PMATH 370 Winter 2024:
Lecture Notes

1 Iteration and Orbits **2**
- 1.1 Orbits . . . 2
- 1.2 Real analysis review . . . 4
- 1.3 Orbits, revisited . . . 7

2 Graphical Analysis **11**
- 2.1 Cobweb plots . . . 11

3 Fixed Points **14**
- 3.1 Attracting/repelling fixed point theorems . . . 14
- 3.2 Neutral fixed points . . . 17

4 Bifurcations **21**
- 4.1 Quadratic family . . . 21

Back Matter **23**
- List of Named Results . . . 23
- Index of Defined Terms . . . 24

Lecture notes taken, unless otherwise specified, by myself during the Winter 2024 offering of PMATH 370, taught by Blake Madill.

Lectures

Lecture 1	Jan 8	2
Lecture 2	Jan 10	4
Lecture 3	Jan 12	6
Lecture 4	Jan 15	9
Lecture 5	Jan 17	12
Lecture 6	Jan 19	17
Lecture 7	Jan 22	19

Chapter 1

Iteration and Orbits

1.1 Orbits

Lecture 1
Jan 8

Definition 1.1.1 (iteration)
Let $f : A \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$. For $a \in A$ we may iterate the function at a:

$$x_1 = a, x_2 = f(a), x_3 = \underbrace{f(f(a))}_{f^2(a)}, \dots, x_i = f^{i-1}(a), \dots .$$

The sequence $(x_n)_{n=1}^{\infty}$ is the orbit of a under f (abbreviated (x_n) without limits).

Example 1.1.2. Let $f(x) = x^4 + 2x^2 - 2$, $a = -1$. What is the orbit of a under f?

Solution. $a = -1$, $f(a) = 1$, $f(f(a)) = f(1) = 1$, so we have $-1, 1, 1, 1, \dots$. We call this eventually constant. $\square$

Example 1.1.3. Let $f(x) = -x^2 - x + 1$, $a = 0$. What is the orbit of a under f?

Solution. Calculate: $0, 1, -1, 1, -1, 1, \dots$. We call this eventually periodic (with period 2). $\square$

Example 1.1.4. Let $f(x) = x^3 - 3x + 1$, $a = 1$. What is the orbit of a under f?

Solution. Calculate the first few terms: $1, -1, 3, 19, \dots$ (too big). This is a divergence to infinity. $\square$

Example 1.1.5. Let $f(x) = x^2 + 2x$, $a = -0.5$. What is the orbit of a under f?

Solution. Calculate: $-0.5, -0.75, -0.9375, -0.9961\dots$ and we make an educated guess that this converges to -1 since $f(-1) = -1$, a fixed point. $\square$

Example 1.1.6. Let $f(x) = x^3 - 3x$, $a = 0.75$. What is the orbit of a under f?

Solution. Calculate: $0.75, -1.828, -0.625, 1.631, -0.552, \dots$. There is no clear pattern, so we call this chaotic. In fact, the orbit is dense in a neighbourhood of 0. □

We can start to formalize the examples.

Definition 1.1.7 (fixed point)

Let $f : A \to \mathbb{R}$ such that $f(A) \subseteq A$. A point $a \in A$ is fixed if $f(a) = a$.

Then, the orbit of a under f is $(a, a, a, \dots)$ which is constant.

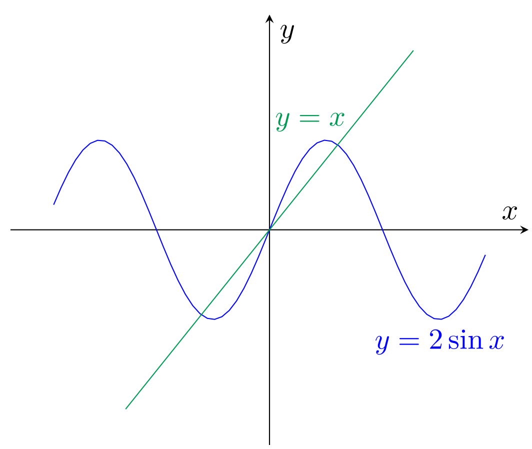
Example 1.1.8. Find all fixed points of $f(x) = x^2 + x - 4$.

Solution. We find points where $f(x) = x$, i.e., $x^2 + x - 4 = x$.

$$x^2 + x - 4 = x \iff x^2 = 4 \iff x = \pm 2$$ □

Example 1.1.9. How many fixed points does $f(x) = 2\sin x$ have?

Solution. Consider where the curve $y = 2\sin x$ meets $y = x$:



We can see there are three fixed points. □

Example 1.1.10. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Proof. We must show there is a solution to $x^4 - 3x + 1 \iff x^4 - 4x + 1 = 0$. Let $g(x) = x^4 - 4x + 1$. Since $g(x)$ is continuous, $g(0) = 1 > 0$, and $g(1) = -2 < 0$, by the Intermediate Value Theorem, there must exist a root of g on the interval $(0, 1)$. That is, a fixed point of f. □

Definition 1.1.11 (periodicity)

Let $f : A \to \mathbb{R}, f(A) \subseteq A$.

1. A point $a \in A$ is <u>periodic</u> for f if its orbit is periodic. An orbit is <u>periodic</u> if for some $n \in \mathbb{N}$, $f^n(a) = a$. The smallest n is the <u>period</u> of (the orbit of) a.
2. An orbit (of a point) is <u>eventually periodic</u> if there exists $n < m$ such that $f^n(a) = f^m(a)$. The smallest difference $m - n$ is the period of the orbit.

Lecture 2
Jan 10

Definition 1.1.12 (doubling function)

$D : [0, 1) \to [0, 1) : x \mapsto 2x - \lfloor 2x \rfloor$ returns the fractional part of $2x$.

