac{d\mathbf{u}}{dt} dx + \int_{\Omega_k} \frac{\phi_{k,j}}{\partial x} F dx + [\phi_{k,i} F]_{x_{k-1/2}}^{x_{k+1/2}} = 0 \end{gathered} \tag{28}$$

Let us consider different orders of basis function. Assuming order $p = 0$, this would give $\phi_1 = 1$. Plugging this basis function into the above equation, we would have the following:

$$\begin{gathered} \int_{\Omega_k} (1) \frac{d\mathbf{u}}{dt} dx + \int_{\Omega_k} \frac{(1)}{\partial x} F dx + [(1)F]_{x_{k-1/2}}^{x_{k+1/2}} = 0 \\ \int_{\Omega_k} \frac{d\mathbf{u}}{dt} dx + [F]_{x_{k-1/2}}^{x_{k+1/2}} = 0 \end{gathered} \tag{29}$$

We see that we recover the equation for Finite Volume Method from Section 2, which by telescoping properties, we already know is conservative. Let us then consider basis functions of arbitrary order p. We maintain the equation as follow:

$$\int_{\Omega_k} \phi_{k,i} \frac{d\mathbf{u}}{dt} dx + \underbrace{\int_{\Omega_k} \frac{\phi_{k,j}}{\partial x} F dx}_{\text{Interior Flux}} + \underbrace{[\phi_{k,i} F]_{x_{k-1/2}}^{x_{k+1/2}}}_{\text{Edge Flux}} = 0 \tag{30}$$

We see that we now have an interior flux term that could not be cancelled out. We could rearrange the equations to make it look more familiar:

$$\int_{\Omega_k} \phi_{k,i} \frac{d\mathbf{u}}{dt} dx + \underbrace{[\phi_{k,i} F]_{x_{k-1/2}}^{x_{k+1/2}}}_{\text{Edge Flux}} = - \underbrace{\int_{\Omega_k} \frac{\phi_{k,j}}{\partial x} F dx}_{\text{Source Term}} \tag{31}$$

We see that with the rearrangement, the equation then looks structurally similar to the linear advection equation with a source term q. We then refer to Appendix A on the definition for conservation. The derivative of the basis functions could be considered as constants as they DO NOT depend on the underlying states u. Therefore, the system could be classified as conservative.

The implicit assumption within Discontinous Galerkin at being conservative, is that a numerical flux that links state from both sides must be used. DG method generally implies that the basis function interpolation are discontinuous from each other (Figure 4). Therefore, we would require a numerical flux that resolves the double state issue at the boundary. Fortunately, this is a resolved issue as all Riemann fluxes such Roe, HLLE and Rusanov flux could all be used to resolve this discontinuity.

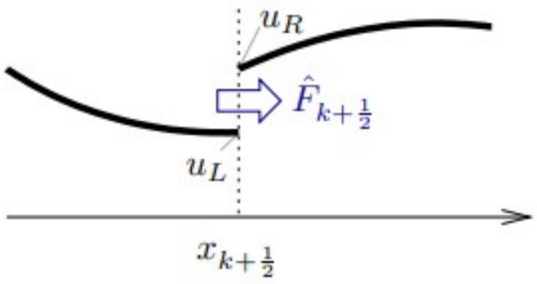

Figure 4: Discontinuous state between cells

4 Numerical Investigation

Consider an 1D advection equation where:

$$0 = \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} \tag{32}$$

where $\mathbf{u} = u$, and $\mathbf{f} = au$, where $a = 1$. The domain is defined to be $x = [0, 1]$, $CFL = 1$, and we have a Gaussian pulse centered at $x = 0.5$ as an initial condition. Figure 5 illustrates the initial condition for this simulation.

We consider three numerical integration schemes: Forward Euler, Backward Euler and Midpoint Rule introduced in Section 2.5. In addition, we also consider two types of fluxes: upwind and central. Their formulae are given as follow.
Upwind:

$$F_{j+1/2} = \frac{1}{2}(au_j + au_{j+1}) - \frac{1}{2}|a|(u_{j+1} - u_j) \tag{33}$$

Central:

$$F_{j+1/2} = \frac{1}{2}(f_j + f_{j+1}) \tag{34}$$

Combining and mixing all time integration and fluxes together, we obtain six possible combined schemes. The results are illustrated as follow: We immediately notice several phenomenons:

