MATH 135 Fall 2020: Extra Practice 10

Warm-Up Exercises

WE01. Express $\dfrac{2-i}{3+4i}$ in standard form.

Solution. Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{2-i}{3+4i} = \frac{(2-i)(3-4i)}{9+16} = \frac{2-11i}{25} = \frac{2}{25} - \frac{11}{25}i \qquad \square$$

WE02. Write $x = \dfrac{9+i}{5-4i}$ in polar form, $r(\cos\theta + i\sin\theta)$, with $0 \le \theta < 2\pi$.

Solution. We express first in standard form by multiplying through the conjugate:

$$\frac{9+i}{5-4i} = \frac{(9+i)(5+4i)}{41} = \frac{41+41i}{41} = 1+i$$

We can geometrically interpret this as $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$. $\square$

WE03. Write $(\sqrt{3}+i)^4$ in standard form.

Solution. We first place the quantity within the brackets in polar form. By inspection, this is $2\operatorname{cis}\frac{\pi}{6}$. Now, applying DMT, we have $(2\operatorname{cis}\frac{\pi}{6})^4 = 2^4\operatorname{cis}^4\frac{\pi}{6} = 16\operatorname{cis}\frac{2\pi}{3}$.

Expressing in standard form, $16(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = 16(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -8 + 8\sqrt{3}i$ $\square$

WE04. Find all $z \in \mathbb{C}$ such that $z^5 = 1$ and plot the solutions in the complex plane. (You may state values in polar form.)

Solution. Note that $1 = 1\operatorname{cis}0$. Applying the CRNT, we have that the five roots are given by $\sqrt[5]{1}\operatorname{cis}\left(\frac{2k\pi}{n}\right)$ for $k = 0,1,2,3,4$. These values are $\{1, \operatorname{cis}\frac{\pi}{5}, \operatorname{cis}\frac{4\pi}{5}, \operatorname{cis}\frac{6\pi}{5}, \operatorname{cis}\frac{8\pi}{5}\}$. I am too lazy to learn `tikz` to draw the diagram. $\square$

WE05. Find all $z \in \mathbb{C}$ such that $z^2 = \dfrac{1+i}{1-i}$.

Solution. Simplifying the fraction on the right-hand side, $\frac{(1+i)(1+i)}{2} = \frac{1+2i-1}{2} = i$. On the complex plane, $i = 1\operatorname{cis}\frac{\pi}{2}$. Then, by CRNT, the solutions are $\operatorname{cis}\frac{\pi}{4}$ and $\operatorname{cis}\frac{5\pi}{4}$. Evaluating to get standard form, we have $z = \pm(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)$. $\square$

Recommended Problems

RP01. Express the following complex numbers in standard form.

(a) $\dfrac{(\sqrt{2}-i)^2}{(\sqrt{2}+i)(1-\sqrt{2}i)}$

Solution. Multiply through conjugates of the denominator:

$$\begin{aligned}\frac{(\sqrt{2}-i)^2}{(\sqrt{2}+i)(1-\sqrt{2}i)} &= \frac{(1-2\sqrt{2}i)(\sqrt{2}-i)(1+\sqrt{2}i)}{(3)(3)} \\ &= -\frac{(5-\sqrt{2}i)(\sqrt{2}-i)}{9} \\ &= -\frac{4\sqrt{2}-7i}{9} \\ &= -\frac{4\sqrt{2}}{9}+\frac{7}{9}i\end{aligned}$$

□

(b) $(\sqrt{5}-i\sqrt{3})^4$

Solution. Let $z=\sqrt{5}-i\sqrt{3}$. We have $z^2=5-2\sqrt{15}i-3=2-2\sqrt{15}i$. Finally, $z^4=(z^2)^2=4-8\sqrt{15}i-60=-56-8\sqrt{15}i$. □

RP02. Prove all of the Properties of Complex Arithmetic that were not proved in the notes or in class.

Proof. Let $u=a+bi$, $v=c+di$, and $z=f+gi$ be complex numbers. We must show the Properties of Complex Arithmetic, i.e., that

(a) Complex addition is associative.

