# Modeling and Optimization with OPL 3 Methods of binary programming

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## Inhalt

## 3.1 Modeling of logical expressions

## 3.2 Decision dependent constraints

The Big-M-Method

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Disjunctive Constraints

## 3.3 OPL: Compact implementation

#### 3.4 Piecewise functions

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3.1 Modeling of logical expressions

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Step function

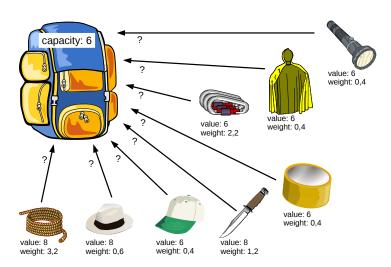
Piecewise linear fu

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#### 3.1 Modeling of logical expressions

# 3.1 Modeling of logical expressions

# Example: Adventure Inc.



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# 3.1 Modeling of logical expressions

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Step function

Piecewise linear functi OPL: the piecewise

#### Index sets:

Set of items

#### Parameters:

weight of item  $i \in I$ 

value of item  $i \in I$ 

capactiy of the knapsack

#### Decision variables:

binary decision variable; indicates if item  $i \in I$  is packed Xi

#### Model description:

$$\max \sum_{i \in I} u_i \cdot x_i$$

$$s.t. \sum_{i \in I} w_i \cdot x_i \le c$$

$$x_i \in \{0, 1\} \qquad \forall i \in I$$

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#### 3.1 Modeling of logical expressions

# Logical operators

- ¬ logical **negation**
- ∧ logical and
- ∨ logical or
- ✓ logical exklusive or ("xor")
- ⇒ logical implication
- ⇔ logical equivalence

## Truth table in numerical representation

_	Α	В	$\neg A$	$\neg B$	$A \wedge B$	$A \vee B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
	1	1	0	0	1	1	0	1	1
	1	0	0	1	0	1	1	0	0
	0	1	1	0	0	1	1	1	0
	0	0	1	1	0	0	0	1	1

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# 3.1 Modeling of logical expressions

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# Logical operators in binary optimization models

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Example: Let  $x_1$  and  $x_2$  be binary decision variables of a knapsack problem, representing items  $I_1$  and  $I_2$ .

 $\neg l_1$ : Get the value of  $l_1$  not being packed.

▶ 
$$1 - x_1$$

 $I_1 \wedge I_2$ : Both  $I_1$  and  $I_2$  must be packed.

$$x_1 + x_2 = 2$$

 $l_1 \vee l_2$ : At least one of the items has to be packed.

► 
$$x_1 + x_2 \ge 1$$

 $\neg(I_1 \land I_2)$ : At most one of the items may be packed.

► 
$$x_1 + x_2 \le 1$$

# 3.1 Modeling of logical expressions

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# Logical operators in binary optimization models

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Example: Let  $x_1$  and  $x_2$  be binary decision variables of a knapsack problem, representing items  $I_1$  and  $I_2$ .

 $\neg (I_1 \lor I_2)$ : None of the items may be packed.

$$x_1 + x_2 = 0$$

 $l_1 \vee l_2$ : Exactly one of the items must be packed.

$$x_1 + x_2 = 1$$

 $l_1 \Rightarrow l_2$ : If  $l_1$  is packed,  $l_2$  must also be packed.

▶ 
$$x_1 \le x_2$$

 $l_1 \Leftrightarrow l_2$ : The decision is identical for both items.

# 3.2 Decision dependent constraints

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implementation

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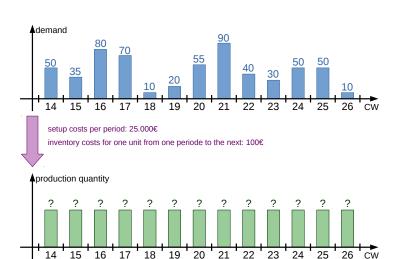
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Piecewise linear function

PPL: the piecewise

# 3.2 Decision dependent constraints

# Example: Lewig Wakuxi



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3.1 Modeling of logical expression:

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#### Parameters:

