MATH 135 Fall 2020: Extra Practice 4

Warm-Up Exercises

WE01. Evaluate $\sum_{i=3}^{8} 2^i$ and $\prod_{j=1}^{5} \frac{j}{3}$.

Solution. Simply expand along the sum/product:

$$\sum_{i=3}^{8} 2^i = 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 = 8 + 16 + 32 + 64 + 128 + 256 = 504$$

and

$$\prod_{j=1}^{5} \frac{j}{3} = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{3} \cdot \frac{4}{3} \cdot \frac{5}{3} = \frac{120}{243} = \frac{40}{81}$$

□

WE02. Let x be a real number. Using the Binomial Theorem, expand $\left(x - \frac{1}{x}\right)^7$.

Solution. Recall the Binomial Theorem, that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Now, substitute $a = x$ and $b = -\frac{1}{x}$.

$$\begin{aligned}
\left(x - \frac{1}{x}\right)^7 &= \sum_{k=0}^{7} \binom{7}{k} x^{7-k} \left(-\frac{1}{x}\right)^k \\
&= \sum_{k=0}^{7} \binom{7}{k} x^{7-k} x^{-k} (-1)^k \\
&= \sum_{k=0}^{7} \binom{7}{k} (-1)^k x^{7-2k} \\
&= x^7 - 7x^{7-2} + 21x^{7-4} - 35x^{7-6} + 35x^{7-8} - 21x^{7-10} + 7x^{7-12} - x^{7-14} \\
&= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}
\end{aligned}$$

□

Recommended Problems

RP01. Prove the following statements by induction.

(a) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} (2i-1) = n^2$.

Proof. We will induct the statement $P(n) \equiv \sum_{i=1}^{n} (2i-1) = n^2$ on n.

(Base Case) When $n = 1$, the left-hand side is

$$\begin{aligned}
\sum_{i=1}^{1} (2i-1) &= 2(1) - 1 \\
&= 1 \\
&= 1^2
\end{aligned}$$

which is the right-hand side, so $P(1)$ holds.

(Inductive Step) Now, suppose that $P(k)$ holds for an arbitrary k. Then, we take the left-hand side of $P(k+1)$

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) &= (2(k+1)-1) + \sum_{i=1}^{k}(2i-1) \\ &= (2k+1) + k^2 && \text{by inductive hypothesis} \\ &= (k+1)^2\end{aligned}$$

as desired to show that if $P(k)$ holds, then $P(k+1)$ holds.

Therefore, by induction, $P(n)$ holds for all n. ☐

(b) For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} r^i = \dfrac{1-r^{n+1}}{1-r}$ where r is any real number such that $r \neq 1$.

Proof. Let r be an arbitrary real other than 1. We will induct the statement $P(n) \equiv \sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ on n.

(Base Case) For $n = 1$, substitute into the LHS and expand the summation:

$$\sum_{i=1}^{1} r^i = r^0 + r^1 = 1 + r = (1+r)\frac{1-r}{1-r} = \frac{1-r^2}{1-r}$$

This is precisely the RHS of the equality, so $P(1)$ holds.

(Inductive Step) Now, suppose that $P(k)$ holds for an arbitrary k. Again, expand the summation but for the LHS of $P(k+1)$:

$$\begin{aligned}\sum_{i=0}^{k+1} r^i &= r^{k+1} + \sum_{i=0}^{k} r^i \\ &= r^{k+1} + \frac{1-r^{k+1}}{1-r} && \text{by inductive hypothesis} \\ &= \frac{(r^{k+1})(1-r) + 1 - r^{k+1}}{1-r} \\ &= \frac{r^{k+1} - r^{k+2} + 1 - r^{k+1}}{1-r} \\ &= \frac{1-r^{k+2}}{1-r}\end{aligned}$$

which is the other side of the equality. We have proved that if $P(n)$ holds, then $P(n+1)$ holds. Therefore, by induction, $P(n)$ holds for all natural n. ☐

(c) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} \dfrac{i}{(i+1)!} = 1 - \dfrac{1}{(n+1)!}$.

