Modeling and Optimization with OPL

5 Problems with multiple objective functions

Andreas Popp

Inhalt

5.1 Soft constraints

Example: production problem

$$
\begin{aligned}
\max \quad & \sum_{i \in I} p_i \cdot x_i & & \\
s.t. \quad & \sum_{i \in I} v_{ri} \cdot x_i \leq c_i & \forall r \in R \quad & (\mathrm{I}) \\
& x_i \geq 0 & \forall i \in I &
\end{aligned}
$$

Constraint (I) is a “hard” constraint and must be fulfilled completely.

Soft inequality constraint

$$
\begin{aligned}
\max \quad & \sum_{i \in I} p_i \cdot x_i \\
s.t. \quad & \sum_{i \in I} v_{ri} \cdot x_i \leq c_r + \boxed{o_r} & \forall r \in R \qquad \text{(I)} \\
& x_i, \boxed{o_r} \geq 0 & \forall i \in I, r \in R
\end{aligned}
$$

Problem: no optimal solution, because the solution space is unbound in the direction of optimization.

Soft constraints with penalty costs and bounding

$$
\begin{aligned}
\max \quad & \sum_{i\in I} p_i \cdot x_i - \sum_{r\in R} \boxed{k_r} \cdot o_r & & \\
s.t. \quad & \sum_{i\in I} v_{ri} \cdot x_i \leq c_r + o_r & \forall r \in R \qquad & \text{(I)} \\
& o_r \leq \boxed{m_r} & \forall r \in R \qquad & \text{(II)} \\
& x_i, o_r \geq 0 & \forall i \in I, r \in R &
\end{aligned}
$$

Example: production problem with complete utilisation

$$
\begin{aligned}
\max \quad & \sum_{i \in I} p_i \cdot x_i \\
s.t. \quad & \sum_{i \in I} v_{ri} \cdot x_i = c_i & \forall r \in R \quad (\mathrm{I}) \\
& x_i \geq 0 & \forall i \in I
\end{aligned}
$$

Constraint (I) is a “hard” constraint and must be fulfilled completely.

Soft equality constraints

$$
\begin{aligned}
\max \quad & \sum_{i \in I} p_i \cdot x_i - \sum_{r \in R} k_r \cdot |o_r| & & \\
s.t. \quad & \sum_{i \in I} v_{ri} \cdot x_i = c_i + o_r & \forall r \in R & \qquad \text{(I)} \\
& x_i \geq 0,\ o_r \lesseqgtr 0 & \forall i \in I, r \in R &
\end{aligned}
$$

Problem: The absolute value is not a linear function.

Soft equality constraints

Solution: Substitute $o_r = o_r^+ - o_r^-$

$$\begin{aligned}
\max \quad & \sum_{i \in I} p_i \cdot x_i - \sum_{r \in R} k_r \cdot (\, o_r^+ + o_r^- \,) & & \\
s.t. \quad & \sum_{i \in I} v_{ri} \cdot x_i = c_i + o_r^+ - o_r^- & \forall r \in R & \qquad \text{(I)} \\
& x_i,\ o_r^+,\ o_r^- \geq 0 & \forall i \in I, r \in R &
\end{aligned}$$

Be aware when eliminating absolute values

The decomposition of a variable in two summands is not unique. We have to make sure that one of the summands is zero.

5.2 Maximizing vs. minimizing

Maximizing vs. minimizing

Minimizing and maximizing are identical procedures.
It holds:

$$\max_{x \in X} f(x) = -\min_{x \in X} -f(x)$$

Only the sign of the optimal value changes.

5.3 Multiple objective functions and Pareto optimality

A. Popp

Example: Lewbrandt GmbH

Total capacity: 120 h

Job	1	2	3	4	5
Gross margin	150 k€	100 k€	150 k€	50 k€	70 k€
Revenue	340 k€	190 k€	220 k€	85 k€	215 k€
Waste water	6.2 t	3.5 t	5.8 t	2.4 t	4.8 t
Capacity consumption	65 h	35 h	65 h	15 h	25 h

Which jobs should be accepted?
$\rightarrow$ knapsack problem

Problem

There are three objective functions, so there is no unique optimal solution.

Pareto optimality

Definition: Pareto optimality

A solution is called Pareto optimal, if there is no other solution, which is better in one objective and at least as good in all other objectives.

