

# PMATH 370 Winter 2024:

## Lecture Notes

<b>1</b>	<b>Iteration and Orbits</b>	<b>3</b>
1.1	Orbits . . . . .	3
1.2	Real analysis review . . . . .	5
1.3	Orbits, revisited . . . . .	8
<b>2</b>	<b>Graphical Analysis</b>	<b>12</b>
2.1	Cobweb plots . . . . .	12
<b>3</b>	<b>Fixed Points</b>	<b>15</b>
3.1	Attracting/repelling fixed point theorems . . . . .	15
3.2	Neutral fixed points . . . . .	18
<b>4</b>	<b>Bifurcations</b>	<b>22</b>
<b>5</b>	<b>Cantor set</b>	<b>25</b>
<b>6</b>	<b>Symbolic dynamics</b>	<b>28</b>
6.1	Intro to topology . . . . .	29
6.2	Revisiting the itinerary . . . . .	31
<b>7</b>	<b>Chaos</b>	<b>34</b>
7.1	Prerequisites to chaos . . . . .	34
7.2	Defining chaos . . . . .	36
<b>8</b>	<b>Sarkovskii's Theorem</b>	<b>38</b>
<b>9</b>	<b>Fractals</b>	<b>41</b>
9.1	Definitions and dimensions . . . . .	41
9.2	Fractal gallery . . . . .	43
9.3	Iterated function systems . . . . .	45
9.4	Generated iterated function systems . . . . .	47
<b>10</b>	<b>Complex Functions</b>	<b>53</b>
<b>11</b>	<b>Julia Sets</b>	<b>55</b>
11.1	Construction . . . . .	55

<b>Back Matter</b>	<b>60</b>
List of Named Results . . . . .	60
Index of Defined Terms . . . . .	61

Lecture notes taken, unless otherwise specified, by myself during the Winter 2024 offering of PMATH 370, taught by Blake Madill.

<b>Lectures</b>			
		Lecture 13	Feb 5 . . . . . 34
		Lecture 14	Feb 7 . . . . . 36
Lecture 1	Jan 8 . . . . . 3	Lecture 15	Feb 9 . . . . . 38
Lecture 2	Jan 10 . . . . . 5	Lecture 16	Feb 12 . . . . . 39
Lecture 3	Jan 12 . . . . . 7	Lecture 17	Feb 14 . . . . . 41
Lecture 4	Jan 15 . . . . . 10	Lecture 18	Feb 16 . . . . . 43
Lecture 5	Jan 17 . . . . . 13	Lecture 19	Feb 26 . . . . . 44
Lecture 6	Jan 19 . . . . . 18	Lecture 20	Feb 28 . . . . . 47
Lecture 7	Jan 22 . . . . . 20	Lecture 21	Mar 4 . . . . . 49
Lecture 8	Jan 24 . . . . . 23	Lecture 22	Mar 6 . . . . . 50
Lecture 9	Jan 26 . . . . . 25	Lecture 23	Mar 8 . . . . . 52
Lecture 10	Jan 29 . . . . . 28	Lecture 24	Mar 11 . . . . . 54
Lecture 11	Jan 31 . . . . . 30	Lecture 25	Mar 13 . . . . . 57
Lecture 12	Feb 2 . . . . . 32		

# Chapter 1

## Iteration and Orbits

### 1.1 Orbits

**Definition 1.1.1** (iteration)

Let  $f : A \rightarrow \mathbb{R}$  such that  $A \subseteq \mathbb{R}$  and  $f(A) \subseteq A$ . For  $a \in A$  we may iterate the function at  $a$ :

$$x_1 = a, x_2 = f(a), x_3 = \underbrace{f(f(a))}_{f^2(a)}, \dots, x_i = f^{i-1}(a), \dots$$

The sequence  $(x_n)_{n=1}^\infty$  is the orbit of  $a$  under  $f$  (abbreviated  $(x_n)$  without limits).

Lecture 1  
Jan 8

**Example 1.1.2.** Let  $f(x) = x^4 + 2x^2 - 2$ ,  $a = -1$ . What is the orbit of  $a$  under  $f$ ?

*Solution.*  $a = -1$ ,  $f(a) = 1$ ,  $f(f(a)) = f(1) = 1$ , so we have  $-1, 1, 1, 1, \dots$ . We call this eventually constant.  $\square$

**Example 1.1.3.** Let  $f(x) = -x^2 - x + 1$ ,  $a = 0$ . What is the orbit of  $a$  under  $f$ ?

*Solution.* Calculate:  $0, 1, -1, 1, -1, 1, \dots$ . We call this eventually periodic (with period 2).  $\square$

**Example 1.1.4.** Let  $f(x) = x^3 - 3x + 1$ ,  $a = 1$ . What is the orbit of  $a$  under  $f$ ?

*Solution.* Calculate the first few terms:  $1, -1, 3, 19, \dots$  (too big). This is a divergence to infinity.  $\square$

**Example 1.1.5.** Let  $f(x) = x^2 + 2x$ ,  $a = -0.5$ . What is the orbit of  $a$  under  $f$ ?

*Solution.* Calculate:  $-0.5, -0.75, -0.9375, -0.9961 \dots$  and we make an educated guess that this converges to  $-1$  since  $f(-1) = -1$ , a fixed point.  $\square$

**Example 1.1.6.** Let  $f(x) = x^3 - 3x$ ,  $a = 0.75$ . What is the orbit of  $a$  under  $f$ ?

*Solution.* Calculate:  $0.75, -1.828, -0.625, 1.631, -0.552, \dots$ . There is no clear pattern, so we call this chaotic. In fact, the orbit is dense in a neighbourhood of 0.  $\square$

We can start to formalize the examples.

