# Modeling and Optimization with OPL 1 Introduction to linear Optimization

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### 1.1 Modeling

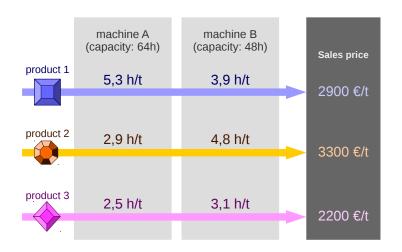
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# 1.1 Modeling

# Example: production problem (Lewig Sanstetten)



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#### 1.1 Modeling

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How much of each product should be produced?

## What is a model?

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- abstract description of a real system
- for exmaple useful for
  - system design
  - performance analysis
  - decision support
- guideline: as exact as necessary, as simple as possible

# Mathematical optimization models for decision support

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Components of a mathematical optimization model<sup>1</sup>:

decision variables system items which can be set by the decision maker

in the example: the production quantities of the products

constraints restrictions which have to be satisfied by the decision variables to get a proper solution

in the example: the capacity of the machines may not be exceeded

objective function a function of the decision variables, which the decision maker has to optimize, i.e. minimize or maximise

▶ in the example: the total revenue

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<sup>&</sup>lt;sup>1</sup>also: mathematical program

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 $x_1$  production quantity of product 1

 $x_2$  production quantity of product 2

x<sub>3</sub> production quantity of product 3

## Definition: Solution of an optimization model

By assigning specific values to the decision variables, we get a solution for the model. Capacity of machine A may not be exceeded:

$$5, 3 \cdot x_1 + 2, 9 \cdot x_2 + 2, 5 \cdot x_3 \le 64$$

Capacity of machine B may not be exceeded:

$$3,9 \cdot x_1 + 4,8 \cdot x_2 + 3,1 \cdot x_3 \leq 48$$

## Definition: Feasible Solution of an optimization model

A solution which satisfies all constraints is called a feasible solution. The set of all feasible solution is called solution space.

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Maximize the revenue (in  $k \in$ ):

 $max U(x_1, x_2, x_3) = 2, 9 \cdot x_1 + 3, 3 \cdot x_2 + 2, 2 \cdot x_3$ 

Definition: Optimal solution and optimal value of an optimization model

If there is a feasible solution, in which the objective function assumes its maximum resp. its minimum (which might not be the case at all), this solution is an optimal solution and its objective function value is the optimal value.

# Example: production problem – complete optimization model

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# 1.2 Linear Optimization

### Linear functions

$$f(x_1,\ldots,x_N)=\sum_{n=1}^N c_n\cdot x_n$$

### Linear constraints

Let f be a linear function:

$$f(x_1,...,x_N) = b$$
  

$$f(x_1,...,x_N) \le b$$
  

$$f(x_1,...,x_N) \ge b$$

## Linear optimization model

objective function and constraints linear in the decision variables ⇒ linear optimization model

# Properties of linear functions

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Proportionality Each variable contributes a proportional value to the function.

Independence The value each variable contributes to the function is independent of the manifestation of the other variables.

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- variables have other exponents than 1
  - other natural exponents, e.g.:  $x^2$
  - ▶ roots, e.g.:  $\sqrt{x} = x^{\frac{1}{2}}$
  - ▶ variables in the denominator, e.g.:  $\frac{1}{x} = x^{-1}$
- ightharpoonup variables are multiplied with each other, e.g.:  $x_1 \cdot x_2$
- exponential functions, e.g.: 2<sup>x</sup>
- ightharpoonup absolute values, e.g. |x|

## Anomaly: constants

While constants are non-linear by definition, they do not interfere with linear optimization models, since they can be eliminated easily.

## deterministic vs. stochastic optimization models

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deterministic optimization models: all parameters and function values are always exactly known stochastic optimization models: parameters and function values succumb to random deviations

Linear optimization models are deterministic in general.