Proof. We will induct the statement $P(n) \equiv \sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$ on n.

First, verify the base case, $P(1)$. Then, we let $n = 1$ and have

$$\sum_{i=1}^{1} \frac{i}{(i+1)!} = 1 - \frac{1}{2!}$$

Expanding the summation, we can show that $P(1)$ holds:

$$\sum_{i=1}^{1} \frac{i}{(i+1)!} = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2!}$$

Now, suppose $P(k)$ is true for some k, and consider $P(k+1)$:

$$\sum_{i=1}^{n+1} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+2)!}$$

Like above, we take out a term of the summation and simplify, so we have

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{i}{(i+1)!} &= \frac{k+1}{(k+2)!} + \sum_{i=1}^{k} \frac{i}{(i+1)!} \\
&= \frac{k+1}{(k+2)!} + 1 - \frac{1}{(k+1)!} && \text{by inductive hypothesis} \\
&= 1 + \frac{(k+1)-(k+2)}{(k+2)!} \\
&= 1 - \frac{1}{(k+2)!}
\end{aligned}$$

as required. We have proven $P(1)$ and that $P(k)$ implies $P(k+1)$, so, by induction, $P(n)$ is true for all natural n. ☐

(d) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

Proof. For induction on n, let $P(n) \equiv \sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.
Verify the base case $P(1)$:

$$\sum_{i=1}^{1} \frac{i}{2^i} = \frac{1}{2} = 2 - \frac{3}{2} = 2 - \frac{1+2}{2^1}$$

Suppose that $P(k)$ holds for some k, and consider $P(k+1)$. Now,

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{i}{2^i} &= \frac{k+1}{2^{k+1}} + \sum_{i=1}^{n} \frac{i}{2^i} \\
&= \frac{k+1}{2^{k+1}} + 2 - \frac{k+2}{2^k} && \text{by inductive hypothesis} \\
&= 2 + \frac{k+1-2(k+2)}{2^{k+1}} \\
&= 2 - \frac{k+3}{2^{k+1}}
\end{aligned}$$

as required. Because $P(1)$ holds and $P(k)$ implies $P(k+1)$, by induction, $P(n)$ holds for all n. ☐

(e) For all $n \in \mathbb{N}$, where $n \geq 4$, $n! > n^2$.

Proof. We will prove by induction on n. Let $P(n)$ be the statement $n! > n^2$.

To verify the base case $P(4)$, notice that $4! = 24$, that $4^2 = 16$, and that $24 > 16$.

Now, suppose that $P(k)$ is true for some $k \geq 4$. We must show that $P(k+1)$ holds, i.e., $(k+1)! > (k+1)^2$.

First, notice that $x^2 > x + 1$ for all $x \geq 4$. Then, we can state the inductive hypothesis as $k! > k + 1$. Multiplying both sides by $k + 1$ gives $(k+1)! > (k+1)^2$, as desired.

Therefore, by induction, $n! > n^2$ for all $n \geq 4$. □

(f) For all $n \in \mathbb{N}$, $4^n - 1$ is divisible by 3.

Proof. Induct the statement "$4^n - 1$ is divisible by 3" on n.

For the base case, let $n = 1$ so $4^1 - 1 = 3$ and 3 is obviously divisible by 3.

Now, suppose that $4^k - 1$ is divisible by 3 for some natural number k. By definition, there exists an integer a where $4^k - 1 = 3a$.

Consider when $n = k+1$. Rearranging, $4^{k+1} - 1 = (4^{k+1} - 4) + 3 = 4(4^k - 1) + 3$. By our inductive hypothesis, this is equal to $4(3a) + 3 = 3(4a+1)$. Then, since $4^{k+1} - 1$ can be written as $3b$ for some integer b (namely, $b = 4a + 1$), it is by definition divisible by 3.

Therefore, by induction, $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$. □

RP02. Let x be a real number. Find the coefficient of x^{19} in the expansion of $(2x^3 - 3x)^9$.