**Definition 1.1.7** (fixed point)

Let  $f : A \rightarrow \mathbb{R}$  such that  $f(A) \subseteq A$ . A point  $a \in A$  is fixed if  $f(a) = a$ .

Then, the orbit of  $a$  under  $f$  is  $(a, a, a, \dots)$  which is constant.

**Example 1.1.8.** Find all fixed points of  $f(x) = x^2 + x - 4$ .

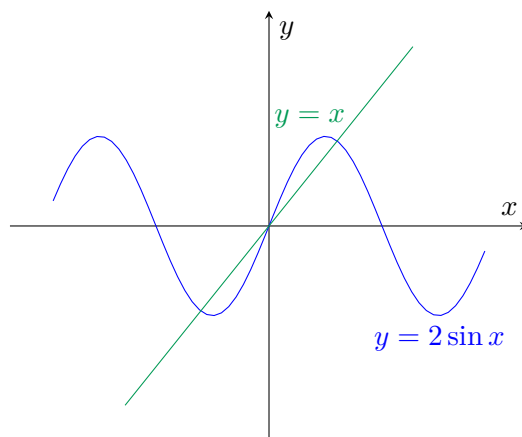
*Solution.* We find points where  $f(x) = x$ , i.e.,  $x^2 + x - 4 = x$ .

$$x^2 + x - 4 = x \iff x^2 = 4 \iff x = \pm 2$$

$\square$

**Example 1.1.9.** How many fixed points does  $f(x) = 2 \sin x$  have?

*Solution.* Consider where the curve  $y = 2 \sin x$  meets  $y = x$ :



We can see there are three fixed points.  $\square$

**Example 1.1.10.** Prove that  $f(x) = x^4 - 3x + 1$  has a fixed point.

*Proof.* We must show there is a solution to  $x^4 - 3x + 1 \iff x^4 - 4x + 1 = 0$ . Let  $g(x) = x^4 - 4x + 1$ . Since  $g(x)$  is continuous,  $g(0) = 1 > 0$ , and  $g(1) = -2 < 0$ , by the Intermediate Value Theorem, there must exist a root of  $g$  on the interval  $(0, 1)$ . That is, a fixed point of  $f$ .  $\square$

**Definition 1.1.11** (periodicity)

Let  $f : A \rightarrow \mathbb{R}, f(A) \subseteq A$ .

1. A point  $a \in A$  is periodic for  $f$  if its orbit is periodic. An orbit is periodic if for some  $n \in \mathbb{N}$ ,  $f^n(a) = a$ . The smallest  $n$  is the period of (the orbit of)  $a$ .
2. An orbit (of a point) is eventually periodic if there exists  $n < m$  such that  $f^n(a) = f^m(a)$ . The smallest difference  $m - n$  is the period of the orbit.

**Definition 1.1.12** (doubling function)

$D : [0, 1) \rightarrow [0, 1) : x \mapsto 2x - \lfloor 2x \rfloor$  returns the fractional part of  $2x$ .

Lecture 2  
Jan 10

**Example 1.1.13.**  $D(0.4) = 0.8$ ,  $D(0.6) = 0.2$ ,  $D(0.8) = 0.6$ ,  $D(0.5) = 0$ .

This is a nice function that gives lots of periodic orbits for funsies.

**Example 1.1.14.** Find the orbit of  $a = \frac{1}{5}$  under  $D$ .

*Solution.* Double until we pass 1:  $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{8}{5} \rightarrow \frac{3}{5}, \frac{6}{5} \rightarrow \frac{1}{5}$ . The period is  $|\{\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\}| = 4$ . □

**Example 1.1.15.** Find the orbit of  $a = \frac{1}{20}$  under  $D$ .

*Solution.* Double:  $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}$  and we can stop because ex. 1.1.14 showed  $\frac{1}{5}$  is periodic.

So this is eventually periodic with period 4. □

**Problem 1.1.16**

Given  $f$  and  $a$ , does  $f^n(a)$  tend towards some limit  $L$ ?

To solve this problem, we need to rigorously define “tend” and “limit”.

## 1.2 Real analysis review

**Notation.** If  $(x_n)_{n=1}^\infty$  is a sequence of real numbers, we write  $(x_n) \subseteq \mathbb{R}$ .

**Definition 1.2.1** (convergence of a sequence)

Let  $(x_n) \subseteq \mathbb{R}$ ,  $x \in \mathbb{R}$ .

We say  $(x_n)$  converges to  $x$  if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $|x_n - x| < \varepsilon$  for all  $n \geq N$ .

Then, we write  $x_n \rightarrow x$  or  $\lim x_n = x$ .

