mathematics

Article

A Review of Wrapped Distributions for Circular Data

William Bell and Saralees Nadarajah *

Department of Mathematics, University of Manchester, Manchester M13 9PL, UK; william.bell-4@postgrad.manchester.ac.uk
* Correspondence: mbbsssn2@manchester.ac.uk

Abstract: The wrapped method is the most widely used method for constructing distributions for circular data. In this paper, we provide a review of all known wrapped distributions, including 45 distributions for continuous circular data and 10 distributions for discrete circular data. For each wrapped distribution, we state its nth trigonometric moment, mean direction, mean resultant length, skewness, and kurtosis. We also discuss data applications and limitations of each wrapped distribution. This review could be a useful reference and encourage the development of more wrapped distributions. We also mention an R package available for fitting all of the reviewed distributions and illustrate its applications.

Keywords: kurtosis; mean direction; mean resultant length; skewness; trigonometric moment

MSC: 62E99

check for updates

Citation: Bell, W.; Nadarajah, S. A Review of Wrapped Distributions for Circular Data. *Mathematics* **2024**, *12*, 2440. https://doi.org/10.3390/math12162440

Academic Editors: Domma Filippo and Francesca Condino

Received: 4 July 2024
Revised: 30 July 2024
Accepted: 4 August 2024
Published: 6 August 2024

1. Introduction

Circular data, also known as directional data, refer to measurements where the values are cyclical and repeat over a defined interval. This kind of data arises naturally in situations where the end of the scale reconnects with the beginning, such as angles, time, and compass directions. For example, angles measured in degrees or radians are circular because 0 is the same as 360, and similarly, 0 radians equates to 2π radians. This cyclical nature poses unique challenges for statistical analysis because traditional linear methods are not suitable for data that wrap around.

Practical examples of circular data abound in various fields. In meteorology, wind directions recorded over time create circular datasets, as the direction can be anywhere between 0 and 360. In biology, the study of animal movement, such as the migratory patterns of birds or the rotational behavior of certain animals, often involves circular data. Time-related data are another common example; for instance, the times of day at which certain events occur (like sleep cycles or peak traffic hours) are inherently circular because they repeat every 24 h. Analyzing such data requires careful consideration of its circular nature to avoid misinterpretation and to uncover meaningful patterns and insights.

There have been many distributions proposed for circular data. Refs. [1–6] provide excellent reviews. Most of the distributions for circular data are based on the method of wrapping. Hence, we feel it is appropriate to provide a review of all known wrapped distributions to date, which is the aim of this paper. No such review is known to date. Such a review could be a useful reference for those interested in both the theory and applications of wrapped distributions. It could also enhance the development of more wrapped distributions.

The method of wrapping can be described as follows. Suppose g is a valid probability density function (PDF). If g is defined on the positive real line, then the PDF of the wrapped distribution is

$$f(\theta) = \sum_{k=0}^{\infty} g(\theta + 2k\pi) \tag{1}$$

for $0 \leq \theta < 2\pi$. If g is defined on the entire real line, then the PDF of the wrapped distribution is

$$f(\theta) = \sum_{k=-\infty}^{\infty} g(\theta + 2k\pi) \tag{2}$$

for $0 \leq \theta < 2\pi$.

On the other hand, now suppose g is a valid probability mass function (PMF). If g is defined on the positive real line, then the PMF of the wrapped distribution is

$$\Pr\left(\Theta = \frac{2\pi r}{m}\right) = \sum_{k=0}^{\infty} g(r + km) \tag{3}$$

for $r = 0, 1, \ldots, m-1$, and $m \geq 1$. If g is defined on the entire real line, then the PMF of the wrapped distribution is

$$\Pr\left(\Theta = \frac{2\pi r}{m}\right) = \sum_{k=-\infty}^{\infty} g(r + km) \tag{4}$$

for $r = 0, 1, \ldots, m-1$, and $m \geq 1$.

The most important properties of a circular random variable, say Θ, are its nth trigonometric moment denoted by m_n, mean direction denoted by μ, mean resultant length denoted by ρ, skewness denoted by γ_1, and kurtosis denoted by γ_2, defined by

$$m_n = E[\cos(n\Theta)] + \mathrm{i}E[\sin(n\Theta)], \tag{5}$$

$$\mu = \arcsin \frac{E[\sin(\Theta)]}{\sqrt{\{E[\cos(\Theta)]\}^2 + \{E[\sin(\Theta)]\}^2}}, \tag{6}$$

$$\rho = \sqrt{\{E[\cos(\Theta)]\}^2 + \{E[\sin(\Theta)]\}^2}, \tag{7}$$

$$\gamma_1 = \frac{\exp(-2\mu)}{(1-\rho)^{\frac{3}{2}}} E[\sin(2\Theta)], \tag{8}$$

and

$$\gamma_1 = \frac{\exp(-2\mu)}{1-\rho^2} E[\cos(2\Theta)] - \frac{\rho^4}{1-\rho^2}, \tag{9}$$

respectively. For each reviewed distribution, we list expressions for (5)–(9) as well as expressions for the cumulative distribution function (CDF) corresponding to (1)–(4), if they are available. We also discuss data applications as well as the limitations of each reviewed distribution. Section 2 lists all known expressions corresponding to (1)–(2). Section 3 lists all known expressions corresponding to (3)–(4). The properties (5)–(9) not listed in Sections 2 and 3 could provide readers with avenues for future work.

The expressions listed in Sections 2 and 3 involve several special functions, including the Lerch function defined by

$$\Phi(z; s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s};$$

the gamma function defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt;$$

the lower incomplete gamma function defined by

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt;$$

the upper incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt;$$

the standard normal probability density function defined by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}};$$

the standard normal cumulative distribution function defined by

$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt;$$

and the modified Bessel function of the second kind defined by

$$K_\nu(x) = \begin{cases} \dfrac{\pi \csc(\pi\nu)}{2} [I_{-\nu}(x) - I_\nu(x)], & \text{if } \nu \notin \mathbb{Z}, \\ \lim\limits_{\mu \to \nu} K_\mu(x), & \text{if } \nu \in \mathbb{Z}; \end{cases}$$

the modified Bessel function of the first kind of order ν defined by

$$I_\nu(x) = \sum_{k=0}^\infty \frac{1}{\Gamma(k+\nu+1)k!} \left(\frac{x}{2}\right)^{2k+\nu}.$$

The properties of these special functions can be found in [7,8]. We also set $\mathrm{i} = \sqrt{-1}$ throughout.

An R software package (version 4.4.1) for fitting of all of the reviewed distributions has been created by the second author [9]. Three data applications illustrating the R package are given in Section 4.

2. A Review of Continuous Wrapped Distributions

In this section, we review wrapped Akash, wrapped Aradhana, wrapped binormal, wrapped Birnbaum–Saunders, wrapped Cauchy, wrapped chi-square, wrapped exponential, wrapped exponentiated inverted Weibull, wrapped gamma, wrapped generalized geometric stable, wrapped generalized Gompertz, wrapped generalized normal Laplace, wrapped generalized skew normal [10], wrapped generalized skew normal [11], wrapped half-logistic, wrapped half-normal, wrapped [12]'s skew Laplace, wrapped hypoexponential, wrapped Ishita, wrapped Laplace, wrapped length-biased weighted exponential, wrapped Levy, wrapped Lindley, wrapped Linnik, wrapped Lomax, wrapped modified Lindley, wrapped new Weibull–Pareto, wrapped normal, wrapped Pareto, wrapped quasi-Lindley, wrapped Rama, wrapped Richard, wrapped Shanker, wrapped skew Laplace, wrapped skew normal, wrapped stable, wrapped Student's t, wrapped transmuted exponential, wrapped two-parameter Lindley, wrapped two-sided Lindley, wrapped variance gamma, wrapped weighted exponential, wrapped Weibull, wrapped XGamma and wrapped XLindley distributions.

2.1. Wrapped Akash Distribution

Ref. [13] took g to be the PDF of the Akash distribution to obtain the wrapped Akash distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda^3 e^{-\lambda\theta}\left[\left(1+\theta^2\right)\left(1-e^{-2\pi\lambda}\right)^2 + 4\pi e^{-2\pi\lambda}\left(\theta+\pi+\pi e^{-2\pi\lambda}-\theta e^{-2\pi\lambda}\right)\right]}{\left(\lambda^2+2\right)\left(1-e^{-2\pi\lambda}\right)^3}$$

and

$$\begin{aligned} F(\theta) = {} & \frac{4\pi\lambda e^{-2\pi\lambda}\left\{\left[1-(1+\lambda\theta)e^{-\lambda\theta}\right]\left(1-e^{-2\pi\lambda}\right)+\pi\lambda\left(1-e^{-\lambda\theta}\right)\left(1+e^{-2\pi\lambda}\right)\right\}}{\left(\lambda^2+2\right)\left(1-e^{-2\pi\lambda}\right)^3} \\ & + \frac{\left(1-e^{-\lambda\theta}\right)\left(\lambda^2+2\right)-\lambda\theta(2+\lambda\theta)e^{-\lambda\theta}}{\left(\lambda^2+2\right)\left(1-e^{-2\pi\lambda}\right)}, \end{aligned}$$

respectively, for $0 \le \theta < 2\pi$ and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \frac{\lambda^3}{\lambda^2+2}\frac{\lambda^2 - 2\mathrm{i}n\lambda - n^2 + 2}{(\lambda - \mathrm{i}n)^3}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = 3\arctan\left(\frac{1}{\lambda}\right) - \arctan\left(\frac{2\lambda}{\lambda^2+1}\right),$$

$$\rho = \frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{\left(\lambda^2+1\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^2}},$$

$$\gamma_1 = \frac{\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{\left(\lambda^2-2\right)^2+16\lambda^2}{\left(\lambda^2+4\right)^3}}\sin(\kappa_{\lambda,2})}{\left[1-\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{\left(\lambda^2+1\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^3}}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{\left(\lambda^2-2\right)^2+16\lambda^2}{\left(\lambda^2+4\right)^3}}\cos(\kappa_{\lambda,2}) - \left[\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{\left(\lambda^2+1\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^3}}\right]^4}{\left[1-\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{\left(\lambda^2+1\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^3}}\right]^2},$$

respectively, where $\kappa_{\lambda,2} = 3\arctan\left(\frac{2}{\lambda}\right) - \arctan\left(\frac{4\lambda}{\lambda^2-2}\right) - 6\arctan\left(\frac{1}{\lambda}\right) + 2\arctan\left(\frac{2\lambda}{\lambda^2+1}\right)$. The wrapped Akash distribution was used on a dataset from [1] regarding the long-axis orientations of 60 feldspar laths in basalt rock. It demonstrated a better fit to the data compared to the wrapped exponential distribution and the wrapped Lindley distribution. Despite having only one parameter, the wrapped Akash distribution proved to be a flexible model. Additionally, it allows for closed form expressions for both its PDF and its CDF.

2.2. Wrapped Aradhana Distribution

Ref. [14] took g to be the PDF of the Aradhana distribution to obtain the wrapped Aradhana distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda^3 e^{-\lambda\theta}}{\lambda^2 + 2\lambda + 2}\left[\frac{(1+\theta)^2}{1-e^{-2\pi\lambda}} + \frac{4(\theta+1)\pi e^{-2\pi\lambda}}{\left(1-e^{-2\pi\lambda}\right)^2} + \frac{4\pi^2 e^{-2\pi\lambda}\left(1+e^{-2\pi\lambda}\right)}{\left(1-e^{-2\pi\lambda}\right)^3}\right]$$

and

$$\begin{aligned} F(\theta) = {} & \frac{1-e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} - \frac{\lambda\theta(\lambda\theta+2\lambda+2)e^{-\lambda\theta}}{(\lambda^2+2\lambda+2)\left(1-e^{-2\pi\lambda}\right)} + \frac{4\pi\lambda e^{-2\pi\lambda}\left[(1+\lambda)\left(1-e^{-\lambda\theta}\right) - \lambda\theta e^{-\lambda\theta}\right]}{(\lambda^2+2\lambda+2)\left(1-e^{-2\pi\lambda}\right)^2} \\ & + \frac{4\pi^2\lambda^2\left(1-e^{-\lambda\theta}\right)\left(1+e^{-2\pi\lambda}\right)e^{-2\pi\lambda}}{(\lambda^2+2\lambda+2)\left(1-e^{-2\pi\lambda}\right)^3}, \end{aligned}$$

respectively, for $0 \le \theta < 2\pi$ and $\lambda > 0$. The nth trigonometric moment is given by

$$m_n = \frac{\lambda^3\left[\lambda^2 + 2\lambda - n^2 + 2 - 2\mathrm{i}n(1+\lambda)\right]}{(\lambda^2+2)(\lambda - \mathrm{i}n)^3}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = 3\arctan\left(\frac{1}{\lambda}\right) - \arctan\left[\frac{2(1+\lambda)}{\lambda^2+2\lambda+1}\right],$$

$$\rho = \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{\left(\lambda^2+2\lambda+1\right)^2 + 4(1+\lambda^2)}{\left(\lambda^2+1\right)^3}},$$

$$\gamma_1 = \frac{\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{\left(\lambda^2+2\lambda-2\right)^2+16(1+\lambda^2)}{\left(\lambda^2+4\right)^3}}\sin(\mu_2 - 2\mu)}{\left[1-\sqrt{\frac{\left(\lambda^2+2\lambda+1\right)^2+4(1+\lambda^2)}{\left(\lambda^2+1\right)^3}}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{\left(\lambda^2+2\lambda-2\right)^2+16(1+\lambda^2)}{\left(\lambda^2+4\right)^3}}\cos(\mu_2 - 2\mu) - \left[\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{\left(\lambda^2+2\lambda-2\right)^2+16(1+\lambda^2)}{\left(\lambda^2+4\right)^3}}\right]^4}{\left[1-\sqrt{\frac{\left(\lambda^2+2\lambda+1\right)^2+4(1+\lambda^2)}{\left(\lambda^2+1\right)^3}}\right]^2},$$

respectively, where $\mu_2 = 3\arctan\left(\frac{2}{\lambda}\right) - \arctan\left(\frac{4(1+\lambda)}{\lambda^2+2\lambda-2}\right)$. The wrapped Aradhana distribution has only one parameter, which might restrict its versatility and relevance to real-world datasets. However, it does have closed form expressions for the PDF and CDF.

2.3. Wrapped Binormal Distribution

Ref. [15] took g to be the PDF of the binormal distribution to obtain the wrapped binormal distribution. Its PDF is

$$f(\theta) = \begin{cases} \sum_{k=-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_1 + \sigma_2} e^{-\frac{(\theta+2k\pi-\mu)^2}{2\sigma_1^2}}, & \theta \le \mu, \\ \sum_{k=-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_1 + \sigma_2} e^{-\frac{(\theta+2k\pi-\mu)^2}{2\sigma_2^2}}, & \theta > \mu \end{cases}$$

for $0 \le \theta < 2\pi$, $\theta > \mu$, and $\sigma > 0$. The nth trigonometric moment is

$$m_n = 2[b_n \cos(p\mu) - c_n \sin(p\mu)] + 2\mathrm{i}[b_n \sin(p\mu) + c_n \cos(p\mu)],$$

where

$$b_p = \frac{1}{2}\left(\frac{\sigma_1}{\sigma_1+\sigma_2} e^{\frac{-p^2\sigma_1^2}{2}} + \frac{\sigma_2}{\sigma_1+\sigma_2} e^{\frac{-p^2\sigma_2^2}{2}}\right)$$

and

$$c_p = -\frac{1}{\sqrt{\pi}} \frac{\sigma_1}{\sigma_1+\sigma_2} e^{\frac{-p^2\sigma_1^2}{2}} \sum_{n=1}^{\infty} \frac{\left(\frac{p\sigma_1}{\sqrt{2}}\right)^{2n-1}}{(2n-1)(n-1)!} + \frac{2}{\sqrt{\pi}} \frac{\sigma_2}{\sigma_1+\sigma_2} e^{\frac{-p^2\sigma_2^2}{2}} \sum_{n=1}^{\infty} \frac{\left(\frac{p\sigma_2}{\sqrt{2}}\right)^{2n-1}}{(2n-1)(n-1)!}.$$

The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left[\frac{b_1 \sin(\mu) + c_1 \cos(\mu)}{b_1 \cos(\mu) - c_1 \sin(\mu)}\right],$$

$$\rho = 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2},$$

$$\gamma_1 = \frac{\sqrt{[b_2 \cos(2\mu) - c_2 \sin(2\mu)]^2 + [b_2 \sin(2\mu) + c_2 \cos(2\mu)]^2} \sin(\mu_2 - 2\mu)}{\left[1 - 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{2\sqrt{[b_2 \cos(2\mu) - c_2 \sin(2\mu)]^2 + [b_2 \sin(2\mu) + c_2 \cos(2\mu)]^2} \sin(\mu_2 - 2\mu)}{\left[1 - 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2}\right]^2} - \frac{\left[2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2}\right]^4}{\left[1 - 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2}\right]^2},$$

respectively, where $\mu_2 = \arctan\left[\frac{b_2 \sin(2\mu)+c_2 \cos(2\mu)}{b_2 \cos(2\mu)-c_2 \sin(2\mu)}\right]$. The wrapped binormal distribution lacks practicality due to its non-closed-form PDF.

2.4. Wrapped Birnbaum–Saunders Distribution

Ref. [16] took g to be the PDF of the Birnbaum-Saunders distribution to obtain the wrapped Birnbaum–Saunders distribution. Its PDF and CDF are

$$f(\theta) = \frac{e^{\frac{\delta}{2}}\sqrt{\delta+1}}{4\sqrt{\pi\mu}} \sum_{k=0}^{\infty} \frac{\theta + 2k\pi + \frac{\delta\mu}{\delta+1}}{(\theta+2k\pi)^{\frac{3}{2}}} e^{-\frac{\delta}{4}\left[\frac{(\theta+2k\pi)(\delta+1)}{\delta\mu} + \frac{\delta\mu}{(\theta+2k\pi)(\delta+1)}\right]}$$

and

$$\begin{aligned} F(\theta) = {} & \sum_{k=0}^{\infty} \Phi\left(\sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{(\theta+2k\pi)(\delta+1)}{\delta\mu}} - \sqrt{\frac{\delta\mu}{(\theta+2k\pi)(\delta+1)}}\right]\right) \\ & - \sum_{k=0}^{\infty} \Phi\left(\sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{2k\pi(\delta+1)}{\delta\mu}} - \sqrt{\frac{\delta\mu}{2k\pi(\delta+1)}}\right]\right), \end{aligned}$$

respectively, for $0 \le \theta < 2\pi$, $0 \le \mu < 2\pi$, and $\delta > 0$. The nth trigonometric moment is

$$m_n = \frac{1}{2}\left[1 + \frac{\sqrt{\delta+1}}{\sqrt{\delta+1-4ni\mu}}\right] e^{\frac{\delta\left[\sqrt{\delta+1}-\sqrt{\delta+1-4ni\mu}\right]}{2\sqrt{\delta+1}}}$$

for $n = 1, 2, \ldots$. The wrapped Birnbaum–Saunders distribution was used on a dataset examined by [1] and originally gathered by [17]. The dataset includes 100 ant directions in reaction to a uniformly lit black target. The wrapped Birnbaum–Saunders distribution fit the data better than both symmetric and asymmetric von Mises distributions.

2.5. Wrapped Cauchy Distribution

Ref. [18] took g to be the PDF of the Cauchy distribution to obtain the wrapped Cauchy distribution. Its PDF is

$$f(\theta) = \sum_{k=-\infty}^{\infty} \frac{\gamma}{\mu(\gamma^2 + (\theta - \mu + 2\pi n)^2)} = \frac{1}{2\pi} \frac{\sinh\gamma}{\cosh\gamma - \cos(\theta-\mu)}$$

for $0 \le \theta < 2\pi$, $\gamma > 0$, and $-\infty < \mu < \infty$. The nth trigonometric moment is

$$m_n = e^{\mathrm{i}n\mu - n\gamma}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu,$$

$$\rho = e^{-\gamma},$$

$$\gamma_1 = 0,$$

and

$$\gamma_2 = \frac{e^{-2\gamma} - e^{-8\gamma}}{(1-e^{-\gamma})^2},$$

respectively. Notably, the wrapped Cauchy distribution has a closed-form expression for its PDF. However, it is a symmetric distribution and, as such, has limited applicability to skewed datasets.