Selected solutions of the example „Lewbrandt GmbH“

x_1	x_2	x_3	x_4	x_5	profit	revenue	waste water	p. o.
0	1	0	1	0	150	275	5.9	yes
0	1	0	1	1	220	490	10,7	no
1	1	0	0	0	250	530	9.7	yes
1	1	0	1	0	300	615	12.1	yes

A. Popp

5.4 Multicriteria optimization

Example objective function

Objective function as in example "Lewbrandt GmbH":

- Profit:
  $\max f_G(\bar{\mathbf{x}}) = 150 \cdot x_1 + 100 \cdot x_2 + 150 \cdot x_3 + 50 \cdot x_4 + 70 \cdot x_5$
- Revenue:
  $\max f_U(\bar{\mathbf{x}}) = 340 \cdot x_1 + 190 \cdot x_2 + 220 \cdot x_3 + 85 \cdot x_4 + 215 \cdot x_5$
- Waste water:
  $\max f_A(\bar{\mathbf{x}}) = -6{,}2 \cdot x_1 - 3{,}5 \cdot x_2 - 5{,}8 \cdot x_3 - 2{,}4 \cdot x_4 - 4{,}8 \cdot x_5$

Weighted objectives

Compose one comprehensive objective function by weighing the objectives and adding them together.

Weighted objectives in example "Lewbrandt GmbH"

weights: $a_g = 5,\ a_U = 1,\ a_A = 50$

new objective function:

$$\begin{aligned} \max f(\bar{\mathbf{x}}) &= a_g \cdot f_G(\bar{\mathbf{x}}) + a_U \cdot f_U(\bar{\mathbf{x}}) + a_A \cdot f_A(\bar{\mathbf{x}}) \\ &= 5 \cdot f_G(\bar{\mathbf{x}}) + 1 \cdot f_U(\bar{\mathbf{x}}) + 50 \cdot f_A(\bar{\mathbf{x}}) \end{aligned}$$

Model: Multicriteria knapsack problem (weighted objectives)

Index sets:

I	set of items
O	set of objectives

Parameters:

w_i	weight of item $i \in I$
u_{oi}	value of item $i \in I$ w.r.t. objective $o \in O$
c	knapsack's capacity
a_o	weight of objective $o \in O$

Decision variables:

x_i	binary decision variable; represents item $i \in I$ being packed

Model description:

$$\begin{aligned} \max \quad & \sum_{o \in O} a_o \sum_{i \in I} u_{oi} \cdot x_i \\ s.t. \quad & \sum_{i \in I} w_i \cdot x_i \leq c & \text{(I)} \\ & x_i \in \{0,1\} \qquad \forall i \in I \end{aligned}$$

5 Problems with multiple objective functions

Main objective & aspiration levels

Choose one objective as main objective. Define aspiration levels for the other objectives, which will be asserted by constraints.

Main objective & aspiration levels in example "Lewbrandt GmbH"

Let the waster water emission be the main objective. We want to achieve at least 225 k€ of profit and 480 k of revenue:

$$\begin{aligned} \max \quad & f_A(\bar{\mathbf{x}}) \\ s.t. \quad & f_A(\bar{\mathbf{x}}) \geq 225 \\ & f_U(\bar{\mathbf{x}}) \geq 480 \end{aligned}$$

Model: Multicriteria knapsack problem (main objective)

Index sets:
I set of items
O set of objectives
Parameters:
w_i weight of item $i \in I$
u_{oi} value of item $i \in I$ w.r.t. objective $o \in O$
c knapsack's capacity
h main objective $h \in O$
a_o aspiration level of objective $o \in O \backslash \{h\}$
Decision variables:
x_i binary decision variable; represents item $i \in I$ being packed

Model description:

$$
\begin{array}{llll}
\max & \sum_{i \in I} u_{hi} \cdot x_i & & \\
s.t. & \sum_{i \in I} w_i \cdot x_i \leq c & & \text{(I)} \\
& \sum_{i \in I} u_{oi} \cdot x_i \geq a_o & \forall o \in O \backslash \{h\} & \text{(II)} \\
& x_i \in \{0,1\} & \forall i \in I &
\end{array}
$$

Goal programming (classic)

Choose a goal value for all objective function and penalize deviation from those target values.

