# Modeling and Optimization with OPL

## 4 Optimization of Graph Problems

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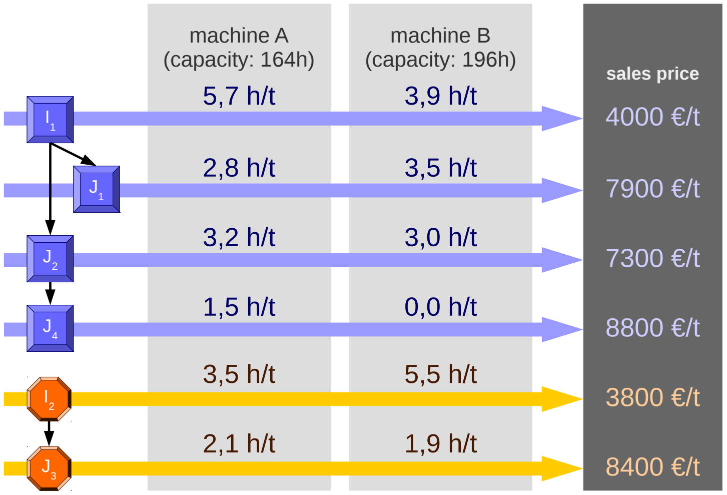
## Inhalt

## 4.1 Short introduction into graph theory

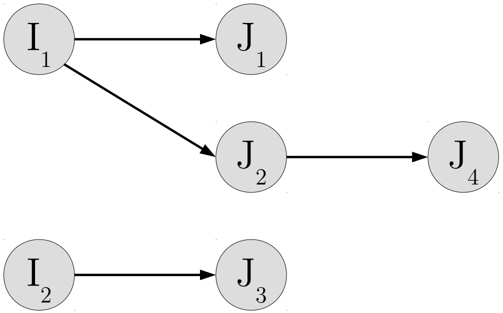
# Example: Lewig Adelburg



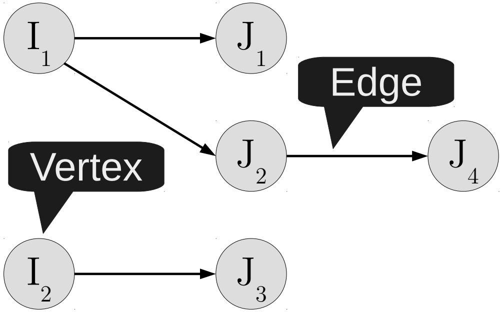
|  | machine A (capacity: 164h) | machine B (capacity: 196h) | sales price |
|---|---|---|---|
| $I_1$ | 5,7 h/t | 3,9 h/t | 4000 €/t |
| $J_1$ | 2,8 h/t | 3,5 h/t | 7900 €/t |
| $J_2$ | 3,2 h/t | 3,0 h/t | 7300 €/t |
| $J_4$ | 1,5 h/t | 0,0 h/t | 8800 €/t |
| $I_2$ | 3,5 h/t | 5,5 h/t | 3800 €/t |
| $J_3$ | 2,1 h/t | 1,9 h/t | 8400 €/t |

$$\begin{tikzcd} I_1 \arrow[r] \arrow[d] & J_1 \\ J_2 \arrow[d] & \\ J_4 & \end{tikzcd} \qquad \begin{tikzcd} I_2 \arrow[d] \\ J_3 \end{tikzcd}$$

## Concept of a graph: components



# Concept of a graph: components



## Concept of a graph: formal description

- **Directed** Graphs are defined as a tuple $G = (V, E)$ with a set of vertices $V$ and a set of edges $E \subseteq V \times V$.

  in example:
  $G = (\{I_1, I_2, J_1, J_2, J_3, J_4\}, \{(I_1, J_1), (I_1, J_2), (I_2, J_3), (J_2, J_4)\})$
- **Undirected** graphs are graphs whose edges do not have a specific direction.
- **Weighted** graphs are defined as a tuple $G = (V, E, g)$ with a set of vertices $V$, a set of edges $E \subseteq V \times V$ and a weight function $g : E \to \mathbb{R}$.