2.6. Wrapped Chi-Square Distribution

Ref. [19] took g to be the PDF of a chi-square distribution to obtain the wrapped chi-square distribution. Its PDF is

$$f(\theta) = \frac{e^{-\frac{\theta}{2}} \pi^{\frac{k}{2}-1}}{2\Gamma\left(\frac{k}{2}\right)} \Phi\left(e^{-\pi}; 1 - \frac{k}{2}, \frac{\theta}{2\pi}\right)$$

for $0 \leq \theta < 2\pi$ and $k > 0$. The nth trigonometric moment is

$$m_n = (1 - 2\mathrm{i}n)^{-\frac{k}{2}}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \frac{k}{2} \arctan(2),$$

$$\rho = 5^{-\frac{k}{4}},$$

$$\gamma_1 = \frac{17^{-\frac{k}{4}} \sin\left[\frac{k}{2} \arctan(4) - k \arctan(2)\right]}{\left(1 - 5^{-\frac{k}{4}}\right)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{17^{-\frac{k}{4}} \cos\left[\frac{k}{2} \arctan(4) - k \arctan(2)\right] - 5^{-k}}{\left(1 - 5^{-\frac{k}{4}}\right)^{2}},$$

respectively. The wrapped chi-square distribution faces limitations due to its non-closed-form PDF, as defined in terms of the Lerch function.

2.7. Wrapped Exponential Distribution

Ref. [20] took g to be the PDF of the exponential distribution to obtain the wrapped exponential distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}$$

and

$$F(\theta) = \frac{1 - e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}},$$

respectively, for $0 \leq \theta < 2\pi$ and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \frac{\lambda}{\lambda - \mathrm{i}n}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left(\frac{1}{\lambda}\right),$$

$$\rho = \frac{\lambda}{\sqrt{1+\lambda^2}},$$

$$\gamma_1 = \frac{-2\lambda}{(1+\lambda^2)^{\frac{1}{4}}(4+\lambda^2)\left(\sqrt{1+\lambda^2}-\lambda\right)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{3\lambda^2}{(1+\lambda^2)(4+\lambda^2)\left(\sqrt{1+\lambda^2}-\lambda\right)^{2}},$$

respectively. The wrapped exponential distribution adequately fits various real-life distributions, as reported by [1]. It offers straightforward closed-form expressions for the PDF and CDF. Nevertheless, its limitation lies in being a one-parameter distribution, restricting its flexibility.

2.8. Wrapped Exponentiated Inverted Weibull Distribution

Ref. [21] took g to be the PDF of the exponentiated inverted Weibull distribution to obtain the wrapped exponential inverted Weibull distribution. Its PDF is

$$f(\theta) = c\sum_{k=0}^{\infty} \lambda(\theta+2\pi k)^{-(c+1)}\left[e^{-(\theta+2\pi k)^{-c}}\right]^{\lambda}$$

for $0 \le \theta < 2\pi$, $c > 0$, and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \sum_{k=0}^{\infty} \frac{\left(\mathrm{i}t\lambda^{\frac{1}{c}}\right)^k}{k!}\Gamma\left(1-\frac{k}{c}\right) = b_t + \mathrm{i}c_t$$

for $n = 1, 2, \ldots$, where

$$b_n = \sum_{k=0}^{\infty} \frac{(-1)^k\left(n\lambda^{\frac{1}{c}}\right)^{2k}}{(2k)!}\Gamma\left(1-\frac{2k}{c}\right)$$

and

$$c_n = \sum_{k=0}^{\infty} \frac{(-1)^k\left(n\lambda^{\frac{1}{c}}\right)^{2k+1}}{(2k+1)!}\Gamma\left(1-\frac{2k+1}{c}\right).$$

The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left(\frac{c_1}{b_1}\right),$$

$$\rho = \sqrt{b_1^2+c_1^2},$$

$$\gamma_1 = \frac{\sqrt{b_2^2+c_2^2}\sin(\mu_2-2\mu)}{\left(1-\sqrt{b_1^2+c_1^2}\right)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\sqrt{b_2^2 + c_2^2}\cos(\mu_2 - 2\mu) - \left(\sqrt{b_1^2 + c_1^2}\right)^4}{\left(1 - \sqrt{b_1^2 + c_1^2}\right)^{\frac{3}{2}}},$$

respectively, where $\mu_2 = \arctan\left(\frac{c_2}{b_2}\right)$. The wrapped exponential inverted Weibull distribution was used to analyze the orientation data of 76 turtles post egg-laying, as documented by [1]. This distribution exhibited a superior fit compared to the wrapped new Weibull–Pareto distribution. Nonetheless, a limitation is noted due to the absence of a closed-form PDF.

2.9. Wrapped Gamma Distribution

Ref. [22] took g to be the PDF of the gamma distribution to obtain the wrapped gamma distribution. Its PDF is

$$f(\theta) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda\theta} (2\pi)^{r-1} \Phi\left(e^{-2\lambda\pi}; 1 - r, \frac{\theta}{2\pi}\right)$$

for $0 \leq \theta < 2\pi$, $r > 0$, and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \lambda^r (\lambda - \mathrm{i}n)^{-r}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = r \arctan\left(\frac{1}{\lambda}\right),$$

$$\rho = \lambda^r \left(\lambda^2 + 1\right)^{-\frac{r}{2}},$$

$$\gamma_1 = \frac{\lambda^r \left(\lambda^2 + 4\right)^{-\frac{r}{2}} \sin(\mu_2 - 2\mu)}{\left[1 - \lambda^r (\lambda^2 + 1)^{-\frac{r}{2}}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\lambda^r \left(\lambda^2 + 4\right)^{-\frac{r}{2}} \sin(\mu_2 - 2\mu) - \left[\lambda^r \left(\lambda^2 + 1\right)^{-\frac{r}{2}}\right]^4}{\left[1 - \lambda^r (\lambda^2 + 1)^{-\frac{r}{2}}\right]^2},$$

respectively, where $\mu_2 = r \arctan\left(\frac{2}{\lambda}\right)$. The wrapped gamma distribution is constrained by its non-closed-form PDF, expressed in terms of the Lerch function.

2.10. Wrapped Generalized Geometric Stable Distribution

Ref. [23] took g to be the PDF of the generalized geometric stable distribution to obtain the wrapped generalized geometric stable distribution. Its PDF and CDF are

$$f(\theta) = \frac{1}{2\pi} \left\{ 1 + 2 \sum_{k=1}^{\infty} [\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)] \right\}$$

and

$$F(\theta) = \frac{1}{2\pi}\left\{1 + 2\sum_{k=1}^{\infty}\left[\frac{\alpha_k}{k}\sin(k\theta) + \frac{\beta_k}{k} - \frac{\beta_k}{k}\cos(k\theta)\right]\right\},$$

respectively, for $0 \le \theta < 2\pi$, where

$$\alpha_k = \begin{cases} \left\{(1+\sigma^\alpha k^\alpha)^2 + \left[\sigma^\alpha k^\alpha \beta \tan\left(\frac{\pi\alpha}{2}\right) + \mu^* k\right]^2\right\}^{-\frac{\lambda}{2}} \cos\left\{\lambda \arctan\left[\frac{\sigma^\alpha k^\alpha \beta \tan\left(\frac{\pi\alpha}{2}\right) + \mu^* k}{1+\sigma^\alpha k^\alpha}\right]\right\}, & \text{if } \alpha \ne 1, \\ \left\{(1+\sigma^\alpha k^\alpha)^2 + \left[\mu^* k - \frac{2\sigma^\alpha k^\alpha \beta}{\pi}\log\mid k\mid\right]^2\right\}^{-\frac{\lambda}{2}} \cos\left\{\lambda \arctan\left(\frac{\mu^* k - \frac{2\sigma^\alpha k^\alpha \beta}{\pi}\log\mid k\mid}{1+\sigma^\alpha k^\alpha}\right)\right\}, & \text{if } \alpha = 1 \end{cases}$$

and

$$\beta_k = \begin{cases} \left\{(1+\sigma^\alpha k^\alpha)^2 + \left[\sigma^\alpha k^\alpha \beta \tan\left(\frac{\pi\alpha}{2}\right) + \mu^* k\right]^2\right\}^{-\frac{\lambda}{2}} \sin\left\{\lambda \arctan\left[\frac{\sigma^\alpha k^\alpha \beta \tan\left(\frac{\pi\alpha}{2}\right) + \mu^* k}{1+\sigma^\alpha k^\alpha}\right]\right\}, & \text{if } \alpha \ne 1, \\ \left\{(1+\sigma^\alpha k^\alpha)^2 + \left[\mu^* k - \frac{2\sigma^\alpha k^\alpha \beta}{\pi}\log\mid k\mid\right]^2\right\}^{-\frac{\lambda}{2}} \sin\left\{\lambda \arctan\left(\frac{\mu^* k - \frac{2\sigma^\alpha k^\alpha \beta}{\pi}\log\mid k\mid}{1+\sigma^\alpha k^\alpha}\right)\right\}, & \text{if } \alpha = 1 \end{cases}$$

for $0 < \alpha \le 2$, $\lambda > 0$, $\sigma > 0$, $-1 \le \beta \le 1$, $-\infty\mu < \infty$, and $\mu^* = \mu \bmod 2\pi$. The nth trigonometric moment is

$$m_n = \alpha_n + \mathrm{i}\beta_n$$

for $n = 1, 2, \ldots$. The mean direction and mean resultant length are

$$\mu = \begin{cases} \lambda \arctan\left[\frac{\mu^* + \sigma^\alpha \beta \tan\left(\frac{\pi\alpha}{2}\right)}{1+\sigma^\alpha}\right] \text{ and } 2\pi, & \text{if } \alpha \ne 1, \\ \lambda \arctan\left[\frac{\mu^*}{1+\sigma^\alpha}\right] \text{ and } 2\pi, & \text{if } \alpha = 1 \end{cases}$$

and

$$\rho = \begin{cases} \left\{(1+\sigma^\alpha)^2 + \left[\mu^* \sigma^\alpha \beta \tan\left(\frac{\pi\alpha}{2}\right)\right]^2\right\}^{-\frac{\lambda}{2}}, & \text{if } \alpha \ne 1, \\ \left\{(1+\sigma)^2 + (\mu^*)^2\right\}^{-\frac{\lambda}{2}}, & \text{if } \alpha = 1, \end{cases}$$

respectively. The wrapped generalized geometric stable distribution was used to analyze hourly wind direction data collected over three days at a location on Black Mountain, ACT, Australia, as reported by [24] and discussed by [1]. Although the distribution was found to fit the data effectively, its drawback lies in its high number of parameters. Simplified distributions with fewer parameters might prove more practical.

2.11. Wrapped Generalized Gompertz Distribution

Ref. [25] took g to be the PDF of the generalized Gompertz distribution to obtain the wrapped generalized Gompertz distribution. Its PDF and CDF are

$$f(\theta) = \frac{1}{b\Gamma(c)} \sum_{k=-\infty}^{\infty} \exp\left(c\frac{\theta + 2\pi k - a}{b} - e^{\frac{\theta+2\pi k-a}{b}}\right)$$

and

$$F(\theta) = \sum_{k=-\infty}^{\infty} \frac{1}{\Gamma(c)} \left[\Gamma\left(c, e^{\frac{2\pi k-a}{b}}\right) - \Gamma\left(c, e^{\frac{\theta+2\pi k-a}{b}}\right)\right],$$

respectively, for $0 \leq \theta < 2\pi$, $0 \leq a < 2\pi$, $b > 0$, $c > 0$, and $\theta + 2\pi k > a$. The nth trigonometric moment is

$$m_n = e^{\mathrm{i}na} \frac{\Gamma(\mathrm{i}nb + c)}{\Gamma(c)}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = a,$$

$$\rho = \frac{\Gamma(\mathrm{i}b + c)}{\Gamma(c)},$$

$$\gamma_1 = 0,$$

and

$$\gamma_2 = \frac{\frac{\Gamma(2\mathrm{i}b+c)}{\Gamma(c)} - \left[\frac{\Gamma(\mathrm{i}b+c)}{\Gamma(c)}\right]^4}{\left[1 - \frac{\Gamma(\mathrm{i}b+c)}{\Gamma(c)}\right]^2},$$

respectively. The wrapped generalized Gompertz distribution was used to analyze the orientations of 50 noisy scrub bird nests along a creek bank, as reported by [1]. While this distribution was found to fit the data effectively, it involves several parameters. Thus, there might be alternative distributions with fewer parameters that offer a better fit to the data.

2.12. Wrapped Generalized Normal Laplace Distribution

Ref. [26] took g to be the PDF of the generalized normal Laplace distribution to obtain the wrapped generalized normal Laplace distribution. Its PDF is

$$f(\theta) = \frac{1}{2\pi} \left\{1 + 2\sum_{k=1}^{\infty} [\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)]\right\}$$

for $0 \leq \theta < 2\pi$, where

$$\alpha_k = \left[\frac{e^{-\tau^2 k^2}}{(1 + a^2k^2)(1 + b^2k^2)}\right]^{\frac{\zeta}{2}} \cos\left\{\arctan\left[\frac{(1 + abk^2)\cos(\eta k) + (b - a)k\sin(\eta k)}{(1 + abk^2)\sin(\eta k) + (b - a)k\cos(\eta k)}\right]\right\}$$

and

$$\beta_k = \left[\frac{e^{-\tau^2 k^2}}{(1+a^2k^2)(1+b^2k^2)}\right]^{\frac{\zeta}{2}} \sin\left\{\arctan\left[\frac{(1+abk^2)\cos(\eta k)+(b-a)k\sin(\eta k)}{(1+abk^2)\sin(\eta k)+(b-a)k\cos(\eta k)}\right]\right\}$$

for $-\infty < \eta < \infty$, $-\infty < \tau < \infty$, $a > 0$, $b > 0$, and $\zeta > 0$. The nth trigonometric moment is

$$m_n = \left[\frac{e^{\mathrm{i}\eta n - \frac{\tau^2 n^2}{2}}}{(1-\mathrm{i}an)(1-\mathrm{i}bn)}\right]^{\zeta}$$

for $n = 1, 2, \ldots$. The mean direction and mean resultant length are

$$\mu = \zeta\left[\eta + \arctan\left(\frac{a-b}{1+ab}\right)\right] \bmod 2\pi$$

and

$$\rho = \left[\frac{e^{-\tau^2}}{(1+a^2)(1+b^2)}\right]^{\frac{\zeta}{2}},$$

respectively. The wrapped generalized normal Laplace distribution was used to analyze a dataset containing 1827 flight headings of migrating birds, as documented by [27]. It was found that this distribution offers a superior fit compared to a five-parameter mixture of two von Mises distributions but that it is not as effective as a four-parameter mixture incorporating circular uniform and skew normal components.

2.13. *Wrapped Generalized Skew Normal Distribution [10]*

Ref. [10] took g to be the PDF of the generalized skew normal distribution to obtain the wrapped generalized skew normal distribution. Its PDF is

$$f(\theta) = \frac{2}{\sigma(\alpha+2)} \sum_{k=-\infty}^{\infty} \phi\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\left[1+\alpha\Phi\left(\frac{\lambda}{\sigma}(\theta+2\pi k-\mu)\right)\right]$$

for $0 \le \theta < 2\pi$, $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \lambda < \infty$, and $\alpha \ge -1$. The nth trigonometric moment is

$$m_n = e^{\mathrm{i}\mu n}\left\{\frac{\gamma}{\left[\alpha^2-(\beta+\mathrm{i}n)^2\right]^{\frac{1}{2}}}\right\}^{2\lambda}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\frac{\sin\mu + \frac{\alpha}{\alpha+2}G(\delta\sigma)\cos\mu}{\cos\mu - \frac{\alpha}{\alpha+2}G(\delta\sigma)\sin\mu},$$

$$\rho = \eta e^{-\frac{\sigma^2}{2}},$$

$$\gamma_1 = \frac{e^{-\frac{p^2\sigma^2}{2}}\left\{\sin[p(2\mu-\omega)] + \frac{\alpha}{\alpha+2}G(p\delta\sigma)\cos[p(2\mu-\omega)]\right\}}{\left(1-\eta e^{-\frac{\sigma^2}{2}}\right)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{e^{-\frac{p^2\sigma^2}{2}}\left[\cos p\omega - \frac{\alpha}{\alpha+2}G(\delta\sigma p)\sin p\omega\right] - \eta^4 e^{-2\sigma^2}}{\left(1-\eta e^{-\frac{\sigma^2}{2}}\right)^2},$$

respectively, where $\eta = \sqrt{1+\frac{\alpha}{2+\alpha}G(\delta\sigma)}$, $G(d) = \sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{d^{2n+1}}{2^n n!(2n+1)}$ for d real, and $\omega = \mu - \frac{\sin\mu+\frac{\alpha}{\alpha+2}G(p\delta\sigma)\cos\mu}{\cos\mu-\frac{\alpha}{\alpha+2}G(p\delta\sigma)\sin\mu}$. The wrapped generalized skew normal distribution is hindered by its non-closed-form PDF and excessive parameters, potentially outperformed by simpler models.

2.14. Wrapped Generalized Skew Normal Distribution [11]

Ref. [11] took g to be the PDF of the generalized skew normal distribution to obtain the wrapped generalized skew normal distribution. Its PDF is

$$f(\theta) = \frac{2}{w}\sum_{k=-\infty}^{\infty}\phi\left(\frac{\theta+2\pi k-\mu}{w}\right)\Phi\left(\alpha\left(\frac{\theta+2\pi k-\mu}{w}\right)+\beta\left(\frac{\theta+2\pi k-\mu}{w}\right)^3\right)$$

for $0 \le \theta < 2\pi$, $-\infty < \mu < \infty$, $w > 0$, $-\infty < \alpha < \infty$, and $-\infty < \beta < \infty$. No properties were derived for this distribution. The wrapped generalized skew normal distribution was utilized to analyze wind direction data from a meteorological station in Villena, Alicante, Spain. The data, collected in June 2009 using an Oregon Scientific WMR928NX automatic weather station, were split into sea breeze and mountain breeze subsets. The wrapped generalized skew normal distribution appeared to be the most suitable fit for both the entire dataset and the mountain breeze period. However, for the sea breeze period, the Jones and Pewsey sine-skewed distribution demonstrated the best fit.

2.15. Wrapped Half-Logistic Distribution

Ref. [15] took g to be the PDF of the half-logistic distribution to obtain the wrapped half-logistic distribution. Its PDF is

$$f(\theta) = \sum_{k=-\infty}^{\infty}\frac{1}{2\sigma}\text{sech}^2\left(\frac{\theta+2k\pi-\mu}{2\sigma}\right)$$

for $0 \le \theta < 2\pi$, $0 \le \mu < 2\pi$, and $\sigma > 0$. The nth trigonometric moment is

$$m_n = 2[b_n\cos(p\mu) - c_n\sin(p\mu)] + 2\mathrm{i}[b_n\sin(p\mu) + c_n\cos(p\mu)],$$

where

$$b_n = \int_0^a \cos(p\sigma y)\frac{e^{-y}}{(1+e^{-y})^2}dy + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{ne^{-na}}{n^2+p^2\sigma^2}[n\cos(p\sigma a) - p\sigma\sin(p\sigma a)]$$

and

$$c_n = \int_0^a \sin(p\sigma y)\frac{e^{-y}}{(1+e^{-y})^2}dy + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{ne^{-na}}{n^2+p^2\sigma^2}[n\sin(p\sigma a) + p\sigma\cos(p\sigma a)].$$

The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left[\frac{b_1\sin(\mu)+c_1\cos(\mu)}{b_1\cos(\mu)-c_1\sin(\mu)}\right],$$

$$\rho = 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2},$$

$$\gamma_1 = \frac{2\sqrt{[b_2\cos(2\mu) - c_2\sin(2\mu)]^2 + [b_2\sin(2\mu) + c_2\cos(2\mu)]^2}\cos(\mu_2 - 2\mu)}{\left[1 - 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{2\sqrt{[b_2\cos(2\mu) - c_2\sin(2\mu)]^2 + [b_2\sin(2\mu) + c_2\cos(2\mu)]^2}\sin(\mu_2 - 2\mu)}{\left[1 - 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2}\right]^2} - \frac{\left[2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2}\right]^4}{\left[1 - 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2}\right]^2},$$

respectively, where $\mu_2 = \arctan\left[\frac{b_2\sin(2\mu)+c_2\cos(2\mu)}{b_2\cos(2\mu)-c_2\sin(2\mu)}\right]$. The utility of the wrapped half-logistic distribution is restricted by its non-closed-form PDF.

