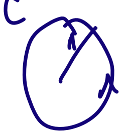
Chap 20 & Chap 19 review

Stokes theorem

given $\underline{F}$

$$\text{circulation} = \oint_C \underline{F}\cdot \underline{dr} = \int_S \text{curl}\, F \cdot dA$$

C  assume c is circle of radius 1 in counter-clockwise dir,

review:

$$\text{grad } f = \underline{\nabla} f$$

$$\text{div } \underline{F} = \underline{\nabla}\cdot \underline{F}$$

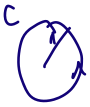
$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F}$$

$$F = Pi + Qj + Rk$$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

1. prob ex not textbook

$$F = y\,\underline{i} + xz\,\underline{j} + \underline{k}$$

C  assume c is circle of radius 1 in counter-clockwise dir

$$x^2 + y^2 = 1$$

$$z = 0$$

$$r = \cos\theta\, i + \sin\theta\, j + 0k$$

$$dr = (-\sin\theta\, i + \cos\theta\, j)\, d\theta$$

$$x = \cos\theta \qquad z = 0$$

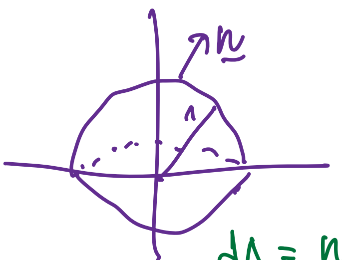
$$y = \sin\theta$$

$$\oint F \cdot dr = \int_0^{2\pi} (\sin\theta\, i + 0\, j + \underline{k}) \cdot (-\sin\theta\, i + \cos\theta\, j + k)\, d\theta$$

$$= -\int_0^{2\pi} \sin^2\theta\, d\theta = -\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$\left[\ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\ \right]$$

$$= -\frac{1}{\cancel{2}}\cancel{2}\pi = \boxed{-\pi}$$



$$d\underline{A} = \underline{n}\, dA \qquad \underline{n} = \frac{\underline{r}}{\|\underline{r}\|}$$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y & x+z & 1 \end{vmatrix}$$

$$= i\,(-x) - j\,(0) + \underline{k}\,(z-1)$$

$$= x\underline{i} + (z-1)\underline{k}$$

$$\oint \underline{F} \cdot d\underline{r} = \int_S \operatorname{curl} F \cdot d\underline{A} = \int (-x\, i + (z-1)\, \underline{k}) \cdot dA$$

[spherical coordinates

$x = \rho \sin\varphi \cos\theta \quad y = \rho \sin\varphi \sin\theta \quad z = \rho\cos\phi$

not volume $\Rightarrow$ surface $\Rightarrow \rho = 1$

$\Rightarrow \underline{n} = \sin\varphi\cos\theta\, \underline{i} + \sin\varphi\sin\theta\, \underline{j} + \cos\varphi\, \underline{k}$

$dA = \sin\varphi \, d\varphi \, d\theta$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \frac{\pi}{2}$

$\Rightarrow = \int [-\sin\varphi\cos\theta\, i + (\cos\varphi - 1)k] \cdot$

$[\sin\varphi\cos\theta\, i + \cos\varphi\, \underline{k}] \cdot \sin\varphi\, d\varphi\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \int_{\theta=0}^{2\pi} [\underbrace{-\sin^2\varphi\cos^2\theta}_{\pi} + \underbrace{\cos^2\varphi}_{2\pi} - \underbrace{\cos\varphi}_{2\pi}] \sin\varphi\, d\varphi\, d\theta$$

no θ term

$$\int_0^{2\pi} \cos^2\theta = d\theta = \frac{1}{2} \int_0^{2\pi} (1 + \cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\cancel{2}} \cancel{2}\pi = \pi$$

$$= \int_0^{\frac{\pi}{2}} \left[-\pi \sin^2\varphi \, d\varphi + 2\pi \cos^2\varphi - 2\pi \cos\varphi\right] \sin\varphi \, d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \left[-\pi \left(1 - \cos^2\varphi\right) + 2\pi \cos^2\varphi - 2\pi \cos\varphi\right] \sin\varphi \, d\varphi$$

$$= \pi \int_0^{\frac{\pi}{2}} \left[\cos^2\varphi - 1 + 2\cos^2\varphi - \cos\varphi\right] \sin\varphi \, d\varphi$$

$$= \pi \int (3\cos^2\varphi - 2\cos\varphi - 1) \sin\varphi \, d\varphi$$

$$= \pi \cdot (-1) = \boxed{-\pi}$$

substitution

$u = \cos\varphi$

$du = -\sin\varphi \, d\varphi$

$\cos 0 = 1$

$\cos\frac{\pi}{2} = 0$

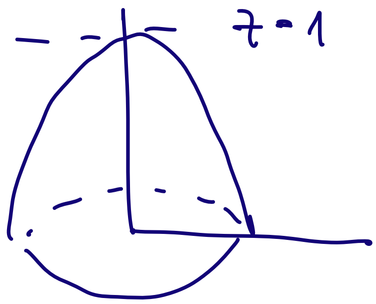
$$\int_1^0 (3u^2 - 2u - 1)(-du) =$$

$$\int_0^1 (3u^2 - 2u - 1)(du) =$$

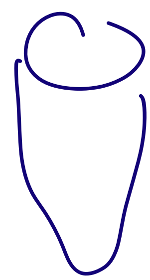
$$\left(\frac{3u^3}{3} - 2\frac{u^2}{2} - u\right)\Bigg|_0^1 = \boxed{-1}$$

conclusion $\int F\cdot dr = \int \nabla\times F\cdot dA$

paraboloid



usually:



$z = 1-(x^2+y^2)$ $\quad z-1 = -(x^2+y^2)$

$$\underline{dA} = -\frac{df}{dx}\underline{i} - \frac{df}{dy}\underline{j} + \underline{k}$$

$$= +2x\underline{i} + 2y\underline{j} + \underline{k}$$

$$\int F\cdot dr = \int_S (-x\underline{i} + (z-1)\underline{k})\cdot(2x i + 2y j + k)\,dx\,dy$$

$$= \int [-2x^2 + (z-1)]\,dx\,dy$$

$$= \int [-2x^2 - (x^2+y^2)]\,dx\,dy$$

$$= \int (-3x^2 - y^2)\, dx\, dy$$

over circle

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 (-3r^2\cos^2\theta - r^2\sin^2\theta)\, r\, dr\, d\theta$$

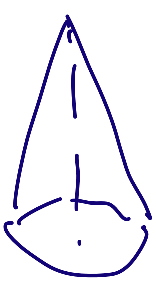
$$= \int_0^{2\pi}\int_0^1 (-3r^3\cos^2\theta - r^3\sin^2\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(-3\frac{r^4}{4}\cos^2\theta - \frac{r^4}{4}\sin^2\theta\right)\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \left(-\frac{3}{4}\cos^2\theta - \frac{1}{4}\sin^2\theta\right) d\theta$$

$$= -\frac{3}{4}\pi - \frac{1}{4}\pi = \boxed{-\pi}$$

Stokes theorem works in 2 cases:



$z = 1$

$z = 1 - \sqrt{x^2+y^2}$

$$dA = \left(-\frac{df}{dx}\, i - \frac{df}{dy}\, j + k \right) dx\, dy$$

$$= \Big(\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}} 2x\, i +$$

$$\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}} 2y\, j$$

$$+ k \Big)\, dx\, dy$$

$$\int F \cdot dr = \int (-x i + (z-1) k) \cdot dA$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\int \left(-\frac{x^2}{\sqrt{x^2+y^2}} + z - 1 \right) dx\, dy$$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2$

$$= \int \left(-\frac{x^2}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2} \right) dx\, dy$$

$$= \int_0^{2\pi}\int_0^1 \left(-\frac{r^2\cos^2\theta}{r} - r \right) r\, dr\, d\theta$$

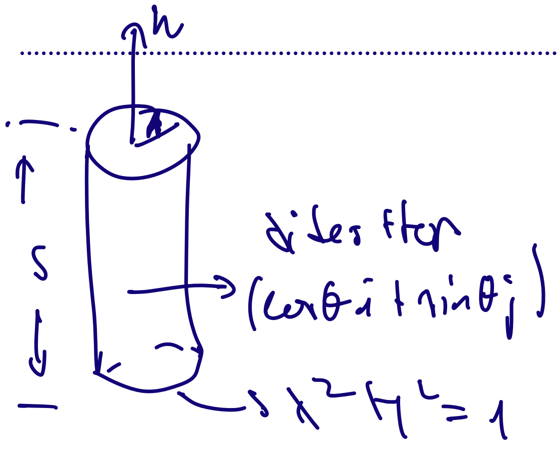
$$= \int_0^{2\pi}\int_0^1 \left(-r^2\cos^2\theta - r^2\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(-\frac{r^3}{3}\cos^2\theta - \frac{r^3}{3}\right)\Bigg|_0^1 d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left(-\cos^2\theta - 1\right) d\theta$$

$$= \frac{1}{3}\left(-3\pi\right) = \boxed{-\pi}$$

Stokes theorem ✓



$x^2+y^2=1$

$x = \cos\theta$

$y = \sin\theta$

sides: $\int \left(-x\,i + (z-1)\,\underline{k}\right)\cdot\left(\cos\theta\, i + \sin\theta\, j\right) r\, d\theta\, dz$

$$= \int (-\cos\theta\, i + (z-1)k)(\cos\theta\, i + \sin\theta\, j)\, d\theta\, dz$$

$$= \int_0^{2\pi} \int_0^5 -\cos^2\theta \, d\theta\, dz = -5\pi$$

top: $\underline{n} = \underline{k}$

$$\int (-x i + (z-1)k) \cdot \underline{k} \cdot dA =$$

$$\int (z-1)\, dA =$$

$$4 \int dA \xrightarrow{\text{circle}} = 4 \cdot \pi 1^2$$

$$= 4\pi$$

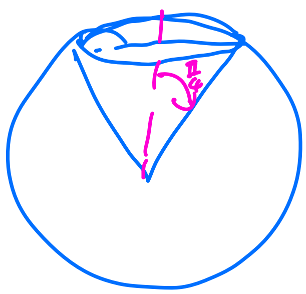
$$\int_{\text{sides + top}} \nabla \times F \cdot dA = -5\pi + 4\pi = -\pi$$

not only about closed surfaces!

generalization of green's theorem

as long as same opening it is same

wiley like last Q



$z = \sqrt{x^2+y^2}$

$R = 6$

volume of region

$V = \int dV$

$$\text{Volume} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^6 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$B$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$$

2x3

3x2

C A

3x2

3x3

A C