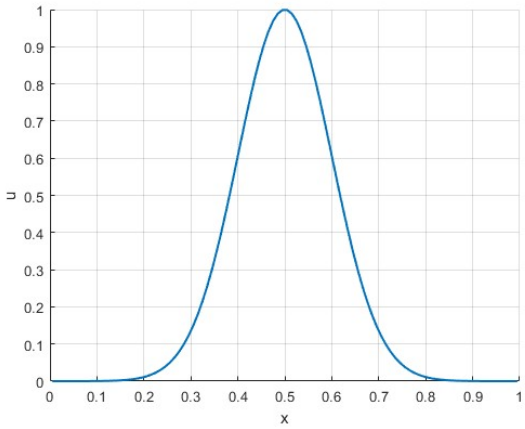

Figure 5: Initial Condition

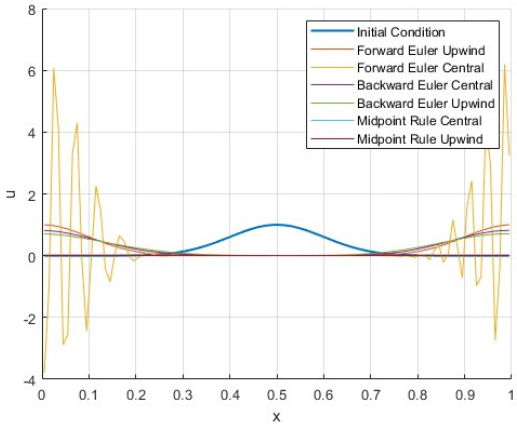

Figure 6: Results for all schemes

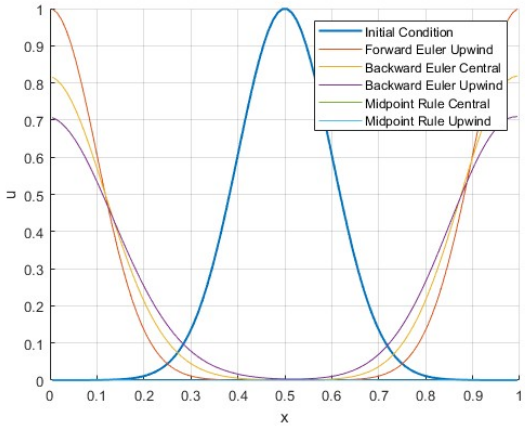

Figure 7: Results for all schemes except for Forward Euler Central in Space

- **Unstable Schemes**. We immediately observe that for Forward Euler Central Flux scheme, the solution immediately becomes unstable. This is due to the inherent stability issue regarding this scheme, or more widely known as Forward in Time Central in Space (FTCS). FTCS is *unconditionally* unstable because Forward Euler method has a small stability *radius*. We could prove this using Von Neumann stability analysis, where we would ultimately find that:

$$g = 1 - \sigma i \sin\phi \tag{35}$$

  Since the magnitude of $|g|$ is always bigger than 1, a.k.a:

$$|g|^2 = 1 + \sigma^2 \sin^2\phi \leq 1 \tag{36}$$

  This scheme is **always** unstable unless artificial viscosity is introduced. In order to use central flux effectively, we require a much more stable time integration scheme such as *Backward Euler*.

- **Numerical Diffusion**. We also observe the effect of numerical diffusion. As time progresses, the peaks for some schemes progressively dwindles. Numerical diffusion arises from higher order terms of the Taylor series expansion when attempting to capture a continuous equation with a discrete set of systems. This effect is especially noticeable for Backward Euler Central in Space and Backward Euler Upwind. In this case, central flux is the more accurate scheme since it is a higher order scheme (2nd order) compared to upwind (1st order).
  Another interesting phenomenon lies in the fact that Forward Euler Upwind does not experience any numerical diffusion. This is actually a special case since both the wavespeed = $CFL = 1$. CFL is defined as follow:

$$CFL = \frac{a\Delta t}{\Delta x} \tag{37}$$

  and it generally represents how well the solution is captured considering the current wavespeed, timestep, and grid spacing. Intuitively, if a wave is moving across a discrete spatial grid and we want to compute its amplitude at discrete time steps of equal duration then this duration must be less than the time for the wave to travel to adjacent grid points. By extension, if the grid spacing is reduced (mesh becomes more fine), the time at which the wave spends in each cell is reduced and a slower timestep is required to maintain accuracy. For the case of Forward Euler Upwind, the wavespeed a, and the CFL number matches equals, and at each timestep the state evolution is captured exactly by the integration scheme. As we change the CFL to a smaller value, such as 0.8, we will see that numerical diffusion still occurs (Figure 8)

- **Midpoint Rule** Midpoint rule also exhibits interesting phenomenons. It seems that the state is diffused so much such that the peaks almost become unnoticeable (Figure 9). In addition, at $t = 0.5s$, it seems that the wave has barely advected. I am honestly unsure on why this is happening and I heavily suspect that it is an implementation error on my part or this scheme is so diffusive such that all variations within states are smoothened out.

4.1 Stability and Conservation

In this section, we explore the relationship between stability and conservation. We could investigate state conservation by summing all states within the grid per each timestep. If the scheme is conservative, then as we track the state across time evolution, the total state should remain constant, and vice versa should occur for un-conservative schemes.

We notice a few surprising results. Figure 10 illustrates total state evolution as a function of time, and Figure 11 indicates a zoomed in view focusing only on Forward Euler and Backward Euler schemes. We observe that, although FTCS is unconditionally unstable, it is a *conservative* scheme, as its total state remained constant throughout time evolution. This directly proves the statement that we introduced in Section 2.5, where stability does not affect conservation. Although there exists some slight fluctuations in backward Euler schemes, the magnitude of the y-axis indicates these fluctuations occurred at almost a machine precision level, implying that all Forward Euler and Backward Euler schemes showcased here are conservative.