 $d_t$  demand in period  $t \in T$  $s_t$  setup costs in period  $t \in T$ 

 $h_t$  inventory costs per item in period  $t \in T$ 

 $i_{t_{min}-1}$  initial inventory

M a big number

#### **Decision variables:**

 $x_t$  production quantity in period  $t \in T$   $i_t$  inventory at the end of period  $t \in T$  $y_t$  production decision in period  $t \in T$ 

#### Model description:

$$\min \quad \sum_{t \in T} s_t \cdot y_t + h_t \cdot i_t$$

$$\begin{array}{lll} s.t. & i_t = i_{t-1} + x_t - d_t & \forall t \in T & \text{(I)} \\ & x_t \leq M \cdot y_t & \forall t \in T & \text{(II)} \\ & x_t, i_t > 0; \ y_t \in \{0, 1\} & \forall t \in T \end{array}$$

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iecewise linear functions

# The Big-M-Method

Let  $\bar{\mathbf{x}}$  be the vector of decision variables and f be a linear function. The constraint

$$f(\overline{\mathbf{x}}) \le b$$
 resp.  $f(\overline{\mathbf{x}}) \ge b$ 

shall only be constraining if a decision represented by the binary variable y assuming the value 0 has been made.

## Decision dependent constraint

Let M be a sufficiently big number.

$$f(\overline{\mathbf{x}}) \leq b \quad \rightarrow \quad f(\overline{\mathbf{x}}) \leq b + M \cdot y$$

$$f(\overline{\mathbf{x}}) \geq b \quad \rightarrow \quad f(\overline{\mathbf{x}}) \geq b - M \cdot y$$

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#### The Big-M-Method

# Constraint of example "Lewig Wakuxi"

$$i_t = i_{t-1} + x_t - d_t \qquad \forall t \in T$$
 (I)

### Implementation attempt 1

```
\{\text{string}\}\ T = \{\text{"KW14"}, \text{"KW15"}, \text{"KW16"}, \text{"KW17"}\};
dvar float+ i[T];
  forall(t in T) i[t] == i[t-1] + x[t] - d[t];
```

Operator for string - int not available.

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- OPL: modeling of time neriods

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Constraint of example "Lewig Wakuxi"

$$i_t = i_{t-1} + x_t - d_t \qquad \forall t \in T$$
 (I)

## Implementation attempt 2

```
{int} T = \{14, 15, 16, 17\};
dvar float+ i[T];
  forall(t in T) i[t] == i[t-1] + x[t] - d[t];
Olimber Index out of bound for array "i": 13.
```

OPL: modeling of time neriods

# Problem with implementation of time periods

# Constraint of example "Lewig Wakuxi"

$$i_t = i_{t-1} + x_t - d_t \qquad \forall t \in \mathcal{T}$$
 (I)

### Implementation attempt 3

```
{int} T = {14, 15, 16, 17};
{int} T0 = {13, 14, 15, 16, 17};
dvar float+ i[T0];
  forall(t in T) i[t] == i[t-1] + x[t] - d[t];
```

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# Problem with implementation of time periods

# Constraint of example "Lewig Wakuxi"

$$i_t = i_{t-1} + x_t - d_t \quad \forall t \in T$$
 (I)

### Implementation attempt 4

```
int Tmin = 14;
int Tmax = 17;
range T = Tmin..Tmax;
dvar float+ i[Tmin-1..Tmax];
  forall(t in T) i[t] == i[t-1] + x[t] - d[t];
```

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# Disjunctive Constraints I

A model shall have the following constraints:

$$f(\overline{\mathbf{x}}) \le b$$
$$g(\overline{\mathbf{x}}) \le d$$

It is enough to only fullfil one constraint.

# Disjunctive Constraints

Let M be a sufficiently large number and y be a binary auxiliary variable.

$$f(\overline{\mathbf{x}}) \le b + M \cdot y$$
  
 $g(\overline{\mathbf{x}}) \le d + M \cdot (1 - y)$ 

≥-constraint analog

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# Disjunctive Constraints II

A model shall have to following constraint:

$$g(\overline{\mathbf{x}}) \leq d$$

This constraint only needs to be fullfiled if it holds:

$$f(\overline{\mathbf{x}}) > b$$

## Disjunctive Constraints

Let M be a sufficiently large number and y be a binary auxiliary variable.

