Polytechnique Montréal
Département de Mathématiques et de Génie Industriel

MTH1102D - Calcul II
Été 2023

Devoir 3

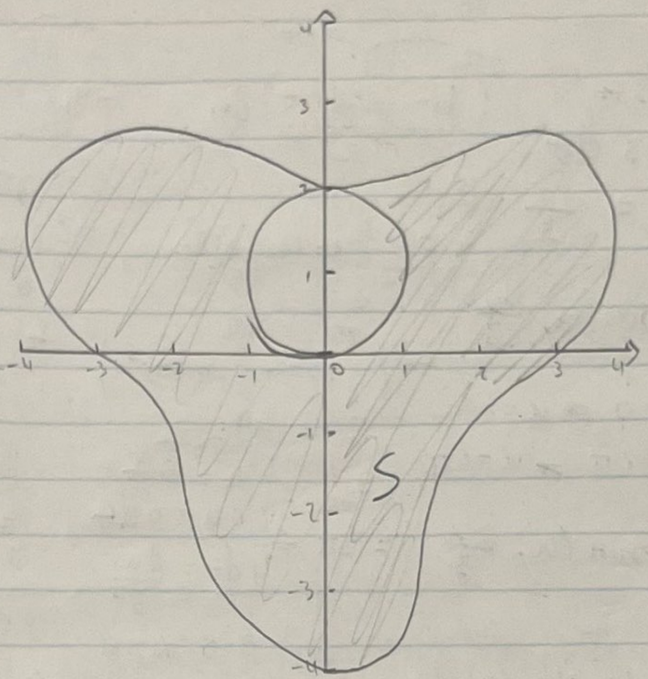
Nom : ______________________ **Prénom :** ______________________

Matricule : ____________________ **Groupe :** 01

Question corrigée	Autres questions	Total
3	4	7 /10

#1 $r = 2\sin(\theta) \quad r = 3 + \sin(3\theta)$

3



a) Aire de S

$r = 2\sin(\theta)$ Borne inf. $\quad r = 3+\sin(3\theta)$ Borne sup.

$$S = \{(r,\theta) \mid 2\sin(\theta) \le r \le 3+\sin(3\theta),\ 0 \le \theta \le 2\pi\}$$

$$\text{aire}(S) = \iint_D 1\, dA$$

séparer le domaine

$$\text{aire}(S) = \int_0^{\cancel{2\pi}} \int_{2\sin(\theta)}^{3+\sin(3\theta)} \cancel{c}\, dr\, d\theta$$

$(3+\sin(3\theta))(3+\sin(3\theta))$

$\to 9 + 6\sin(3\theta) + \sin^2(3\theta)$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{2\sin(\theta)}^{3+\sin(3\theta)} = \frac{1}{2}\int_0^{2\pi} (3+\sin(3\theta))^2 - (2\sin(\theta))^2\, d\theta$$

$\to (2\sin(\theta))(2\sin(\theta))$

$4\sin^2(\theta)$

$$= \frac{1}{2}\int_0^{2\pi} \left(9 + 6\sin(3\theta) + \sin^2(3\theta)\right) - \left(4\sin^2(\theta)\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 9 + 6\sin(3\theta) + \sin^2(3\theta) - 4\sin^2(\theta)\, d\theta$$

$$= \frac{1}{2}\left(\int_0^{2\pi} 9\,d\theta + 6\int_0^{2\pi} \sin(3\theta)\,d\theta + \int_0^{2\pi} \sin^2(3\theta)\,d\theta - 4\int_0^{2\pi} \sin^2(\theta)\,d\theta \right)$$

$$[9\theta]_0^{2\pi}$$
$$= 18\pi$$

$$u = 3\theta$$
$$du = 3\,d\theta$$
$$d\theta = \frac{du}{3}$$
$$= \int \sin(u)\,\frac{du}{3}$$
$$\theta = 0 \Rightarrow u = 0$$
$$\theta = 2\pi \Rightarrow u = 6\pi$$
$$= \int_0^{6\pi} \sin(u)\,\frac{du}{3}$$
$$= 6 \cdot \frac{1}{3}\int_0^{6\pi} \sin(u)\,du$$
$$= 2\left[-\cos(u)\right]_0^{6\pi}$$
$$= 2\left[-\underbrace{\cos(3\cdot 6\pi)}_{+1} - \underbrace{\cos(3\cdot 0)}_{-1}\right]$$
$$= 2 \cdot 0$$
$$= 0$$