Example 1.1.13. $D(0.4) = 0.8,\ D(0.6) = 0.2,\ D(0.8) = 0.6,\ D(0.5) = 0$.

This is a nice function that gives lots of periodic orbits for funsies.

Example 1.1.14. Find the orbit of $a = \frac{1}{5}$ under D.

Solution. Double until we pass 1: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{8}{5} \to \frac{3}{5}, \frac{6}{5} \to \frac{1}{5}$. The period is $\left|\{\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\}\right| = 4$. □

Example 1.1.15. Find the orbit of $a = \frac{1}{20}$ under D.

Solution. Double: $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}$ and we can stop because Example 1.1.14 showed $\frac{1}{5}$ is periodic.

So this is eventually periodic with period 4. □

Problem 1.1.16

Given f and a, does $f^n(a)$ tend towards some limit L?

To solve this problem, we need to rigorously define "tend" and "limit".

1.2 Real analysis review

Notation. If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers, we write $(x_n) \subseteq \mathbb{R}$.

Definition 1.2.1 (convergence of a sequence)
Let $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$.

We say (x_n) converges to x if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

Then, we write $x_n \to x$ or $\lim x_n = x$.

Example 1.2.2. Show that $\frac{1}{n} \to 0$.

Proof. Let $\varepsilon > 0$. Consider $N = \frac{2}{\varepsilon} > \frac{1}{\varepsilon}$. For $n \geq N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

Therefore, $\frac{1}{n} \to 0$. □

Example 1.2.3. Prove that $\frac{2n}{n+3} \to 2$.

Proof. Let $\varepsilon > 0$. Since we know $\frac{1}{n} \to 0$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{6}$.

For $n \geq N$,

$$\left|\frac{2n}{n+3} - 2\right| = \left|\frac{2n}{n+3} - \frac{2n+6}{n+3}\right| = \left|\frac{-6}{n+3}\right| = \frac{6}{n+3} < \frac{6}{n} \leq \frac{6}{N} < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

Therefore, $\frac{2n}{n+3} \to 2$. □

Definition 1.2.4 (bounded sequence)
A sequence (x_n) is bounded (by M) if there exists $M > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq M$.

Proposition 1.2.5 (convergence implies boundedness)
If (x_n) is convergent, then (x_n) is bounded.

Proof. Suppose $x_n \to x$. Then, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - x| < 1$.

For $n \geq N$, $|x_n| - |x| \leq |x_n - x| < 1$. That is, $|x_n| < 1 + |x|$.

Let $M = \max\{|x_1|, \dots, |x_{n-1}|, 1 + |x|\}$. Then, for both all $n < N$ and $n \geq N$, we have $|x_n| \leq M$. □

Remark 1.2.6. The converse is not true. Notice that $x_n = (-1)^n$ is bounded by 1 but obviously not convergent.

Proposition 1.2.7 (limit laws)

Let $x_n \to x$ and $y_n \to y$. Then:

(1) $x_n + y_n \to x + y$

(2) $x_n y_n \to xy$

Proof. (1) Let $\varepsilon > 0$. Then, since $x_n \to x$ and $y_n \to y$, there exist $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies |x_n - x| < \frac{\varepsilon}{2}$ and $n \geq N_2 \implies |y_n - y| < \frac{\varepsilon}{2}$.

For $N = \max\{N_1, N_2\}$ and $n \geq N$,

$$\begin{aligned} |(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

That is, $x_n + y_n \to x + y$.

(2) Let $\varepsilon > 0$. Notice that:

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \tag{$*$}$$

Since x_n is bounded, there exists $M > 0$ such that $|x_n| \leq M$ for all n.

Let $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 &\implies |x_n - x| \leq \frac{\varepsilon}{2(|y| + 1)} \text{ and} \\ n \geq N_2 &\implies |y_n - y| < \frac{\varepsilon}{2M}. \end{aligned}$$

Then, for $n \geq N := \max\{N_1, N_2\}$, $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by $(*)$. □

Lecture 3
Jan 12

Definition 1.2.8 (Cauchy sequence)

We say $(x_n) \in \mathbb{R}$ is <u>Cauchy</u> if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all n and m,

$$n, m \geq N \implies |x_n - x_m| < \varepsilon$$

Proposition 1.2.9

Every convergent sequence is Cauchy.

Proof. Intuitively: if the terms get arbitrarily close to some limit, they must get arbitrarily close to each other.

Formally: Let $x_n \to x$ be a convergent sequence and $\varepsilon > 0$. Since x_n converges, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |x_n - x| < \frac{\varepsilon}{2}$.

Then, when $n, m \geq N$, we have:

$$\begin{aligned} |x_n - x_m| &= |x_n - x_m + x - x| \\ &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

as desired. □

We take the following theorem from real analysis without proof.

Theorem 1.2.10 (completeness of $\mathbb{R}$)

A sequence is Cauchy if and only if it is convergent.

The big idea here: To prove (x_n) is Cauchy, you do not have to guess the limit first. That is, if you must prove convergence but do not care about the limit's value, prove that it is Cauchy instead.

Definition 1.2.11 (continuity of a function)

Let $f : A \to \mathbb{R}, A \subseteq \mathbb{R}, a \in A$. We say f is continuous at a if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in A$ and $|x - a| < \delta$.

If f is continuous at all $a \in A$, we say it is continuous.

We also take this theorem from MATH 137 without proof.

Theorem 1.2.12

A function $f : A \to \mathbb{R}$ is continuous at $a \in A$ if and only if for all sequences $(x_n) \subseteq A$ with $x_n \to a$, we have $f(x_n) \to a$.

1.3 Orbits, revisited

Proposition 1.3.1

If $f : [a, b] \to [a, b]$ is continuous, then $f(x)$ has a fixed point.