2.16. Wrapped Half-Normal Distribution

If g is the pdf of the half-normal distribution we obtain the wrapped half-normal distribution. Its pdf is

$$f(\theta) = \sum_{k=0}^{\infty} \frac{1}{2\sigma}\operatorname{sech}^2\left(\frac{\theta + 2k\pi - \mu}{2\sigma}\right)$$

for $0 \le \theta < 2\pi$ and $\sigma > 0$. The nth trigonometric moment is

$$m_n = 2[b_n \cos(p\mu) - c_n \sin(p\mu)] + 2\mathrm{i}[b_n \sin(p\mu) + c_n \cos(p\mu)],$$

where

$$b_n = \int_0^a \cos(p\sigma y)\frac{e^{-y}}{(1+e^{-y})^2}dy + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{ne^{-m}}{n^2 + p^2\sigma^2}\begin{pmatrix} n\cos(p\sigma a) \\ -p\sigma\sin p\sigma a\end{pmatrix}$$

and

$$c_n = \int_0^a \cos(p\sigma y)\frac{e^{-y}}{(1+e^{-y})^2}dy + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{ne^{-m}}{n^2 + p^2\sigma^2}\begin{pmatrix} n\sin(p\sigma a) \\ p\sigma\cos p\sigma a\end{pmatrix}.$$

The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left[\frac{b_1\sin(\mu) + c_1\cos(\mu)}{b_1\cos(\mu) - c_1\sin(\mu)}\right],$$

$$\rho = 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2},$$

$$\gamma_1 = \frac{2\sqrt{[b_2\cos(2\mu) - c_2\sin(2\mu)]^2 + [b_2\sin(2\mu) + c_2\cos(2\mu)]^2}\sin(\mu_2 - 2\mu)}{\left[1 - 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2}\right]^{\frac{3}{2}}}$$

and

$$\gamma_2 = \frac{2\sqrt{[b_2\cos(2\mu) - c_2\sin(2\mu)]^2 + [b_2\sin(2\mu) + c_2\cos(2\mu)]^2}\sin(\mu_2 - 2\mu)}{\left[1 - 2\sqrt{(b_1\cos\mu - c_1\sin\mu)^2 + (b_1\sin\mu + c_1\cos\mu)^2}\right]^2}$$
$$- \frac{\left[2\sqrt{(b_1\cos\mu - c_1\sin\mu)^2 + (b_1\sin\mu + c_1\cos\mu)^2}\right]^4}{\left[1 - 2\sqrt{(b_1\cos\mu - c_1\sin\mu)^2 + (b_1\sin\mu + c_1\cos\mu)^2}\right]^2},$$

respectively, where $\mu_2 = \arctan\left[\frac{b_2\sin(2\mu)+c_2\cos(2\mu)}{b_2\cos(2\mu)-c_2\sin(2\mu)}\right]$.

2.17. Wrapped [12]'s Skew Laplace Distribution

Ref. [28] took g to be the PDF of [12]'s skew Laplace distribution to obtain the wrapped [12]'s skew Laplace distribution. Its PDF and CDF are

$$f(\theta) = \frac{pce^{c(\beta+2\pi-2\beta\pi-\theta)} + (1-p)e^{c+(2\pi-1)c\beta}}{e^{2\pi c} - 1}$$

and

$$F(\theta) = \frac{pe^{c(\beta+2\pi-2\beta\pi)}\left(1 - e^{-c\theta}\right) + (1-p)e^{(2\pi-1)c\beta}\left(e^{c\theta} - 1\right)}{e^{2\pi c} - 1},$$

respectively, for $0 \leq \theta < 2\pi$, $-\infty < \beta < \infty$, $c > 0$, and $0 < p < 1$. The nth trigonometric moment is

$$m_n = \alpha_n + \mathrm{i}\beta_n$$

for $n = 1, 2, \ldots$, where

$$\alpha_n = \frac{c^2}{n^2 + c^2}\left[ne^{(1-2\pi)\beta c} + (1-n)e^{(2\pi-1)\beta c}\right]$$

and

$$\beta_n = \frac{nc}{n^2 + c^2}\left[(n-1)e^{(2\pi-1)\beta c} - ne^{(1-2\pi)\beta c}\right].$$

The distribution was tested on the Black Mountain wind direction dataset [1]. It showed a superior fit compared to other distributions, specifically the wrapped variance gamma distribution and the generalized von Mises distribution, which have more parameters.

2.18. Wrapped Hypoexponential Distribution

Ref. [29] took g to be the PDF of the hypoexponential distribution to obtain the wrapped hypoexponential distribution. Its PDF is

$$f(\theta) = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1\theta}\sum_{k=0}^{\infty} e^{-2k\pi\lambda_1} - e^{-\lambda_2\theta}\sum_{k=0}^{\infty} e^{-2k\pi\lambda_2}\right)$$

for $0 \leq \theta < 2\pi$, $\lambda_1 > 0$, and $\lambda_2 > 0$. The nth trigonometric moment is

$$m_n = \frac{\lambda_1\lambda_2}{(\lambda_1 - \mathrm{i}n)(\lambda_2 - \mathrm{i}n)}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right),$$

$$\rho = \frac{\lambda_1 \lambda_2}{\sqrt{\left(1+\lambda_1^2\right)\left(1+\lambda_2^2\right)}},$$

$$\gamma_1 = \frac{\frac{\lambda_1\lambda_2}{\sqrt{\left(4+\lambda_1^2\right)\left(4+\lambda_2^2\right)}} \sin\left[\arctan\left(\frac{2}{\lambda_1} + \frac{2}{\lambda_2}\right) - 2\arctan\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\right]}{\left[1 - \frac{\lambda_1\lambda_2}{\sqrt{\left(1+\lambda_1^2\right)\left(1+\lambda_2^2\right)}}\right]^{\frac{3}{4}}},$$

and

$$\gamma_2 = \frac{\frac{\lambda_1\lambda_2}{\sqrt{\left(4+\lambda_1^2\right)\left(4+\lambda_2^2\right)}} \sin\left[\arctan\left(\frac{2}{\lambda_1} + \frac{2}{\lambda_2}\right) - 2\arctan\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\right] - \frac{\lambda_1^4\lambda_2^4}{\left(1+\lambda_1^2\right)^2\left(1+\lambda_2^2\right)^2}}{\left[1 - \frac{\lambda_1\lambda_2}{\sqrt{\left(1+\lambda_1^2\right)\left(1+\lambda_2^2\right)}}\right]^{2}},$$

respectively, The wrapped hypoexponential distribution has a PDF that can be recast into a closed form. It also offers additional flexibility compared to one-parameter models.

2.19. Wrapped Ishita Distribution

Ref. [30] took g to be the PDF of the Ishita distribution to obtain the wrapped Ishita distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda^3 e^{-\lambda\theta}}{\lambda+2}\left[\frac{\lambda+\theta^2}{1-e^{-2\pi\lambda}} + \frac{4\pi\theta e^{-2\pi\lambda}}{\left(1-e^{-2\pi\lambda}\right)^2} + \frac{4\pi^2 e^{-2\pi\lambda}\left(1+e^{-2\pi\lambda}\right)}{\left(1-e^{-2\pi\lambda}\right)^3}\right]$$

and

$$\begin{aligned} F(\theta) = \frac{1}{\lambda^3+2}\Bigg\{ & \frac{\lambda^3+2-\left[\lambda^3+2+\lambda\theta(\lambda\theta+2)\right]e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \\ & + \frac{4\pi\lambda\left[1-(\lambda\theta+1)e^{-\lambda\theta}\right]e^{-2\lambda\pi}}{\left(1-e^{-2\pi\lambda}\right)^2} \\ & + \frac{4\pi^2\lambda^2\left(1-e^{-\lambda\theta}\right)\left(1+e^{-2\pi\lambda}\right)e^{-2\lambda\pi}}{\left(1-e^{-2\pi\lambda}\right)^3}\Bigg\}, \end{aligned}$$

respectively, for $0 \le \theta < 2\pi$ and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \frac{\lambda^3}{\lambda^3+2} \frac{\lambda^3 - n^2\lambda + 2 - 2in\lambda^2}{(\lambda - in)^3}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = 3\arctan\left(\frac{1}{\lambda}\right) - \arctan\left(\frac{2\lambda^2}{\lambda^3 - \lambda + 2}\right),$$

$$\rho = \frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{\left(\lambda^3-\lambda+2\right)^2+4\lambda^4}{\left(\lambda^2+1\right)^3}},$$

$$\gamma_1 = \frac{\frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{\left(\lambda^3-4\lambda+2\right)^2+16\lambda^4}{\left(\lambda^2+4\right)^3}}\sin(\kappa_{\lambda,2})}{\left[1-\frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{\left(\lambda^3-\lambda+2\right)^2+4\lambda^4}{\left(\lambda^2+1\right)^3}}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\lambda^3}{\lambda^3+2}\sqrt{\frac{\left(\lambda^3-4\lambda+2\right)^2+16\lambda^4}{\left(\lambda^2+4\right)^3}}\cos(\kappa_{\lambda,2}) - \left[\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{\left(\lambda^2-\lambda+2\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^3}}\right]^4}{\left[1-\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{\left(\lambda^2-\lambda+2\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^3}}\right]^2},$$

respectively, where $\kappa_{\lambda,2} = 3\arctan\left(\frac{2}{\lambda}\right) - \arctan\left(\frac{4\lambda^2}{\lambda^3-4\lambda+2}\right) - 6\arctan\left(\frac{1}{\lambda}\right) + 2\arctan\left(\frac{2\lambda^2}{\lambda^3-\lambda+2}\right)$. The wrapped Ishita distribution, despite having closed-form expressions for PDF and CDF, might be unsuitable for some real-life datasets, as its flexibility is restricted by the fact it only has one parameter.

2.20. Wrapped Laplace Distribution

Ref. [31] took g to be the PDF of the Laplace distribution to obtain the wrapped Laplace distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda\kappa}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\theta}}{1-e^{-2\pi\lambda\kappa}} + \frac{e^{\frac{\lambda\theta}{\kappa}}}{e^{\frac{2\pi\lambda}{\kappa}}-1}\right)$$

and

$$F(\theta) = \frac{1}{1+\kappa^2}\frac{1-e^{-\kappa\lambda\theta}}{1-e^{-2\pi\kappa\lambda}} + \frac{\kappa^2}{1+\kappa^2}\frac{e^{\frac{\lambda\theta}{\kappa}}-1}{1-e^{-\frac{2\pi\lambda}{\kappa}}},$$

respectively, for $0 \le \theta < 2\pi$, $\kappa > 0$, and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \frac{1}{\left(1-\frac{in}{\lambda\kappa}\right)\left[1+\frac{in}{\left(\frac{\lambda}{\kappa}\right)}\right]}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \begin{cases} \arctan\left(\frac{1}{\lambda\kappa}\right) - \arctan\left(\frac{\kappa}{\lambda}\right), & \text{for } \kappa \le 1, \\ 2\pi + \arctan\left(\frac{1}{\lambda\kappa}\right) - \arctan\left(\frac{\kappa}{\lambda}\right), & \text{for } \kappa > 1, \end{cases}$$

$$\rho = \frac{\lambda^2}{\sqrt{1+(\lambda\kappa)^2}\sqrt{\frac{\lambda^2}{\kappa^2}+1}},$$

$$\gamma_1 = \frac{-2\lambda}{(1+\lambda^2)^{\frac{1}{4}}(4+\lambda^2)\left(\sqrt{1+\lambda^2}-\lambda\right)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{3\lambda^2}{(1+\lambda^2)(4+\lambda^2)\left(\sqrt{1+\lambda^2}-|\lambda|\right)^2},$$

respectively. The ant orientation data from [17] were analyzed using the wrapped Laplace distribution, which demonstrated a superior fit compared to the von Mises distribution. However, the scope of conclusions is constrained by the limited number of distributions considered. To better evaluate the goodness of fit, it is essential to compare the wrapped Laplace distribution against a broader range of distributions.

2.21. Wrapped Length-Biased Weighted Exponential Distribution

Ref. [32] took g to be the PDF of the length-biased weighted exponential distribution to obtain the wrapped length-biased weighted exponential distribution. Its PDF and CDF are

$$f(\theta) = \frac{[\lambda(\alpha+1)]^2}{\alpha(\alpha+2)} e^{-\lambda\beta} \left\{ \frac{1}{1-e^{-2\pi\lambda}} \left(\beta + \frac{2\pi e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}} \right) - \frac{e^{-\alpha\beta\lambda}}{1-e^{-2\pi\lambda(1+\alpha)}} \left[\beta + \frac{2\pi e^{-2\pi\lambda(1+\alpha)}}{1-e^{-2\pi\lambda(1+\alpha)}} \right] \right\}$$

and

$$F(\theta) = \frac{1}{\alpha(\alpha+2)} \left\{ \frac{(1+\alpha)^2\left[1-(1+\beta)e^{-\beta}\right]}{1-e^{-2\pi}} + \frac{(1+\alpha)^2 2\pi\left(1-e^{-\beta}\right)e^{-2\pi}}{\left(1-e^{-2\pi}\right)^2} \right.$$
$$\left. + \frac{e^{-\beta(1+\alpha)}[1+\beta(1+\alpha)]-1}{1-e^{-2\pi(1+\alpha)}} + \frac{2\pi(1+\alpha)e^{-2\pi(1+\alpha)}\left[e^{-\beta(1+\alpha)}-1\right]}{\left[1-e^{-2\pi(1+\alpha)}\right]^2} \right\},$$

respectively, for $0 \le \theta < 2\pi$, $\alpha > 0$, $\lambda > 0$, and $0 < \beta \le 2\pi$. The nth trigonometric moment is

$$m_n = \frac{(1-\mathrm{i}n)^{-2}\left(1-\frac{\mathrm{i}n}{1+\alpha}\right)^{-2}}{\left(1-\frac{2\mathrm{i}n}{2+\alpha}\right)^{-1}}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan(1) + \arctan\left(\frac{1}{1+\alpha}\right) - \arctan\left(\frac{2}{2+\alpha}\right),$$

$$\rho = \frac{\sqrt{1+(2+\alpha)^{-2}}}{2[1+(1+\alpha)^{-2}]},$$

$$\gamma_1 = \frac{\sin(\mu_2 - 2\mu)}{5\left\{1 - \frac{[1+(2+\alpha)^{-2}]^{-\frac{1}{2}}}{2[1+(1+\alpha)^{-2}]}\right\}^{\frac{3}{2}} [1+4(1+\alpha)^{-2}]\sqrt{1+\left(\frac{4}{2+\alpha}\right)^2}},$$

and

$$\gamma_2 = \frac{\cos(\mu_2 - 2\mu)}{5\left\{1 - \frac{[1+(2+\alpha)^{-2}]^{-\frac{1}{2}}}{2[1+(1+\alpha)^{-2}]}\right\}^2 [1 + 4(1+\alpha)^{-2}]\sqrt{1 + \left(\frac{4}{2+\alpha}\right)^2}} - \frac{\left\{\frac{[1+(2+\alpha)^{-2}]^{-\frac{1}{2}}}{2[1+(1+\alpha)^{-2}]}\right\}^4}{\left\{1 - \frac{[1+(2+\alpha)^{-2}]^{-\frac{1}{2}}}{2[1+(1+\alpha)^{-2}]}\right\}^2},$$

respectively, where $\mu_2 = \arctan(2) + \arctan\left(\frac{2}{1+\alpha}\right) - \arctan\left(\frac{4}{2+\alpha}\right)$. The wrapped length-biased weighted exponential distribution was used to analyze the feldspar laths dataset obtained from [33] and published by [1]. Watson's U^2 test indicated a good fit of the distribution to the data. No comparisons with other distributions were performed.

2.22. Wrapped Levy Distribution

Ref. [1] took g to be the PDF of the Levy distribution to obtain the wrapped Levy distribution. Its PDF is

$$f(\theta) = \sum_{k=-\infty}^{\infty} \sqrt{\frac{c}{2\pi}} \frac{e^{-\frac{c}{2}(\theta+2\pi n-\mu)}}{(\theta + 2\pi n - \mu)^{\frac{3}{2}}}$$

for $0 \leq \theta < 2\pi$, the summand is zero if $\theta + 2\pi n - \mu \leq 0$, $c > 0$, and $-\infty < \mu < \infty$. The nth trigonometric moment is

$$m_n = e^{\mathrm{i}n\mu - \sqrt{c|n|}[1-\mathrm{i}\,\mathrm{sgn}(n)]} = e^{\mathrm{i}n\mu - \sqrt{cn}(1-\mathrm{i})}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu + \sqrt{c},$$

$$\rho = e^{-\sqrt{c}},$$

$$\gamma_1 = \frac{e^{-\sqrt{2c}} \sin\left(\sqrt{2c} - 2\sqrt{c}\right)}{\left(1 - e^{-\sqrt{c}}\right)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{e^{-\sqrt{2c}} \cos\left(\sqrt{2c} - 2\sqrt{c}\right) - e^{-4\sqrt{c}}}{\left(1 - e^{-\sqrt{c}}\right)^2},$$

respectively. The practical utility of the wrapped Levy distribution is hampered by its non-closed-form PDF.

2.23. Wrapped Lindley Distribution

Ref. [34] took g to be the PDF of the Lindley distribution to obtain the wrapped Lindley distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda^2}{1+\lambda} e^{-\lambda\theta} \left[\frac{1+\theta}{1 - e^{-2\pi\lambda}} + \frac{2\pi e^{-2\pi\lambda}}{\left(1 - e^{-2\pi\lambda}\right)^2}\right]$$

and

$$F(\theta)=\frac{1}{1-e^{-2\pi\lambda}}\left(1-e^{-\lambda\theta}-\frac{\lambda\theta}{\lambda+1}e^{-\lambda\theta}\right)-\frac{2\pi\lambda}{\lambda+1}\frac{e^{-2\pi\lambda}\left(1-e^{-\lambda\theta}\right)}{\left(1-e^{-2\pi\lambda}\right)^{2}},$$

respectively, for $0 \leq \theta < 2\pi$ and $\lambda > 0$. The nth trigonometric moment is given by

$$m_n=\frac{\lambda^2}{1+\lambda}\frac{(1+\lambda-\mathrm{i}n)}{(\lambda-\mathrm{i}n)^2}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu=2\arctan\left(\frac{1}{\lambda}\right)-\arctan\left(\frac{1}{\lambda+1}\right),$$

$$\rho=\frac{\lambda^2\sqrt{(\lambda+1)^2+1}}{(1+\lambda)(\lambda^2+1)},$$

$$\gamma_1=\frac{\lambda^2\sqrt{4+(1+\lambda)^2}\sin(\mu_2-2\mu)}{(1+\lambda)(4+\lambda^2)\left\{1-\frac{\lambda^2[1+(1+\lambda)^2]^{\frac{1}{2}}}{(1+\lambda)(1+\lambda^2)}\right\}^{\frac{3}{2}}},$$

and

$$\gamma_2=\frac{\lambda^2\sqrt{4+(1+\lambda)^2}\cos(\mu_2-2\mu)-\left[\frac{\lambda^2\sqrt{1+(1+\lambda)^2}}{(1+\lambda)(1+\lambda^2)}\right]^4}{(1+\lambda)(4+\lambda^2)\left[1-\frac{\lambda^2\sqrt{1+(1+\lambda)^2}}{(1+\lambda)(1+\lambda^2)}\right]^4},$$

respectively, where $\mu_2 = 2\arctan\left(\frac{2}{\lambda}\right) - \arctan\left(\frac{2}{\lambda+1}\right)$. The wrapped Lindley distribution was used to analyze a dataset concerning the orientations of 76 turtles after laying eggs, as documented by [1]. This distribution seemed to match the data effectively and performed better than the alternative distribution it was contrasted with, the wrapped exponential distribution.

2.24. Wrapped Linnik Distribution

Ref. [35] took g to be the PDF of the Linnik distribution to obtain the wrapped Linnik distribution. Its PDF and CDF are

$$f(\theta)=\frac{1}{2\pi}\left[1+2\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{1+\sigma k^{\alpha}}\right]$$

and

$$F(\theta)=\frac{1}{2\pi}\left[\theta+2\sum_{k=1}^{\infty}\frac{\sin(k\theta)}{k(1+\sigma k^{\alpha})}\right],$$

respectively, for $0 \leq \theta < 2\pi$, $\sigma > 0$, and $0 < \alpha \leq 2$. The nth trigonometric moment is

$$m_n=\frac{1}{1+\sigma\mid n\mid^{\alpha}}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = 0,$$

$$\rho = \frac{1}{1+\sigma},$$

$$\gamma_1 = 0,$$

and

$$\gamma_2 = \frac{(1+\sigma)^4 - (1+2^\alpha \sigma)}{\sigma^{2\alpha}(1+2^\alpha \sigma)(1+\sigma)^2},$$

respectively. The wrapped Linnik distribution was used to analyze the frequency of traffic accidents throughout the day in Srinagar, India, in 2016, sourced from reports from the National Crime Records Bureau, India. This distribution was also applied to ant data collected by [17]. Results indicated that the wrapped Linnik distribution provided a better fit for both datasets compared to the wrapped stable distribution.

