Polytechnique Montréal
Département de Mathématiques et de Génie Industriel

MTH1102D - Calcul II
Été 2023

Devoir 5

Nom : ______________________ **Prénom :** ______________________

Matricule : ____________________ **Groupe :** 01

Question corrigée	Autres questions	Total
6	4	10 /10

Devoir 5

#1

a) sphère : $x^2+y^2+z^2=4$

sphère : $x^2+y^2+(z-2)^2=4$

$$J_1 = \iiint_E \frac{z}{x^2+y^2+z^2}\,dV$$

On sait que : $\iiint_E f(x,y,z)\,dV =$

$$\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

avec $E=\{(\rho,\theta,\phi)\mid \rho_1\le\rho\le\rho_2, \theta_1\le\theta\le\theta_2, \phi_1\le\phi\le\phi_2\}$

θ : $\boxed{0\le\theta\le 2\pi}$

ρ : $x^2+y^2+(z-2)^2=4$

On sait que :
$x^2+y^2+(z-c)^2=c^2$
$\rho = 2c\cos\phi$

$\rho = 2\cdot 2\cdot\cos\phi$
$= 4\cos\phi$

$x^2+y^2+z^2=4$

On sait que :
$\rho=\sqrt{x^2+y^2+z^2}$

$\rho^2 = 4$
$\rho = 2$ $\quad\boxed{2\le\rho\le 4\cos\phi}$

ϕ : $2 = 4\cos\phi$
$\frac{2}{4} = \cos\phi$
$\frac{1}{2} = \cos\phi$ $\quad \phi = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$ $\quad\boxed{0\le\phi\le\frac{\pi}{3}}$

$$E = \left\{(\rho, \theta, \phi) \mid 2 \leq \rho \leq 4\cos\phi,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{3}\right\}$$

Évaluer intégrale :

$$\int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_2^{4\cos\phi} \frac{\rho\cos\phi}{\cancel{\rho^2}} \cancel{\rho^2} \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_2^{4\cos\phi} \rho\cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\rightarrow \int_2^{4\cos\phi} \rho \cos\phi \sin\phi \, d\rho$$

$$= \cos\phi \sin\phi \int_2^{4\cos\phi} \rho \, d\rho$$

$$= \cos\phi \sin\phi \left[\frac{\rho^2}{2}\right]_2^{4\cos\phi}$$

$$= \cos\phi \sin\phi \cdot \frac{1}{2} \left[\rho^2\right]_2^{4\cos\phi}$$

$$(4\cos\phi)(4\cos\phi)$$
$$16\cos^2\phi$$

$$= \frac{\cos\phi \sin\phi}{2} \left[(4\cos\phi)^2 - (2)^2\right]$$

$$= \frac{\cos\phi \sin\phi}{2} \left[16\cos^2\phi - 4\right]$$

$= \cos\phi \sin\phi \,(8\cos^2\phi - 2)$

$\rightarrow \int_0^{\frac{\pi}{3}} \cos\phi \sin\phi \,(8\cos^2\phi - 2)\, d\phi$

$= \int_0^{\frac{\pi}{3}} \cos\phi \sin\phi \cdot 8\cos^2\phi - 2\cos\phi\sin\phi \; d\phi$

$= 8\int_0^{\frac{\pi}{3}} \cos^3\phi \sin\phi \; d\phi - 2\int_0^{\frac{\pi}{3}} \cos\phi \sin\phi \; d\phi$

