8.4 Rectangular Waveguides

Compared to parallel-plate waveguides, rectangular guides have the advantage of no fringing fields because of their enclosed nature. Unlike parallel-plate waveguides, though, rectangular waveguides do not support TEM waves since they are single-conductor waveguides. Only TM and TE modes can be supported. In the following, we consider a rectangular guide with four conducting walls at $x = 0$, $x = a$, $y = 0$, and $y = b$.

8.4.1 TM Waves in Rectangular Waveguides

For TM waves, $H_z = 0$, and $E_z = e_z(x, y)e^{-jk_z z}$ satisfies (8.25) as well as the boundary conditions on the conducting walls:

$$E_z = E_y = 0, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a, \tag{8.75a}$$

$$E_z = E_x = 0, \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b. \tag{8.75b}$$

Using the method of separation of variables, equations (8.25), (8.75a) and (8.75b) can be solved to yield

$$e_z(x, y) = E_0 \sin(k_{xm}x)\sin(k_{yn}y), \tag{8.76}$$

where the eigenvalues are

$$k_{xm} = \frac{m\pi}{a}, \qquad k_{yn} = \frac{n\pi}{b}, \qquad m, n = 1, 2, 3, \cdots \tag{8.77a}$$

and

$$k_{tmn}^2 = k_{xm}^2 + k_{yn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \tag{8.77b}$$

Other field components can be obtained from E_z by equations (8.12)–(8.15):

$$e_x(x, y) = -\frac{\gamma}{k_{tmn}^2} k_{xm} E_0 \cos(k_{xm}x)\sin(k_{yn}y), \tag{8.78}$$

$$e_y(x, y) = -\frac{\gamma}{k_{tmn}^2} k_{yn} E_0 \sin(k_{xm}x)\cos(k_{yn}y), \tag{8.79}$$

$$h_x(x, y) = \frac{j\omega\epsilon}{k_{tmn}^2} k_{yn} E_0 \sin(k_{xm}x)\cos(k_{yn}y), \tag{8.80}$$

$$h_y(x, y) = -\frac{j\omega\epsilon}{k_{tmn}^2} k_{xm} E_0 \cos(k_{xm}x)\sin(k_{yn}y), \tag{8.81}$$

where the propagation constant is

$$\gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \tag{8.82}$$

• **Designation of Modes:** TM_{mn} mode is designated for every possible combination of integers m and n. m and n denote the numbers of half-cycle variations of the fields in x and y directions, respectively.

- **Cutoff Frequency and Cutoff Wavelength of TM_{mn} Mode:**

$$(f_c)_{mn} = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{(m/a)^2 + (n/b)^2}, \tag{8.83a}$$

$$(\lambda_c)_{mn} = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}}. \tag{8.83b}$$

- **Dominant TM Mode:** Since for TM modes, neither m nor n can be zero, the dominant mode for TM waves is TM_{11} mode.

8.4.2 TE Waves in Rectangular Waveguides

For TE waves, $E_z = 0$ and H_z satisfies the following equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_t^2\right) h_z(x, y) = 0, \tag{8.84a}$$

as well as the boundary conditions on the conducting walls:

$$\frac{\partial h_z}{\partial x} = 0 \quad (E_y = 0), \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a, \tag{8.84b}$$

$$\frac{\partial h_z}{\partial y} = 0 \quad (E_x = 0), \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b. \tag{8.84c}$$

The field solutions are then obtained as

$$h_z(x, y) = H_0 \cos(k_{xm}x)\cos(k_{yn}y), \tag{8.85}$$

where the eigenvalues are

$$k_{xm} = \frac{m\pi}{a}, \qquad k_{yn} = \frac{n\pi}{b}, \qquad m, n = 0, 1, 2, 3, \cdots \tag{8.86a}$$

and

$$k_{tmn}^2 = k_{xm}^2 + k_{yn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \tag{8.86b}$$

Other field components can be obtained from E_z by equations

$$e_x(x, y) = \frac{j\omega\mu}{k_{tmn}^2} k_{yn} H_0 \cos(k_{xm}x)\sin(k_{yn}y), \tag{8.87}$$

$$e_y(x, y) = -\frac{j\omega\mu}{k_{tmn}^2} k_{xm} H_0 \sin(k_{xm}x)\cos(k_{yn}y), \tag{8.88}$$

$$h_x(x, y) = \frac{\gamma}{k_{tmn}^2} k_{xm} H_0 \sin(k_{xm}x)\cos(k_{yn}y), \tag{8.89}$$

$$h_y(x, y) = \frac{\gamma}{k_{tmn}^2} k_{yn} H_0 \cos(k_{xm}x)\sin(k_{yn}y), \tag{8.90}$$

where γ is the same as for the TM modes in equation (8.80).

