Essential Laws of Propositional Logic

$$\text{Double Negation}\left\{\neg(\neg p) \vDash\!\!\dashv p\right.$$

$$\text{Excluded Middle}\left\{p \lor \neg p \vDash\!\!\dashv 1\right.$$

$$\text{Contradiction}\left\{p \land \neg p \vDash\!\!\dashv 0\right.$$

$$\text{Idempotence}\begin{cases} p \land p \vDash\!\!\dashv p \\ p \lor p \vDash\!\!\dashv p \end{cases}$$

$$\text{Identity}\begin{cases} p \land 1 \vDash\!\!\dashv p \\ p \lor 0 \vDash\!\!\dashv p \end{cases}$$

$$\text{Domination}\begin{cases} p \land 0 \vDash\!\!\dashv 0 \\ p \lor 1 \vDash\!\!\dashv 1 \end{cases}$$

$$\text{Commutativity}\begin{cases} p \land q \vDash\!\!\dashv q \land p \\ p \lor q \vDash\!\!\dashv q \lor p \\ p \leftrightarrow q \vDash\!\!\dashv q \leftrightarrow p \end{cases}$$

$$\text{Associativity}\begin{cases} p \land (q \land r) \vDash\!\!\dashv (p \land q) \land r \\ p \lor (q \lor r) \vDash\!\!\dashv (p \lor q) \lor r \end{cases}$$

$$\text{Distributivity}\begin{cases} p \land (q \lor r) \vDash\!\!\dashv (p \land q) \lor (q \land r) \\ p \lor (q \land r) \vDash\!\!\dashv (p \lor q) \land (q \lor r) \end{cases}$$

$$\text{Implication}\left\{p \to q \vDash\!\!\dashv \neg p \lor q\right.$$

$$\text{Contrapositive}\left\{p \to q \vDash\!\!\dashv \neg q \to \neg p\right.$$

$$\text{Equivalence}\left\{p \leftrightarrow q \vDash\!\!\dashv (p \to q) \land (q \to p)\right.$$

$$\text{De Morgan}\begin{cases} \neg(p \land q) \vDash\!\!\dashv \neg p \lor \neg q \\ \neg(p \lor q) \vDash\!\!\dashv \neg p \land \neg q \end{cases}$$

$$\text{Absorption I}\begin{cases} p \land (p \lor q) \vDash\!\!\dashv p \\ p \lor (p \land q) \vDash\!\!\dashv p \end{cases}$$

$$\text{Absorption II}\begin{cases} (p \lor q) \land (\neg p \lor q) \vDash\!\!\dashv q \\ (p \land q) \lor (\neg p \land q) \vDash\!\!\dashv q \end{cases}$$

Eleven Rules of Formal Deduction

(Abbr.)	From	Conclude	Rule
(Ref)	$\emptyset$	$A \vdash A$	Reflexivity
$(+)$	$\Sigma \vdash A$	$\Sigma, \Sigma' \vdash A$	Addition of premises
$(\neg -)$	$\Sigma, \neg A \vdash B$ $\Sigma, \neg A \vdash \neg B$	$\Sigma \vdash A$	$\neg$ elimination
$(\to -)$	$\Sigma \vdash A \to B$ $\Sigma \vdash A$	$\Sigma \vdash B$	$\to$ elimination (modus ponens)
$(\to +)$	$\Sigma, A \vdash B$	$\Sigma \vdash A \to B$	$\to$ introduction
$(\land -)$	$\Sigma \vdash A \land B$	$\Sigma \vdash A$ $\Sigma \vdash B$	$\land$ elimination
$(\land +)$	$\Sigma \vdash A$ $\Sigma \vdash B$	$\Sigma \vdash A \land B$	$\land$ introduction
$(\lor -)$	$\Sigma, A \vdash C$ $\Sigma, B \vdash C$	$\Sigma, A \lor B \vdash C$	$\lor$ elimination
$(\lor +)$	$\Sigma \vdash A$	$\Sigma \vdash A \lor B$ $\Sigma \vdash B \lor A$	$\lor$ introduction
$(\leftrightarrow -)$	$\Sigma \vdash A \leftrightarrow B$ $\Sigma \vdash A$	$\Sigma \vdash B$	$\leftrightarrow$ elimination
	$\Sigma \vdash A \leftrightarrow B$ $\Sigma \vdash B$	$\Sigma \vdash A$	
$(\leftrightarrow +)$	$\Sigma, A \vdash B$ $\Sigma, B \vdash A$	$\Sigma \vdash A \leftrightarrow B$	$\leftrightarrow$ introduction

Proved Theorems

(Abbr.)	From	Conclude	Theorem
$(\in)$	$A \in \Sigma$	$\Sigma \vdash A$	Membership
(Tr.)	$\Sigma \vdash \Sigma'$ $\Sigma' \vdash A$	$\Sigma \vdash A$	Transitivity
$(\neg +)$	$\Sigma, A \vdash B$ $\Sigma, A \vdash \neg B$	$\Sigma \vdash \neg A$	Reductio ad absurdum
(Repl.)	$A \dashv\vdash A'$ $C = B$ with A' for A	$B \dashv\vdash C$	Replaceability
	$A \vdash B$	$\neg B \vdash \neg A$	Flip-Flop

Proved Theorems (cont.)

