Q01. Evaluate the following integrals

(a) $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-16}}$

Solution. Let $x = 4\sec\theta$ so $\mathrm{d}x = 4\sec\theta\tan\theta\,\mathrm{d}\theta$:

$$\begin{aligned}\int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\tan^2\theta}}\,\mathrm{d}\theta &= \int \frac{\mathrm{d}\theta}{16\sec\theta}\\ &= \frac{1}{16}\int \cos\theta\,\mathrm{d}\theta\\ &= -\frac{\sin\theta}{16} + C\\ &= \frac{\sqrt{x^2-16}}{16x} + C\end{aligned}$$

□

(b) $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}\,\mathrm{d}x$ using a trigonometric substitution

Solution. Let $x = 6\sin\theta$ so $\mathrm{d}x = 6\cos\theta\,\mathrm{d}\theta$:

$$\begin{aligned}\int_0^{\pi/3} \frac{(6\sin\theta)6\cos\theta}{6\cos\theta}\,\mathrm{d}\theta &= \int_0^{\pi/3} 6\sin\theta\,\mathrm{d}\theta\\ &= -6\cos\theta\Big|_0^{\pi/3}\\ &= -3\sqrt{3}+6\end{aligned}$$

□

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,\mathrm{d}x$

Solution. Let $u = \sin x$ so $\mathrm{d}u = \cos x\,\mathrm{d}x$:

$$\int_0^1 \frac{\mathrm{d}u}{\sqrt{1+u^2}}$$

Now, let $u = \tan\theta$ so $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$:

$$\begin{aligned}\int_0^{\pi/4} \frac{\sec^2\theta\,\mathrm{d}\theta}{\sqrt{\sec^2\theta}} &= \int_0^{\pi/4} \sec\theta\,\mathrm{d}\theta\\ &= \ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4}\\ &= \ln\left|\sqrt{2}+1\right| - \ln|1+0|\\ &= \ln\left(\sqrt{2}+1\right)\end{aligned}$$

□

(d) $\displaystyle\int \frac{x^5}{\sqrt{x^2+2}}\,\mathrm{d}x$

Solution. Let $x = \sqrt{2}\tan\theta$ so $\mathrm{d}x = \sqrt{2}\sec^2\theta$:

$$\int \frac{8\tan^5\theta\sec^2\theta}{\sqrt{2}\sec\theta}\,\mathrm{d}\theta = 4\sqrt{2}\int \tan^4\theta\sec\theta\tan\theta\,\mathrm{d}\theta$$

Now let $u = \sec\theta$ so $\mathrm{d}u = \tan\theta\sec\theta\,\mathrm{d}\theta$:

$$\begin{aligned}
4\sqrt{2}\int \tan^4\theta\,\mathrm{d}u &= 4\sqrt{2}\int (u^2-1)^2\,\mathrm{d}u \\
&= 4\sqrt{2}\int u^4 - 2u^2 - 1\,\mathrm{d}u \\
&= 4\sqrt{2}\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 - u\right) + C \\
&= 4\sqrt{2}\left(\frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta - \sec\theta\right) + C \\
&= 4\sqrt{2}\left(\frac{(x^2+4)^{5/2}}{20\sqrt{2}} - \frac{2(x^2+4)^{3/2}}{6\sqrt{2}} - \frac{(x^2+4)^{1/2}}{\sqrt{2}}\right) + C \\
&= \frac{(x^2+4)^{5/2}}{5} - \frac{4(x^2+4)^{3/2}}{3} - 4(x^2+4)^{1/2} + C
\end{aligned}$$

□

(e) $\displaystyle\int_1^3 x^5 \ln x^2\,\mathrm{d}x$

Solution. Let $u = \ln x^2 = 2\ln x$ and $\mathrm{d}v = x^5\,\mathrm{d}x$. Then, $\mathrm{d}u = \frac{2}{x}\,\mathrm{d}x$ and $v = \frac{1}{6}x^6$:

$$\begin{aligned}
\left.\frac{x^6\ln x}{3}\right|_1^3 - \int_1^3 \frac{1}{3}x^5\,\mathrm{d}x &= \left.\frac{x^6\ln x}{3} - \frac{1}{18}x^6\right|_1^3 \\
&= 243\ln 3 - \frac{81}{2} + \frac{1}{18} \\
&= 243\ln 3 - \frac{364}{9}
\end{aligned}$$

