PMATH 370 Winter 2024:
Lecture Notes

Lecture notes taken, unless otherwise specified, by myself during the Winter 2024 offering of PMATH 370, taught by Blake Madill.

Lectures

Chapter 1

Iteration and Orbits

1.1 Orbits

Lecture 1
Jan 8

Definition 1.1.1 (iteration)
Let $f : A \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$. For $a \in A$ we may iterate the function at a:

$$x_1 = a, x_2 = f(a), x_3 = \underbrace{f(f(a))}_{f^2(a)}, \dots, x_i = f^{i-1}(a), \dots .$$

The sequence $(x_n)_{n=1}^{\infty}$ is the orbit of a under f (abbreviated (x_n) without limits).

Example 1.1.2. Let $f(x) = x^4 + 2x^2 - 2$, $a = -1$. What is the orbit of a under f?

Solution. $a = -1$, $f(a) = 1$, $f(f(a)) = f(1) = 1$, so we have $-1, 1, 1, 1, \dots$. We call this eventually constant. $\square$

Example 1.1.3. Let $f(x) = -x^2 - x + 1$, $a = 0$. What is the orbit of a under f?

Solution. Calculate: $0, 1, -1, 1, -1, 1, \dots$. We call this eventually periodic (with period 2). $\square$

Example 1.1.4. Let $f(x) = x^3 - 3x + 1$, $a = 1$. What is the orbit of a under f?

Solution. Calculate the first few terms: $1, -1, 3, 19, \dots$ (too big). This is a divergence to infinity. $\square$

Example 1.1.5. Let $f(x) = x^2 + 2x$, $a = -0.5$. What is the orbit of a under f?

Solution. Calculate: $-0.5, -0.75, -0.9375, -0.9961 \dots$ and we make an educated guess that this converges to -1 since $f(-1) = -1$, a fixed point. $\square$

Example 1.1.6. Let $f(x) = x^3 - 3x$, $a = 0.75$. What is the orbit of a under f?

Solution. Calculate: $0.75, -1.828, -0.625, 1.631, -0.552, \dots$. There is no clear pattern, so we call this chaotic. In fact, the orbit is dense in a neighbourhood of 0. $\square$

We can start to formalize the examples.

Definition 1.1.7 (fixed point)

Let $f : A \to \mathbb{R}$ such that $f(A) \subseteq A$. A point $a \in A$ is fixed if $f(a) = a$.

Then, the orbit of a under f is $(a, a, a, \dots)$ which is constant.

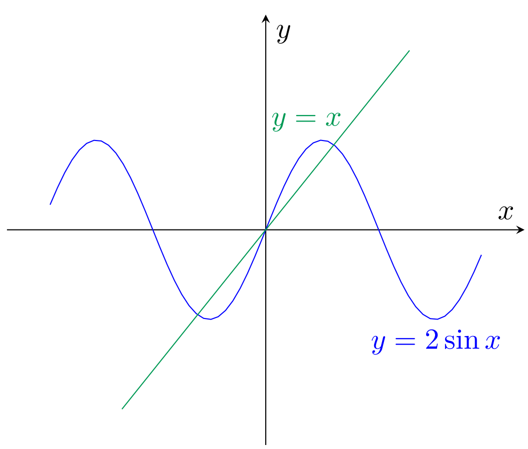
Example 1.1.8. Find all fixed points of $f(x) = x^2 + x - 4$.

Solution. We find points where $f(x) = x$, i.e., $x^2 + x - 4 = x$.

$$x^2 + x - 4 = x \iff x^2 = 4 \iff x = \pm 2$$ $\square$

Example 1.1.9. How many fixed points does $f(x) = 2\sin x$ have?

Solution. Consider where the curve $y = 2\sin x$ meets $y = x$:



We can see there are three fixed points. $\square$

Example 1.1.10. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Proof. We must show there is a solution to $x^4 - 3x + 1 \iff x^4 - 4x + 1 = 0$. Let $g(x) = x^4 - 4x + 1$. Since $g(x)$ is continuous, $g(0) = 1 > 0$, and $g(1) = -2 < 0$, by the Intermediate Value Theorem, there must exist a root of g on the interval $(0, 1)$. That is, a fixed point of f. $\square$

Definition 1.1.11 (periodicity)

Let $f : A \to \mathbb{R}, f(A) \subseteq A$.

1. A point $a \in A$ is periodic for f if its orbit is periodic. An orbit is periodic if for some $n \in \mathbb{N}$, $f^n(a) = a$. The smallest n is the period of (the orbit of) a.
2. An orbit (of a point) is eventually periodic if there exists $n < m$ such that $f^n(a) = f^m(a)$. The smallest difference $m - n$ is the period of the orbit.

Lecture 2
Jan 10

Definition 1.1.12 (doubling function)

$D : [0, 1) \to [0, 1) : x \mapsto 2x - \lfloor 2x \rfloor$ returns the fractional part of $2x$.

Example 1.1.13. $D(0.4) = 0.8,\ D(0.6) = 0.2,\ D(0.8) = 0.6,\ D(0.5) = 0.$

This is a nice function that gives lots of periodic orbits for funsies.

Example 1.1.14. Find the orbit of $a = \frac{1}{5}$ under D.

Solution. Double until we pass 1: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{8}{5} \to \frac{3}{5}, \frac{6}{5} \to \frac{1}{5}$. The period is $\left|\{\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\}\right| = 4$. □

Example 1.1.15. Find the orbit of $a = \frac{1}{20}$ under D.

Solution. Double: $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}$ and we can stop because ex. 1.1.14 showed $\frac{1}{5}$ is periodic.

So this is eventually periodic with period 4. □

Problem 1.1.16

Given f and a, does $f^n(a)$ tend towards some limit L?

To solve this problem, we need to rigorously define "tend" and "limit".

1.2 Real analysis review

Notation. If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers, we write $(x_n) \subseteq \mathbb{R}$.

Definition 1.2.1 (convergence of a sequence)
Let $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$.

We say (x_n) converges to x if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

Then, we write $x_n \to x$ or $\lim x_n = x$.

Example 1.2.2. Show that $\frac{1}{n} \to 0$.

Proof. Let $\varepsilon > 0$. Consider $N = \frac{2}{\varepsilon} > \frac{1}{\varepsilon}$. For $n \geq N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

Therefore, $\frac{1}{n} \to 0$. □

Example 1.2.3. Prove that $\frac{2n}{n+3} \to 2$.

Proof. Let $\varepsilon > 0$. Since we know $\frac{1}{n} \to 0$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{6}$.

For $n \geq N$,

$$\left|\frac{2n}{n+3} - 2\right| = \left|\frac{2n}{n+3} - \frac{2n+6}{n+3}\right| = \left|\frac{-6}{n+3}\right| = \frac{6}{n+3} < \frac{6}{n} \leq \frac{6}{N} < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

Therefore, $\frac{2n}{n+3} \to 2$. □

Definition 1.2.4 (bounded sequence)
A sequence (x_n) is bounded (by M) if there exists $M > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq M$.

Proposition 1.2.5 (convergence implies boundedness)
If (x_n) is convergent, then (x_n) is bounded.

Proof. Suppose $x_n \to x$. Then, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - x| < 1$.

For $n \geq N$, $|x_n| - |x| \leq |x_n - x| < 1$. That is, $|x_n| < 1 + |x|$.

Let $M = \max\{|x_1|, \ldots, |x_{n-1}|, 1 + |x|\}$. Then, for both all $n < N$ and $n \geq N$, we have $|x_n| \leq M$. □

Remark 1.2.6. The converse is not true. Notice that $x_n = (-1)^n$ is bounded by 1 but obviously not convergent.

Proposition 1.2.7 (limit laws)

Let $x_n \to x$ and $y_n \to y$. Then:

(1) $x_n + y_n \to x + y$

(2) $x_n y_n \to xy$

Proof. (1) Let $\varepsilon > 0$. Then, since $x_n \to x$ and $y_n \to y$, there exist $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies |x_n - x| < \frac{\varepsilon}{2}$ and $n \geq N_2 \implies |y_n - y| < \frac{\varepsilon}{2}$.

For $N = \max\{N_1, N_2\}$ and $n \geq N$,

$$\begin{aligned} |(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

That is, $x_n + y_n \to x + y$.

(2) Let $\varepsilon > 0$. Notice that:

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \tag{$*$}$$

Since x_n is bounded, there exists $M > 0$ such that $|x_n| \leq M$ for all n.

Let $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 &\implies |x_n - x| \leq \frac{\varepsilon}{2(|y| + 1)} \text{ and} \\ n \geq N_2 &\implies |y_n - y| < \frac{\varepsilon}{2M}. \end{aligned}$$

Then, for $n \geq N := \max\{N_1, N_2\}$, $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by $(*)$. $\square$

Lecture 3
Jan 12

Definition 1.2.8 (Cauchy sequence)

We say $(x_n) \in \mathbb{R}$ is Cauchy if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all n and m,

$$n, m \geq N \implies |x_n - x_m| < \varepsilon$$

Proposition 1.2.9

Every convergent sequence is Cauchy.

Proof. Intuitively: if the terms get arbitrarily close to some limit, they must get arbitrarily close to each other.

Formally: Let $x_n \to x$ be a convergent sequence and $\varepsilon > 0$. Since x_n converges, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |x_n - x| < \frac{\varepsilon}{2}$.

Then, when $n, m \geq N$, we have:

$$\begin{aligned} |x_n - x_m| &= |x_n - x_m + x - x| \\ &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

as desired. □

We take the following theorem from real analysis without proof.

Theorem 1.2.10 (completeness of $\mathbb{R}$)
A sequence is Cauchy if and only if it is convergent.

The big idea here: To prove (x_n) is Cauchy, you do not have to guess the limit first. That is, if you must prove convergence but do not care about the limit's value, prove that it is Cauchy instead.

