Useful Proofs/Exercises

1 Logic06

1.1 Theorems

Theorem (Double Negation)**.** $A \dashv\vdash \neg\neg A$

Proof. In the forwards direction:

(1)	$\neg\neg A \vdash \neg\neg A$	(Ref)
(2)	$\neg\neg A, \neg A \vdash \neg\neg A$	(+, 1)
(3)	$\neg A \vdash \neg A$	(+)
(4)	$\neg\neg A, \neg A \vdash \neg A$	(+, 3)
(5)	$A \vdash \neg\neg A$	($\neg -$, 2, 4)

In the backwards direction:

(1)	$A, \neg A \vdash A$	($\in$)
(2)	$A, \neg A \vdash \neg A$	($\in$)
(3)	$A \vdash \neg\neg A$	($\neg +$, 1, 2)

□

Theorem (Inconsistency Rule)**.** $\neg A, A \vdash B$

Proof.

(1)	$A, \neg A, \neg B \vdash A$	($\in$)
(2)	$A, \neg A, \neg B \vdash \neg A$	($\in$)
(3)	$A, \neg A, \vdash B$	($\neg -$, 1, 2)

□

Theorem (Disjunctive Syllogism)**.** $A \vee B, \neg A \vdash B$

Proof.

(1)	$\neg A, B \vdash B$	($\in$)
(2)	$\neg A, A \vdash B$	(Inconsistency Rule)
(3)	$\neg A, A \vee B \vdash B$	($\vee -$, 1, 2)

□

Theorem (Modus Tollens). $\neg B, A \to B \vdash \neg A$

Proof.

(1)	$\neg B, \neg A \vee B \vdash \neg A$	(Disjunctive Syllogism)
(2)	$\neg A \vee B \vdash A \to B$	(Implication)
(3)	$\neg B, A \to B \vdash \neg A$	(Repl., 1, 2)

□

Theorem (Contrapositive). $A \to B \vdash \neg A \to \neg B$

Proof. In the forwards direction:

(1)	$A \to B, \neg B \vdash \neg A$	(Modus Tollens)
(2)	$A \to B \vdash \neg B \to \neg A$	($\to +$, 1)

In the backwards direction:

(1)	$B \vee \neg A, \neg\neg A \vdash B$	(Disjunctive Syllogism)
(2)	$B \vee \neg A, A \vdash B$	(Repl., Double Negation, 1)
(3)	$\neg\neg B \vee \neg A, A \vdash B$	(Repl., Double Negation, 2)
(4)	$\neg B \to \neg A, A \vdash B$	(Repl., Implication, 3)
(5)	$\neg B \to \neg A \vdash A \to B$	($\to +$, 4)

□

Theorem (Affirmation). *$A \vdash B$ if and only if $\emptyset \vdash A \to B$*

Proof. In the forwards direction:

(1)	$\emptyset \vdash A \to B$	(Premise)
(2)	$A \vdash A \to B$	($+$, 1)
(3)	$A \vdash A$	(Ref)
(4)	$A \vdash B$	($\to -$, 3, 2)

The backwards direction is just ($\to +$) on the premise. □

Theorem (Flip-Flop). *If $A \vdash B$, then $\neg B \vdash \neg A$.*

Proof. Suppose $A \vdash B$. Then,

(1)	$A \vdash B$	(Premise)
(2)	$\emptyset \vdash A \to B$	($\to +$, 1)
(3)	$\emptyset \vdash \neg B \to \neg A$	(Repl., Contrapositive, 2)
(4)	$\neg B \vdash \neg A$	(Affirmation, 3)

as desired. □

Theorem (De Morgan 1). $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$

Proof. In the forwards direction:

(1)	$\neg(A \vee B), A \vdash A$	$(\in)$
(2)	$\neg(A \vee B), A \vdash A \vee B$	$(\vee +, 1)$
(3)	$\neg(A \vee B), A \vdash \neg(A \vee B)$	$(\in)$
(4)	$\neg(A \vee B) \vdash \neg A$	$(\neg +, 2, 3)$
(5)	$\neg(A \vee B), B \vdash B$	$(\in)$
(6)	$\neg(A \vee B), B \vdash A \vee B$	$(\vee +, 5)$
(7)	$\neg(A \vee B), B \vdash \neg(A \vee B)$	$(\in)$
(8)	$\neg(A \vee B) \vdash \neg B$	$(\neg +, 6, 7)$
(9)	$\neg(A \vee B) \vdash \neg A \wedge \neg B$	$(\wedge +, 5, 8)$