• **TE$_{mn}$ Mode:** Note that m or n (but not both) can be zero. The cutoff frequency has the same expression as (8.81).
• **Dominant TE Mode:** If $a > b$, then the dominant TE mode is TE$_{10}$; otherwise TE$_{01}$ mode is the dominant TE mode.
• **Dominant Mode for Rectangular Waveguides:** Since $m \neq 0$, $n \neq 0$ for TM modes, the dominant mode for rectangular waveguides (with lowest cutoff frequency) is TE$_{10}$ mode if $a > b$.

8.4.3 Attenuation in Rectangular Waveguides

For the all TE and TM modes, the attenuation due to dielectric losses is

$$\alpha_d = \frac{\sigma\eta}{2\sqrt{1-(f_c/f)^2}}. \tag{8.91}$$

The calculation for α_c due to the ohmic loss at the conducting wall is more involved. For the most important mode—TE$_{01}$ mode,

$$(\alpha_c)_{TE_{10}} = \frac{1}{\eta b}\sqrt{\frac{\pi f \mu_c}{\sigma_c[1-(f_c/f)^2]}}\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right]. \tag{8.92}$$

Similarly, for TM$_{11}$ mode,

$$(\alpha_c)_{TM_{11}} = \frac{2(b/a^2+a/b^2)\sqrt{\frac{\pi f\mu_c}{\sigma_c}}}{\eta ab(1/a^2+1/b^2)\sqrt{1-(f_c/f)^2}}. \tag{8.93}$$

8.5 Circular Waveguides

Waveguides are those with uniform circular cross sections. Similar to rectangular ones, circular waveguides do not support TEM waves since they are single-conductor hollow waveguides. However, both TM and TE waves can be supported by circular waveguides.

The partial differential equations for TM and TE waves are essentially the same as those for rectangular guides. The only difference is that they are now expressed in cylindrical coordinates since it is easier to impose boundary conditions on the circular wall with these coordinates.

Instead of sinusoidal functions for rectangular waveguides, wave solutions in circular waveguides are expressed in terms of special functions known as Bessel functions. We now first introduce Bessel's differential equation and Bessel functions before discussing on TM and TE waves in circular guides.

8.5.1 Bessel's Differential Equations and Bessel Functions

In cylindrical coordinates, the homogeneous Helmholtz equation for e_z can be written as

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial e_z}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 e_z}{\partial\phi^2}+k_t^2 e_z = 0, \tag{8.94}$$

where (ρ, ϕ, z) are the cylindrical coordinates. The equation for h_z is the same as (8.94).

The solution to equation (8.94) can be obtained by using the method of separation of variables as illustrated in the textbook. Here we will obtain the solution by making the following observation: since the solution $e_z(\rho, \phi)$ should be the same as $e_z(\rho, \phi + 2k\pi)$ where k is an integer (this is the periodicity of the solution), it can be written as

$$e_z(\rho, \phi) = R(\rho) \cos n\phi, \tag{8.95}$$

where n is an integer. We have chosen $\cos n\phi$ instead of $\sin n\phi$ or a combination of them because these other choices change only the location of the reference $\phi = 0$ angle. Substituting equation (8.95) into (8.94) yields the **Bessel's differential equation**

$$\frac{d^2 R(\rho)}{dr^2} + \frac{1}{\rho} \frac{dR(\rho)}{d\rho} + (k_t^2 - \frac{n^2}{\rho^2}) R(\rho) = 0. \tag{8.96}$$