Proof. We know by the domain and codomain that $f(a) \geq a$ and $f(b) \leq b$. This means $f(a) - a \geq 0$ and $f(b) - b \leq 0$. By the IVT on the continuous function $g(x) = f(x) - x$, we know there exists an $x \in [a, b]$ such that $g(x) = f(x) - x = 0 \iff f(x) = x$, i.e., x is a fixed point. □

Definition 1.3.2 (contraction)

Let $f : A \to \mathbb{R}, A \subseteq \mathbb{R}$. We say f is a contraction if there exists $C \in [0, 1)$ such that for all $x, a \in A$,

$$|f(x) - f(y)| \leq C|x - y|$$

This is just a Lipschitz function with Lipschitz constant less than 1.

Proposition 1.3.3

Contractions are continuous.

Proof. Let $\varepsilon > 0$. Suppose f is a contraction such that $|f(x) - f(y)| \leq C|x - y|$.

Consider $y \in A$. Let $\delta = \frac{\varepsilon}{C+1}$ and assume that $x \in A$ and $|x - y| < \delta$. But we have:

$$|f(x) - f(y)| \leq C|x - y| \leq C\delta < \varepsilon$$

as desired. □

Definition 1.3.4 (closure of an interval)

We say $A \in \mathbb{R}$ is closed if whenever $(x_n) \subseteq A$ with $x_n \to x$, then $x \in A$.

Example 1.3.5. $[a, b]$ is closed but $(0, 1]$ is not because $\frac{1}{n} \to 0 \notin (0, 1]$.

Theorem 1.3.6 (Banach contraction mapping theorem)

Suppose $A \subseteq \mathbb{R}$ is closed and $f : A \to A$ is a contraction. Then, there exists a unique fixed point $a \in A$ for f.

Moreover, for all $x \in A$, $f^n(x) \to a$.

Example 1.3.7. Analyze the orbit of $f : [0, 1] \to [0, 1]$, $f(x) = \frac{1}{3-x}$.

Solution. We can observe that $\frac{1}{3} \leq \frac{1}{3-x} \leq \frac{1}{2}$.

Also, $f'(x) = \frac{1}{(3-x)^2}$. Notice that $\frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$. So by the mean value theorem, for all $x, y \in [0, 1]$, there exists $c \in (0, 1)$ such that:

$$\begin{aligned}
f(x) - f(y) &= f'(c)(x - y) \\
|f(x) - f(y)| &= |f'(c)| \cdot |x - y| \\
&\leq \frac{1}{4}|x - y|
\end{aligned}$$

Then, identifying $C = \frac{1}{4}$, f is a contraction. Now,

$$\frac{1}{3-x} = x \iff 1 = 3x - x^2 \iff x^2 - 3x + 1 = 0 \iff x = \frac{3 \pm \sqrt{9-4}}{2} \iff x = \frac{3-\sqrt{5}}{2}$$

where we pick the negative root because we need $x \in [0,1]$.

Therefore, by the Banach contraction mapping theorem, for all $x \in [0,1]$, $f^n(x) \to \frac{3-\sqrt{5}}{2}$. □

Lecture 4
Jan 15

Definition 1.3.8
A sequence $(a_n) \subseteq \mathbb{R}$ is strongly-Cauchy if there exists $(\varepsilon_n) \subseteq [0,\infty)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and for all n, $|a_n - a_{n+1}| < \varepsilon_n$.

Informally, "far enough along the sequence, the *neighbours* must get close". This is distinct from Cauchy, which is "far enough along the sequence, the *terms* must get close".

Remark 1.3.9 (assignment hint!). Let $\sum_{n=1}^{\infty} a_n = L$. This means that $\sum_{k=1}^{n} a_k \xrightarrow{n\to\infty} L$.

That is, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\sum_{k=1}^{n} a_k - L\right| < \varepsilon$.

But $\left|\sum_{k=1}^{n} a_k - L\right| = \left|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{\infty} a_k\right| < \varepsilon$.

We can now prove the Banach contraction mapping theorem.

Proof. Let $A \subseteq \mathbb{R}$ be closed and suppose there exists $f : A \to A$ and $C \in [0,1)$ such that $|f(x) - f(y)| \leq C|x-y|$ for all x and y in A.

Fix $x_0 \in A$ and construct the orbit $x_1 = f(x_0)$, $x_2 = f(x_1)$, ..., $x_n = f(x_{n-1}) = f^n(x_0)$.

For $n \in \mathbb{N}$, since f is a contraction,

$$\begin{aligned}
|x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \\
&\leq C|x_n - x_{n-1}| \\
&= C|f(x_{n-1}) - f(x_{n-2})| \\
&\leq C^2|x_{n-1} - x_{n-2}| \\
&\vdots \\
&\leq C^n|x_1 - x_0|
\end{aligned}$$

Since $\sum_{n=1}^{\infty} C^n|x_1 - x_0| = |x_1 - x_0| \sum_{n=1}^{\infty} C^n$ is a convergent geometric series, we have that the sequence (x_n) is strongly-Cauchy.

Hence, by Assignment 1, $x_n \to a$ for some limit point $a \in A$ since A is closed.

Since f is continuous (Proposition 1.3.3), we have that $f(x_n) \to f(a)$. By definition, $f(x_n) = x_{n+1}$, so $x_n \to f(a)$. But we already know $x_n \to a$, so $a = f(a)$. That is, a is a fixed point of f.

It remains to show uniqueness.

Suppose $a, b \in A$ such that $f(a) = a$ and $f(b) = b$.

$$\begin{aligned} |f(a) - f(b)| &\leq C|a - b| \\ |a - b| &\leq C|a - b| \end{aligned}$$

Since $C < 1$, we must have $|a - b| = 0$, that is, $a = b$. $\square$

Chapter 2

Graphical Analysis

2.1 Cobweb plots

Recall Example 1.1.9. To visualize the orbit of a under f, we can:

1. Superimpose $y = f(x)$ over the line $y = x$.
2. Connect a vertical line $(a, a) - (a, f(a))$
3. Connect a horizontal line $(a, f(a)) - (f(a), f(a))$
4. Connect a vertical line $(f(a), f(a)) - (f(a), f(f(a)))$
5. Connect a horizontal line $(f(a), f(f(a))) - (f(f(a)), f(f(a)))$
   etc.

