Chap 6

6.1 Inner product

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad A\underline{x} = \cdot\sim$$

$$\begin{bmatrix} - & - & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$$

$$\underline{u}^T \cdot \underline{v} = [u_1 \; u_2 \; \cdots \; u_n] \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [\;]_{1\times 1}$$

scalar

$\underline{u}^T \cdot \underline{v} = \underline{u} \cdot \underline{v} \rightarrow$ dot product

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Th1 $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

$\underline{v} \cdot \underline{u} = \underline{v}^T \underline{u}$

$(\underline{u} + \underline{v}) \cdot \underline{w} = u^T \underline{w} + v^T \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$

$(\underline{u} + \underline{v})^T \cdot \underline{w}$

$(A+B)\underline{x} = A\underline{x} + B\underline{x}$

The length of vector $\|\underline{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$

euclidian norm

$\|u\|^2 = \underline{u} \cdot \underline{u}$

ex3 W subspace spanned

$\underline{x} = \left(\frac{2}{3}, 1\right)$ find unit vector

basis

S is a subspace if:

1 $\underline{0} \in S$ (zero is a subspace of the set)

2 $\underline{x} \in S \Rightarrow c\underline{x} \in S \quad \forall c$

3 $\underline{x} \in S,\ y \in S \Rightarrow \underline{x} + \underline{y} \in S$

Spanset: all linear combinations of a vector make subspace

linear combo: constant · vector

$$W = \{ \underline{y} \mid \underline{y} = c\,\underline{x}, \ \forall c \}$$

__Basis__: set of vectors is a basis if:

1) linearly independent

2) Span { } = W

$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is a basis

$\underline{v} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ also basis

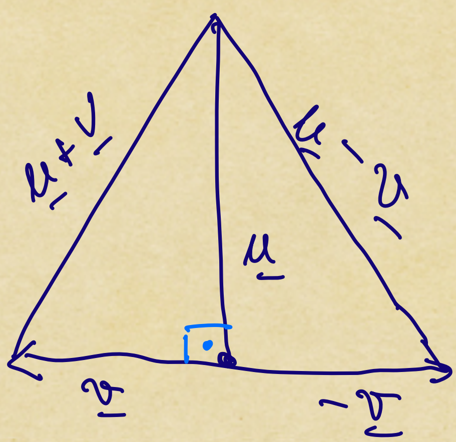
$$\|\underline{v}\| = \sqrt{2^2+3^2} = \sqrt{13}$$

$$u = \frac{1}{\sqrt{13}} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

distance between $\quad \text{dist}(u, v) = \| u - v \|$

$$= \sqrt{(u_1 - v_1)^2 + \dots}$$

Orthogonal Vectors



$$\| u - v \|^2 = (u - v) \cdot (u - v)$$

$$= \underline{u} \cdot \underline{u} - \underline{u} \cdot \underline{v} - \underline{v} \cdot \underline{u} + \underline{v} \cdot \underline{v}$$

equal $= \| \underline{u} \|^2 - 2\, \underline{u} \cdot \underline{v} + \| \underline{v} \|^2$

$$\| u + v \|^2 = \| u \|^2 + 2\, \underline{u} \cdot \underline{v} + \| \underline{v} \|^2$$

$$2 \cdot \underline{u} \cdot \underline{v} = -2 \cdot \underline{u} \cdot \underline{v}$$

$$\Rightarrow \boxed{\underline{u} \cdot \underline{v} = 0}$$

→ orthogonal!!

2 vectors are perpendicular if their dot product is equal to 0

Orthogonal Complements

W subspace

$\underline{z}$ not in subspace W

$\underline{z}$ is orthogonal to W if it is orthogonal to all vectors in W

$\forall$ $\underline{z}$ orthogonal to W

$u \in W$

$\underline{z} \cdot \underline{u} = \underline{0}$

<u>Ex</u> 6 $\quad W$ plane thru origin

L line thru origin, $\perp$ to W

orthogonal complement $W^{\perp}$

$\underline{w} \in W$

$\underline{z} \in L$

$\underline{z} \cdot \underline{w} = 0$

$L = W^{\perp}$ $\qquad W = L^{\perp}$

Symetric relationship

$x \in W^{\perp}$ if $\underline{x} \cdot \underline{w} = 0 \quad \forall\, \underline{w} \in W$

Proof $W^{\perp}$ is a subspace

1. $\underline{0} \cdot \underline{w} = 0 \Rightarrow \underline{0} \in W^{\perp}$