**Example 1.2.2.** Show that  $\frac{1}{n} \rightarrow 0$ .

*Proof.* Let  $\varepsilon > 0$ . Consider  $N = \frac{2}{\varepsilon} > \frac{1}{\varepsilon}$ . For  $n \geq N$ , we have

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon$$

Therefore,  $\frac{1}{n} \rightarrow 0$ . □

**Example 1.2.3.** Prove that  $\frac{2n}{n+3} \rightarrow 2$ .

*Proof.* Let  $\varepsilon > 0$ . Since we know  $\frac{1}{n} \rightarrow 0$ , let  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \frac{\varepsilon}{6}$ .

For  $n \geq N$ ,

$$\left| \frac{2n}{n+3} - 2 \right| = \left| \frac{2n}{n+3} - \frac{2n+6}{n+3} \right| = \left| \frac{-6}{n+3} \right| = \frac{6}{n+3} < \frac{6}{n} \leq \frac{6}{N} < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

Therefore,  $\frac{2n}{n+3} \rightarrow 2$ . □

**Definition 1.2.4** (bounded sequence)

A sequence  $(x_n)$  is bounded (by  $M$ ) if there exists  $M > 0$  such that  $\forall n \in \mathbb{N}$ ,  $|x_n| \leq M$ .

**Proposition 1.2.5** (convergence implies boundedness)

If  $(x_n)$  is convergent, then  $(x_n)$  is bounded.

*Proof.* Suppose  $x_n \rightarrow x$ . Then, there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $|x_n - x| < 1$ .

For  $n \geq N$ ,  $|x_n| - |x| \leq |x_n - x| < 1$ . That is,  $|x_n| < 1 + |x|$ .

Let  $M = \max\{|x_1|, \dots, |x_{N-1}|, 1 + |x|\}$ . Then, for both all  $n < N$  and  $n \geq N$ , we have  $|x_n| \leq M$ . □

**Remark 1.2.6.** The converse is not true. Notice that  $x_n = (-1)^n$  is bounded by 1 but obviously not convergent.

**Proposition 1.2.7** (limit laws)

Let  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . Then:

$$(1) \quad x_n + y_n \rightarrow x + y$$

$$(2) \quad x_n y_n \rightarrow xy$$

*Proof.* (1) Let  $\varepsilon > 0$ . Then, since  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , there exist  $N_1, N_2 \in \mathbb{N}$  such that  $n \geq N_1 \implies |x_n - x| < \frac{\varepsilon}{2}$  and  $n \geq N_2 \implies |y_n - y| < \frac{\varepsilon}{2}$ .

For  $N = \max\{N_1, N_2\}$  and  $n \geq N$ ,

$$\begin{aligned} |(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

That is,  $x_n + y_n \rightarrow x + y$ .

(2) Let  $\varepsilon > 0$ . Notice that:

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \quad (*)$$

Since  $x_n$  is bounded, there exists  $M > 0$  such that  $|x_n| \leq M$  for all  $n$ .

Let  $N_1, N_2 \in \mathbb{N}$  such that

$$\begin{aligned} n \geq N_1 &\implies |x_n - x| \leq \frac{\varepsilon}{2(|y| + 1)} \text{ and} \\ n \geq N_2 &\implies |y_n - y| < \frac{\varepsilon}{2M}. \end{aligned}$$

Then, for  $n \geq N := \max\{N_1, N_2\}$ ,  $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$  by (\*). □

**Definition 1.2.8** (Cauchy sequence)

We say  $(x_n) \in \mathbb{R}$  is Cauchy if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n$  and  $m$ ,

$$n, m \geq N \implies |x_n - x_m| < \varepsilon$$

**Proposition 1.2.9**

Every convergent sequence is Cauchy.

*Proof.* Intuitively: if the terms get arbitrarily close to some limit, they must get arbitrarily close to each other.

Formally: Let  $x_n \rightarrow x$  be a convergent sequence and  $\varepsilon > 0$ . Since  $x_n$  converges, there exists  $N \in \mathbb{N}$  such that  $n \geq N \implies |x_n - x| < \frac{\varepsilon}{2}$ .

Lecture 3  
Jan 12

Then, when  $n, m \geq N$ , we have:

$$\begin{aligned}
 |x_n - x_m| &= |x_n - x_m + x - x| \\
 &= |(x_n - x) + (x - x_m)| \\
 &\leq |x_n - x| + |x_m - x| \\
 &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
 &= \varepsilon
 \end{aligned}$$

as desired. □

We take the following theorem from real analysis without proof.

**Theorem 1.2.10** (completeness of  $\mathbb{R}$ )

A sequence is Cauchy if and only if it is convergent.

The big idea here: To prove  $(x_n)$  is Cauchy, you do not have to guess the limit first. That is, if you must prove convergence but do not care about the limit's value, prove that it is Cauchy instead.

**Definition 1.2.11** (continuity of a function)

Let  $f : A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}$ ,  $a \in A$ . We say  $f$  is continuous at  $a$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x) - f(a)| < \varepsilon$  whenever  $x \in A$  and  $|x - a| < \delta$ .

If  $f$  is continuous at all  $a \in A$ , we say it is continuous.

We also take this theorem from MATH 137 without proof.

