PMATH 370 Winter 2024:
Lecture Notes

Lecture notes taken, unless otherwise specified, by myself during the Winter 2024 offering of PMATH 370, taught by Blake Madill.

Lectures

Chapter 1

Iteration and Orbits

1.1 Orbits

Lecture 1
Jan 8

Definition 1.1.1 (iteration)
Let $f : A \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$. For $a \in A$ we may iterate the function at a:

$$x_1 = a, x_2 = f(a), x_3 = \underbrace{f(f(a))}_{f^2(a)}, \dots, x_i = f^{i-1}(a), \dots .$$

The sequence $(x_n)_{n=1}^{\infty}$ is the orbit of a under f (abbreviated (x_n) without limits).

Example 1.1.2. Let $f(x) = x^4 + 2x^2 - 2$, $a = -1$. What is the orbit of a under f?

Solution. $a = -1$, $f(a) = 1$, $f(f(a)) = f(1) = 1$, so we have $-1, 1, 1, 1, \dots$. We call this eventually constant. ☐

Example 1.1.3. Let $f(x) = -x^2 - x + 1$, $a = 0$. What is the orbit of a under f?

Solution. Calculate: $0, 1, -1, 1, -1, 1, \dots$. We call this eventually periodic (with period 2). ☐

Example 1.1.4. Let $f(x) = x^3 - 3x + 1$, $a = 1$. What is the orbit of a under f?

Solution. Calculate the first few terms: $1, -1, 3, 19, \dots$ (too big). This is a divergence to infinity. ☐

Example 1.1.5. Let $f(x) = x^2 + 2x$, $a = -0.5$. What is the orbit of a under f?

Solution. Calculate: $-0.5, -0.75, -0.9375, -0.9961 \dots$ and we make an educated guess that this converges to -1 since $f(-1) = -1$, a fixed point. ☐

Example 1.1.6. Let $f(x) = x^3 - 3x$, $a = 0.75$. What is the orbit of a under f?

Solution. Calculate: $0.75, -1.828, -0.625, 1.631, -0.552, \dots$. There is no clear pattern, so we call this chaotic. In fact, the orbit is dense in a neighbourhood of 0. □

We can start to formalize the examples.

Definition 1.1.7 (fixed point)

Let $f : A \to \mathbb{R}$ such that $f(A) \subseteq A$. A point $a \in A$ is fixed if $f(a) = a$.

Then, the orbit of a under f is $(a, a, a, \dots)$ which is constant.

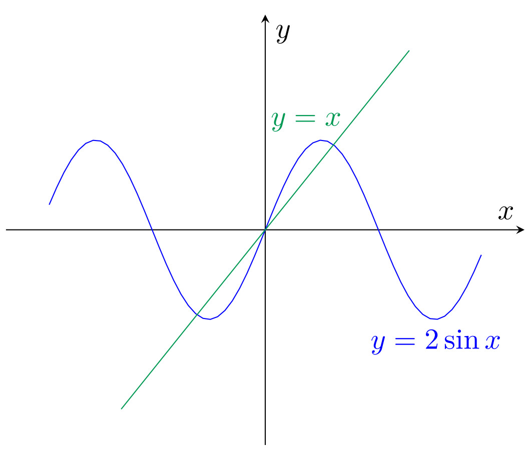
Example 1.1.8. Find all fixed points of $f(x) = x^2 + x - 4$.

Solution. We find points where $f(x) = x$, i.e., $x^2 + x - 4 = x$.

$$x^2 + x - 4 = x \iff x^2 = 4 \iff x = \pm 2$$ □

Example 1.1.9. How many fixed points does $f(x) = 2\sin x$ have?

Solution. Consider where the curve $y = 2\sin x$ meets $y = x$:



We can see there are three fixed points. □

Example 1.1.10. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Proof. We must show there is a solution to $x^4 - 3x + 1 \iff x^4 - 4x + 1 = 0$. Let $g(x) = x^4 - 4x + 1$. Since $g(x)$ is continuous, $g(0) = 1 > 0$, and $g(1) = -2 < 0$, by the Intermediate Value Theorem, there must exist a root of g on the interval $(0, 1)$. That is, a fixed point of f. □

Definition 1.1.11 (periodicity)

Let $f : A \to \mathbb{R}, f(A) \subseteq A$.

1. A point $a \in A$ is periodic for f if its orbit is periodic. An orbit is periodic if for some $n \in \mathbb{N}$, $f^n(a) = a$. The smallest n is the period of (the orbit of) a.
2. An orbit (of a point) is eventually periodic if there exists $n < m$ such that $f^n(a) = f^m(a)$. The smallest difference $m - n$ is the period of the orbit.