First, $u+v=(a+c)+(b+d)i$ and $(u+v)+z=((a+c)+f)+((b+d)+g)i$. Then, $v+z=(c+f)+(d+g)i$, so $u+(v+z)=(a+(c+f))+(b+(d+g))i$. The result follows by the associativity of real addition.

(b) Complex addition is commutative.

We have $u+v=(a+c)+(b+d)i=(c+a)+(d+b)i=v+u$ by the commutativity of real addition.

(c) The complex additive identity is $0=0+0i$. (Example 3, p. 159)

(d) A complex additive inverse $-z$ exists. (Example 3, p. 159)

(e) Complex multiplication is associative.

By definition, $uv=(ac-bd)+(ad+bc)i$, so we have

$$(uv)w=((ac-bd)f-(ad+bc)g)+((ac-bd)g+(ad+bc)f)i$$

We also have $vw = (cf - dg) + (cg + df)i$ and by extension

$$\begin{aligned} u(vw) &= (a(cf - dg) - b(cg + df)) + (a(cg + df) + b(cf - dg))i \\ &= (acf - adg - bcg - bdf) + (acg + adf + bcf - bdg)i \\ &= (acf - bdf - adg - bcg) + (acg - bdg + adf + bcf)i \\ &= ((ac - bd)f - (ad + bc)g) + ((ac - bd)g + (ad + bc)f)i \\ &= (uv)w \end{aligned}$$

as desired.

(f) Complex multiplication is commutative.

Again, $uv = (ac-bd)+(ad+bc)i$ and $vu = (ca-db)+(cb+da)i$. The result follows from the commutativity of real multiplication and addition.

(g) The complex multiplicative identity is $1 = 1 + 0i$. (Example 3, p. 159)

(h) A complex multiplicative inverse z^{-1} exists iff $z \neq 0$. (Proposition 1, p. 159)

(i) Complex multiplication distributes over addition.

We have $u + v = (a + c) + (b + d)i$. Then,

$$z(u + v) = (f(a + c) - g(b + d)) + (f(b + d) + g(a + c))i$$

Now, $zu = (fa - gb) + (fb + ga)i$ and $zv = (fc - gd) + (fd + gc)i$, so by definition,

$$\begin{aligned} zu + zv &= ((fa - gb) + (fc - gd)) + ((fb + ga) + (fd + gc))i \\ &= (fa + fc - gb - gd) + (fb + fd + ga + gc)i \\ &= (f(a + c) - g(b + d)) + (f(b + d) + g(a + c))i \\ &= z(u + v) \end{aligned}$$

completing the proof. □

RP03. Let $n \in \mathbb{N}$. Prove that if $n \equiv 1 \pmod 4$, then $i^n = i$.

Proof. Let n be a natural number congruent to 1 modulo 4. Then, we may write $n = 4k+1$ for some integer k. Notice that $i^4 = (i^2)^2 = (-1)^2 = 1$.

Therefore, $i^{4k+1} = (i^4)^k i^1 = (1)^k i = i$, as desired. □

RP04. Find all $z \in \mathbb{C}$ which satisfy

(a) $z^2 + 2z + 1 = 0$

Solution. Factor: $z^2 + 2z + 1 = (z + 1)^2$ so $z = -1 + 0i$ (by RP06) □

(b) $z^2 + 2\bar{z} + 1 = 0$

Solution. Let $z = a + bi$ so $\bar{z} = a - bi$ for two real numbers a and b. Then,

$$\begin{aligned} 0 &= z^2 + 2\bar{z} + 1 \\ 0 &= (a + bi)^2 + 2(a - bi) + 1 \\ 0 &= (a^2 + 2a - b^2 + 1) + (2ab - 2b)i \end{aligned}$$

which is true if and only if both $a^2 + 2a - b^2 + 1 = 0$ and $2ab - 2b = 0$.

The second equation implies $2ab = 2b$ so $a = 1$ or $b = 0$.

If $a = 1$ then $a^2 + 2a - b^2 + 1 = 4 - b^2 = 0$, so $b = \pm 2$.