□

(f) $\displaystyle\int e^{2x}\cos x\,\mathrm{d}x$

Solution. Let $u = e^{2x}$ and $\mathrm{d}v = \cos x\,\mathrm{d}x$. Then, $\mathrm{d}u = 2e^{2x}\,\mathrm{d}x$ and $v = -\sin x$:

$$\int e^{2x}\cos x\,\mathrm{d}x = -e^{2x}\sin x + 2\int e^{2x}\sin x\,\mathrm{d}x$$

If we integrate by parts again, with $u = e^{2x}$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$, we have $\mathrm{d}u = 2e^{2x}\,\mathrm{d}x$ and $v = -\cos x$:

$$\begin{aligned}
\int e^{2x}\cos x\,\mathrm{d}x &= -e^{2x}\sin x - 2e^{2x}\cos x - 4\int e^{2x}\cos x\,\mathrm{d}x \\
5\int e^{2x}\cos x\,\mathrm{d}x &= -e^{2x}\sin x - 2e^{2x}\cos x \\
\int e^{2x}\cos x\,\mathrm{d}x &= -\frac{1}{5}e^{2x}\sin x - \frac{2}{5}e^{2x}\cos x + C
\end{aligned}$$

□

(g) $\displaystyle\int_0^2 e^{2x}\cos e^x\,\mathrm{d}x$

Solution. First, let $u = e^x$, so $\mathrm{d}u = e^x\,\mathrm{d}x$:

$$\int_0^2 e^{2x}\cos e^x\,\mathrm{d}x = \int_1^{e^2} u\cos u\,\mathrm{d}u$$

Now, integrate by parts:

$$\int u \cos u \, \mathrm{d}u = u \sin u - \int \sin u \, \mathrm{d}u = u \sin u + \cos u + C$$

and evaluate at the bounds:

$$\begin{aligned} \int_1^{e^2} u \cos u \, \mathrm{d}u &= u \sin u + \cos u \Big|_1^{e^2} \\ &= e^2 \sin e^2 + \cos e^2 - \sin 1 - \cos 1 \end{aligned}$$

□

(h) $\displaystyle\int \arcsin x \, \mathrm{d}x$

Solution. Let $x = \sin u$, so $\mathrm{d}x = \cos u \, \mathrm{d}u$ and

$$\int \arcsin x \, \mathrm{d}x = \int \arcsin(\sin u) \cos u \, \mathrm{d}u = \int u \cos u \, \mathrm{d}u = \cos u + u \sin u + C$$

by part (g). Now, $\cos u = \sqrt{1 - \sin^2 u} = \sqrt{1 - x^2}$, so

$$\int \arcsin x \, \mathrm{d}x = \sqrt{1 - x^2} + x \arcsin x + C$$

□

(i) $\displaystyle\int \frac{x^2 - x + 6}{x^3 + 3x} \, \mathrm{d}x$

Solution. First, factor: $x^3 + 3x = x(x^2 + 3)$. We must decompose the fraction:

$$\frac{x^2 - x + 6}{x^3 + 3x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}$$

Now, if $x^2 - x + 6 = A(x^2 + 3) + Bx^2 + Cx = (A + B)x^2 + Cx + 3A$, we can equate coefficients and determine $A = 2$, $C = -1$, and $A + B = 2 + B = 1$ so $B = -1$. Therefore,

$$\begin{aligned} \int \frac{x^2 - x + 6}{x^3 + 3x} \, \mathrm{d}x &= \int \frac{2}{x} - \frac{x}{x^2 + 3} - \frac{1}{x^2 + 3} \, \mathrm{d}x \\ &= 2 \ln |x| - \frac{1}{2} \ln\left(x^2 + 3\right) - \frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + C \end{aligned}$$

□

(j) $\displaystyle\int \frac{x^2 - 5x + 16}{(2x + 1)(x - 2)^2} \, \mathrm{d}x$

Solution. The denominator is factored, so we directly apply decomposition:

$$\frac{x^2 - 5x + 16}{(2x + 1)(x - 2)^2} = \frac{A}{2x + 1} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2}$$

Now, if $x^2 - 5x + 16 = A(x - 2)^2 + B(2x + 1)(x - 2) + C(2x + 1)$, we can substitute $x = 2$ to find $10 = 5C \iff C = 2$ and $x = -\frac{1}{2}$ to find $\frac{75}{4} = \frac{25}{4}A \iff A = 3$. Finally, we can deduce that $(A + 2B)x^2 = x^2$, so $B = -1$. Therefore,

$$\begin{aligned} \int \frac{x^2 - 5x + 16}{(2x + 1)(x - 2)^2} \, \mathrm{d}x &= \int \frac{3}{2x + 1} - \frac{1}{x - 2} + \frac{2}{(x - 2)^2} \, \mathrm{d}x \\ &= \frac{3}{2} \ln |2x + 1| - \ln |x - 2| - \frac{2}{x - 2} + C \end{aligned}$$