# continuous vs. integer optimization models

continuous optimization models: the values of the decision variables are continuous (real) values integer optimization models: the values of the decision

integer optimization models: the values of the decision variables can only be integer

Types of linear optimization models by possible values for decision variables:

- ▶ continuous decision variables ⇒ (continuous) linear optimization model
- ▶ integer decision variables ⇒ integer linear optimization model

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# Solution structure of linear optimization models

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Solution of linear problems

- there is exactly one optimal solution
- there is an unlimited number of optimal solutions
- there is no optimal solution
  - the solution space is empty
  - the solution space is unbound and the objective function approaches infinity

# Well known solution methods for linear optimization models

Solution methods for (continuous) linear optimization models

- Dantzig's simplex method
- Karmarkar's inner point method

Solution methods for (mixed) integer linear optimization models

- ▶ Branch and bound method
- Cutting planes methods

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# Optimization model of example "Lewig Sanstetten"

Model: production problem

$$\begin{array}{ll} \max & 2, 9 \cdot x_1 + 3, 3 \cdot x_2 + 2, 2 \cdot x_3 \\ s.t. & 5, 3 \cdot x_1 + 2, 9 \cdot x_2 + 2, 5 \cdot x_3 \leq 64 \\ & 3, 9 \cdot x_1 + 4, 8 \cdot x_2 + 3, 1 \cdot x_3 \leq 48 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array}$$

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# Substitution of data with parameters

Model: production problem

max 
$$p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3$$
  
s.t.  $v_{A1} \cdot x_1 + v_{A2} \cdot x_2 + v_{A3} \cdot x_3 \le c_A$   
 $v_{B1} \cdot x_1 + v_{B2} \cdot x_2 + v_{B3} \cdot x_3 \le c_B$   
 $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$ 

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# Index sets as a type of parameter

## Version 1: maximal index as parameter

Let  $I \in \mathbb{N}$  by the number of products, so e.g. the objective functions becomes:

$$\sum_{i=1}^{l} p_i \cdot x_i$$

## Version 2: Index set as parameter

Let *I* be the set of products, so e.g. the objective functions becomes:

$$\sum_{i \in I} p_i \cdot x$$

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# Summing up over parameterized index sets

Model: production problem

$$\max \sum_{i \in I} p_i \cdot x_i$$

$$s.t. \sum_{i \in I} v_{Ai} \cdot x_i \le c_A$$

$$\sum_{i \in I} v_{Bi} \cdot x_i \le c_B$$

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$$

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# Application of the universal quantifier ("for all" quantifier)

$$\sum_{i \in I} v_{Ai} \cdot x_i \le c_A$$

$$\sum_{i \in I} v_{Bi} \cdot x_i \le c_B$$

 $\downarrow$  index set R of resources  $\downarrow$ 

$$\sum_{i=1}^{r} v_{ri} \cdot x_i \le c_r \qquad \forall r \in R$$

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Model: production problem

$$\max \sum_{i \in I} p_i \cdot x_i$$
s.t. 
$$\sum_{i \in I} v_{ri} \cdot x_i \le c_r \qquad \forall r \in R$$

$$x_i \ge 0 \qquad \forall i \in I$$

## Model: production problem

#### Index Sets:

set of products

R set of resources

#### Parameters:

 $p_i$  sales price of product  $i \in I$ 

 $c_r$  capacity of resource  $r \in R$ 

 $v_{ri}$  capacity consumption of product  $i \in I$  on resource  $r \in R$ 

#### Decision variables:

 $x_i$  production quantity of product  $i \in I$ 

### Model description:

$$\max \sum_{i \in I} p_i \cdot x_i$$
s.t. 
$$\sum_{i \in I} v_{ri} \cdot x_i \le c_r \qquad \forall r \in R \quad (I)$$

$$x_i > 0 \qquad \forall i \in I$$

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## Model vs. model instanz

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### Model

The general description of the problem structure with general index sets and parameters.

## Model instance

A specific problem, in which specific values are assigned to the index sets and parameters.