

# CS 341 Spring 2023:

## Lecture Notes

|          |                                  |           |
|----------|----------------------------------|-----------|
| <b>1</b> | <b>Introduction</b>              | <b>2</b>  |
| 1.1      | Asymptotic Review . . . . .      | 2         |
| 1.2      | Types of Algorithms . . . . .    | 3         |
| 1.3      | Recurrence Relations . . . . .   | 5         |
| <b>2</b> | <b>Divide and Conquer</b>        | <b>7</b>  |
| 2.1      | Examples . . . . .               | 7         |
| <b>3</b> | <b>Graph Algorithms</b>          | <b>12</b> |
| 3.1      | Graph Theory Review . . . . .    | 12        |
| 3.2      | Breadth-First Search . . . . .   | 13        |
| 3.3      | Shortest Path by BFS . . . . .   | 14        |
| 3.4      | Bipartiteness by BFS . . . . .   | 15        |
| 3.5      | Depth-First Search . . . . .     | 15        |
| 3.6      | Cut Vertices by DFS . . . . .    | 17        |
| 3.7      | Directed Graphs . . . . .        | 19        |
|          | <b>Back Matter</b>               | <b>21</b> |
|          | List of Problems . . . . .       | 21        |
|          | List of Named Results . . . . .  | 21        |
|          | Index of Defined Terms . . . . . | 23        |

Lecture notes taken, unless otherwise specified, by myself during section 001 of the Spring 2023 offering of CS 350, taught by Armin Jamshidpey.

### Lectures

|           |                   |    |
|-----------|-------------------|----|
| Lecture 1 | (05/09) . . . . . | 2  |
| Lecture 2 | (05/11) . . . . . | 3  |
| Lecture 3 | (05/16) . . . . . | 6  |
| Lecture 4 | (05/18) . . . . . | 8  |
| Lecture 5 | (05/25) . . . . . | 10 |
| Lecture 6 | (05/30) . . . . . | 13 |
| Lecture 7 | (06/01) . . . . . | 15 |
| Lecture 8 | (06/06) . . . . . | 17 |
| Lecture 9 | (06/08) . . . . . | 19 |

# Chapter 1

## Introduction

### 1.1 Asymptotic Review

Recall from CS 240, that given a problem with instances  $I$  of size  $n$ :

*Lecture 1  
(05/09)*

**Definition** (*runtime*)

The runtime of an instance  $I$  is  $T(I)$ .  
The worst-case runtime is  $T(n) = \max_{\{I: |I|=n\}} T(I)$ .  
The average runtime is  $T_{\text{avg}}(n) = \frac{\sum_{\{I: |I|=n\}} T(I)}{|\{I: |I|=n\}|}$

Recall also the asymptotic comparison of functions  $f(n)$  and  $g(n)$  with values in  $\mathbb{R}_{>0}$ :

**Definition** (*big-O*)

$f(n) \in O(g(n))$  if there exists  $C > 0$  and  $n_0$  such that  $n \geq n_0 \implies f(n) \leq Cg(n)$ .

**Definition** (*big-Ω*)

$f(n) \in \Omega(g(n))$  if there exists  $C > 0$  and  $n_0$  such that  $n \geq n_0 \implies f(n) \geq Cg(n)$ .  
Equivalently,  $g(n) \in O(f(n))$ .

**Definition** (*big-Θ*)

$f(n) \in \Theta(g(n))$  if there exists  $C, C' > 0$  and  $n_0$  with  $n \geq n_0 \implies Cg(n) \leq f(n) \leq C'g(n)$ . Equivalently,  $f(n) \in O(g(n)) \cap \Omega(g(n))$ .  
Recall also that if  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}$  is finite, then  $f(n) \in \Theta(g(n))$ .

**Definition** (*little-o*)

$f(n) \in o(g(n))$  if for all  $C > 0$ , there exists  $n_0$  such that  $n \geq n_0 \implies f(n) \leq Cg(n)$ .  
Equivalently,  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ .

**Definition** (*little-ω*)

$f(n) \in \omega(g(n))$  if for all  $C > 0$ , there exists  $n_0$  such that  $n \geq n_0 \implies f(n) > Cg(n)$ .  
Equivalently,  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$  or  $g(n) \in o(f(n))$ .

Also, recall that any polynomial of degree  $k$  is in  $\Theta(n^k)$ .<sup>1</sup>

<sup>1</sup>As long as  $n$  is eventually increasing, i.e., the  $n^k$  term dominates.

We write  $n^{O(1)}$  to mean at most polynomial (i.e.,  $O(n^k(n))$  where  $k \in O(1)$ )

**Exercise 1.1.1.** Is  $2^{n-1}$  in  $\Theta(2^n)$ ?

*Proof.* Notice that  $2^{n-1} = \frac{1}{2}2^n$ . If we let  $C = \frac{1}{2} = C'$ ,  $n_0 = 1$ , notice that for  $n \geq n_0$ , we have  $C2^n = 2^{n-1} \leq 2^{n-1} \leq 2^{n-1} = C'2^n$ . That is,  $2^{n-1} \in \Theta(2^n)$ .  $\square$

**Exercise 1.1.2.** Is  $(n-1)!$  in  $\Theta(n!)$ ?

*Solution.* No. Notice that  $\lim_{n \rightarrow \infty} \frac{(n-1)!}{n!} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . Therefore,  $(n-1)! \in o(n!)$ , which contradicts  $(n-1)! \in \Theta(n!)$ .  $\square$

Consider now multivariate functions  $f(n, m)$  and  $g(n, m)$  with values in  $\mathbb{R}_{>0}$ . Then,

**Definition** (*multivariate big-O*)

$f(n, m)$  is in  $O(g(n, m))$  if there exist  $C, n_0, m_0$  such that  $f(n, m) \leq Cg(n, m)$  for  $n \geq n_0$  **or**  $m \geq m_0$ .

We similarly define the other asymptotic analysis functions. We could alternatively define using  $n \geq n_0$  **and**  $m \geq m_0$  but they both give the same results.

Notice that all basic operations are not equal. For example, multiplication may take  $O(b)$  time for a  $b$ -bit word.

*Lecture 2  
(05/11)*

Warning: big- $O$  is only an upper bound, so  $1 \in O(n^2)$  and  $n \in O(n)$ , but we know that  $1 \ll n$ .