**Theorem 1.2.12**

A function  $f : A \rightarrow \mathbb{R}$  is continuous at  $a \in A$  if and only if for all sequences  $(x_n) \subseteq A$  with  $x_n \rightarrow a$ , we have  $f(x_n) \rightarrow f(a)$ .

## 1.3 Orbits, revisited

**Proposition 1.3.1**

If  $f : [a, b] \rightarrow [a, b]$  is continuous, then  $f(x)$  has a fixed point.

*Proof.* We know by the domain and codomain that  $f(a) \geq a$  and  $f(b) \leq b$ . This means  $f(a) - a \geq 0$  and  $f(b) - b \leq 0$ . By the IVT on the continuous function  $g(x) = f(x) - x$ , we know there exists an  $x \in [a, b]$  such that  $g(x) = f(x) - x = 0 \iff f(x) = x$ , i.e.,  $x$  is a fixed point. □



**Definition 1.3.2** (contraction)

Let  $f : A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}$ . We say  $f$  is a contraction if there exists  $C \in [0, 1)$  such that for all  $x, y \in A$ ,

$$|f(x) - f(y)| \leq C|x - y|$$

This is just a Lipschitz function with Lipschitz constant less than 1.

**Proposition 1.3.3**

Contractions are continuous.

*Proof.* Let  $\varepsilon > 0$ . Suppose  $f$  is a contraction such that  $|f(x) - f(y)| \leq C|x - y|$ .

Consider  $y \in A$ . Let  $\delta = \frac{\varepsilon}{C+1}$  and assume that  $x \in A$  and  $|x - y| < \delta$ . But we have:

$$|f(x) - f(y)| \leq C|x - y| \leq C\delta < \varepsilon$$

as desired. □

**Definition 1.3.4** (closure of an interval)

We say  $A \subseteq \mathbb{R}$  is closed if whenever  $(x_n) \subseteq A$  with  $x_n \rightarrow x$ , then  $x \in A$ .

**Example 1.3.5.**  $[a, b]$  is closed but  $(0, 1]$  is not because  $\frac{1}{n} \rightarrow 0 \notin (0, 1]$ .

**Theorem 1.3.6** (Banach contraction mapping theorem)

Suppose  $A \subseteq \mathbb{R}$  is closed and  $f : A \rightarrow A$  is a contraction. Then, there exists a unique fixed point  $a \in A$  for  $f$ .

Moreover, for all  $x \in A$ ,  $f^n(x) \rightarrow a$ .

**Example 1.3.7.** Analyze the orbit of  $f : [0, 1] \rightarrow [0, 1]$ ,  $f(x) = \frac{1}{3-x}$ .

*Solution.* We can observe that  $\frac{1}{3} \leq \frac{1}{3-x} \leq \frac{1}{2}$ .

Also,  $f'(x) = \frac{1}{(3-x)^2}$ . Notice that  $\frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$ . So by the mean value theorem, for all  $x, y \in [0, 1]$ , there exists  $c \in (0, 1)$  such that:

$$\begin{aligned} f(x) - f(y) &= f'(c)(x - y) \\ |f(x) - f(y)| &= |f'(c)| \cdot |x - y| \\ &\leq \frac{1}{4}|x - y| \end{aligned}$$

Then, identifying  $C = \frac{1}{4}$ ,  $f$  is a contraction. Now,

$$\frac{1}{3-x} = x \iff 1 = 3x - x^2 \iff x^2 - 3x + 1 = 0 \iff x = \frac{3 \pm \sqrt{9-4}}{2} \iff x = \frac{3 - \sqrt{5}}{2}$$

where we pick the negative root because we need  $x \in [0, 1]$ .

Therefore, by the [Banach contraction mapping theorem](#), for all  $x \in [0, 1]$ ,  $f^n(x) \rightarrow \frac{3-\sqrt{5}}{2}$ .  $\square$

### Definition 1.3.8

A sequence  $(a_n) \subseteq \mathbb{R}$  is strongly-Cauchy if there exists  $(\varepsilon_n) \subseteq [0, \infty)$  such that  $\sum_{n=1}^{\infty} \varepsilon_n < \infty$  and for all  $n$ ,  $|a_n - a_{n+1}| < \varepsilon_n$ .

Informally, “far enough along the sequence, the *neighbours* must get close”. This is distinct from Cauchy, which is “far enough along the sequence, the *terms* must get close”.

**Remark 1.3.9** (assignment hint!). Let  $\sum_{n=1}^{\infty} a_n = L$ . This means that  $\sum_{k=1}^n a_k \xrightarrow{n \rightarrow \infty} L$ .

That is, for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies  $|\sum_{k=1}^n a_k - L| < \varepsilon$ .

But  $|\sum_{k=1}^n a_k - L| = |\sum_{k=1}^{\infty} a_k - \sum_{k=1}^n a_k| = |\sum_{k=n+1}^{\infty} a_k| < \varepsilon$ .

We can now prove the [Banach contraction mapping theorem](#).

*Proof.* Let  $A \subseteq \mathbb{R}$  be closed and suppose there exists  $f : A \rightarrow A$  and  $C \in [0, 1)$  such that  $|f(x) - f(y)| \leq C|x - y|$  for all  $x$  and  $y$  in  $A$ .

