Polytechnique Montréal
Département de Mathématiques et de Génie Industriel

MTH1102D - Calcul II

Été 2023

Devoir 10

Nom : ______________________ **Prénom :** ______________________

Matricule : ____________________ **Groupe :** 01

Question corrigée	Autres questions	Total
4.5	4	8.5 /10

Devoir 10

#1

$z = 11 - 4x^2 - 4y^2$ située entre plan : $z = -1$ et plan $z = 2$

Évaluer intégrale :

$$J_1 = \iint_S (x^2+y^2)^{3/2} \, dS$$

On sait que l'intégrale de surface de f sur S est

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{R}(u,v)) \|\vec{R}_u \times \vec{R}_v\| \, dA$$

$$f(\vec{R}(u,v)) = f(x(u,v), y(u,v), z(u,v))$$

$$\Rightarrow \iint_S f(x,y,z)\, dS = \iint_D f(\vec{R}(r,\theta)) \|\vec{R}_r \times \vec{R}_\theta\| \, dA \quad D \in (r,\theta)$$

$$f(x,y,z) = (x^2+y^2)^{3/2}$$

$$\vec{R}(u,v) = (r^2)^{3/2} = r^3$$

$$\vec{R}(r,\theta) = r\cos\theta\, \vec{i} + r\sin\theta\, \vec{j} + g(r,\theta)\vec{k}$$

$$\vec{R}(r,\theta) = r\cos\theta\, \vec{i} + r\sin\theta\, \vec{j} + (11 - 4(r^2))\vec{k}$$

$$S : \begin{cases} \vec{R}_z = 0\vec{i} + 0\vec{j} + \vec{k} & r = \frac{\sqrt{z-11}}{2} \\ \vec{R}_\theta = -r\sin\theta\, \vec{i} + r\cos\theta\, \vec{j} + 0\vec{k} \end{cases}$$

$$\|\vec{R}_{\cancel{\theta}} \times \vec{R}_\theta\| = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

pas les bons paramètres

$$= -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + 0\vec{k}$$

$$\|\vec{R}_z \times \vec{R}_\theta\| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2}$$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

$$\int_0^{2\pi}\int_{-1}^{2} r^3 \cdot r \, dz\, d\theta = \int_0^{2\pi}\int_{-1}^{2} r^4 \, dz\, d\theta$$

$$\int_0^{2\pi}\int_{-1}^{2} \left(\frac{\sqrt{z-11}}{2}\right)^4 dz\, d\theta = \int_0^{2\pi}\int_{-1}^{2} \frac{(z-11)^2}{16}\, dz\, d\theta$$

$$= \frac{1}{16}\int_0^{2\pi}\int_{-1}^{2} z^2 - 22z + 121 \, dz\, d\theta$$

$$= \frac{1}{16}\left(\int_0^{2\pi} \underbrace{\left[\frac{z^3}{3}\right]_{-1}^{2}}_{3} + \int_0^{2\pi} \underbrace{22\left[\frac{z^2}{2}\right]_{-1}^{2}}_{-33} + \int_0^{2\pi} \underbrace{\left[121z\right]_{-1}^{2}}_{363}\right) d\theta$$

$$= \frac{1}{16}\int_0^{2\pi} 333 \, d\theta$$

X

$$= \frac{1}{16}\left[333\theta\right]_0^{2\pi} = \frac{1}{16}\cdot 666\pi = \boxed{\frac{333\pi}{8}}$$

Roppel

$$x^2+y^2=r^2$$
$$x=r\cos\theta$$
$$y=r\sin\theta$$

- $Z = -1 \leq z \leq 2$

- r : $-1 = 11-4x^2-4y^2$
  $-12 = -4(x^2+y^2)$
  $3 = x^2+y^2$
  $3 = r^2$
  $r = \sqrt{3}$