Definition 1.2.11 (continuity of a function)
Let $f : A \to \mathbb{R}, A \subseteq \mathbb{R}, a \in A$. We say f is continuous at a if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in A$ and $|x - a| < \delta$.

If f is continuous at all $a \in A$, we say it is continuous.

We also take this theorem from MATH 137 without proof.

Theorem 1.2.12
A function $f : A \to \mathbb{R}$ is continuous at $a \in A$ if and only if for all sequences $(x_n) \subseteq A$ with $x_n \to a$, we have $f(x_n) \to a$.

1.3 Orbits, revisited

Proposition 1.3.1
If $f : [a, b] \to [a, b]$ is continuous, then $f(x)$ has a fixed point.

Proof. We know by the domain and codomain that $f(a) \geq a$ and $f(b) \leq b$. This means $f(a) - a \geq 0$ and $f(b) - b \leq 0$. By the IVT on the continuous function $g(x) = f(x) - x$, we know there exists an $x \in [a, b]$ such that $g(x) = f(x) - x = 0 \iff f(x) = x$, i.e., x is a fixed point. □

Definition 1.3.2 (contraction)

Let $f : A \to \mathbb{R}, A \subseteq \mathbb{R}$. We say f is a contraction if there exists $C \in [0, 1)$ such that for all $x, a \in A$,

$$|f(x) - f(y)| \leq C|x - y|$$

This is just a Lipschitz function with Lipschitz constant less than 1.

Proposition 1.3.3

Contractions are continuous.

Proof. Let $\varepsilon > 0$. Suppose f is a contraction such that $|f(x) - f(y)| \leq C|x - y|$.

Consider $y \in A$. Let $\delta = \frac{\varepsilon}{C+1}$ and assume that $x \in A$ and $|x - y| < \delta$. But we have:

$$|f(x) - f(y)| \leq C|x - y| \leq C\delta < \varepsilon$$

as desired. □

Definition 1.3.4 (closure of an interval)

We say $A \in \mathbb{R}$ is closed if whenever $(x_n) \subseteq A$ with $x_n \to x$, then $x \in A$.

Example 1.3.5. $[a, b]$ is closed but $(0, 1]$ is not because $\frac{1}{n} \to 0 \notin (0, 1]$.

Theorem 1.3.6 (Banach contraction mapping theorem)

Suppose $A \subseteq \mathbb{R}$ is closed and $f : A \to A$ is a contraction. Then, there exists a unique fixed point $a \in A$ for f.

Moreover, for all $x \in A$, $f^n(x) \to a$.

Example 1.3.7. Analyze the orbit of $f : [0, 1] \to [0, 1]$, $f(x) = \frac{1}{3-x}$.

Solution. We can observe that $\frac{1}{3} \leq \frac{1}{3-x} \leq \frac{1}{2}$.

Also, $f'(x) = \frac{1}{(3-x)^2}$. Notice that $\frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$. So by the mean value theorem, for all $x, y \in [0, 1]$, there exists $c \in (0, 1)$ such that:

$$\begin{aligned} f(x) - f(y) &= f'(c)(x - y) \\ |f(x) - f(y)| &= |f'(c)| \cdot |x - y| \\ &\leq \frac{1}{4}|x - y| \end{aligned}$$

Then, identifying $C = \frac{1}{4}$, f is a contraction. Now,

$$\frac{1}{3-x} = x \iff 1 = 3x - x^2 \iff x^2 - 3x + 1 = 0 \iff x = \frac{3 \pm \sqrt{9-4}}{2} \iff x = \frac{3-\sqrt{5}}{2}$$

where we pick the negative root because we need $x \in [0,1]$.

Therefore, by the Banach contraction mapping theorem, for all $x \in [0,1]$, $f^n(x) \to \frac{3-\sqrt{5}}{2}$. □

Lecture 4
Jan 15

Definition 1.3.8
A sequence $(a_n) \subseteq \mathbb{R}$ is strongly-Cauchy if there exists $(\varepsilon_n) \subseteq [0,\infty)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and for all n, $|a_n - a_{n+1}| < \varepsilon_n$.

Informally, "far enough along the sequence, the *neighbours* must get close". This is distinct from Cauchy, which is "far enough along the sequence, the *terms* must get close".

Remark 1.3.9 (assignment hint!). Let $\sum_{n=1}^{\infty} a_n = L$. This means that $\sum_{k=1}^{n} a_k \xrightarrow{n\to\infty} L$.

That is, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\sum_{k=1}^{n} a_k - L\right| < \varepsilon$.

But $\left|\sum_{k=1}^{n} a_k - L\right| = \left|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{\infty} a_k\right| < \varepsilon$.

We can now prove the Banach contraction mapping theorem.

Proof. Let $A \subseteq \mathbb{R}$ be closed and suppose there exists $f : A \to A$ and $C \in [0,1)$ such that $|f(x) - f(y)| \leq C|x-y|$ for all x and y in A.

Fix $x_0 \in A$ and construct the orbit $x_1 = f(x_0)$, $x_2 = f(x_1)$, ..., $x_n = f(x_{n-1}) = f^n(x_0)$.

For $n \in \mathbb{N}$, since f is a contraction,

$$\begin{aligned}
|x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \\
&\leq C|x_n - x_{n-1}| \\
&= C|f(x_{n-1}) - f(x_{n-2})| \\
&\leq C^2|x_{n-1} - x_{n-2}| \\
&\vdots \\
&\leq C^n|x_1 - x_0|
\end{aligned}$$

Since $\sum_{n=1}^{\infty} C^n|x_1 - x_0| = |x_1 - x_0| \sum_{n=1}^{\infty} C^n$ is a convergent geometric series, we have that the sequence (x_n) is strongly-Cauchy.

Hence, by Assignment 1, $x_n \to a$ for some limit point $a \in A$ since A is closed.

Since f is continuous (prop. 1.3.3), we have that $f(x_n) \to f(a)$. By definition, $f(x_n) = x_{n+1}$, so $x_n \to f(a)$. But we already know $x_n \to a$, so $a = f(a)$. That is, a is a fixed point of f.

It remains to show uniqueness.

Suppose $a, b \in A$ such that $f(a) = a$ and $f(b) = b$.

$$\begin{aligned} |f(a) - f(b)| &\leq C|a - b| \\ |a - b| &\leq C|a - b| \end{aligned}$$

Since $C < 1$, we must have $|a - b| = 0$, that is, $a = b$. $\square$

Chapter 2

Graphical Analysis

2.1 Cobweb plots

Recall ex. 1.1.9. To visualize the orbit of a under f, we can:

1. Superimpose $y = f(x)$ over the line $y = x$.
2. Connect a vertical line $(a, a) - (a, f(a))$
3. Connect a horizontal line $(a, f(a)) - (f(a), f(a))$
4. Connect a vertical line $(f(a), f(a)) - (f(a), f(f(a)))$
5. Connect a horizontal line $(f(a), f(f(a))) - (f(f(a)), f(f(a)))$
   etc.

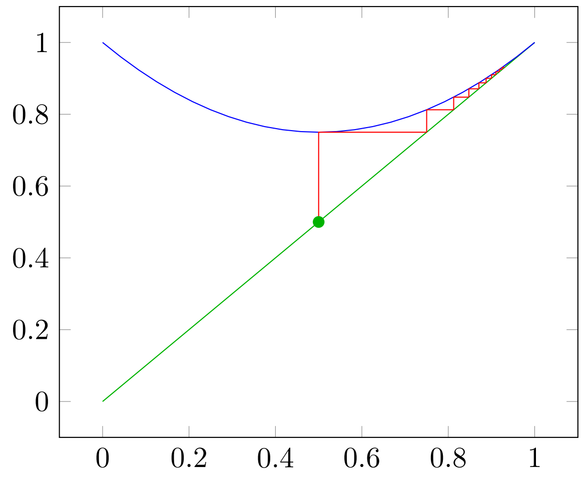
This is sometimes called a cobweb plot. We will be using https://marksmath.org/visualization/cobwebs/ to make cobweb plots.

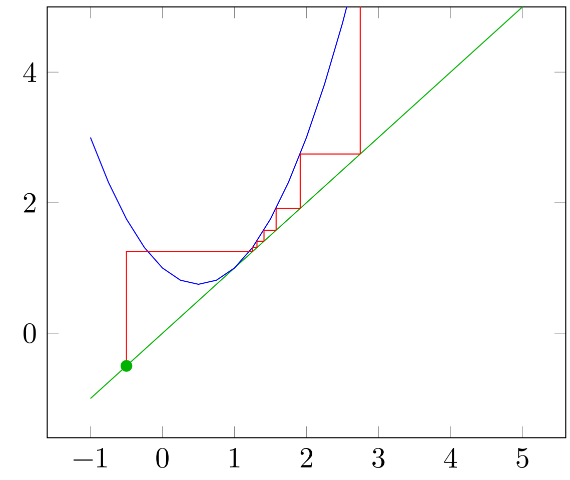
Within these lecture notes, I use a LaTeX macro to draw plots defined here.

> **Example 2.1.1.** Conduct a complete orbit analysis of $f(x) = x^2 - x + 1$

Solution. Playing around, we find that there is one fixed point $x = 1$.

When $x \in [0, 1]$, $f^n(x) \to 1$. Otherwise, $f^n(x) \to \infty$.