$$\int_0^{2\pi} \frac{1-\cos(6\theta)}{2}\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} 1 - \cos(6\theta)\,d\theta$$
$$= \frac{1}{2}\left(\int_0^{2\pi} 1\,d\theta - \int_0^{2\pi} \cos(6\theta)\,d\theta\right)$$
$$[\theta]_0^{2\pi} = 2\pi$$
$$u = 6\theta$$
$$du = 6\,d\theta$$
$$d\theta = \frac{du}{6}$$
$$\theta = 0 \quad u = 0$$
$$\theta = 2\pi \quad u = 12\pi$$
$$\int_0^{12\pi} \cos(u)\,\frac{du}{6}$$
$$= \frac{1}{6}\left[\underbrace{\sin(12\pi)}_{0} - \underbrace{\sin(0)}_{0}\right]$$
$$= 0$$
$$\Rightarrow \frac{1}{2}(2\pi)$$
$$= \pi$$

$$-4\int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\,d\theta$$
$$-2\int_0^{2\pi} 1 - \cos(2\theta)\,d\theta$$
$$-2\left(\int_0^{2\pi} 1\,d\theta - \int_0^{2\pi} \cos(2\theta)\,d\theta\right)$$
$$[\theta]_0^{2\pi} = 2\pi$$
$$u = 2\theta$$
$$du = 2\,d\theta$$
$$d\theta = \frac{du}{2}$$
$$\theta = 0 \quad u = 0$$
$$\theta = 2\pi \quad u = 4\pi$$
$$-\int_0^{4\pi} \cos(u)\,\frac{du}{2}$$
$$-\frac{1}{2}\left[\sin(8\pi) - \sin(0)\right]$$
$$= 0$$
$$= -2 \cdot 2\pi$$
$$= -4\pi$$

$$= \frac{1}{2}\left(18\pi + 0 + \pi - 4\pi\right)$$
$$= \cancel{\frac{15\pi}{2}}$$

2/4

b) $\rho(r) = k \cdot r^2$

$\sqrt{x^2+y^2}$

$\sqrt{r^2} = r$ $\quad \rho(x,y,z) = k\sqrt{x^2+y^2} = k \cdot r^2$

$$\int_0^{\cancel{2\pi}} \int_{2\sin\theta}^{3+\sin(3\theta)} k\, r^2 \, dr\, d\theta \quad \text{integrale} = \frac{r^3}{3}$$

$$= \frac{k}{3}\int_0^{2\pi} (3+\sin(3\theta))^3 - (2\sin\theta)^3 \, d\theta$$

$$= \frac{k}{3}\int_0^{2\pi} \left[27 + 27\sin(3\theta) + 9\sin^2(3\theta) + \sin^3(3\theta)\right] - \left[8\sin^3(\theta)\right] d\theta$$

$$= \frac{k}{3}\int_0^{2\pi} \underbrace{27}_{=27\theta} + \underbrace{27\sin(3\theta)}_{27\cdot\frac{1}{3}\int \sin(u)\,du} + \underbrace{9\sin^2(3\theta)}_{9\cdot\frac{1}{2}\left(\int d\theta - \int\cos(6\theta)d\theta\right)} + \underbrace{\sin^3(3\theta)}_{\frac{1}{3}\int (1-\cos^2(u))\sin(u)\,du} - 8\sin^3(\theta)\, d\theta$$

$= -9\cos(3\theta)$ $\quad = \frac{9}{2}\left(\theta - \frac{1}{6}\sin(6\theta)\right)$

$\frac{1}{3}\int -1 + v^2\, dv$ $\quad u = \cos(\theta)$, $\frac{du}{d\theta} = -\sin(\theta)$

$\frac{1}{3}\int -1\, dv + \int v^2 dv$

$\frac{1}{3}\left(-v + \frac{v^3}{3}\right)$

$= \frac{1}{3}\left(-\cos(3\theta) + \frac{\cos^3(3\theta)}{3}\right)$

$\frac{1}{2}$

$8\int (1-\cos^2(\theta))\sin(\theta)\,d\theta$ $\quad u = \cos\theta$, $du = -\sin(\theta)d\theta$

$8\int -\int du + \int u^2 du$

$8\left(-u + \frac{u^3}{3}\right)$

$= 8\left(-\cos(\theta) + \frac{\cos^3(\theta)}{3}\right)$

$$= \frac{k}{3}\left[27\theta - 9\cos(3\theta) + \frac{9}{2}\left(\theta - \frac{1}{6}\sin(6\theta)\right) + \frac{1}{3}\left(-\cos(3\theta) + \frac{\cos^3(3\theta)}{3}\right) - 8\left(-\cos(\theta) + \frac{\cos^3(\theta)}{3}\right)\right]_0^{2\pi}$$

$$= \frac{k}{3}\left[63\pi - \frac{35}{9} - \left(-\frac{35}{9}\right)\right] = \frac{k}{3}\cdot 63\pi = 21\cancel{\pi}k$$