$u = \cos\phi$

$du = -\sin\phi \, d\phi$

$d\phi = \frac{-du}{\sin\phi}$

$\int u^3 \cancel{\sin\phi} \cdot \frac{-du}{\cancel{\sin\phi}}$

$\int -u^3 \, du$

$\phi = 0 \quad u = 1$

$\phi = \frac{\pi}{3} \quad u = \frac{1}{2}$

$\int_1^{\frac{1}{2}} -u^3 du$

$8\int_1^{\frac{1}{2}} -u^3 du$

$8\left(-\int_{\frac{1}{2}}^{1} -u^3 du\right)$

$8\left[\frac{u^4}{4}\right]_{\frac{1}{2}}^{1}$

$8\left[\cos^4(1) - \cos^4(\tfrac{1}{2})\right]$

$8 \cdot \frac{15}{64} = \boxed{\frac{15}{8}}$

$u = \sin\phi$

$du = \cos\phi \, d\phi$

$d\phi = \frac{du}{\cos\phi}$

$\int \cancel{\cos\phi} \frac{u}{\cancel{\cos\phi}} du$

$\int u \, du \qquad \phi = 0 \quad u = 0$

$\phi = \frac{\pi}{3} \quad u = \frac{\sqrt{3}}{2}$

$\int_0^{\frac{\sqrt{3}}{2}} u \, du$

$\left[\frac{u^2}{2}\right]_0^{\frac{\sqrt{3}}{2}}$

$\left[\frac{\left(\frac{\sqrt{3}}{2}\right)^2 - 0}{2}\right] = \frac{3}{8}$

$-2 \cdot \left(\frac{3}{8}\right)$

$= \boxed{-\frac{3}{4}}$

$\Rightarrow \frac{15}{8} - \frac{3}{4} = \frac{9}{8}$

$$\rightarrow \int_0^{2\pi} \frac{9}{8}\, d\theta = \frac{9}{8}\int_0^{2\pi} d\theta = \frac{9}{8}\left[\theta\right]_0^{2\pi}$$

$$= \boxed{\frac{9\pi}{4}}$$

b) Masse B ?

Solide B

sphère : $x^2+y^2+z^2=100$ $\qquad x \geq 0 \quad y \geq 0 \quad z \geq 0$

cylindre : $x^2+y^2=25$ $\qquad x^2+y^2 \geq 25$

Selon le devoir 4, j'avais calculé la région de B en coordonnées sphériques :

$$B = \left\{(\rho, \phi, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, \frac{\pi}{6} \leq \phi \leq \frac{5\pi}{6}, \underbrace{\frac{5}{\sin\phi}}_{5\csc\phi} \leq \rho \leq 10\right\}$$

Densité proportionnelle au carré de la distance à l'axe z.

Densité $\propto \left(\sqrt{x^2+y^2}\right)^2$

$$\rho(x,y,z) \propto (x^2+y^2)$$

$\rightarrow \rho(x,y,z)$ en coordonnées sphériques

$$\rho(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) = (\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)$$

$$m = \iiint_E \rho(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,dV$$

$$= \iiint_E \rho^2\sin^2\phi\cdot\rho^2\sin\phi\,dV$$

$$= \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{5\csc\phi}^{10} \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$$

$$\to \sin^3\phi\int_{5\csc\phi}^{10}\rho^4\,d\rho = \sin^3\phi\left[\frac{\rho^5}{5}\right]_{5\csc\phi}^{10}$$

$$= \frac{\sin^3\phi}{5}\left[\rho^5\right]_{5\csc\phi}^{10} = \sin^3\phi\left(20\,000 - 625\csc^5\phi\right)$$

$$\to \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 20\,000\sin^3\phi - 625\,\cancel{\sin^3\phi}\cdot\frac{1}{\sin^5\phi}\,d\phi$$

$$= 20\,000\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\sin^3\phi\,d\phi - 625\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\csc^2\phi\,d\phi$$

$$20\,000\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(1-\cos^2\phi)\sin\phi\,d\phi$$

$u = \cos\phi \quad du = -\sin\phi\,d\phi$

$d\phi = -\frac{du}{\sin\phi}$

$\int(1-u^2)\cdot\cancel{\sin\phi}\cdot\frac{-du}{\cancel{\sin\phi}}$

$\int -1+u^2\,du \qquad \phi = \frac{\pi}{6} \quad u = \frac{\sqrt{3}}{2}$

$\qquad\qquad\qquad \phi = \frac{5\pi}{6} \quad u = -\frac{\sqrt{3}}{2}$

$$\int_{\frac{\sqrt{3}}{2}}^{-\frac{\sqrt{3}}{2}} -1+u^2\,du$$

$$20\,000\left(-\left(-\underbrace{\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} 1\,du}_{[u]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}} + \underbrace{\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} u^2\,du}_{\left[\frac{u^3}{3}\right]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}}\right)\right)$$

$$-625\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\frac{1}{\sin^2\phi}\,d\phi$$

$$-625\left[\cot\phi\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$-625\left(2\sqrt{3}\right)$$

$$1250\sqrt{3}$$

$20\ 000\left(-\left(-\sqrt{3}+\frac{\sqrt{3}}{4}\right)\right)$

$15\ 000\sqrt{3}$

$= 15\ 000\sqrt{3} - 1\ 250\sqrt{3}$

$= 13750\sqrt{3}$

$\longrightarrow \int_0^{\frac{\pi}{2}} 13750\sqrt{3}\ d\theta = 13750\sqrt{3}\int_0^{\frac{\pi}{2}} 1\ d\theta$

