

## Essential Laws of Propositional Logic

Double Negation	$\{\neg(\neg p) \models p\}$
Excluded Middle	$\{p \vee \neg p \models 1\}$
Contradiction	$\{p \wedge \neg p \models 0\}$
Idempotence	$\begin{cases} p \wedge p \models p \\ p \vee p \models p \end{cases}$
Identity	$\begin{cases} p \wedge 1 \models p \\ p \vee 0 \models p \end{cases}$
Domination	$\begin{cases} p \wedge 0 \models 0 \\ p \vee 1 \models 1 \end{cases}$
Commutativity	$\begin{cases} p \wedge q \models q \wedge p \\ p \vee q \models q \vee p \\ p \leftrightarrow q \models q \leftrightarrow p \end{cases}$
Associativity	$\begin{cases} p \wedge (q \wedge r) \models (p \wedge q) \wedge r \\ p \vee (q \vee r) \models (p \vee q) \vee r \end{cases}$
Distributivity	$\begin{cases} p \wedge (q \vee r) \models (p \wedge q) \vee (p \wedge r) \\ p \vee (q \wedge r) \models (p \vee q) \wedge (p \vee r) \end{cases}$
Implication	$\{p \rightarrow q \models \neg p \vee q\}$
Contrapositive	$\{p \rightarrow q \models \neg q \rightarrow \neg p\}$
Equivalence	$\{p \leftrightarrow q \models (p \rightarrow q) \wedge (q \rightarrow p)\}$
De Morgan	$\begin{cases} \neg(p \wedge q) \models \neg p \vee \neg q \\ \neg(p \vee q) \models \neg p \wedge \neg q \end{cases}$
Absorption I	$\begin{cases} p \wedge (p \vee q) \models p \\ p \vee (p \wedge q) \models p \end{cases}$
Absorption II	$\begin{cases} (p \vee q) \wedge (\neg p \vee q) \models q \\ (p \wedge q) \vee (\neg p \wedge q) \models q \end{cases}$

## Eleven Rules of Formal Deduction (+ Six of First Order Logic)

(Abbr.)	From	Conclude	Rule
(Ref)	$\emptyset$	$A \vdash A$	Reflexivity
(+)	$\Sigma \vdash A$	$\Sigma, \Sigma' \vdash A$	Addition of premises
( $\neg$ -)	$\begin{matrix} \Sigma, \neg A \vdash B \\ \Sigma, \neg A \vdash \neg B \end{matrix}$	$\Sigma \vdash A$	$\neg$ elimination
( $\rightarrow$ -)	$\begin{matrix} \Sigma \vdash A \rightarrow B \\ \Sigma \vdash A \end{matrix}$	$\Sigma \vdash B$	$\rightarrow$ elimination (modus ponens)
( $\rightarrow$ +)	$\Sigma, A \vdash B$	$\Sigma \vdash A \rightarrow B$	$\rightarrow$ introduction
( $\wedge$ -)	$\Sigma \vdash A \wedge B$	$\begin{matrix} \Sigma \vdash A \\ \Sigma \vdash B \end{matrix}$	$\wedge$ elimination
( $\wedge$ +)	$\begin{matrix} \Sigma \vdash A \\ \Sigma \vdash B \end{matrix}$	$\Sigma \vdash A \wedge B$	$\wedge$ introduction
( $\vee$ -)	$\begin{matrix} \Sigma, A \vdash C \\ \Sigma, B \vdash C \end{matrix}$	$\Sigma, A \vee B \vdash C$	$\vee$ elimination
( $\vee$ +)	$\Sigma \vdash A$	$\begin{matrix} \Sigma \vdash A \vee B \\ \Sigma \vdash B \vee A \end{matrix}$	$\vee$ introduction
( $\leftrightarrow$ -)	$\begin{matrix} \Sigma \vdash A \leftrightarrow B \\ \Sigma \vdash A \end{matrix}$	$\Sigma \vdash B$	$\leftrightarrow$ elimination
( $\leftrightarrow$ +)	$\begin{matrix} \Sigma, A \vdash B \\ \Sigma, B \vdash A \end{matrix}$	$\Sigma \vdash A \leftrightarrow B$	$\leftrightarrow$ introduction
( $\forall$ -)	$\Sigma \vdash \forall x A(x)$	$\Sigma \vdash A(t)$	$\forall$ elimination
( $\forall$ +)	$\begin{matrix} \Sigma \vdash A(u) \\ u \text{ not in } \Sigma \end{matrix}$	$\Sigma \vdash \forall x A(x)$	$\forall$ introduction
( $\exists$ -)	$\begin{matrix} \Sigma, A(u) \vdash B \\ u \text{ not in } \Sigma, B \end{matrix}$	$\Sigma, \exists x A(x) \vdash B$	$\exists$ elimination
( $\exists$ +)	$\Sigma \vdash A(u)$	$\Sigma \vdash \exists x A(x)$	$\exists$ introduction
( $\approx$ -)	$\begin{matrix} \Sigma \vdash A(t_1) \\ \Sigma \vdash t_1 = t_2 \end{matrix}$	$\Sigma \vdash A(t_2)$	$\approx$ elimination
( $\approx$ +)	$\emptyset$	$\emptyset \vdash u = u$	$\approx$ introduction