Asymptotic notation hides constants. Any  $\Theta(n^2)$  algorithm will beat a  $\Theta(n^3)$  algorithm eventually. A galactic algorithm is practically irrelevant because the crossing point is stupidly large.

## 1.2 Types of Algorithms

**Problem 1.1** (*contiguous subarrays*)

Given an array  $A[1..n]$ , find a contiguous subarray  $A[i..j]$  that maximizes the sum  $A[i] + \dots + A[j]$ .

Consider the brute-force attempt

**Algorithm 1.2.1** BRUTEFORCE( $A$ )

```

1:  $opt \leftarrow 0$ 
2: for  $i \leftarrow 1, \dots, n$  do
3:   for  $j \leftarrow i, \dots, n$  do
4:      $sum \leftarrow 0$ 
5:     for  $k \leftarrow i, \dots, j$  do
6:        $sum \leftarrow sum + A[k]$ 
7:     if  $sum > opt$  then
8:        $opt \leftarrow sum$ 
9: return  $opt$ 
```

which has a runtime  $\Theta(n^3)$ . This is inefficient. We are recomputing the same sum in the  $j$  loop, so if we instead keep the running sum:

**Algorithm 1.2.2** BETTERBRUTEFORCE( $A$ )

---

```

1:  $opt \leftarrow 0$ 
2: for  $i \leftarrow 1, \dots, n$  do
3:    $sum \leftarrow 0$ 
4:   for  $j \leftarrow i, \dots, n$  do
5:      $sum \leftarrow sum + A[j]$ 
6:     if  $sum > opt$  then
7:        $opt \leftarrow sum$ 
8: return  $opt$ 

```

---

we get  $\Theta(n^2)$ .

We can develop a divide-and-conquer algorithm by noticing that the optimal subarray (if not empty) is either (1) completely in  $A[1..n/2]$ , (2) completely in  $A[n/2 + 1..n]$ , or (3) contains  $A[n/2]$  and  $A[n/2 + 1]$ .

**Algorithm 1.2.3** DIVIDEANDCONQUER( $A$ )

---

```

1: if  $n = 1$  then return  $\max(A[1], 0)$ 
2:  $opt_{lo} \leftarrow \text{DIVIDEANDCONQUER}(A[1..n/2])$ 
3:  $opt_{hi} \leftarrow \text{DIVIDEANDCONQUER}(A[n/2 + 1..n])$ 
4: function MAXIMIZELOWERHALF()
5:    $opt \leftarrow A[n/2]$ 
6:    $sum \leftarrow A[n/2]$ 
7:   for  $i \leftarrow n/2 - 1, \dots, 1$  do
8:      $sum \leftarrow sum + A[i]$ 
9:     if  $sum > opt$  then  $opt \leftarrow sum$ 
10:  return  $opt$ 
11: function MAXIMIZEUPPERHALF()
12:  ...
13:  $opt_{mid} \leftarrow \text{MAXIMIZELOWERHALF}() + \text{MAXIMIZEUPPERHALF}()$ 
14: return  $\max(opt_{lo}, opt_{hi}, opt_{mid})$ 

```

---

Each of MAXIMIZEUPPERHALF and MAXIMIZELOWERHALF have runtime  $\Theta(n)$ , so DIVIDEANDCONQUER has runtime  $2T(n/2) + \Theta(n) \in \Theta(n \log n)$ .

Finally, notice that we can instead solve the problem in nested subarrays  $A[1..j]$  of sizes  $1, \dots, n$ . The optimal subarray is either a subarray of  $A[1..n-1]$  or contains  $A[n]$ .

Write  $M(j)$  for the maximum sum for subarrays of  $A[1..j]$ . Then,

$$M(n) = \max(M(n-1), \bar{M}(n)) = A[n] + \max(\bar{M}(n-1), 0)$$

where  $\bar{M}(j)$  is the maximum sum for subarrays of  $A[1..j]$  that include  $j$ . Notice that the optimal subarray containing  $A[n]$  is either  $A[i..n]$  for  $i \leq n-1$  or exactly  $A[n]$ .

**Algorithm 1.2.4** DYNAMICPROGRAMMING( $A$ )

---

```

1:  $\bar{M} \leftarrow A[1]$ 
2:  $M \leftarrow \max(\bar{M}, 0)$ 
3: for  $i = 2, \dots, n$  do
4:    $\bar{M} \leftarrow A[i] + \max(\bar{M}, 0)$ 
5:    $M \leftarrow \max(M, \bar{M})$ 
6: return  $M$ 

```

---

which has runtime  $\Theta(n)$ . We cannot do better than this (proof beyond the scope of the course, but intuitively notice that we cannot find a max without knowing the entire array).

### 1.3 Recurrence Relations

Recall merge sort.

The recurrence relation is  $T(n) = \begin{cases} T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + \Theta(n) & n > 1 \\ \Theta(1) & n = 1 \end{cases}$

If we let  $c$  and  $d$  be the constants, we get  $T(n) = \begin{cases} T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + cn & n > 1 \\ d & n = 1 \end{cases}$

Equivalently, we can sloppily remove floors and ceilings to get  $T(n) = \begin{cases} 2T(\frac{n}{2}) + cn & n > 1 \\ d & n = 1 \end{cases}$

Construct now a recursion tree, assuming  $n = 2^j$ . Notice that we will end up with  $j$  layers where layer  $i$  has  $2^i$  nodes where each node takes  $cn$  time (the last layer nodes take  $d$  time).

#### **Theorem 1 (*master theorem*)**

Suppose  $a \geq 1$  and  $b > 1$ . Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^y)$$

in sloppy or exact form. Let  $x = \log_b(a)$ . Then,

$$T(n) = \begin{cases} \Theta(n^x) & y < x \\ \Theta(n^y \log n) & y = x \\ \Theta(n^y) & y > x \end{cases}$$

*Proof.* Let  $a \geq 1$  and  $b \geq 2$ . Then, let  $T(n) = aT(\frac{n}{b}) + cn^y$  and  $T(1) = d$ . Also, write for convenience  $n = b^j$ . We can now consider the recurrence tree.