If $b = 0$, then $a^2 + 2a + 1 = (a + 1)^2 = 0$, so $a = -1$.

Therefore, the solutions are $-1 + 0i$, $1 + 2i$, and $1 - 2i$. $\square$

(c) $z^2 = \dfrac{1+i}{1-i}$

Solution. Simplify: $z^2 = \frac{(1+i)^2}{2} = \frac{2i}{2} = i$. The square roots of i are $\pm(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)$. $\square$

RP05.

(a) Find all $w \in \mathbb{C}$ satisfying $w^2 = -15 + 8i$.

Solution. We rewrite $w = a+bi$ for some reals a and b. Then, $(a+bi)^2 = (a^2-b^2) + (2ab)i = -15 + 8i$. Equating real and complex parts, $a^2 - b^2 = -15$ and $2ab = 8$.

Now, $|w^2| = |ww| = |w||w| = |w|^2$ by PM4. Then, $a^2 + b^2 = \sqrt{(-15)^2 + (8)^2} = 17$. Solving the system in a^2 and b^2, $a^2 = 1$ and $b^2 = 16$.

Therefore, $a = \pm 1$ and $b = \pm 4$. To satisfy $2ab = 8$, we must have $z = \pm(1 + 4i)$. $\square$

(b) Find all $z \in \mathbb{C}$ satisfying $z^2 - (3 + 2i)z + 5 + i = 0$.

Solution. We apply the quadratic formula. The discriminant is a solution to $w^2 = (3 + 2i)^2 - 4(1)(5 + i) = (5 + 12i) - (20 + 4i) = -15 + 8i$. From above, a solution is $w = 1 + 4i$. Therefore, the solutions are $z = \frac{(3+2i)\pm(1+4i)}{2(1)}$.

The first is $z = \frac{(3+2i)+(1+4i)}{2} = 2 + 3i$ and the second is $z = \frac{(3+2i)-(1+4i)}{2} = 1 - i$. $\square$

RP06. Let $z, w \in \mathbb{C}$. Prove that if $zw = 0$ then $z = 0$ or $w = 0$.

Proof. Let z and w be complex numbers such that $zw = 0$. Suppose for a contradiction that both z and w are non-zero. Then, by PM1, $|z| \neq 0$ and $|w| \neq 0$. However, by PM4, $|zw| = |z||w| \neq 0$, which is a contradiction, since $zw = 0$.

Therefore, z or w is zero. $\square$

RP07. Let $a, b, c \in \mathbb{C}$. Prove: if $|a| = |b| = |c| = 1$, then $\overline{a + b + c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Proof. First, consider some arbitrary complex number $z = a + bi$ with modulus 1. By definition, $a^2 + b^2 = 1^2 = 1$. Then, $z^{-1} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{1} = a - bi = \bar{z}$

Let a, b, and c be complex numbers with modulus 1. From above, $a^{-1} = \bar{a}$, $b^{-1} = \bar{b}$, and $c^{-1} = \bar{c}$. The conclusion immediately follows from PCJ2:

$$\begin{aligned}\overline{a + b + c} &= \bar{a} + \bar{b} + \bar{c} \\ &= \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\end{aligned}$$

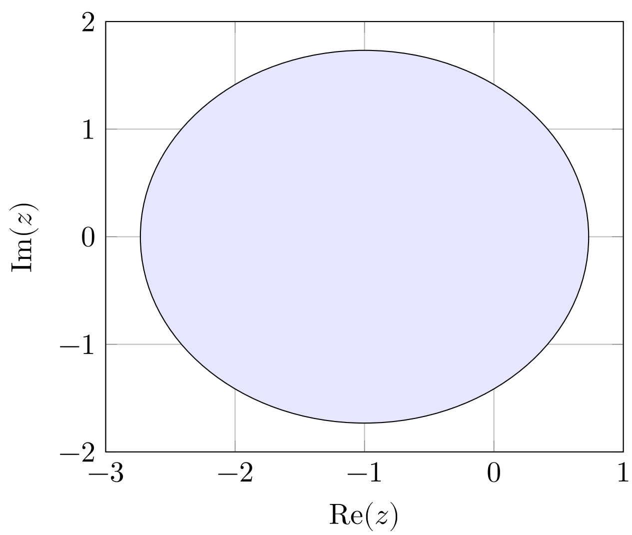
$\square$

RP08. Find all $z \in \mathbb{C}$ satisfying $z^2 = |z|^2$.