Backwards:

(1)	$\neg A \wedge \neg B, A \vee B \vdash A \vee B$	$(\in)$
(2)	$\neg A \wedge \neg B, A \vee B \vdash \neg A \wedge \neg B$	$(\in)$
(3)	$\neg A \wedge \neg B, A \vee B \vdash \neg A$	$(\wedge -, 2)$
(4)	$\neg A \wedge \neg B, A \vee B \vdash \neg B$	$(\wedge -, 2)$
(5)	$\neg A \wedge \neg B, A \vee B \vdash A$	(Disjunctive Syllogism, 1, 4)
(6)	$\neg A \wedge \neg B \vdash \neg(A \vee B)$	$(\neg +, 3, 5)$

□

Theorem (De Morgan 2)**.** $\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B$

Proof. Forwards:

(1)	$\neg(A \wedge B), \neg(\neg A \vee \neg B) \vdash \neg(\neg A \vee \neg B)$	$(\in)$
(2)	$\neg(A \wedge B), \neg(\neg A \vee \neg B) \vdash \neg\neg A \wedge \neg\neg B$	(Tr., De Morgan 1, 1)
(3)	$\neg(A \wedge B), \neg(\neg A \vee \neg B) \vdash A \wedge B$	(Repl., Double Negation, 2)
(4)	$\neg(A \wedge B), \neg(\neg A \vee \neg B) \vdash \neg(A \wedge B)$	$(\in)$
(5)	$\neg(A \wedge B) \vdash \neg A \vee \neg B$	$(\neg -, 3, 4)$

Backwards:

(1)	$\neg A \vee \neg B, A \wedge B \vdash A \wedge B$	$(\in)$
(2)	$\neg A \vee \neg B, A \wedge B \vdash A$	$(\wedge -, 1)$
(3)	$\neg A \vee \neg B, A \wedge B \vdash B$	$(\wedge -, 1)$
(4)	$\neg A \vee \neg B, A \wedge B \vdash \neg\neg B$	(Double Negation, 3)
(5)	$\neg A \vee \neg B, A \wedge B \vdash \neg A \vee \neg B$	$(\in)$
(6)	$\neg A \vee \neg B, A \wedge B \vdash \neg A$	(Disjunctive Syllogism, 5, 4)
(7)	$\neg A \vee \neg B \vdash \neg(A \wedge B)$	$(\neg +, 2, 6)$

□

Theorem (Implication)**.** $A \to B \dashv\vdash \neg A \vee B$

Proof. Forwards:

(1)	$A \to B, \neg(\neg A \vee B) \vdash \neg(\neg A \vee B)$	$(\in)$
(2)	$A \to B, \neg(\neg A \vee B) \vdash \neg\neg A \wedge \neg B$	(Tr., De Morgan 1, 1)
(3)	$A \to B, \neg(\neg A \vee B) \vdash A \wedge \neg B$	(Repl., Double Negation, 2)
(4)	$A \to B, \neg(\neg A \vee B) \vdash A$	$(\wedge -$, 3)
(5)	$A \to B, \neg(\neg A \vee B) \vdash A \to B$	$(\in)$
(6)	$A \to B, \neg(\neg A \vee B) \vdash B$	$(\to -$, 5, 4)
(7)	$A \to B, \neg(\neg A \vee B) \vdash \neg B$	$(\wedge -$, 3)
(8)	$A \to B \vdash \neg A \vee B$	$(\neg -$, 6, 7)

Backwards:

(1)	$\neg A \vee B, A \vdash A$	$(\in)$
(2)	$\neg A \vee B, A \vdash \neg\neg A$	(Double Negation, 1)
(3)	$\neg A \vee B, A \vdash \neg A \vee B$	$(\in)$
(4)	$\neg A \vee B, A \vdash B$	(Disjunctive Syllogism, 3, 2)
(5)	$\neg A \vee B \vdash A \to B$	$(\to +$, 4)

□

Theorem (Non-Contradiction)**.** $\emptyset \vdash \neg(A \wedge \neg A)$

Proof.

