PMATH 370 Winter 2024:
Lecture Notes

Lecture notes taken, unless otherwise specified, by myself during the Winter 2024 offering of PMATH 370, taught by Blake Madill.

Lectures

Chapter 1

Iteration and Orbits

1.1 Orbits

Definition 1.1.1 (iteration)
Let $f : A \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$. For $a \in A$ we may iterate the function at a:

$$x_1 = a, x_2 = f(a), x_3 = \underbrace{f(f(a))}_{f^2(a)}, \dots, x_i = f^{i-1}(a), \dots .$$

The sequence $(x_n)_{n=1}^{\infty}$ is the orbit of a under f (abbreviated (x_n) without limits).

Lecture 1
Jan 8

Example 1.1.2. Let $f(x) = x^4 + 2x^2 - 2$, $a = -1$. What is the orbit of a under f?

Solution. $a = -1$, $f(a) = 1$, $f(f(a)) = f(1) = 1$, so we have $-1, 1, 1, 1, \dots$. We call this eventually constant. ☐

Example 1.1.3. Let $f(x) = -x^2 - x + 1$, $a = 0$. What is the orbit of a under f?

Solution. Calculate: $0, 1, -1, 1, -1, 1, \dots$. We call this eventually periodic (with period 2). ☐

Example 1.1.4. Let $f(x) = x^3 - 3x + 1$, $a = 1$. What is the orbit of a under f?

Solution. Calculate the first few terms: $1, -1, 3, 19, \dots$ (too big). This is a divergence to infinity. ☐

Example 1.1.5. Let $f(x) = x^2 + 2x$, $a = -0.5$. What is the orbit of a under f?

Solution. Calculate: $-0.5, -0.75, -0.9375, -0.9961\dots$ and we make an educated guess that this converges to -1 since $f(-1) = -1$, a fixed point. ☐

Example 1.1.6. Let $f(x) = x^3 - 3x$, $a = 0.75$. What is the orbit of a under f?

Solution. Calculate: $0.75, -1.828, -0.625, 1.631, -0.552, \dots$. There is no clear pattern, so we call this chaotic. In fact, the orbit is dense in a neighbourhood of 0. □

We can start to formalize the examples.

Definition 1.1.7 (fixed point)
Let $f : A \to \mathbb{R}$ such that $f(A) \subseteq A$. A point $a \in A$ is fixed if $f(a) = a$.

Then, the orbit of a under f is $(a, a, a, \dots)$ which is constant.

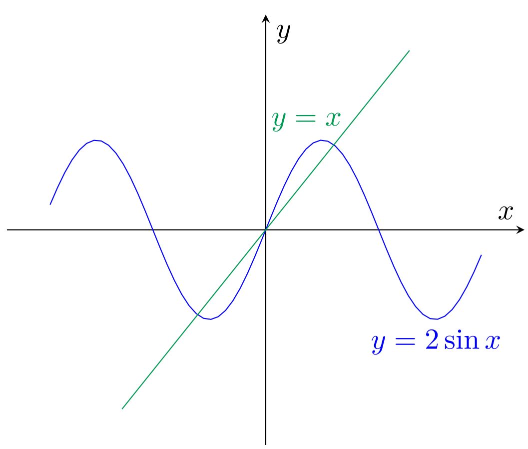
Example 1.1.8. Find all fixed points of $f(x) = x^2 + x - 4$.

Solution. We find points where $f(x) = x$, i.e., $x^2 + x - 4 = x$.

$$x^2 + x - 4 = x \iff x^2 = 4 \iff x = \pm 2$$ □

Example 1.1.9. How many fixed points does $f(x) = 2\sin x$ have?

Solution. Consider where the curve $y = 2\sin x$ meets $y = x$:



We can see there are three fixed points. □

Example 1.1.10. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Proof. We must show there is a solution to $x^4 - 3x + 1 \iff x^4 - 4x + 1 = 0$. Let $g(x) = x^4 - 4x + 1$. Since $g(x)$ is continuous, $g(0) = 1 > 0$, and $g(1) = -2 < 0$, by the Intermediate Value Theorem, there must exist a root of g on the interval $(0, 1)$. That is, a fixed point of f. □

Definition 1.1.11 (periodicity)

Let $f : A \to \mathbb{R}, f(A) \subseteq A$.