The general solution to equation (8.96) is the combination of $J_n(k_t\rho)$ and $Y_n(k_t\rho)$,

$$R(\rho) = C_n J_n(k_t\rho) + D_n Y_n(k_t\rho), \tag{8.97}$$

where $J_n(k_t\rho)$ is **Bessel function of the first kind**, $Y_n(k_t\rho)$ is **Bessel function of the second kind**, and C_n and D_n are arbitrary constants to be determined by boundary conditions. The Bessel functions are given by

$$J_n(k_t\rho) = \sum_{m=0}^{\infty} \frac{(-1)^m (k_t\rho)^{n+2m}}{m!(n+m)! 2^{n+2m}}, \tag{8.98}$$

$$Y_n(k_t\rho) = \frac{(\cos n\pi) J_n(k_t\rho) - J_{-n}(k_t\rho)}{\sin n\pi}. \tag{8.99}$$

For hollow circular waveguides, $D_n = 0$ since $Y_n(k_t\rho)$ is singular (goes to infinity) at $r = 0$. Hence, for circular waveguides, the solution is expressed in terms of Bessel function of the first kind.

- $J_n(k_t\rho)$ **at** $r = 0$**:** At the origin $r = 0$, $J_0(0) = 1$ (for $n = 0$) and $J_n(0) = 0$ for $n \neq 0$.
- x_{np}—**Zeros of** $J_n(x)$**:** x_{np} are the arguments at which $J_n(x_{np}) = 0$.
- x'_{np}—**Zeros of** $J'_n(x)$**:** x'_{np} are the arguments at which $J'_n(x'_{np}) = 0$.

8.5.2 TM Waves in Circular Waveguides

For TM waves, $H_z = 0$, and

$$E_z(\rho, \phi, z) = e_z(\rho, \phi) e^{-\gamma z}, \tag{8.100}$$

where $e_z(\rho, \phi)$ satisfies (8.94). The solution for e_z is then given by

$$e_z(\rho, \phi) = C_n J_n(k_t\rho)) \cos n\phi. \tag{8.101}$$

Similar to fields in Cartesian coordinates, the traverse field components in cylindrical coordinates can be obtained from the z components. For TM modes, the transverse electric field $\mathbf{e}_t = \hat{r}e_\rho + \hat{\phi}e_\phi$ is related to e_z by

$$\mathbf{e}_t = -\frac{\gamma}{k_t^2}\Big(\hat{r}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big)e_z, \tag{8.102}$$

or specifically

$$e_\rho = -\frac{j\beta}{k_t}C_n J_n'(k_t\rho)\cos n\phi, \tag{8.103}$$

$$e_\phi = \frac{j\beta n}{k_t^2 r}C_n J_n(k_t\rho)\sin n\phi. \tag{8.104}$$

The magnetic field can be obtained by using equation (8.20) which gives

$$h_\rho = -\frac{j\omega\epsilon n}{k_t^2 r}C_n J_n(k_t\rho)\sin n\phi, \tag{8.105}$$

$$h_\phi = -\frac{j\omega\epsilon}{k_t}C_n J_n'(k_t\rho)\cos n\phi. \tag{8.106}$$

In the above, $\gamma = j\beta$, J_n' is the derivative of J_n with respect to its argument $(k_t\rho)$, and C_n is a coefficient which depends on the strength of the excitation.

The transverse wavenumber k_t takes discrete numbers, as in rectangular waveguides. This can be determined by the boundary conditions

$$e_z(\rho = a, \phi) = e_\phi(\rho = a, \phi) = 0. \tag{8.107}$$

Using (8.107) in (8.101), we have

$$J_n(k_t a) = 0. \tag{8.108}$$

Therefore,

$$k_t = \frac{x_{np}}{a}, \qquad p = 1, 2, 3, \cdots \tag{8.109}$$

where x_{np} is the p-th zero of the Bessel function $J_n(x)$.

• **TM$_{np}$ Mode:** The mode corresponding to the zero x_{np}.

• **Cutoff Frequency of TM$_{np}$ Mode:**

$$(f_c)_{TM_{np}} = \frac{x_{np}}{2\pi a\sqrt{\mu\epsilon}}. \tag{8.110}$$

Some of the roots x_{np} are given in the textbook. For examples, $x_{01} = 2.405$, $x_{02} = 5.520$, $x_{11} = 3.832$, $x_{12} = 7.016$.

• **Significance of Mode Indices np:** n represents the number of half-wave field variations in the ϕ direction, and p represents the number of half-wave field variations in the r direction.