  $z = 11-4x^2-4y^2$
  $-9 = -4(x^2+y^2)$
  $\frac{9}{4} = r^2$
  $r = \frac{3}{2}$

$$\frac{3}{3} \leq r \leq \sqrt{3}$$

- θ : $0 \leq \theta \leq 2\pi$

Surface représentée et paramétrée par :

#2 $\vec{R}(u,v) = (4-u^2)\cos(v)\vec{i} + (4-u^2)\sin(v)\vec{j} + 2u\vec{k}$

$(u,v) \in [0,2] \times [0,2\pi]$

Orienté point $(3,0,2)$ par $\vec{n} = \frac{1}{\sqrt{2}}(\vec{i}+\vec{k})$

a) $J_z = \iint_S z\, dS$

$R_v = (u^2-4)\sin v\,\vec{i} + (4-u^2)\cos v\,\vec{j}$

$R_u = -2u\cos v\,\vec{i} - 2u\sin v\,\vec{j} + 2\vec{k}$

$$R_u \times R_v = \begin{vmatrix} i & j & k \\ -2u\cos(v) & -2u\sin(v) & 2 \\ (u^2-4)\sin v & (4-u^2)\cos v & 0 \end{vmatrix}$$

$= -2(4-u^2)\cos v\,\vec{i} + 2(u^2-4)\sin v\,\vec{j} + (-2u(-u^2+4)(\cos^2(v)+\sin^2(v)))\vec{k}$

$= -2(4-u^2)\cos v\,\vec{i} + 2(u^2-4)\sin v\,\vec{j} + (-2u(-u^2+4))\vec{k}$

$= -2(4-u^2)\cos v\,\vec{i} + 2(u^2-4)\sin v\,\vec{j} \underbrace{-2u(4-u^2)\vec{k}}_{+u(2u^2-8)}$

$\|R_u \times R_v\| = \sqrt{((2u^2-8)\cos v)^2 + ((2u^2-8)\sin v)^2 + (u(2u^2-8))^2}$

$= \sqrt{(2u^2-8)^2(\cos v)^2 + (2u^2-8)^2(\sin v)^2 + (u^2(2u^2-8)^2)}$

$= \sqrt{(2u^2-8)^2(\cos^2 v + \sin^2 v + u^2)}$

$= \sqrt{(2u^2-8)^2(1+u^2)}$

$= (2u^2-8)(\sqrt{1+u^2})$

$$\iint_D z\|R_u \times R_v\| \, dA \qquad z = 2u$$

$$\int_0^{2\pi}\int_0^2 2u(2u^2-8)(\sqrt{1+u^2})\,du\,dv$$

$$= \int_0^{2\pi}\int_0^2 (4u^3-16u)(\sqrt{1+u^2})\,du\,dv$$

$$= \int_0^{2\pi}\int_0^2 \underbrace{4u^3(\sqrt{1+u^2})}_{S_1} - \underbrace{16u(\sqrt{1+u^2})}_{S_2}\,du\,dv$$

$$\int_0^{2\pi}\int_0^2 4u^3(\sqrt{1+u^2})\,du\,dv - \int_0^{2\pi}\int_0^2 16u(\sqrt{1+u^2})\,du\,dv$$

$$\Rightarrow \int_0^2 4u^3\sqrt{1+u^2}\,du - \int_0^2 16u\sqrt{1+u^2}\,du$$

$w = 1+u^2$

$\frac{dw}{du} = 2u$

$\int u^3\sqrt{w}\,\frac{1}{2u}\,dw$

$4\int_1^5 \frac{(w-1)\sqrt{w}}{2}\,dw$

$4\cdot\frac{1}{2}\left(\int_1^5 w^{\frac{3}{2}}\,dw - \int_1^5 \sqrt{w}\,dw\right)$

$4\cdot\frac{1}{2}\left(10\sqrt{5} - \frac{2}{5} - \frac{10\sqrt{5}-2}{3}\right)$