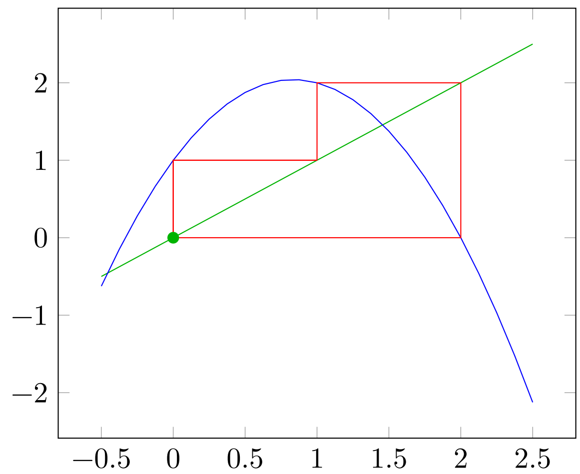


□

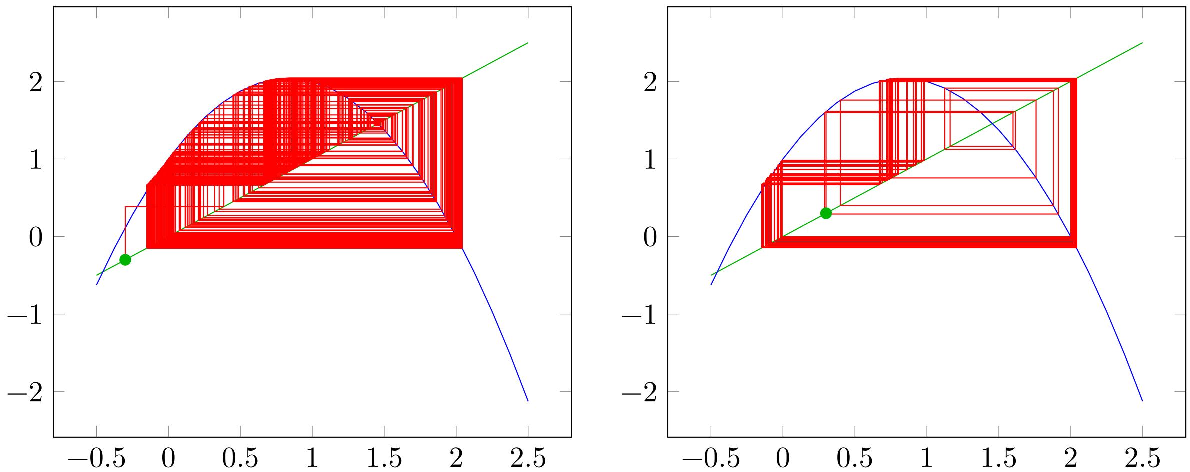
↓ *Lectures 5 and 6 adapted from Rosie* ↓ *Lecture 5 Jan 17*

> **Example 2.1.2.** Conduct a complete orbit analysis of $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$.

Solution. At $x = 0$, we can see there is a cycle going from $0 \to 1 \to 2 \to 0$:



At points near 0, like $x = -0.3$ or $x = 0.3$, the graph becomes chaotic:



It appears that the cobweb densely covers the graph. ☐

As an aside, note that we cannot actually hit every point in the interval because the orbit is countable (i.e., has the same size as the naturals) but the interval is uncountable. We will later show that the points are dense (as the rationals are).

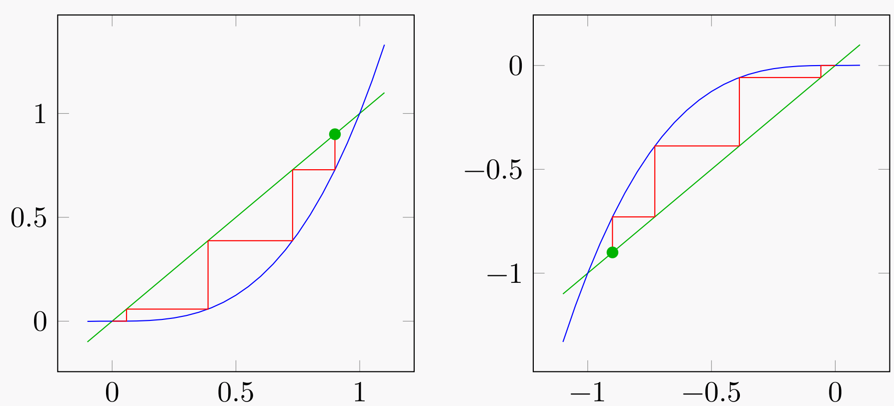
Chapter 3

Fixed Points

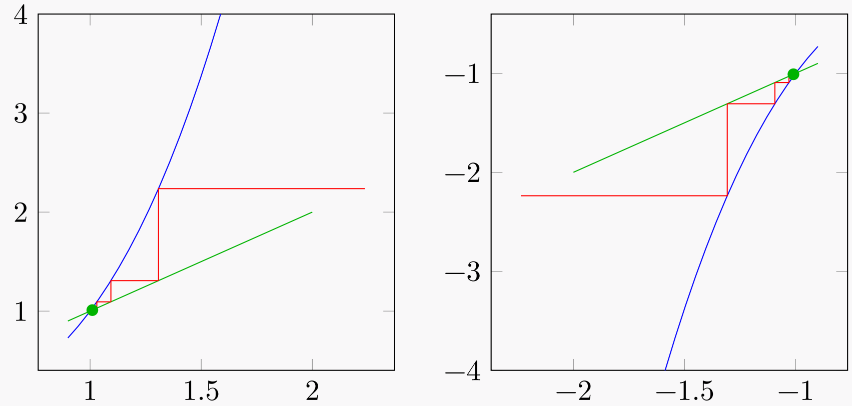
3.1 Attracting/repelling fixed point theorems

Remark 3.1.1. If $f(x)$ is continuous and $f^n(a) \to L$, then $f^{n+1}(a) \to f(L)$. Therefore, $f(L) = L$ is a fixed point.

Example 3.1.2. The function $f(x) = x^3$ has three fixed points: $0, \pm 1$. For $x \in (-1, 1)$, we see that $f^n(x) \to 0$:

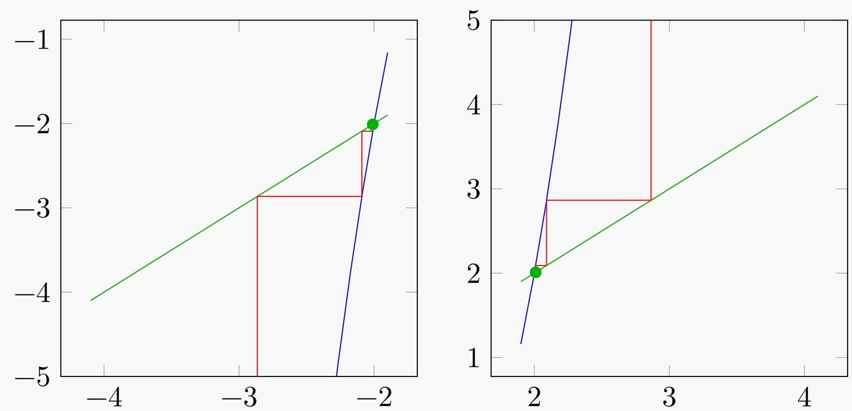


It looks like point 0 is <u>attracting</u> the orbit. For $x \in (-\infty, -1) \cup (1, \infty)$, we see $f^n(x) \to \infty$:

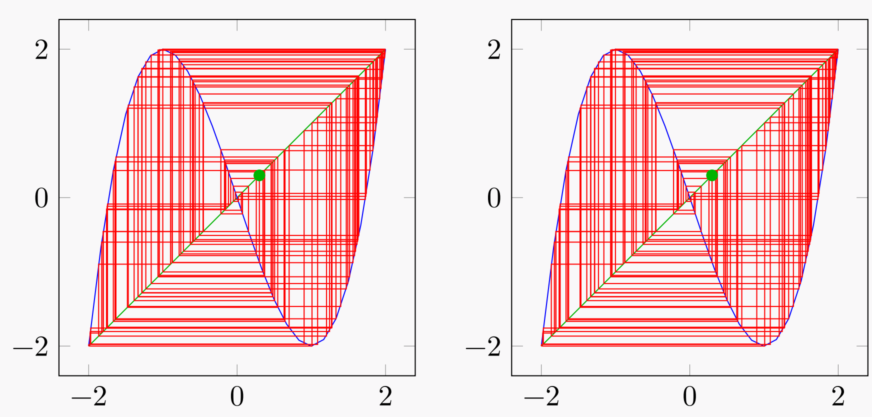


so the points ± 1 are <u>repelling</u> the orbit.

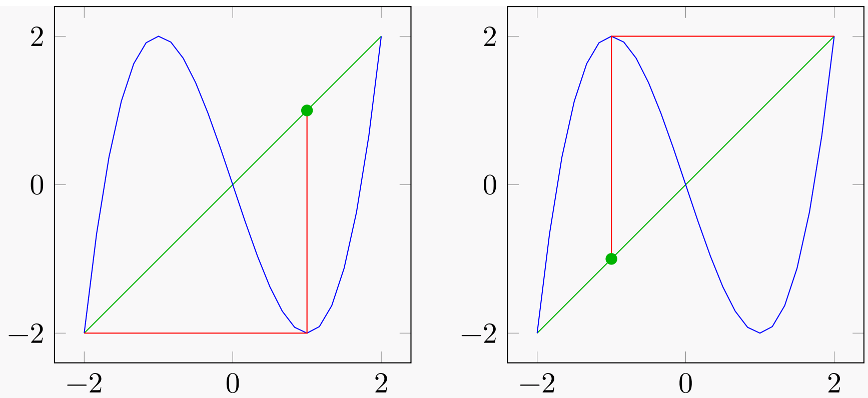
Example 3.1.3. The function $f(x) = x^3 - 3x$ also has three fixed points: $0, \pm 2$. To the right (left) of ± 2, orbits go to infinity:



The point 0 is repelling (in a different sense) since we get chaos:



At $x_0 = \pm 1$, the orbit is eventually constant, jumping to the fixed point ∓ 2:



Definition 3.1.4

Let a be a fixed point of $f(x)$.

1. If $|f'(a)| > 1$, we call a a <u>repelling</u> fixed point
2. If $|f'(a)| < 1$, we call a a <u>attracting</u> fixed point
3. If $|f'(a)| = 1$, we call a a <u>neutral</u> fixed point

Neutral fixed points can be a lot of different things.