• **TM$_{01}$ Mode:** Since $x_{01} = 2.405$, TM$_{01}$ mode has a cutoff frequency of

$$(f_c)_{TM_{01}} = \frac{2.405}{2\pi a\sqrt{\mu\epsilon}}. \tag{8.111}$$

8.5.3 TE Waves in Circular Waveguides

For TE waves, $E_z = 0$, and

$$H_z(\rho, \phi, z) = h_z(\rho, \phi)e^{-\gamma z}, \tag{8.112}$$

where $h_z(\rho, \phi)$ satisfies (8.94) with e_z replaced by h_z. The solution for h_z is then given by

$$h_z(\rho, \phi) = C'_n J_n(k_t \rho) \cos n\phi. \tag{8.113}$$

For TE modes, the transverse magnetic field $\mathbf{h}_t = \hat{r}h_\rho + \hat{\phi}h_\phi$ and electric field components are

$$h_\rho = -\frac{j\beta}{k_t} C'_n J'_n(k_t\rho) \cos n\phi, \tag{8.114}$$

$$h_\phi = \frac{j\beta n}{k_t^2 r} C'_n J_n(k_t\rho) \sin n\phi, \tag{8.115}$$

$$e_\rho = \frac{j\omega\mu n}{k_t^2 r} C'_n J_n(k_t\rho) \sin n\phi, \tag{8.116}$$

$$e_\phi = \frac{j\omega\mu}{k_t} C'_n J'_n(k_t\rho) \cos n\phi, \tag{8.117}$$

where C'_n is a coefficient which depends on the strength of the excitation.

The transverse wavenumber k_t takes discrete numbers, as in rectangular waveguides. This can be determined by the boundary conditions

$$e_\phi(\rho = a, \phi) = 0. \tag{8.118}$$

Using (8.118) in (8.117), we have

$$J'_n(k_t a) = 0. \tag{8.119}$$

Therefore,

$$k_t = \frac{x'_{np}}{a}, \qquad p = 1, 2, 3, \cdots \tag{8.120}$$

where x'_{np} is the p-th zero of the derivative of Bessel function $J'_n(x)$.

• **TE$_{np}$ Mode:** The mode corresponding to the zero x'_{np}.

• **Cutoff Frequency of TE$_{np}$ Mode:**

$$(f_c)_{TE_{np}} = \frac{x'_{np}}{2\pi a\sqrt{\mu\epsilon}}. \tag{8.121}$$

Some of the roots x'_{np} are given in the textbook. For examples, $x'_{01} = 3.832$, $x'_{02} = 7.016$, $x'_{11} = 1.841$, $x'_{12} = 5.331$.

• **Significance of Mode Indices np:** n represents the number of half-wave field variations in the ϕ direction, and p represents the number of half-wave field variations in the r direction.

• **TE$_{11}$ Mode:** Since $x'_{11} = 1.841$, TE$_{11}$ mode has a cutoff frequency of

$$(f_c)_{TE_{11}} = \frac{1.841}{2\pi a\sqrt{\mu\epsilon}}. \tag{8.122}$$

• **Dominant Mode in a Circular Waveguide:** From equations (8.110) and (8.121), it it seen that the smallest x_{np} or x'_{np} corresponds to the dominant mode since it has the lowest cutoff frequency. Since x'_{11} is the smallest, TE$_{11}$ mode is the dominant mode in a circular waveguide.

8.6 Dielectric Waveguides

In addition to metallic waveguides, dielectric waveguides can also support electromagnetic waves. In this section we will discuss the simplest dielectric waveguides, i.e., dielectric slabs. For simplicity, we assume that the dielectric slabs is infinite in size in both x and z directions, and the parameters for 3 layers of dielectric media are (μ_0, ϵ_0), (μ_d, ϵ_d), and (μ_0, ϵ_0) respectively. The interfaces of the layers are located at $y = -d/2$ and $y = d/2$ respectively. This dielectric slab is a special case of the class of dielectric waveguides.