Solution. Recall the Binomial Theorem, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Let $a = 2x^3$, $b = -3x$, and $n = 9$. Then, we have $(2x^3 - 3x)^9 = \sum_{k=0}^{9} \binom{9}{k} 2^{9-k} (-3)^k x^{27-2k}$. We only care about when the exponent on x is 19, i.e., $27 - 2k = 19 \implies k = 4$. On this term of the summation, we have $\binom{9}{4} 2^5 (-3)^4 x^{19}$.

The coefficient is $\binom{9}{4} 2^5 (-3)^4 = 126 \cdot 32 \cdot 81 = 326592$. □

RP03. Let n be a non-negative integer. Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof. We will induct the statement $P(n) \equiv \sum_{k=0}^{n} \binom{n}{k} = 2^n$ on $n \geq 0$.

For the base case, $P(0)$, we have

$$\sum_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0.$$

Now, suppose $P(m)$ is true for some $m \geq 0$. Consider the summation in $P(m+1)$:

$$\begin{aligned}
\sum_{k=0}^{m+1} \binom{m+1}{k} &= \binom{m+1}{m+1} + \sum_{k=0}^{m} \binom{m+1}{k} \\
&= \binom{m+1}{m+1} + \sum_{k=0}^{m} \left(\binom{m}{k} + \binom{m}{k-1} \right) && \text{by Pascal's identity} \\
&= \binom{m+1}{m+1} + \sum_{k=0}^{m} \binom{m}{k} + \sum_{k=0}^{m} \binom{m}{k-1} \\
&= 1 + 2^k + \sum_{k=0}^{m} \binom{m}{k-1} && \text{by inductive hypothesis}
\end{aligned}$$

Recall that negative binomial coefficients are undefined, so we can change the variable in the summation with $j = k+1$ and ignore the $k = 0$ term. Add and subtract a $\binom{m}{m}$ term to round out the summation and apply the IH once more:

$$\begin{aligned}
\sum_{k=0}^{m+1} \binom{m+1}{k} &= 1 + 2^k + \sum_{j=0}^{m-1} \binom{m}{j} \\
&= 1 + 2^k + \sum_{j=0}^{m-1} \binom{m}{j} + \binom{m}{m} - \binom{m}{m} \\
&= 1 + 2^k + \sum_{j=0}^{m} \binom{m}{j} - 1 \\
&= 1 + 2^k + 2^k - 1 && \text{by inductive hypothesis} \\
&= 2^{k+1}
\end{aligned}$$

which is what we wanted to show that $P(m+1)$ is true.

Therefore, by induction, $P(n)$ is true for all non-negative integer n. □

RP04. Let n be a non-negative integer. Prove by induction on k that $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$ for all non-negative integers k.

Proof. Let $n \geq 0$ be an integer, and let $P(k)$ be the statement $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$. We will induct $P(k)$ on k.

For the base case, let $k = 0$. Then, $P(k)$ reads $\sum_{j=0}^{0} \binom{n+j}{j} = \binom{n+1}{0}$. The summation only has one term, so we have $\binom{n}{0} = \binom{n+1}{0}$ which is true for all n (since $\binom{a}{0} = 1$ for all a).

Now, suppose that $P(s)$ holds for some non-negative integer s.

This means that $\sum_{j=0}^{s} \binom{n+j}{j} = \binom{n+s+1}{s}$. Now, consider the left-hand side of $P(s+1)$:

$$\begin{aligned}
\sum_{j=0}^{s+1} \binom{n+j}{j} &= \binom{n+s+1}{s+1} + \sum_{j=0}^{s} \binom{n+j}{j} \\
&= \binom{n+s+1}{s+1} + \binom{n+s+1}{s} && \text{by inductive hypothesis} \\
&= \binom{n+s+2}{s+1} && \text{by Pascal's identity}
\end{aligned}$$

which is exactly the right-hand side. Since $P(n)$ is true for $n = 0$ and $P(s)$ implies $P(s+1)$, it holds for all $n \geq 0$ by induction. $\square$

RP05. The sequence $x_1, x_2, x_3, \ldots$ is defined recursively by $x_1 = 8$, $x_2 = 32$, and $x_i = 2x_{i-1} + 3x_{i-2}$ for all integers $i \geq 3$. Prove that for all $n \in \mathbb{N}$, $x_n = 2 \times (-1)^n + 10 \times 3^{n-1}$.