Theorem 3.1.5 (attracting fixed point theorem)

Suppose a is an attracting fixed point of $f(x)$. Then, there exists an open interval I containing a such that

1. for all $x \in I$, $n \in \mathbb{N}$, $f^n(x) \in I$
2. for all $x \in I$, $f^n(x) \to a$

Recall the ε-δ definition of a limit.

Definition 3.1.6 (limit of a function at a point)

Let $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$.

We say a point $a \in A$ is <u>non-isolated</u> if for each $\varepsilon > 0$ there exists $b \in A$, $b \neq a$ with $b \in (a - \varepsilon, a + \varepsilon)$.

Suppose a is non-isolated. We say $\lim_{x\to a} f(x) = L$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $a \in A$ and $0 < |x - a| < \delta$.

It is important to define non-isolation. If a is isolated, we can choose a δ where $|x - a| < \delta$ is false. Then, every point is vacuously a limit point.

We now give the proof of the attracting fixed point theorem:

Proof. Assume $|f'(a)| < 1$. Then, there exists $c \in \mathbb{R}$ such that $|f'(a)| < c < 1$. By definition of the derivative, this means we can write

$$\lim_{x \to a} \frac{|f(x) - f(a)|}{x - a} < c$$

and by the definition of the limit, we know there exists $\delta > 0$ such that

$$\frac{|f(x) - f(a)|}{|x - a|} \leq c, \quad \forall x \in (a - \delta, a + \delta)$$

Hence, for $x \in I := (a - \delta, a + \delta)$, we have $|f(x) - f(a)| \leq c|x - a|$ and f is a contraction.

Lecture 6 Jan 19

In particular, for $x \in I$, we have

$$\begin{aligned} |f(x) - a| &= |f(x) - f(a)| && (a \text{ is a fixed point}) \\ &\leq c|x - a| \leq |x - a| && (c \in (0, 1)) \\ &< \delta \end{aligned}$$

That is, $f(x) \in (a - \delta, a + \delta) = I$. Continuing for the rest of the orbit, for all $n \in \mathbb{N}$,

$$|f^n(x) - a| \leq c^n|x - a| \leq |x - a| < \delta$$

so we also have $f^n(x) \in I$.

Finally, notice that $0 \leq |f^n(x) - a| \leq c^n|x - a|$ and $c^n|x - a| \to 0$ since $c \in (0, 1)$. By the squeeze theorem, $|f^n(x) - a| \to 0$. □

Theorem 3.1.7 (repelling fixed point theorem)
Suppose a is a repelling fixed point for $f(x)$. Then, there exists an open interval I containing a such that for all $x \in I$, $x \neq a$, there exists $n \in \mathbb{N}$ such that $f^n(x) \notin I$.

Proof. Say $|f'(a)| > c > 1$. Then, as above, there exists a δ such that

$$\lim_{x \to a} \frac{|f(x) - f(a)|}{x - a} > c \implies |f(x) - f(a)| \geq c|x - a|$$

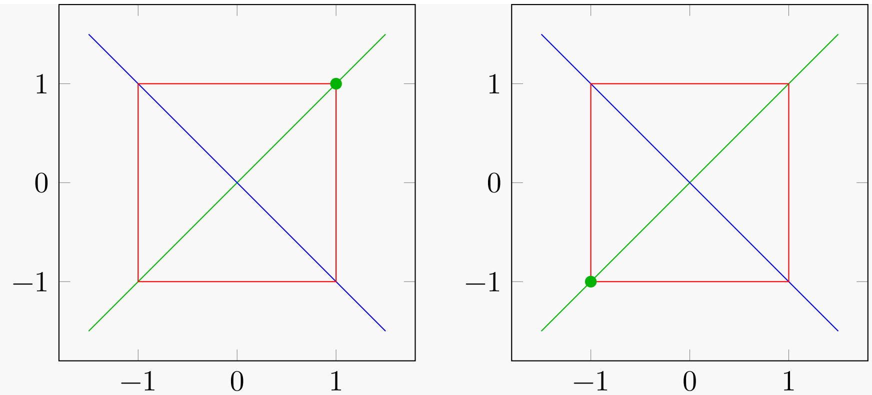
for all $x \in I := (a - \delta, a + \delta)$.

Since a is a fixed point, $|f(x) - f(a)| = |f(x) - a|$. Suppose for a contradiction that for all n, $f^n(x) \in I$. But since $c > 1$, $|f(n) - a| \geq c^n|x - a| \to \infty$. That is, δ must be arbitrarily large, which it is not. □

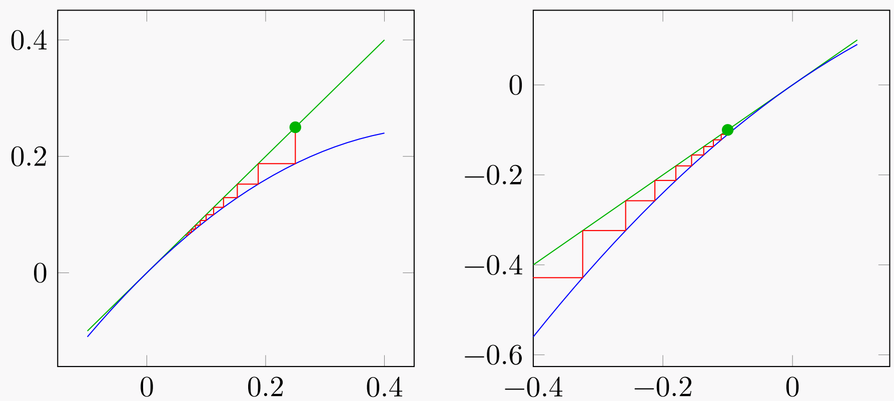
3.2 Neutral fixed points

Neutral fixed points can exhibit a lot of different behaviours.

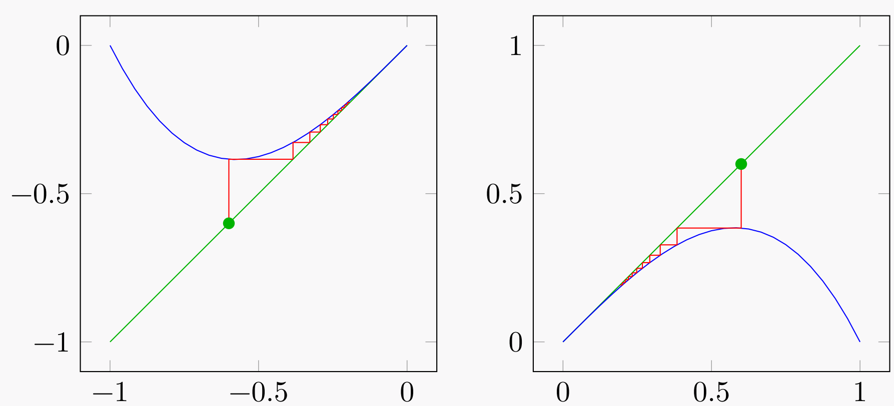
Example 3.2.1. For $f(x) = -x$, 0 is a fixed point with $|f'(0)| = 1$. The orbit bounces:



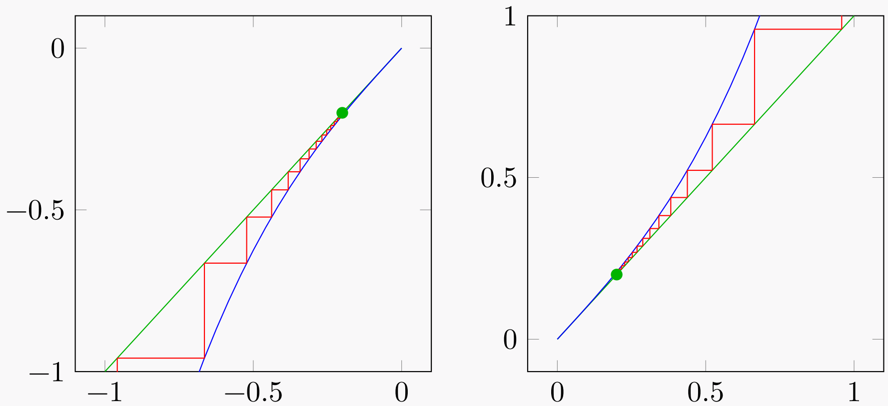
Example 3.2.2. For $f(x) = x - x^2$, $|f'(1)| = 1$ is a neutral fixed point. It is attracting from the right and repelling from the left:



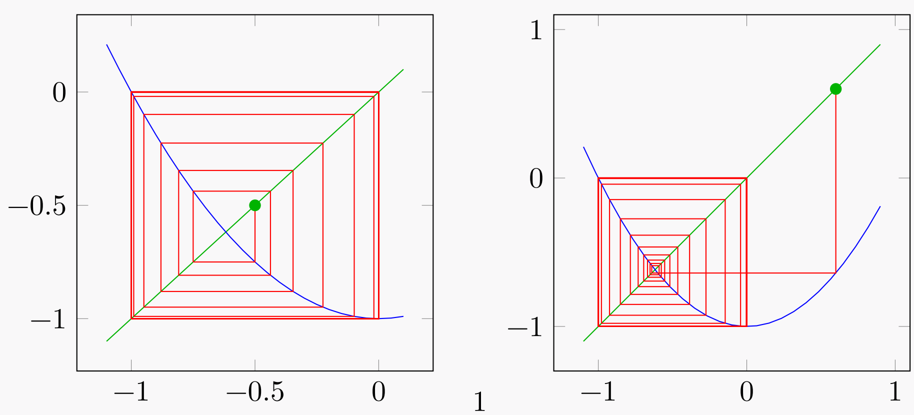
Example 3.2.3. For $f(x) = x - x^3$, $|f'(0)| = 1$ is a neutral fixed point. It is <u>weakly attracting</u>, attracting but too slowly.



Example 3.2.4. For $f(x) = x + x^3$, $|f'(0)| = 1$ is a neutral fixed point. It is <u>weakly repelling</u>, repelling but too slowly:



Example 3.2.5. Consider $f(x) = x^2 - 1$. The orbit at $a = 0$ is periodic $(0, -1, 0, -1, \dots)$ with period 2. Near 0, the orbit tends to the $(0, -1)$-cycle:



We will call 0 an <u>attracting periodic point</u> because 0 is an attracting point of $f^2(x)$.