(1)	$A \wedge \neg A \vdash A \wedge \neg A$	(Ref)
(2)	$A \wedge \neg A \vdash A$	(Tr., $\wedge -$, 1)
(3)	$A \wedge \neg A \vdash \neg A$	(Tr., $\wedge -$, 1)
(4)	$\emptyset \vdash \neg(A \wedge \neg A)$	$(\neg +$, 2, 3)

□

Theorem (Excluded Middle)**.** $\emptyset \vdash A \vee \neg A$

Proof. Apply Transitivity, De Morgan 2 to Non-Contradiction. □

Theorem (Rule of Cases)**.** $A \to B, \neg A \to B \vdash B$

Proof.

(1)	$A \to B, \neg A \to B, \neg B \vdash A \to B$	$(\in)$
(2)	$A \to B, \neg A \to B, \neg B \vdash \neg A \to B$	$(\in)$
(3)	$A \to B, \neg A \to B, \neg B \vdash \neg B$	$(\in)$
(4)	$A \to B, \neg A \to B, \neg B \vdash \neg A$	(Modus Tollens, 1, 3)
(5)	$A \to B, \neg A \to B, \neg B \vdash \neg\neg A$	(Modus Tollens, 2, 3)
(6)	$A \to B, \neg A \to B \vdash B$	$(\neg -$, 4, 5)

□

1.2 Exercises

Exercise 1.2.1. $A \to (B \vee C), A \to \neg B, C \to \neg D \vdash A \to \neg D$

Proof. Let $\Sigma = \{A \to (B \vee C), A \to \neg B, C \to \neg D\}$.

(1)	$\Sigma, A \vdash A$	$(\in)$
(2)	$\Sigma, A \vdash A \to (B \vee C)$	$(\in)$
(3)	$\Sigma, A \vdash B \vee C$	$(\to -$, 2, 1)
(4)	$\Sigma, A \vdash A \to \neg B$	$(\in)$
(5)	$\Sigma, A \vdash \neg B$	$(\to -$, 4, 1)
(6)	$\Sigma, A \vdash C$	(Disjunctive Syllogism, 3, 5)
(7)	$\Sigma, A \vdash C \to \neg D$	$(\in)$
(8)	$\Sigma, A \vdash \neg D$	$(\to -$, 7, 6)
(9)	$\Sigma \vdash A \to \neg D$	$(\to +$, 8)

□

Exercise 1.2.2. $A \to (B \to C), C \to \neg D, \neg E \to D, A \wedge B \vdash E$

Proof. Let $\Sigma = \{A \to (B \to C), C \to \neg D, \neg E \to D, A \wedge B\}$.

(1)	$\Sigma \vdash A \wedge B$	$(\in)$
(2)	$\Sigma \vdash A$	$(\wedge -$, 1)
(3)	$\Sigma \vdash A \to (B \to C)$	$(\in)$
(4)	$\Sigma \vdash B \to C$	$(\to -$, 3, 2)
(5)	$\Sigma \vdash B$	$(\wedge -$, 1)
(6)	$\Sigma \vdash C$	$(\to -$, 4, 5)
(7)	$\Sigma \vdash C \to \neg D$	$(\in)$
(8)	$\Sigma \vdash \neg D$	$(\to -$, 7, 6)
(9)	$\Sigma \vdash \neg E \to D$	$(\in)$
(10)	$\Sigma \vdash E$	(Modus Tollens, 8, 9)

□

Exercise 1.2.3. $\neg A \to C \vee D, B \to E \wedge F, E \to D, \neg D \vdash (A \to B) \to C$