The  $i^{\text{th}}$  row in the tree (except the bottom) will have  $a^i$  subproblems of size  $n/b^i$  which each have cost  $c(n/b^i)^y = cn^y b^{-iy}$ . The  $j^{\text{th}}$  row will have  $a^j$  nodes with cost  $d$ . Then,

$$T(n) = da^j + cn^y \sum_{i=0}^{j-1} \left(\frac{a}{b^y}\right)^i$$

We know that  $x = \log_b a$  which gives  $b^x = a$ . Assume  $r = \frac{a}{b^y} = \frac{b^x}{b^y} = b^{x-y}$ . Then, we have *Lecture 3*  
(05/16)

$$\begin{aligned}
 da^{\log_b n} + cn^y \sum_{i=0}^{j-1} r^i &= dn^{\log_b a} + cn^y \sum_{i=0}^{j-1} r^i \\
 &= dn^x + cn^y \begin{cases} j & r = 1 \\ \Theta(1) & r < 1 \\ \frac{r^j - 1}{r - 1} \in \Theta(r^j) & r > 1 \end{cases} \\
 &= \begin{cases} dn^x + cn^y \log_b n \in \Theta(n^y \log n) & x = y \\ dn^x + c'n^y \in \Theta(n^y) & x < y \\ dn^x + c''n^x \in \Theta(n^x) & x > y \end{cases}
 \end{aligned}$$

noting that  $r^j = r^{\log_b n} = n^{\log_b r} = n^{x-y}$ , so in the latter case  $cn^y \Theta(r^j) \in \Theta(n^x)$ . □

When  $n^x$  dominates, we call it “heavy leaves”. When  $n^y$  dominates, we call it “heavy top”. Otherwise, we call it “balanced”.

## Chapter 2

# Divide and Conquer

In general, we want to:

- divide: split a problem into subproblems;
- conquer: solve the subproblems recursively; and
- combine: use subproblem results to derive problem result

This is possible when:

- the original problem is easily decomposable into subproblems;
- combining solutions is not costly; and
- subproblems are not overly unbalanced

### 2.1 Examples

#### Problem 2.1 (*counting inversions*)

Given an unsorted array  $A[1..n]$ , find the number of inversions in it, i.e., pairs  $(i, j)$  such that  $A[i] > A[j]$ .

**Example 2.1.1.** Given  $A = [1, 5, 2, 6, 3, 8, 7, 4]$ , we get  $(2,3)$ ,  $(2,5)$ ,  $(2,8)$ ,  $(4,5)$ ,  $(4,8)$ ,  $(6,7)$ ,  $(6,8)$ , and  $(7,8)$ .

The naive algorithm checks all pairs and takes  $\Theta(n^2)$  time. We can do better.

Let  $c_\ell$  be the number of inversions in  $A[1..n/2]$ ,  $c_r$  be the number of inversions in  $A[n/2 + 1..n]$ , and  $c_t$  be the number of transverse inversions, i.e., inversions where  $i$  is on the left and  $j$  is on the right.

We can find  $c_\ell$  and  $c_r$  by recursion.

To find  $c_t$ , we must count the number of left indices greater than each right index. This can be done by sorting and then binary searching, since the binary search result index gives exactly what we want. The sort takes  $O(n \log n)$  and each of the  $n$  binary searches takes  $O(\log n)$ .

This gives us  $T(n) \leq 2T(n/2) + O(n \log n) = O(n \log^2 n)$ .

We can instead modify MERGESORT and find  $c_t$  using a modified MERGE:

**Algorithm 2.1.1** Modified MERGE( $A[1..n]$ ) (additions in green)**Require:** both halves of  $A$  are sorted

```

1: copy  $A$  into a new array  $S$ ;  $c = 0$ 
2:  $i \leftarrow 1$ ;  $j \leftarrow n/2 + 1$ 
3: for  $k \leftarrow 1, \dots, n$  do
4:   if  $i > n/2$  then  $A[k] \leftarrow S[j++]$ 
5:   else if  $j > n$  then
6:      $A[k] \leftarrow S[i++]$ 
7:      $c \leftarrow c + \frac{n}{2}$ 
8:   else if  $S[i] < S[j]$  then
9:      $A[k] \leftarrow S[i++]$ 
10:     $c \leftarrow c + j - (\frac{n}{2} + 1)$ 
11:   else  $A[k] \leftarrow S[j++]$ 

```

Here, every time we merge in an element from the left, we add to  $c$  the number of elements on the right which are greater than it. This will run in  $\Theta(n \log n)$  time because the modified MERGE is still  $\Theta(n)$ .

**Problem 2.2** (*polynomial multiplication*)

Given  $F = f_0 + \dots + f_{n-1}x^{n-1}$  and  $G = g_0 + \dots + g_{n-1}x^{n-1}$ , calculate  $H = FG$ .

The naive algorithm takes  $\Theta(n^2)$  time to expand.

Notice that we can split  $F = F_0 + F_1x^{n/2}$  and  $G = G_0 + G_1x^{n/2}$ . Then, we have  $H = F_0G_0 + (F_0G_1 + F_1G_0)x^{n/2} + F_1G_1x^n$ . If we divide and conquer, we make 4 recursive calls with size  $n/2$  and  $\Theta(n)$  extra work for the additions.

However,  $T(n) = 4T(n/2) + \Theta(n) \in \Theta(n^2)$  which is not an improvement.

**Lemma 2.1.1** (*Karatsuba's identity*)

$$(x + y)(a + b) - xa - yb = xb + ya$$

But if we already have  $F_0G_0$  and  $F_1G_1$ , we can use Karatsuba's identity to instead calculate  $(F_0 + F_1)(G_0 + G_1) - F_0G_0 - F_1G_1 = F_0G_1 + F_1G_0$ . That is, we will calculate:

Lecture 4  
(05/18)

$$\begin{aligned} H &= (F_0 + F_1x^{n/2})(G_0 + G_1x^{n/2}) \\ &= F_0G_0 + ((F_0 + F_1)(G_0 + G_1) - F_0G_0 - F_1G_1)x^{n/2} + F_1G_1x^n \end{aligned}$$

This means we only need to make 3 recursive calls instead of 4.

Then,  $T(n) = 3T(n/2) + \Theta(n) \in \Theta(n^{\lg 3})$  which is an improvement.

Based on this observation, Toom-Cook created a family of algorithms that for  $k \geq 2$  make  $2k - 1$  recursive calls in size  $n/k$ , i.e.,  $T(n) \in \Theta(n^{\log_k(2k-1)})$  which gets arbitrarily close to linear (but with increasingly massive constants).