□

(k) $\displaystyle\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x$

Solution. Let $u = \sin^2 x$ and $\mathrm{d}u = 2\sin x \cos x\,\mathrm{d}x$:

$$\begin{aligned}\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x &= \int \frac{\mathrm{d}u}{2(u^2 + (1-u)^2)}\\ &= \int \frac{\mathrm{d}u}{4u^2 - 4u + 2}\\ &= \int \frac{\mathrm{d}u}{(2u-1)^2 + 1}\\ &= \frac{1}{2}\arctan(2u-1) + C\\ &= \frac{1}{2}\arctan\bigl(2\sin^2 x - 1\bigr) + C \qquad \square\end{aligned}$$

(l) $\displaystyle\int \frac{x \ln x}{\sqrt{x^2-1}}\,\mathrm{d}x$

Solution. Let $u = \sqrt{x^2-1}$ so $\mathrm{d}u = \frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x$:

$$\int \frac{x \ln x}{\sqrt{x^2-1}}\,\mathrm{d}x = \int \ln\Bigl(\sqrt{u^2+1}\Bigr)\,\mathrm{d}u = \int \frac{1}{2}\ln\bigl(u^2+1\bigr)\,\mathrm{d}u$$

We now integrate by parts, with $u = \ln\bigl(u^2+1\bigr)$ and $\mathrm{d}v = \mathrm{d}u$:

$$\begin{aligned}\int \frac{1}{2}\ln\bigl(u^2+1\bigr)\,\mathrm{d}u &= \frac{1}{2}u\ln\bigl(u^2+1\bigr) - \int \frac{2u(u)}{2(u^2+1)}\,\mathrm{d}u\\ &= \frac{1}{2}u\ln\bigl(u^2+1\bigr) - \int \frac{u^2}{u^2+1}\,\mathrm{d}u\\ &= \frac{1}{2}u\ln\bigl(u^2+1\bigr) - \int 1 - \frac{1}{u^2+1}\,\mathrm{d}u\\ &= \frac{1}{2}u\ln\bigl(u^2+1\bigr) - u + \arctan u + C\\ &= \frac{1}{2}\sqrt{x^2-1}\ln x^2 - \sqrt{x^2-1} + \arctan\Bigl(\sqrt{x^2-1}\Bigr) + C\\ &= \sqrt{x^2-1}\ln x - \sqrt{x^2-1} + \operatorname{arcsec}\Bigl(\sqrt{x^2-1}\Bigr) + C\end{aligned}$$

since we can draw the triangle to deduce that if $\theta = \arctan\sqrt{x^2-1}$, then $\cos\theta = \frac{1}{x}$ and $\theta = \operatorname{arcsec} x$. $\square$

(m) $\displaystyle\int \frac{\sec x \cos 2x}{\sin x + \sec x}\,\mathrm{d}x$

Solution. We apply trig identities to simplify:

$$\int \frac{\sec x \cos 2x}{\sin x + \sec x}\,\mathrm{d}x = \int \frac{\cos 2x}{\sin x \cos x + 1}\,\mathrm{d}x = \int \frac{2\cos 2x}{\sin 2x + 2}\,\mathrm{d}x$$

Now let $u = \sin 2x + 2$ so $\mathrm{d}u = 2\cos 2x\,\mathrm{d}x$:

$$\int \frac{2\cos 2x}{\sin 2x + 2}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u}\,\mathrm{d}x = \ln|u| + C = \ln|\sin 2x + 2| + C \qquad \square$$

Q02. An integrand with trigonometric functions in the numerator and denominator can often be converted to a rational integrand using the substitution $u = \tan(x/2)$ or $x = 2\tan^{-1} u = 2\arctan u$.

(a) With this substitution, prove that $\cos x = \frac{1-u^2}{1+u^2}$ and $\sin x = \frac{2u}{1+u^2}$.

Proof. If $u = \tan\frac{x}{2}$ then $\sec^2\frac{x}{2} = u^2+1$, so $\cos\frac{x}{2} = \frac{1}{\sqrt{u^2+1}}$.

But $\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{u^2+1} - 1 = \frac{1-u^2}{1+u^2}$.