Proof. Let $\Sigma = \neg A \to C \vee D, B \to E \wedge F, E \to D, \neg D$

(1)	$\Sigma, A \to B \vdash E \to D$	$(\in)$
(2)	$\Sigma, A \to B \vdash \neg D$	$(\in)$
(3)	$\Sigma, A \to B \vdash \neg E$	(Modus Tollens, 2, 1)
(4)	$\Sigma, A \to B \vdash \neg E \vee \neg F$	$(\vee +$, 3)
(5)	$\Sigma, A \to B \vdash \neg(E \wedge F)$	(Tr., De Morgan, 4)
(6)	$\Sigma, A \to B \vdash B \to E \wedge F$	$(\in)$
(7)	$\Sigma, A \to B \vdash \neg B$	(Modus Tollens, 5, 6)
(8)	$\Sigma, A \to B \vdash A \to B$	$(\in)$
(9)	$\Sigma, A \to B \vdash \neg A$	(Modus Tollens, 7, 8)
(10)	$\Sigma, A \to B \vdash \neg A \to C \vee D$	$(\in)$
(11)	$\Sigma, A \to B \vdash C \vee D$	$(\to -$, 10)
(12)	$\Sigma, A \to B \vdash C$	(Disjunctive Syllogism, 11, 2)
(13)	$\Sigma \vdash (A \to B) \to C$	$(\to +$, 12)

□

Exercise 1.2.4. $\neg(A \vee B) \to (C \to D), \neg A \wedge \neg D \vdash \neg B \to \neg C$

Proof. Let $\Sigma = \neg(A \vee B) \to (C \to D), \neg A \wedge \neg D$.

(1)	$\Sigma, \neg B \vdash \neg(A \vee B) \to (C \to D)$	$(\in)$
(2)	$\Sigma, \neg B \vdash (\neg A \wedge \neg B) \to (C \to D)$	(Repl., De Morgan, 1)
(3)	$\Sigma, \neg B \vdash \neg A \wedge \neg D$	$(\in)$
(4)	$\Sigma, \neg B \vdash \neg A$	$(\wedge -$, 3)
(5)	$\Sigma, \neg B \vdash \neg B$	$(\in)$
(6)	$\Sigma, \neg B \vdash \neg A \wedge \neg B$	$(\wedge +$, 4, 5)
(7)	$\Sigma, \neg B \vdash C \to D$	$(\to -$, 2, 6)
(8)	$\Sigma, \neg B \vdash \neg D$	$(\wedge -$, 3)
(9)	$\Sigma, \neg B \vdash \neg C$	(Modus Tollens, 8, 7)
(10)	$\Sigma \vdash \neg B \to \neg C$	$(\to +$, 9)

□

Exercise 1.2.5. $B \vee A, B \to A \vdash \neg(A \to \neg A)$

Proof. Let $\Sigma = \{B \vee A, B \to A\}$

(1)	$\Sigma, A \to \neg A, \neg B \vdash \neg B$	$(\in)$
(2)	$\Sigma, A \to \neg A, \neg B \vdash B \vee A$	$(\in)$
(3)	$\Sigma, A \to \neg A, \neg B \vdash A$	(Disjunctive Syllogism, 1, 2)
(4)	$\Sigma, A \to \neg A \vdash \neg B \to A$	$(\to +$, 3)
(5)	$\Sigma, A \to \neg A \vdash B \to A$	$(\in)$
(6)	$\Sigma, A \to \neg A \vdash A$	(Tr., Rule of Cases, 4, 5)
(7)	$\Sigma, A \to \neg A \vdash A \to \neg A$	$(\in)$
(8)	$\Sigma, A \to \neg A \vdash \neg A$	$(\to -$, 7)
(9)	$\Sigma \vdash \neg(A \to \neg A)$	$(\neg +$, 6, 8)

□

2 Logic14

2.1 Exercises

Exercise 2.1.1. $\forall x\, \forall y\, P(x,y) \vdash \forall y\, \forall x\, P(x,y)$

Proof.

(1)	$\forall x\, \forall y\, P(x,y) \vdash \forall x\, \forall y\, P(x,y)$	(Ref)
(2)	$\forall x\, \forall y\, P(x,y) \vdash \forall y\, P(a,y)$	($\forall -$, 1)
(3)	$\forall x\, \forall y\, P(x,y) \vdash P(a,b)$	($\forall -$, 2)
(4)	$\forall x\, \forall y\, P(x,y) \vdash \forall x\, P(x,b)$	($\forall +$, 3, a not elsewhere)
(5)	$\forall x\, \forall y\, P(x,y) \vdash \forall y\, P(x,y)$	($\forall +$, 4, b not elsewhere)

□

Exercise 2.1.2. $\forall x\, P(x) \vdash \forall y\, P(y)$

Proof.