Proof. We will strongly induct the statement $P(n)$, $x_n = 2(-1)^n + 10(3)^{n-1}$, on n.

For a base case, let $n = 1$. Then, $2(-1)^1 + 10(3)^0 = -2 + 10 = 8$, which is the defined value of x_1. For another, let $n = 2$. Then, $2(-1)^2 + 10(3)^1 = 2 + 30 = 32$, which is the defined value of x_2. Therefore, $P(1)$ and $P(2)$ hold.

Now, for some $m \geq 3$, suppose $P(n)$ holds for all $n < m$. Specifically, $P(m-1)$ and $P(m-2)$ hold.

Consider the definition of x_m:

$$\begin{aligned}
x_m &= 2x_{m-1} + 3x_{m-2} \\
&= 2\left(2(-1)^{m-1} + 10(3)^{m-2}\right) + 3\left(2(-1)^{m-2} + 10(3)^{m-3}\right) \\
&= 4(-1)^{m-1} + 20(3)^{m-2} + 6(-1)^{m-2} + 30(3)^{m-3} \\
&= 4(-1)(-1)^{m-2} + 6(-1)^{m-2} + 20(3)(3)^{m-3} + 30(3)^{m-3} \\
&= 2(-1)^{m-2} + 90(3)^{m-3} \\
&= 2(-1)^2(-1)^{m-2} + 10(3)^2(3)^{m-3} \\
&= 2(-1)^m + 10(3)^{m-1}
\end{aligned}$$

which is precisely $P(m)$.

Therefore, by strong induction, $P(n)$ is true for all n. $\square$

RP06. The sequence $t_1, t_2, t_3, \ldots$ is defined recursively by $t_1 = 2$ and $t_n = 2t_{n-1} + n$ for all integers $n > 1$. Prove that for all $n \in \mathbb{N}$, $t_n = 5 \times 2^{n-1} - 2 - n$.

Proof. Let $P(n)$ be the statement $t_n = 5 \times 2^{n-1} - 2 - n$. We will induct $P(n)$ on n.

We first verify base cases: $n = 1$, hypothesized as $t_1 = 5(2)^0 - 2 - 1 = 2$, which matches the defined value; and $n = 2$, for which t_2 is defined as $2(2) + 2 = 6$ and we hypothesize $t_2 = 5(2)^1 - 2 - 2 = 6$.

Now, let m be an integer above 2 and suppose that $P(m-1)$ holds. Consider the definition of t_m:

$$\begin{aligned}
t_m &= 2t_{m-1} + m \\
&= 2\left(5(2)^{m-2} - 2 - (m-1)\right) + m && \text{by inductive hypothesis} \\
&= 2\left(5(2)^{m-2} - m - 1\right) + m \\
&= 5(2)^{m-1} - 2m - 2 + m \\
&= 5(2)^{m-1} - 2 - m
\end{aligned}$$

This is exactly $P(m)$, so $P(m-1)$ implies $P(m)$.

Therefore, by induction, $P(n)$ is true for all natural n. $\square$

RP07. The Fibonacci sequence is defined as the sequence $f_1, f_2, f_3, \ldots$ where $f_1 = 1$, $f_2 = 1$ and $f_i = f_{i-1} + f_{i-2}$ for $i \geq 3$. Use induction to prove the following statements:

(a) For $n \geq 2$, $f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1$.

Proof. We will induct the statement $P(n)$, $\sum_{i=1}^{n-1} f_i = f_{n+1} - 1$ on n.

To verify the base case, $n = 2$, substitute and notice $f_1 = 1 = 2 - 1 = f_3 - 1$.