Proof. Let z be a complex number. Recall that $|z|^2 = \bar{z}z$ by PM3. Then, we have $z^2 = \bar{z}z$ so $z = \bar{z}$, that is, $z - \bar{z} = 0$. By PCJ3, this is true if $2\,\mathrm{Im}(z)i = 0$, which means that z is purely real. Therefore, z is any purely real number. ☐

RP09. Find all $z \in \mathbb{C}$ satisfying $|z+1|^2 \leq 3$ and shade the corresponding region in the complex plane.

Solution. We write $z = a+bi$, so $|z+1|^2 = |(a+1)+bi|^2 = (\sqrt{(a+1)^2+b^2})^2 = (a+1)^2+b^2$. Then, we are shading the inside of the circle defined by $(a+1)^2 + b^2 = 3$.



This is the circle centered at $(-1, 0)$ with radius $\sqrt{3}$. ☐

RP10. Let $z, w \in \mathbb{C}$ such that $\overline{z}w \neq 1$. Prove that if $|z| = 1$ or $|w| = 1$, then $\left|\dfrac{z-w}{1-\overline{z}w}\right| = 1$.

Proof (by sooshi). Let z and w be complex numbers such that $\overline{z}w \neq 1$. Suppose that $|z| = 1$ or $|w| = 1$. If $z = w$ and $|z| = |w| = 1$, then $\overline{z}w = \overline{z}z = |z|^2 = 1$. Therefore, $z \neq w$.

Now, consider the case when $|z| = 1$. Then,

$$\left|\frac{z-w}{1-\overline{z}w}\right| = \frac{|z-w|}{|1-\overline{z}w|} = \frac{|z||z-w|}{|z||1-\overline{z}w|} = \frac{(1)|z-w|}{|z-z\overline{z}w|} = \frac{|z-w|}{|z-w|} = 1$$

Likewise, if $|w| = 1$, then

$$\left|\frac{z-w}{1-\overline{z}w}\right| = \frac{|z-w|}{|1-\overline{z}w|} = \frac{|z-w|}{|w\overline{w}-\overline{z}w|} = \frac{|z-w|}{|w||\overline{w}-\overline{z}|} = \frac{|z-w|}{|w-z|} = 1$$

since $|w-z| = |-(z-w)| = |-1||z-w| = |z-w|$, completing the proof. ☐

RP11. Show that for all complex numbers z, $|\,\mathrm{Re}(z)| + |\,\mathrm{Im}(z)| \leq \sqrt{2}|z|$.

Proof. Let $z = r\operatorname{cis}\theta$ be a complex number. Then, $|z| = r$, $\operatorname{Re}(z) = r\cos\theta$ and $\operatorname{Im}(z) = r\sin\theta$. Due to the symmetry of sine and cosine, instead of taking absolute values, we restrict without loss of generality to the first quadrant $0 \leq \theta \leq \frac{\pi}{2}$. Now,

$$\begin{aligned}
\operatorname{Re}(z) + \operatorname{Im}(z) &= r(\cos\theta + \sin\theta) \\
&= r\sqrt{2}\frac{\sqrt{2}}{2}(\cos\theta + \sin\theta) \\
&= r\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right) \\
&= r\sqrt{2}\left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right) \\
&= r\sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \\
&\leq r\sqrt{2}(1) \\
&= \sqrt{2}|z|
\end{aligned}$$

completing the proof. □

RP12. Use *De Moivre's Theorem* (DMT) to prove that $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$ for all $\theta \in \mathbb{R}$.