2.25. Wrapped Lomax Distribution

Ref. [36] took g to be the PDF of the Lomax distribution to obtain the wrapped Lomax distribution. Its PDF and CDF are

$$f(\theta) = \frac{\alpha}{\sigma} \sum_{k=0}^{\infty} \left(1 + \frac{\theta + 2k\pi}{\sigma}\right)^{-\alpha-1}$$

and

$$F(\theta) = \sum_{k=0}^{\infty} \left(1 + \frac{2k\pi}{\sigma}\right)^{-\alpha} - \sum_{k=0}^{\infty} \left(1 + \frac{\theta + 2k\pi}{\sigma}\right)^{-\alpha},$$

respectively, for $0 \le \theta < 2\pi$, $\sigma > 0$, and $\alpha > 0$. The wrapped Lomax distribution faces limitations due to its non-closed-form PDF.

2.26. Wrapped Modified Lindley Distribution

Ref. [37] took g to be the PDF of the modified Lindley distribution to obtain the wrapped modified Lindley distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda e^{-\lambda\theta}}{1 - e^{-4\lambda\pi}} \left\{1 + e^{-2\lambda\pi} + \frac{e^{-\lambda\theta}}{1+\lambda}\left[2\lambda\theta - 1 + \frac{4\lambda\pi e^{-4\lambda\pi}}{1 - e^{-4\lambda\pi}}\right]\right\}$$

and

$$F(\theta) = \frac{1 - e^{-\lambda\theta}}{1 - e^{-2\lambda\pi}} + \frac{2\lambda\pi e^{-4\lambda\pi}\left(1 - e^{-2\lambda\theta}\right)}{(1+\lambda)\left(1 - e^{-4\lambda\theta}\right)^2} - \frac{\lambda\theta e^{-2\lambda\theta}}{(1+\lambda)\left(1 - e^{-4\lambda\theta}\right)},$$

respectively, for $0 \le \theta < 2\pi$ and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \frac{\lambda}{\lambda - n\mathrm{i}} + \frac{\lambda n\mathrm{i}}{(1+\lambda)(2\lambda - n\mathrm{i})^2}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \begin{cases} \pi + \arctan\left(\dfrac{\beta_1}{\alpha_1}\right), & \text{if } \alpha_1 < 0, \beta_1 \geq 0, \\ \dfrac{\pi}{2}, & \text{if } \alpha_1 < 0, \beta_1 > 0, \\ \arctan\left(\dfrac{\beta_1}{\alpha_1}\right), & \text{if } \alpha_1 > 0, \\ \text{undefined}, & \text{if } \alpha_1 = 0, \beta_1 = 0 \end{cases}$$

and

$$\rho = \lambda^2 \sqrt{\frac{16\lambda^4 + 32\lambda^3 + 24\lambda^2 + 16\lambda + 10}{(1+\lambda)^2(\lambda^2+1)(4\lambda^2+1)^2}},$$

respectively, where

$$\alpha_1 = \frac{\lambda^2}{\lambda^2+1} + \frac{4\lambda^2}{(1+\lambda)(4\lambda^2+1)^2}$$

and

$$\beta_1 = \frac{\lambda}{\lambda^2+1} + \frac{4\lambda^3}{(1+\lambda)(4\lambda^2+1)^2} - \frac{\lambda}{(1+\lambda)(4\lambda^2+1)^2}.$$

The wrapped modified Lindley distribution was used on two real-life datasets from [1]. The first set involves 76 turtles' egg-laying orientations, while the second set has 133 measurements of feldspar lath orientations in basalt. This distribution was compared with the wrapped exponential distribution, transmuted wrapped exponential distribution, and wrapped Lindley distribution for goodness of fit. The wrapped modified Lindley distribution provided a competitive fit for both datasets. It offered the best fit for the turtle dataset but did not outperform the transmuted wrapped exponential distribution for the feldspar lath dataset.

2.27. Wrapped New Weibull–Pareto Distribution

Ref. [38] took g to be the PDF of the new Weibull–Pareto distribution to obtain the wrapped new Weibull–Pareto distribution. Its PDF is

$$f(\theta) = \sum_{k=-\infty}^{\infty} \frac{1}{4\sigma} \operatorname{sech}^2\left(\frac{\theta + 2k\pi - \mu}{2\sigma}\right)$$

for $0 \leq \theta < 2\pi$, $\theta > \mu$, and $\sigma > 0$. The nth trigonometric moment is

$$m_n = \frac{\pi\sigma n}{\sinh \pi\sigma n} e^{in\mu}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu,$$

$$\rho = \frac{\pi\sigma p}{\sinh \pi\sigma p},$$

$$\gamma_1 = 0,$$

and

$$\gamma_2 = \frac{\frac{2\pi\sigma}{\sinh(2\pi\sigma)} - \left[\frac{\pi\sigma}{\sinh(\pi\sigma)}\right]^4}{\left[1 - \frac{\pi\sigma}{\sinh(\pi\sigma)}\right]^2},$$

respectively. The wrapped new Weibull–Pareto distribution and the wrapped exponentiated inverted Weibull distribution were tested on the turtle dataset from [1]. Both models fit the data well, but the wrapped exponentiated inverted Weibull distribution slightly outperformed the wrapped new Weibull–Pareto distribution.

2.28. Wrapped Normal Distribution

Ref. [39] took g to be the PDF of the normal distribution to obtain the wrapped normal distribution. Its PDF and CDF are

$$f(\theta) = \frac{1}{\sqrt{2\pi}\sigma} \sum_{k=-\infty}^{\infty} e^{-\frac{(\theta-\mu+2\pi k)^2}{2\sigma^2}}$$

and

$$F(\theta) = \sum_{k=-\infty}^{\infty} \left[\Phi\left(\frac{\theta + 2\pi k - \mu}{\sigma}\right) - \Phi\left(\frac{2\pi k - \mu}{\sigma}\right)\right],$$

respectively, for $0 \leq \theta < 2\pi$, $-\infty < \mu < \infty$, and $\sigma > 0$. The nth trigonometric moment is

$$m_n = e^{\mathrm{i}n\mu - \frac{n^2\sigma^2}{2}}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu,$$

$$\rho = e^{-\frac{\sigma^2}{2}},$$

$$\gamma_1 = 0,$$

and

$$\gamma_2 = 0,$$

respectively. The wrapped normal distribution, commonly used in literature, adequately fits real-life data but is restricted by its non-closed-form PDF.

2.29. Wrapped Pareto Distribution

Ref. [40] took g to be the PDF of a Pareto distribution to obtain the wrapped Pareto distribution. Its PDF is

$$f(\theta) = \frac{\alpha^{-\frac{1}{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right)} \int_0^{\infty} \frac{\lambda s e^{-\lambda s\theta}}{1 - e^{-2\pi\lambda s}} s^{\frac{1}{\alpha}-1} e^{-\frac{s}{\alpha}} ds$$

for $0 \leq \theta < 2\pi$, $\lambda > 0$, and $\alpha > 0$. The nth trigonometric moment is

$$m_n = \frac{\alpha^{-\frac{1}{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right)} \int_0^{\infty} \left(1 - \frac{\mathrm{i}n}{\lambda s}\right)^{-1} s^{\frac{1}{\alpha}-1} e^{-\frac{s}{\alpha}} ds$$

for $n = 1, 2, \ldots$. The wrapped Pareto distribution is constrained by its non-closed-form PDF.

2.30. Wrapped Quasi-Lindley Distribution

Ref. [41] took g to be the PDF of the quasi-Lindley distribution to obtain the wrapped quasi-Lindley distribution. Its PDF is

$$f(\theta) = \frac{be^{-b\theta}}{a+1}\left[\frac{a+b\theta}{1-e^{-2\pi b}} + \frac{2\pi b e^{-2\pi b}}{\left(1-e^{-2\pi b}\right)^2}\right]$$

for $0 \leq \theta < 2\pi$, $a > 0$, and $b > 0$. The nth trigonometric moment is

$$m_n = \alpha_n + \mathrm{i}\beta_n$$

for $n = 1, 2, \ldots$, where

$$\alpha_n = \frac{\sqrt{b^4(a+1)^2 + n^2a^2b^2}}{(a+1)(b^2+n^2)} \cos\left\{2\arctan\left(\frac{n}{b}\right) - \arctan\left[\frac{na}{b(1+a)}\right]\right\}$$

and

$$\beta_n = \frac{\sqrt{b^4(a+1)^2 + n^2a^2b^2}}{(a+1)(b^2+n^2)} \sin\left\{2\arctan\left(\frac{n}{b}\right) - \arctan\left[\frac{na}{b(1+a)}\right]\right\}.$$

The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = 2\arctan\left(\frac{1}{b}\right) - a\arctan\left[\frac{a}{b(1+a)}\right],$$

$$\rho = \frac{\sqrt{b^4(a+1)^2 + a^2b^2}}{(a+1)(b^2+1)},$$

$$\gamma_1 = \frac{\overline{\beta}_2}{\rho^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\overline{\alpha}_2 - \rho^4}{\rho^2},$$

respectively, where

$$\overline{\alpha}_2 = \frac{\sqrt{b^4(a+1)^2 + 4a^2b^2}}{(a+1)(b^2+4)} \cos\left\{2\arctan\left(\frac{2}{b}\right) - \arctan\left[\frac{2a}{b(1+a)}\right] - 4\arctan\left(\frac{1}{b}\right) + \arctan\left[\frac{a}{b(1+a)}\right]\right\}$$

and

$$\overline{\beta}_2 = \frac{\sqrt{b^4(a+1)^2 + 4a^2b^2}}{(a+1)(b^2+4)} \sin\left\{2\arctan\left(\frac{2}{b}\right) - \arctan\left[\frac{2a}{b(1+a)}\right] - 4\arctan\left(\frac{1}{b}\right) + \arctan\left[\frac{a}{b(1+a)}\right]\right\}.$$

2.31. Wrapped Rama Distribution

Ref. [42] took g to be the PDF of the Rama distribution to obtain the wrapped Rama distribution. Its PDF and CDF are

$$f(\theta) = Ae^{-\lambda\theta}\left(\theta^3 + B\theta^2 + C\theta + D\right)$$

and

$$F(\theta) = \frac{A}{\lambda^4}\left[\gamma(4, \lambda\theta) + B\lambda\gamma(3, \lambda\theta) + C\lambda^2\gamma(2, \lambda\theta) + D\lambda^3\gamma(1, \lambda\theta)\right]$$

for $0 \leq \theta < 2\pi$ and $\lambda > 0$, where

$$A = \frac{\lambda^4}{\left(\lambda^3 + 6\right)\left(1 - e^{-2\lambda\pi}\right)},$$

$$B = \frac{6\pi e^{-2\lambda\pi}}{1 - e^{-2\lambda\pi}},$$

$$C = \frac{2\pi\left(e^{-2\lambda\pi} + 1\right)B}{1 - e^{-2\lambda\pi}},$$

and

$$D = 1 + \frac{8\pi^3 e^{-2\lambda\pi}\left(e^{-4\lambda\pi} + 4e^{-2\lambda\pi} + 1\right)}{\left(1 - e^{-2\lambda\pi}\right)^3}.$$

The nth trigonometric moment is

$$m_n = \frac{A}{\left(\lambda^2 + n^2\right)^4}\left[\left(c_0 + c_2 n^2 + c_4 n^4 + c_6 n^6\right) + \mathrm{i}\left(d_1 n + d_3 n^2 + d_5 n^5 + d_7 n^7\right)\right]$$

for $n = 1, 2, \ldots$, where

$$\begin{aligned} c_0 = {} & \lambda^7 - 4\frac{\lambda^6 B\pi}{\left(\mathrm{e}^{\pi\lambda}\right)^2} - 4\frac{\lambda^7 B\pi^2}{\mathrm{e}^{2\pi\lambda}} - 2\frac{\lambda^7 C\pi}{\mathrm{e}^{2\pi\lambda}} - 2\frac{\lambda^5 B}{\mathrm{e}^{2\pi\lambda}} - \frac{\lambda^6 C}{\mathrm{e}^{2\pi\lambda}} - 6\frac{\lambda^4}{\mathrm{e}^{2\pi\lambda}} - 12\frac{\pi^2\lambda^6}{\mathrm{e}^{2\pi\lambda}} \\ & - 8\frac{\pi^3\lambda^7}{\mathrm{e}^{2\pi\lambda}} - 12\frac{\pi\lambda^5}{\mathrm{e}^{2\pi\lambda}} + 6\lambda^4 + \lambda^6 C - \frac{\lambda^7}{\mathrm{e}^{2\pi\lambda}} + 2\lambda^5 B, \end{aligned}$$

$$\begin{aligned} c_2 = {} & -\frac{\lambda^4 C}{\mathrm{e}^{2\pi\lambda}} + 4\frac{\lambda^3 B}{\mathrm{e}^{2\pi\lambda}} + 24\frac{\pi\lambda^3}{\mathrm{e}^{2\pi\lambda}} - 12\frac{\pi^2\lambda^4}{\mathrm{e}^{2\pi\lambda}} - 24\frac{\pi^3\lambda^5}{\mathrm{e}^{2\pi\lambda}} + \lambda^4 C - 3\frac{\lambda^5}{\mathrm{e}^{2\pi\lambda}} - 4\lambda^3 B - 36\lambda^2 \\ & + 36\frac{\lambda^2}{\mathrm{e}^{2\pi\lambda}} + 3\lambda^5 - 4\frac{\lambda^4 B\pi}{\mathrm{e}^{2\pi\lambda}} - 6\frac{\lambda^5 C\pi}{\mathrm{e}^{2\pi\lambda}} - 12\frac{\lambda^5 B\pi^2}{\mathrm{e}^{2\pi\lambda}}, \end{aligned}$$

$$\begin{aligned} c_4 = {} & \frac{\lambda^2 C}{\mathrm{e}^{2\pi\lambda}} + 6\frac{\lambda B}{\mathrm{e}^{2\pi\lambda}} + 36\frac{\pi\lambda}{\mathrm{e}^{2\pi\lambda}} + 12\frac{\pi^2\lambda^2}{\mathrm{e}^{2\pi\lambda}} - 24\frac{\pi^3\lambda^3}{\mathrm{e}^{2\pi\lambda}} - 3\frac{\lambda^3}{\mathrm{e}^{2\pi\lambda}} - \lambda^2 C - 6\lambda B - \frac{6}{\mathrm{e}^{2\pi\lambda}} \\ & + 6 + 3\lambda^3 - 6\frac{C\pi\lambda^3}{\mathrm{e}^{2\pi\lambda}} - 12\frac{B\pi^2\lambda^3}{\mathrm{e}^{2\pi\lambda}} + 4\frac{B\pi\lambda^2}{\mathrm{e}^{2\pi\lambda}}, \end{aligned}$$

$$c_6 = 4\frac{B\pi}{\mathrm{e}^{2\pi\lambda}} - 8\frac{\pi^3\lambda}{\mathrm{e}^{2\pi\lambda}} - \frac{\lambda}{\mathrm{e}^{2\pi\lambda}} + \frac{C}{\mathrm{e}^{2\pi\lambda}} + 12\frac{\pi^2}{\mathrm{e}^{2\pi\lambda}} + \lambda - C - 2\frac{\lambda C\pi}{\mathrm{e}^{2\pi\lambda}} - 4\frac{\lambda B\pi^2}{\mathrm{e}^{2\pi\lambda}},$$

$$d_1 = 2\lambda^5 C - \frac{\lambda^6}{e^{2\pi\lambda}} + 6\lambda^4 B - 6\frac{\lambda^4 B}{e^{2\pi\lambda}} - 2\frac{\lambda^5 C}{e^{2\pi\lambda}} - 24\frac{\pi^2\lambda^5}{e^{2\pi\lambda}} - 8\frac{\pi^3\lambda^6}{e^{2\pi\lambda}} - 3\frac{\pi\lambda^4}{e^{2\pi\lambda}} + 24\lambda^3 - 24\frac{\lambda^3}{e^{2\pi\lambda}} + \lambda^6 - 8\frac{\lambda^5 B\pi}{e^{2\pi\lambda}} - 4\frac{\lambda^6 B\pi^2}{e^{2\pi\lambda}} - 2\frac{\lambda^6 C\pi}{e^{2\pi\lambda}},$$

$$d_3 = -3\frac{\lambda^4}{e^{2\pi\lambda}} + 4\lambda^3 C + 4\lambda^2 B - 4\frac{\lambda^3 C}{e^{2\pi\lambda}} - 4\frac{\lambda^2 B}{e^{2\pi\lambda}} - 24\frac{\pi\lambda^2}{e^{2\pi\lambda}} - 48\frac{\pi^2\lambda^3}{e^{2\pi\lambda}} - 24\frac{\pi^3\lambda^4}{e^{2\pi\lambda}} - 24\lambda + 24\frac{\lambda}{e^{2\pi\lambda}} + 3\lambda^4 - 6\frac{\lambda^4 C\pi}{e^{2\pi\lambda}} - 12\frac{\lambda^4 B\pi^2}{e^{2\pi\lambda}} - 16\frac{\lambda^3 B\pi}{e^{2\pi\lambda}},$$

$$d_5 = 2\lambda C + 2\frac{B}{e^{2\pi\lambda}} - 3\frac{\lambda^2}{e^{2\pi\lambda}} - 2\frac{\lambda C}{e^{2\pi\lambda}} - 24\frac{\pi^2\lambda}{e^{2\pi\lambda}} - 24\frac{\pi^3\lambda^2}{e^{2\pi\lambda}} + 12\frac{\pi}{e^{2\pi\lambda}} + 3\lambda^2 - 2B - 6\frac{\lambda^2 C\pi}{e^{2\pi\lambda}} - 12\frac{\lambda^2 B\pi^2}{e^{2\pi\lambda}} - 8\frac{\lambda B\pi}{e^{2\pi\lambda}},$$

and

$$d_7 = 1 - 2\frac{C\pi}{e^{2\pi\lambda}} - 4\frac{B\pi^2}{e^{2\pi\lambda}} - 8\frac{\pi^3}{e^{2\pi\lambda}} - \frac{1}{e^{2\pi\lambda}}.$$

The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arcsin\left[\frac{d_1 + d_3 + d_5 + d_7}{\sqrt{(c_0 + c_2 + c_4 + c_6)^2 + (d_1 + d_3 + d_5 + d_7)^2}}\right],$$

$$\rho = \frac{A}{(\lambda^2 + 1)^4}\sqrt{(c_0 + c_2 + c_4 + c_6)^2 + (d_1 + d_3 + d_5 + d_7)^2},$$

$$\gamma_1 = \frac{Ae^{-2\mu}}{v^{\frac{3}{2}}(\lambda^2 + 4)^4}(2d_1 + 4d_3 + 32d_5 + 128d_7),$$

and

$$\gamma_2 = \frac{Ae^{-2\mu}}{(1 - \rho^2)(\lambda^2 + 4)^4}(c_0 + 4c_2 + 16c_4 + 64c_6) - \frac{\rho^4}{1 - \rho^2},$$

respectively. The wrapped Rama distribution boasts closed-form expressions for its PDF and CDF. This feature, coupled with its good fit to various datasets, makes it a compelling choice for statistical analysis. In fact, it outperformed six other widely used distributions (with up to three parameters) when applied to two specific datasets. The first dataset, initially obtained by [33] and later published by [1], consists of long-axis orientation measurements for 60 feldspar laths in basalt, recorded in degrees. The second dataset, also reported by [1], comprises horizontal axis values for 100 outwash pebbles collected from a late Wisconsin outwash terrace near Cary, Illinois, along the Fox River.

2.32. Wrapped Richard Distribution

Ref. [43] introduced a wrapped distribution based on the Richard link function. Its PDF is

$$f(\theta) = \sum_{k=0}^{\infty} \frac{ke^{-k(\theta+2\pi n)}}{1 - m^{\frac{1}{1-m}}}\left[1 + (m-1)e^{-k(\theta+2\pi n)}\right]^{\frac{m}{1-m}}$$

for $0 \leq \theta < 2\pi$, $m \geq 1$, and $n \geq 1$. The wrapped Richard distribution is limited by its non-closed-form PDF.