If  $F, G \in \mathbb{C}[x]$ , then we can use FFT to get  $T(n) = 2T(n/2) + \Theta(n) \in \Theta(n \log n)$  time.

**Problem 2.3** (*matrix multiplication*)

Given  $A = [a_{i,j}] \in M_{n \times n}$  and  $B = [b_{j,k}] \in M_{n \times n}$ , calculate  $C = AB$ .

The naive algorithm takes inputs of size  $\Theta(n^2)$  in  $\Theta(n^3)$  time.



Consider instead breaking into block matrices:  $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$  and  $B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$ .

Then,  $C = \begin{pmatrix} A_{1,1}B_{1,1} + A_{1,2}B_{2,1} & A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ A_{2,1}B_{1,1} + A_{2,2}B_{2,1} & A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{pmatrix}$

This makes 8 recursive calls of size  $n/2$  and  $\Theta(n^2)$  additions, which resolves to  $\Theta(n^3)$  (no improvement). However, due to Strassen, we can reduce this to 7, giving  $\Theta(n^{\lg 7})$  time.

We can generalize to do  $k$  multiplications of  $\ell \times \ell$  matrices in  $\Theta(n^{\log_\ell k})$  time and  $k$  multiplications of  $\ell \times m$  by  $m \times p$  in  $\Theta(n^{3 \log_{\ell mp} k})$  time.

#### Problem 2.4 (*closest pairs*)

Given  $n$  distinct points  $(x_i, y_i)$ , find a pair  $(i, j)$  that minimizes the distance  $d_{i,j} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ . Equivalently, minimize  $d_{i,j}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$ .

Separate the space of points into  $L$  and  $R$  halfspaces based on the median  $x$  value. The closest pair is either entirely in  $L$ , entirely in  $R$ , or transverse.

We can recursively find minimum distances  $\delta_L$  and  $\delta_R$ . Then, if we let  $\delta = \min\{\delta_L, \delta_R\}$ , any closer transverse points must be within  $\delta$  units of the median  $x$  value.

Now, if we start from the bottom point  $P \in L$  by  $y$ -value in that band, we only have to compare  $P$  with points  $Q \in R$  with  $y_P \leq y_Q < y_P + \delta$ .

We can only have a maximum of 8 points inside the  $2\delta \times \delta$  rectangle of possible  $Q$  points, because the points must be at least  $\delta$  apart.

Therefore, we are doing  $\Theta(1)$  work for each  $P$ , so we do  $\Theta(n)$  work to find transverse pairs.

For this to work, we must first sort the points by  $x$  and by  $y$  (in  $O(n \log n)$  time). We can find the median in  $O(1)$  time. We split the sorted points in  $O(n)$  time for the two recursive calls and find the  $\delta$  band in  $O(n)$  time. Again, it takes  $O(n)$  time to find transverse pairs. Therefore,  $T(n) = 2T(n/2) + O(n) = O(n \log n)$ .

#### Problem 2.5 (*selection*)

Given  $A[0..n-1]$ , find the entry that would be at index  $k$  if  $A$  were sorted.

Recall from CS 240 that selection by sorting takes  $O(n \log n)$  time or  $O(n)$  randomized expected time using QUICKSELECT( $A, k$ ):

---

#### Algorithm 2.1.2 QUICKSELECT( $A, k$ )

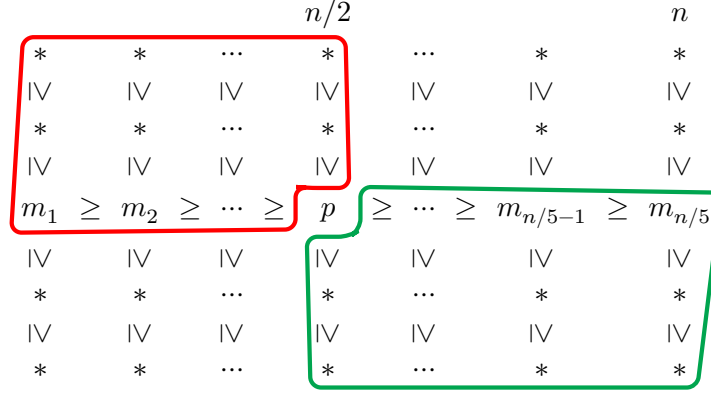
---

```

1:  $p \leftarrow \text{CHOOSEPIVOT}(A)$ 
2:  $i \leftarrow \text{PARTITION}(A, p)$   $\triangleright i$  is the correct index of  $p$ 
3: if  $i = k$  then return  $A[i]$ 
4: else if  $i > k$  then return QUICKSELECT( $A[0..i-1], k$ )
5: else return QUICKSELECT( $A[i+1..n-1], k-i-1$ )
```

---

Consider splitting  $A$  into groups  $G_i$  of size 5. Then, find the medians  $m_i$  of each group. We can choose the pivot  $p$  as the median of medians:



*Solution.* Notice that if we draw a tree, each layer sums to  $n$  (this makes sense inductively since we pass  $\frac{2}{3}$  of  $n$  to the left and  $\frac{1}{3}$  of  $n$  to the right). There will be  $O(\log_{3/2} n)$  layers in the tree, so it should resolve to  $O(n \log n)$ .  $\square$

# Chapter 3

## Graph Algorithms

### 3.1 Graph Theory Review

Recall graph theory from MATH 239, specifically: ms

**Definition** (*graph*)

A graph  $G$  is a pair  $(V, E)$  where  $V$  is a finite set of vertices and  $E$  is a set of unordered pairs of distinct vertices, called edges. By convention, we write  $n = |V|$  and  $m = |E|$ .

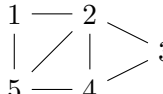
Now, we can define some structures on a graph:

**Definition** (*adjacency list*)

An array  $A[1..n]$  such that  $A[v]$  is a linked list containing all edges connected to  $v$ . This contains  $2m$  list cells with total size  $\Theta(n + m)$  but takes more than  $O(1)$  time to test if an edge exists.

**Definition** (*adjacency matrix*)

A matrix  $M \in M_{n \times n}(\{0, 1\})$  such that  $M[v, w] = 1$  if and only if  $\{v, w\} \in E$ . Size is  $\Theta(n^2)$  but testing if an edge exists is  $O(1)$ .