8.6.1 TM Waves Along A Dielectric Slab

For transverse magnetic waves, $H_z = 0$, and $E_z \neq 0$ satisfies the following equation

$$\frac{d^2 e_z(y)}{dy^2} + k_t^2 e_z(y) = 0, \tag{8.123}$$

where

$$k_t^2 = \gamma^2 + \omega^2\mu\epsilon. \tag{8.124}$$

In the above, for lossless dielectric waveguides, $\gamma = j\beta$. Equation (8.123) governs the field in the three layers in the dielectric slab. The general solution to this equation is exponential functions or sinusoidal functions. However, in order for the waves not to radiate away from the waveguide, the field must decay exponentially away from the slab. Therefore, inside the slab,

$$e_z(y) = E_o \sin k_y y + E_e \cos k_y y, \qquad |y| \leq d/2, \tag{8.125}$$

where

$$k_y^2 = \omega^2\mu_d\epsilon_d - \beta^2. \tag{8.126}$$

Outside the slab,

$$e_z(y) = \begin{cases} \left(E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\right) e^{-\alpha(y-d/2)}, & y \geq d/2, \\ \left(-E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\right) e^{\alpha(y+d/2)}, & y \leq -d/2, \end{cases} \tag{8.127}$$

where

$$\alpha^2 = \beta^2 - \omega^2\mu_0\epsilon_0 = \omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0) - k_y^2. \tag{8.128}$$

In equation (8.127), the coefficients for the exponential functions are chosen so the E_z is continuous at the slab interfaces.

The other field components can be found from the e_z component using equations (8.2)–(8.5). The nonzero transverse components are given by

$$e_y(y) = \begin{cases} -\frac{j\beta}{\alpha}\Big(E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\Big)e^{-\alpha(y-d/2)}, & y \geq d/2, \\ -\frac{j\beta}{k_y}(E_o \cos k_y y - E_e \sin k_y y), & |y| \leq d/2, \\ \frac{j\beta}{\alpha}\Big(-E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\Big)e^{\alpha(y+d/2)} & y \leq -d/2 \end{cases} \quad (8.129)$$

$$h_x(y) = \begin{cases} \frac{j\omega\epsilon_0}{\alpha}\Big(E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\Big)e^{-\alpha(y-d/2)}, & y \geq d/2, \\ \frac{j\omega\epsilon_d}{k_y}(E_o \cos k_y y - E_e \sin k_y y), & |y| \leq d/2, \\ -\frac{j\omega\epsilon_0}{\alpha}\Big(-E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\Big)e^{\alpha(y+d/2)}, & y \leq -d/2. \end{cases} \quad (8.130)$$

From (8.130), the continuity condition for the tangential component of magnetic field h_x at $y = d/2$ and at $y = -d/2$ requires

$$\begin{cases} \Big(\frac{\epsilon_d}{k_y} - \frac{\epsilon_0}{\alpha}\tan\frac{k_y d}{2}\Big)E_o - \Big(\frac{\epsilon_d}{k_y}\tan\frac{k_y d}{2} + \frac{\epsilon_0}{\alpha}\Big)E_e & = 0, \\ \Big(\frac{\epsilon_d}{k_y} - \frac{\epsilon_0}{\alpha}\tan\frac{k_y d}{2}\Big)E_o + \Big(\frac{\epsilon_d}{k_y}\tan\frac{k_y d}{2} + \frac{\epsilon_0}{\alpha}\Big)E_e & = 0. \end{cases} \quad (8.131)$$

Only under the following two conditions do this set of equations can be satisfied:

- **(i) Odd TM Modes:**

In this case, $E_e = 0$ but $E_o \neq 0$, and

$$\Big(\frac{\epsilon_d}{k_y} - \frac{\epsilon_0}{\alpha}\tan\frac{k_y d}{2}\Big) = 0,$$

or equivalently

$$[\omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0) - k_y^2]^{1/2} = \frac{\epsilon_0}{\epsilon_d}k_y \tan\frac{k_y d}{2}. \quad (8.132)$$

The roots for this transcendental equation (8.132) give the discrete values of k_y. In contrast to metallic waveguides, there are only a finite number of possible modes. Since $E_o = 0$, the E_z component is an odd function of y.

For odd TM modes, since the tangential electric field components are zero at $y = 0$, a perfectly conducting plane can be introduced at $y = 0$ without affecting the field distribution. Hence, odd TM modes propagating along a dielectric slab of thickness d are the same as those of the corresponding TM modes supported by a dielectric slab of thickness $d/2$ that is backed by a PEC plane.