N.B.: In ( $\forall$  +) and ( $\exists$  +),  $A(x)$  is  $A(u)$  with some but not necessarily all occurrences of  $u$  replaced by  $x$ .

## Proved Theorems

(Abbr.)	From	Conclude	Theorem
( $\in$ )	$A \in \Sigma$	$\Sigma \vdash A$	Membership
(Tr.)	$\Sigma \vdash \Sigma'$ $\Sigma' \vdash A$	$\Sigma \vdash A$	Transitivity
( $\neg +$ )	$\Sigma, A \vdash B$ $\Sigma, A \vdash \neg B$	$\Sigma \vdash \neg A$	Reductio ad absurdum
(Repl.)	$A \vdash A'$ $C = B$ with $A'$ for $A$	$B \vdash C$	Replaceability
	$A \vdash B$	$\neg B \vdash \neg A$	Flip-Flop
( $\approx -$ )'	$\Sigma \vdash A(t_1)$ $\Sigma \vdash t_2 = t_1$	$\Sigma \vdash A'(t_2)$	Partial substitution

$$\neg \neg A \vdash A$$

(Double Negation)

$$\emptyset \vdash A \vee \neg A$$

(Excluded Middle)

$$\emptyset \vdash \neg(A \wedge \neg A)$$

(Non-Contradiction)

$$A, \neg A \vdash B$$

(Inconsistency Rule)

$$A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$$

(Hypothetical Syllogism)

$$A \vee B, \neg B \vdash A$$

(Disjunctive Syllogism)

$$A \rightarrow B \vdash \neg A \vee B$$

(Implication)

$$A \rightarrow B \vdash \neg B \rightarrow \neg A$$

(Contrapositive)

$$\neg(A \wedge B) \vdash \neg A \vee \neg B$$

$$\neg(A \vee B) \vdash \neg A \wedge \neg B$$

(De Morgan)

$$\emptyset \vdash \forall x (x = x)$$

(Reflexivity)

$$\emptyset \vdash \forall x \forall y ((x = y) \rightarrow (y = x))$$

(Symmetry)

$$\emptyset \vdash \forall x \forall y \forall z (((x = y) \wedge (y = z)) \rightarrow (x = z))$$

(Transitivity)

## Axioms of Peano Arithmetic

$$\text{PA1. } \forall x \neg(s(x) = 0)$$

$$\text{PA2. } \forall x \forall y ((s(x) = s(y)) \rightarrow (x = y))$$

$$\text{PA3. } \forall x (x + 0 = x)$$

$$\text{PA4. } \forall x \forall y (x + s(y) = s(x + y))$$

$$\text{PA5. } \forall x (x \cdot 0 = 0)$$

$$\text{PA6. } \forall x \forall y (x \cdot s(y) = x \cdot y + x)$$

$$\text{PA7. } (A(0) \wedge \forall x (A(x) \rightarrow A(s(x)))) \rightarrow \forall x A(x) \text{ (induction)}$$

## Definitions

*Alphabet* (§16a, 13). Finite set of symbols  $\Sigma$ .  $\Sigma^*$  contains all possible strings, including empty string  $\lambda$ . Subset of  $\Sigma^*$  is language.