2. Let $\underline{x} \in W^{\perp}$ $\quad \underline{x} \cdot \underline{w} = 0$

then $c(\underline{x} \cdot \underline{w}) = c \cdot (0) = 0$

$(c\underline{x}) \cdot \underline{w} = 0$

$c\underline{x} \in W^{\perp}$

3. Let $\underline{x} \in W^{\perp}$ $\quad y \in W^{\perp}$

$\underline{x} \cdot \underline{w} = 0$

$\underline{y} \cdot \underline{w} = 0$

$x \cdot \underline{w} + y \cdot \underline{w} = 0 + 0 = 0$

property $(\underline{x}+\underline{y})\cdot\underline{w} = 0\}$

$\underline{x}+\underline{y} \in W^\perp$

Thm 3

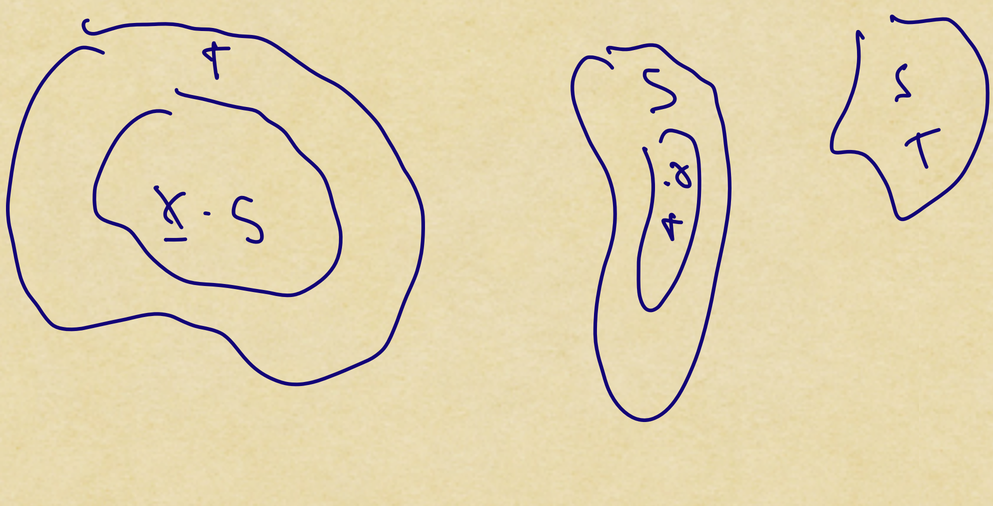
A $m\times n$

$$(\text{Row } A)^\perp = \text{Nul } A$$

Strategy Proof $S = T$

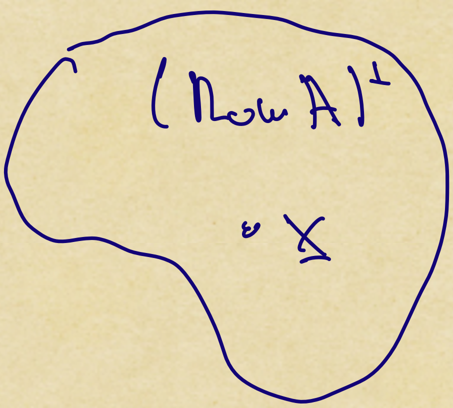
1. $\underline{x} \in S \overset{\text{show}}{\Rightarrow} \underline{x} \in T$

2. $\underline{x} \in T \Rightarrow \underline{x} \in S$



$$\underline{A} = \begin{bmatrix} r_1 & - - - \\ r_2 & - \\ \vdots & \\ r_m & - - - \end{bmatrix}$$

$$\text{Row } A = \{ \underline{r} \mid \underline{r} = c_1 \underline{r_1} + c_2 r_2 + \dots + c_m \underline{r_m} \}$$



$x \in (\text{Row } A)^\perp$

if $\underline{x} \cdot \underline{r} = 0$

$\forall\, \underline{r} \in \text{Row } A$

In particular

$$\begin{aligned} \underline{x} \cdot \underline{r_1} &= 0 \\ \underline{x} \cdot \underline{r_2} &= 0 \\ &\vdots \\ \underline{x} \cdot r_{\underline{m}} &= 0 \end{aligned} \Rightarrow A\underline{x} = \underline{0} \Rightarrow \underline{x} \in \text{Nul } A$$

$$A\underline{x} = 0$$

$$\begin{bmatrix} \underline{r_1} \\ \underline{r_2} \\ \end{bmatrix} x = \begin{bmatrix} 0 \\ \vdots \end{bmatrix}$$