Proof. Let $\theta \in \mathbb{R}$ and note that by DMT, we have

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$$

so we may say that $\sin 4\theta = \operatorname{Im}((\cos\theta + i\sin\theta)^4)$. Expanding this quantity by hand,

$$\begin{aligned}
(\cos\theta + i\sin\theta)^4 &= (\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta)^2 \\
&= \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta + 4i\cos^3\theta\sin\theta - 4i\sin^3\theta\cos\theta \\
&= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + (4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta)i
\end{aligned}$$

and we have that

$$\sin 4\theta = \operatorname{Im}((\cos\theta + i\sin\theta)^4) = 4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta$$

as desired. □

RP13. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$. Show that $z = (a + bi)^n + (a - bi)^n$ is real.

Proof. Let n be a natural number and $u = a + bi$ be a complex number. Then, $\overline{u} = a - bi$. It inductively follows from PCJ4 and the associativity of multiplication that $(\overline{u})^n = \overline{u^n}$.

Now, the fact that $z = u^n + \overline{u^n}$ is real follows immediately from PCJ3. □

RP14. An *n-th root of unity* is any complex solution to $z^n = 1$. Prove that if w is an n-th root of unity, $\frac{1}{w}$ is also an n-th root of unity.

Proof. Let n be a natural number and w be an n-th root of unity, so $w^n = 1$. Knowing that $1 = \operatorname{cis} 0$, the CNRT states that $w = \operatorname{cis}(\frac{2k\pi}{n})$ for some $0 \leq k < n$.

By PM$\mathbb{C}$, notice that $w\operatorname{cis}(-\frac{2k\pi}{n}) = \operatorname{cis}(\frac{2k\pi}{n} - \frac{2k\pi}{n}) = \operatorname{cis} 0 = 1$, so $\operatorname{cis}(-\frac{2k\pi}{n})$ is the multiplicative inverse w^{-1} of w. Now, since cis is 2π-periodic, we have

$$\operatorname{cis}\left(-\frac{2k\pi}{n}\right) = \operatorname{cis}\left(2\pi - \frac{2k\pi}{n}\right) = \operatorname{cis}\left(\frac{2n\pi - 2k\pi}{n}\right) = \operatorname{cis}\left(\frac{2(n-k)\pi}{n}\right)$$

but since $0 \leq k < n$, we also have that $0 \leq n - k < n$. Therefore, by the CNRT, w^{-1} is an n-th root of unity. □

RP15. A complex number z is called a *primitive* n-th root of unity if $z^n = 1$ and $z^k \neq 1$ for all $1 \leq k \leq n-1$.

(a) For each $n = 1, 3, 5, 6$ list all the primitive n-th roots of unity.

Solution. Recall that $1^x = 1$ for any real x. Applying the CNRT, there are n n-th roots of unity, of the form

$$z = \operatorname{cis}\left(\frac{2\pi k}{n}\right)$$

for some integer $0 \leq k < n$. Note that 1 is always an n-th root of unity but only a primitive first root of unity. Therefore, we can ignore the case $k = 0$.

The only primitive 1st root of unity is 1.

The primitive 3rd roots of unity are $\operatorname{cis}\frac{2\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ and $\operatorname{cis}\frac{4\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

For this, we remain in polar form as calculating sines and cosines of fractions over 5 is *pain*. The primitive 5th roots of unity are $\operatorname{cis} 0 = 1$, $\operatorname{cis}\frac{2\pi}{5}$, $\operatorname{cis}\frac{4\pi}{5}$, $\operatorname{cis}\frac{6\pi}{5}$, and $\operatorname{cis}\frac{8\pi}{5}$.

The 6th roots of unity are $\operatorname{cis}\frac{2\pi k}{6} = \operatorname{cis}\frac{\pi k}{3}$. However, when $k = 2$, $k = 3$, and $k = 4$, these are also 2nd/3rd roots of unity. Thus, the primitive roots of unity are $\operatorname{cis}\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\operatorname{cis}\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. □

(b) Let z be a primitive n-th root of unity. Prove the following statements:

i. For any $j \in \mathbb{Z}$, $z^j = 1$ if and only if $n \mid j$.

Proof. Let n be a natural number, j be an integer, and z be a primitive n-th root of unity so $z^n = 1$. Proceed by mutual implication.