(1)	$\forall x\, P(x) \vdash \forall x\, P(x)$	(Ref)
(2)	$\forall x\, P(x) \vdash P(a)$	($\forall -$, 1)
(3)	$\forall x\, P(x) \vdash \forall y\, P(y)$	($\forall +$, 2, a not elsewhere)

□

Exercise 2.1.3. $\neg\exists x\, P(x) \vdash \forall x\, \neg P(x)$

Proof.

(1)	$\Sigma, P(t) \vdash P(t)$	($\in$)
(2)	$\Sigma, P(t) \vdash \exists x\, P(x)$	($\exists +$, 1)
(3)	$\Sigma, P(t) \vdash \neg\exists x\, P(x)$	($\in$)
(4)	$\Sigma \vdash \neg P(t)$	($\neg +$, 2, 3)
(5)	$\Sigma \vdash \forall x\, \neg P(x)$	($\forall +$, t not elsewhere)

□

Exercise 2.1.4. $\forall x\, \neg P(x) \vdash \neg\exists x\, P(x)$

Proof.

(1)	$\Sigma, \exists x\, P(x), P(u) \vdash P(u)$	($\in$)
(2)	$\Sigma, \exists x\, P(x), P(u) \vdash \forall x\, \neg P(x)$	($\in$)
(3)	$\Sigma, \exists x\, P(x), P(u) \vdash \neg P(u)$	(Tr., $\forall -$, 2)
(4)	$\Sigma, P(u) \vdash \neg\exists x\, P(x)$	($\neg +$, 1, 3)
(5)	$\Sigma, \exists x\, P(x) \vdash \neg\exists x\, P(x)$	($\exists -$, u not elsewhere)
(6)	$\Sigma, \exists x\, P(x) \vdash \exists x\, P(x)$	($\in$)
(7)	$\Sigma \vdash \neg\exists x\, P(x)$	($\neg +$, 5, 6)

□

Exercise 2.1.5. $\forall x\,(P(x) \to Q(x)), \exists x\,(R(x) \wedge \neg Q(x)), \forall x\,(R(x) \to P(x) \vee S(x)) \vdash \exists x\,(R(x) \vee S(x))$

Proof. Let $\Sigma = \{\forall x\,(P(x) \to Q(x)), \forall x\,(R(x) \to P(x) \vee S(x))\}$

(1)	$\Sigma, R(u) \wedge \neg Q(u) \vdash R(u) \wedge \neg Q(u)$	$(\in)$
(2)	$\Sigma, R(u) \wedge \neg Q(u) \vdash \neg Q(u)$	($\wedge -$, 1)
(3)	$\Sigma, R(u) \wedge \neg Q(u) \vdash \forall x\,(P(x) \to Q(x))$	$(\in)$
(4)	$\Sigma, R(u) \wedge \neg Q(u) \vdash P(u) \to Q(u)$	($\forall -$, 3)
(5)	$\Sigma, R(u) \wedge \neg Q(u) \vdash \neg P(u)$	(modus tollens, 2, 4)
(6)	$\Sigma, R(u) \wedge \neg Q(u) \vdash \forall x\,(R(x) \to P(x) \vee S(x))$	$(\in)$
(7)	$\Sigma, R(u) \wedge \neg Q(u) \vdash R(u) \to P(u) \vee S(u)$	($\forall -$, 6)
(8)	$\Sigma, R(u) \wedge \neg Q(u) \vdash R(u)$	($\wedge -$, 1)
(9)	$\Sigma, R(u) \wedge \neg Q(u) \vdash P(u) \vee S(u)$	($\to -$, 8, 7)
(10)	$\Sigma, R(u) \wedge \neg Q(u) \vdash S(u)$	(Disjunctive Syllogism, 9, 5)
(11)	$\Sigma, R(u) \wedge \neg Q(u) \vdash R(u) \vee S(u)$	($\vee +$, 8, 10)
(12)	$\Sigma, R(u) \wedge \neg Q(u) \vdash \exists x\,(R(x) \vee S(x))$	($\exists +$, 11)
(13)	$\Sigma, \exists x\,(R(x) \wedge \neg Q(x)) \vdash \exists x\,(R(x) \vee S(x))$	($\exists -$, 12, u not elsewhere)

□