<u>Let</u>

$$\underline{x} \in \text{Nul} A$$

$$A\underline{x} = 0$$

$$\begin{bmatrix} r_1 & \text{---} \\ r_2 & \text{---} \\ \vdots & \\ r_m & \text{---} \end{bmatrix} \begin{bmatrix} \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{r_1} \cdot \underline{x} = 0$$

$$r_2 \cdot \underline{x} = 0$$

$$r_m \cdot x = 0$$

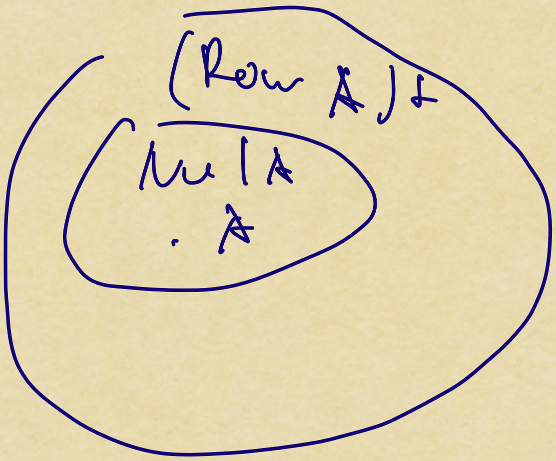
Row A

$$\underline{r} \cdot \underline{x} = x \cdot (c_1 r_1 + c_2 r_2 + \dots + c_m r_m)$$

$$= c_1 \underline{x} \cdot r_1 + c_2 x \cdot r_2 + \dots + c_m \underline{x} \cdot r_m$$

$$= 0$$

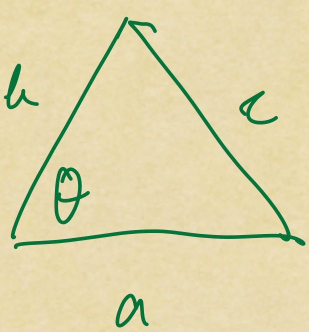
$$\underline{x} \in (\text{Row} A)^\perp$$



1. $(\text{Row } A^T)^\perp = \text{Nul } A^T$ ✓
2. $(\text{Col } A^T)^\perp = \text{Nul } A^T$

$$\text{Col } A^T = \text{Row } A$$

$$\underline{u} \cdot \underline{v} = \|u\| \|v\| \cos\theta$$



$$\Rightarrow c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$\| \underline{v} - \underline{u} \|^2 = \cancel{\|\underline{u}\|^2} + \cancel{\|\underline{v}\|^2} - 2 \|u\| \|v\| \cos\theta$$

$$(\underline{v} - \underline{u}) \cdot (\underline{v} - \underline{u}) =$$

$$\cancel{\|\underline{v}\|^2} + \cancel{\|\underline{u}\|^2} - 2\, \underline{u} \cdot \underline{v}$$

$$\cancel{-}2\, \underline{u} \cdot \underline{v} = \cancel{-2} \|\underline{u}\| \|\underline{v}\| \cos\theta$$

$$\underline{u} \cdot \underline{v} = \|u\| \|v\| \cos\theta$$

$$\theta = \frac{\pi}{2} \quad \underline{u} \cdot \underline{v} = 0$$

abstract qs from text book

no true/False

21 $(c\,\underline{u})\cdot\underline{v} = c\,(\underline{u}\cdot\underline{v})$

Prove using transpose def.

$$(c\,\underline{u})^T\underline{v} = c(u^T\underline{v}) = c\,(\underline{u}\cdot\underline{v})$$

$$(cA)x = c(A\underline{x})$$

22 $\underline{u} = (u_1, u_2, u_3)$

$\underline{u}\cdot\underline{u} \geq 0$ $\quad u_1^2 + u_2^2 + u_3^2$

$\underline{u}\cdot\underline{u} \rightarrow$ square
always positive

$$u_1^2 + u_2^2 + u_3^2 = 0$$

$$\Rightarrow \underline{u} = 0$$

25. Let $\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \perp \underline{v} \right\}$

$ax + by = 0$

$by = -ax$

$y = -\frac{a}{b}x$

$\underline{x} = \begin{bmatrix} -b \\ a \end{bmatrix}$