($\Rightarrow$) Suppose $z^j = 1$. By the Division Algorithm, $j = qn + r$ for integers q and $0 \leq r < n$. Then, $1 = z^j = z^{qn+r} = z^{qn}z^r = (z^n)^q z^r = 1^q z^r = z^r$.

If $r = 0$, then $j = qn$ and $j \mid n$. Otherwise, we have $1 \leq r \leq n-1$ and $z^r = 1$, which is a contradiction to the fact that z is a primitive n-th root of unity.

Therefore, $r = 0$ and $j \mid n$.

($\Leftarrow$) If $n \mid j$ and $j = nk$ for an integer k, then $z^j = z^{nk} = (z^n)^k = 1^k = 1$. □

ii. For any $m \in \mathbb{Z}$, if $\gcd(m, n) = 1$, then z^m is a primitive n-th root of unity.

Proof (new and improved by sooshi). Let z be a primitive n-th root of unity and m an integer coprime to n.

Suppose for a contradiction that z^m is a k-th root of unity for some $1 \leq k < n$. Then, $(z^m)^k = z^{mk} = 1$. From above, this implies that $n \mid mk$ and by CAD, $n \mid k$. However, BBD gives that $n \leq k$, which is a contradiction.

Therefore, z^m is a primitive n-th root of unity. □

RP16. Let u and v be fixed complex numbers. Let ω be a non-real cube root of unity. For each $k \in \mathbb{Z}$, define $y_k \in \mathbb{C}$ by the formula

$$y_k = \omega^k u + \omega^{-k} v$$

(a) Compute y_1, y_2, and y_3 in terms of u, v, and ω.

Solution. From RP15(a), the only real cube root of unity is 1, so $\omega \neq 1$. In fact, $\omega = \operatorname{cis} \frac{n\pi}{3}$ for either $n = 2$ or $n = 4$.

If $n = 2$, then $\omega^{-1} = \operatorname{cis} \frac{-2\pi}{3} = \operatorname{cis} \frac{4\pi}{3}$. If $n = 4$, then $\omega^{-1} = \operatorname{cis} \frac{-4\pi}{3} = \operatorname{cis} \frac{2\pi}{3}$.

However, using the standard form from RP15(a), $\operatorname{cis} \frac{2\pi}{3} = \overline{\operatorname{cis} \frac{4\pi}{3}}$. Therefore, $\omega^{-1} = \overline{\omega}$.

Now, $y_1 = \omega u + \overline{\omega} v$, $y_2 = \omega^2 u + \overline{\omega}^2 v$, and $y_3 = \omega^3 u + \overline{\omega}^3 v = u + v$. □

(b) Show that $y_k = y_{k+3}$ for any $k \in \mathbb{Z}$.

Proof. Let k be an integer. Then, knowing that both ω and $\overline{\omega}$ are cube roots of unity,

$$\begin{aligned} y_{k+3} &= \omega^{k+3} u + \overline{\omega}^{k+3} v \\ &= \omega^k \omega^3 u + \overline{\omega}^k \overline{\omega}^3 v \\ &= \omega^k u + \overline{\omega}^k v \\ &= y_k \end{aligned}$$

completing the proof. □

(c) Show that for any $k \in \mathbb{Z}$,

$$y_k - y_{k+1} = \omega^k (1 - \omega)(u - \omega^{k-1} v)$$

Proof. Let k be an integer. Expand the right-hand side:

$$\begin{aligned} \omega^k (1-\omega)(u - \omega^{k-1} v) &= (\omega^k - \omega^{k+1})(u - \omega^{k-1} v) \\ &= \omega^k u - \omega^{2k+1} v - \omega^{k+1} u + \omega^{2k+2} v \\ &= (\omega^k u + \omega^{2k+2} v) - (\omega^{k+1} u + \omega^{2k+1} v) \end{aligned}$$