Cutoff Frequency: The cutoff frequency for dielectric waveguides has to be determined by the wave behavior outside the slab based on the attenuation constant α in (8.128). When this attenuation becomes zero, the frequency is called the **cutoff frequency** since the waves are no longer bound to the slab. Hence, at the cutoff frequency f_{co} for the odd TM modes, $\alpha = 0$ and $\beta = (2\pi f_{co})\sqrt{\mu_0\epsilon_0}$, $k_y = (2\pi f_{co})\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}$. From (8.132), we have

$$\tan\Big(\frac{\omega_{co}d}{2}\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}\Big) = 0,$$

or

$$f_{co} = \frac{(n-1)}{d\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}}, \qquad n = 1, 2, 3, \cdots \tag{8.133}$$

The lowest odd TM mode, TM_1 mode ($n = 1$), has a zero cutoff frequency. Therefore, TM_1 mode can propagate along a dielectric-slab waveguide regardless of the thickness of the slab.

• **(ii) Even TM Modes:**

The other solution for equation (8.131) is that $E_o = 0$ but $E_e \neq 0$, and

$$\left(\frac{\epsilon_d}{k_y} \tan\frac{k_y d}{2} + \frac{\epsilon_0}{\alpha}\right) = 0,$$

or equivalently

$$[\omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0) - k_y^2]^{1/2} = -\frac{\epsilon_0}{\epsilon_d} k_y \cot\frac{k_y d}{2}. \tag{8.134}$$

Since $E_o = 0$, the E_z component is an even function of y.

Similarly, the cutoff frequency is

$$f_{ce} = \frac{(n-\frac{1}{2})}{d\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}}, \qquad n = 1, 2, 3, \cdots \tag{8.135}$$

Only the waves with frequencies higher than the cutoff frequency can propagate in the slab waveguide.

8.6.2 TE Waves Along A Dielectric Slab

For transverse electric waves, $E_z = 0$, and h_z satisfies the following equation

$$\frac{d^2 h_z(y)}{dy^2} + k_t^2 h_z(y) = 0, \tag{8.136}$$

where k_t^2 is given by (8.124). Similar to the TM waves, the solution for h_z can be written as

$$h_z(y) = \begin{cases} \left(H_o \sin\frac{k_y d}{2} + H_e \cos\frac{k_y d}{2}\right) e^{-\alpha(y-d/2)}, & y \geq d/2, \\ H_o \sin k_y y + H_e \cos k_y y, & |y| \leq d/2, \\ \left(-H_o \sin\frac{k_y d}{2} + H_e \cos\frac{k_y d}{2}\right) e^{\alpha(y+d/2)}, & y \leq -d/2, \end{cases} \tag{8.137}$$

where k_y and α are given by (8.126) and (8.128) respectively.

The other field components can be found from the h_z component using equations (8.2)–(8.5). The nonzero transverse components are given by

$$h_y(y) = \begin{cases} -\frac{j\beta}{\alpha}\left(H_o \sin\frac{k_y d}{2} + H_e \cos\frac{k_y d}{2}\right) e^{-\alpha(y-d/2)}, & y \geq d/2, \\ -\frac{j\beta}{k_y}(H_o \cos k_y y - H_e \sin k_y y), & |y| \leq d/2, \\ \frac{j\beta}{\alpha}\left(-H_o \sin\frac{k_y d}{2} + H_e \cos\frac{k_y d}{2}\right) e^{\alpha(y+d/2)}, & y \leq -d/2, \end{cases} \tag{8.138}$$

$$e_x(y) = \begin{cases} -\frac{j\omega\mu_0}{\alpha}\Big(H_o \sin\frac{k_y d}{2} + H_e \cos\frac{k_y d}{2}\Big)e^{-\alpha(y-d/2)}, & y \geq d/2, \\ -\frac{j\omega\mu_d}{k_y}(H_o \cos k_y y - H_e \sin k_y y), & |y| \leq d/2, \\ \frac{j\omega\mu_0}{\alpha}\Big(-H_o \sin\frac{k_y d}{2} + H_e \cos\frac{k_y d}{2}\Big)e^{\alpha(y+d/2)}, & y \leq -d/2. \end{cases} \tag{8.139}$$