$$f(\overline{\mathbf{x}}) \le b + M \cdot y$$
  
 $g(\overline{\mathbf{x}}) \le d + M \cdot (1 - y)$ 

≥-constraint analog

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## Decision expressions

#### Objective function of the Wagner-Whitin-problem:

```
// objective function
minimize sum(t in T)(s[t]*y[t] + h[t]* i[t]);
```

#### Structuring with decision expressions:

```
// decision expressions
dexpr float setupCost = sum(t in T)(s[t]*y[t]);
dexpr float inventoryCost = sum(t in T)(h[t]*i[t]);
// objective function
minimize setupCost + inventoryCost;
```

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# Arrays of decision expressions

### Objective function of the Wagner-Whitin-problem:

```
// objective function
minimize sum(t in T)(s[t]*y[t] + h[t]* i[t]);
```

#### Structuring with decision expressions:

```
//decision expressions
dexpr float periodCost[t in T]
 = s[t]*y[t] + h[t]*i[t];
// objective function
minimize sum (t in T)(periodCost[t]);
```

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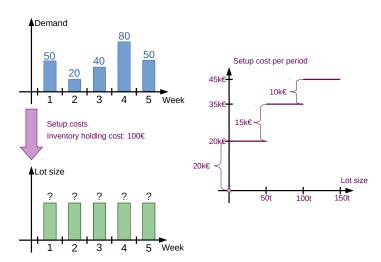
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# Example: Lewig Xanxi



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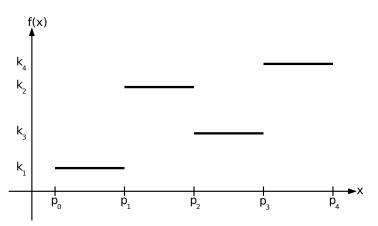
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#### Step functions

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# Step functions

Let x be a continous decision variable:



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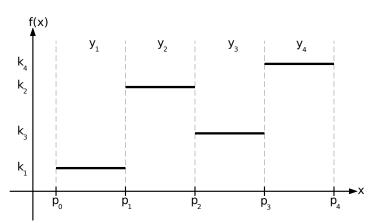
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## Step functions

Let x be a continous decision variable:



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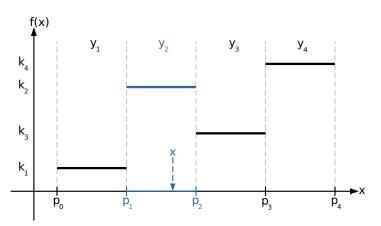
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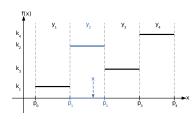
Step functions

Piecewise linear functions

L: the piecewise mmand

• e.g.:  $x = \frac{1}{3} \cdot p_1 + \frac{2}{3} \cdot p_2$ 

# Decision variable as convex combination of the supporting points



$$x = \sum_{n=0}^{N} z_n \cdot p_n$$

$$\sum_{n=0}^{N} z_n = 1$$

$$0 \le z_n \le 1 \quad \forall n \in \{1, \dots, N\}$$

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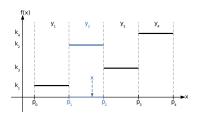
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command

## Choice of the correct interval



$$\sum_{n=1}^{N} y_n = 1$$

$$z_0 \le y_1$$

$$z_n \le y_n + y_{n+1} \quad \forall n \in \{1, \dots, N-1\}$$

$$z_N \le y_N$$

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#### Step runc

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command

# Complete modeling

 $z_N < y_N$ 

$$f(x) = \sum_{n=1}^{N} y_n \cdot k_n$$

$$x = \sum_{n=0}^{N} z_n \cdot p_n$$

$$\sum_{n=0}^{N} z_n = 1$$

$$0 \le z_n \le 1 \quad \forall n \in \{1, \dots, N\}$$

$$\sum_{n=1}^{N} y_n = 1$$

$$z_0 \le y_1$$

 $z_n \leq y_n + y_{n+1} \quad \forall n \in \{1, \dots, N-1\}$ 

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У,

 $y_4$ 

# Example: Lewig Tadelbach

Assume a production problem.