*Propositional language* (§02, 3).  $\mathcal{L}^p$  has  $\Sigma = \{p, q, r, \dots, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, (, )\}$ . *First-order language* (§11, 4).  $\mathcal{L}$  extends above with quantifier  $(\forall, \exists)$ , free  $(u, v, w)$  and bound variable  $(x, y, z)$ , individual  $(a, b, c, \dots)$ , relation  $(F, G, H)$ , and function  $(f, g, h)$  symbols.

*Expression* (§02, 4). Element of language, including empty expression  $\epsilon$ .

*Segment* (§02, 5).  $V$  segment of  $U$  if  $U = W_1 V W_2$ . Initial if  $W_1 = \epsilon$ , final if  $W_2 = \epsilon$ . Proper if  $V \neq U$ .

*Term* (§11, 6).  $\text{Term}(\mathcal{L})$  is smallest set that contains all individual symbols, free variable symbols, and functions of terms.

*Atom* (§02, 6).  $\text{Atom}(\mathcal{L}^p)$  has expressions that are one proposition symbol.

*Atom* (§11, 8).  $\text{Atom}(\mathcal{L})$  has  $F(t_1, t_2, \dots, t_n)$  and  $t_1 \approx t_2$  for  $t_i \in \text{Term}(\mathcal{L})$ .

*Formula* (§02, 6).  $\text{Form}(\mathcal{L}^p)$  defined by:  $\text{Atom}(\mathcal{L}^p) \subset \text{Form}(\mathcal{L}^p)$ .

$A, B \in \text{Form}(\mathcal{L}^p) \Rightarrow (\neg A), (A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B) \in \text{Form}(\mathcal{L}^p)$ .

No other expressions are in  $\text{Form}(\mathcal{L}^p)$ .

*Formula* (§11, 9).  $\text{Form}(\mathcal{L})$  has same formation rules but also includes

$A(u) \in \text{Form}(\mathcal{L}), x \notin A(u) \Rightarrow \forall x A(x), \exists x A(x) \in \text{Form}(\mathcal{L})$

*Sentence* (§11, 17).  $\text{Sent}(\mathcal{L}) \subset \text{Form}(\mathcal{L})$  with no free variables.

*Scope* (§02, 45). In  $(\neg A)$ ,  $A$  is the scope of  $\neg$ . In  $(A \star B)$ ,  $A$  is the left scope and  $B$  is the right scope of  $\star$ .

*Truth valuation* (§03, 6). A function  $t : \text{Atom}(\mathcal{L}^p) \rightarrow \{0, 1\}$ .

*Valuation* (§12, 7). A domain  $D$  and function  $v$  such that

$a^v, u^v \in D$  for all individual symbols  $a$  and free variables  $u$

$w^{v(u/d)} = d$  if  $w = u$  and  $w^v$  otherwise for free variables  $u, w, d$

$F^v \subseteq D^n$  for all  $n$ -ary relations  $F$  with  $\approx^v = \{(x, x), x \in D\}$

$f^v : D^n \rightarrow D$  for all  $n$ -ary functions  $f$

*Satisfiability* (§03, 9).  $t$  satisfies  $A$  if  $A^t = 1$ . Set satisfied if members satisfied.

*Tautology/Universally valid* (§03, 14, §12, 23). For all  $t$ ,  $A^t = 1$ .

*Contradiction/Unsatisfiability* (§03, 14). For all  $t$ ,  $A^t = 0$ .

*Contingent* (§03, 14).  $A$  is neither a tautology nor contradiction.

*Tautological consequence* (§03, 21).  $\Sigma \models A$  if for all  $t$ ,  $\Sigma^t = 1$  gives  $A^t = 1$ .

*Tautological equivalence* (§03, 25).  $A \models B$  if  $A \models B$  and  $B \models A$ .

*Literal* (§04, 10). A formula of the form  $p$  or  $\neg p$ .