Fix  $x_0 \in A$  and construct the orbit  $x_1 = f(x_0)$ ,  $x_2 = f(x_1)$ ,  $\dots$ ,  $x_n = f(x_{n-1}) = f^n(x_0)$ .

For  $n \in \mathbb{N}$ , since  $f$  is a contraction,

$$\begin{aligned} |x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \\ &\leq C|x_n - x_{n-1}| \\ &= C|f(x_{n-1}) - f(x_{n-2})| \\ &\leq C^2|x_{n-1} - x_{n-2}| \\ &\vdots \\ &\leq C^n|x_1 - x_0| \end{aligned}$$

Since  $\sum_{n=1}^{\infty} C^n|x_1 - x_0| = |x_1 - x_0| \sum_{n=1}^{\infty} C^n$  is a convergent geometric series, we have that the sequence  $(x_n)$  is strongly-Cauchy.

Hence, by Assignment 1,  $x_n \rightarrow a$  for some limit point  $a \in A$  since  $A$  is closed.

Since  $f$  is continuous (prop. 1.3.3), we have that  $f(x_n) \rightarrow f(a)$ . By definition,  $f(x_n) = x_{n+1}$ , so  $x_n \rightarrow f(a)$ . But we already know  $x_n \rightarrow a$ , so  $a = f(a)$ . That is,  $a$  is a fixed point of  $f$ .

It remains to show uniqueness.

Lecture 4  
Jan 15

Suppose  $a, b \in A$  such that  $f(a) = a$  and  $f(b) = b$ .

$$\begin{aligned} |f(a) - f(b)| &\leq C|a - b| \\ |a - b| &\leq C|a - b| \end{aligned}$$

Since  $C < 1$ , we must have  $|a - b| = 0$ , that is,  $a = b$ . □

## Chapter 2

# Graphical Analysis

### 2.1 Cobweb plots

Recall ex. 1.1.9. To visualize the orbit of  $a$  under  $f$ , we can:

1. Superimpose  $y = f(x)$  over the line  $y = x$ .
  2. Connect a vertical line  $(a, a) - (a, f(a))$
  3. Connect a horizontal line  $(a, f(a)) - (f(a), f(a))$
  4. Connect a vertical line  $(f(a), f(a)) - (f(a), f(f(a)))$
  5. Connect a horizontal line  $(f(a), f(f(a))) - (f(f(a)), f(f(a)))$
- etc.

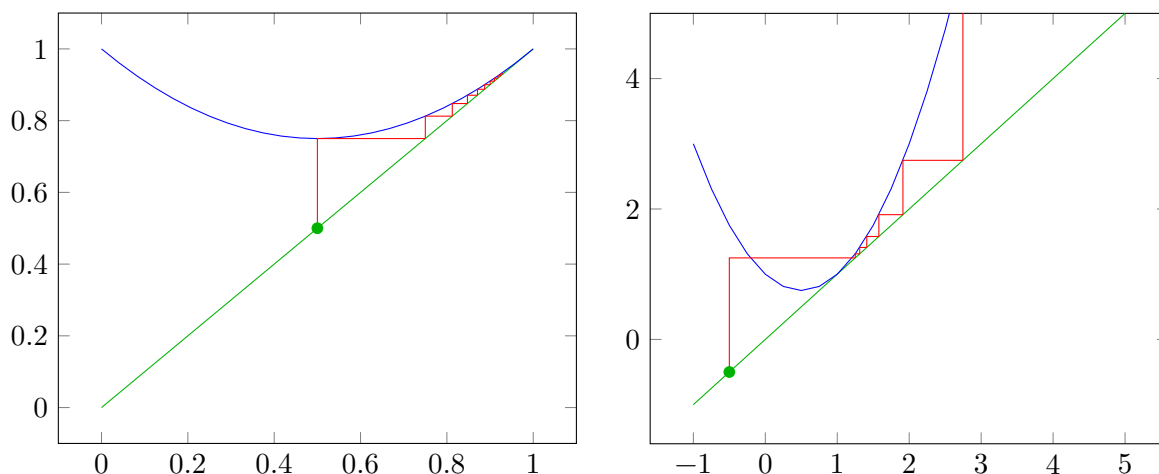
This is sometimes called a cobweb plot. We will be using <https://marksmath.org/visualization/cobwebs/> to make cobweb plots.

Within these lecture notes, I use a  $\text{\LaTeX}$  macro to draw plots [defined here](#).

**Example 2.1.1.** Conduct a complete orbit analysis of  $f(x) = x^2 - x + 1$

*Solution.* Playing around, we find that there is one fixed point  $x = 1$ .

When  $x \in [0, 1]$ ,  $f^n(x) \rightarrow 1$ . Otherwise,  $f^n(x) \rightarrow \infty$ .