**Example 3.1.1.** Write the adjacency list and matrix for 

*Solution.* The adjacency list is:

$1 \rightarrow 2 \rightarrow 5$   
 $2 \rightarrow 1 \rightarrow 3 \rightarrow 4 \rightarrow 5$   
 $3 \rightarrow 2 \rightarrow 4$   
 $4 \rightarrow 2 \rightarrow 3 \rightarrow 5$   
 $5 \rightarrow 1 \rightarrow 2 \rightarrow 4$

and the matrix is  $M = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

□

**Definition** (*graph terminology*)

We also recall some terms from MATH 239:

- A path is a sequence of vertices  $v_1, \dots, v_k$  such that  $\{v_i, v_{i+1}\} \in E$  for all  $i$ . If a path from  $v$  to  $w$  exists, we write  $v \rightsquigarrow w$ .
- A connected graph has  $v \rightsquigarrow w$  for all  $v, w \in V$ .
- A cycle is a path  $v \rightsquigarrow v$  of length at least 3 with all elements pairwise distinct.
- A tree is a graph with no cycles.
- A rooted tree is a tree with a vertex chosen to be the root.
- A subgraph of  $G = (V, E)$  is a graph  $G' = (V', E')$  where  $V' \subseteq V$ ,  $E' \subseteq E$ , and  $u, v \in V'$  for all  $uv \in E'$ .
- A connected component of  $G$  is a connected subgraph of  $G$  that is not a subset of any other connected subgraph.

**3.2 Breadth-First Search****Problem 3.1**

Search a graph  $G$  starting from a vertex  $s$  in order of distance from  $s$ .

**Algorithm 3.2.1** BFS( $G, s$ )

```

1: let  $Q$  be an empty queue
2: let visited be a boolean array of size  $n$  with all entries set to  $\perp$ 
3: ENQUEUE( $s, Q$ )
4: visited[ $s$ ]  $\leftarrow \top$ 
5: while  $Q$  is not empty do
6:    $v \leftarrow$  DEQUEUE( $Q$ )
7:   for  $w$  neighbours of  $v$  do
8:     if visited[ $w$ ] =  $\perp$  then
9:       ENQUEUE( $w, Q$ )
10:    visited[ $w$ ]  $\leftarrow \top$ 

```

Each vertex is enqueued at most once and dequeued at most once, which has cost  $O(n)$ . Therefore, each adjacency list is read at most once. The cost for the for loop is  $O(\sum \deg v) = O(m)$  by the Handshaking Lemma.

Therefore, the total cost of BFS is  $O(n + m)$ .

**Lemma 3.2.1**

**visited**[ $v$ ] is true for some vertex  $v$  if and only if  $s \rightsquigarrow v$  in  $G$ .

*Proof.* Let  $s = v_0, \dots, v_K$  be the vertices with **visited** $v_i = \top$ , in order of discovery. By induction, we show that  $s \rightsquigarrow v_i$ .

For  $i = 0$ ,  $v_0 = s$ , so trivially  $s \rightsquigarrow s$ .

Otherwise, suppose  $s \rightsquigarrow v_j$  for all  $j < i$ . We are currently in the for loop for some vertex  $w$  already visited. Therefore, by assumption,  $s \rightsquigarrow w$ . But since  $v_i$  is a neighbour of  $w$ ,  $s \rightsquigarrow v_i$ .  $\square$

**Exercise 3.2.1.** For a connected graph,  $m \geq n - 1$ .

*Proof.* Recall from MATH 239 that if a graph  $G$  is connected, then it has a spanning tree  $T$ . The spanning tree of  $n$  vertices has exactly  $n - 1$  edges. Then, since the spanning tree is a subgraph of  $G$ ,  $m \geq |E(T)| = n - 1$ , as desired.  $\square$

### 3.3 Shortest Path by BFS

#### Problem 3.2

What is the shortest path from  $s$  to  $v$  in  $G$ ?

Consider now how we can keep track of parents (predecessors) and levels (depths):

---

#### Algorithm 3.3.1 BFS( $G, s$ ) with parents and levels

---

```

1: let  $Q$  be an empty queue
2: let  $\text{parent}$  be an array of size  $n$  with all entries set to  $\perp$ 
3: let  $\text{level}$  be an array of size  $n$  with all entries set to  $\infty$ 
4: ENQUEUE( $s, Q$ )
5:  $\text{parent}[s] \leftarrow s$ 
6:  $\text{level}[s] \leftarrow 0$ 
7: while  $Q$  is not empty do
8:    $v \leftarrow \text{DEQUEUE}(Q)$ 
9:   for  $w$  neighbours of  $v$  do
10:    if  $\text{parent}[w] = \perp$  then
11:      ENQUEUE( $w, Q$ )
12:       $\text{parent}[w] \leftarrow v$ 
13:       $\text{level}[w] \leftarrow \text{level}[v] + 1$ 

```

---

We can define a BFS tree  $T$  as the subgraph of  $G$  made of all  $w$  such that  $\text{parent}[w] \neq \perp$  and all edges  $\{w, \text{parent}[w]\}$  between those vertices.

*Claim 3.3.1.* The BFS tree  $T$  is in fact a tree.

*Proof.* Proceed by induction on the vertices for which  $\text{parent}[v]$  is not  $\perp$ .

When we set  $\text{parent}[s] \leftarrow s$ , we have one vertex and no edges.

Suppose  $T$  is a tree and we are adding  $\text{parent}[w] \leftarrow v$ . Then,  $v$  must have already been in  $T$  because it came from  $Q$ , so we are extending  $T$  by adding (1) the vertex  $w$  and (2) the edge  $\{v, w\}$ . This does not create a cycle because  $\text{parent}[w] = \perp$ , so  $T$  remains a tree.

Therefore, by induction, at the end of BFS,  $T$  is a tree.  $\square$

*Claim 3.3.2.* The levels in the queue  $Q$  are non-decreasing.

*Proof.* Exercise (TODO).  $\square$

*Claim 3.3.3.* For all vertices  $u$  and  $v$ , if there is an edge  $\{u, v\}$ , then  $\text{level}[v] \leq \text{level}[u] + 1$ .

*Proof.* Suppose that  $u$  and  $v$  are adjacent and visited.