*Disjunctive clause* (§04, 12). Disjunction with literal disjuncts.

*Conjunctive clause* (§04, 12). Conjunction with literal conjuncts.

*DNF* (§04, 13). Disjunction with conjunctive clause disjuncts.

*CNF* (§04, 13). Conjunction with disjunctive clause conjuncts.

*Definability/Reducibility* (§05, 2). Connective  $\star$  reducible to set  $\mathcal{S}$  if  $A \star B \models C$  where  $C$  uses only  $A$ ,  $B$ , and connectives in  $\mathcal{S}$ .

*Adequate* (§05, 7). Connectives which express any truth table/connective.

*Formal deducibility* (§06, 15).  $\Sigma \vdash A$  generated by finite deduction rules.

*Syntactic equivalence* (§06, 15).  $A \vdash B$  if  $A \vdash B$  and  $B \vdash A$ .

*Consistency* (§06, 67). There is no  $F$  such that  $\Sigma \vdash F$  and  $\Sigma \vdash \neg F$ .

*Resolution* (§07, 8).  $C \vee p, D \vee \neg p \vdash_r C \vee D$  if  $C$  and  $D$  disjunctive clauses. Resolve parent clauses over  $p$  to resolvent  $C \vee D$ . Commutativity and idempotence allowed. Note  $p, \neg p \vdash_r \{\}$  (empty clause, representing contradiction).

*Set-of-Support Strategy* (§07, 19). Let  $\Sigma' = \neg C$ . Resolve all premises against  $\Sigma'$  and add resolvents to  $\Sigma'$ . Repeat until  $\{\}$ .

*Davis-Putnam Procedure* (§07, 29). For each  $p$ : discard tautologies from  $\Sigma$ , resolve all pairs over  $p$ , discard clauses with  $p$ . Always outputs  $\emptyset$  or  $\{\}$ .

*Prenex Normal Form* (§15, 4). All quantifiers at start. Prefix (quantifiers) + matrix (rest). If no  $\exists$ , it is  $\exists$ -free PNF.

*Skolem function* (§15, 14). Given  $\forall x_i \exists y$ , a function  $f(x_i) = y$ . Note that Skolemized sentence is not equivalent to original.

*Unification* (§15, 26).  $A$  unifies with  $B$  if exists unifiers  $x_i := t_i$  (replacing variable  $x_i$  with term  $t_i$ ) that make  $A$  equal  $B$ .

*Turing machine* (§16a, 13). Finite set of states  $S$ , alphabet  $I$  containing blank  $B$ , start state  $s_0 \in S$ , transition function  $f : S \times I \rightarrow S \times I \times \{L, R\}$  or transition rules  $(s, x, s', x', d)$ . State  $s$  final if no rules for  $s$ . Halts if no rule for  $(s, x)$ , accepts if  $s$  final (reject otherwise). TM total if always halts.

*Decidability* (§16a, 22). Decision problem can be solved by TM that accepts “yes” answers. Function that can be computed by TM is computable.

*P and NP* (§16b, 42). Problem in  $P$  if DTM can solve in polynomial time. In  $NP$  if NDTM can solve (or DTM can verify) in polynomial time. Problem

$NP$ -complete if any  $NP$  can be reduced to it in polynomial time.

## Theorems (... the general kind)

*Lemma* (§02, 29). Every formula has equal number of left/right parentheses.

**Unique Readability Theorem** (§02, 32). Every formula in  $\text{Form}(\mathcal{L}^p)$  is exactly one form of exactly one of  $(\neg A)$ ,  $(A \wedge B)$ ,  $(A \vee B)$ ,  $(A \rightarrow B)$ ,  $(A \leftrightarrow B)$ .

**Theorem** (§11, 15). The same applies to  $\text{Form}(\mathcal{L})$  plus  $(\forall x A(x))$ ,  $(\exists x A(x))$ .

**Theorem** (§11, 15). Also to  $\text{Term}(\mathcal{L})$  with free/individual/function symbols.

*Lemma* (§03, 23). Equivalent statements of  $\{A_i\} \models C$ :

Argument with premises  $A_i$  and conclusion  $C$  is valid.