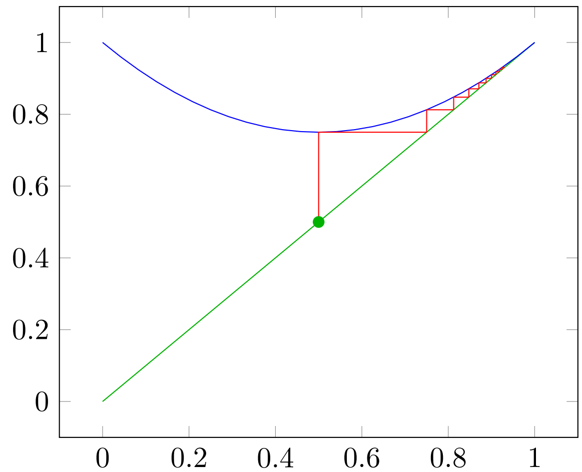
This is sometimes called a cobweb plot. We will be using https://marksmath.org/visualization/cobwebs/ to make cobweb plots.

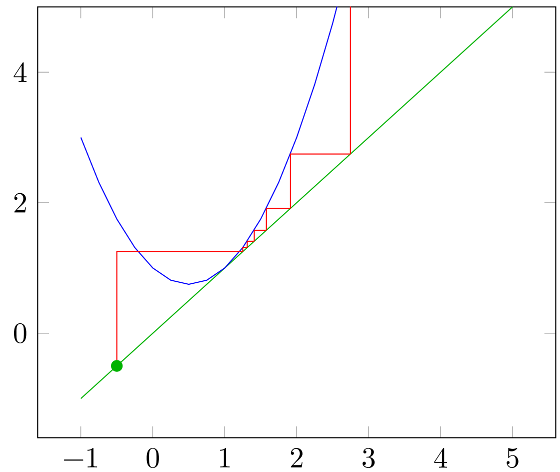
Within these lecture notes, I use a LaTeX macro to draw plots defined here.

Example 2.1.1. Conduct a complete orbit analysis of $f(x) = x^2 - x + 1$

Solution. Playing around, we find that there is one fixed point $x = 1$.

When $x \in [0, 1]$, $f^n(x) \to 1$. Otherwise, $f^n(x) \to \infty$.





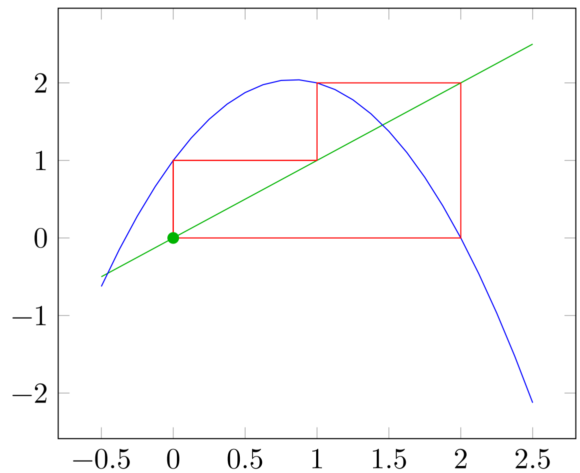
□

↓ *Lectures 5 and 6 adapted from Rosie* ↓

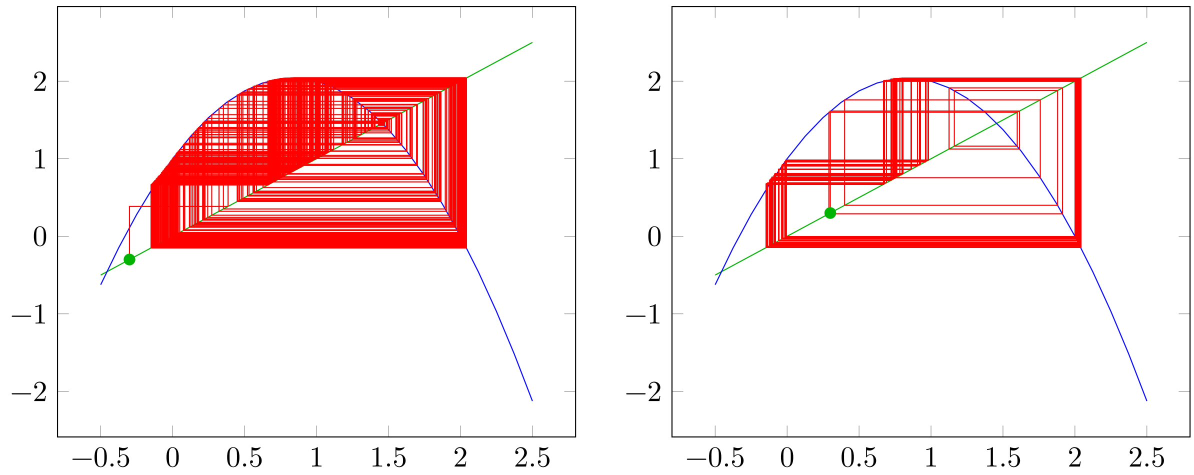
Lecture 5
Jan 17

Example 2.1.2. Conduct a complete orbit analysis of $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$.

Solution. At $x = 0$, we can see there is a cycle going from $0 \to 1 \to 2 \to 0$:



At points near 0, like $x = -0.3$ or $x = 0.3$, the graph becomes chaotic:



It appears that the cobweb densely covers the graph. □

As an aside, note that we cannot actually hit every point in the interval because the orbit is countable (i.e., has the same size as the naturals) but the interval is uncountable. We will later show that the points are dense (as the rationals are).