1. A point $a \in A$ is periodic for f if its orbit is periodic. An orbit is periodic if for some $n \in \mathbb{N}$, $f^n(a) = a$. The smallest n is the period of (the orbit of) a.
2. An orbit (of a point) is eventually periodic if there exists $n < m$ such that $f^n(a) = f^m(a)$. The smallest difference $m - n$ is the period of the orbit.

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Jan 10

Definition 1.1.12 (doubling function)

$D : [0, 1) \to [0, 1) : x \mapsto 2x - \lfloor 2x \rfloor$ returns the fractional part of $2x$.

Example 1.1.13. $D(0.4) = 0.8,\ D(0.6) = 0.2,\ D(0.8) = 0.6,\ D(0.5) = 0.$

This is a nice function that gives lots of periodic orbits for funsies.

Example 1.1.14. Find the orbit of $a = \frac{1}{5}$ under D.

Solution. Double until we pass 1: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{8}{5} \to \frac{3}{5}, \frac{6}{5} \to \frac{1}{5}$. The period is $\left|\{\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\}\right| = 4$. ☐

Example 1.1.15. Find the orbit of $a = \frac{1}{20}$ under D.

Solution. Double: $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}$ and we can stop because Example 1.1.14 showed $\frac{1}{5}$ is periodic.

So this is eventually periodic with period 4. ☐

Problem 1.1.16

Given f and a, does $f^n(a)$ tend towards some limit L?

To solve this problem, we need to rigorously define "tend" and "limit".

1.2 Real analysis review

Notation. If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers, we write $(x_n) \subseteq \mathbb{R}$.

Definition 1.2.1 (convergence of a sequence)
Let $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$.

We say (x_n) converges to x if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

Then, we write $x_n \to x$ or $\lim x_n = x$.

Example 1.2.2. Show that $\frac{1}{n} \to 0$.

Proof. Let $\varepsilon > 0$. Consider $N = \frac{2}{\varepsilon} > \frac{1}{\varepsilon}$. For $n \geq N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

Therefore, $\frac{1}{n} \to 0$. □

Example 1.2.3. Prove that $\frac{2n}{n+3} \to 2$.

Proof. Let $\varepsilon > 0$. Since we know $\frac{1}{n} \to 0$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{6}$.

For $n \geq N$,

$$\left|\frac{2n}{n+3} - 2\right| = \left|\frac{2n}{n+3} - \frac{2n+6}{n+3}\right| = \left|\frac{-6}{n+3}\right| = \frac{6}{n+3} < \frac{6}{n} \leq \frac{6}{N} < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

Therefore, $\frac{2n}{n+3} \to 2$. □

Definition 1.2.4 (bounded sequence)
A sequence (x_n) is bounded (by M) if there exists $M > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq M$.

Proposition 1.2.5 (convergence implies boundedness)
If (x_n) is convergent, then (x_n) is bounded.

Proof. Suppose $x_n \to x$. Then, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - x| < 1$.

For $n \geq N$, $|x_n| - |x| \leq |x_n - x| < 1$. That is, $|x_n| < 1 + |x|$.

Let $M = \max\{|x_1|, \dots, |x_{n-1}|, 1 + |x|\}$. Then, for both all $n < N$ and $n \geq N$, we have $|x_n| \leq M$. □

Remark 1.2.6. The converse is not true. Notice that $x_n = (-1)^n$ is bounded by 1 but obviously not convergent.

Proposition 1.2.7 (limit laws)

Let $x_n \to x$ and $y_n \to y$. Then:

(1) $x_n + y_n \to x + y$

(2) $x_n y_n \to xy$

Proof. (1) Let $\varepsilon > 0$. Then, since $x_n \to x$ and $y_n \to y$, there exist $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies |x_n - x| < \frac{\varepsilon}{2}$ and $n \geq N_2 \implies |y_n - y| < \frac{\varepsilon}{2}$.

For $N = \max\{N_1, N_2\}$ and $n \geq N$,

$$\begin{aligned} |(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

That is, $x_n + y_n \to x + y$.

(2) Let $\varepsilon > 0$. Notice that:

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \tag{$*$}$$

Since x_n is bounded, there exists $M > 0$ such that $|x_n| \leq M$ for all n.