Midpoint rule however illustrates a drop in total state as the simulation progresses. This implies that midpoint rule schemes are inherently non-conservative. I am not sure if this is an inherent error with the scheme or a implementation error on my part (which I heavily suspect so)

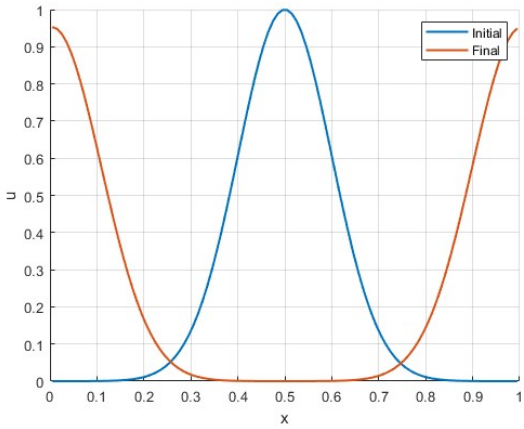

Figure 8: State evolution for Forward Euler Upwind at $t = 0.5s$ with $CFL = 0.8$

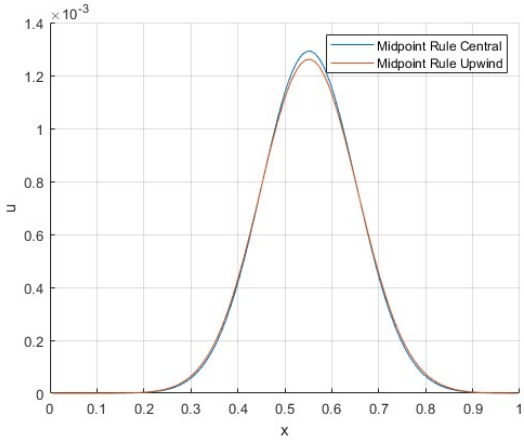

Figure 9: Midpoint Rule Upwind and Midpoint Rule Central

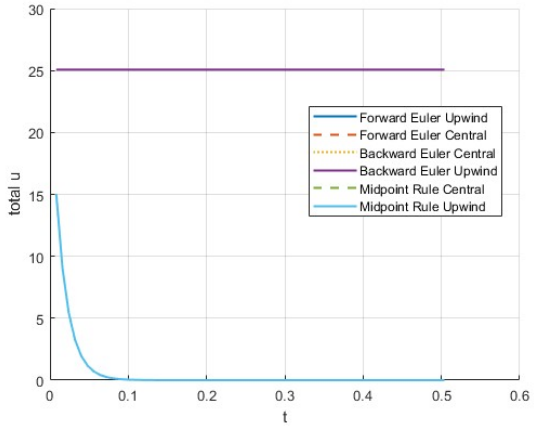

Figure 10: Conservation of all schemes

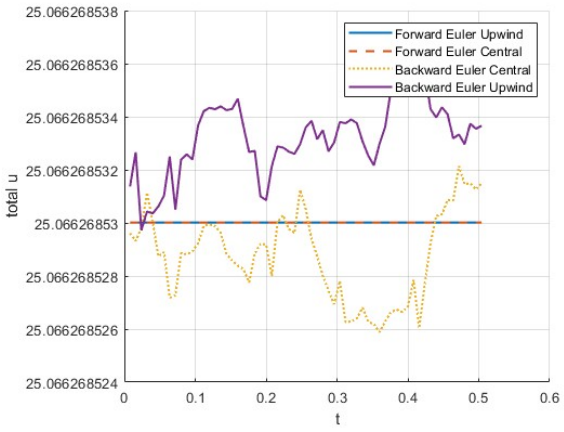

Figure 11: Conservation of all Forward Euler and Backward Euler schemes

4.2 Conservation with u^2

In this section, we prove that the square of the state u^2, is also conserved when plugged into the linear advection equation. We first make the following connection:

$$\frac{\partial u^2}{\partial t} = 2u\frac{\partial u}{\partial x} \tag{38}$$

We could then modify the linear advection equation to account for u^2 as:

$$2u\frac{\partial u^2}{\partial t} + 2au\frac{\partial u}{\partial x} = 0 \tag{39}$$

At this form, it is clear that this equation is non-conservative as we did not write everything in terms of a flux and there exists explicit dependency on state u outside of the derivatives. By applying product rule, we are able to obtain the following:

$$\frac{\partial u^2}{\partial t} + \frac{\partial au^2}{\partial x} = 0 \tag{40}$$

Since all states are now casted inside a flux function, $F = au^2$, and there is no explicit dependency on the states outside of the derivatives, this equation could then be classified as conservative.

A What is Conservation

Conservation, in short, could be expressed as each edge having a *single-valued flux* on each side, meaning that two sides of an edge could not have different fluxes entering and leaving the domain. Mathematically, it could be stated as follow: an equation is classified as conservative when it could be written in the following form:

$$\frac{\partial \mathbf{u}_i}{\partial t} + \frac{\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}}{\Delta x} = q_i(\mathbf{u}_i) \tag{41}$$

In this equation, $\mathbf{u}_i$ is a vector representing the state of a physical system at position x_i and time t, $\mathbf{F}_{i+1/2}$ and $\mathbf{F}_{i-1/2}$ are the *numerical* fluxes of $\mathbf{u}$ at the cell interfaces between cell i and $i+1$ and between cell $i-1$ and i, respectively, and Δx is the spatial discretization. The term q_i represents a source term that may depend on the state $\mathbf{u}$.