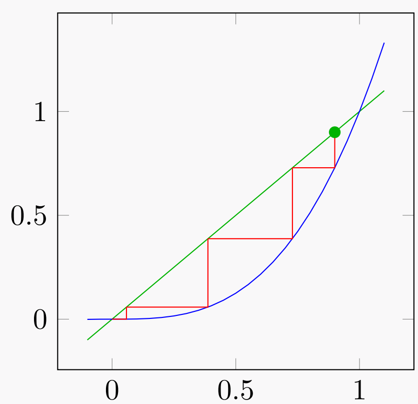
Chapter 3

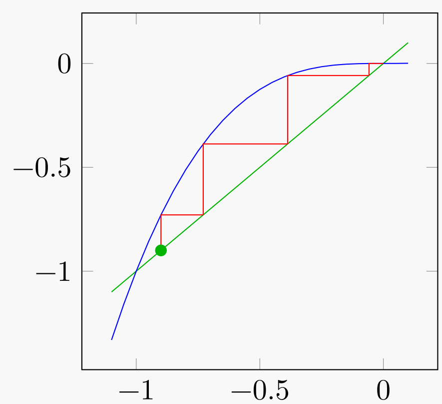
Fixed Points

3.1 Attracting/repelling fixed point theorems

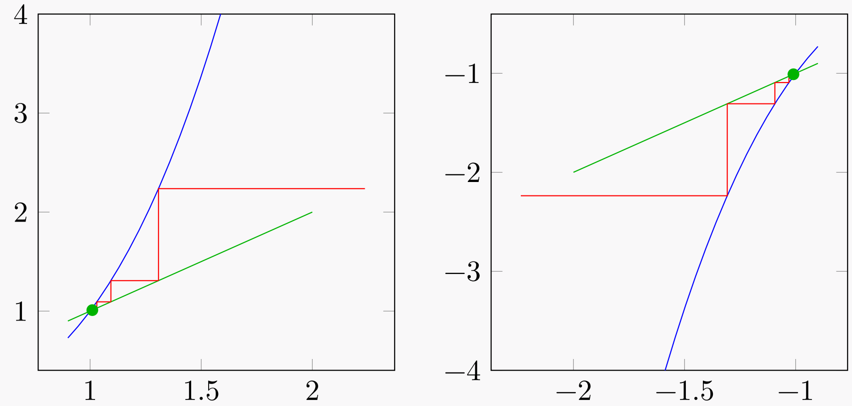
Remark 3.1.1. If $f(x)$ is continuous and $f^n(a) \to L$, then $f^{n+1}(a) \to f(L)$. Therefore, $f(L) = L$ is a fixed point.

Example 3.1.2. The function $f(x) = x^3$ has three fixed points: $0, \pm 1$. For $x \in (-1, 1)$, we see that $f^n(x) \to 0$:



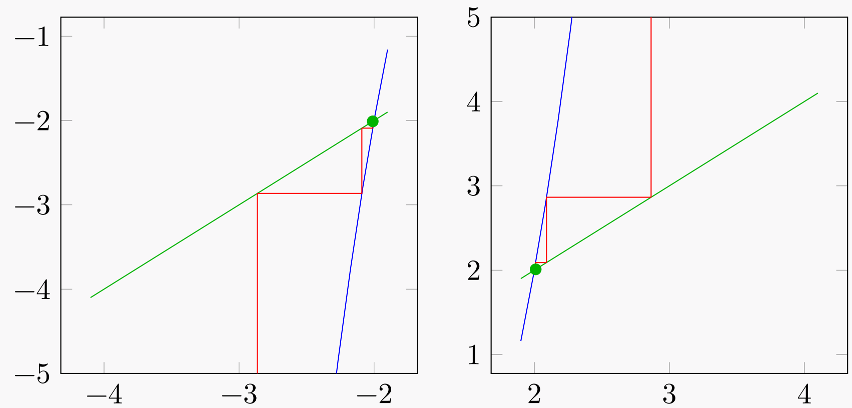


It looks like point 0 is <u>attracting</u> the orbit. For $x \in (-\infty, -1) \cup (1, \infty)$, we see $f^n(x) \to \infty$:

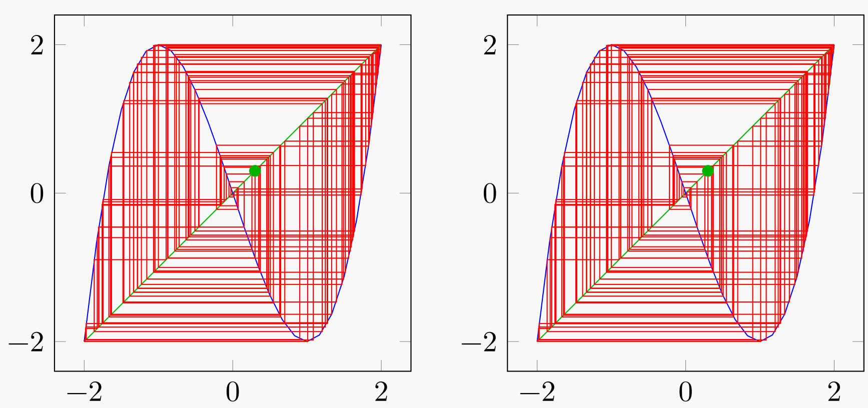


so the points ± 1 are <u>repelling</u> the orbit.