↑ *Lectures 5 and 6 adapted from Rosie* ↑

Lecture 7
Jan 22

Definition 3.2.6

Let a be a periodic point for $f(x)$ with period n.

We say a is an <u>attracting</u>/<u>repelling</u>/<u>neutral periodic point</u> if a is an attracting/repelling/neutral fixed point of f^n

Finding a closed form expression for something like $f^{10}(x)$ is a nightmare, so we need a better way.

Proposition 3.2.7

Let $f(x)$ be a differentiable function. Then, $(f^n)'(x) = f'(x) \cdot f'(f(x)) \cdots f'(f^{n-1}(x))$.

Proof. Proceed by induction on n.

If $n = 1$, we have $f'(x) = f'(x)$ and we are done.

Suppose $(f^n)'(x) = \prod_{k=0}^{n-1} f'(f^k(x))$ for some $n \geq 1$. Consider f^{n+1}:

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{n+1}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(f^n(x)) = f'(f^n(x)) \cdot (f^n)'(x)$$

by the chain rule. Then,

$$\begin{aligned}
(f^{n+1})'(x) &= f'(f^n(x)) \cdot (f^n)'(x) \\
&= f'(f^n(x)) \cdot \prod_{k=0}^{n-1} f'(f^k(x)) \\
&= \prod_{k=0}^{n} f'(f^k(x))
\end{aligned}$$

completing the proof. □

Example 3.2.8. Analyze the periodic point $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$, $a = 0$

Solution. The orbit is $(0, 1, 2, 0, 1, 2, \dots)$ with period 3.

We have $f'(x) = -3x + \frac{5}{2}$. Then, $(f^3)'(0) = f'(0)f'(1)f'(2) = (-\frac{7}{2})(-\frac{1}{2})(\frac{5}{2}) = \frac{35}{8} > 1$.

Therefore, the point is repelling. □

Chapter 4

Bifurcations

In general, bifurcation theory is the study of how a family of curves can change when a defining parameter is changed.

Consider the quadratic family:

$$Q_C(x) = x^2 + C$$

defined by the parameter $C \in \mathbb{R}$.

Problem 4.0.1

How does the behaviour (fixed points, orbits, etc.) of Q_C change based on C?

First, we can find the fixed points (if they exist) by solving

$$Q_C(x) = x \iff x^2 - x + C = 0 \iff x = \frac{1 \pm \sqrt{1-4C}}{2}$$

and note that $Q_C(x)$ has 2 fixed points when $C < \frac{1}{4}$, 1 fixed point when $C = \frac{1}{4}$, and no fixed points when $C > \frac{1}{4}$.

Suppose $C > \frac{1}{4}$. Then, we must have $Q_C^n(x) \to \infty$ for all x.

Instead, if $C = \frac{1}{4}$, $Q_C(x)$ has the unique fixed point $p = \frac{1}{2}$. Since $Q_C'(x) = 2x$ and $Q_C'(p) = 1$, this is a neutral fixed point. In fact, it attracts to one side and repels from the other.

Finally, if $C < \frac{1}{4}$, $Q_C(x)$ has two fixed points $p_+ = \frac{1+\sqrt{1-4C}}{2}$ and $p_- = \frac{1-\sqrt{1-4C}}{2}$. Then, $Q_C'(p_+) =$

$1 + \sqrt{1-4C} > 1$ is repelling. Next,

$$\begin{aligned}
& -1 < Q_C'(p_-) < 1 \\
\iff & -1 < 1 - \sqrt{1-4C} < 1 \\
\iff & -2 < -\sqrt{1-4C} < 0 \\
\iff & 0 < \sqrt{1-4C} < 2 \\
\iff & -\frac{3}{4} < C < \frac{1}{4}
\end{aligned}$$

and in fact if $C < -\frac{3}{4}$, $Q_C'(p_-) < -1$ and if $C = -\frac{3}{4}$, $Q_C'(p_-) = -1$.

Theorem 4.0.2

For the family

$$Q_C(x) = x^2 + C,$$

depending on C:

1. All orbits tend to ∞ if $C > \frac{1}{4}$.
2. When $C = \frac{1}{4}$, $Q_C(x)$ has a unique fixed point $\frac{1}{2}$ and it is neutral.
3. If $C < \frac{1}{4}$, $Q_C(x)$ has two fixed points p_+ and p_-. The point p_+ is repelling. Moreover,
   (a) if $-\frac{3}{4} < C < \frac{1}{4}$, p_- is attracting;
   (b) if $C = -\frac{3}{4}$, p_- is neutral; and
   (c) if $C < -\frac{3}{4}$, p_- is repelling.

Lecture 8
Jan 24

Definition 4.0.3 (bifurcation)

We say a family of functions $F_\lambda(x)$ undergoes a bifurcation at λ_0 if there is a change in fixed point structure at λ_0.

Example 4.0.4. The quadratic family $Q_C(x) = x^2 + C$ undergoes a bifurcation at $\lambda_0 = \frac{1}{4}$.

Definition 4.0.5 (tangent bifurcation)

A family $F_\lambda(x)$ undergoes a tangent bifurcation at λ_0 if there is an open interval I and an $\varepsilon > 0$ such that:

1. for $\lambda_0 - \varepsilon < \lambda < \lambda_0$, $F_\lambda(x)$ has no fixed points on I;
2. for $\lambda = \lambda_0$, $F_\lambda(x)$ has one fixed point and it is neutral; and
3. for $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, $F_\lambda(x)$ has two fixed points in I, one of which is attracting and the other repelling.

(or with all inequalities flipped)

Visually, you have situations like

TODO: graphs

for $\lambda < \lambda_0$, $\lambda = \lambda_0$, and $\lambda > \lambda_0$.

Example 4.0.6. Consider the exponential family $E_\lambda(x) = e^x + \lambda$ at $\lambda_0 = -1$.

This is a tangent bifurcation.

Example 4.0.7. $F_\lambda(x) = \lambda x(1 - x), \lambda_0 = 1$

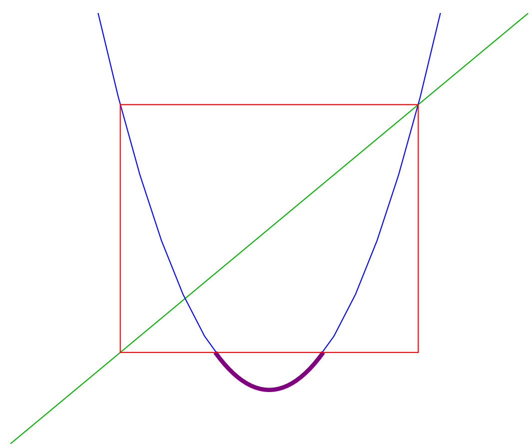
Here, we have two fixed points on one side of λ_0 and one fixed point on the other. So this is a bifurcation but not a tangent bifurcation.

Chapter 5

Cantor set

Recall the quadratic family $Q_C(x) = x^2 + C$ for $C < -2$. Then, $p_+ = \frac{1+\sqrt{1-4C}}{2} > 2$ and $-p_+ < -2$. Consider the interval/region $I = [-p_+, p_+]$ and $I \times I$.

Draw the picture of $y = x$, $y = Q_C(x)$, and the box $I \times I$:



Let $J_1 \subseteq I$ be the interval such that $Q_C(x) \notin I$ for all $x \in J_1$.

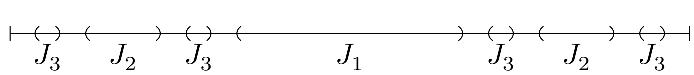
For $x \in J_1$, $Q_C^n(x) \to \infty$. Moreover, if there exists n such that $Q_C^n(x) \in J_1$, then $Q_C^n(x) \to \infty$.

Consider the set of points $\Lambda = \{x \in I : \forall n, Q_C^n(x) \in I\}$ with "interesting" orbits staying inside I.

Now, let $J_2 = \{x \in I : Q_C(x) \in J_1\} = \{x \in I : Q_C^2(x) \notin I\}$ and define higher J_n likewise.

Then, $\Lambda = I \setminus (J_1 \cup J_2 \cup \cdots)$ is a Cantor set, that is, a fractal. (roll credits!)

Drawing Λ on the x-axis, we get something that looks like



↓ *Lecture 9 adapted from Imaad* ↓

Lecture 9
Jan 26

Definition 5.0.1 (Cantor middle thirds set)
Let $C_0 = [0, 1]$. Remove the open middle third interval each time.

That is, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so on.

The set $K = \bigcap_{n=1}^{\infty} C_n$ is the Cantor (middle thirds) set.

Proposition 5.0.2
Suppose a bunch of sets $A_n \subseteq \mathbb{R}$ are closed. Then, $\bigcap A_n$ is also closed.

Proof. Let $(a_k) \subseteq \cap A_n$ where $(a_k) \to a$.

Note that for all n, $(a_k) \subseteq A_n \implies a \in A_n \implies a \in \bigcap A_n$ □

Proposition 5.0.3
Let $A, B \subseteq \mathbb{R}$ be closed. Then, $A \cup B$ is closed.

Proof. Let $(a_n) \subseteq A \cup B$ where $a_n \to a$.

WLOG, $\{n : a_n \in A\}$ is infinite. This allows us to construct $(b_n) \subseteq A$ such that $b_n \to a$.

Since A is closed, $a \in A \subseteq A \cup B$. □

Theorem 5.0.4 (Cantor sets are closed)
Any Cantor set, in particular K, is closed.

Theorem 5.0.5
K contains no non-empty open intervals.

Proof. Consider $I \subseteq K$. Then $\forall n, I \subseteq C_n$.