Similar to TM waves, the continuity condition for the tangential component of electric field e_x at $y = d/2$ and at $y = -d/2$ requires

$$\begin{cases} \Big(\frac{\mu_d}{k_y} - \frac{\mu_0}{\alpha}\tan\frac{k_y d}{2}\Big)E_o - \Big(\frac{\mu_d}{k_y}\tan\frac{k_y d}{2} + \frac{\mu_0}{\alpha}\Big)E_e & = 0, \\ \Big(\frac{\mu_d}{k_y} - \frac{\mu_0}{\alpha}\tan\frac{k_y d}{2}\Big)E_o + \Big(\frac{\mu_d}{k_y}\tan\frac{k_y d}{2} + \frac{\mu_0}{\alpha}\Big)E_e & = 0. \end{cases} \tag{8.140}$$

Only under the following two conditions do this set of equations can be satisfied:

- **(i) Odd TE Modes:**

In this case, $H_e = 0$ but $H_o \neq 0$, and

$$\Big(\frac{\mu_d}{k_y} - \frac{\mu_0}{\alpha}\tan\frac{k_y d}{2}\Big) = 0,$$

or equivalently

$$[\omega^2(\mu_d\mu_d - \mu_0\mu_0) - k_y^2]^{1/2} = \frac{\mu_0}{\mu_d}k_y \tan\frac{k_y d}{2}. \tag{8.141}$$

The roots for this transcendental equation (8.141) give the discrete values of k_y. Since $H_o = 0$, the H_z component is an odd function of y.

Cutoff Frequency: The cutoff frequency for odd TE mode is the same as that for odd TM mode given by (8.133). The lowest odd TE mode, TE_1 mode ($n = 1$), has a zero cutoff frequency. Therefore, TE_1 mode can propagate along a dielectric-slab waveguide regardless of the thickness of the slab.

- **(ii) Even TE Modes:**

The other solution for equation (8.140) is that $H_o = 0$ but $H_e \neq 0$, and

$$\Big(\frac{\mu_d}{k_y}\tan\frac{k_y d}{2} + \frac{\mu_0}{\alpha}\Big) = 0,$$

or equivalently

$$[\omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0) - k_y^2]^{1/2} = -\frac{\mu_0}{\mu_d}k_y \cot\frac{k_y d}{2}. \tag{8.142}$$

Since $H_o = 0$, the H_z component is an even function of y.

Similarly, the cutoff frequency for even TE modes is the same as for even TM modes given by (8.135).

8.7 Cavity Resonators

At UHF and higher frequencies, resonant circuits using R, L, and C elements become difficult because of the skin effect and radiation of the elements. In this section rectangular and circular cavity resonators are discussed.

8.7.1 Rectangular Cavity Resonators

If a rectangular resonator has dimensions a, b, and d, we can define a Cartesian coordinate system such that the cavity walls are located at $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = d$. With respect to the z direction, we can have both TM_{mnp} and TE_{nmnp} modes, where mnp refer to the wave pattern in x, y, and z directions.

The difference between rectangular cavity and rectangular waveguides is the waves are no longer propagating in the z direction in cavities. They become standing waves. Therefore, instead of the $e^{\pm\gamma z}$ factor, the possible factor in z are $\cos\beta z$ and $\sin\beta z$, depending on the appropriate boundary conditions (zero tangential electric field components) at $z = 0$ and $z = d$.

• **TM**$_{mnp}$ **Modes:** For TM modes, $H_z = 0$ but $E_z \neq 0$. The boundary conditions require that the form of TM_{mnp} modes to be

$$E_z(x, y, z) = E_0 \sin\left(\frac{m\pi}{a}\right) \sin\left(\frac{n\pi}{b}\right) \cos\left(\frac{p\pi}{d}\right). \tag{8.143}$$

The other field components can be found from E_z as

$$E_x(x, y, z) = -\frac{1}{k_t^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right) E_0 \cos\left(\frac{m\pi}{a}\right) \sin\left(\frac{n\pi}{b}\right) \sin\left(\frac{p\pi}{d}\right), \tag{8.144}$$

$$E_y(x, y, z) = -\frac{1}{k_t^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right) E_0 \sin\left(\frac{m\pi}{a}\right) \cos\left(\frac{n\pi}{b}\right) \sin\left(\frac{p\pi}{d}\right), \tag{8.145}$$

$$H_x(x, y, z) = \frac{j\omega\epsilon}{k_t^2}\left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}\right) \cos\left(\frac{n\pi}{b}\right) \cos\left(\frac{p\pi}{d}\right), \tag{8.146}$$

$$H_y(x, y, z) = -\frac{j\omega\epsilon}{k_t^2}\left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}\right) \sin\left(\frac{n\pi}{b}\right) \cos\left(\frac{p\pi}{d}\right), \tag{8.147}$$

where

$$k_t^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \tag{8.148}$$

In the above, $m, n = 1, 2, 3, \cdots$ and $p = 0, 1, 2, \cdots$.