We demonstrate this property by deriving the kinetic energy from both compressible Euler's and Navier-Stokes equation, and demonstrate numerically that it is not a conserved property.

A.1 Kinetic Energy from Euler's Equation

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) &= 0 \\
\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho uv) &= 0 \\
\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + p) &= 0 \\
\frac{\partial}{\partial t}(E) + \frac{\partial}{\partial x}((E+p)u) + \frac{\partial}{\partial y}((E+p)v) &= 0
\end{aligned} \tag{42}$$

where ρ is the density, u and v are the velocities in the x and y directions, respectively, p is the pressure, and E is the total energy per unit volume. Note that these equations form a system of partial differential equations and are nonlinear.

In 2-D, kinetic energy in flow is defined as:

$$\frac{\partial \rho k}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{(\rho u)^2}{\rho} + \frac{(\rho v)^2}{\rho}\right) \tag{43}$$

where we used the definition of k in 2D: $k = \frac{(\rho u)^2}{\rho} + \frac{(\rho v)^2}{\rho}$. Then, we have the following:

$$\frac{\partial \rho k}{\partial t} = u\frac{\partial}{\partial t}(\rho u) + v\frac{\partial}{\partial t}(\rho v) - \frac{1}{2}(u^2 + v^2)\frac{\partial \rho}{\partial t} \tag{44}$$

We use the conservation mass, x-momentum and y-momentum equation to convert the time derivative in the above equation into spatial derivatives:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) \\
\frac{\partial}{\partial t}(\rho u) &= -\frac{\partial}{\partial x}(\rho u^2 + p) - \frac{\partial}{\partial y}(\rho uv) \\
\frac{\partial}{\partial t}(\rho v) &= -\frac{\partial}{\partial x}(\rho uv) - \frac{\partial}{\partial y}(\rho v^2 + p)
\end{aligned} \tag{45}$$

We substitute the equations into the formula for kinetic energy and obtain:

$$\frac{\partial \rho k}{\partial t} = \underbrace{u(-\frac{\partial}{\partial x}(\rho u^2 + p) - \frac{\partial}{\partial y}(\rho uv))}_{\text{Term 1}} + \underbrace{v(-\frac{\partial}{\partial x}(\rho uv) - \frac{\partial}{\partial y}(\rho v^2 + p))}_{\text{Term 2}} - \underbrace{\frac{1}{2}(u^2 + v^2)(-\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v))}_{\text{Term 3}} \tag{46}$$

The above equation is conveniantly divided into three different terms, we expand each term and then piece everything together:

Term 1:

$$\begin{aligned} u(-\frac{\partial}{\partial x}(\rho u^2+p)-\frac{\partial}{\partial y}(\rho uv)) &= -u\frac{\partial(\rho u^2)}{\partial x}-u\frac{\partial p}{\partial x}-u\frac{\partial(\rho uv)}{\partial y} \\ &= -u^2\frac{\partial \rho u}{\partial x}-\rho u^2\frac{\partial u}{\partial x}-u\frac{\partial p}{\partial x}-u^2\frac{\partial \rho v}{\partial y}-\rho uv\frac{\partial u}{\partial y} \\ &= -u^2\frac{\partial \rho u}{\partial x}-\rho u^2\frac{\partial u}{\partial x}-u\frac{\partial p}{\partial x}-u^2\frac{\partial \rho v}{\partial y}-\frac{1}{2}\rho v\frac{\partial u^2}{\partial y} \end{aligned} \tag{47}$$

Term 2:

$$\begin{aligned} v(-\frac{\partial}{\partial x}(\rho uv)-\frac{\partial}{\partial y}(\rho v^2+p)) &= -v\frac{\partial}{\partial x}(\rho uv)-v\frac{\partial}{\partial y}(\rho v^2+p) \\ &= -v\frac{\partial \rho uv}{\partial x}-v\frac{\partial(\rho v^2)}{\partial y}-v\frac{\partial p}{\partial y} \\ &= -v^2\frac{\partial \rho u}{\partial x}-\frac{1}{2}\rho u\frac{\partial v^2}{\partial x}-v^2\frac{\partial(\rho v)}{\partial y}-\rho v^2\frac{\partial v}{\partial y}-v\frac{\partial p}{\partial y} \end{aligned} \tag{48}$$

Term 3:

$$\frac{1}{2}(u^2+v^2)(-\frac{\partial}{\partial x}(\rho u)-\frac{\partial}{\partial y}(\rho v))=\frac{1}{2}u^2\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v)+\frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}v^2\frac{\partial}{\partial y}(\rho v) \tag{49}$$