If we dequeue  $v$  before  $u$ , then  $\text{level}[v] \leq \text{level}[u] + 1$  by Claim 3.3.2.

If  $u$  is dequeued before  $v$ , then the parent of  $v$  is either  $u$  or something else before  $u$ . This is because while visiting  $u$ , we must either have enqueued  $v$  or already visited  $v$ . Therefore,  $v$ 's parent must be at or before  $u$ . Then, by Claim 3.3.2,  $\text{level}[\text{parent}[v]] \leq \text{level}[u]$ .

That is,  $\text{level}[v] = \text{level}[\text{parent}[v]] + 1 \leq \text{level}[u] + 1$ .  $\square$

### Lemma 3.3.1

For all  $v$  in  $G$ , there is a path  $s \rightsquigarrow v$  in  $G$  if and only if there is a path  $s \rightsquigarrow v$  in  $T$ .  
If so, the path in  $T$  is a shortest path and  $\text{level}[v]$  is the distance from  $s$  to  $t$ .

*Proof.* By Lemma 3.2.1,  $s \rightsquigarrow_G v$  if and only if  $v$  is visited. That is, all such  $v$  are in  $T$ . But  $T$  is connected as a tree, therefore  $s \rightsquigarrow_G v \iff s \rightsquigarrow_T v$ .

Let  $\delta$  be the distance from  $s$  to  $v$ . We must show  $\text{level}[v] \leq \delta$  and  $\delta \leq \text{level}[v]$ .

Trivially,  $\delta \leq \text{level}[v]$  because  $\text{level}[v]$  is the length of the path  $s \rightsquigarrow_T v$ .

We will prove by induction that for all  $i$ , if there is a path of length  $i$  from  $s$  to  $v$ , then  $\text{level}[v] \leq i$ . For the base case  $i = 0$ , there are no such paths.

Suppose this is true for  $i - 1$ , and consider a path  $P = s \cdots uv$  with length  $i$ . Then, we can decompose  $P$  as  $P' = s \cdots u$  and  $uv$ . But  $P'$  has length  $i - 1$ , so  $\text{level}[u] \leq i - 1$ . Then, by Claim 3.3.3,  $\text{level}[v] \leq \text{level}[u] + 1 \leq i$ .

Therefore, since this is true for all  $i$ , it is true for  $i = \delta$ .

Finally, we have that  $\delta = \text{level}[v]$  and  $s \rightsquigarrow_T v$  is a shortest path.  $\square$

## 3.4 Bipartiteness by BFS

### Definition (*bipartite*)

A graph  $G = (V, E)$  is bipartite if there exists a partition  $U_1 \sqcup U_2 = V$  such that for every  $uv \in E$ ,  $u \in U_1$  and  $v \in U_2$  (or vice versa).

### Problem 3.3

Is  $G$  bipartite?

### Lemma 3.4.1

Suppose  $G$  is connected and we run  $\text{BFS}(G, s)$  for some  $s$ . Let  $V_1$  and  $V_2$  be vertex sets with odd and even level respectively. Then,  $G$  is bipartite if and only if all edges have one end in  $V_1$  and one end in  $V_2$ .

*Proof.* Suppose all edges have one end in  $V_1$  and one end in  $V_2$ . Then,  $G$  is bipartite by definition.

Suppose  $G$  has bipartition  $(W_1, W_2)$ . Then, wlog say that  $s \in W_2$ . Since  $s \rightsquigarrow v$  for all  $v \in V$  and all paths alternate between  $W_1$  and  $W_2$ , odd depth vertices will fall in  $W_1 = V_1$  and even ones in  $W_2 = V_2$ .  $\square$

This is nice because we can test in  $O(m)$  time.

## 3.5 Depth-First Search

Analogous to BFS, but we use a stack (implicitly with recursion, or explicitly with a stack data structure) to follow neighbours until we cannot.

Lecture 7  
(06/01)

**Algorithm 3.5.1** DFS( $G$ )**Require:**  $G$  is a graph on  $n$  vertices given by adjacency lists

---

```

1: visited  $\leftarrow$  array of size  $n$  initialized to  $\perp$ 
2: procedure EXPLORE( $v$ )
3:   visited[ $v$ ]  $\leftarrow \top$ 
4:   for  $w$  neighbour of  $v$  do
5:     if visited[ $w$ ] =  $\perp$  then
6:       EXPLORE( $w$ )
7: for  $v \in G$  do
8:   if visited[ $v$ ] =  $\perp$  then
9:     EXPLORE( $v$ )

```

---

**Lemma 3.5.1** (*white path lemma*)

When we start exploring  $v$ , any  $w$  connected to  $v$  by an unvisited path will be visited during EXPLORE( $v$ ).

*Proof.* Let  $v_0 = v \dots v_k = w$  be a path  $v \rightsquigarrow w$  with  $v_1, \dots, v_k$  all not visited. We prove all  $v_i$  are visited before EXPLORE( $v$ ) is finished.

Obviously holds for  $i = 0$ . Suppose it holds for  $i < k$ . When we visit  $v_i$ , EXPLORE( $v$ ) is not finished and  $v_{i+1}$  is one of the neighbours.

If **visited**[ $v_{i+1}$ ] is already true (because  $v \rightsquigarrow v_{i+1}$  by some other path), we are done. Otherwise, we are going to visit it now, which is before EXPLORE( $v$ ) is finished.

Therefore,  $v_{i+1}$  is visited during EXPLORE( $v$ ), as desired.  $\square$

*Corollary.* After we call EXPLORE at  $v_1, \dots, v_k$ , we have visited exactly the connected components containing  $v_1, \dots, v_k$ .

Note: we cannot find shortest paths using a DFS tree without customization. For example, the DFS tree for a cycle will be a path even though the root and leaf are adjacent.

The runtime is still  $O(n + m)$ .

**Definition**

Let  $T_1, \dots, T_k$  be a DFS forest with vertices  $u$  and  $v$ . Then,  $u$  is an ancestor of  $v$  if  $u, v \in T_i$  for some  $i$  and  $u$  is on the path from the root of  $T_i$  to  $v$ . Equivalently, we write that  $v$  is a descendant of  $u$ .

**Lemma 3.5.2** (*key property*)

All edges in  $G$  connect a vertex to one of its descendants or ancestors.

