MATH 135 Fall 2020: Extra Practice 1

Warm-Up Exercises

WE01. Determine if the following quantified statements are true or false. No justification is needed.

(a) $\forall x \in \mathbb{R}, \sin^2 x + \cos^2 x = 1$
True, by the Pythagorean identity.

(b) $\exists y \in \mathbb{Z}, 6y - 3 = 28$
False, since $6y - 3 = 28 \implies 6y = 25 \implies y = \frac{25}{6}$, which is undefined in $\mathbb{Z}$.

(c) $\forall p \in \mathbb{Q}, \exists q \in \mathbb{Z}, |p - q| \leq 1$
True, select $q = \lfloor p \rfloor$.

Recommended Problems

RP01. Which of the following are statements? If it is a statement, determine if it is true or false. No justification is needed.

(a) $3 \leq \pi$
Statement, true.

(b) $2x - 3 \geq -1$
Not a statement, depends on x.

(c) $x^2 - y^3 = 1$
Not a statement, depends on x and y.

(d) N is a perfect square.
Not a statement, depends on N.

(e) $x^2 + 5x - 2$
Not a statement or an open sentence.

(f) $x \leq x + 1$
Not a statement, depends on x.

(g) There is a largest real number.
Statement, false.

(h) There is a smallest positive number.
Statement, true.

(i) Every real number is either positive or negative.
Statement, false.

(j) Some triangles are right triangles.
Statement, true.

RP02. For each of the following statements, identify the four parts of the quantified statement (quantifier, variables, domain, and open sentence). Next, express the statement in symbolic form using as few words as possible and then write down the negation of the statement (when possible, without using any negative words such as "not" or the $\neg$ symbol, but negative math symbols like $\neq$ are okay). Finally, determine if the original statement is true or false. No justification is needed.

(a) The equation $x^2 + 2x - 3 = 0$ has a real solution.

$$\exists x \in \mathbb{R}, x^2 + 2x - 3 = 0$$

Quantifier: existential; variable: x; domain: $\mathbb{R}$; open sentence: $x^2 + 2x - 3 = 0$. Negation:

$$\forall x \in \mathbb{R}, x^2 + 2x - 3 \neq 0$$

The statement is *true*.

(b) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x \neq y \implies x^2 + y^2 > 0$$

Quantifier: universal; variable: x; domain: $\mathbb{R}$; open sentence: $x \neq y \Rightarrow x^2 + y^2 > 0$. Negation:

$$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x \neq y \wedge x^2 + y^2 \leq 0$$

The statement is *true*.

(c) For every even integer a and odd integer b, a rational number c can always be found such that $a < c < b$ or $b < c < a$.

$$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, \exists c \in \mathbb{Q}, \left(\frac{a}{2} \in \mathbb{Z} \wedge \frac{b-1}{2} \in \mathbb{Z}\right) \implies (a < c < b \vee b < c < a)$$

Quantifier: universal/existential; variables: a, b, c; domain: $\mathbb{Z}$, $\mathbb{Q}$; open sentence: $(\frac{a}{2} \in \mathbb{Z} \wedge \frac{b-1}{2} \in \mathbb{Z}) \Rightarrow (a < c < b \vee b < c < a)$. Negation:

$$\exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}, \forall c \in \mathbb{Q}, \left(\frac{a}{2} \in \mathbb{Z} \wedge \frac{b-1}{2} \in \mathbb{Z}\right) \wedge ((c \leq a \vee c \geq b) \wedge (c \leq b \vee c \geq a))$$

The statement is *true*.

(d) There is a perfect square which is also a perfect cube.

$$\exists n \in \mathbb{Z}, \exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, n = x^2 = y^3$$

Quantifier: existential; variables: n, x, y; domain: $\mathbb{Z}$; open sentence: $n = x^2 = y^3$. Negation:

$$\forall n \in \mathbb{Z}, \forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, n \neq x^2 \wedge n \neq y^2$$

The statement is *true*.

RP03. Negate the following statements without using words or the $\neg$ symbol. For each statement determine whether it or its negation is true.

(a) $\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, 3a = b$: false
$\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, 3a \neq b$: true

(b) $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, 3a = b$: true
$\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, 3a \neq b$: false

(c) $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \exists c \in \mathbb{R}, \frac{a}{c} = b$: true
$\exists a \in \mathbb{R}, \exists b \in \mathbb{R}, \forall c \in \mathbb{R}, \frac{a}{c} \neq b$: false

RP04. Express the following statement symbolically without using any words: *Every integer is a perfect square.*

$$\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, n = m^2$$