## Capacities

	Machine A	Machine B	Machine C
$I_1$ $I_2$	2,2 1,2	1,6 1,9	2,8 2,3
-	72	48	60

## Sales prices

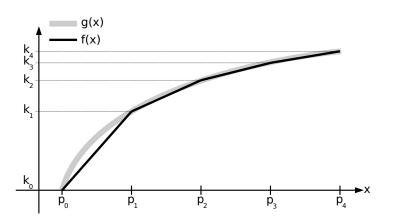
$$f_1(x_1) = 2000 \cdot \sqrt{x_1}$$
  
 $f_2(x_2) = 1800 \cdot \sqrt{x_2}$ 

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Piecewise linear functions

## Piecewise linear functions



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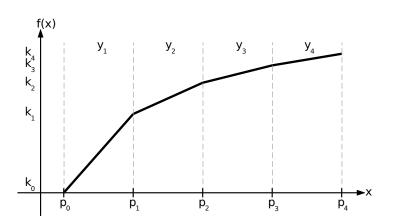
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## Piecewise linear functions



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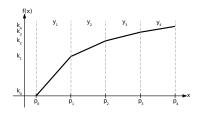
3.4 Piecew functions

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Piecewise linear functions

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## Function values as convex combination



$$x = \sum_{n=0}^{N} z_n \cdot p_n$$

$$f(x) = \sum_{n=0}^{N} z_n \cdot f(p_n)$$

$$\sum_{n=0}^{N} z_n = 1$$

$$0 \le z_n \le 1 \quad \forall n \in \{1, \dots, N\}$$

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# Complete modeling

 $z_0 \leq y_1$ 

 $z_N < y_N$ 

$$f(x) = \sum_{n=0}^{N} z_n \cdot k_n$$

$$x = \sum_{n=0}^{N} z_n \cdot p_n$$

$$\sum_{n=0}^{N} z_n = 1$$

$$0 \le z_n \le 1 \quad \forall n \in \{1, \dots, N\}$$

$$\sum_{n=1}^{N} y_n = 1$$

 $z_n \leq y_n + y_{n+1} \quad \forall n \in \{1, \dots, N-1\}$ 

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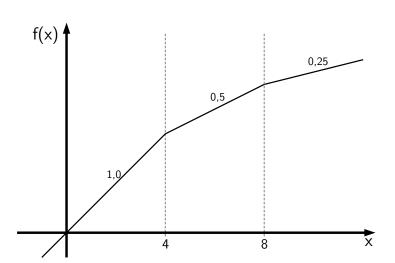
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# Piecewise linear functions by slope



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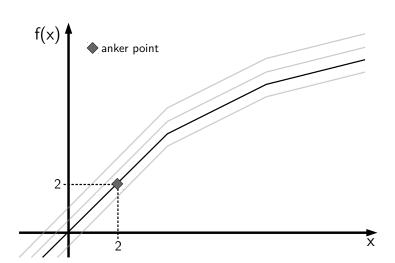
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# Ankering of piecewise linear functions



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Array p of supporting points and array s of slopes:

```
piecewise(i in 1..N){
  s[i] \rightarrow p[i];
  s[N+1]
} (anker point) x;
```

## Example of figure above

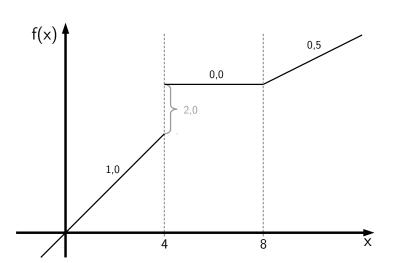
```
int N = 2:
float p[1..N] = [4, 8];
float s[1..N+1] = [1.0, 0.5, 0.25];
dvar float+ x:
piecewise(i in 1..N){
  s[i] \rightarrow p[i];
  s[N+1]
} (2, 2) x:
```

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# Step functions and general discontinuities



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# Step functions and general discontinuities

Second slope value at the same supporting point in the piecewise command becomes step value.

## Example of figure above

```
int N = 3;
float p[1..N] = [4, 4, 8];
float s[1..N+1] = [1.0, 2.0, 0.0, 0.5];
dvar float+ x;
piecewise(i in 1..N){
  s[i] -> p[i];
  s[N+1]
} x:
```

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