From the definition $k_t^2 = k^2 - \beta^2$, we obtain the **resonant frequency** for TM_{mnp} mode of the cavity

$$f_{mnp} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}. \tag{8.149}$$

• **TE**$_{mnp}$ **Modes:** For TE modes, $E_z = 0$ but $H_z \neq 0$. The boundary conditions require that the form of TE_{mnp} modes to be

$$H_z(x, y, z) = H_0 \cos\left(\frac{m\pi}{a}\right) \cos\left(\frac{n\pi}{b}\right) \sin\left(\frac{p\pi}{d}\right). \tag{8.150}$$

The other field components can be found from H_z as

$$H_x(x, y, z) = -\frac{1}{k_t^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right) H_0 \sin\left(\frac{m\pi}{a}\right) \cos\left(\frac{n\pi}{b}\right) \cos\left(\frac{p\pi}{d}\right), \tag{8.151}$$

$$H_y(x,y,z) = -\frac{1}{k_t^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)H_0 \cos\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi}{b}\right)\cos\left(\frac{p\pi}{d}\right), \qquad (8.152)$$

$$E_x(x,y,z) = \frac{j\omega\mu}{k_t^2}\left(\frac{n\pi}{b}\right)H_0 \cos\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi}{b}\right)\sin\left(\frac{p\pi}{d}\right), \qquad (8.153)$$

$$E_y(x,y,z) = -\frac{j\omega\mu}{k_t^2}\left(\frac{m\pi}{a}\right)H_0 \sin\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}\right)\sin\left(\frac{p\pi}{d}\right), \qquad (8.154)$$

where k_t^2 is given by (8.148). In the above, $m, n = 0, 1, 2, 3, \cdots$ (but m and n cannot be zero simultaneously) and $p = 1, 2, 3, \cdots$.

The **resonant frequency** for TE_{mnp} mode is the same as that for TM_{mnp} mode as given by (8.149).

- **Dominant Mode:** The modes with lowest resonant frequency.
- **Degenerate Modes:** different modes having the same resonant frequency. If m, n, p are all nonzero, then TM_{mnp} and TE_{mnp} modes are always degenerate.

B. Quality Factor of Cavity Resonators

The finite conductivity in the conducting walls of cavities results in power loss which causes a decay in the stored energy inside the cavities. We define a **quality factor** Q as the ratio

$$Q = \frac{\omega W}{P_L}, \qquad (8.155)$$

where ω is the resonant frequency, W is the time-averaged energy stored at the resonant frequency, and P_L is the time-averaged power dissipated in the cavity.

For TE_{101} mode, we can find that

$$Q_{101} = \frac{\pi f_{101}\mu_0 abd(a^2+d^2)}{R_s[2b(a^3+d^3)+ad(a^2+d^2)]}, \qquad (8.156)$$

where $R_s = \sqrt{\pi f \mu_c/\sigma_c}$ is the intrinsic resistance of the conducting wall.

8.7.2 Circular Cavity Resonators

TM_{mnp} and TE_{mnp} modes for circular cavities can also be analyzed with the same procedures as for the rectangular cavities. The resonant frequency for TM_{mnp} mode is

$$f_{TM_{mnp}} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}, \qquad (8.157)$$

where d is the dimension of the cavity in z direction, and x_{mn} is the n-th zero of $J_m(x)$.

Similarly, the resonant frequency for TE_{mnp} mode is

$$f_{TE_{mnp}} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}, \qquad (8.158)$$

where x'_{mn} is the n-th zero of $J'_m(x)$.

The quality factor for TM_{010} is given by

$$Q_{TM_{010}} = \left(\frac{\eta_0}{R_s}\right)\frac{2.405}{2(1+a/d)}. \tag{8.159}$$