$\Rightarrow 2\left(10\sqrt{5} - \frac{2}{5} - \frac{10\sqrt{5}-2}{3}\right)$

$w = \sqrt{1+u^2}$ $\qquad dw = \frac{u}{\sqrt{1+u^2}}\,du$

$\frac{dw}{du} = \frac{u}{\sqrt{1+u^2}}$ $\qquad du = \frac{\sqrt{1+u^2}}{u}\,dw$

$\int u\,w\,\frac{\sqrt{1+u^2}}{u}\,dw$

$\int u\,w\,\frac{w}{u}\,dw$

$\int w^2\,dw$

$\int_1^{\sqrt{5}} w^2\,dw$

$16\int_1^{\sqrt{5}} w^2\,dw$

$16\left[\frac{w^3}{3}\right]_1^{\sqrt{5}}$

$\Rightarrow \frac{16(5\sqrt{5}-1)}{3}$

$$= 2\left(10\sqrt{5} - \frac{2}{5} - \frac{10\sqrt{5}-2}{3}\right) - \frac{16(5\sqrt{5}-1)}{3} = \frac{88-200\sqrt{5}}{15}$$

$$\Rightarrow \int_0^{2\pi} \frac{88-200\sqrt{5}}{15}\,dv$$

$$\left[\frac{88-200\sqrt{5}}{15}\,v \right]_0^{2\pi} = \boxed{\frac{16\pi(25\sqrt{5}-11)}{15}}$$

2.5/2.5

b) $J_3 = \iint_S \vec{F} \cdot d\vec{S} \qquad \vec{F}(x,y,z) = y\vec{i} - x\vec{j} + z^2\vec{k}$.

$= \iint_S \vec{F} \cdot \vec{n}\, dS = \iint \vec{F}(\vec{R}(u,v)) \cdot \pm \{R_u \times R_v\}\, dA$

$F(x,y,z) = y\vec{i} - x\vec{j} + z^2\vec{k}$

$\vec{F}(\vec{R}(u,v)) = (4-u^2)\sin v\,\vec{i} - (4-u^2)\cos v\,\vec{j} + 4u^2\vec{k}$

$\vec{F}(\vec{R}(u,v)) \cdot \cancel{\vec{n}} = [(4-u^2)\sin v, -(4-u^2)\cos v, 4u^2] \cdot [\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]$

$= \dfrac{(4-u^2)\sin v + 4u^2}{\sqrt{2}}$

$\vec{n}$ n'est pas constant sous S

$J_3 = \int_0^{2\pi}\int_0^2 \dfrac{(4-u^2)\sin v}{\sqrt{2}} + \dfrac{4u^2}{\sqrt{2}}\, du\, dv$

$= \int_0^{2\pi} \frac{\sin v}{\sqrt{2}} \int_0^2 (4-u^2)\, du\, dv + \int_0^{2\pi} \frac{1}{\sqrt{2}} \int_0^2 4u^2\, du\, dv$

$\frac{1}{2.5}$

$= \int_0^{2\pi} \left[\frac{\sin v}{\sqrt{2}}\left(4u - \frac{u^3}{3}\right)\right]_0^2 dv + \int_0^{2\pi} \left[\frac{1}{\sqrt{2}} \frac{4u^3}{3}\right]_0^2 dv$

$= \left[\frac{1}{\sqrt{2}}(-\cos v)\right]_0^{2\pi} \cdot \left[4u - \frac{u^3}{3}\right]_0^2 + \left[\frac{2\pi}{\sqrt{2}} \frac{4u^3}{3}\right]_0^2$

$= \left[\frac{1}{\sqrt{2}}(0)\right]_0^{2\pi} \cdot \left(8 - \frac{2^3}{3}\right) + \left(\frac{2\pi}{\sqrt{2}} \frac{4(2)^3}{3}\right)$

$= 0 + \dfrac{64\pi}{3\sqrt{2}} = \boxed{\dfrac{64\pi}{3\sqrt{2}}}$ ✗