□

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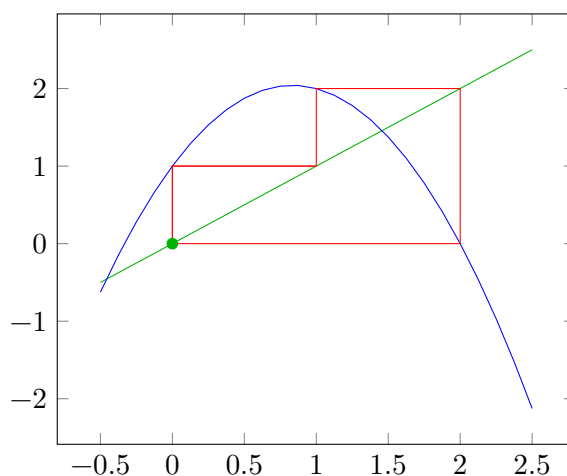
↓ Lectures 5 and 6 adapted from *Rosie* ↓

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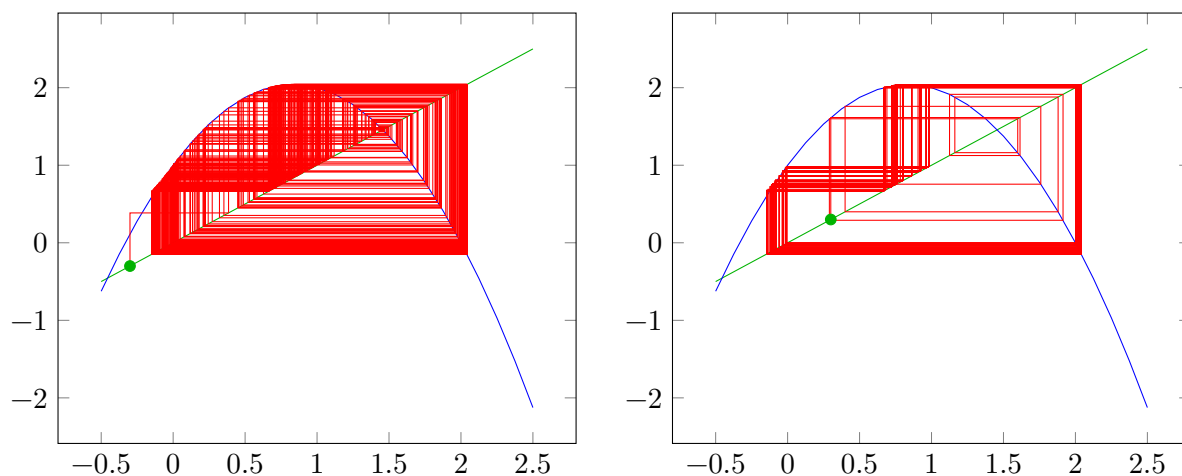
Lecture 5  
Jan 17

**Example 2.1.2.** Conduct a complete orbit analysis of  $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$ .

*Solution.* At  $x = 0$ , we can see there is a cycle going from  $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ :



At points near 0, like  $x = -0.3$  or  $x = 0.3$ , the graph becomes chaotic:



It appears that the cobweb densely covers the graph. □

As an aside, note that we cannot actually hit every point in the interval because the orbit is countable (i.e., has the same size as the naturals) but the interval is uncountable. We will later show that the points are dense (as the rationals are).

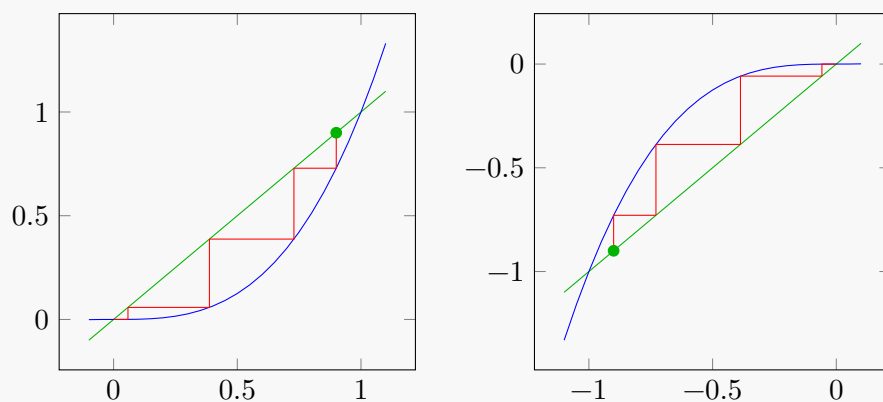
## Chapter 3

# Fixed Points

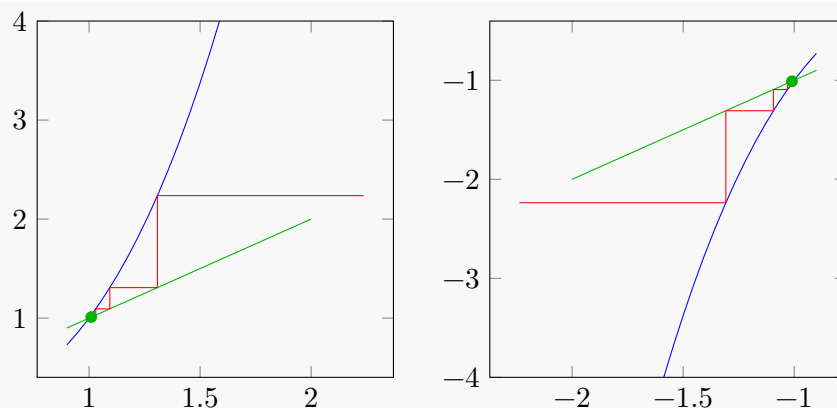
### 3.1 Attracting/repelling fixed point theorems

**Remark 3.1.1.** If  $f(x)$  is continuous and  $f^n(a) \rightarrow L$ , then  $f^{n+1}(a) \rightarrow f(L)$ . Therefore,  $f(L) = L$  is a fixed point.