Then $\ell(I) \leq \frac{1}{3^n} \implies \ell(I) = 0 \implies I = \varnothing$, contradiction. □

Now, let's consider the base-3 expansion of $x \in [0, 1]$. $x = 0.s_1 s_2 s_3, \cdots, s_i \in \{0, 1, 2\}$

Consider $\underbrace{[0, 1/3]}_{s_1=0}$ and $\underbrace{[2/3, 1]}_{s_1=2}$ and $\underbrace{[0, 1/9]}_{s_1=0, s_2=0}$ $[2/9, 1/3]$ $[2/3, 7/9]$ $[8/9, 1]$.

Remark 5.0.6. $x \in K$ if and only if x can be written in base 3 using only 0s and 2s

Example 5.0.7. $\frac{1}{3} \in K$. $\frac{1}{3} = 0.1_3 = 0.02222\ldots_3$

Theorem 5.0.8

K is uncountable and $|K| = |\mathbb{R}|$.

↑ *Lecture 9 adapted from Imaad* ↑

Chapter 6

Symbolic dynamics

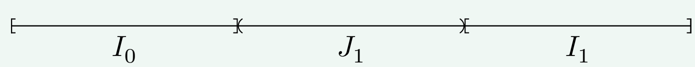
Lecture 10
Jan 29

Recall the construction of the Cantor set from the quadratic family:

> Fix $C < -2$ and consider $Q_C(x) = x^2 + C$. Define an interval $I = [-p_+, p_+]$ for a fixed point $p_+ = \frac{1+\sqrt{1-4C}}{2}$. Then, let
>
> $$\begin{aligned} J_1 &= \{x \in I : Q_C(x) \notin I\} \\ J_2 &= \{x \in I : Q_C(x) \in J_1\} \\ J_3 &= \{x \in I : Q_C(x) \in J_2\} \\ &\vdots \end{aligned}$$
>
> and define $\Lambda = I \setminus (\bigcup J_i) = \{x \in I : \forall n, Q_C^n(x) \in I\}$.

We proceed to do some analysis of Λ by translating into some sort of sequence space, doing analysis, and then going back to the Cantor set.

Notation. Define closed intervals $I_0 \cup I_1 := I \setminus J_1$ on the left/right of J_1:



Definition 6.0.1

For $x \in \Lambda$, the <u>itinerary</u> of x is the sequence $S(x) = (x_0 x_1 x_2 x_3 \cdots)$ with $x_i \in \{0, 1\}$ where $x_i = 0 \iff Q_C^i(x) \in I_0$ and $x_i = 1 \iff Q_C^i(x) \in I_1$.

Our goal is to understand $S(x)$ better so that we can glean information about Λ.

Notation. Let $\Sigma = \{(x_0 x_1 x_2 \cdots) : x_i \in \{0, 1\}\}$ be the sequence space. Write elements of Σ as binary strings. Then, $S : \Lambda \to \Sigma$ is a function.

It would be helpful to define some PMATH 351/topology shit.

6.1 Intro to topology

Definition 6.1.1 (metric space)

Let X be a set. A function $d : X \times X \to [0, \infty)$ is a metric if

1. $d(x, y) = 0 \iff x = y$ (positive definiteness),
2. $d(x, y) = d(y, x)$ (symmetry), and
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

The pair (X, d) is a metric space.

Once we have a metric space with a notion d of distance, we can adapt all our definitions from real analysis to an abstract space.

Example 6.1.2. The following are all metrics:

- $X = \mathbb{R}$, $d(x, y) = |x - y|$
- $X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$
- For any set X, the discrete metric $d(x, y) = [x \neq y]$ (but is not particularly useful).
- For a subset $A \subseteq R$, $d(x, y) = |x - y|$ is a metric.

Extremely helpfully, we can define a metric on the sequence space.

Definition 6.1.3 (Cantor space)

Let $X = \Sigma$. Define $d(x, y) = \sum_{i=0}^{\infty} 2^{-i} |x_i - y_i|$.

This is well-defined (converges) since $|x_i - y_i| \leq 1$ and $\sum 2^{-i}$ converges.

Example 6.1.4. Let $x = (1111\cdots)$ and $y = (1010\cdots)$. Calculate $d(x, y)$.

Solution. By definition,

$$\begin{aligned} d(x, y) &= \sum_{i=0}^{\infty} \frac{x_i - y_i}{2^i} \\ &= \sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} && \text{(even indices cancel)} \\ &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^i} \\ &= \frac{1}{2}\left(\frac{1}{1 - \frac{1}{4}}\right) = \frac{1}{2}\left(\frac{4}{3}\right) = \frac{4}{6} = \frac{2}{3} \end{aligned}$$

$\square$

We don't want to do this manual calculation every time.

Proposition 6.1.5

Let $x, y \in \Sigma$.

1. If $x_i = y_i$ for $i \leq n$, then $d(x,y) \leq \frac{1}{2^n}$.
2. If $d(x,y) < \frac{1}{2^n}$, then $x_i = y_i$ for $i \leq n$.

Proof. Suppose $x_i = y_i$ for $i \leq n$. Then, $d(x,y) \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k}$ since the first n terms will be 0 and $|x_i - y_i| \leq 1$. That is, $d(x,y) \leq \frac{1/2^{n+1}}{1-\frac{1}{2}} = \frac{1}{2^n}$.

Conversely, suppose $d(x,y) < \frac{1}{2^n}$ and for a contradiction that there exists $k \leq n$ where $x_k \neq y_k$. Then, there will be a $\frac{1}{2^k}$ term in the sum, so $d(x,y) \geq \frac{1}{2^k} \geq \frac{1}{2^n}$. Contradiction. □

Example 6.1.6. Let $x = (0000\cdots)$ and $y = (1000\cdots)$. Then, the distance is $\frac{1}{2^0} = 1$. However, $x_0 \neq y_0$.

Definition 6.1.7 (shift map)

The map $\sigma : \Sigma \to \Sigma : (x_0 x_1 x_2 \cdots) \mapsto (x_1 x_2 x_3 \cdots)$ that shifts a bitstring one bit to the left.

Remark 6.1.8. $\sigma^k(x_0 x_1 x_2 \cdots) = x_k x_{k+1} x_{k+2} \cdots$

Lecture 11
Jan 31

Definition 6.1.9 (continuity in metric spaces)

Suppose (X, d) and (Y, d') are (possibly distinct) metric spaces.

A function $f : X \to Y$ is <u>continuous at $y \in X$</u> if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$,

$$d(x,y) < \delta \implies d'(f(x), f(y)) < \varepsilon$$

We say f is <u>continuous</u> if it is continuous at every $y \in X$

Proposition 6.1.10

The shift map $\sigma : \Sigma \to \Sigma$ is continuous.

Proof. Fix $y = (y_0 y_1 y_2 \cdots) \in \Sigma$ and let $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$.

Consider $\delta = \frac{1}{2^{n+1}}$. Let $x = (x_0 x_1 x_2 \cdots) \in \Sigma$ such that $d(x,y) < \delta$.

Therefore, by prop. 6.1.5, $x_i = y_i$ for $i = 0, 1, \dots, n+1$. Then, $\sigma(x) = (x_1 x_2 x_3 \cdots)$ and $\sigma(y) = (y_1 y_2 y_3 \cdots)$ agree for the first n terms.

Again by prop. 6.1.5, $d(\sigma(x), \sigma(y)) \leq \frac{1}{2^n} < \varepsilon$. □

Definition 6.1.11 (convergence in metric spaces)
Let (X, d) be a metric space, $(x_n) \subseteq X$, and $x \in X$.

We say (x_n) converges to x ($x_n \to x$) if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_n, x) < \varepsilon.$$

Proposition 6.1.12 (sequential characterization of continuity in metric spaces)
Let (X, d) and (Y, d') be metric spaces and $f : X \to Y$. Then, f is continuous if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.

Definition 6.1.13 (homeomorphism)
Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is a homeomorphism if

1. f is injective,
2. f is surjective,
3. f is continuous, and
4. f^{-1} is continuous.

Suppose $f : X \to Y$ is a homeomorphism. Then, if $(x_n) \subseteq X$ with $x_n \to x$, then $f(x_n) \to f(x)$.

Likewise, suppose $(y_n) \subseteq Y$ with $y_n \to y$. Say $y_n = f(x_n)$ and $y = f(x)$. Then, $f(x_n) \to f(x)$, so $f^{-1}(f(x_n)) \to f^{-1}(f(x))$ and $x_n \to x$.

That is, f is a *relabelling* of X to Y. We think of X and Y as the "same metric space".

6.2 Revisiting the itinerary

Remark 6.2.1. Suppose we have $x \in \Lambda$ with $S(x) = (x_0 x_1 \cdots)$. Then, by definition, $x \in I_{x_0}$, $Q_c(x) \in I_{x_1}$, $Q_c^2(x) \in I_{x_2}$, etc. Therefore, $S(Q_c(x)) = (x_1 x_2 \cdots) = \sigma(S(x))$.

Iterating, $S(Q_c^n(x)) = \sigma^n(x)$.

Theorem 6.2.2
$S : \Lambda \to \Sigma$ is a homeomorphism.

We will prove this with some more tools. Recall from MATH 137:

Theorem 6.2.3 (monotone convergence theorem)
If $(a_n) \subseteq \mathbb{R}$ is increasing/decreasing and bounded, then (a_n) converges.

Instead of using this directly, we use a lemma:

Lemma 6.2.4 (nested intervals lemma)
If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ are closed intervals, then $\bigcap_{i=1}^{\infty} I_n \neq \varnothing$.

Proof. Let $I_k = [a_k, b_k]$.

That is, $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \cdots$.

Then, (a_n) is increasing and $(a_n) \subseteq [a_1, b_1]$. Likewise, (b_n) is decreasing and $(b_n) \subseteq [a_1, b_1]$. By the monotone convergence theorem, $a_n \to a$ and $b_n \to b$ for some limit points a and b.