2.33. Wrapped Shanker Distribution

Ref. [44] took g to be the PDF of the Shanker distribution to obtain the wrapped Shanker distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda^2}{\lambda^2+1} e^{-\lambda\theta} \left[\frac{\lambda+\theta}{1-e^{-2\pi\lambda}} + \frac{2\pi e^{-2\pi\lambda}}{\left(1-e^{-2\pi\lambda}\right)^2} \right]$$

and

$$F(\theta) = \frac{1}{1-e^{-2\pi\lambda}} \left(1 - e^{-\lambda\theta} - \frac{\lambda\theta e^{-\lambda\theta}}{\lambda^2+1} \right) + \frac{2\pi\lambda\left(1-e^{-\lambda\theta}\right)e^{-2\pi\lambda}}{\left(\lambda^2+1\right)\left(1-e^{-2\pi\lambda}\right)^2},$$

respectively, for $0 \leq \theta < 2\pi$ and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \frac{\lambda^2\left(1-e^{-2\pi\lambda}\right)\left(e^{2\pi(in-\lambda)}-1\right)[\lambda(in-\lambda)-1] + 2\pi\lambda^2(in-\lambda)e^{2\pi(in-\lambda)}}{\left(\lambda^2+1\right)\left(1-e^{-2\pi\lambda}\right)^2(in-\lambda)^2}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = 2\arctan\left(\frac{1}{\lambda}\right) - \arctan\left(\frac{\lambda}{\lambda^2+1}\right),$$

$$\rho = \frac{\lambda^2\sqrt{\left(\lambda^2+1\right)^2+\lambda^2}}{\left(\lambda^2+1\right)^2},$$

$$\gamma_1 = \frac{\frac{\lambda^2\sqrt{\left(\lambda^2+1\right)^2+4\lambda^2}}{\left(\lambda^2+1\right)\left(\lambda^2+4\right)} \sin\kappa_{2,\lambda}}{\left[1-\frac{\lambda^2\sqrt{\left(\lambda^2+1\right)^2+\lambda^2}}{\left(\lambda^2+1\right)^2}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\lambda^2\sqrt{\left(\lambda^2+1\right)^2+4\lambda^2}}{\left(\lambda^2+1\right)\left(\lambda^2+4\right)} \cos\kappa_{2,\lambda} - \left[\frac{\lambda^2\sqrt{\left(\lambda^2+1\right)^2+\lambda^2}}{\left(\lambda^2+1\right)^2}\right]^4}{\left[1-\frac{\lambda^2\sqrt{\left(\lambda^2+1\right)^2+\lambda^2}}{\left(\lambda^2+1\right)^2}\right]^2},$$

respectively, where $\kappa_{\lambda,2} = 2\arctan\left(\frac{2}{\lambda}\right) - \arctan\left(\frac{2\lambda}{\lambda^2+1}\right) - 4\arctan\left(\frac{1}{\lambda}\right) + 2\arctan\left(\frac{\lambda}{\lambda^2+1}\right)$. The wrapped Shanker distribution has closed-form expressions for the PDF and CDF, but it might have limited utility for real-life data modeling due to the fact it only has one parameter.

2.34. Wrapped Skew Laplace Distribution

Ref. [45] took g to be the PDF of a skew Laplace distribution to obtain a wrapped skew Laplace distribution. Its PDF is

$$f(\theta) = \frac{1}{2\sigma}\left[e^{-\frac{|\theta-\mu|}{\sigma}} + \frac{e^{\frac{\theta-\mu}{\sigma}} + e^{\frac{\mu-\theta}{\sigma}}}{e^{\frac{2\pi}{\sigma}} - 1} + A(\theta; \lambda_1, \lambda_2, \lambda_3)\right]$$

for $0 \leq \theta < 2\pi$, $-\infty < \lambda_1 < \infty$, $\lambda_2 > 0$, $-\infty < \lambda_3 < \infty$, $-\infty < \mu < \infty$, and $\sigma > 0$, where

$$A(\theta) = \sum_{k=-\infty}^{\infty} e^{-\frac{|\theta+2\pi k-\mu|}{\sigma}}\left[1 - e^{-\left|g\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\right|}\right] \operatorname{sign}\left[g\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\right]$$

and

$$g(x) = \frac{\lambda_1 x + \lambda_3 x^3}{\sqrt{1+\lambda_2 x^2}}.$$

The nth trigonometric moment is

$$m_n = \alpha_n + \mathrm{i}\beta_n$$

for $n = 1, 2, \ldots$, where

$$\alpha_n = \frac{\cos(n\mu) + 2\nabla e^{-\xi}\sin(n\mu)\xi_n}{n^2\sigma^2+1} - \frac{\Delta n\sigma\sin(n\mu)}{n^2\sigma^2+1} - \Delta^2 C_n,$$

$$\beta_n = \frac{\sin(n\mu) - 2\nabla e^{-\xi}\cos(n\mu)\xi_n}{n^2\sigma^2+1} - \frac{\Delta n\sigma\cos(n\mu)}{n^2\sigma^2+1} - \Delta^2 S_n,$$

$$\nabla = \begin{cases} \operatorname{sign}(\lambda_1), & \text{if } \lambda_1\lambda_3 < 0, \\ 0, & \text{if } \lambda_1\lambda_3 \geq 0, \end{cases}$$

$$\Delta = \begin{cases} 0, & \text{if } \lambda_1 = \lambda_3 = 0, \\ \operatorname{sign}(\lambda_3), & \text{if } \lambda_1 = 0, \lambda_3 \neq 0, \\ \operatorname{sign}(\lambda_1), & \text{if } \lambda_1 \geq 0, \end{cases}$$

$$C_n = \frac{1}{2\sigma}\sum_{k=-\infty}^{\infty}\int_0^{2\pi}\cos(n\theta)e^{-\frac{|\theta+2\pi k-\mu|}{\sigma}}e^{-\left|g\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\right|}\operatorname{sign}\left[g\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\right]d\theta,$$

$$S_n = \frac{1}{2\sigma}\sum_{k=-\infty}^{\infty}\int_0^{2\pi}\sin(n\theta)e^{-\frac{|\theta+2\pi k-\mu|}{\sigma}}e^{-\left|g\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\right|}\operatorname{sign}\left[g\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\right]d\theta,$$

$$\xi_n = \sin(\xi n\sigma) + n\sigma\cos(\xi n\sigma),$$

and

$$\xi = \begin{cases} \sqrt{-\frac{\lambda_1}{\lambda_3}}, & \text{if } \lambda_1\lambda_3 < 0, \\ 0, & \text{if } \lambda_1\lambda_3 \geq 0. \end{cases}$$

The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left[\frac{1}{\sigma\Delta - \Delta^2 S_1(1+\sigma^2) - 2\nabla e^{-\xi}\xi_1}\right],$$

$$\rho = \left(\frac{\sigma\Delta}{\sigma^2+1} - \Delta^2 S_1 - \frac{2\nabla e^{-\xi}\xi_1}{\sigma^2+1}\right)\sin\mu + \frac{\cos\mu}{\sigma^2+1},$$

$$\gamma_1 = \frac{\left[2\sigma\Delta - \Delta^2 S_2\left(4\sigma^2+1\right) - 2\nabla e^{-\xi}\xi_2\right]\cos(2\mu) - \sin(2\mu)}{(4\sigma^2+1)(1-\rho)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\left[2\sigma\Delta - \Delta^2 S_2\left(4\sigma^2+1\right) - 2\nabla e^{-\xi}\xi_2\right]\sin(2\mu) - \cos(2\mu) - \left(4\sigma^2+1\right)\rho^4}{(4\sigma^2+1)(1-\rho)^2},$$

respectively. The wrapped skew Laplace distribution was used on the turtle and ant datasets discussed by [1]. It was observed that this distribution provided a better fit for the first dataset compared to the wrapped Lindley, exponential, transmuted wrapped exponential, and non-negative trigonometric sums distributions [46]. Similarly, for the second dataset, the wrapped skew Laplace distribution outperformed the symmetric wrapped Laplace and non-negative trigonometric sums distributions.

2.35. Wrapped Skew Normal Distribution

Ref. [47] took g to be the PDF of the skew normal distribution to obtain the wrapped skew normal distribution. Its PDF is

$$f(\theta) = \frac{2}{\eta}\sum_{k=-\infty}^{\infty}\phi\left(\frac{\theta+2\pi r-\xi}{\eta}\right)\Phi\left(\lambda\left(\frac{\theta+2\pi r-\xi}{\eta}\right)\right)$$

for $0 \leq \theta < 2\pi$, $-\infty < \xi < \infty$, $\eta > 0$, and $-\infty < \lambda < \infty$. The nth trigonometric moment is

$$m_n = e^{\mathrm{i}n\xi - \frac{1}{2}n^2\eta^2}\left[1 + \mathrm{i}\mathcal{J}(\delta\eta n)\right]$$

for $n = 1, 2, \ldots$, where

$$\mathcal{J}(x) = \int_0^x b e^{\frac{u^2}{2}}\, du$$

and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left[\frac{\sin\xi + \mathcal{J}(\delta\eta)\cos\xi}{\cos\xi - \mathcal{J}(\delta\eta)\sin\xi}\right],$$

$$\rho = e^{-\frac{\eta^2}{2}}\left[1 + \mathcal{J}^2(\delta\eta)\right]^{\frac{1}{2}},$$

$$\gamma_1 = \frac{\frac{\omega^4\left\{\mathcal{J}(2\delta\eta)\left[1-\mathcal{J}^2(\delta\eta)\right]-2\mathcal{J}(\delta\eta)\right\}}{1+\mathcal{J}^2(\delta\eta)}}{\left\{1-e^{-\frac{\eta^2}{2}}\left[1+\mathcal{J}^2(\delta\eta)\right]^{\frac{1}{2}}\right\}^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\omega^4\left[1-\mathcal{J}^2(\delta\eta)+2\mathcal{J}(\delta\eta)\mathcal{J}(2\delta\eta)\right]}{1+\mathcal{J}^2(\delta\eta)} - \left\{e^{-\frac{\eta^2}{2}}\left[1+\mathcal{J}^2(\delta\eta)\right]^{\frac{1}{2}}\right\}^4}{\left\{1-e^{-\frac{\eta^2}{2}}\left[1+\mathcal{J}^2(\delta\eta)\right]^{\frac{1}{2}}\right\}^2},$$

respectively. The wrapped skew normal distribution was used to analyze bird heading data from the autumn migration of 1987 [27]. Although it fit the data well, its PDF does not have a simple mathematical expression, which might restrict its practical utility.

2.36. Wrapped Stable Distribution

Ref. [48] took g to be the PDF of the stable distribution to obtain the wrapped stable distribution. The nth trigonometric moment is

$$m_n = \begin{cases} e^{-\gamma^\alpha n^\alpha}\left\{1+\mathrm{i}\beta\left[(\gamma n)^{1-\alpha}-1\right]\tan\left(\frac{\pi\alpha}{2}\right)\right\}+\mathrm{i}\delta_0^* n, & \text{if } \alpha \neq 1, \\ e^{-\gamma n}\left[1+\frac{2\beta\mathrm{i}}{\pi}\log(\gamma n)\right]+\mathrm{i}\delta_0^* n, & \text{if } \alpha = 1 \end{cases}$$

for $n = 1, 2, \ldots$, where $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma > 0$, $-\infty < \delta_0 < \infty$, and $\delta_0^* = \delta_0 \bmod 2\pi$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \begin{cases} \delta_0^* + \beta(\gamma^\alpha - \gamma)\tan\left(\frac{\pi\alpha}{2}\right) \bmod 2\pi, & \text{if } \alpha \neq 1, \\ \delta_0^* - \frac{2\beta\gamma}{\pi}\log\gamma \bmod 2\pi, & \text{if } \alpha = 1, \end{cases}$$

$$\rho = e^{-\gamma^\alpha},$$

$$\gamma_1 = \frac{\overline{\beta}_2}{(1-\rho)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\overline{\alpha}_2 - \rho^4}{(1-\rho)^2},$$

respectively, where

$$\overline{\alpha}_2 = \begin{cases} \rho^{2^\alpha}\cos\left[\beta\gamma^\alpha(2^\alpha-2)\tan\left(\frac{\pi\alpha}{2}\right)\right], & \text{if } \alpha \neq 1, \\ \rho^{2^\alpha}\cos\left(-\frac{2\beta\gamma\log 2}{\pi}\right), & \text{if } \alpha = 1 \end{cases}$$

and

$$\overline{\beta}_2 = \begin{cases} \rho^{2^\alpha} \sin\left[\beta\gamma^\alpha (2^\alpha - 2) \tan\left(\frac{\pi\alpha}{2}\right)\right], & \text{if } \alpha \neq 1, \\ \rho^{2^\alpha} \sin\left(-\frac{2\beta\gamma \log 2}{\pi}\right), & \text{if } \alpha = 1. \end{cases}$$

The wrapped stable distribution demonstrated a slightly superior fit compared to a mixture distribution incorporating circular uniform and wrapped skew normal components when analyzing the bird heading data from [27].

2.37. Wrapped Student's t Distribution

Ref. [49] took g to be the PDF of the Student's t distribution to obtain the wrapped Student's t distribution. Its PDF is

$$f(\theta) = \frac{c}{\lambda} \sum_{k=-\infty}^{\infty} \left[1 + \frac{(\theta + 2\pi k - \mu_0)^2}{\lambda^2 \nu}\right]^{-\frac{\nu+1}{2}}$$

for $0 \leq \theta < 2\pi$, $0 \leq \mu < 2\pi$, $\lambda > 0$, and $\nu > 0$, where

$$c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}.$$

The nth trigonometric moment is

$$m_n = \frac{K_{\frac{\nu}{2}}\left(n\sqrt{\nu}\right)\left(n\sqrt{\nu}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}-1}}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu,$$

$$\rho = \frac{K_{\frac{\nu}{2}}\left(\lambda\sqrt{\nu}\right)\left(\lambda\sqrt{\nu}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}-1}},$$

$$\gamma_1 = \frac{e^{-\sqrt{2c}} \sin\left(\sqrt{2c} - 2\sqrt{c}\right)}{\left(1 - e^{-\sqrt{c}}\right)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{e^{-\sqrt{2c}} \cos\left(\sqrt{2c} - 2\sqrt{c}\right) - e^{-4\sqrt{c}}}{\left(1 - e^{-\sqrt{c}}\right)^2},$$

respectively. The wrapped Student's t distribution was used on a dataset of 104 cross-bed measurements from Himalayan molasse in Pakistan, as discussed by [1]. The wrapped Student's t distribution yielded a better fit to this dataset compared to the von Mises distribution.

2.38. Wrapped Transmuted Exponential Distribution

Ref. [50] took g to be the PDF of the transmuted exponential distribution to obtain the wrapped transmuted exponential distribution. Its PDF and CDF are

$$f(\theta) = \frac{2\lambda\Lambda e^{-\lambda\theta}\left(e^{-\theta\lambda}-1\right)}{c^2} - \frac{\lambda e^{-\lambda\theta}(\Lambda+1)}{c}$$

and

$$F(\theta) = \frac{\left(e^{-\lambda\theta}-1\right)\left[c+\Lambda\left(1+c-e^{-\lambda\theta}\right)\right]}{c^2},$$

respectively, for $0 \leq \theta < 2\pi$, $\lambda > 0$, $|\Lambda| \leq 1$, and $c = e^{-2\pi\lambda} - 1$. The nth trigonometric moment is

$$m_n = \frac{\lambda(\lambda + in)(2\Lambda + c + c\Lambda)\left[(c+1)e^{2\pi n i} - 1\right]}{c^2(\lambda^2+n^2)} - \frac{2\lambda\Lambda(2\lambda + in)\left[(c+1)^2 e^{2i\pi n} - 1\right]}{c^2(4\lambda^2+n^2)}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan \frac{c - 2\Lambda + 4\lambda^2\Lambda + 4c\lambda^2 - c\Lambda + 2c\lambda^2\Lambda}{\lambda(4c\lambda^2 - 6\Lambda + c - 3c\Lambda)},$$

$$\rho = \sqrt{\frac{\lambda^2(2\Lambda - c + c\Lambda)^2 + 4c^2\lambda^4}{c^2(4\lambda^2+1)(\lambda^2+1)}},$$

$$\gamma_1 = -\frac{\frac{\lambda(\lambda\sin 2\mu - 2\cos 2\mu)(2\Lambda+c+c\Lambda))}{c(\lambda^2+4)} - \frac{\lambda\Lambda(c+2)(\cos 2\mu - \lambda\sin 2\mu)}{c(\lambda^2+1)}}{\left[1 - \sqrt{\frac{\lambda^2(2\Lambda-c+c\Lambda)^2+4c^2\lambda^4}{c^2(4\lambda^2+1)(\lambda^2+1)}}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\lambda^2(2\Lambda+2c+\Lambda c)+2c(1-\Lambda)-4\Lambda}{c\left(\lambda^4+5\lambda^2+4\right)}\lambda\sin 2\mu + \frac{c\lambda\left(1+\lambda^2-3\Lambda\right)-6\lambda\Lambda}{c\left(\lambda^4+5\lambda^2+4\right)}\lambda\cos 2\mu - \frac{\left[(2\Lambda-c+\Lambda c)^2+4\lambda^4c^2\right]^2}{c^4\left(4\lambda^2+1\right)^2\left(\lambda^2+1\right)^2}}{\left[1 - \sqrt{\frac{\lambda^2(2\Lambda-c+c\Lambda)^2+4c^2\lambda^4}{c^2(4\lambda^2+1)(\lambda^2+1)}}\right]^2},$$

respectively. The transmuted wrapped exponential distribution was used on the turtle dataset from [1]. It outperformed the wrapped exponential and wrapped Lindley distributions. It offers closed-form expressions for the PDF and CDF. Additionally, it is more flexible than commonly used one-parameter distributions.

2.39. Wrapped Two-Parameter Lindley Distribution

Ref. [51] took g to be the PDF of the two-parameter Lindley distribution to obtain the wrapped two-parameter Lindley distribution. Its PDF and CDF are

$$f(\theta) = \left(\frac{\xi^2}{\xi+\alpha}e^{-\xi\theta}\right)\left[\frac{1+\alpha\theta}{1-e^{-2\pi\xi}} + \frac{2\pi\alpha e^{-2\pi\xi}}{\left(1-e^{-2\pi\xi}\right)^2}\right]$$

and

$$F(\theta) = \frac{1}{1-e^{-2\pi\xi}}\left(1-e^{-\xi\theta}-\frac{\alpha\xi\theta}{\alpha+\xi}\right)+\frac{2\pi\alpha\xi}{\alpha+\xi}\left(1-e^{-\xi\theta}\right)\frac{e^{-2\pi\xi}}{\left(1-e^{-2\pi\xi}\right)^2},$$

respectively, for $0 \leq \theta < 2\pi$, $\xi > 0$, and $\alpha > -\xi$. The nth trigonometric moment is

$$m_n = \frac{\xi^2(\xi+\alpha-\mathrm{i}n)}{(\xi+\alpha)(\xi-\mathrm{i}n)}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = 2\arctan\left(\frac{1}{\xi}\right) - \arctan\left(\frac{1}{\xi+\alpha}\right),$$

$$\rho = \frac{\xi^2\left[(\xi+\alpha)^2+1\right]^{\frac{1}{2}}}{(\xi+\alpha)(\xi^2+1)},$$

$$\gamma_1 = \frac{\frac{\xi^2\sqrt{(\alpha+\xi)^2+4}}{(\alpha+\xi)(4+\xi^2)}\sin(\mu_2-2\mu)}{\left[1-\frac{\xi^2\sqrt{(\alpha+\xi)^2+1}}{(\alpha+\xi)(1+\xi^2)}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\xi^2\sqrt{(\alpha+\xi)^2+4}}{(\alpha+\xi)(4+\xi^2)}\cos(\mu_2-2\mu) - \left[\frac{\xi^2\sqrt{(\alpha+\xi)^2+1}}{(\alpha+\xi)(1+\xi^2)}\right]^4}{\left(1-\frac{\xi^2\sqrt{(\alpha+\xi)^2+1}}{(\alpha+\xi)(1+\xi^2)}\right]^2},$$

respectively, where$\mu_2 = 2\arctan\left(\frac{2}{\xi}\right) - \arctan\left(\frac{2}{\alpha+\xi}\right)$. The two-parameter wrapped Lindley distribution was used for analyzing two datasets: the feldspar lath dataset from [1], Appendix B5, and a dataset on wind directions observed at Gorleston, England, during summer Sundays in 1968 [52]. It was found that this distribution provided a better fit for the first dataset compared to the wrapped exponential distribution and the wrapped Lindley distribution. Additionally, for the second dataset, it was observed that the wrapped two-parameter Lindley distribution outperformed the wrapped Lindley distribution in terms of fitting.