Piecing everything together into one super long equation, we have the following:

$$\begin{aligned} \frac{\partial \rho k}{\partial t} &= -u^2\frac{\partial \rho u}{\partial x}-\rho u^2\frac{\partial u}{\partial x}-u\frac{\partial p}{\partial x}-u^2\frac{\partial \rho v}{\partial y} \\ &\quad -\frac{1}{2}\rho v\frac{\partial u^2}{\partial y}-v^2\frac{\partial \rho u}{\partial x}-\frac{1}{2}\rho u\frac{\partial v^2}{\partial x}-v^2\frac{\partial(\rho v)}{\partial y}-\rho v^2\frac{\partial v}{\partial y}-v\frac{\partial p}{\partial y} \\ &\quad +\frac{1}{2}u^2\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v)+\frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}v^2\frac{\partial}{\partial y}(\rho v) \\ &= -\frac{1}{2}\frac{\partial \rho uu^2}{\partial x}-u\frac{\partial \rho}{\partial x}-\frac{1}{2}\frac{\partial \rho v\dot{u}^2}{\partial y}-\frac{1}{2}\frac{\partial \rho uv^2}{\partial x}-\frac{1}{2}\frac{\partial \rho v^3}{\partial y}-v\frac{\partial p}{\partial y} \end{aligned} \tag{50}$$

Grouping together spatial derivatives, we obtain the following:

$$\begin{aligned} \frac{\rho k}{\partial t} &= -(\frac{1}{2}\frac{\partial(\rho u^3+\rho uv^2+pu)}{\partial x})+p\frac{\partial u}{\partial x}-(\frac{1}{2}\frac{\partial(\rho v^3+\rho vu^2+pv)}{\partial y})+p\frac{\partial v}{\partial y} \\ \frac{\rho k}{\partial t}+(\frac{1}{2}\frac{\partial(\rho u^3+\rho uv^2+pu)}{\partial x})+(\frac{1}{2}\frac{\partial(\rho v^3+\rho vu^2+pv)}{\partial y}) &= p\frac{\partial u}{\partial x}+p\frac{\partial v}{\partial y} \\ \frac{\rho k}{\partial t}+\frac{1}{2}\frac{\partial(\rho uk+pu)}{\partial x}+\frac{1}{2}\frac{\partial(\rho vk+pv)}{\partial y} &= p\frac{\partial u}{\partial x}+p\frac{\partial v}{\partial y} \end{aligned} \tag{51}$$

We have arrived at the equation for kinetic energy in Euler's equation. For incompressible flows, the pressure terms on the right hand side equals to zero and kinetic energy is conserved. For compressible flows, however, that is not the case. We could not write the right hand side of the equation within the spatial derivatives (as flux), which means that the kinetic energy is not conserved within compressible flow.

Consider the case where the flow is incompressible:

$$\frac{\rho k}{\partial t}+\frac{1}{2}\frac{\partial(\rho uk+pu)}{\partial x}+\frac{1}{2}\frac{\partial(\rho vk+pv)}{\partial y}=0 \tag{52}$$

We see that we could express the left hand side as:

$$\frac{\rho k}{\partial t}+\frac{1}{2}\frac{\partial(\rho uk+pu)}{\partial x}+\frac{1}{2}\frac{\partial(\rho vk+pv)}{\partial y}=\frac{\rho k}{\partial t}+\frac{1}{2}\frac{\partial(\mathbf{F}_x)}{\partial x}+\frac{1}{2}\frac{\partial(\mathbf{F}_y)}{\partial y} \tag{53}$$

where $\mathbf{F}_x = \rho uk + pu$ and $\mathbf{F}_y = \rho vk + pv$. If we integrate the above expression for an element within the grid, we would obtain results similar to Equation 5 and arrive at a conserved equation.

Again, we consider the right-hand side of the kinetic energy equation:

$$\text{RHS} = p(u)\frac{\partial u}{\partial x} + p(u)\frac{\partial v}{\partial y} \tag{54}$$

We represent p as $p(u)$ in order to emphasize the term's dependency on the current state. In addition, we place u with flux term F Consider the below diagram:

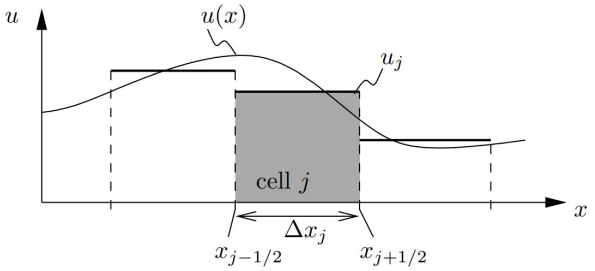

Figure 12: Mesh Illustration

We take the integral of the above equation from $x_{j-1/2}$ to $x_{j+1/2}$, and obtain the following expression:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} p(u_j)\frac{\partial F}{\partial x} + \int_{x_{j-1/2}}^{x_{j+1/2}} p(u_j)\frac{\partial F}{\partial y} \tag{55}$$

We focus our attention exclusively on $\int_{x_{j-1/2}}^{x_{j+1/2}} p(u_j)\frac{\partial F}{\partial x}$ to simplify mathematics and we would have the following integrals:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} p(u_j)\frac{\partial F}{\partial x} = p(u_j)F|_{x_{j-1/2}}^{x_{j+1/2}} + \int_{x_{j-1/2}}^{x_{j+1/2}} F\frac{\partial u_j}{\partial x}dx \tag{56}$$