Goal programmimg in example “Lewbrandt GmbH”

Goal values: $a_G = 220$, $a_U = 480$, $a_A = -11$

$$\begin{aligned} \min \quad & |z_G| + |z_U| + |z_A| \\ s.t. \quad & f_G(\overline{\mathbf{x}}) = 220 + z_G \\ & f_U(\overline{\mathbf{x}}) = 480 + z_U \\ & f_A(\overline{\mathbf{x}}) = -11 + z_A \end{aligned}$$

Model: Multicriteria knapsack problem (GP1)

Index sets:

I	set of items
O	set of objectives

Parameters:

w_i	weight of item $i \in I$
u_{oi}	value of item $i \in I$ w.r.t. objective $o \in O$
c	knapsack's capacity
a_o	goal value for objective $o \in O$

Decision variables:

x_i	binary decision variable; represents item $i \in I$ being packed
z_o	deviation from goal value of objective $o \in O$

Model description:

$$
\begin{aligned}
\min \quad & \sum_{o \in O} |z_o| & & \\
s.t. \quad & \sum_{i \in I} w_i \cdot x_i \leq c & & \text{(I)} \\
& \sum_{i \in I} u_{oi} \cdot x_i = a_o + z_o & \forall o \in O & \text{(II)} \\
& x_i \in \{0,1\}, z_o \lesseqgtr 0 & \forall i \in I, o \in O &
\end{aligned}
$$

Goal Programming (extended version)

Penalize only unwanted deviation and use weights for deviations.

Goal programming in example "Lewbrandt GmbH"

$$\begin{array}{ll} \min & w_G \cdot z_G + w_U \cdot z_U + w_A \cdot z_A \\ s.t. & f_G(\bar{\mathbf{x}}) \geq 220 - z_G \\ & f_U(\bar{\mathbf{x}}) \geq 480 - z_U \\ & f_A(\bar{\mathbf{x}}) \geq -11 - z_A \end{array}$$

Modell: Multicriteria knapsack problem (GP2)

Index sets:

I	set of items
O	set of objectives

Parameters:

w_i	weight of item $i \in I$
u_{oi}	value of item $i \in I$ w.r.t. objective $o \in O$
c	knapsack's capacity
a_o	goal value of objective $o \in O$
b_o	Abweichungskosten für Ziel $o \in O$

Decision variables:

x_i	binary decision variable; represents item $i \in I$ being packed
z_o	deviation from goal value of objective $o \in O$

Model description:

$$\begin{array}{llll}
\min & \sum_{o \in O} b_o \cdot z_o & & \\
s.t. & \sum_{i \in I} w_i \cdot x_i \leq c & & \text{(I)} \\
& \sum_{i \in I} u_{oi} \cdot x_i \geq a_o - z_o & \forall o \in O & \text{(II)} \\
& x_i \in \{0,1\}, z_o \geq 0 & \forall i \in I, o \in O &
\end{array}$$

Lexicographical ordering of solutions

With a strict objective hierarchy it is possible to achieve a Lexicographical ordering of the solutions.

Selected lexicographically ordered solutions of the example “Lewbrandt GmbH”

Let the objective hierarchy be: profit $>$ revenue $>$ waste water

x_1	x_2	x_3	x_4	x_5	profit	revenue	waste water
1	1	0	1	0	300	615	12,1
0	1	1	1	0	300	495	11,7
1	0	0	1	1	270	640	13,4
1	1	0	0	0	250	530	9,7
0	1	1	0	0	250	410	9,3

Preemptive Goal Programming

Algorithmn: Preemptive Goal Programming

1. Let $i = 1$
2. Solve the problem with the objective function f_i of objective i. Get the optimal solution $\mathbf{x}^*$ with the optimal value f_i^*.
3. if $i = n$: $\mathbf{x}^*$ is the lexicographically optimal solution. Stop.
4. Add the following costraint to the model:
$$f_i(\mathbf{x}) = f_i^*$$
5. Let $i = i + 1$ and go to step 2.

5.5 Bottleneck objectives

Example: Arabasta County

town	concert hall	water park	museum
Alubarna	1,45 M$	1,25 M$	1,10 M$
Nanohana	1,00 M$	0,95 M$	0,90 M$
Erumalu	0,32 M$	0,28 M$	0,24 M$

Each facility can only be built once. Which facility should be built in which town?