## 4.2 Representation of graphs in OPL

# Sequence dependet production problem

**Index sets**:

| | |
|---|---|
| $I$ | set of products |
| $R$ | set of ressources |

**Parameters**:

| | |
|---|---|
| $p_i$ | price of product $i \in I$ |
| $c_r$ | capacity of ressource $r \in R$ |
| $v_{ri}$ | capacity consumption of product $i \in I$ on ressource $r \in R$ |
| $E$ | set of edges in the sequence graph |

**Decision variables**:

| | |
|---|---|
| $x_i$ | production quantity of product $i \in I$ |

**Modellbeschreibung**:

$$\begin{aligned}
\max \quad & \sum_{i \in I} p_i \cdot x_i \\
s.t. \quad & \sum_{i \in I} v_{ri} \cdot x_i \le c_i && \forall r \in R \quad \text{(I)} \\
& x_i \ge \sum_{(i,j) \in E} x_j && \forall i \in I \quad \text{(II)} \\
& x_i \ge 0 && \forall i \in I
\end{aligned}$$

## Graphs in optimization problems

### Application example for graphs

$$x_i \geq \sum_{(i,j)\in E} x_j \qquad \forall i \in I$$

Question: How can the graph be represented in the optimization model?

## The adjacency matrix

### Definition: adjacency matrix

The adjacency matrix of a grap $G = (V, E)$ with $V = \{V_1, \ldots, V_N\}$ is a quadratic $N \times N$-Matrix $(a_{ij})$, which holds:

$$a_{ij} = \begin{cases} 1 & \text{Edge } V_i \to V_j \text{ exists} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

## Adjacency matrix in the example

$$\begin{tikzcd}
I_1 \arrow[r] \arrow[dr] & J_1 & \\
 & J_2 \arrow[r] & J_4 \\
I_2 \arrow[r] & J_3 &
\end{tikzcd}$$

↓ Translation into adjacency matrix ↓

$$\begin{array}{c|cccccc}
 & I_1 & I_2 & J_1 & J_2 & J_3 & J_4 \\
\hline
I_1 & 0 & 0 & 1 & 1 & 0 & 0 \\
I_2 & 0 & 0 & 0 & 0 & 1 & 0 \\
J_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
J_2 & 0 & 0 & 0 & 0 & 0 & 1 \\
J_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
J_4 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

## Application of adjacency matrixes in optimization problems

```
{string} I = ...;
int a [I,I] = [
  [0, 0, 1, 1, 0, 0],
  [0, 0, 0, 0, 1, 0],
  [0, 0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0, 1],
  [0, 0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0, 0],
];
```

$$x_i \geq \sum_{(i,j)\in E} x_j \qquad \forall i \in I$$

$\downarrow$ OPL $\downarrow$

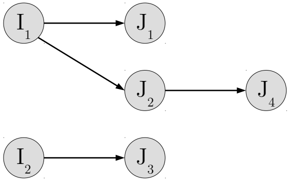
```
forall(i in I)
  x[i] >= sum (j in I)(a[i,j]*x[j]);
```

## Adjacency lists

### Definition: adjacency lists

The adjacency list of a vertex $v \in V$ of a graph $G = (V, E)$ is a set $A_v \subseteq V$, which contains all successors of $v$.

### Adjacency lists in the example



$$\begin{array}{ll} A_{I_1} = \{J_1, J_2\} & A_{I_2} = \{J_3\} \\ A_{J_1} = \{\} & A_{J_2} = \{J_4\} \\ A_{J_3} = \{\} & A_{J_4} = \{\} \end{array}$$

## Application of adjacency lists in optimization problems

```
{string} I = ...;
{string} A[I] = [
  {"J1", "J2"},
  {"J3"},
  {},
  {"J4"},
  {},
  {}
];
```

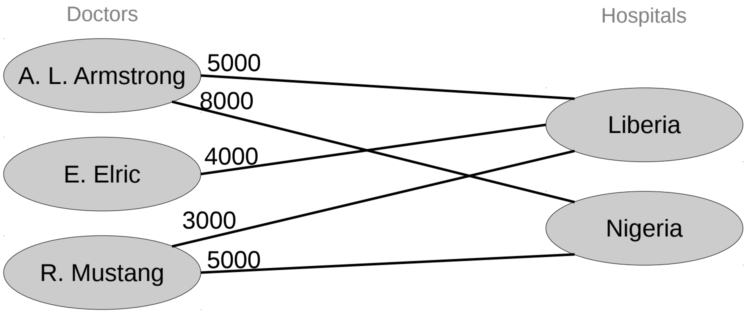
$$x_i \geq \sum_{(i,j)\in E} x_j \qquad \forall i \in I$$