**Example 3.1.2.** The function  $f(x) = x^3$  has three fixed points:  $0, \pm 1$ . For  $x \in (-1, 1)$ , we see that  $f^n(x) \rightarrow 0$ :

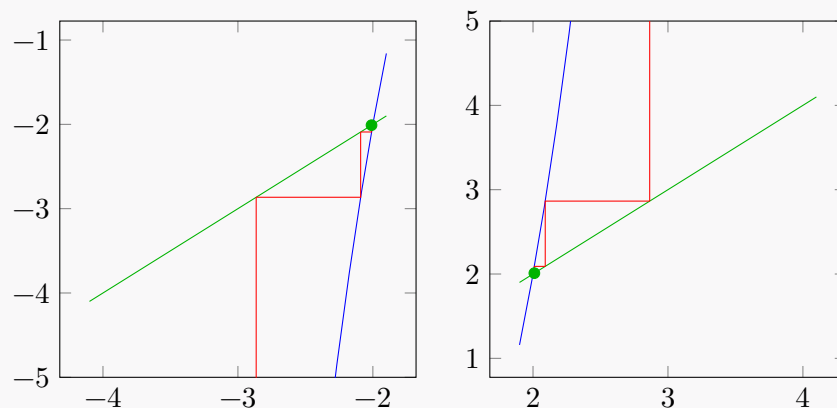


It looks like point 0 is attracting the orbit. For  $x \in (-\infty, -1) \cup (1, \infty)$ , we see  $f^n(x) \rightarrow \infty$ :

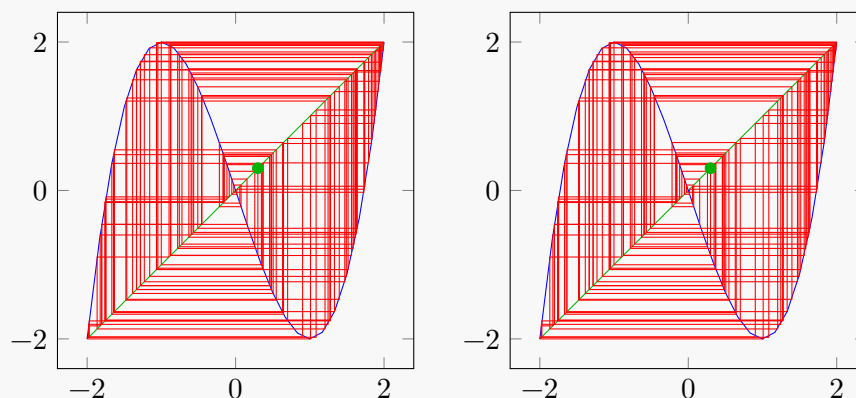


so the points  $\pm 1$  are repelling the orbit.

**Example 3.1.3.** The function  $f(x) = x^3 - 3x$  also has three fixed points:  $0, \pm 2$ . To the right (left) of  $\pm 2$ , orbits go to infinity:

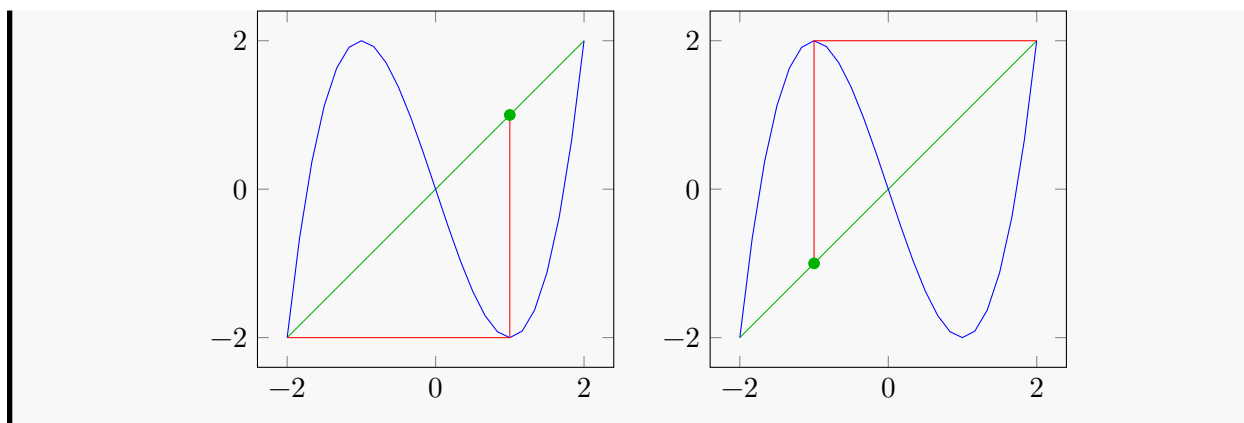


The point  $0$  is repelling (in a different sense) since we get chaos:



At  $x_0 = \pm 1$ , the orbit is eventually constant, jumping to the fixed point  $\mp 2$ :



**Definition 3.1.4**

Let  $a$  be a fixed point of  $f(x)$ .