*Proof.* Let  $\{v, w\}$  be an edge and suppose WLOG we visit  $v$  first.

Then, when we visit  $v$ ,  $(v, w)$  is an unvisited path  $v \rightsquigarrow w$ , so by the [white path lemma](#),  $w$  must become a descendant of  $v$ .  $\square$

**Definition** (*back edge*)

An edge in  $G$  connecting an ancestor to a descendant which is not in the BFS forest.

*Corollary.* All edges are either tree edges or back edges.

*Proof.* Equivalent statement of the [3.5.2](#).  $\square$



We can extend BFS with `start` and `finish` arrays:

---

**Algorithm 3.5.2** DFS( $G$ ) with timing
 

---

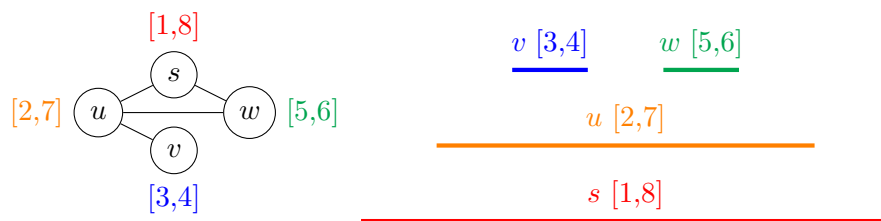
**Require:**  $G$  is a graph on  $n$  vertices given by adjacency lists

```

1: visited  $\leftarrow$  array of size  $n$  initialized to  $\perp$ 
2: start, finish  $\leftarrow$  array of size  $n$ 
3:  $t \leftarrow 1$ 
4: procedure EXPLORE( $v$ )
5:   visited[ $v$ ]  $\leftarrow \top$ 
6:   start[ $v$ ]  $\leftarrow t$ ;  $t++$ 
7:   for  $w$  neighbour of  $v$  do
8:     if visited[ $w$ ] =  $\perp$  then
9:       EXPLORE( $w$ )
10:  start[ $v$ ]  $\leftarrow t$ ;  $t++$ 
11: for  $v \in G$  do
12:   if visited[ $v$ ] =  $\perp$  then
13:     EXPLORE( $v$ )
  
```

---

For example, we can draw a graph with  $[\text{start}[v], \text{finish}[v]]$  labelled:



Notice that the intervals shrink with depth and follow a structure similar to the well-formed parenthesis problem. We can in fact prove:

**Lemma 3.5.3 (parentheses theorem)**

If  $\text{start}[u] < \text{start}[v]$ , then either  $\text{finish}[u] < \text{start}[v]$  or  $\text{finish}[u] < \text{finish}[v]$ .

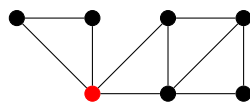
*Proof.* If  $\text{start}[u] < \text{start}[v]$ , we push  $v$  on the stack while  $u$  is still there, so we pop  $v$  before we pop  $u$  since stacks are LIFO.  $\square$

### 3.6 Cut Vertices by DFS

We define a cut vertex analogous to a bridge edge from MATH 239.

**Definition** (*cut vertex*)

Given a connected graph  $G$ , a vertex  $v \in V(G)$  is a cut vertex (or articulation point) if removing  $v$  and its edges makes  $G$  disconnected.



**Example 3.6.1.** The graph has a cut vertex in red.

**Problem 3.4**

Which of the vertices in  $G$  are cut vertices?

Lecture 8  
(06/06)

Consider a rooted DFS tree  $T$  with known **parent** and **level**.

**Proposition 3.6.1**

The root  $s$  is a cut vertex if and only if it has more than one child.

*Proof.* Suppose  $s$  has one child  $v$ . Then,  $T - s$  is a rooted DFS tree with root  $v$  (i.e., it remains connected).

Suppose  $s$  has subtrees  $S_1, \dots, S_k$ . Let  $u \in S_i$  and  $v \in S_j$  for  $i \neq j$ . Then, there does not exist a path  $u \rightsquigarrow v$  in  $T - s$  by the **key property** since it would involve a non-tree, non-back cross edge. Therefore, the subtrees are disconnected in  $T - s$ .  $\square$

**Proposition 3.6.2**

Let  $a(v) = \min\{\text{level}[w] : vw \in E(G)\}$  and  $m(v) = \min\{a(w) : w \text{ descendant of } v\}$ . Any non-root vertex  $v$  is a cut vertex if and only if it has a child  $w$  with  $m(w) \geq \text{level}[v]$ .

*Proof.* Let  $w$  be a child of  $v$  with subtrees  $T_w$  and  $T_v$ , respectively.

Suppose  $m(w) < \text{level}[v]$  and we have removed  $v$ . Then, there is a vertex  $w'$  in  $T_w$  to some vertex  $v'$  above  $v$ . That is, for any vertex  $u \in V(T_w)$ , we have that  $u \rightsquigarrow w \rightsquigarrow w' \rightsquigarrow v' \rightsquigarrow s$  and  $T_w$  is still connected.

Therefore, for  $v$  to be a cut vertex, we must have  $m(w) \geq \text{level}[v]$ .

Suppose  $m(w) \geq \text{level}[v]$ . Then, by the **key property**, all edges from  $T_w$  end in  $T_v$ . They are either the tree edge  $vw$  or a back edge going to an ancestor at or below  $v$ . Therefore, removing  $v$  will cause  $T_w$  to be disconnected and  $v$  is a cut vertex.  $\square$

Therefore, we can solve the cut vertex problem by calculating  $m(v)$  for every vertex.

We can compute  $a(v)$  in  $O(\deg v)$ . Notice that if  $v$  has children  $w_1, \dots, w_k$ , then  $m(v) = \min\{a(v), m(w_1), \dots, m(w_k)\}$ . Then, if we have the  $m$  of the children, we get  $m(v)$  in  $O(\deg v)$ .

By traversing the DFS tree, we get every  $m(v)$  in  $O(n + m) = O(m)$  (since  $G$  connected). Then, we can test the cut vertex condition for each vertex  $v$  and each of its children in  $O(\deg v)$ .

Therefore, we can test all vertices in  $O(m)$  time.