$(\bigwedge A_i) \rightarrow C$  is a tautology;  $(\neg C \wedge \bigwedge A_i)$  is a contradiction.

Formula  $(\neg C \wedge \bigwedge A_i)$  or set  $\{\neg C, A_i\}$  is not satisfiable.

**Replaceability of tautologically equivalent formulas** (§03, 44, §13, 16).

If  $A \models A'$  and  $A$  is a subformula of  $B$ , then  $B \models B'$  where  $B'$  is  $B$  with some of the  $A$ s replaced by  $A'$ .

**Duality Theorem** (§03, 44). If  $A$  has only  $\neg$ ,  $\wedge$ ,  $\vee$  and  $\Delta(A)$  replaces atoms with negations and swaps  $\wedge$  with  $\vee$ , then  $\neg A \models \Delta(A)$ .

**Duality in FoL** (§13, 16). If  $A$  is as above plus  $\forall$ ,  $\exists$ , and  $\Delta(A)$  replaces atoms with negations, swaps  $\wedge$  with  $\vee$ , and swaps  $\forall$  with  $\exists$ , then  $\neg A \models \Delta(A)$ .

**Theorem** (§04, 22). All formulas have  $F \models \text{DNF}(F)$  based on truth table.

**Theorem** (§04, 24). All formulas have  $F \models \text{CNF}(F) = \Delta(\text{DNF}(\neg F))$ .

**Theorem** (§05, 8). The set  $S_0 = \{\neg, \wedge, \vee\}$  is adequate.

**Finiteness of Premise Set** (§06, 31). If  $\Sigma \vdash A$ ,  $\Sigma^0 \vdash A$  with finite  $\Sigma^0 \subseteq \Sigma$ .

**Soundness Theorem** (§06, 45). If  $\Sigma \vdash A$  then  $\Sigma \models A$ .

**Completeness Theorem** (§06, 49). If  $\Sigma \models A$  then  $\Sigma \vdash A$ .

*Lemma* (§06, 67).  $\Sigma$  is satisfiable if and only if  $\Sigma$  is consistent.

**Theorem** (§07, 20).  $\vdash_r$  is complete with the set of support strategy.

**Theorem** (§07, 35).  $\vdash_r$  is sound and complete with DPP.

**Theorem** (§15, 31).  $\vdash_r$  in FoL is sound and complete with unification.

**Theorem** (§12, 17). All terms in  $\text{Term}(\mathcal{L})$  have  $t^v \in D$ .

**Theorem** (§12, 20). All formulas in  $\text{Form}(\mathcal{L})$  have  $A^v \in \{0, 1\}$ .

**Theorem** (§12, 28). There is no algorithm to decide validity or satisfiability

in first-order logic.

**Theorem** (§14, 33). First order formal deduction is sound and complete.

**Theorem** (§15, 19). All sentences  $F$  in  $\text{Sent}(\mathcal{L})$  have a corresponding  $\exists$ -free PNF  $F'$  such that  $F$  satisfiable iff  $F'$  satisfiable.

**Theorem** (§15, 20). All  $\exists$ -free PNFs  $F$  can construct a finite set of disjunctive clauses  $C_F$  such that  $F$  satisfiable iff  $C_F$  satisfiable.

**Theorem** (§15, 21). Argument  $\Sigma \models A$  valid iff  $C_{\neg A} \cup \bigcup_{F \in \Sigma} C_F$  unsatisfiable.

**Theorem** (§16a, 7). The halting problem is undecidable.

*Lemma* (§16a, 28). Satisfiability and validity problems for FoL undecidable.

*Lemma* (§16b, 50). Satisfiability for propositional logic  $NP$ -complete.

**Theorem** (§16a, 31). If  $P_1$  reduces to  $P_2$  and  $P_1$  undecidable,  $P_2$  undecidable.

**Gödel's Incompleteness Theorem** (§17, 32). Given axiomatic theory  $T$  with decidable axioms and capacity to express Peano Arithmetic, there exists statement  $G_T$  which cannot be proved or disproved by  $T$