Let us write a similar equation for the adjacent cell u_{j+1}:

$$\int_{x_{j+1/2}}^{x_{j+3/2}} p(u_{j+1})\frac{\partial F}{\partial x} = p(u_{j+1})F|_{x_{j+1/2}}^{x_{j+3/2}} + \int_{x_{j+1/2}}^{x_{j+3/2}} F\frac{\partial u_{j+1}}{\partial x}dx \tag{57}$$

We then consider the common edge: $x_{j+1/2}$. In order to maintain conservation, the flux across the edge must equal for both elements. This means mathematically:

$$p(u_j)F|_{x_{j+1/2}} + \int_{x_{j-1/2}}^{x_{j+1/2}} F\frac{\partial u_j}{\partial x}dx = p(u_{j+1})F|_{x_{j+1/2}} + \int_{x_{j+1/2}}^{x_{j+3/2}} F\frac{\partial u_{j+1}}{\partial x}dx \tag{58}$$

F would maintain conservation due to the property of numerical fluxes (i.e: numerical fluxes are made to be equal on the edges). However, there is no guarantee that the *states on each side u_j and u_{j+1} are identical*! Therefore, this equation is NOT conservative because the total flux calculated at the same edge from both cells are different.

B Numerical Investigation

We propose the following problem with Euler's equation. Consider a pressure tube with periodic boundary condition containing a region of high pressure at its center (Figure **??**:

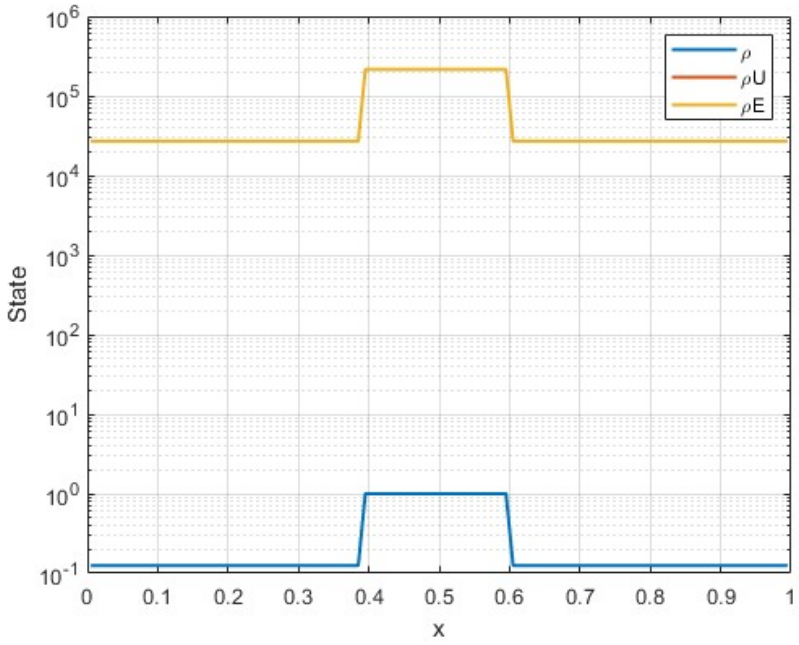

Figure 13: Initial condition for the shock tube problem

where there is zero velocity initially $\rho U = 0$. We use a 1D first order finite volume solver and numerically integrate the time to $T = 0.04s$. Due to this being a region of high pressure at the center, we should expect to see the wave advect toward both sides symmetrically. In addition, we should see discontinuities forming on both sides due to the presence of shocks. We choose Roe flux due to its low diffusivity. We obtain the following state plots (Figure 14) for ρ after at $T = 0.04s$.

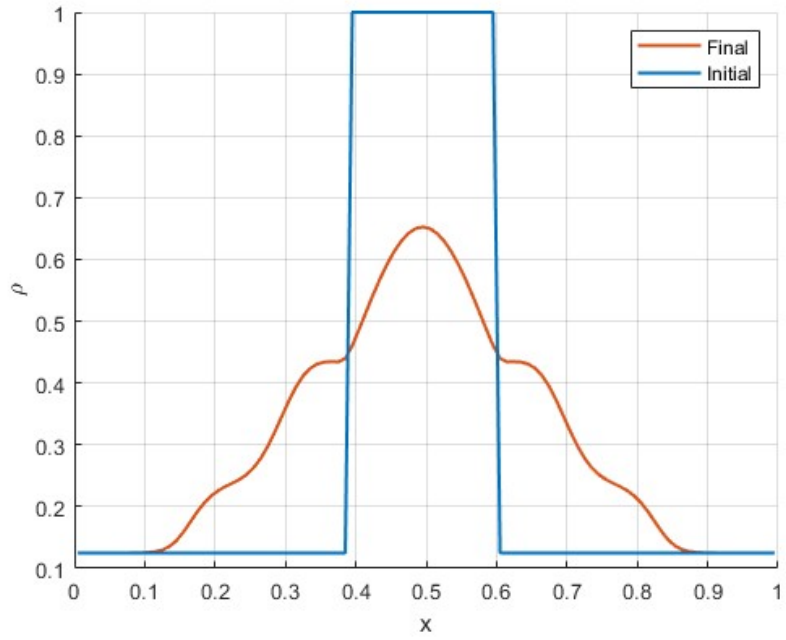

Figure 14: States at $T = 0.04s$

We see that there are two compression waves that advects towards both sides. We then examine the state in addition to the kinetic energy as a function of time. It is important to note that Euler's equations are a set of conservative equation; meaning the major states (density ρ, momentum ρu and total energy ρE) are conserved quantities, a.k.a their total quantity DOES NOT change

throughout evolution. Kinetic energy is a specific quantity within the fluid, and we will notice that it is not conserved through time. Refer to Figure 15:

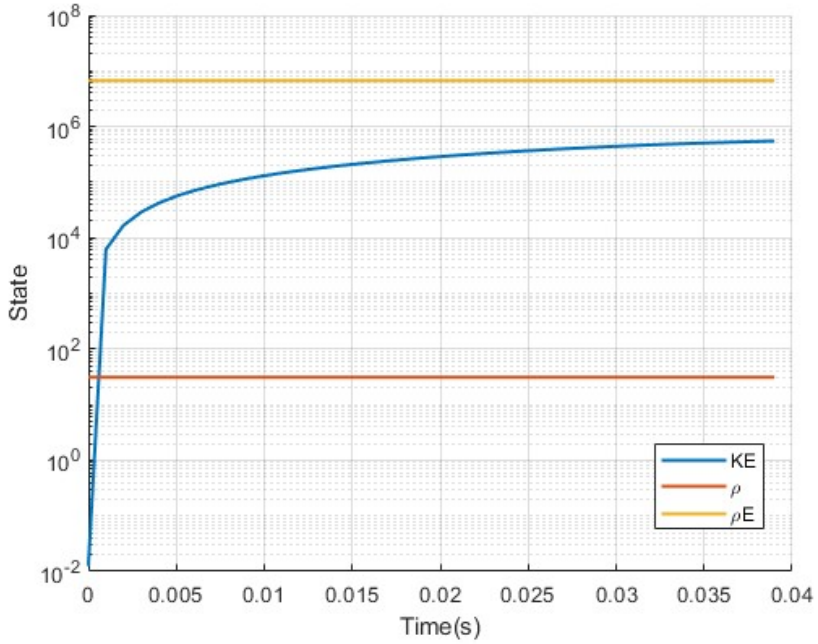

Figure 15: State plots from $T = 0 sto T = 0.04s$

Since kinetic energy is defined as $\rho u^2/2$, the KE at $T = 0$ must equal to zero due to the fact that all states starts at rest. However, as simulation progresses, fluids start to move and KE begins to increase, showing that this quantity is NOT conserved. In contrast, Euler states such as mass and energy remains a consistent straight line throughout the entire simulation.

C Kinetic Energy in Navier-Stokes

We now look at Navier-Stokes equation, which is basically Euler's equations with viscous term:

$$\begin{gathered}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \\
\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho uv) - \frac{1}{Re_r}(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}) = 0 \\
\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + p) - \frac{1}{Re_r}(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}) = 0 \\
\frac{\partial}{\partial t}(E) + \frac{\partial}{\partial x}((E + p)u) + \frac{\partial}{\partial y}((E + p)v) = 0
\end{gathered} \tag{59}$$

The kinetic energy is then again:

$$\frac{\rho k}{\partial t} + \frac{1}{2}\frac{\partial(\rho uk + pu)}{\partial x} + \frac{1}{2}\frac{\partial(\rho vk + pv)}{\partial y} = 0 \tag{60}$$

We again substitute the time derivative with conservation of mass, x-momentum and y-momentum, which we then have the following:

$$\begin{aligned}\frac{\partial \rho k}{\partial t} =& \underbrace{u(-\frac{\partial}{\partial x}(\rho u^2+p)-\frac{\partial}{\partial y}(\rho uv)+\frac{1}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}))}_{\text{Term 1}} \\ &+\underbrace{v(-\frac{\partial}{\partial x}(\rho uv)-\frac{\partial}{\partial y}(\rho v^2+p)+\frac{1}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}))}_{\text{Term 2}} \\ &-\underbrace{\frac{1}{2}(u^2+v^2)(-\frac{\partial}{\partial x}(\rho u)-\frac{\partial}{\partial y}(\rho v))}_{\text{Term 3}}\end{aligned} \tag{61}$$

We again split them into different terms.
Term 1:

$$\begin{aligned}u(-\frac{\partial}{\partial x}(\rho u^2+p)-\frac{\partial}{\partial y}(\rho uv)+\frac{1}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y})) &= -u\frac{\partial(\rho u^2)}{\partial x}-u\frac{\partial p}{\partial x}-u\frac{\partial(\rho uv)}{\partial y}+\frac{u}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}) \\ &= -u^2\frac{\partial \rho u}{\partial x}-\rho u^2\frac{\partial u}{\partial x}-u\frac{\partial p}{\partial x}-u^2\frac{\partial \rho v}{\partial y}-\rho uv\frac{\partial u}{\partial y}+\frac{u}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}) \\ &= -u^2\frac{\partial \rho u}{\partial x}-\rho u^2\frac{\partial u}{\partial x}-u\frac{\partial p}{\partial x}-u^2\frac{\partial \rho v}{\partial y}-\frac{1}{2}\rho v\frac{\partial u^2}{\partial y}+\frac{u}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y})\end{aligned} \tag{62}$$