---

**Algorithm 3.6.1** FINDCUTVERTICES( $G, s$ )

---

```

1:  $T \leftarrow \text{DFS}(G, s)$   $\triangleright$  DFS tree for  $G$  with root and level
2:  $a, m \leftarrow$  arrays of size  $|V(G)|$  initialized to  $\infty$ 
3:  $\text{cut} \leftarrow$  array of size  $|V(G)|$  initialized to  $\perp$ 
4: procedure EXPLORE( $v$ )
5:   for  $w$  child of  $v$  do
6:      $a[v] \leftarrow \min\{a[v], \text{level}[w]\}$ 
7:     EXPLORE( $w$ )
8:      $m[v] \leftarrow \min\{m[v], a[w]\}$ 
9:    $m[v] \leftarrow \min\{a[v], m[v]\}$ 
10:  for  $w$  child of  $v$  do
11:    if  $m[w] \geq T.\text{level}[v]$  then  $\text{cut}[v] \leftarrow \top$ 
12: EXPLORE( $T.\text{root}$ )
```

---

### 3.7 Directed Graphs

We can define a directed graph similar to an ordinary graph:

**Definition** (*directed graph*)

A graph  $G = (V, E)$  where edges are *ordered* pairs  $(u, v)$ . If  $G$  has no cycles, it is a directed acyclic graph (DAG).

Note that we allow loops  $(v, v)$ . Paths and cycles have the ordinary meaning.

**Definition** (*topological ordering*)

An ordering  $<$  of  $V$  in a DAG such that  $(a, b) \in E$  implies  $a < b$ .

**Proposition 3.7.1**

A directed graph is acyclic if and only if there is a topological ordering on it.

Lecture 9  
(06/08)

*Proof.* The backwards direction is clear.

Assume we have a DAG. There exists at least one vertex with in-degree 0, because otherwise there would be a cycle. We can inductively remove the vertex with in-degree 0 to get a topological ordering.

In fact, if run DFS and we order  $V$  with the ordering  $v < w \iff \text{finish}[w] < \text{finish}[v]$ , then we can show that  $<$  is a topological order.

Suppose that  $(v, w) \in E$ .

If we discover  $v$  before  $w$ , then  $w$  is a descendant of  $v$  by the [white path lemma](#) so we must finish exploring it before we finish  $v$ .

Otherwise, if we discover  $w$  before  $v$ , then there cannot exist a path  $w \rightsquigarrow v$  because otherwise  $w \rightsquigarrow vw$  is a cycle. Therefore,  $\text{finish}[w] < \text{start}[v] < \text{finish}[v]$ .

Therefore,  $<$  is a topological order whose existence is necessary and sufficient for a DAG.  $\square$

**Definition** (*strong connectivity*)

A directed graph  $G$  is strongly connected if for all  $v$  and  $w$  in  $G$ , there is a path  $v \rightsquigarrow w$  (and  $w \rightsquigarrow v$ )

*Corollary.*  $G$  is strongly connected if and only if there exists  $s$  such that for all  $w$  there exist paths  $s \rightsquigarrow w$  and  $w \rightsquigarrow s$ .

*Proof.* The forwards direction is trivial. In the backwards direction, notice that for any two vertices  $v$  and  $w$ , we have  $v \rightsquigarrow s \rightsquigarrow w$  and  $w \rightsquigarrow s \rightsquigarrow v$ .  $\square$

**Problem 3.5**

How can we test if a graph is strongly connected?

*Solution.* Call EXPLORE twice, starting from the same vertex  $s$ . On the second run, reverse all the edges. Then, if every vertex  $v$  is explored in both runs, we know that  $s \rightsquigarrow v$  and  $v \rightsquigarrow s$ , i.e., the graph is strongly connected.

We can reverse the edges using an adjacency list in  $O(n + m)$  time, so this algorithm runs in  $O(n + m)$  time.  $\square$

Notice that any directed graph is a DAG of its strongly connected components.

# List of Problems

|     |   |   |
|-----|---|---|
| 1.1 | Problem (contiguous subarrays) . . . . .      | 3 |
| 2.1 | Problem (counting inversions) . . . . .       | 7 |
| 2.2 | Problem (polynomial multiplication) . . . . . | 8 |
| 2.3 | Problem (matrix multiplication) . . . . .     | 8 |
| 2.4 | Problem (closest pairs) . . . . .             | 9 |
| 2.5 | Problem (selection) . . . . .                 | 9 |

# List of Named Results

|       |                              |    |
|-------|------------------------------|----|
| 1     | Theorem (master theorem)     | 5  |
| 2.1.1 | Lemma (Karatsuba's identity) | 8  |
| 3.5.1 | Lemma (white path lemma)     | 16 |
| 3.5.2 | Lemma (key property)         | 16 |
| 3.5.3 | Lemma (parentheses theorem)  | 17 |

# Index of Defined Terms

|   |  |  |
|---|--|--|
| adjacency list, <a href="#">12</a>      | descendant, <a href="#">16</a>             | root, <a href="#">13</a>                 |
| adjacency matrix, <a href="#">12</a>    | directed acyclic graph, <a href="#">19</a> | rooted tree, <a href="#">13</a>          |
| ancestor, <a href="#">16</a>            | directed graph, <a href="#">19</a>         | runtime                                  |
| articulation point, <a href="#">17</a>  | edges, <a href="#">12</a>                  | average, <a href="#">2</a>               |
| back edge, <a href="#">16</a>           | galactic algorithm, <a href="#">3</a>      | of an instance, <a href="#">2</a>        |
| BFS tree, <a href="#">14</a>            | graph, <a href="#">12</a>                  | worst-case, <a href="#">2</a>            |
| big- $\Omega$ , <a href="#">2</a>       | inversion, <a href="#">7</a>               | sloppy recurrence, <a href="#">5</a>     |
| big- $\Theta$ , <a href="#">2</a>       | little- $\omega$ , <a href="#">2</a>       | strongly connected, <a href="#">19</a>   |
| big- $O$ , <a href="#">2</a>            | little- $o$ , <a href="#">2</a>            | subgraph, <a href="#">13</a>             |
| bipartite, <a href="#">15</a>           | path, <a href="#">13</a>                   | topological ordering, <a href="#">19</a> |
| connected component, <a href="#">13</a> |  | tree, <a href="#">13</a>                 |
| connected graph, <a href="#">13</a>     |  | vertices, <a href="#">12</a>             |
| cycle, <a href="#">13</a>               |  |  |