Let $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 &\implies |x_n - x| \leq \frac{\varepsilon}{2(|y| + 1)} \text{ and} \\ n \geq N_2 &\implies |y_n - y| < \frac{\varepsilon}{2M}. \end{aligned}$$

Then, for $n \geq N := \max\{N_1, N_2\}$, $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by $(*)$. $\square$

Lecture 3
Jan 12

Definition 1.2.8 (Cauchy sequence)

We say $(x_n) \in \mathbb{R}$ is <u>Cauchy</u> if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all n and m,

$$n, m \geq N \implies |x_n - x_m| < \varepsilon$$

Proposition 1.2.9

Every convergent sequence is Cauchy.

Proof. Intuitively: if the terms get arbitrarily close to some limit, they must get arbitrarily close to each other.

Formally: Let $x_n \to x$ be a convergent sequence and $\varepsilon > 0$. Since x_n converges, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |x_n - x| < \frac{\varepsilon}{2}$.

Then, when $n, m \geq N$, we have:

$$\begin{aligned} |x_n - x_m| &= |x_n - x_m + x - x| \\ &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

as desired. □

We take the following theorem from real analysis without proof.

Theorem 1.2.10 (completeness of $\mathbb{R}$)

A sequence is Cauchy if and only if it is convergent.

The big idea here: To prove (x_n) is Cauchy, you do not have to guess the limit first. That is, if you must prove convergence but do not care about the limit's value, prove that it is Cauchy instead.

Definition 1.2.11 (continuity of a function)

Let $f : A \to \mathbb{R}, A \subseteq \mathbb{R}, a \in A$. We say f is continuous at a if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in A$ and $|x - a| < \delta$.

If f is continuous at all $a \in A$, we say it is continuous.

We also take this theorem from MATH 137 without proof.

Theorem 1.2.12

A function $f : A \to \mathbb{R}$ is continuous at $a \in A$ if and only if for all sequences $(x_n) \subseteq A$ with $x_n \to a$, we have $f(x_n) \to a$.

1.3 Orbits, revisited

Proposition 1.3.1

If $f : [a, b] \to [a, b]$ is continuous, then $f(x)$ has a fixed point.

Proof. We know by the domain and codomain that $f(a) \geq a$ and $f(b) \leq b$. This means $f(a) - a \geq 0$ and $f(b) - b \leq 0$. By the IVT on the continuous function $g(x) = f(x) - x$, we know there exists an $x \in [a, b]$ such that $g(x) = f(x) - x = 0 \iff f(x) = x$, i.e., x is a fixed point. □

Definition 1.3.2 (contraction)

Let $f : A \to \mathbb{R}, A \subseteq \mathbb{R}$. We say f is a contraction if there exists $C \in [0, 1)$ such that for all $x, a \in A$,

$$|f(x) - f(y)| \leq C|x - y|$$

This is just a Lipschitz function with Lipschitz constant less than 1.

Proposition 1.3.3

Contractions are continuous.

Proof. Let $\varepsilon > 0$. Suppose f is a contraction such that $|f(x) - f(y)| \leq C|x - y|$.

Consider $y \in A$. Let $\delta = \frac{\varepsilon}{C+1}$ and assume that $x \in A$ and $|x - y| < \delta$. But we have:

$$|f(x) - f(y)| \leq C|x - y| \leq C\delta < \varepsilon$$

as desired. □

Definition 1.3.4 (closure of an interval)

We say $A \in \mathbb{R}$ is closed if whenever $(x_n) \subseteq A$ with $x_n \to x$, then $x \in A$.

Example 1.3.5. $[a, b]$ is closed but $(0, 1]$ is not because $\frac{1}{n} \to 0 \notin (0, 1]$.

Theorem 1.3.6 (Banach contraction mapping theorem)

Suppose $A \subseteq \mathbb{R}$ is closed and $f : A \to A$ is a contraction. Then, there exists a unique fixed point $a \in A$ for f.

Moreover, for all $x \in A$, $f^n(x) \to a$.

Example 1.3.7. Analyze the orbit of $f : [0, 1] \to [0, 1]$, $f(x) = \frac{1}{3-x}$.

Solution. We can observe that $\frac{1}{3} \leq \frac{1}{3-x} \leq \frac{1}{2}$.