To simplify, we show that $\omega^{2k+2} = \omega^{-k}$. Equivalently, $\omega^{2k+2}\omega^k = \omega^{3k+2} = 1$. Let $j = k + 1$. Then,

$$\omega^{3k+2} = \omega^{3(j-1)+2} = \omega^{3j-1} = (\omega^3)^j \omega^{-1} = 1^j \omega^{-1} = \omega^{-1}$$

as desired. Now, we have $\omega^{2k+2} = \omega^{-k}$ and $\omega^{2k+1} = \omega^{-(k+1)}$ so

$$\begin{aligned} \omega^k (1-\omega)(u - \omega^{k-1} v) &= (\omega^k u + \omega^{2k+2} v) - (\omega^{k+1} u + \omega^{2k+1} v) \\ &= (\omega^k u + \omega^{-k} v) - (\omega^{k+1} u + \omega^{-(k+1)} v) \\ &= y_k - y_{k+1} \end{aligned}$$

□

Challenges

C01. Let $z, w \in \mathbb{C}$.

(a) Prove that $|z + w| \leq |z| + |w|$.

Proof. This is the Triangle Inequality, for which a geometric proof is provided in Chapter 10.3. In short, for complex numbers $z = a + bi$ and $w = c + di$, we consider a triangle $\triangle OZW$ with points $O(0, 0)$, $Z(a, b)$, and $W(c, d)$ in the complex plane. Then, $|z| = \ell_{OZ}$, $|w| = \ell_{OW}$, and $|z + w| = \ell_{ZW}$. The length of one side of a triangle cannot exceed the sum of the lengths of the other two sides.

Equivalently, $\ell_{ZW} \leq \ell_{OZ} + \ell_{OW}$. □

(b) Prove that $||z| - |w|| \leq |z - w| \leq |z| + |w|$.

Proof. Let z and w be complex numbers. We prove the inequalities separately.

We apply the Triangle Inequality with z and $-w$. Then, $|z + (-w)| \leq |z| + |-w|$ but $|-w| = |-1||w| = |w|$ by PM4, so we have $|z - w| \leq |z| + |w|$.

Now, notice that $|z| = |(z - w) + w| \leq |z - w| + |w|$ so $|z| - |w| \leq |z - w|$.

Likewise, $|w| = |(w - z) + z| \leq |w - z| + |z|$ so $|z| - |w| \geq -|w - z|$.

Like the absolute value in $\mathbb{R}$, we have by PM4 $|w-z| = |-1||z-w| = 1|z-w| = |z-w|$, so if we combine the above two inequalities, we have $||z| - |w|| \leq |z - w|$.

Equivalently, using the same triangle from above, this follows from the fact that any one side of a triangle is longer than the difference of the other two sides. □

C02. Let $a, b, c \in \mathbb{C}$. Show that if $\dfrac{b-a}{a-c} = \dfrac{a-c}{c-b}$ then $|b - a| = |a - c| = |c - b|$.

C03. Let $n \geq 2$ be an integer. Prove that

$$\sum_{k=0}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = 0 = \sum_{k=0}^{n-1} \sin\left(\frac{2k\pi}{n}\right)$$

Proof (with help from Ainsley, Kenson, Mabel). Let $n \neq 1$ be a natural number. Then, we have that the n-th roots of unity are given by

$$\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

for $k = 0, 1, 2, \ldots, n - 1$. Let z be the sum of the n-th roots of unity. Then,

$$z = \sum_{k=0}^{n-1} \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)$$

The conclusion can equivalently be stated as that $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) = 0$. The only complex number that satisfies this is $z = 0$.

Now, let $a = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$, the root of unity with $k = 1$. Then, we have that each root of unity is given by a^j for $j = 1, 2, \ldots, n$. Since $n \neq 1$, $a = \operatorname{cis}\frac{2\pi}{n} \neq 1$ and $z = 1 + a + a^2 + \cdots + a^{n-1}$.

Recall that the polynomial $a^n - 1$ for $n \geq 2$ factors as $(a-1)(a^{n-1} + a^{n-2} + \cdots + a^2 + a + 1)$. It follows that $a^n - 1 = 1 - 1 = 0$ and $0 = (a - 1)z$ so, from above, $a \neq 1$ so $z = 0$. □