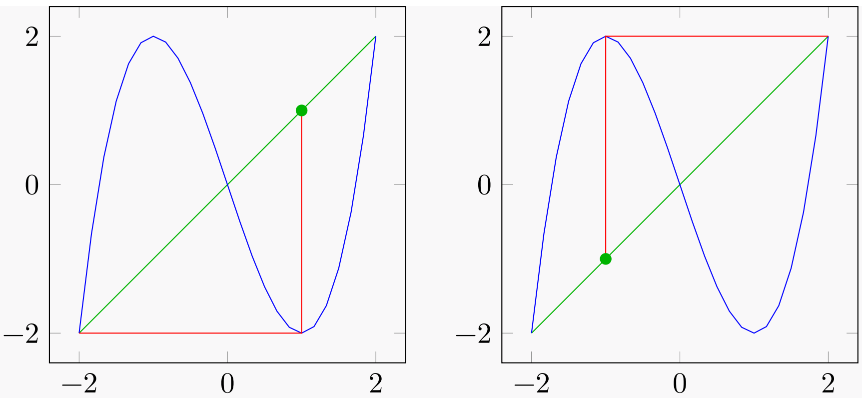
Example 3.1.3. The function $f(x) = x^3 - 3x$ also has three fixed points: $0, \pm 2$. To the right (left) of ± 2, orbits go to infinity:



The point 0 is repelling (in a different sense) since we get chaos:



At $x_0 = \pm 1$, the orbit is eventually constant, jumping to the fixed point ∓ 2:



Definition 3.1.4

Let a be a fixed point of $f(x)$.

1. If $|f'(a)| > 1$, we call a a repelling fixed point
2. If $|f'(a)| < 1$, we call a a attracting fixed point
3. If $|f'(a)| = 1$, we call a a neutral fixed point

Neutral fixed points can be a lot of different things.

Theorem 3.1.5 (attracting fixed point theorem)

Suppose a is an attracting fixed point of $f(x)$. Then, there exists an open interval I containing a such that

1. for all $x \in I$, $n \in \mathbb{N}$, $f^n(x) \in I$
2. for all $x \in I$, $f^n(x) \to a$

Recall the ε-δ definition of a limit.

Definition 3.1.6 (limit of a function at a point)

Let $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$.

We say a point $a \in A$ is non-isolated if for each $\varepsilon > 0$ there exists $b \in A$, $b \neq a$ with $b \in (a - \varepsilon, a + \varepsilon)$.

Suppose a is non-isolated. We say $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $a \in A$ and $0 < |x - a| < \delta$.

It is important to define non-isolation. If a is isolated, we can choose a δ where $|x - a| < \delta$ is false. Then, every point is vacuously a limit point.

We now give the proof of the attracting fixed point theorem:

Proof. Assume $|f'(a)| < 1$. Then, there exists $c \in \mathbb{R}$ such that $|f'(a)| < c < 1$. By definition of the derivative, this means we can write

$$\lim_{x \to a} \frac{|f(x) - f(a)|}{x - a} < c$$

and by the definition of the limit, we know there exists $\delta > 0$ such that

$$\frac{|f(x) - f(a)|}{|x - a|} \leq c, \quad \forall x \in (a - \delta, a + \delta)$$

Hence, for $x \in I := (a - \delta, a + \delta)$, we have $|f(x) - f(a)| \leq c|x - a|$ and f is a contraction.

Lecture 6
Jan 19

In particular, for $x \in I$, we have

$$\begin{aligned} |f(x) - a| &= |f(x) - f(a)| && (a \text{ is a fixed point}) \\ &\leq c|x - a| \leq |x - a| && (c \in (0, 1)) \\ &< \delta \end{aligned}$$

That is, $f(x) \in (a - \delta, a + \delta) = I$. Continuing for the rest of the orbit, for all $n \in \mathbb{N}$,

$$|f^n(x) - a| \leq c^n |x - a| \leq |x - a| < \delta$$

so we also have $f^n(x) \in I$.

Finally, notice that $0 \leq |f^n(x) - a| \leq c^n|x - a|$ and $c^n|x - a| \to 0$ since $c \in (0, 1)$. By the squeeze theorem, $|f^n(x) - a| \to 0$. □

Theorem 3.1.7 (repelling fixed point theorem)
Suppose a is a repelling fixed point for $f(x)$. Then, there exists an open interval I containing a such that for all $x \in I$, $x \neq a$, there exists $n \in \mathbb{N}$ such that $f^n(x) \notin I$.

Proof. Say $|f'(a)| > c > 1$. Then, as above, there exists a δ such that

$$\lim_{x \to a} \frac{|f(x) - f(a)|}{x - a} > c \implies |f(x) - f(a)| \geq c|x - a|$$

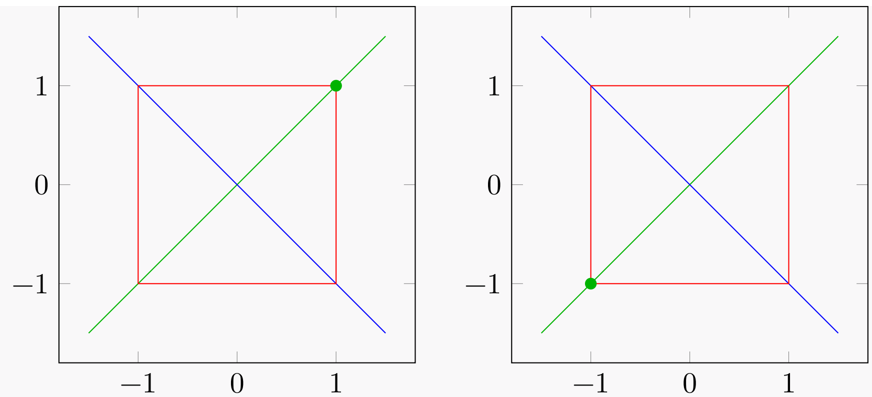
for all $x \in I := (a - \delta, a + \delta)$.