$= 13750\sqrt{3}\ \left[\theta\right]_0^{\frac{\pi}{2}}$

$= \boxed{6875\pi\sqrt{3}}$ } la masse du solide B

✓

#2

$y = x^2 \quad y = 16x^2 \quad y = \frac{1}{x^2} \quad y = \frac{81}{x^2}$

6

a) Calculer jacobien

$$x = uv \,, \quad y = \frac{u^2}{v^2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{2u}{v^2} & \frac{-2u^2}{v^3} \end{vmatrix}$$

$$\left(\frac{1}{v^2}u^2\right)' = \frac{1}{v^2}\cdot 2u \qquad (u^2 v^{-2})' = \frac{-2u^2}{v^3}$$

$$= \left(\cancel{v} \cdot \frac{-2u^2}{v^3}\right) - \left(\frac{2u}{v^2}\cdot u\right) = \left(\frac{-2u^2}{v^2}\right) - \left(\frac{2u^2}{v^2}\right)$$

$$= -\frac{4u^2}{v^2}$$

b) Montrer que le changement de variable transformé D en un rectangle dans le plan des nouvelles variables (u, v). u, v sont positives

- $y = x^2$

$\frac{u^2}{v^2} = (uv)^2$

$\frac{u^2}{v^2} = u^2v^2$

$u^2 = u^2v^2\cdot v^2$

$\frac{u^2}{u^2} = v^4$

$1 = v^4$

$\boxed{v = 1}$

- $y = 16x^2$

$\frac{u^2}{v^2} = 16\cdot(uv)^2$

$\frac{u^2}{v^2} = 16u^2v^2$

$u^2 = 16u^2v^4$

$\frac{\cancel{u^2}}{16\cancel{u^2}} = v^4$

$\frac{1}{16} = v^4$

$\boxed{v = \frac{1}{2}}$

- $y = \frac{1}{x^2}$

$\frac{u^2}{v^2} = \frac{1}{u^2 v^2}$

$\frac{u^4 \cancel{v^2}}{\cancel{v^2}} = 1$

$u^4 = 1$

$\boxed{u = 1}$

- $y = \frac{81}{x^2}$

$\frac{u^2}{v^2} = \frac{81}{u^2 v^2}$

$\frac{u^4 \cancel{v^2}}{\cancel{v^2}} = 81$

$u^4 = 81$

$\boxed{u = 3}$

$\underbrace{[1, 3]}_{u} \quad \underbrace{[\frac{1}{2}, 1]}_{v}$

$D = \{(u,v) \mid 1 \le u \le 3, \frac{1}{2} \le v \le 1\}$

Cette région c'est un rectangle. Les bornes sont constantes pour les variables u et v.

c) $$J_2 = \iint_D \frac{x^2}{y}\, dA$$

On sait que :

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$f(x,y) = f\left(uv, \frac{u^2}{v^2}\right) = \frac{(uv)^2}{\frac{u^2}{v^2}} = \frac{u^2v^2}{\frac{u^2}{v^2}} = \frac{\cancel{u^2}v^2}{1} \cdot \frac{v^2}{\cancel{u^2}}$$

$$= v^4$$

Jacobien : $-\frac{4u^2}{v^2}$

$$J_2 = \int_1^3 \int_{1/2}^1 v^4 \cdot \left| -\frac{4u^2}{v^2} \right| du\,dv = \int_1^3 \int_{1/2}^1 4u^2v^2\,du\,dv$$

$$\rightarrow 4v^2 \int_1^3 u^2\,du = \frac{4}{3}v^2 \left[u^3\right]_1^3 = \frac{4v^2}{3}[26]$$

$$= \frac{104v^2}{3}$$

$$\rightarrow \frac{104}{3} \int_{1/2}^1 v^2\,dv = \frac{104}{9}\left[v^3\right]_{1/2}^1 = \frac{104}{9}\left[\frac{7}{8}\right]$$

$$= \boxed{\frac{91}{9}}$$