

# Modeling and Optimization with OPL

## 1 Introduction to linear Optimization

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## Properties

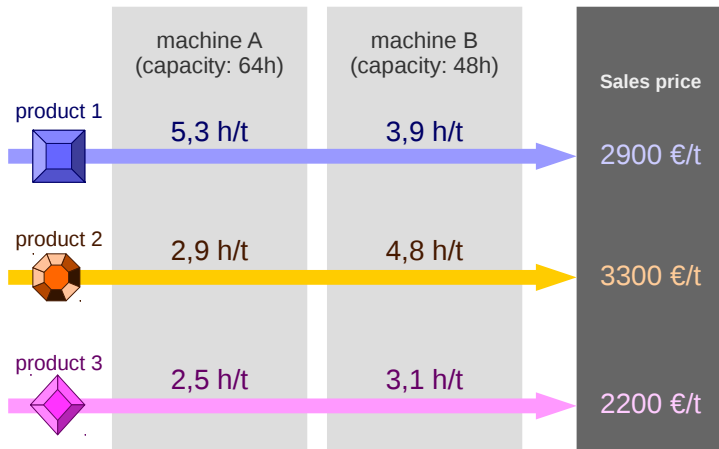
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### Solution of linear problems

# 1.1 Modeling

# Example: production problem (Lewig Sanstetten)



How much of each product should be produced?

# What is a model?

- ▶ abstract description of a real system
- ▶ for example useful for
  - ▶ system design
  - ▶ performance analysis
  - ▶ **decision support**
- ▶ guideline: as exact as necessary, as simple as possible

# Mathematical optimization models for decision support

## Components of a mathematical optimization model<sup>1</sup>:

**decision variables** system items which can be set by the decision maker

- ▶ in the example: the production quantities of the products

**constraints** restrictions which have to be satisfied by the decision variables to get a proper solution

- ▶ in the example: the capacity of the machines may not be exceeded

**objective function** a function of the decision variables, which the decision maker has to optimize, i.e. minimize or maximise

- ▶ in the example: the total revenue

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<sup>1</sup>also: mathematical program

# Example: production problem – decision variables

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$x_1$  production quantity of product 1

$x_2$  production quantity of product 2

$x_3$  production quantity of product 3

## Definition: Solution of an optimization model

By assigning specific values to the decision variables, we get a solution for the model.

# Example: production problem – constraints

Capacity of machine A may not be exceeded:

$$\blacktriangleright 5,3 \cdot x_1 + 2,9 \cdot x_2 + 2,5 \cdot x_3 \leq 64$$

Capacity of machine B may not be exceeded:

$$\blacktriangleright 3,9 \cdot x_1 + 4,8 \cdot x_2 + 3,1 \cdot x_3 \leq 48$$

## Definition: Feasible Solution of an optimization model

A solution which satisfies all constraints is called a feasible solution. The set of all feasible solution is called solution space.



# Example: production problem – objective function

Maximize the revenue (in k€):

$$\blacktriangleright \max U(x_1, x_2, x_3) = 2,9 \cdot x_1 + 3,3 \cdot x_2 + 2,2 \cdot x_3$$

**Definition:** Optimal solution and optimal value of an optimization model

If there is a feasible solution, in which the objective function assumes its maximum resp. its minimum (which might not be the case at all), this solution is an optimal solution and its objective function value is the optimal value.

# Example: production problem – complete optimization model

$$\begin{array}{ll} \max & 2,9 \cdot x_1 + 3,3 \cdot x_2 + 2,2 \cdot x_3 & (\text{objective function}) \\ \text{s.t.} & 5,3 \cdot x_1 + 2,9 \cdot x_2 + 2,5 \cdot x_3 \leq 64 & (\text{constraint I}) \\ & 3,9 \cdot x_1 + 4,8 \cdot x_2 + 3,1 \cdot x_3 \leq 48 & (\text{constraint II}) \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 & (\text{non negativity constraint}) \end{array}$$

## 1.2 Linear Optimization

# Linear functions and constraints

## Linear functions

$$f(x_1, \dots, x_N) = \sum_{n=1}^N c_n \cdot x_n$$

## Linear constraints

Let  $f$  be a linear function:

$$f(x_1, \dots, x_N) = b$$

$$f(x_1, \dots, x_N) \leq b$$

$$f(x_1, \dots, x_N) \geq b$$

## Linear optimization model

objective function and constraints linear in the decision variables  
 $\implies$  linear optimization model

# Properties of linear functions

**Proportionality** Each variable contributes a proportional value to the function.

**Independence** The value each variable contributes to the function is independent of the manifestation of the other variables.



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### Solution of linear problems





## Solution structure of linear optimization models

- ▶ there is exactly one optimal solution
- ▶ there is an unlimited number of optimal solutions
- ▶ there is no optimal solution
  - ▶ the solution space is empty
  - ▶ the solution space is unbound and the objective function approaches infinity

## Well known solution methods for linear optimization models

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## Solution methods for (continuous) linear optimization models

- ▶ Dantzig's simplex method
- ▶ Karmarkar's inner point method

## Solution methods for (mixed) integer linear optimization models

- ▶ Branch and bound method
- ▶ Cutting planes methods

### Properties

### Solution of linear problems

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# Optimization model of example “Lewig Sanstetten”

Model: production problem

$$\begin{array}{ll}\max & 2,9 \cdot x_1 + 3,3 \cdot x_2 + 2,2 \cdot x_3 \\s.t. & 5,3 \cdot x_1 + 2,9 \cdot x_2 + 2,5 \cdot x_3 \leq 64 \\& 3,9 \cdot x_1 + 4,8 \cdot x_2 + 3,1 \cdot x_3 \leq 48 \\& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\end{array}$$

# Substitution of data with parameters

Model: production problem

$$\begin{aligned} \max \quad & p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3 \\ \text{s.t.} \quad & v_{A1} \cdot x_1 + v_{A2} \cdot x_2 + v_{A3} \cdot x_3 \leq c_A \\ & v_{B1} \cdot x_1 + v_{B2} \cdot x_2 + v_{B3} \cdot x_3 \leq c_B \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$



# Summing up over parameterized index sets

Model: production problem

$$\begin{aligned} \max \quad & \sum_{i \in I} p_i \cdot x_i \\ \text{s.t.} \quad & \sum_{i \in I} v_{Ai} \cdot x_i \leq c_A \\ & \sum_{i \in I} v_{Bi} \cdot x_i \leq c_B \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

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# Application of the universal quantifier (“for all” quantifier)

$$\sum_{i \in I} v_{Ai} \cdot x_i \leq c_A$$

$$\sum_{i \in I} v_{Bi} \cdot x_i \leq c_B$$

↓ index set  $R$  of resources ↓

$$\sum_{i \in I} v_{ri} \cdot x_i \leq c_r \quad \forall r \in R$$



## Model: production problem

$$\begin{array}{ll} \max & \sum_{i \in I} p_i \cdot x_i \\ \text{s.t.} & \sum_{i \in I} v_{ri} \cdot x_i \leq c_r \quad \forall r \in R \\ & x_i \geq 0 \quad \forall i \in I \end{array}$$

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## Model

The **general** description of the problem structure with **general** index sets and parameters.

## Model instance

A **specific** problem, in which **specific** values are assigned to the index sets and parameters.