MTH1102D Calcul II

Chapitre 7, section 5 :

Changement de variables : exemple 1

Introduction

- Calcul du jacobien en coordonnées cylindriques

Jacobien en coordonnées cylindriques

Pour les coordonnées cylindriques, on a

$$x = x(r, \theta, z) = r\cos\theta, \quad y = y(r, \theta, z) = r\sin\theta, \quad z = z(r, \theta, z) = z$$

ou encore

$$(x, y, z) = T(r, \theta, z) = (r\cos\theta, r\sin\theta, z).$$

Jacobien en coordonnées cylindriques (suite)

Le jacobien est donc

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

Jacobien en coordonnées cylindriques (suite)

Le jacobien est donc

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\[2ex] \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial r\cos\theta}{\partial r} & \dfrac{\partial r\cos\theta}{\partial \theta} & \dfrac{\partial r\cos\theta}{\partial z} \\[2ex] \dfrac{\partial r\sin\theta}{\partial r} & \dfrac{\partial r\sin\theta}{\partial \theta} & \dfrac{\partial r\sin\theta}{\partial z} \\[2ex] \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix}$$

Jacobien en coordonnées cylindriques (suite)

Le jacobien est donc

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} & \frac{\partial r\cos\theta}{\partial z} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial \theta} & \frac{\partial r\sin\theta}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos(\theta) & -r\sin\theta & 0 \\ \sin(\theta) & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Jacobien en coordonnées cylindriques (suite)

Le jacobien est donc

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} & \frac{\partial r\cos\theta}{\partial z} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial \theta} & \frac{\partial r\sin\theta}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos(\theta) & -r\sin\theta & 0 \\ \sin(\theta) & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \cdot \left(r\cos^2\theta + r\sin^2\theta \right) = r.$$

Résumé

- Calcul du jacobien en coordonnées cylindriques.