↓ OPL ↓

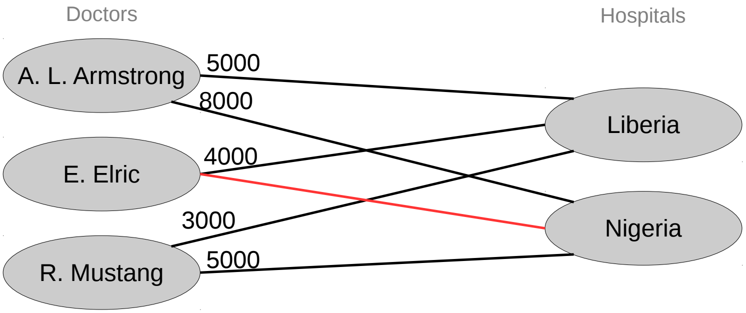
```
forall (i in I)
  x[i] >= sum(j in A[i])(x[j]);
```

## 4.3 OPL: custom tupels as data structure

# Example: Relieve Doctors



## Example: Relieve Doctors



# Model: Assignment problem

**Index sets**:

| | |
|---|---|
| $R$ | set of ressources |
| $T$ | set of tasks |

**Parameters**:

| | |
|---|---|
| $E$ | set of egdes in the assignment graph |
| $c_{rt}$ | cost of each edge $(r,t) \in E$ |

**Decision variables**:

| | |
|---|---|
| $x_{rt}$ | binary variable representing the choice of edge $(r,t) \in E$ |

**Model description**:

$$\begin{aligned}
\min \quad & \sum_{(r,t)\in E} c_{rt} \cdot x_{rt} & & \\
s.t. \quad & \sum_{(r,t)\in E} x_{rt} = 1 & \forall t \in T \qquad & \text{(I)} \\
& \sum_{(r,t)\in E} x_{rt} \leq 1 & \forall r \in R \qquad & \text{(II)} \\
& x_{rt} \in \{0,1\} & \forall (r,t) \in E &
\end{aligned}$$

## Representation of missing edges in OPL

- Assign prohibitively high costs to missing edges. Disadvantages:
  - unnecessary binary variables
  - susceptible to machine rounding errors
  - only applicable to weighted graphs (if at all)
- Adjacency matrix. Disadvantages:
  - unnecessary binary variables
- Adjacency lists. Disadvantages:
  - `x[r in R][t in A[r]]` $\rightarrow$ Error: „Variable indexer size not allowed for a generic array.“

## Tuple data structure

Tuples are custom data structurs, consisting of elements of other data types.

### Definition of a new tuple data type

```
tuple Name_of_the_tuple_data_type {
  Data_type_of_1st_Element Name_of_1st_Element;
  Data_type_of_2nd_Element Name_of_2nd_Element;
  ...
}
```

### Example: Edges as tuple data type

```
{string} V = {"A", "B", "C"};
tupel edge {
  string start;
  string end;
};
```

# Tuple literals and elements

In a tuple data type's literals the elements are sorted into angle brackets.

## Example: Definition of an edge as literal

```
edge e = <"A", "B">;
```

Single elements of a tuple data type are adressed with a dot.

## Example: getting the starting vertex of an edge

e.start $\rightarrow$ "A"

## Application of tuple data type (Alternative 1)

Vertices and Edges shall be defined as above.

### Application example

$$\sum_{(r,t)\in E} x_{rt} = 1 \qquad \forall t \in T$$

↓ OPL ↓

```
forall(t in T)
  sum(<r,t> in E)(x[<r,t>]) == 1;
```

## 4.4 OPL: conditional operators

## Conditional operators

Using a colon, we can apply conditions to iteration indexes, which have to be fulfilled for an index to be incorporated by the operator:

`sum(`*iteration index* `in` *index set* `:` *condition*`)`

resp.

`forall(`*iteration index* `in` *index set* `:` *condition*`)`

Conditions are logical expressions (not boolean decision variables!)

## Construtction of conditions

### Literals for logical values

`true`, `false`

### Comparison operators for logical values

| math. notation | $=$ | $\neq$ | $\leq$ | $<$ | $\geq$ | $>$ |
|---|---|---|---|---|---|---|
| OPL Syntax | `==` | `!=` | `<=` | `<` | `>=` | `>` |

### Logical operators for logical values

| math. notation | $\neg$ | $\wedge$ | $\vee$ | $\veebar$ |
|---|---|---|---|---|
| OPL syntax | `!` | `&&` | `\|\|` | `!=` |

## Application of tuple data type (Alternative 2)

Vertices and Edges shall be defined as above.

### Application example

$$\sum_{(r,t)\in E} x_{rt} = 1 \qquad \forall t \in T$$

↓ OPL ↓

```
forall(t in T)
  sum(e in E : e.task == t)(x[e]) == 1;
```