2.40. Wrapped Two-Sided Lindley Distribution

Ref. [53] took g to be the PDF of the two-sided Lindley distribution to obtain the wrapped two-sided Lindley distribution. Its PDF and CDF are

$$f(\theta) = \frac{e^{-\alpha\theta}}{2\Lambda_\alpha\Lambda_\beta}\left\{\alpha^2 e^{2\pi\alpha}\Lambda_\beta\left[(\theta+1)e^{2\pi\alpha}-\theta+2\pi-1\right]+\beta^2 e^{\theta(\alpha+\beta)}\Lambda_\alpha\left[(2\pi+1-\theta)e^{2\pi\beta}+\theta-1\right]\right\}$$

and

$$\begin{aligned} F(\theta) = {} & \frac{e^{\alpha(2\pi-\theta)}}{2\Lambda_\alpha}\left\{(\alpha+1)e^{\alpha(\theta+2\pi)}+[(2\pi-1)\alpha-1]e^{\alpha\theta}+\alpha(\theta-2\pi+1)-e^{2\pi\alpha}(\alpha\theta+\alpha+1)+1\right\} \\ & + \frac{1}{2\Lambda_\beta}\left\{e^{\beta\theta}[\beta(\theta-1)-1]+\beta-e^{2\pi\beta}(2\pi\beta+\beta+1)+e^{\beta(\theta+2\pi)}(-\beta\theta+2\pi\beta+\beta+1)+1\right\}, \end{aligned}$$

respectively, for $0 \le \theta < 2\pi$, $\alpha > 0$, and $\beta > 0$, where $\Lambda_\alpha = \left(e^{2\pi\alpha} - 1\right)^2(\alpha + 1)$ and $\Lambda_\beta = \left(e^{2\pi\beta} - 1\right)^2(\beta + 1)$. The nth trigonometric moment is

$$
\begin{aligned}
m_n = {} & \frac{\alpha^2(n + i\alpha)^2}{2\Lambda_\alpha} e^{2i\pi n}\left(e^{2\pi\alpha} - 1\right)\left[2\pi\alpha - e^{2\pi(\alpha - in)} - 2i\pi n + 1\right] \\
& + \frac{\alpha^2}{2(n + i\alpha)\Lambda_\alpha}\left(e^{2\pi\alpha} - e^{2ipn}\right)\left(2\pi + e^{2\pi\alpha} - 1\right) \\
& + \frac{\beta^2}{2\Lambda_\beta(\beta + in)^2}\left(1 - e^{2\pi\beta}\right)\left[1 + e^{2\pi(\beta + in)}(2\pi\beta + 2i\pi n - 1)\right] \\
& + \frac{\beta^2}{2(\beta + in)\Lambda_\beta}\left[1 - e^{2\pi(\beta + in)}\right]\left[1 - (1 + 2\pi)e^{2\pi\beta}\right]
\end{aligned}
$$

for $n = 1, 2, \ldots$. The wrapped two-sided Lindley distribution has been applied to the ant dataset discussed by [1] and was shown to provide a good fit to this dataset.

2.41. *Wrapped Variance Gamma Distribution*

Ref. [54] took g to be the PDF of the variance gamma distribution to obtain the wrapped variance gamma distribution. Its PDF is

$$
f(\theta) = \frac{\gamma^{2\lambda} e^{\beta(\theta - \mu)}}{\sqrt{\pi}\Gamma(\lambda)(2\alpha)^{\lambda - \frac{1}{2}}} \sum_{m=-\infty}^{\infty} \frac{e^{2\pi m\beta} K_{\lambda - \frac{1}{2}}(\alpha|\theta + 2m\pi - \mu|)}{|\theta + 2m\pi - \mu|^{\lambda - \frac{1}{2}}}
$$

for $0 \le \theta < 2\pi$, $\alpha > 0$, $\beta > 0$, $0 \le |\beta| < \alpha$, and $\lambda \in \mathbb{R}$. The nth trigonometric moment is

$$
m_n = e^{i\mu n}\left(\frac{\gamma}{\sqrt{\alpha^2 - (\beta + in)^2}}\right)^{2\lambda}
$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$
\mu,
$$

$$
\rho = \left[\frac{\gamma}{(\alpha^2 - (\beta + i)^2)^{\frac{1}{2}}}\right]^{2\lambda},
$$

$$
\gamma_1 = 0,
$$

and

$$
\gamma_2 = \frac{\left[\frac{\gamma}{(\alpha^2 - (\beta + 2i)^2}\right]^{2\lambda} - \left[\frac{\gamma}{(\alpha^2 - (\beta + i)^2)}\right]^{8\lambda}}{\left[1 - \left\{\frac{\gamma}{[\alpha^2 - (\beta + i)^2]}\right\}^{2\lambda}\right]^2},
$$

respectively. The wrapped variance gamma distribution was used to analyze the Black Mountain wind direction dataset from [1]. It fit the data well, but its practical utility is limited because it lacks a closed-form PDF or CDF.

2.42. Wrapped Weighted Exponential Distribution

Ref. [55] took g to be the PDF of the weighted exponential distribution to obtain the wrapped weighted exponential distribution. Its PDF is

$$f(\theta) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda\theta}\sum_{k=0}^{\infty} e^{-2k\pi\lambda}\left[1 - e^{-\alpha\lambda(\theta+2k\pi)}\right]$$

for $0 \le \theta < 2\pi$, $\alpha > 0$, and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \alpha_n + \mathrm{i}\beta_n,$$

where

$$\alpha_n = \frac{\lambda^2 \cos\left[\arctan\left(\frac{n}{\lambda}\frac{2+\alpha}{1+\alpha}\right)\right]}{\sqrt{\lambda^2+n^2}\sqrt{\lambda^2+\frac{n^2}{(1+\alpha)^2}}}$$

and

$$\beta_n = \frac{\lambda^2 \sin\left[\arctan\left(\frac{n}{\lambda}\frac{2+\alpha}{1+\alpha}\right)\right]}{\sqrt{\lambda^2+n^2}\sqrt{\lambda^2+\frac{n^2}{(1+\alpha)^2}}}.$$

The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left[\frac{2+\alpha}{\lambda(1+\alpha)}\right],$$

$$\rho = \frac{\lambda^2}{\sqrt{\lambda^2+1}\sqrt{\lambda^2+\frac{1}{(1+\alpha)^2}}},$$

$$\gamma_1 = \frac{\lambda^2 \sin\left[\arctan\left(\frac{2}{\lambda}\frac{2+\alpha}{1+\alpha}\right) - 2\arctan\left(\frac{1}{\lambda}\frac{2+\alpha}{1+\alpha}\right)\right]}{\sqrt{\lambda^2+4}\sqrt{\lambda^2+\frac{4}{(1+\alpha)^2}}(1-\rho)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\lambda^2 \cos\left[\arctan\left(\frac{2}{\lambda}\frac{2+\alpha}{1+\alpha}\right) - 2\arctan\left(\frac{1}{\lambda}\frac{2+\alpha}{1+\alpha}\right)\right] - \rho^4}{\sqrt{\lambda^2+4}\sqrt{\lambda^2+\frac{4}{(1+\alpha)^2}}(1-\rho)^2},$$

respectively. The wrapped weighted exponential distribution is a flexible model with a PDF that can be recast into a closed form.

2.43. Wrapped Weibull Distribution

Ref. [40] took g to be the PDF of the Weibull distribution to obtain the wrapped Weibull distribution. Its PDF is

$$f(\theta) = \sum_{k=0}^{\infty} \frac{1}{2\sigma}\mathrm{sech}^2\left(\frac{\theta+2k\pi-\mu}{2\sigma}\right)$$

for $0 \le \theta < 2\pi$ and $c > 0$. The nth trigonometric moment is

$$m_n = \sum_{k=0}^{\infty} \frac{\mathrm{i}^k n^k}{k!}\Gamma\left(1+\frac{k}{c}\right) = b_n + \mathrm{i}c_n,$$

where

$$b_n = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \Gamma\left(1 + \frac{2k}{c}\right)$$

and

$$c_n = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \Gamma\left(1 + \frac{2k+1}{c}\right).$$

The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left(\frac{c_1}{b_1}\right),$$

$$\rho = 2\sqrt{(b_1 \cos\mu - c_1 \sin\mu)^2 + (b_1 \sin\mu + c_1 \cos\mu)^2},$$

$$\gamma_1 = \frac{\sqrt{b_1^2 + c_1^2} \sin(\mu_2 - 2\mu)}{\left(1 - \sqrt{b_1^2 + c_1^2}\right)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\sqrt{b_1^2 + c_1^2} \cos(\mu_2 - 2\mu) - \left(\sqrt{b_1^2 + c_1^2}\right)^4}{\left(1 - \sqrt{b_1^2 + c_1^2}\right)^2},$$

respectively, where $\mu_2 = \arctan\left(\frac{c_2}{b_2}\right)$. The wrapped Weibull distribution is limited by the fact that it does not admit a closed-form PDF.

2.44. Wrapped XGamma Distribution

Ref. [56] took g to be the PDF of the XGamma distribution to obtain the wrapped XGamma distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda^2 e^{-\theta\lambda}}{(\lambda+1)\left(1 - e^{-2\pi\lambda}\right)} \left\{1 + \frac{\lambda\theta^2}{2} + 2\pi\lambda\left[(\pi - \theta)e^{-2\pi\lambda} + (\theta + \pi)\right] \frac{e^{-2\pi\lambda}}{\left(1 - e^{-2\pi\lambda}\right)^2}\right\}$$

and

$$F(\theta) = \left[1 - \frac{1 + \lambda(1+\theta) + \frac{\theta^2\lambda^2}{2}}{\lambda+1} e^{-\theta\lambda}\right] \frac{1}{1 - e^{-2\pi\lambda}} + \frac{2\pi\lambda}{\lambda+1}\left[1 - (1 + \theta\lambda)e^{-\theta\lambda}\right] \frac{e^{-2\pi\lambda}}{(1 - e^{-2\pi\lambda})^2} + \frac{2\pi^2\lambda^2}{\lambda+1}\left(1 - e^{-\theta\lambda}\right) \frac{e^{-2\pi\lambda}\left(1 + e^{-2\pi\lambda}\right)}{\left(1 - e^{-2\pi\lambda}\right)^3},$$

respectively, for $0 \leq \theta < 2\pi$ and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \frac{\lambda^2\left[\lambda^2 + \lambda(1 - 2\mathrm{i}n) - n^2\right]}{(\lambda+1)(\lambda - \mathrm{i}n)^3}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = 3\arctan\left(\frac{1}{\lambda}\right) - \arctan\left(\frac{2\lambda}{\lambda^2+\lambda-1}\right),$$

$$\rho = \frac{\lambda^2}{\lambda+1}\sqrt{\frac{\left(\lambda^2+\lambda-1\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^3}},$$

$$\gamma_1 = \frac{\frac{\lambda^2}{\lambda+1}\sqrt{\frac{\left(\lambda^2+\lambda-4\right)^2+16\lambda^2}{\left(\lambda^2+4\right)^2}}\sin(\kappa_{\lambda,2})}{\left[1-\frac{\lambda^2}{\lambda+1}\sqrt{\frac{\left(\lambda^2+\lambda-1\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^3}}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\lambda^2}{\lambda+1}\sqrt{\frac{\left(\lambda^2+\lambda-4\right)^2+16\lambda^2}{\left(\lambda^2+4\right)^2}}\cos(\kappa_{\lambda,2}) - \left[\frac{\lambda^2}{1+\lambda}\sqrt{\frac{\left(\lambda^2+\lambda-1\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^3}}\right]^4}{\left[1-\frac{\lambda^2}{\lambda+1}\sqrt{\frac{\left(\lambda^2+\lambda-1\right)^2+4\lambda^2}{\left(\lambda^2+1\right)^3}}\right]^2},$$

respectively, where $\kappa_{\lambda,2} = \arctan\left(\frac{2}{\lambda}\right) - 6\arctan\left(\frac{1}{\lambda}\right) + 2\arctan\left(\frac{2\lambda}{\lambda^2+\lambda-1}\right) - \arctan\left(\frac{4\lambda}{\lambda^2+\lambda-4}\right)$. The wrapped XGamma distribution was used to analyze the feldspar lath dataset from [1]. It demonstrated superior fit compared to the wrapped exponential and the wrapped Lindley distributions.

2.45. Wrapped XLindley Distribution

Ref. [57] took g to be the PDF of the XLindley distribution to obtain the wrapped XLindley distribution. Its PDF and CDF are

$$f(\theta) = \frac{\lambda^2 e^{-\theta\lambda}}{(\lambda+1)^2\left(1-e^{-2\pi\lambda}\right)^2}\left[\left(1-e^{-2\pi\lambda}\right)(\lambda+\theta+2)+2\pi e^{-2\pi\lambda}\right]$$

and

$$F(\theta) = \frac{1-e^{-\theta\lambda}\left(1+\frac{\lambda\theta}{(\lambda+1)^2}\right)}{1-e^{-2\pi\lambda}} + \frac{2\pi\lambda e^{-2\pi\lambda}\left(1-e^{-\theta\lambda}\right)}{(\lambda+1)^2\left(1-e^{-2\pi\lambda}\right)^2},$$

respectively, for $0 \le \theta < 2\pi$ and $\lambda > 0$. The nth trigonometric moment is

$$m_n = \frac{\lambda^2}{(\lambda+1)(\lambda-\mathrm{i}n)} + \frac{\lambda^2(1+\lambda-\mathrm{i}n)}{(\lambda+1)^2(\lambda-\mathrm{i}n)^2}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness and kurtosis are

$$\mu = 2\arctan\left(\frac{1}{\lambda}\right) - \arctan\left(\frac{\lambda+2}{(\lambda+1)^2}\right),$$

$$\rho = \frac{\lambda^2\sqrt{(\lambda+1)^4+(\lambda+2)^2}}{(\lambda+1)^2(\lambda^2+1)},$$

$$\gamma_1 = \frac{\lambda^2 \sqrt{4(\lambda^2+2)^2 + (\lambda+1)^4} \sin(\mu_2 - 2\mu)}{(\lambda+1)^2(\lambda^2+4)\left[1 - \frac{\lambda^2 \sqrt{(\lambda+1)^4+(\lambda+2)^2}}{(\lambda+1)^2(\lambda^2+1)}\right]^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\lambda^2 \sqrt{4(\lambda^2+2)^2+(\lambda+1)^4} \cos(\mu_2-2\mu)}{(\lambda+1)^2(\lambda^2+4)} - \frac{\lambda^8\left[(\lambda+2)^2+(\lambda+1)^4\right]^2}{(\lambda+1)^8(\lambda^2+1)^4}}{\left[1 - \frac{\lambda^2 \sqrt{(\lambda+1)^4+(\lambda+2)^2}}{(\lambda+1)^2(\lambda^2+1)}\right]^2},$$

respectively, where $\mu_2 = 2\arctan\left(\frac{2}{\lambda}\right) - \arctan\left[\frac{2(\lambda+2)}{(\lambda+1)^2}\right]$. The wrapped XLindley distribution was used to analyze two datasets: one consisted of sun directions recorded from 50 starhead topminnows under overcast conditions [1], and the other included 349 transactions occurring between 1 January 2020 and 29 July 2020 [58]. Although the wrapped XLindley distribution outperformed some distributions based on certain goodness-of-fit measures, it was not consistently superior in all cases.

3. A Review of Discrete Wrapped Distributions

In this section, we review wrapped binomial, wrapped discrete Cauchy, wrapped discrete exponential, wrapped discrete Mittag–Leffler, wrapped discrete skew Laplace, wrapped geometric, wrapped negative binomial, wrapped Poisson, wrapped Poisson–Lindley, and wrapped zero-inflated Poisson distributions.

3.1. Wrapped Binomial Distribution

Ref. [59] took g to be the PDF of the binomial distribution to obtain the wrapped binomial distribution. Its PMF is

$$P\left(\Theta = \frac{2\pi r}{m}\right) = \sum_{k=0}^{\left[\frac{n-r}{m}\right]} \binom{n}{r+km} p_1^{r+km} q_1^{n-r-km}$$

for $0 \leq \theta < 2\pi$, $r = 0, 1, \ldots, m-1$, $m \geq 1$, $n \geq m-1$, $0 < p_1 < 1$, and $q_1 = 1 - p_1$. The nth trigonometric moment is

$$m_n = \left[q_1 + p_1 \cos\left(\frac{2n\pi}{m}\right) + i p_1 \sin\left(\frac{2n\pi}{m}\right)\right]^n$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = n \arctan\left[\frac{p_1 \sin\left(\frac{2\pi}{m}\right)}{q_1 + p_1 \sin\left(\frac{2\pi}{m}\right)}\right],$$

$$\rho = \left[p_1^2 + q_1^2 + 2p_1 q_1 \sin\left(\frac{2\pi}{m}\right)\right]^{\frac{n}{2}},$$

$$\gamma_1 = \frac{\beta_2 \cos(2\mu) - \alpha_2 \sin(2\mu)}{(1-\rho)^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\alpha_2 \cos(2\mu) + \beta_2 \sin(2\mu) - \rho^4}{(1-\rho)^2},$$

respectively, where

$$\alpha_2 = \left[p_1^2 + q_1^2 + 2p_1q_1\cos\left(\frac{4\pi}{m}\right)\right]^{\frac{n}{2}} \cos\left\{2\arctan\left[\frac{p_1\sin\left(\frac{4\pi}{m}\right)}{q_1 + p_1\cos\left(\frac{4\pi}{m}\right)}\right]\right\}$$

and

$$\beta_2 = \left[p_1^2 + q_1^2 + 2p_1q_1\cos\left(\frac{4\pi}{m}\right)\right]^{\frac{n}{2}} \sin\left\{2\arctan\left[\frac{p_1\sin\left(\frac{4\pi}{m}\right)}{q_1 + p_1\cos\left(\frac{4\pi}{m}\right)}\right]\right\}.$$

The wrapped binomial distribution faces practical limitations due to its non-closed-form PMF.

3.2. Wrapped Discrete Cauchy Distribution

Ref. [60] introduced the wrapped discrete Cauchy distribution. Its PMF is

$$P\left(\Theta = \frac{2\pi r}{m}\right) = \frac{\left(1-a^2\right)\left[1+a^{2m}-2a^m\cos(m\mu)\right]}{m(1-a^{2m})\left[1+a^2-2a\cos(\theta-\mu)\right]}$$

for $0 < a < 1$, $0 \leq \theta < 2\pi$, $r = 0,1,\ldots,m-1$, and m is a positive integer. The nth trigonometric moment is

$$m_n = \frac{\left(1-a^2\right)\left[1+a^{2m}-2a^m\cos(m\mu)\right]}{m(1-a^{2m})} \sum_{r=0}^{m-1} \frac{e^{n\theta}}{1+a^2-2a\cos(\theta-\mu)}$$

for $n = 1,2,\ldots$. The wrapped discrete Cauchy distribution has the advantage of having a closed-form PMF.