Since a is a fixed point, $|f(x) - f(a)| = |f(x) - a|$. Suppose for a contradiction that for all n, $f^n(x) \in I$. But since $c > 1$, $|f(n) - a| \geq c^n|x - a| \to \infty$. That is, δ must be arbitrarily large, which it is not. □

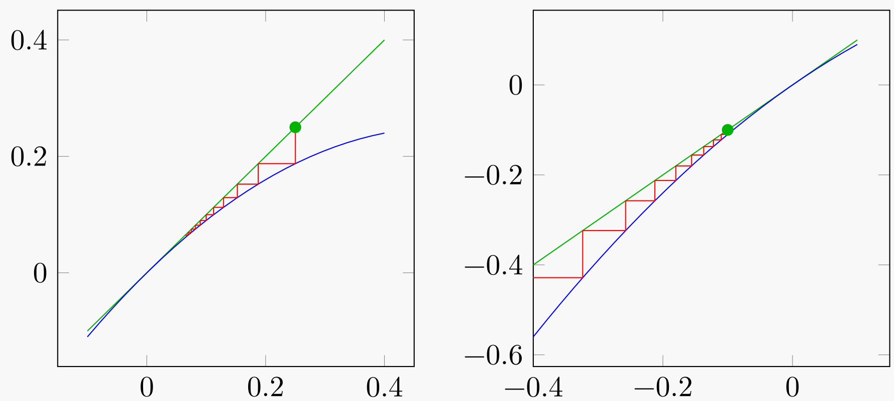
3.2 Neutral fixed points

Neutral fixed points can exhibit a lot of different behaviours.

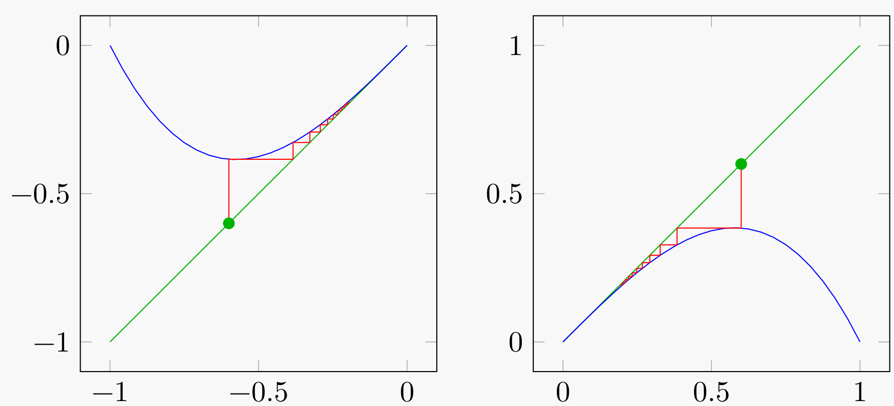
Example 3.2.1. For $f(x) = -x$, 0 is a fixed point with $|f'(0)| = 1$. The orbit bounces:



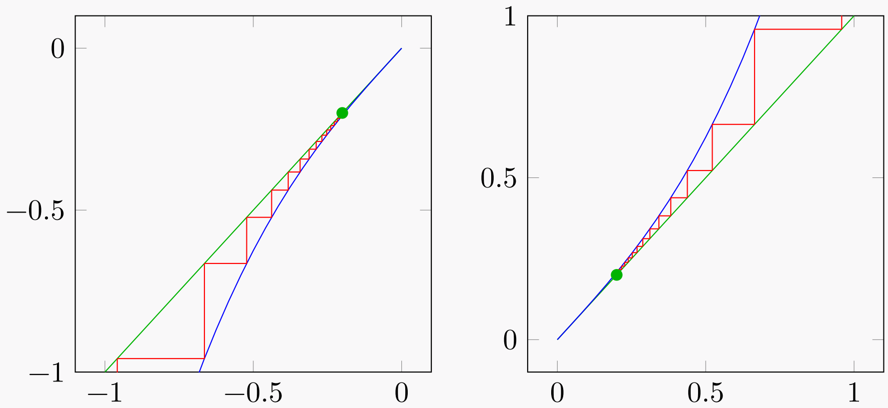
Example 3.2.2. For $f(x) = x - x^2$, $|f'(1)| = 1$ is a neutral fixed point. It is attracting from the right and repelling from the left:



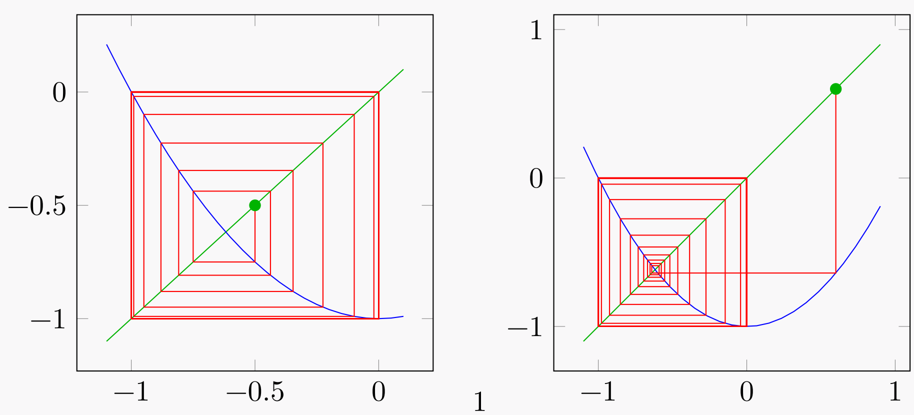
Example 3.2.3. For $f(x) = x - x^3$, $|f'(0)| = 1$ is a neutral fixed point. It is <u>weakly attracting</u>, attracting but too slowly.



Example 3.2.4. For $f(x) = x + x^3$, $|f'(0)| = 1$ is a neutral fixed point. It is weakly repelling, repelling but too slowly:



Example 3.2.5. Consider $f(x) = x^2 - 1$. The orbit at $a = 0$ is periodic $(0, -1, 0, -1, \dots)$ with period 2. Near 0, the orbit tends to the $(0, -1)$-cycle:



We will call 0 an attracting periodic point because 0 is an attracting point of $f^2(x)$.