Therefore (handwavey), $\varnothing \neq [a, b] \subseteq \bigcap_{n=1}^{\infty} I_n$. □

We will now prove thm. 6.2.2. It is true for $c < -2$, but we will show it for $c < -\frac{5+2\sqrt{5}}{4}$. *Lecture 12 Feb 2*

Proof. (injective) Suppose $x, y \in \Lambda$ with $S(x) = S(y)$ but $x \neq y$. Then, for all n, $Q_c^n(x)$ and $Q_c^n(y)$ live in the same I_0 or I_1. Recall from Assignment 2 that for all $x \in I \setminus J_1 = I_0 \cup I_1$, we have $|Q_c'(x)| \geq \mu > 1$. By the mean value theorem,

$$|Q_c(x) - Q_c(y)| \geq \mu |x - y|.$$

Since Q_c is injective on I_0 and I_1, we have that $Q_c(x) \neq Q_c(y)$. Thus,

$$\begin{gathered} \left|Q_c^2(x) - Q_c^2(y)\right| \geq \mu^2 |x - y| \\ \vdots \\ |Q_c^n(x) - Q_c^n(y)| \geq \mu^n |x - y| \end{gathered}$$

Since $\mu > 1$, we have $\mu^n |x - y| \to \infty$. However, $|Q_c^n(x) - Q_c^n(y)| \leq \max\{\ell(I_0), \ell(I_1)\}$, so it cannot blow up to infinity. Contradiction, so we have injectivity.

(surjective) Let $y = (y_0 y_1 \cdots) \in \Sigma$. For $n \in \mathbb{N}$, define

$$I_{y_0 y_1 \cdots y_n} := \{x \in I : x \in I_{y_0}, Q_c(x) \in I_{y_1}, \ldots, Q_c^n(x) \in I_{y_n}\}.$$

It is enough to show there exists

$$x \in \bigcap_{n=1}^{\infty} I_{y_0 y_1 \cdots y_n}$$

which would imply $S(x) = y$. Clearly, by definition, $I_{y_0} \supseteq I_{y_0 y_1} \supseteq I_{y_0 y_1 y_2} \supseteq \cdots$

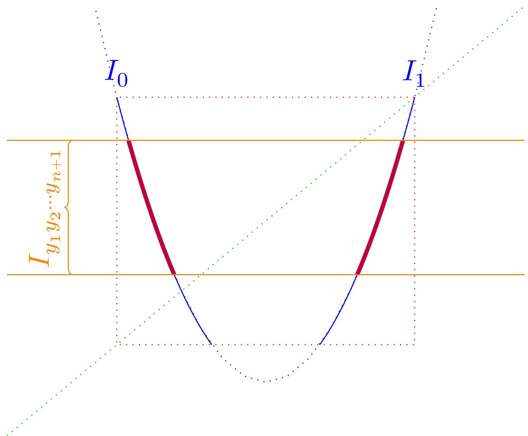
We claim that each $I_{y_0 y_1 \cdots y_n}$ is a closed interval. Proceed by induction.

First, $I_{y_0} \in \{I_0, I_1\}$ so it is closed. Assume $I_{y_0 y_1 \cdots y_n}$ is closed for some $n \geq 0$. Note:

$$\begin{aligned} & x \in I_{y_0 y_1 \cdots y_{n+1}} \\ \iff & x \in I_{y_0}, Q_c(x) \in I_{y_1}, Q_c(Q_c(x)) \in I_{y_2}, Q_c(Q_c^2(x)) \in I_{y_3}, \ldots, Q_c(Q_c^n(x)) \in I_{y_{n+1}} \\ \iff & x \in I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \cdots y_{n+1}}) \end{aligned} \tag{$\star$}$$

By the inductive hypothesis, $I_{y_1 y_2 \cdots y_{n+1}}$ is a closed interval (the subscript has length n).

We have



That is, $Q_c^{-1}(I_{y_1 y_2 \cdots y_{n+1}})$ is a union of two disjoint closed intervals, one in I_0 and one in I_1.

In particular, returning to $(\star)$, $I_{y_0 y_1 \cdots y_{n+1}} = I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \cdots y_{n+1}})$ is one of these closed intervals.

By the nested intervals lemma, there must exist $x \in \bigcap_{n=1}^{\infty} I_{y_0 y_1 \cdots y_n}$. Hence, $S(x) = y$ and we have surjectivity.

(continuous) Fix $y \in \Lambda$ and say $S(y) = (y_0 y_1 y_2 \cdots)$. Let $\varepsilon > 0$ and choose n such that $\frac{1}{2^n} < \varepsilon$.

Consider the 2^{n+1} disjoint, closed intervals $I_{t_0 t_1 \cdots t_n}$.

Pick $\delta > 0$ such that $(y - \delta, y + \delta)$ only overlaps with $I_{y_0 y_1 \cdots y_n}$. We know δ exists since we have a finite set of disjoint closed intervals.

For $x \in \Lambda$ with $|x - y| < \delta$, $x \in I_{y_0 y_1 \cdots y_n}$ and so $d(S(x), S(y)) \le \frac{1}{2^n} < \varepsilon$.

(continuous inverse) Similar. $\square$

Chapter 7

Chaos

7.1 Prerequisites to chaos

Lecture 13
Feb 5

Definition 7.1.1 (density)

Let (X, d) be a metric space. We say $A \subseteq X$ is dense in X if for all $x \in X$ and $\varepsilon > 0$, there exists $a \in A$ such that $d(a, x) < \varepsilon$.

Informally, there is always something "that close" to any point.

Example 7.1.2. $\mathbb{Q}$ is dense in $\mathbb{R}$. Given a real number, there is always a decimal approximation with arbitrary accuracy.

$\mathbb{Z}$ is not dense in $\mathbb{R}$. Given $x = \frac{1}{2} \in \mathbb{R}$, there are no integers within $\varepsilon = \frac{1}{4}$.

Example 7.1.3. Let $A = \{x \in \Sigma : \exists N, \forall i > N, x_i = 0\}$, i.e., the sequences which are eventually constant 0s. This is dense in Σ.

Proof. Let $x = (x_0 x_1 x_2 \cdots) \in \Sigma$ and let $\varepsilon > 0$. As usual, take $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$.

Consider $y = (x_0 x_1 x_2 \cdots x_n 0000 \cdots) \in A$. Then, by prop. 6.1.5, $d(x, y) \leq \frac{1}{2^n} < \varepsilon$. $\square$

Exercise 7.1.4. Let $A = \{x \in \Sigma : x \text{ is periodic}\}$. Show that this is dense in Σ.

Remark 7.1.5. A in exercise 7.1.4 is exactly the set of periodic points for the shift map $\sigma : \Sigma \to \Sigma$.

Proposition 7.1.6

There exists $z \in \Sigma$ such that $\{\sigma^k(z) : K \in \mathbb{N} \cup \{0\}\}$ is dense in Σ.

Proof. Take $z = (0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ \cdots)$ to contain all possible blocks of increasing sizes.

Let $x \in \Sigma$ and $\varepsilon > 0$. Again, let $\frac{1}{2^n} < \varepsilon$.

For some k, $\sigma^k(z)$ and x agree on the first n terms. This must exist because z has *every possible* sequence of n terms. That is, by prop. 6.1.5, $d(\sigma^k(z), x) \leq \frac{1}{2^n} < \varepsilon$. □

Warning: definition 7.1.7 is not the normal definition from applied math textbooks, but it is what we will use in the course.

Definition 7.1.7 (dynamical system)
A metric space (X, d) together with a continuous function $f : X \to X$.

This is an abstract space in which we can do orbit analysis and all our fun stuff.

Example 7.1.8. $\sigma : \Sigma \to \Sigma$ is a dynamical system (see thm. 6.2.2).

Definition 7.1.9 (transitivity)
We say a dynamical system $f : X \to X$ is transitive if for all $x, y \in X$ and $\varepsilon > 0$, there exists $z \in X$ and $n, m \in \mathbb{N} \cup \{0\}$ such that $d(x, f^n(z)) < \varepsilon$ and $d(y, f^m(z)) < \varepsilon$.

Informally, given any two points, there is a special point whose orbit gets arbitrarily close to both points.

Proposition 7.1.10
$\sigma : \Sigma \to \Sigma$ is transitive.

Proof. Take z from prop. 7.1.6 such that the orbit is dense in Σ.

Then, for all $\varepsilon > 0$ and $x, y \in \Sigma$, there must exist by the definition of density n and m such that $d(x, \sigma^n(z)) < \varepsilon$ and $d(y, \sigma^m(z)) < \varepsilon$. □

Definition 7.1.11 (sensitive dependence on initial conditions)
Let $f : X \to X$ be a dynamical system.

We say f is sensitively dependent on initial conditions (or just sensitive) if

$$\exists \beta > 0,\ \forall \varepsilon > 0,\ \forall x \in X,\ \exists y \in X,\ \exists k \in \mathbb{N}$$

such that $d(x, y) < \varepsilon$ and $d(f^k(x), f^k(y)) \geq \beta$.

Informally, there exists a "wrongness" β that can always be achieved in the orbit no matter how close two starting points are.

Proposition 7.1.12

$\sigma : \Sigma \to \Sigma$ is sensitive.

Proof. Take $\beta = 1$.

Let $\varepsilon > 0$ and let $x \in \Sigma$. Say $\frac{1}{2^n} < \varepsilon$ and pick $y \in \Sigma$ such that $0 < d(x, y) < \frac{1}{2^n}$. That is, x and y must agree on the first n terms by prop. 6.1.5, but they are not equal.

Therefore, there exists $k \geq n$ such that $x_k \neq y_k$.

In the distance $d(\sigma^k(x), \sigma^k(y)) \geq \frac{|x_k - y_k|}{2^0} \geq 1 = \beta$. □

7.2 Defining chaos

↓ *Lectures 14 and 15 adapted from Imaad* ↓ *Lecture 14 Feb 7*

Definition 7.2.1 (chaos)

A dynamical system $f : X \to X$ is chaotic if

1. the periodic points for f are dense in X,
2. f is transitive, and
3. f is sensitive.