3.3. Wrapped Discrete Exponential Distribution

Ref. [61] introduced the wrapped discrete exponential distribution. Its PMF and CDF are

$$P\left(\Theta = \frac{2\pi r}{m}\right) = \frac{e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}\left(1-e^{-\frac{2\pi\lambda}{m}}\right)$$

and

$$F(\Theta \leq k) = \frac{1-e^{-\frac{2\pi\lambda(k+1)}{m}}}{1-e^{-2\pi\lambda}}, \quad k = 0,1,\ldots,m-1,$$

respectively, for $\lambda > 0$, $r = 0,1,\ldots,m-1$, and m is a positive integer. The nth trigonometric moment is

$$m_n = \frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\left(\frac{1-e^{-2\pi\lambda+2\pi i n}}{1-e^{-\frac{2\pi\lambda}{m}+\frac{2\pi i n}{m}}}\right)$$

for $n = 1,2,\ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left[\frac{e^{-\frac{2\pi\lambda}{m}}\sin\left(\frac{2\pi}{m}\right) - e^{-\frac{4\pi\lambda}{m}}\sin\left(\frac{2\pi}{m}\right)}{1-e^{-\frac{2\pi\lambda}{m}}\cos\left(\frac{2\pi}{m}\right) - e^{-\frac{2\pi\lambda}{m}} + e^{-\frac{4\pi\lambda}{m}}\cos\left(\frac{2\pi}{m}\right)}\right],$$

$$\rho = \sqrt{\frac{\left[1 - e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right) - e^{-\frac{2\pi\lambda}{m}} + e^{-\frac{4\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2 + \left[e^{-\frac{2\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right) - e^{-\frac{4\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right)\right]^2}{\left[1 + e^{-\frac{4\pi\lambda}{m}} - 2e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2}},$$

$$\gamma_1 = \frac{\sqrt{\frac{\left[1-e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{4\pi}{m}\right)-e^{-\frac{2\pi\lambda}{m}}+e^{-\frac{4\pi\lambda}{m}} \cos\left(\frac{4\pi}{m}\right)\right]^2+\left[e^{-\frac{2\pi\lambda}{m}} \sin\left(\frac{4\pi}{m}\right)-e^{-\frac{4\pi\lambda}{m}} \sin\left(\frac{4\pi}{m}\right)\right]^2}{\left[1+e^{-\frac{4\pi\lambda}{m}}-2e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{4\pi}{m}\right)\right]^2}} \sin(\mu_2 - 2\mu)}{\left\{1 - \sqrt{\frac{\left[1-e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)-e^{-\frac{2\pi\lambda}{m}}+e^{-\frac{4\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2+\left[e^{-\frac{2\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right)-e^{-\frac{4\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right)\right]^2}{\left[1+e^{-\frac{4\pi\lambda}{m}}-2e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2}}\right\}^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\sqrt{\frac{\left[1-e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{4\pi}{m}\right)-e^{-\frac{2\pi\lambda}{m}}+e^{-\frac{4\pi\lambda}{m}} \cos\left(\frac{4\pi}{m}\right)\right]^2+\left[e^{-\frac{2\pi\lambda}{m}} \sin\left(\frac{4\pi}{m}\right)-e^{-\frac{4\pi\lambda}{m}} \sin\left(\frac{4\pi}{m}\right)\right]^2}{\left[1+e^{-\frac{4\pi\lambda}{m}}-2e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{4\pi}{m}\right)\right]^2}} \cos(\mu_2 - 2\mu)}{\left\{1 - \sqrt{\frac{\left[1-e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)-e^{-\frac{2\pi\lambda}{m}}+e^{-\frac{4\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2+\left[e^{-\frac{2\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right)-e^{-\frac{4\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right)\right]^2}{\left[1+e^{-\frac{4\pi\lambda}{m}}-2e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2}}\right\}^{2}} - \frac{\left\{\frac{\left[1-e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)-e^{-\frac{2\pi\lambda}{m}}+e^{-\frac{4\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2+\left[e^{-\frac{2\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right)-e^{-\frac{4\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right)\right]^2}{\left[1+e^{-\frac{4\pi\lambda}{m}}-2e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2}\right\}^{2}}{\left\{1 - \sqrt{\frac{\left[1-e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)-e^{-\frac{2\pi\lambda}{m}}+e^{-\frac{4\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2+\left[e^{-\frac{2\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right)-e^{-\frac{4\pi\lambda}{m}} \sin\left(\frac{2\pi}{m}\right)\right]^2}{\left[1+e^{-\frac{4\pi\lambda}{m}}-2e^{-\frac{2\pi\lambda}{m}} \cos\left(\frac{2\pi}{m}\right)\right]^2}}\right\}^{2}},$$

respectively. The wrapped discrete exponential distribution is a simple model with a closed-form PMF. For this reason, some may consider it more practical than competing distributions.

3.4. Wrapped Discrete Mittag–Leffler Distribution

Ref. [62] introduced the discrete Mittag–Leffler distribution. Its nth trigonometric moment is given by

$$m_n = \left[1 + \frac{1-\delta}{\delta}\left(1 - e^{\frac{2\pi i n}{m}}\right)^{\alpha}\right]^{-1}$$

for $n = 1, 2, \ldots$, where $0 < \delta < 1$, $0 < \alpha < 1$, and $m = 1, 2, \ldots$. The wrapped discrete Mittag–Leffler distribution has the disadvantage of having a non-closed-form PMF.

3.5. Wrapped Discrete Skew Laplace Distribution

Ref. [63] obtained the wrapped discrete skew Laplace distribution. Its PMF and CDF are

$$P\left(\Theta = \frac{2\pi r}{m}\right) = \frac{(1-p)(1-q)}{1-pq}\left[\frac{q^{m-r}(1-p^m) + p^r(1-q^m)}{(1-p^m)(1-q^m)}\right]$$

and

$$F(\Theta \leq s) = \frac{(1-p)(1-q)}{(1-pq)(1-p^m)(1-q^m)} \left\{ \frac{(1-p^m)(q^m-s^m)q}{q-s} + \frac{[1-(ps)^m](1-q^m)}{1-ps} \right\}, \quad s = 0, 1, \ldots, m-1,$$

respectively, for $m \in \mathbb{N}$, $r = 0, 1, \ldots, m-1$, and $\lambda > 0$. The nth trigonometric moment is given by

$$m_n = \frac{(1-p)(1-q)}{\left(1-pe^{\frac{i2\pi n}{m}}\right)\left(1-qe^{-\frac{2i\pi n}{m}}\right)}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left[\frac{(p-q)\sin\left(\frac{2\pi}{m}\right)}{1+pq-(p+q)\cos\left(\frac{2\pi}{m}\right)}\right],$$

$$\rho = \frac{(1-p)(1-q)}{\sqrt{\left[1+pq-(p+q)\cos\left(\frac{2\pi}{m}\right)\right]^2 + \left[(p-q)\sin\left(\frac{2\pi}{m}\right)\right]^2}},$$

$$\gamma_1 = \frac{\frac{(1-p)(1-q)}{\sqrt{\left[1+pq-(p+q)\cos\left(\frac{4\pi}{m}\right)\right]^2+\left[(p-q)\sin\left(\frac{4\pi}{m}\right)\right]^2}}\sin(\mu_2-2\mu)}{\left\{1-\frac{(1-p)(1-q)}{\sqrt{\left[1+pq-(p+q)\cos\left(\frac{2\pi}{m}\right)\right]^2+\left[(p-q)\sin\left(\frac{2\pi}{m}\right)\right]^2}}\right\}^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{(1-p)(1-q)}{\sqrt{\left[1+pq-(p+q)\cos\left(\frac{4\pi}{m}\right)\right]^2+\left[(p-q)\sin\left(\frac{4\pi}{m}\right)\right]^2}}\cos(\mu_2-2\mu) - \frac{((1-p)(1-q))^2}{\left[1+pq-(p+q)\cos\left(\frac{2\pi n}{m}\right)\right]^2+\left[(p-q)\sin\left(\frac{2\pi n}{m}\right)\right]^2}}{\left\{1-\frac{(1-p)(1-q)}{\sqrt{\left[1+pq-(p+q)\cos\left(\frac{2\pi}{m}\right)\right]^2+\left[(p-q)\sin\left(\frac{2\pi}{m}\right)\right]^2}}\right\}^{2}},$$

respectively, where $\mu_2 = \arctan\left[\frac{(p-q)\sin\left(\frac{4\pi}{m}\right)}{1+pq-(p+q)\cos\left(\frac{4\pi}{m}\right)}\right]$. The wrapped discrete skew Laplace distribution may be considered more practical than certain other models due to it having a closed-form PMF.

3.6. Wrapped Geometric Distribution

Ref. [62] obtained the wrapped geometric distribution. Its PMF and CDF are

$$P\left(\Theta = \frac{2\pi r}{m}\right) = \frac{\delta(1-\delta)^r}{1-(1-\delta)^m}$$

and

$$F(\Theta \leq y) = \sum_{r=0}^{y} \frac{\delta(1-\delta)^r}{1-(1-\delta)^m}, \quad y = 0, 1, \ldots, m-1,$$

respectively, for $m \in \mathbb{N}$, $r = 0, 1, \ldots, m-1$, and $\delta > 0$. The nth trigonometric moment is

$$m_n = \frac{\delta}{x - \mathrm{i}y},$$

where $x = 1 - (1-\delta)\cos\frac{2\pi n}{m}$ and $y = (1-\delta)\sin\frac{2\pi n}{m}$ for $n \neq 0 \pmod m$, $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left[\frac{(1-\delta)\sin\frac{2\pi}{m}}{1-(1-\delta)\cos\frac{2\pi}{m}}\right],$$

$$\rho = \frac{\delta}{\sqrt{\delta^2 + 2(1-\delta)\left(1-\cos\frac{2\pi}{m}\right)}},$$

$$\gamma_1 = \frac{\delta\sin(\mu_2 - 2\mu)}{\sqrt{\left[1-(1-\delta)\cos\left(\frac{4\pi}{m}\right)\right]^2 + \left[(1-\delta)\sin\left(\frac{4\pi}{m}\right)\right]^2}\left\{1 - \frac{\delta}{\sqrt{\delta^2+2(1-\delta)\left[1-\cos\left(\frac{2\pi}{m}\right)\right]}}\right\}^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\frac{\delta\cos(\mu_2 - 2\mu)}{\sqrt{\left[1-(1-\delta)\cos\left(\frac{4\pi}{m}\right)\right]^2+\left[(1-\delta)\sin\left(\frac{4\pi}{m}\right)\right]^2}} - \frac{\delta^4}{\left\{\delta^2+2(1-\delta)\left[1-\cos\left(\frac{2\pi}{m}\right)\right]\right\}^2}}{\left\{1 - \frac{\delta}{\sqrt{\delta^2+2(1-\delta)\left[1-\cos\left(\frac{2\pi}{m}\right)\right]}}\right\}^2},$$

respectively, where $\mu_2 = \arctan\left[\frac{(1-\delta)\sin\frac{4\pi}{m}}{1-(1-\delta)\cos\frac{4\pi}{m}}\right]$. The wrapped geometric distribution is a simple discrete circular model with the practical advantage of having a closed-form PMF.

3.7. Wrapped Negative Binomial Distribution

Ref. [64] introduced the wrapped negative binomial distribution. Its PMF and CDF are

$$P\left(\Theta = \frac{2\pi r}{m}\right) = \sum_{k=0}^{\infty}\binom{r+km+n-1}{n-1}p_1^n q_1^{r+km}$$

and

$$F(\Theta \leq y) = \sum_{r=0}^{y}\binom{r+km+n-1}{n-1}p_1^n q_1^{r+km}, \quad y = 0, 1, \ldots, m-1,$$

respectively, for $p_1 \in [0,1]$, $p_1 + q_1 = 1$, and $n \in \mathbb{Z}^+$. The nth trigonometric moment is

$$m_n = \left\{\frac{p_1\left[1 - q_1\cos\left(\frac{2\pi n}{m}\right)\right] + \mathrm{i}p_1q_1\sin\left(\frac{2\pi n}{m}\right)}{1 + q_1^2 - 2q_1\cos\left(\frac{2\pi n}{m}\right)}\right\}^n$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = n\arctan\left\{\frac{p_1q_1\left(\frac{2\pi}{m}\right)}{p_1\left[1-q_1\cos\left(\frac{2\pi}{m}\right)\right]}\right\},$$

$$\rho = \left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\left(\frac{2\pi}{m}\right)}}\right]^n,$$

$$\gamma_1 = \frac{\left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\left(\frac{4\pi}{m}\right)}}\right]^n \sin(\mu_2 - 2\mu)}{\left\{1 - \left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\left(\frac{2\pi}{m}\right)}}\right]^n\right\}^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{\left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\left(\frac{4\pi}{m}\right)}}\right]^n \cos(\mu_2 - 2\mu) - \left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\left(\frac{4\pi}{m}\right)}}\right]^{4n}}{\left\{1 - \left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\left(\frac{2\pi}{m}\right)}}\right]^n\right\}^2},$$

respectively, where $\mu_2 = n \arctan\left\{\frac{p_1 q_1 \sin\left(\frac{4\pi}{m}\right)}{p_1\left[1-q_1\cos\left(\frac{4\pi}{m}\right)\right]}\right\}$. The wrapped negative binomial distribution does not have a closed-form PMF.

3.8. Wrapped Poisson Distribution

Ref. [39] obtained the wrapped Poisson distribution. Its PMF and CDF are

$$P\left(\Theta = \frac{2\pi r}{m}\right) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{r+km}}{(r+km)!}$$

and

$$F(\Theta \le y) = \sum_{r=0}^{y} P\left(\theta = \frac{2\pi r}{m}\right), \quad y = 0, 1, \ldots, m-1,$$

respectively, for $m \in \mathbb{N}$, $r = 0, 1, \ldots, m-1$, and $\lambda > 0$. The nth trigonometric moment is

$$m_n = e^{-\lambda\left[1-\cos\left(\frac{2\pi n}{m}\right)\right] + \mathrm{i}\sin\left(\frac{\lambda 2\pi n}{m}\right)}$$

for $n = 1, 2, \ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \lambda \sin\left(\frac{2\pi}{m}\right),$$

$$\rho = e^{-\lambda\left[1-\cos\left(\frac{2\pi}{m}\right)\right]},$$

$$\gamma_1 = \frac{e^{-\lambda\left[1-\cos\left(\frac{4\pi}{m}\right)\right]} \sin\left[\lambda \sin\left(\frac{4\pi}{m}\right) - 2\mu\right]}{\left\{1 - e^{-\lambda\left[1-\cos\left(\frac{2\pi}{m}\right)\right]}\right\}^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{e^{-\lambda\left[1-\cos\left(\frac{4\pi}{m}\right)\right]} \cos\left[\lambda \sin\left(\frac{4\pi}{m}\right) - 2\mu\right] - e^{-4\lambda\left[1-\cos\left(\frac{2\pi}{m}\right)\right]}}{\left\{1 - e^{-\lambda\left[1-\cos\left(\frac{2\pi}{m}\right)\right]}\right\}^2},$$

respectively. The wrapped Poisson distribution lacks a closed-form PMF, limiting its practical usefulness.

3.9. Wrapped Poisson–Lindley Distribution

Ref. [65] obtained the wrapped Poisson–Lindley distribution. Its PMF and CDF are

$$P\left(\Theta = \frac{2\pi r}{m}\right) = \frac{\theta^2(1+\theta)^m\{(r+\theta+2)[(1+\theta)^m-1]+m\}}{(1+\theta)^{r+3}[(1+\theta)^m-1]^2}$$

and

$$F(\Theta \le y) = \frac{(1+\theta)^{m-y-3}}{[(1+\theta)^m-1]^2}\Bigg[1-(1+\theta)^m-\theta\{m+(3+y+\theta)[(1+\theta)^m-1]\} + (1+\theta)^{1+y}\left[-1-2\theta+m\theta-\theta^2+(1+\theta)^{2+m}\right]\Bigg], \quad y = 0,1,\ldots,m-1,$$

respectively, for $m \in \mathbb{N}$ and $r = 0,1,\ldots,m-1$. The wrapped Poisson–Lindley distribution was used to analyze the turtle dataset and the Gorleston wind direction dataset discussed by [1]. Additionally, it was applied to a dataset consisting of arrival directions of low showers of cosmic rays, with declination and right ascension as the coordinate system [66]. The goodness of fit of the wrapped Poisson–Lindley distribution was compared to that of the wrapped geometric distribution. The wrapped Poisson–Lindley distribution showed the best fit for the first and third datasets, but not for the second dataset.

3.10. Wrapped Zero-Inflated Poisson Distribution

Ref. [67] introduced the wrapped zero-inflated Poisson distribution. Its PMF and CDF are

$$P\left(\Theta = \frac{2\pi r}{m}\right) = \sum_{k=-\infty}^{\infty} p(r+km)$$

and

$$F(\Theta \le y) = \sum_{r=0}^{y} P\left(\theta = \frac{2\pi r}{m}\right), \quad y = 0,1,\ldots,m-1,$$

respectively, for $m \in \mathbb{N}$, $r = 0,1,\ldots,m-1$, $\lambda > 0$, and $w \in [0,1]$, where

$$p(x) = \begin{cases} w+(1-w)e^{-\lambda}, & \text{if } x = 0, \\ \dfrac{(1-w)e^{-\lambda}\lambda^x}{x!}, & \text{if } x = 1,2,\ldots. \end{cases}$$

The nth trigonometric moment is

$$m_n = w+(1-w)e^{-\lambda+\lambda e^{\mathrm{i}n}}$$

for $n = 1,2,\ldots$. The mean direction, mean resultant length, skewness, and kurtosis are

$$\mu = \arctan\left\{\frac{(1-w)e^{-\lambda\left[1-\cos\left(\frac{2\pi}{m}\right)\right]}\sin\left[\lambda\sin\left(\frac{2\pi}{m}\right)\right]}{w+(1-w)e^{-\lambda\left[1-\cos\left(\frac{2\pi}{m}\right)\right]}\cos\left[\lambda\sin\left(\frac{2\pi}{m}\right)\right]}\right\},$$

$$\rho = \sqrt{w^2+(1-w)^2e^{-2\lambda\left[1-\cos\left(\frac{2\pi p}{m}\right)\right]}+2w(1-w)e^{-\lambda\left[1-\cos\left(\frac{2\pi p}{m}\right)\right]}\cos\left[\lambda\sin\left(\frac{2\pi p}{m}\right)\right]},$$

$$\gamma_1 = \frac{w \sin 2\mu + (1-w)e^{-\lambda\left[1-\cos\left(\frac{4\pi}{m}\right)\right]} \sin\left[\lambda \sin\left(\frac{4\pi}{m}\right) - 2\mu\right]}{\left\{1 - \sqrt{a_1^2 + b_1^2 + 2a_1 b_1 \cos\left[\lambda \sin\left(\frac{2\pi}{m}\right)\right]}\right\}^{\frac{3}{2}}},$$

and

$$\gamma_2 = \frac{w \cos 2\mu + (1-w)e^{-\lambda\left[1-\cos\left(\frac{4\pi}{m}\right)\right]} \cos\left[\lambda \sin\left(\frac{4\pi}{m}\right) - 2\mu\right] - \left\{a_1^2 + b_1^2 + 2a_1 b_1 \cos\left[\lambda \sin\left(\frac{2\pi}{m}\right)\right]\right\}^2}{\left\{1 - \sqrt{a_1^2 + b_1^2 + 2a_1 b_1 \cos\left[\lambda \sin\left(\frac{2\pi}{m}\right)\right]}\right\}^2},$$

respectively, where $a_1 = w$ and $b_1 = (1-w)e^{-\lambda\left[1-\cos\left(\frac{2\pi}{m}\right)\right]}$. The wrapped zero-inflated Poisson distribution, a more flexible alternative to the one-parameter wrapped Poisson distribution, is useful for directional data with a high number of zero counts.

4. Data Applications

All of the distributions reviewed in Sections 2 and 3 can be fit by using the package Wrapped due to [9]. The package computes the PDF, CDF, quantile function, random samples, maximum likelihood estimates, standard errors, confidence intervals, and measures of goodness of fit associated with (1)–(2) for any specified g. In this section, we illustrate three data applications by fitting all of the reviewed distributions using Wrapped. In each data application, the following distributions gave the best fits: wrapped exponential, wrapped gamma, wrapped Weibull, wrapped Pareto, wrapped normal, wrapped Cauchy, wrapped Laplace and, wrapped t distributions. Hence, the tables and figures will be limited to these distributions.

4.1. Dataset 1

This data taken from [1] contain arrival times on a 24 h clock of 254 patients at an intensive care unit over a period of about 12 months. The maximized log-likelihood values as well as the corresponding values of AIC and BIC are given in Table 1.

Table 1. Fitted distributions and values of $\log L$, AIC, and BIC for dataset 1.

Distribution	$\log L$	AIC	BIC
Wrapped exponential	−468.6	939.2	942.8
Wrapped gamma	−436.9	877.8	884.9
Wrapped Weibull	−441.0	886.1	893.1
Wrapped Pareto	−446.1	896.2	903.3
Wrapped normal	−438.5	881.1	888.1
Wrapped Cauchy	−443.6	891.2	898.2
Wrapped Laplace	−443.4	890.8	897.9
Wrapped t	−438.5	883.1	893.7

We see that the wrapped gamma distribution gives the best fit in terms of AIC and BIC. The wrapped exponential distribution gives the worst fit in terms of AIC and BIC. These findings are confirmed by the density plots shown in Figure 1.

