Q01. Determine whether each integral is convergent or divergent. If it is convergent, evaluate it. If divergent, justify why it is divergent.

(a) $\int_{-\infty}^{\infty}(x^3 - 3x^2)\,\mathrm{d}x$

Solution. Notice that we can distribute $\int_{-\infty}^{\infty}(x^3-3x^2)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^3\,\mathrm{d}x - 3\int_{-\infty}^{\infty} x^2\,\mathrm{d}x$. Since x^3 is odd, the term goes to zero. By the p-test, the x^2 term diverges. Therefore, the integral diverges. □

(b) $\int_0^4 \frac{1}{x^2-x-2}\,\mathrm{d}x$

Solution. Notice that $x^2 - x - 2 = (x-2)(x+1)$ so there is an asymptote at $x = 2$. We must find $\int_0^2 \frac{1}{x^2-x-2}\,\mathrm{d}x + \int_2^4 \frac{1}{x^2-x-2}\,\mathrm{d}x$. By partial fractions:

$$\int_0^2 \frac{1/3}{x-2} - \frac{1/3}{x+1}\,\mathrm{d}x = \lim_{t\to 2^-}\left[\frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1|\right]_0^t = -\infty$$

so the integral diverges. □

(c) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,\mathrm{d}x$

Solution. Let $u = \sin x$.

Then, $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}u}{\sqrt{u}} = \lim_{t\to 0^+}[2\sqrt{u}]_t^1$, which converges to 2. □

(d) $\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,\mathrm{d}x$

Solution. After substituting, we have $\int_5^0 \frac{\mathrm{d}x}{\sqrt[3]{x}} = \lim_{t\to 0^+}[\frac{3}{2}x^{2/3}]_5^t$, converging to $\frac{3\sqrt[3]{25}}{2}$. □

(e) $\int_1^{\infty} \frac{e^{1/x}}{x^2}\,\mathrm{d}x$

Solution. Notice that if we let $u = e^{1/x}$, then $\mathrm{d}u = -\frac{e^{1/x}}{x^2}$.

So we have $\lim_{t\to\infty}\int_1^t -\mathrm{d}u = \lim_{t\to\infty}\left[-e^{1/x}\right]_1^t = -e^0 + e^1 = e - 1$ which converges. □

(f) $\int_1^{\infty} \frac{\ln x}{x^2}\,\mathrm{d}x$

Solution. Integrate by parts:

$$\int \frac{\ln x}{x^2}\,\mathrm{d}x = -\frac{\ln x}{x} - \int \frac{-\mathrm{d}x}{x^2} = -\frac{1+\ln x}{x} + C$$

Then, $\lim_{t\to\infty} -\frac{1+\ln x}{x}\Big|_1^t = 1$ converges by the Fundamental Log Limit. □

(g) $\int_0^1 \frac{e^{1/x}}{x^3}\,\mathrm{d}x$

Solution. Use the same substitution as (e). Then, integrating by parts:

$$\int \frac{e^{1/x}}{x^3}\,\mathrm{d}x = -\frac{e^{1/x}}{x} - \int \frac{e^{1/x}}{x^2}\,\mathrm{d}x = -\frac{e^{1/x}}{x} + e^{1/x}$$

And the limit $\lim_{t\to 0^+}\left[-\frac{e^{1/x}}{x} + e^{1/x}\right]_t^1 = \lim_{t\to 0^+}\left[0 + \frac{e^{1/t}}{t} - e^{1/t}\right] = \infty$ diverges. □

Q02. Use the Comparison Theorem to determine whether each integral is convergent or divergent.

(a) $\int_1^\infty \frac{2+e^{-x}}{x}\,\mathrm{d}x$

Proof. By the p-test, $\int_1^\infty \frac{\mathrm{d}x}{x}$ diverges. But $1+e^{-x}$ is positive, so $\frac{2+e^{-x}}{x} > \frac{1}{x} > 0$. Therefore, by the Comparison Theorem, $\int_1^\infty \frac{2+e^{-x}}{x}\,\mathrm{d}x$ diverges. □

(b) $\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,\mathrm{d}x$

Proof. By the p-test, $\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}}$ diverges. For $x > 1$, $\frac{\sin^2 x}{\sqrt{x}} > 0$, so we have $\frac{1+\sin^2 x}{\sqrt{x}} > \frac{1}{\sqrt{x}} > 0$. By the Comparison Theorem, $\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,\mathrm{d}x$ must diverge. □

Q03. Consider the following integrals:

(a) Prove that $\int_e^\infty \frac{\cos x^2}{x^2 \ln x}\,\mathrm{d}x$ is convergent.

Proof. We apply the Absolute Convergence Theorem. Notice that for all $x \geq e$, we have $\ln x > 0$ and

$$0 \leq \left|\frac{\cos x^2}{x^2 \ln x}\right| \leq \frac{1}{x^2 \ln x} \leq \frac{1}{x^2}$$

which, by the p-test, is convergent.

Therefore, by the Absolute Convergence Theorem, the integral converges. □

(b) Prove that $\int_1^\infty \frac{\sin x}{x}\,\mathrm{d}x$ is convergent.