Now, let $m > 2$ and suppose $P(m)$ holds. Then,

$$\begin{aligned} \sum_{i=1}^{m-1} f_i &= f_{m+1} - 1 \\ f_m + \sum_{i=1}^{m-1} f_i &= f_m + f_{m+1} - 1 \\ \sum_{i=1}^{m} f_i &= f_{m+2} - 1 \end{aligned}$$

which is $P(m + 1)$.

Therefore, by induction, $P(n)$ is true for all $n \geq 2$. □

(b) Let $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$. For all $n \in \mathbb{N}$, $f_n = \dfrac{a^n - b^n}{\sqrt{5}}$.

Proof. Let $P(n)$ be the statement $f_n = \frac{a^n - b^n}{\sqrt{5}}$. We will strongly induct $P(n)$ on n.

For the base cases, start with $n = 1$. f_1 is defined to be 1, and $\frac{a-b}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$. Likewise, for $n = 2$, f_2 is defined as 1, and $\frac{a^2-b^2}{\sqrt{5}} = \frac{a-b}{\sqrt{5}}(a + b) = (1)(1) = 1$.

For our inductive step, first notice that a and b are the roots of $x^2 - x - 1 = 0$. Let x be either root.

Notice that for any $n \geq 2$, we have

$$\begin{aligned} 0 &= x^2 - x - 1 \\ 0 &= x^{n-2}(x^2 - x - 1) \\ 0 &= x^n - x^{n-1} - x^{n-2} \\ x^n &= x^{n-1} + x^{n-2} \end{aligned}$$

Therefore, $a^n = a^{n-1} + a^{n-2}$ and $b^n = b^{n-1} + b^{n-2}$ for any $n \geq 2$.

Now, let $m \geq 2$ and suppose $P(m - 1)$ and $P(m - 2)$ hold. Then, f_m is defined by:

$$\begin{aligned} f_m &= f_{m-1} + f_{m-2} \\ &= \frac{a^{m-1} - b^{m-1}}{\sqrt{5}} + \frac{a^{m-2} - b^{m-2}}{\sqrt{5}} \\ &= \frac{(a^{m-1} + a^{m-2}) - (b^{m-1} + b^{m-2})}{\sqrt{5}} \\ &= \frac{a^m - b^m}{\sqrt{5}} \end{aligned}$$

Therefore, by strong induction, $P(n)$ holds for all n. □

RP08. Each of the following "proofs" by induction incorrectly "proved" a statement that is actually false. State what is wrong with each proof.

(a) The proof does not consider the given definition $x_2 = 20$, and $3(5)^1 = 15 \neq 20$. Note that the recursive definition *only* applies to x_i for $i \geq 3$.

(b) The proof erroneously assumes that $n = 2$ always falls within the inductive hypothesis. However, when proving the case $n = 2$ with strong induction, the only given is $n = 1$.

RP09. In a strange country, there are only 4 cent and 7 cent coins. Prove that any integer amount of currency greater than 17 cents can always be formed.

Proof. Let $P(x)$ be the statement that there exist non-negative integer a and b where $x = 4a + 7b$. We will strongly induct on $x > 17$.

Verify a few base cases:
For $P(18)$ (where $18 = 4(4) + 2$), let $a = 1$ and $b = 2$, so $4(1) + 7(2) = 18$.
For $P(19)$ (where $19 = 4(4) + 3$), let $a = 3$ and $b = 1$, so $4(3) + 7(1) = 19$.
For $P(20)$ (where $20 = 4(5) + 0$), let $a = 5$ and $b = 0$, so $4(5) + 7(0) = 20$.
For $P(21)$ (where $21 = 4(5) + 1$), let $a = 0$ and $b = 3$, so $4(0) + 7(3) = 21$.

Now, suppose for some $n > 21$, $P(m)$ holds for all $m < n$. Specifically, $P(n-4)$ holds. That is, $n - 4 = 4a_0 + 7b_0$ for some a_0 and b_0. Equivalently, $n = 4(a_0 + 1) + 7b_0$. If we let $a = a_0 + 1$ and $b = b_0$, it follows that $P(n)$ holds.