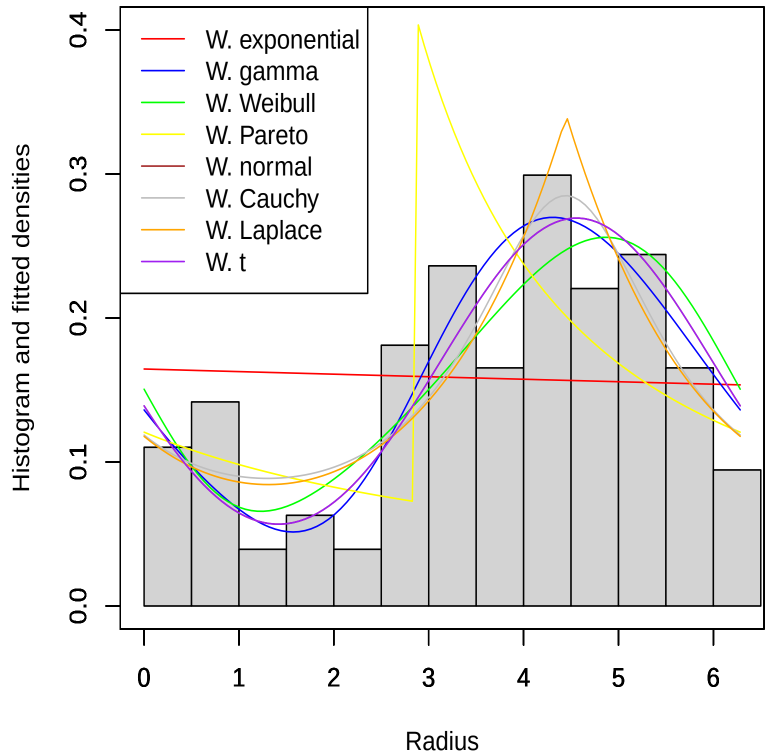

Figure 1. Histogram of the data and fitted densities for dataset 1.

4.2. Dataset 2

These data taken from [1] contain measurements of the directions taken by 76 turtles after treatment. The maximized log-likelihood values as well as the corresponding values of AIC and BIC are given in Table 2.

Table 2. Fitted distributions and values of $\log L$, AIC, and BIC for dataset 2.

Distribution	$\log L$	AIC	BIC
Wrapped exponential	−122.3	246.7	249.0
Wrapped gamma	−119.6	243.2	247.9
Wrapped Weibull	−120.6	245.3	249.9
Wrapped Pareto	−135.9	275.8	280.5
Wrapped normal	−125.1	254.1	258.8
Wrapped Cauchy	−113.9	231.9	236.6
Wrapped Laplace	−116.7	237.3	242.0
Wrapped t	−113.7	233.5	240.5

We see that the wrapped Cauchy distribution gives the best fit in terms of AIC and BIC. The wrapped Pareto distribution gives the worst fit in terms of AIC and BIC. These findings are confirmed by the density plots shown in Figure 2.

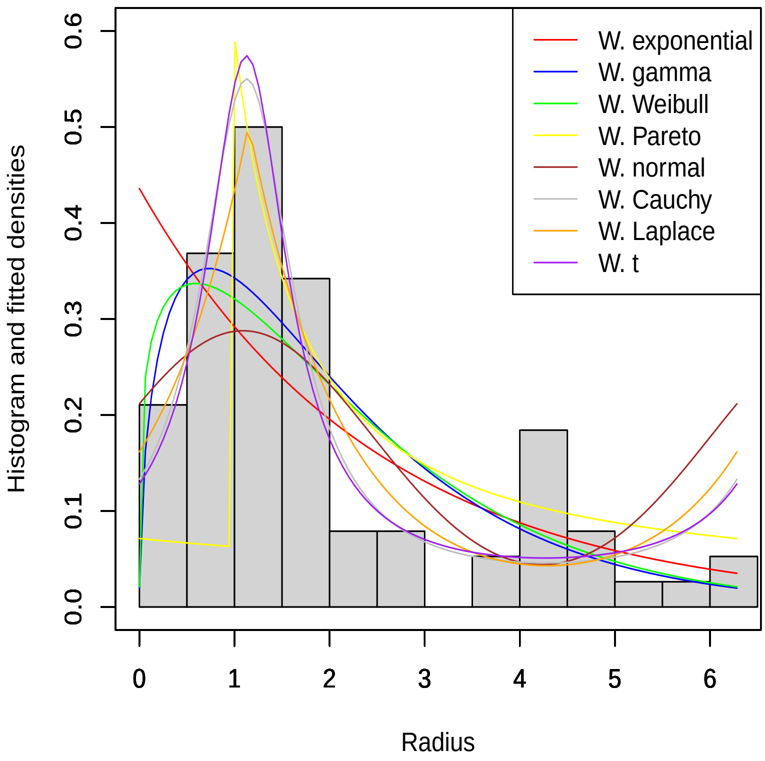

Figure 2. Histogram of the data and fitted densities for dataset 2.

4.3. Dataset 3

These data taken from [1] contain measurements of long-axis orientation of 133 feldspar laths in basalt. The maximized log-likelihood values as well as the corresponding values of AIC and BIC are given in Table 3.

Table 3. Fitted distributions and values of log L, AIC, and BIC for dataset 3.

Distribution	log L	AIC	BIC
Wrapped exponential	−243.9	489.8	492.6
Wrapped gamma	−243.9	491.8	497.5
Wrapped Weibull	−243.9	491.8	497.5
Wrapped Pareto	−247.6	499.2	505.0
Wrapped normal	−242.4	488.7	494.5
Wrapped Cauchy	−242.7	489.5	495.3
Wrapped Laplace	−243.9	491.9	497.6
Wrapped t	−242.4	490.7	499.4

We can see that the wrapped normal distribution gives the best fit in terms of AIC, whereas the wrapped exponential distribution gives the best fit in terms of BIC. The wrapped Pareto distribution gives the worst fit in terms of AIC and BIC. These findings are confirmed by the density plots shown in Figure 3.

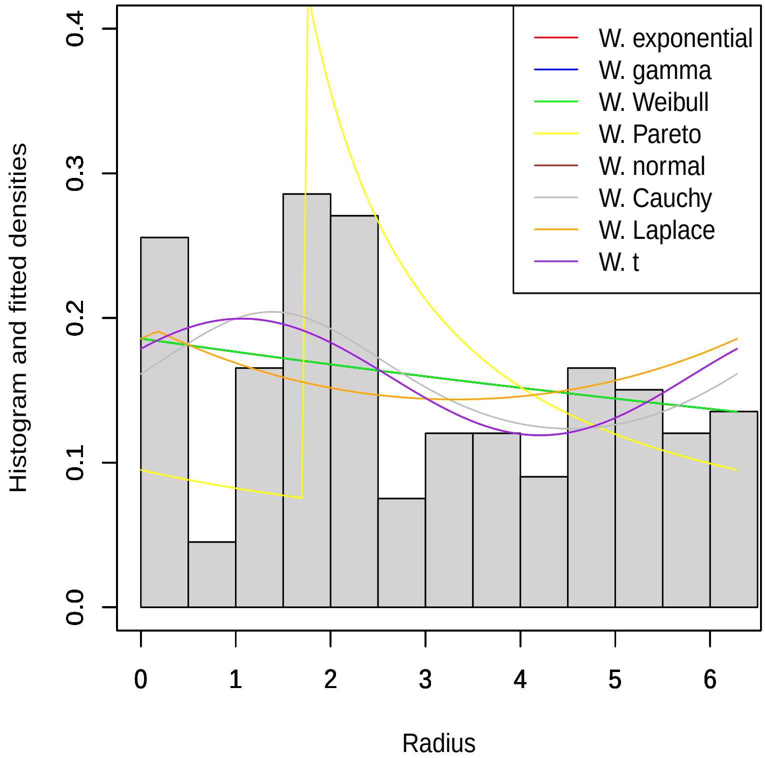

Figure 3. Histogram of the data and fitted densities for dataset 3.

5. Conclusions

In this paper, we have reviewed 45 wrapped distributions for continuous circular data and 10 wrapped distributions for discrete circular data by listing their probability density/mass functions, cumulative distribution functions, trigonometric moments, mean directions, mean resultant lengths, skewness, and kurtosis (whenever they are available). We have also discussed data applications and limitations of the reviewed distributions. This paper could be a source of reference and may encourage further developments in the area of wrapped distributions.

Future work may provide similar reviews for wrapped bivariate distributions, wrapped multivariate distributions, wrapped matrix variate distributions, and wrapped complex variate distributions. Another future possibility is to write R packages for fitting wrapped bivariate distributions, wrapped multivariate distributions, wrapped matrix variate distributions, and wrapped complex variate distributions.

Author Contributions: Investigation, W.B. and S.N.; methodology, W.B. and S.N.; formal analysis, W.B. and S.N.; software, W.B. and S.N.; writing—review and editing, W.B. and S.N. Both authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in the study are included in the article, and further inquiries can be directed to the corresponding author.

Acknowledgments: The authors would like to thank the editor and the two referees for careful reading and comments, which greatly improved the paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Fisher, N.I. *Statistical Analysis of Circular Data*; Cambridge University Press: Cambridge, UK, 1996.
2. Fisher, N.I. *Statistical Analysis of Circular Data*; Cambridge University Press: Cambridge, UK, 2011.
3. Jones, M.C. On families of distributions with shape parameters. *Int. Stat. Rev.* **2015**, *83*, 175–192. [CrossRef]

4. Ley, C.; Verdebout, T. *Modern Directional Statistics*; CRC Press: Boca Raton, FL, USA, 2017.
5. Pewsey, A.; Garcia-Portugues, E. Recent advances in directional statistics. *Test* **2021**, *30*, 1–58. [CrossRef]
6. Ley, C.; Babic, S.; Craens, D. Flexible models for complex data with applications. *Annu. Rev. Stat. Its Appl.* **2021**, *8*, 369–391. [CrossRef]
7. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series*; Gordon and Breach Science Publishers: Amsterdam, The Netherlands, 1986; Volumes 1–3.
8. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*, 6th ed.; Academic Press: San Diego, CA, USA, 2000.
9. Nadarajah, S.; Zhang, Y. Wrapped: An R package for circular data. *PLoS ONE* **2017**, *12*, e0188512. [CrossRef] [PubMed]
10. Kumar, C.S.; Alex, R. Wrapped generalized skew normal distribution. *Stoch. Model. Appl.* **2021**, *25*, 1–7.
11. Hernandez-Sanchez, E.; Scarpa, B. A wrapped flexible generalized skew-normal model for a bimodal circular distribution of wind directions. *Chil. J. Stat.* **2012**, *3*, 129–141.
12. Holla, M.S.; Bhattacharya, S.K. On a compound Gaussian distribution. *Ann. Inst. Stat. Math.* **1968**, *20*, 331–336. [CrossRef]
13. Al-Khazaleh, A.; Al-Meanazel, T.R. Wrapped Akash distribution. *Electron. J. Appl. Stat. Anal.* **2021**, *14*, 305–317.
14. Ganaie, R.; Rajagopalan, V.; Rather, A. Weighted Aradhana distribution: Properties and applications. *J. Inf. Technol. Res.* **2019**, *IX*, 392–406.
15. Ramabhadra Sarma, I.; Dattatreya Rao, A.V.; Girija, S.V.S. On characteristic functions of wrapped half logistic and binormal distributions. *Int. J. Stat. Syst.* **2009**, *4*, 33–45.
16. Saraiva, M.A.; de Azevedo Cysneiros, F.J. Wrapped Birnbaum-Saunders distribution: Definition and estimation. *Nexus Math.* **2022**, *5*, 200010. [CrossRef]
17. Jander, R. Die optische Richtungsorientierung derroten Waldameise (*Formica rufa*). *Z. Fur-Vergle-Ichende Physiol.* **1957**, *40*, 162–238. [CrossRef]
18. Levy, P. L'addition des variables aléatoires définies sur une circonférence. *Bull. SociéTé MathéMatique Fr.* **1939**, *67*, 1–41. [CrossRef]
19. Roy, S.; Adnan, M.A.S. Wrapped Chi-Square Distribution. Available online: https://www.researchgate.net/publication/263080292_Wrapped_Chi-square_distribution (accessed on 3 August 2024).
20. Rao, J.S.; Kozubowski, T.J. A wrapped exponential circular model. *Proc. AP Acad. Sci.* **2001**, *5*, 43–56.
21. Subrahmanyam, P.S.; Dattatreya Rao, A.V.; Girija, S.V.S. On wrapping of exponentiated inverted Weibull distribution. *Int. J. Innov. Res. Sci. Technol.* **2017**, *3*, 18–25.
22. Coelho, C.A. The wrapped gamma distribution and wrapped sums and linear combinations of independent gamma and Laplace distributions. *J. Stat. Theory Pract.* **2007**, *1*, 1–29. [CrossRef]
23. Jayakumar, K.; Sajayan, T. Wrapped generalized geometric stable distributions with an application to wind direction. *Far East J. Theor. Stat.* **2022**, *66*, 147–166.
24. Cameron, M.A. The comparison of time series recorders. *Technometrics* **1983**, *25*, 9–22. [CrossRef]
25. Roy, S.; Adnan, M.A.S. Wrapped generalized Gompertz distribution: An application to ornithology. *J. Biom. Biostat.* **2012**, *3*, 153–156. [CrossRef]
26. Reed, W.J.; Pewsey, A. Two nested families of skew-symmetric circular distributions. *Test* **2009**, *18*, 516–528. [CrossRef]
27. Bruderer, B.; Jenni, L. Migration across the alps. In *Bird Migration*; Springer: Berlin/Heidelberg, Germany, 1990; pp. 60–77.
28. Varghese, J.; Jose, K.K. Wrapped hb-skewed laplace distribution and its application in meteorology. *Biom. Biostat. Int. J.* **2018**, *7*, 525–530.
29. Adnan, M.A.S.; Roy, S. Wrapped hypo-exponential distribution. *J. Stat. Manag. Syst.* **2013**, *16*, 1–11. [CrossRef]
30. Al-Meanazel, T.R.; Al-Khazaleh, A. Wrapped Ishita distribution. *J. Stat. Appl. Probab.* **2021**, *10*, 293–299.
31. Jammalamadaka, S.R.; Kozubowski, T. A new family of circular models: The wrapped Laplace distributions. *Adv. Appl. Stat.* **2003**, *3*, 77–103.
32. Bhattacharjee, S.; Borah, D. Wrapped length biased weighted exponential distribution. *Thail. Stat.* **2019**, *17*, 223–234.
33. Smith, N.M. Reconstruction of the Tertiary Drainage Systems of the Inverell Region. Unpublished B.Sc. Ph.D. Thesis, Department of Geography, University of Sydney, Sydney, Australia, 1988.
34. Joshi, S.; Jose, K.K. Wrapped Lindley distribution. *Commun. -Stat.-Theory Methods* **2018**, *47*, 1013–1021. [CrossRef]
35. Jacob, S. Study on Circular Distributions. Ph.D. Thesis, Department of Statistics, University of Calicut, Calicut, India, 2012.
36. Subba Rao, R.; Ravindranath, V.; Dattatreya Rao, A.V.; Prasad, G.; Ravi Kishore, P. Wrapped Lomax distribution: A new circular probability model. *Int. J. Eng. Technol.* **2018**, *7*, 150–152. [CrossRef]
37. Chesneau, C.; Tomy, L.; Jose, M. Wrapped modified Lindley distribution. *J. Stat. Manag. Syst.* **2021**, *24*, 1025–1040. [CrossRef]
38. Subrahmanyam, P.S., Dattatreya Rao, A.V.; Girija, S.V.S. On wrapping of new Weibull Pareto distribution. *Int. J. Adv. Res. Rev.* **2017**, *2*, 10–20.
39. Mardia, K.V. Statistics of Directional Data. In *Probability and Mathematical Statistics*; Academic Press: London, UK, 1972.
40. Jammalamadaka, S.R.; Kozubowski, T.J. A general approach for obtaining wrapped circular distributions via mixtures. *Sankhyā A* **2017**, *79*, 133–157. [CrossRef]
41. Bhattacharjee, S. Exploration of some distributional properties, parameter estimates and applications of the wrapped quasi Lindley distribution. *J. Math. Comput. Sci.* **2020**, *10*, 2390–2407.
42. Bell, W.; Nadarajah, S. The wrapped Rama distribution. *Preprint* **2023**.
43. Demirelli, A.E.; Gürcan, M. New wrapped distribution via Richard link function. *AIP Conf. Proc.* **2017**, *1863*, 350002.

44. Al-Meanazel, T.R.; Al-Khazaleh, A. Wrapped Shanker distribution. *Ital. J. Pure Appl. Math.* **2021**, *46*, 184–194.
45. Yilmaz, A. Wrapped flexible skew Laplace distribution. *J. Turk. Stat. Assoc.* **2018**, *11*, 53–64.
46. Fernandez-Duran, J.J. Circular distributions based on nonnegative trigonometric sums. *Biometrics* **2004**, *60*, 499–503. [CrossRef] [PubMed]
47. Pewsey, A. The wrapped skew-normal distribution on the circle. *Commun. -Stat.—Theory Methods* **2000**, *29*, 2459–2472. [CrossRef]
48. Pewsey, A. The wrapped stable family of distributions as a flexible model for circular data. *Comput. Stat. Data Anal.* **2008**, *52*, 1516–1523. [CrossRef]
49. Pewsey, A.; Lewis, T.; Jones, M. The wrapped *t* family of circular distributions. *Aust. N. Z. J. Stat.* **2007**, *49*, 79–91.
50. Yilmaz, A.; Bicer, C. A new wrapped exponential distribution. *Math. Sci.* **2018**, *12*, 285–293. [CrossRef]
51. Bhattacharjee, S.; Ahmed, I.; Das, K.K. Wrapped two-parameter Lindley distribution for modelling circular data. *Thail. Stat.* **2020**, *19*, 81–94.
52. Mardia, K.V.; Jupp, P.E. *Directional Statistics*; John Wiley and Sons: New York, NY, USA, 2000.
53. Bicer, C.; Yilmaz, A. Wrapped two sided Lindley distribution. In *Research and Reviews in Science and Mathematics*; Iyit, N., Dogan, H.H., Akgul, H., Eds.; Gece Academy Publishing: Ankara, Turkey, 2022; pp. 1–13.
54. Adnan, M.A.S.; Roy, S. Wrapped variance gamma distribution with an application to wind direction. *J. Environ. Stat.* **2014**, *6*, 1–10.
55. Roy, S.; Adnan, M.A.S. Wrapped weighted exponential distributions. *Stat. Probab. Lett.* **2012**, *82*, 77–83. [CrossRef]
56. Al-Mofleh, H.; Sen, S. The wrapped Xgamma distribution for modeling circular data appearing in geological context. *arXiv* **2019**, arXiv:1903.00177.
57. Zinhom, E.; Nassar, M.M.; Radwan, S.S.; Elmasry, A. The wrapped XLindley distribution. *Environ. Ecol. Stat.* **2023**, *30*, 669–686. [CrossRef]
58. Bahnsen, A.C.; Stojanovic, A.; Aouada, D.; Ottersten, B. Improving credit card fraud detection with calibrated probabilities. In Proceedings of the 2014 SIAM International Conference on Data Mining, Philadelphia, PA, USA, 24–26 April 2014 ; pp. 677–685, SIAM.
59. Girija, S.V.S.; Dattatreya Rao, A.V.; Srihari, V.L.N. On wrapped binomial model characteristics. *Math. Stat.* **2014**, *2*, 231–234. [CrossRef]
60. Girija, S.V.S.; Srihari, G.V.L.N.; Srinivas, R. On discrete wrapped Cauchy model. *Math. Theory Model.* **2019**, *9*, 15–23.
61. Srinivas, R.; Srihari, G.V.L.N.; Girija, S.V.S. On construction of discrete wrapped Lindley distribution. *i-Manag. J. Math.* **2008**, *7*, 10–19. [CrossRef]
62. Jacob, S.; Jayakumar, K. Wrapped geometric distribution: A new probability model for circular data. *J. Stat. Theory Appl.* **2013**, *12*, 348–355. [CrossRef]
63. Jayakumar, K.; Jacob, S. Wrapped skew Laplace distribution on integers: A new probability model for circular data. *Open J. Stat.* **2012**, *2*, 106–114. [CrossRef]
64. Srihari, G.V.L.N.; Girija, S.V.S.; Rao, A.V.D. On wrapped negative binomial model. *Int. J. Math. Arch.* **2018**, *9*, 148–153.
65. Chesneau, C.; Bakouch, H.S.; Hussain, T. Modeling and applications to circular data with a wrapped Poisson-Lindley model. *J. Math. Study* **2022**, *55*, 139–157. [CrossRef]
66. Fisher, N.I.; Lewis, T.; Embleton, B.J.J. *Statistical Analysis of Spherical Data*; Cambridge University Press: New York, NY, USA, 1987.
67. Thavaneswaran, A.; Mandal, S.; Pathmanathan, D. Estimation for wrapped zero inflated Poisson and wrapped Poisson distributions. *Int. J. Stat. Probab.* **2016**, *5*, 1–8. [CrossRef]