↑ *Lectures 5 and 6 adapted from Rosie* ↑

Lecture 7
Jan 22

Definition 3.2.6

Let a be a periodic point for $f(x)$ with period n.

We say a is an attracting/repelling/neutral periodic point if a is an attracting/repelling/neutral fixed point of f^n

Finding a closed form expression for something like $f^{10}(x)$ is a nightmare, so we need a better way.

Proposition 3.2.7

Let $f(x)$ be a differentiable function. Then, $(f^n)'(x) = f'(x) \cdot f'(f(x)) \cdots f'(f^{n-1}(x))$.

Proof. Proceed by induction on n.

If $n = 1$, we have $f'(x) = f'(x)$ and we are done.

Suppose $(f^n)'(x) = \prod_{k=0}^{n-1} f'(f^k(x))$ for some $n \geq 1$. Consider f^{n+1}:

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{n+1}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(f^n(x)) = f'(f^n(x)) \cdot (f^n)'(x)$$

by the chain rule. Then,

$$\begin{aligned} (f^{n+1})'(x) &= f'(f^n(x)) \cdot (f^n)'(x) \\ &= f'(f^n(x)) \cdot \prod_{k=0}^{n-1} f'(f^k(x)) \\ &= \prod_{k=0}^{n} f'(f^k(x)) \end{aligned}$$

completing the proof. □

Example 3.2.8. Analyze the periodic point $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$, $a = 0$

Solution. The orbit is $(0, 1, 2, 0, 1, 2, \dots)$ with period 3.

We have $f'(x) = -3x + \frac{5}{2}$. Then, $(f^3)'(0) = f'(0)f'(1)f'(2) = (-\frac{7}{2})(-\frac{1}{2})(\frac{5}{2}) = \frac{35}{8} > 1$.

Therefore, the point is repelling. □

Chapter 4

Bifurcations

In general, bifurcation theory is the study of how a family of curves can change when a defining parameter is changed.

4.1 Quadratic family

Consider the quadratic family:

$$Q_C(x) = x^2 + C$$

defined by the parameter $C \in \mathbb{R}$.

Problem 4.1.1

How does the behaviour (fixed points, orbits, etc.) of Q_C change based on C?

First, we can find the fixed points (if they exist) by solving

$$Q_C(x) = x \iff x^2 - x + C = 0 \iff x = \frac{1 \pm \sqrt{1-4C}}{2}$$

and note that $Q_C(x)$ has 2 fixed points when $C < \frac{1}{4}$, 1 fixed point when $C = \frac{1}{4}$, and no fixed points when $C > \frac{1}{4}$.

Suppose $C > \frac{1}{4}$. Then, we must have $Q_C^n(x) \to \infty$ for all x.

Instead, if $C = \frac{1}{4}$, $Q_C(x)$ has the unique fixed point $p = \frac{1}{2}$. Since $Q_C'(x) = 2x$ and $Q_C'(p) = 1$, this is a neutral fixed point. In fact, it attracts to one side and repels from the other.

Finally, if $C < \frac{1}{4}$, $Q_C(x)$ has two fixed points $p_+ = \frac{1+\sqrt{1-4C}}{2}$ and $p_- = \frac{1-\sqrt{1-4C}}{2}$. Then, $Q_C'(p_+) =$

$1 + \sqrt{1-4C} > 1$ is repelling. Next,

$$\begin{aligned} & -1 < Q_C'(p_-) < 1 \\ \iff & -1 < 1 - \sqrt{1-4C} < 1 \\ \iff & -2 < -\sqrt{1-4C} < 0 \\ \iff & 0 < \sqrt{1-4C} < 2 \\ \iff & -\frac{3}{4} < C < \frac{1}{4} \end{aligned}$$

and in fact if $C < -\frac{3}{4}$, $Q_C'(p_-) < -1$ and if $C = -\frac{3}{4}$, $Q_C'(p_-) = -1$.

Theorem 4.1.2

For the family

$$Q_C(x) = x^2 + C,$$

depending on C:

1. All orbits tend to ∞ if $C > \frac{1}{4}$.
2. When $C = \frac{1}{4}$, $Q_C(x)$ has a unique fixed point $\frac{1}{2}$ and it is neutral.
3. If $C < \frac{1}{4}$, $Q_C(x)$ has two fixed points p_+ and p_-. The point p_+ is repelling. Moreover,
   (a) if $-\frac{3}{4} < C < \frac{1}{4}$, p_- is attracting;
   (b) if $C = -\frac{3}{4}$, p_- is neutral; and
   (c) if $C < -\frac{3}{4}$, p_- is repelling.

List of Named Results

1.2.5 Proposition (convergence implies boundedness) 5
1.2.7 Proposition (limit laws) 6
1.2.10 Theorem (completeness of $\mathbb{R}$) 7
1.3.6 Theorem (Banach contraction mapping theorem) 8

3.1.5 Theorem (attracting fixed point theorem) 16
3.1.7 Theorem (repelling fixed point theorem) 17

Index of Defined Terms

cobweb plot, 11
contraction, 8

doubling function, 4

fixed point, 3
  attracting, 16
    weakly, 18
  neutral, 16
  repelling, 16
    weakly, 19
function
  continuity, 7

interval
  closure, 8
iteration, 2

non-isolated, 16

orbit, 2
  constant, 3
  eventually periodic, 4
  periodic, 4

period, 4
periodic point, 4
  attracting, 19
  neutral, 19
  repelling, 19
periodicity, 4

quadratic family, 21

sequence
  bounded, 5
  Cauchy, 6
  convergence, 5
strongly-Cauchy, 9