Theorem 7.2.2

$\sigma : \Sigma \to \Sigma$ is chaotic.

Proof. By props. 7.1.6, 7.1.10 and 7.1.12. □

Proposition 7.2.3

Let $(X, d), (Y, d')$ be metric spaces.

Suppose $f : X \to Y$ is continuous and surjective. If $A \subseteq X$ is dense in X, then $f(A)$ is dense in Y.

Proof. Let $y \in Y$ and say $y = f(x)$.

Let $\epsilon > 0$. Since f is continuous at x, there exists $\delta > 0$ such that

$$d(z, x) < \delta \implies d'(f(z), f(x)) < \epsilon$$

for any z. In particular, since A is dense in X, we may find $a \in A$ such that

$$d(a, x) < \delta \implies d'(f(a), f(x)) = d'(f(a), y) < \epsilon$$

as desired. □

Theorem 7.2.4

Let $c < \frac{-(5+2\sqrt{5})}{4}$. Then, $Q_c : \Lambda \to \Lambda$ is chaotic.

Proof. (periodic point density) Note that $Q_c^n(x) = x \iff S(Q_c^n(x)) = S(x) \iff \sigma^n(S(x)) = S(x)$.

By prop. 7.2.3 applied to $S^{-1} : \Sigma \to \Lambda$, the periodic points for Q_c are dense in Λ.

(transitivity) Take $z \in \Sigma$ from prop. 7.1.6 such that $\{\sigma^K(z) : K \in \mathbb{N} \cup \{0\}\}$ is dense in Σ. Again by prop. 7.2.3, $\{S^{-1}(\sigma^K(z)) : K \in \mathbb{N} \cup \{0\}\}$ is dense in Λ.

Note: Say $S(x) = z$, we know $(S(Q_c^K(x))) = \sigma^K(S(x)) \iff Q_c^K(x) = S^{-1}(\sigma^K(S(x)))$

This, $\{Q_c^K(x) : K \in \mathbb{N} \cup \{0\}\}$ is dense in Λ. As in prop. 7.1.10, we have that Q_c is transitive.

(sensitivity) Recall that $\Lambda \subseteq I \setminus J_1 = I_0 \cup I_1$. Let $\beta > 0$ be less than the gap between I_0 and I_1.

For $x, y \in \Lambda$ with $x \neq y$, supppose $S(x) \neq S(y)$. Then, there must exist a k where k^{th} term of $S(x)$ does not equal the k^{th} term of $S(y)$.

Hence, $|Q_c^k(x) - Q_c^k(y)| > \beta$ and Q_c is sensitive. □

Chapter 8

Sarkovskii's Theorem

8.1 Period 3 points

Lecture 15
Feb 9

Theorem 8.1.1 (period 3)
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a point with period 3, then f has a point with period n for all $n \in \mathbb{N}$.

Proposition 8.1.2
Let $I \subseteq J$ be closed intervals and suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. If $f(I) \supseteq J$, then $f(x)$ has a fixed point in I.

Proposition 8.1.3
Let I, J be closed intervals, $f : \mathbb{R} \to \mathbb{R}$ be continuous, and $f(I) \supseteq J$. Then, there exists a closed interval $I' \subseteq I$ such that $f(I') = J$.

We can now prove thm. 8.1.1.

Proof. Let $a \in \mathbb{R}$ be a period 3 point for $f(x)$. Say $f(a) = b, f(b) = c, f(c) = a$. WLOG, suppose $a < b$ and $a < c$.

Suppose $a < b < c$. The case where $a < c < b$ is left as an exercise.

Let $I = [a, b]$ and $J = [b, c]$. Then, $f(a) = b$ and $f(b) = c$ imply by IVT that $[b, c] = J \subseteq f(I)$. Likewise, $f(b) = c$ and $f(c) = a$ imply by IVT that $[a, c] = I \cup J \subseteq f(J)$.

Since $J \subseteq f(J)$, there exists a closed interval $A_1 \subseteq J$ such that $f(A_1) = J$ by prop. 8.1.3. Again, $A_1 \subseteq J = f(A_1)$, so there exists a closed interval $A_2 \subseteq A_1$ such that $f(A_2) = A_1$.

Now, fix $n > 3$. Repeating the above process, we can find $A_{n-2} \subseteq A_{n-3} \subseteq \cdots \subseteq A_2 \subseteq A_1 \subseteq J$ such that $f(A_i) = A_{i-1}$. Now, $f(I) \supseteq J \supseteq A_{n-2}$ means there exists a closed interval $A_{n-1} \subseteq I$ such that $f(A_{n-1}) = A_{n-2}$.

Moreover, $f(J) \supseteq I \supseteq A_{n-1}$ which means there exists a closed interval $A_n \subseteq J$ such that $f(A_n) = A_{n-1}$.

We have $f^n(A_n) = J$ and $A_n \subseteq J$. By prop. 8.1.2, there exists $x_0 \in A_n$ such that $f^n(x_0) = x_0$.

Note: for $x_0 \in A_n$, $f(x_0) \in A_{n-1} \subseteq I$, $f^i(x_0) \in J$ for $i = 2, 3, \dots, n$.

For contradiction, suppose $f^i(x_0) = x_0$ for $i < n$.

Then, $\overbrace{f(x_0)}^{\in I} = \overbrace{f^{i+1}(x_0)}^{\in J} = b$ so $f(x_0) = b$, $f^2(x_0) = c$, and $f^3(x_0) = a$, which is a contradiction because $f^3(x_0) \in J$ but $a \notin J$. Hence, x_0 has period n.

That is, f has a periodic point with period n for all $n > 3$.

Further, $f(J) \supseteq J$ and so by prop. 8.1.2, f has a fixed point (aka period 1) in J.

Finally, $f(I) \supseteq J$ means $J = f(I')$ and $f(J) \supseteq I'$ means $f(J') = I'$. This implies $f^2(j') = f(I') = J \sup J'$. Therefore, we know there exists $x \in J'$ such that $f^2(x) = x$.

If $f(x) = x$, then $x \in J'$ and $f(x) \in I'$, meaning $x = b$. But, $f(b) \neq b = c$, contradiction.

Hence, x has period 2.

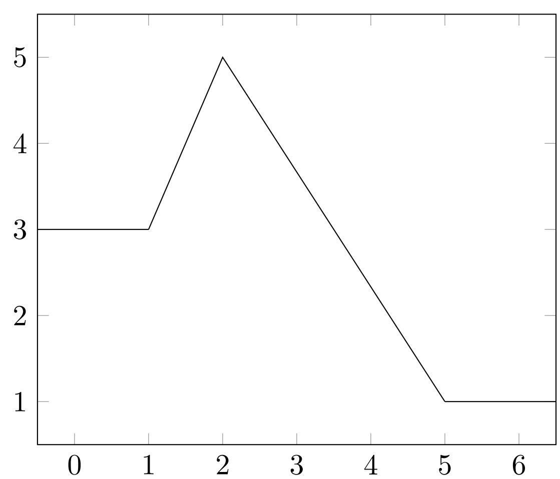
Therefore, since we already supposed f has a period 3 point, f has a period n point for all n. □

Exercise 8.1.4. Complete the proof for the case where $a < c < b$.

↑ *Lectures 14 and 15 adapted from Imaad* ↑

Lecture 16
Feb 12

Draw the continuous function



Then, the orbit of 1 is $1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5 \mapsto 1$ and 1 has period 5.

Claim 8.1.5. f has no point with period 3.

Proof. Suppose that f has a point x with period 3. Then, $1 \leq x \leq 5$.

Suppose $x \in [1, 2]$. Then, $x \in [1, 2] \cap f^3([1, 2])$ since $x = f^3(x)$. But $f^3([1, 2]) = [2, 5]$, so $x = 2$.

However, 2 has period 5 since it is on the same 5-cycle given above.

Suppose instead that $x \in [2,3]$. Then, $x \in [2,3] \cap f^3([2,3]) = [2,3] \cap [3,5] = \{3\}$ which is also on the 5-cycle.

If $x \in [4,5]$, then $x \in [4,5] \cap f^3([4,5]) = [4,5] \cap [1,4] = \{4\}$ which is, again, on the 5-cycle.

Finally, suppose that $x \in [3,4]$. Then, $f([3,4]) = [2,4]$ and it is strictly decreasing. Further, $f([2,4]) = [2,5]$ and it is also strictly decreasing. Once more, $f([2,5]) = [1,5]$ and it is again strictly decreasing. Since f^3 is strictly decreasing, it has a unique fixed point in $[3,4]$, but it is just the fixed point of f.

Since we have covered the entire interval $[1,5]$, x must not exist. $\square$

Example 8.1.6. The function $f(x) = \begin{cases} 1 & x < -1 \\ -x & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$ has a period 1 point at $x = 0$, period 2 points $[-1,1] \setminus \{0\}$, and no other periodic points.

Definition 8.1.7 (Sarkovskii ordering)

Start by ordering the odd numbers $3 \prec 5 \prec 7 \prec 9 \prec \cdots$

Then, all those are $\cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec \cdots$

All those are $\cdots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec \cdots$

Complete the ordering as $\cdots \prec 2^n \prec 2^{n-1} \prec \cdots \prec 2^2 \prec 2 \prec 1$.

This is a total order on the natural numbers.

Example 8.1.8.

- $26 = 2 \cdot 13 \prec 2^2 \cdot 5 = 40$ because the exponent of 2 is smaller.
- $3072 = 2^{10} \cdot 3 \prec 2^5 = 32$ because powers of 2 are big.
- $n \preccurlyeq 1$ for all n.
- $2^{15} \prec 2^3$ since the powers of 2 are ordered backwards.

Theorem 8.1.9 (Sarkovskii's theorem)

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $n \prec m$ in the Sarkovskii ordering. Then, if f has a point with period n, then it has a point with period m.

List of Named Results

Index of Defined Terms