$z = a^2 - x^2$ $\quad z = 10 - \frac{10x^2}{a^2}$ $\quad 0 < a < \sqrt{10}$

#2

$y = \pm 10$

a) $z = 0$

Type I $\quad E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

$E = \{(x,y,z) \mid (x,y) \in D,\ -10 \le y \le 10\}$

$D = \{(x,z) \mid -a \le x \le a,\ a^2 - x^2 \le y \le 10 - \frac{10x^2}{a^2}\}$

Points d'intersection pour $z = a^2 - x^2$ et $z = 10 - \frac{10x^2}{a^2}$

$a^2 - x^2 = 10 - \frac{10x^2}{a^2}$

$a^2 - x^2 + \frac{10x^2}{a^2} = 10$

$\frac{10x^2}{a^2} = 10 - a^2 + x^2$

$10x^2 = a^2x^2 - a^4 + 10a^2$

$x^2\cancel{(10 - a^2)} = a^2\cancel{(10 - a^2)}$

$x^2 = a^2$

$x = \pm a$

$\rho(x,y,z) = k z$ car densité proportionnelle à z

$m = \iiint_E \rho(x,y,z)\, dA$

$m = \iint_D \left[\int_{-10}^{10} k z\, dy\right] dA = \iint_D \left[k z y\right]_{-10}^{10} dA$

$$= \iint_D \left[10kz + 10kz \right] dA = \iint_D 20kz \, dA = \int_{-a}^{a} \int_{a^2-x^2}^{10-\frac{10x^2}{a^2}} 20kz \, dz \, dx$$

$$= \int_{-a}^{a} \left[\frac{20kz^2}{2} \right]_{a^2-x^2}^{10-\frac{10x^2}{a^2}} dx = \int_{-a}^{a} \left[\frac{20k\left(10-\frac{10x^2}{a^2}\right)^2}{2} - \frac{20k\left(a^2-x^2\right)^2}{2} \right] dx$$

$$= 10k \int_{-a}^{a} \left[\left(10 - \frac{10x^2}{a^2}\right)^2 - \left(a^2-x^2\right)^2 \right] dx = 10k \int_{-a}^{a} \left[\left(100 - \frac{200x^2}{a^2} + \frac{100x^4}{a^4} - a^4 + 2a^2x^2 + x^4\right) dx \right.$$

$$= 10k \left[\int_{-a}^{a} (100 - a^4) dx + \int_{-a}^{a} \left(2a^2 - \frac{200}{a^2}\right) x^2 dx + \int_{-a}^{a} x^4 \left(\frac{100}{a^4} + 1\right) dx \right.$$

$$= 10k \left[(100 - a^4)(x)\Big|_{-a}^{a} + \left(2a^2 - \frac{200}{a^2}\right)\left(\frac{x^3}{3}\right)\Big|_{-a}^{a} + \left(\frac{100}{a^4} + 1\right)\left(\frac{x^5}{5}\right)\Big|_{-a}^{a} \right]$$

symbolab

$$= 10k \left[\frac{2a}{5} - \frac{2a^5}{5} - \frac{400a}{3} + \frac{2a^5}{3} + 200a - 2a^5 \right]$$

$$= 10k \left[a\left(\frac{2}{5} - \frac{400}{3} + 200\right) - a^5\left(\frac{1}{3} - \frac{1}{5} + 2\right) \right]$$

$$= k\left(-25{,}3\, a^5 + 67{,}07\, a\right) \quad \text{calculatrice}$$

✓

b) Le solide B est symétrique en x ainsi que y.
Le CM est 0.

- $\bar{x} = \dfrac{\bar{M}_{yz}}{m} = \dfrac{1}{m}\iiint_V \rho(x,z,y)\, dy\, dz\, dx = 0$

- $\bar{y} = \dfrac{\bar{M}_{xz}}{m} = \dfrac{1}{m}\iiint_V \rho(x,z,y)\, dy\, dz\, dx = 0$

- $\bar{z} = \dfrac{\bar{M}_{xy}}{m} = \dfrac{1}{m}\iiint_V z\rho(x,z,y)\, dy\, dz\, dx$

$$= \iiint_V k z^2\, dy\, dz\, dx$$

Wolfram-Alfa

$$= \frac{640k}{105}(1625a - a^7)$$

$\bar{z}$ se situe entre 0 et 10

- $\bar{z} \Rightarrow \dfrac{\dfrac{640k}{105}(1625a - a^7)}{k(67{,}07a - 21{,}3a^5)} = \dfrac{9904{,}76 - \dfrac{640a^6}{105}}{67{,}07 - 21{,}3a^4}$

avec z entre 0 et 10

Condition sur a?