Therefore, by strong induction, $P(x)$ is true for all $x > 17$. □

Challenges

C01. Prove that for every positive integer, there exists a unique way to write the integer as the sum of distinct non-consecutive Fibonacci numbers.

Proof. Let f_i denote the ith Fibonacci number, i.e., $f_1 = 0$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$. Note that we proceed without loss of generality with increasing lists of Fibonacci numbers.

We begin by proving inductively that $f_n > f_{k_1} + \cdots + f_{k_m}$ where $k_1 < \cdots < k_m < n$ and $k_1 + 1 \neq k_2$, $k_2 + 1 \neq k_3$, etc. That is, the k_i are increasing, and non-consecutive. For the cases $n = 0$ and $n = 1$, no such sums can exist. When $n = 2$, the only such sum is f_0, and $0 < f_2 = 1$.

Suppose that $n \geq 2$ and $f_{n-2} > f_{r_1} + \cdots + f_{r_s}$ with the r_i increasing and non-consecutive. Then, since $k_m < n$, $k_m \leq n - 1$ and we have

$$\begin{aligned} f_{k_1} + \cdots + f_{k_{m-1}} + f_{k_m} &\leq f_{k_1} + \cdots + f_{k_{m-1}} + f_{n-1} \\ &< f_{n-2} + f_{n-1} && \text{by inductive hypothesis} \\ &= f_n \end{aligned} \tag{1}$$

Now, let $P(n)$ be the statement that all positive integers $x < f_n$, $x = \sum_{i=1}^{m} f_{k_i}$ for unique, increasing, non-consecutive k_i.

For the base cases $P(1)$, $P(2)$, and $P(3)$ there are no positive integers $x < 0$ or $x < 1$. For the base case $P(4)$, the only positive integer less than $f_4 = 2$ is $x = 1$. Trivially, we can uniquely write $f_1 + f_3 = 0 + 1 = 1$.

For the inductive step, suppose that $P(n)$ holds for some $n \geq 4$. Let $f_n \leq x < f_{n+1}$.

If x is f_n, then we may write $x = f_1 + f_n$. That is, $x = \sum_{i=1}^{2} f_{k_i}$ with increasing, non-consecutive $k_1 = 1$ and $k_2 = n$.

Otherwise, write $x = d + f_n$ where $0 < d < f_{n+1} - f_n = f_{n-1}$. We now have, $d < f_{n-1} < f_n$ with positive integer d. By the inductive hypothesis, $d = \sum_{i=1}^{m} f_{k_i}$ for unique, increasing, non-consecutive k_i. Then, since $d < f_{n-1} < f_n$, none of the k_is can be n or $n-1$. Finally, let $k_{m+1} = n$ so that $x = \sum_{i=1}^{m+1} f_{k_i}$ has increasing, non-consecutive f_{k_i}.

Now, we show that the integers k_i are unique. Suppose $x = \sum_{i=1}^{m+1} f_{k_i} = \sum_{i=1}^{m+1} f_{\ell_i}$. We show that $k_i = \ell_i$ for all i.

Since both sums are increasing, the largest k_{m+1} is n. If $f_{\ell_{m+1}} > f_n$, then the sum is greater than f_{n+1}. But $x < f_{n+1}$, so this is a contradiction. If $f_{\ell_{m+1}} < f_n$, then by eq. (1), the sum is less than f_n. But $x \geq f_n$, so this is again a contradiction. Thus, $\ell_{m+1} = n = k_{m+1}$.

Then, $\sum_{i=1}^{m} f_{\ell_i} = x - f_n = d$. However, the inductive hypothesis gives that $\sum_{i=1}^{m} k_i$ is a unique representation of d. It follows that the remaining $\ell_i = k_i$ for all $i \leq m$.

Therefore, since we have proven $P(n+1)$, by induction, $P(n)$ holds for all n. □

C02. Find a formula for the minimum steps required to solve the Tower of Hanoi puzzle with three pegs with n rings. Prove that your answer is correct.