Likewise, $u = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$ so $\sin\frac{x}{2} = \frac{u}{\sqrt{u^2+1}}$. Then, $\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2} = \frac{2u}{u^2+1}$. □

(b) Using this substitution and part (a), evaluate the following integrals:

i. $\displaystyle\int \frac{1}{1+\cos x}\,\mathrm{d}x$

Solution. Let $u = \tan\frac{x}{2}$ then $\mathrm{d}u = \frac{1}{2}\sec^2\frac{x}{2}\,\mathrm{d}x = \frac{1}{2}(1+u^2)\,\mathrm{d}x$:

$$\int \frac{1}{1+\cos x}\,\mathrm{d}x = \int \frac{1}{1+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,\mathrm{d}u = \int \mathrm{d}u = \tan\frac{x}{2} + C$$ □

ii. $\displaystyle\int \frac{\mathrm{d}x}{1-\cos x + \sin x}$

Solution. Again, $u = \tan\frac{x}{2}$ and $\mathrm{d}x = \frac{2}{1+u^2}\,\mathrm{d}u$:

$$\int \frac{\mathrm{d}x}{1-\cos x+\sin x} = \int \frac{\frac{2}{1+u^2}}{1-\frac{1-u^2}{1+u^2}+\frac{2u}{1+u^2}}\,\mathrm{d}u = \int \frac{\mathrm{d}u}{u^2+u}$$

Now, separate the fraction as $\frac{1}{u^2+u} = \frac{1}{u} - \frac{1}{u+1}$, so

$$\begin{aligned}\int \frac{1}{u^2+u}\,\mathrm{d}u &= \int \frac{1}{u} - \frac{1}{u+1}\,\mathrm{d}u \\ &= \ln|u| - \ln|u+2| + C \\ &= \ln\left|\frac{u}{u+1}\right| + C \\ &= \ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+1}\right| + C\end{aligned}$$ □

Q03. It has been shown that $\int e^{x^2}\,\mathrm{d}x$ and $\int x^2 e^{x^2}\,\mathrm{d}x$ do not have elementary antiderivatives. However, $\int (2x^2+1)e^{x^2}\,\mathrm{d}x$ does. Evaluate

$$\int (2x^2+1)e^{x^2}\,\mathrm{d}x$$

[Hint: integration by parts]

Q04. (a) Evaluate $\displaystyle\int_0^1 \frac{x^4(1-x^4)}{1+x^2}\,\mathrm{d}x$.

(b) Prove, using part (a), that $\frac{22}{7} > \pi$.

Solution. Expand and divide: $x^4(1-x^4) = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$ and

$$
\begin{array}{r|rrrrrrr}
 & & & x^6 & -4x^5 & +5x^4 & -4x^2 & +4 \\
\hline
x^2+1 & x^8 & -4x^7 & +6x^6 & -4x^5 & +x^4 & & \\
 & -x^8 & & -x^6 & & & & \\
\cline{2-4}
 & & -4x^7 & +5x^6 & -4x^5 & & & \\
 & & 4x^7 & & +4x^5 & & & \\
\cline{3-5}
 & & & 5x^6 & & +x^4 & & \\
 & & & -5x^6 & & -5x^4 & & \\
\cline{4-6}
 & & & & & -4x^4 & & \\
 & & & & & 4x^4 & +4x^2 & \\
\cline{6-7}
 & & & & & & 4x^2 & \\
 & & & & & & -4x^2 & -4 \\
\cline{7-8}
 & & & & & & & -4
\end{array}
$$

Therefore,

$$
\begin{aligned}
\int_0^1 \frac{x^4(1-x^4)}{1+x^2}\,\mathrm{d}x &= \int_0^1 x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\,\mathrm{d}x \\
&= \left(\frac{1}{7}x^7 - \frac{2}{3}x^6 + x^5 - \frac{4}{3}x^3 + 2x^2 - 4\arctan x\right)\Big|_0^1 \\
&= \frac{22}{7} - \pi
\end{aligned}
$$

□

Now, on the interval $[0,1]$, the integral is non-negative since the integrand is non-negative. Therefore, $\frac{22}{7} - \pi \geq 0$, i.e., $\frac{22}{7} > \pi$ (since π is irrational).

Q05. Use integration by parts to prove each of the following *reduction formulas*, for integers $n \geq 2$:

(a) $\displaystyle\int (\ln x)^n\,\mathrm{d}x = x(\ln x)^n - n\int (\ln x)^{n-1}\,\mathrm{d}x$

(b) $\displaystyle\int x^n(\ln x)^n\,\mathrm{d}x = \frac{x^{n+1}(\ln x)^n}{n+1} - \frac{n}{n+1}\int x^n(\ln x)^{n-1}\,\mathrm{d}x$