$\neg\neg A \vDash\Dashv A$	(Double Negation)
$\emptyset \vdash A \vee \neg A$	(Excluded Middle)
$\emptyset \vdash \neg(A \wedge \neg A)$	(Non−Contradiction)
$A, \neg A \vdash B$	(Inconsistency Rule)
$A \to B, B \to C \vdash A \to C$	(Hypothetical Syllogism)
$A \vee B, \neg B \vdash A$	(Disjunctive Syllogism)
$A \to B \vDash\Dashv \neg A \vee B$	(Implication)
$A \to B \vDash\Dashv \neg B \to \neg A$	(Contrapositive)
$\neg(A \wedge B) \vdash\dashv \neg A \vee \neg B$ $\neg(A \vee B) \vdash\dashv \neg A \wedge \neg B$	(De M$\vee$ gan)

Definitions

Propositional language (§02, 3). A set of all strings of proposition, connective, and punctuation symbols. $\mathcal{L}^p$ has $p, q, r, \ldots, \neg, \wedge, \vee, \to, \leftrightarrow, (,)$.

Expression (§02, 4). An element of $\mathcal{L}^p$. The empty expression is ϵ.

Segment (§02, 5). V segment of U if $U = W_1 V W_2$. Initial if $W_1 = \epsilon$, final if $W_2 = \epsilon$. Proper if $V \neq U$.

Atom (§02, 6). Atom($\mathcal{L}^p$) has expressions that are one proposition symbol.

Formula (§02, 6). Form($\mathcal{L}^p$) defined by: Atom($\mathcal{L}^p$) $\subset$ Form($\mathcal{L}^p$).
$A, B \in \text{Form}(\mathcal{L}^p) \Rightarrow (\neg A), (A \wedge B), (A \vee B), (A \to B), (A \leftrightarrow B) \in \text{Form}(\mathcal{L}^p)$.
No other expressions are in Form($\mathcal{L}^p$).

Scope (§02, 45). In $(\neg A)$, A is the scope of $\neg$. In $(A \star B)$, A is the left scope and B is the right scope of $\star$.

Truth valuation (§03, 6). A function $t : \text{Atom}(\mathcal{L}^p) \to \{0, 1\}$.

Satisfiablity (§03, 9). The truth valuation t satisfies A if $A^t = 1$. Σ is satisfiable if there exists t such that $\Sigma^t = 1$.

Tautology (§03, 14). For all t, $A^t = 1$.
Contradiction (§03, 14). For all t, $A^t = 0$.
Contingent (§03, 14). A is neither a tautology nor contradiction.

Tautological consequence (§03, 21). $\Sigma \vDash A$ if for all t, $\Sigma^t = 1$ gives $A^t = 1$.
Tautological equivalence (§03, 25). $A \vDash\Dashv B$ if $A \vDash B$ and $B \vDash A$.

Literal (§04, 10). A formula of the form p or $\neg p$.

Disjunctive clause (§04, 12). Disjunction with literal disjuncts.
Conjunctive clause (§04, 12). Conjunction with literal conjuncts.

DNF (§04, 13). Disjunction with conjunctive clause disjuncts.
CNF (§04, 13). Conjunction with disjunctive clause conjuncts.

Definability/Reducibility (§05, 2). Connective $\star$ reducible to set $\mathcal{S}$ if $A \star B \vDash\Dashv C$ where C uses only A, B, and connectives in $\mathcal{S}$.

Adequate (§05, 7). Connectives which express any truth table/connective.

Formal deducibility (§06, 15). $\Sigma \vdash A$ generated by finite deduction rules.
Syntactic equivalence (§06, 15). $A \vDash\Dashv B$ if $A \vdash B$ and $B \vdash A$.

Consistency (§06, 67). There is no F such that $\Sigma \vdash F$ and $\Sigma \vDash \neg F$.

Theorems (...the general kind)

Lemma (§02, 29). Every formula has equal number of left/right parentheses.

Unique Readability Theorem (§02, 32). Every formula is exactly one form of exactly one of $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \leftrightarrow B)$.

Lemma (§03, 23). Equivalent statements of $\{A_i\} \vDash C$:
Argument with premises A_i and conclusion C is valid.
$(\bigwedge A_i) \to C$ is a tautology; $(\neg C \wedge \bigwedge A_i)$ is a contradiction.
Formula $(\neg C \wedge \bigwedge A_i)$ or set $\{\neg C, A_i\}$ is not satisfiable.

Replaceability of tautologically equivalent formulas (§03, 44). If $A \vDash\Dashv A'$ and A is a subformula of B, then $B \vDash\Dashv B'$ where B' is B with some of the As replaced by A'.

Duality Theorem (§03, 44). If A has only $\neg$, $\wedge$, $\vee$ and $\Delta(A)$ replaces atoms with negations and swaps $\wedge$ with $\vee$, then $\neg A \vDash\Dashv \Delta(A)$.

Theorem (§04, 22). All formulas have $F \vDash\Dashv \text{DNF}(F)$ based on truth table.
Theorem (§04, 24). All formulas have $F \vDash\Dashv \text{CNF}(F) = \Delta(\text{DNF}(\neg F))$.

Theorem (§05, 8). The set $S_0 = \{\neg, \wedge, \vee\}$ is adequate.

Finiteness of Premise Set (§06, 31). If $\Sigma \vdash A$, $\Sigma^0 \vdash A$ with finite $\Sigma^0 \subseteq \Sigma$.

Soundness Theorem (§06, 45). If $\Sigma \vdash A$ then $\Sigma \vDash A$.
Completeness Theorem (§06, 49). If $\Sigma \vDash A$ then $\Sigma \vdash A$.

Lemma (§06, 67). Σ is satisfiable if and only if Σ is consistent.