Also, $f'(x) = \frac{1}{(3-x)^2}$. Notice that $\frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$. So by the mean value theorem, for all $x, y \in [0, 1]$, there exists $c \in (0, 1)$ such that:

$$\begin{aligned} f(x) - f(y) &= f'(c)(x - y) \\ |f(x) - f(y)| &= |f'(c)| \cdot |x - y| \\ &\leq \frac{1}{4}|x - y| \end{aligned}$$

Then, identifying $C = \frac{1}{4}$, f is a contraction. Now,

$$\frac{1}{3-x} = x \iff 1 = 3x - x^2 \iff x^2 - 3x + 1 = 0 \iff x = \frac{3 \pm \sqrt{9-4}}{2} \iff x = \frac{3-\sqrt{5}}{2}$$

where we pick the negative root because we need $x \in [0,1]$.

Therefore, by the Banach contraction mapping theorem, for all $x \in [0,1]$, $f^n(x) \to \frac{3-\sqrt{5}}{2}$. $\square$

Lecture 4
Jan 15

Definition 1.3.8
A sequence $(a_n) \subseteq \mathbb{R}$ is strongly-Cauchy if there exists $(\varepsilon_n) \subseteq [0,\infty)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and for all n, $|a_n - a_{n+1}| < \varepsilon_n$.

Informally, "far enough along the sequence, the *neighbours* must get close". This is distinct from Cauchy, which is "far enough along the sequence, the *terms* must get close".

Remark 1.3.9 (assignment hint!). Let $\sum_{n=1}^{\infty} a_n = L$. This means that $\sum_{k=1}^{n} a_k \xrightarrow{n\to\infty} L$.

That is, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\sum_{k=1}^{n} a_k - L\right| < \varepsilon$.

But $\left|\sum_{k=1}^{n} a_k - L\right| = \left|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{\infty} a_k\right| < \varepsilon$.

We can now prove the Banach contraction mapping theorem.

Proof. Let $A \subseteq \mathbb{R}$ be closed and suppose there exists $f : A \to A$ and $C \in [0,1)$ such that $|f(x) - f(y)| \leq C|x-y|$ for all x and y in A.

Fix $x_0 \in A$ and construct the orbit $x_1 = f(x_0)$, $x_2 = f(x_1)$, ..., $x_n = f(x_{n-1}) = f^n(x_0)$.

For $n \in \mathbb{N}$, since f is a contraction,

$$\begin{aligned}
|x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \\
&\leq C|x_n - x_{n-1}| \\
&= C|f(x_{n-1}) - f(x_{n-2})| \\
&\leq C^2|x_{n-1} - x_{n-2}| \\
&\vdots \\
&\leq C^n|x_1 - x_0|
\end{aligned}$$

Since $\sum_{n=1}^{\infty} C^n|x_1 - x_0| = |x_1 - x_0| \sum_{n=1}^{\infty} C^n$ is a convergent geometric series, we have that the sequence (x_n) is strongly-Cauchy.

Hence, by Assignment 1, $x_n \to a$ for some limit point $a \in A$ since A is closed.

Since f is continuous (Proposition 1.3.3), we have that $f(x_n) \to f(a)$. By definition, $f(x_n) = x_{n+1}$, so $x_n \to f(a)$. But we already know $x_n \to a$, so $a = f(a)$. That is, a is a fixed point of f.

It remains to show uniqueness.

Suppose $a, b \in A$ such that $f(a) = a$ and $f(b) = b$.

$$\begin{aligned} |f(a) - f(b)| &\leq C|a - b| \\ |a - b| &\leq C|a - b| \end{aligned}$$

Since $C < 1$, we must have $|a - b| = 0$, that is, $a = b$. $\square$

Chapter 2

Graphical Analysis

2.1 Cobweb plots

Recall Example 1.1.9. To visualize the orbit of a under f, we can:

1. Superimpose $y = f(x)$ over the line $y = x$.
2. Connect a vertical line $(a, a) - (a, f(a))$
3. Connect a horizontal line $(a, f(a)) - (f(a), f(a))$
4. Connect a vertical line $(f(a), f(a)) - (f(a), f(f(a)))$
5. Connect a horizontal line $(f(a), f(f(a))) - (f(f(a)), f(f(a)))$
   etc.

This is sometimes called a cobweb plot. We will be using https://marksmath.org/visualization/cobwebs/ to make cobweb plots.

Example 2.1.1. Conduct a complete orbit analysis of $f(x) = x^2 - x + 1$

Solution. Playing around, we find that there is one fixed point $x = 1$.

When $x \in [0, 1]$, $f^n(x) \to 1$. Otherwise, $f^n(x) \to \infty$. $\square$

List of Named Results

Index of Defined Terms