Term 2:

$$\begin{aligned}v(-\frac{\partial}{\partial x}(\rho uv)-\frac{\partial}{\partial y}(\rho v^2+p)+\frac{1}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y})) &= -v\frac{\partial}{\partial x}(\rho uv)-v\frac{\partial}{\partial y}(\rho v^2+p)+\frac{v}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}) \\ &= -v\frac{\partial \rho uv}{\partial x}-v\frac{\partial(\rho v^2)}{\partial y}-v\frac{\partial p}{\partial y}+\frac{v}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}) \\ &= -v^2\frac{\partial \rho u}{\partial x}-\frac{1}{2}\rho u\frac{\partial v^2}{\partial x}-v^2\frac{\partial(\rho v)}{\partial y}-\rho v^2\frac{\partial v}{\partial y}-v\frac{\partial p}{\partial y}+\frac{v}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y})\end{aligned} \tag{63}$$

Term 3:

$$\frac{1}{2}(u^2+v^2)(-\frac{\partial}{\partial x}(\rho u)-\frac{\partial}{\partial y}(\rho v))=\frac{1}{2}u^2\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v)+\frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}v^2\frac{\partial}{\partial y}(\rho v) \tag{64}$$

Piecing everything together, we see that the equation is almost exact same as that given before, with an additional viscous term on the side

$$\begin{aligned}\frac{\partial \rho k}{\partial t} =& -u^2\frac{\partial \rho u}{\partial x}-\rho u^2\frac{\partial u}{\partial x}-u\frac{\partial p}{\partial x}-u^2\frac{\partial \rho v}{\partial y} \\ &-\frac{1}{2}\rho v\frac{\partial u^2}{\partial y}-v^2\frac{\partial \rho u}{\partial x}-\frac{1}{2}\rho u\frac{\partial v^2}{\partial x}-v^2\frac{\partial(\rho v)}{\partial y}-\rho v^2\frac{\partial v}{\partial y}-v\frac{\partial p}{\partial y} \\ &+\frac{1}{2}u^2\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}u^2\frac{\partial}{\partial y}(\rho v)+\frac{1}{2}v^2\frac{\partial}{\partial x}(\rho u)+\frac{1}{2}v^2\frac{\partial}{\partial y}(\rho v) \\ &+\frac{u}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y})+\frac{v}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y})\end{aligned} \tag{65}$$

Therefore, we follow the same procedure as Euler's equation, where we simplify terms with products rules and group together spatial derivatives. The process is almost exactly the same with the exception that we now have viscous terms at the end of the equation:

$$\begin{aligned}\frac{\partial \rho k}{\partial t} &= -\frac{1}{2}\frac{\partial \rho u u^2}{\partial x}-u\frac{\partial \rho}{\partial x}-\frac{1}{2}\frac{\partial \rho v \dot{u}^2}{\partial y}-\frac{1}{2}\frac{\partial \rho u v^2}{\partial x}-\frac{1}{2}\frac{\partial \rho v^3}{\partial y}-v\frac{\partial p}{\partial y}+\frac{u}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y})+\frac{v}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}) \\ \frac{\rho k}{\partial t} &= -(\frac{1}{2}\frac{\partial(\rho u^3+\rho uv^2+pu)}{\partial x})+p\frac{\partial u}{\partial x}-(\frac{1}{2}\frac{\partial(\rho v^3+\rho vu^2+pv)}{\partial y})+p\frac{\partial v}{\partial y}+\frac{u}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y})+\frac{v}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}) \\ \frac{\rho k}{\partial t}+(\frac{1}{2}\frac{\partial(\rho u^3+\rho uv^2+pu)}{\partial x})+(\frac{1}{2}\frac{\partial(\rho v^3+\rho vu^2+pv)}{\partial y}) &= p\frac{\partial u}{\partial x}+p\frac{\partial v}{\partial y}+\frac{u}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y})+\frac{v}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}) \\ \frac{\rho k}{\partial t}+\frac{1}{2}\frac{\partial(\rho uk+pu)}{\partial x}+\frac{1}{2}\frac{\partial(\rho vk+pv)}{\partial y} &= p\frac{\partial u}{\partial x}+p\frac{\partial v}{\partial y}+\frac{u}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y})+\frac{v}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y})\end{aligned} \tag{66}$$

Moving all terms to the left, we receive the following transport equation for kinetic energy:

$$\frac{\rho k}{\partial t}+\frac{1}{2}\frac{\partial(\rho uk+pu)}{\partial x}+\frac{1}{2}\frac{\partial(\rho vk+pv)}{\partial y}-p\frac{\partial u}{\partial x}-p\frac{\partial v}{\partial y}-\underbrace{(\frac{u}{Re_r}(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y})+\frac{v}{Re_r}(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}))}_{\text{Viscous Diffusion}}=0 \tag{67}$$

We see that the minus sign in front of the equation indicates that the viscous diffusion eats off kinetic energy.