Lecture 2
Jan 10

Definition 1.1.12 (doubling function)

$D : [0,1) \to [0,1) : x \mapsto 2x - \lfloor 2x \rfloor$ returns the fractional part of $2x$.

Example 1.1.13. $D(0.4) = 0.8$, $D(0.6) = 0.2$, $D(0.8) = 0.6$, $D(0.5) = 0$.

This is a nice function that gives lots of periodic orbits for funsies.

Example 1.1.14. Find the orbit of $a = \frac{1}{5}$ under D.

Solution. Double until we pass 1: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{8}{5} \to \frac{3}{5}, \frac{6}{5} \to \frac{1}{5}$. The period is $\left|\{\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\}\right| = 4$. □

Example 1.1.15. Find the orbit of $a = \frac{1}{20}$ under D.

Solution. Double: $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}$ and we can stop because Example 1.1.14 showed $\frac{1}{5}$ is periodic.

So this is eventually periodic with period 4. □

Problem 1.1.16

Given f and a, does $f^n(a)$ tend towards some limit L?

To solve this problem, we need to rigorously define “tend” and “limit”.

1.2 Real Analysis Review

Notation. If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers, we write $(x_n) \subseteq \mathbb{R}$.

Definition 1.2.1 (convergence of a sequence)
Let $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$.

We say (x_n) converges to x if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

Then, we write $x_n \to x$ or $\lim x_n = x$.

Example 1.2.2. Show that $\frac{1}{n} \to 0$.

Proof. Let $\varepsilon > 0$. Consider $N = \frac{2}{\varepsilon} > \frac{1}{\varepsilon}$. For $n \geq N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

Therefore, $\frac{1}{n} \to 0$. □

Example 1.2.3. Prove that $\frac{2n}{n+3} \to 2$.

Proof. Let $\varepsilon > 0$. Since we know $\frac{1}{n} \to 0$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{6}$.

For $n \geq N$,

$$\left|\frac{2n}{n+3} - 2\right| = \left|\frac{2n}{n+3} - \frac{2n+6}{n+3}\right| = \left|\frac{-6}{n+3}\right| = \frac{6}{n+3} < \frac{6}{n} \leq \frac{6}{N} < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

Therefore, $\frac{2n}{n+3} \to 2$. □

Definition 1.2.4 (bounded sequence)
A sequence (x_n) is bounded (by M) if there exists $M > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq M$.

Proposition 1.2.5 (convergence implies boundedness)
If (x_n) is convergent, then (x_n) is bounded.

Proof. Suppose $x_n \to x$. Then, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - x| < 1$.

For $n \geq N$, $|x_n| - |x| \leq |x_n - x| < 1$. That is, $|x_n| < 1 + |x|$.

Let $M = \max\{|x_1|, \dots, |x_{n-1}|, 1 + |x|\}$. Then, for both all $n < N$ and $n \geq N$, we have $|x_n| \leq M$. □

The converse is not true. Notice that $x_n = (-1)^n$ is bounded by 1 but obviously not convergent.

Proposition 1.2.6 (limit laws)

Let $x_n \to x$ and $y_n \to y$. Then:

(1) $x_n + y_n \to x + y$

(2) $x_n y_n \to xy$

Proof. (1) Let $\varepsilon > 0$. Then, since $x_n \to x$ and $y_n \to y$, there exist $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies |x_n - x| < \frac{\varepsilon}{2}$ and $n \geq N_2 \implies |y_n - y| < \frac{\varepsilon}{2}$.

For $N = \max\{N_1, N_2\}$ and $n \geq N$,

$$\begin{aligned}
|(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| \\
&\leq |x_n - x| + |y_n - y| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

That is, $x_n + y_n \to x + y$.

(2) Let $\varepsilon > 0$. Notice that:

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \tag{$*$}$$

Since x_n is bounded, there exists $M > 0$ such that $|x_n| \leq M$ for all n.

Let $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned}
n \geq N_1 &\implies |x_n - x| \leq \frac{\varepsilon}{2(|y| + 1)} \text{ and} \\
n \geq N_2 &\implies |y_n - y| < \frac{\varepsilon}{2M}.
\end{aligned}$$

Then, for $n \geq N := \max\{N_1, N_2\}$, $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by $(*)$. $\square$

List of Named Results

Index of Defined Terms