Maximin problems

Multiple equally scaled single objective functions $f_1, \dots, f_N$.
The main objective function is:

$$\max \min_{n \in \{1,\dots,N\}} f_n(\bar{\mathbf{x}})$$

Linearising of maximin problems

Let $z_{\min} \lesseqgtr 0$ be an auxiliary variable.

$$\begin{aligned} \max \quad & z_{\min} \\ s.t. \quad & f_n(\bar{\mathbf{x}}) \geq z_{\min} \quad \forall n \in \{1, \dots, N\} \end{aligned}$$

Minimax problems

Multiple equally scaled single objective functions $f_1, \ldots, f_N$.
The main objective function is:

$$\min \max_{n \in \{1,\ldots,N\}} f_n(\bar{\mathbf{x}})$$

Linearising of maximin problems

Let $z_{\max} \lesseqgtr 0$ be an auxiliary variable.

$$\begin{aligned} \min \quad & z_{\max} \\ s.t. \quad & f_n(\bar{\mathbf{x}}) \leq z_{\max} \quad \forall n \in \{1, \ldots, N\} \end{aligned}$$

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Model: maximin assignment problem (Alternative 1)

Index sets:

R set of ressources

T set of tasks

Parameters:

p_{tr} profit if Task t is fulfilled by ressource r

Decision variables:

x_{tr} binary variable representing if task t is fulfilled by Ressource r

$p_{\min}$ auxiliary variable for minimal profit

Model description:

$$
\begin{array}{lllr}
\max & p_{\min} & & \\
s.t. & \sum_{r \in R} x_{tr} = 1 & \forall t \in T & \text{(I)} \\
& \sum_{t \in T} x_{tr} \leq 1 & \forall r \in R & \text{(II)} \\
& p_{\min} \leq \sum_{r \in R} x_{tr} \cdot p_{tr} & \forall t \in T & \text{(III)} \\
& x_{rt} \in \{0,1\}, p_{\min} \lessgtr 0 & \forall r \in R, t \in T &
\end{array}
$$

A. Popp

Excplicit modeling of maxima and minima

Excplicit modeling of maxima

$$f_n(\overline{\mathbf{x}}) \leq z_{\max} \qquad \forall n \in \{1, \ldots, N\}$$

$$z_{\max} - f_n(\overline{\mathbf{x}}) \leq M \cdot (1 - y_n) \qquad \forall n \in \{1, \ldots, N\}$$

$$\sum_{n=1}^{N} y_n = 1$$

Excplicit modeling of minima

$$f_n(\overline{\mathbf{x}}) \geq z_{\min} \qquad \forall n \in \{1, \ldots, N\}$$

$$f_n(\overline{\mathbf{x}}) - z_{\min} \leq M \cdot (1 - y_n) \qquad \forall n \in \{1, \ldots, N\}$$

$$\sum_{n=1}^{N} y_n = 1$$

Model: maximin assignment problem (Alternative 2)

Index sets:

R	set of ressources
T	set of tasks

Parameters:

p_{tr}	profit if Task t is fulfilled by ressource r
M	a sufficiently big number

Decision variables:

x_{tr}	binary variable representing if task t is fulfilled by Ressource r
$p_{\min}$	auxiliary variable for minimal profit
y_t	binary selection variable for minimum

Model description:

$$
\begin{array}{lllr}
\max & p_{\min} & & \\
s.t. & \sum_{r \in R} x_{tr} = 1 & \forall t \in T & \text{(I)} \\
& \sum_{t \in T} x_{tr} \leq 1 & \forall r \in R & \text{(II)} \\
& p_{\min} \leq \sum_{r \in R} x_{tr} \cdot p_{tr} & \forall t \in T & \text{(III)} \\
& \sum_{r \in R} (x_{tr} \cdot p_{tr}) - p_{\min} \leq M \cdot (1 - y_t) & \forall t \in T & \text{(IV)} \\
& \sum_{t \in T} y_t = 1 & & \text{(V)} \\
& x_{rt} \in \{0, 1\}, p_{\min} \lessgtr 0 & \forall r \in R, t \in T &
\end{array}
$$