Q04. Prove that if $f(x)$ is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x) = \alpha > 0$ (or $\alpha = \infty$), then $\int_0^\infty f(x)\,\mathrm{d}x$ diverges.

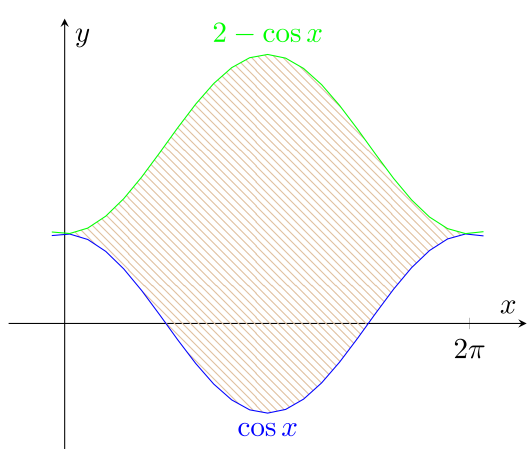
Proof. First, consider when α is finite. By the definition of the infinite limit, given $\frac{\alpha}{2}$, there is a $M > 0$ such that when $x \geq M$, $|f(x) - \alpha| < \frac{\alpha}{2}$. Since α is positive, this implies $f(x) > \frac{\alpha}{2}$. Now, $\int_M^\infty \frac{\alpha}{2}\,\mathrm{d}x = \infty$. By the Comparision Theorem, the integral diverges.

If $\alpha = \infty$, there exists a cutoff $N > 0$ such that when $x > N$, $f(x) > 1$. The same logic applies, and the integral must diverge. □

Q05. Sketch the region enclosed by the given curves and find the area.

(a) $y = \cos x$, $y = 2 - \cos x$, $0 \leq x \leq 2\pi$

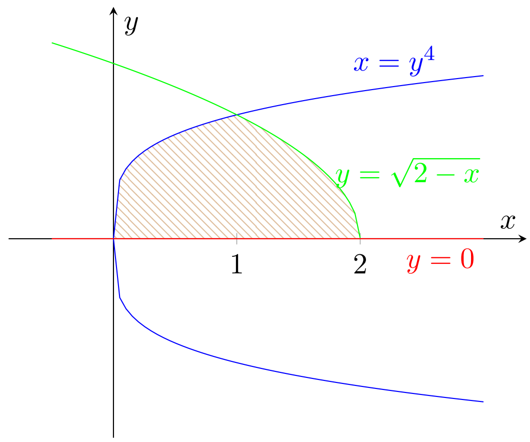
Solution. Doodle with `pgfplots`.



We simply evaluate the integral $\int_0^{2\pi} (2-\cos x) - \cos x\,\mathrm{d}x$. This is $-2\int_0^{2\pi} 1 - \cos x\,\mathrm{d}x = 2[x - \sin x]_0^{2\pi} = 4\pi$. □

(b) $x = y^4$, $y = \sqrt{2-x}$, $y = 0$

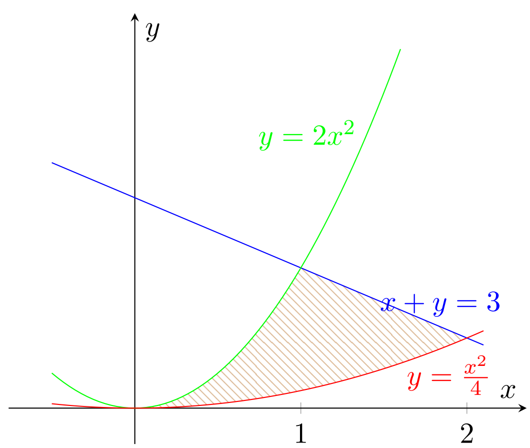
Solution. Doodle, noticing that the POI is $\sqrt[4]{x} = \sqrt{2-x} \iff x = 1$.



The two areas are $\int_0^1 \sqrt[4]{x}\,\mathrm{d}x$ and $\int_1^2 \sqrt{2-x}\,\mathrm{d}x$. The first is $[\frac{4}{5}x^{5/4}]_0^1 = \frac{4}{5}$ and the second is $\int_0^1 \sqrt{x}\,\mathrm{d}x = [\frac{2}{3}x^{3/2}]_0^1 = \frac{2}{3}$, so the total area is $\frac{22}{15}$. □

(c) $y = \frac{x^2}{4}$, $y = 2x^2$, $x + y = 3$, $x \geq 0$

Solution. Doodle, noticing that the POI again at $x = 1$.



Now, we have $\int_0^1 2x^2 - \frac{x^2}{4}\,\mathrm{d}x = [\frac{2}{3}x^3 - \frac{x^3}{12}]_0^1 = \frac{7}{12}$ for the area between 0 and 1, and $\int_1^2 3 - x - \frac{x^2}{4}\,\mathrm{d}x = [3x - \frac{x^2}{2} - \frac{x^3}{12}]_1^2 = \frac{10}{3} - \frac{29}{12} = \frac{11}{12}$ for the remainder.
The sum is $\frac{2}{3}$. □