1. If  $|f'(a)| > 1$ , we call  $a$  a repelling fixed point
2. If  $|f'(a)| < 1$ , we call  $a$  a attracting fixed point
3. If  $|f'(a)| = 1$ , we call  $a$  a neutral fixed point

Neutral fixed points can be a lot of different things.

**Theorem 3.1.5** (attracting fixed point theorem)

Suppose  $a$  is an attracting fixed point of  $f(x)$ . Then, there exists an open interval  $I$  containing  $a$  such that

1. for all  $x \in I$ ,  $n \in \mathbb{N}$ ,  $f^n(x) \in I$
2. for all  $x \in I$ ,  $f^n(x) \rightarrow a$

Recall the  $\varepsilon$ - $\delta$  definition of a limit.

**Definition 3.1.6** (limit of a function at a point)

Let  $f : A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}$ .

We say a point  $a \in A$  is non-isolated if for each  $\varepsilon > 0$  there exists  $b \in A$ ,  $b \neq a$  with  $b \in (a - \varepsilon, a + \varepsilon)$ .

Suppose  $a$  is non-isolated. We say  $\lim_{x \rightarrow a} f(x) = L$  if for all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $|f(x) - L| < \varepsilon$  whenever  $a \in A$  and  $0 < |x - a| < \delta$ .

It is important to define non-isolation. If  $a$  is isolated, we can choose a  $\delta$  where  $|x - a| < \delta$  is false. Then, every point is vacuously a limit point.

We now give the proof of the attracting fixed point theorem:

*Proof.* Assume  $|f'(a)| < 1$ . Then, there exists  $c \in \mathbb{R}$  such that  $|f'(a)| < c < 1$ . By definition of the

derivative, this means we can write

$$\lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{x - a} < c$$

and by the definition of the limit, we know there exists  $\delta > 0$  such that

$$\frac{|f(x) - f(a)|}{|x - a|} \leq c, \quad \forall x \in (a - \delta, a + \delta)$$

Hence, for  $x \in I := (a - \delta, a + \delta)$ , we have  $|f(x) - f(a)| \leq c|x - a|$  and  $f$  is a contraction.

In particular, for  $x \in I$ , we have

$$\begin{aligned} |f(x) - a| &= |f(x) - f(a)| && (a \text{ is a fixed point}) \\ &\leq c|x - a| \leq |x - a| && (c \in (0, 1)) \\ &< \delta \end{aligned}$$

That is,  $f(x) \in (a - \delta, a + \delta) = I$ . Continuing for the rest of the orbit, for all  $n \in \mathbb{N}$ ,

$$|f^n(x) - a| \leq c^n|x - a| \leq |x - a| < \delta$$

so we also have  $f^n(x) \in I$ .

Finally, notice that  $0 \leq |f^n(x) - a| \leq c^n|x - a|$  and  $c^n|x - a| \rightarrow 0$  since  $c \in (0, 1)$ . By the squeeze theorem,  $|f^n(x) - a| \rightarrow 0$ .  $\square$

### **Theorem 3.1.7** (repelling fixed point theorem)

Suppose  $a$  is a repelling fixed point for  $f(x)$ . Then, there exists an open interval  $I$  containing  $a$  such that for all  $x \in I$ ,  $x \neq a$ , there exists  $n \in \mathbb{N}$  such that  $f^n(x) \notin I$ .

*Proof.* Say  $|f'(a)| > c > 1$ . Then, as above, there exists a  $\delta$  such that

$$\lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{x - a} > c \implies |f(x) - f(a)| \geq c|x - a|$$

for all  $x \in I := (a - \delta, a + \delta)$ .

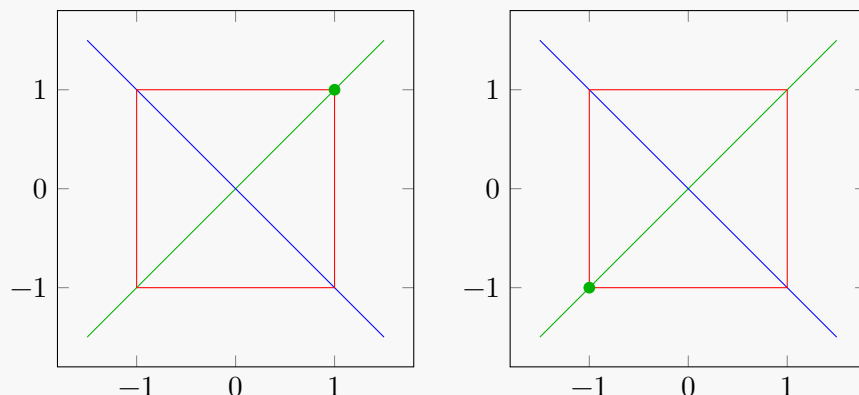
Since  $a$  is a fixed point,  $|f(x) - f(a)| = |f(x) - a|$ . Suppose for a contradiction that for all  $n$ ,  $f^n(x) \in I$ . But since  $c > 1$ ,  $|f^n(x) - a| \geq c^n|x - a| \rightarrow \infty$ . That is,  $\delta$  must be arbitrarily large, which it is not.  $\square$

## **3.2 Neutral fixed points**

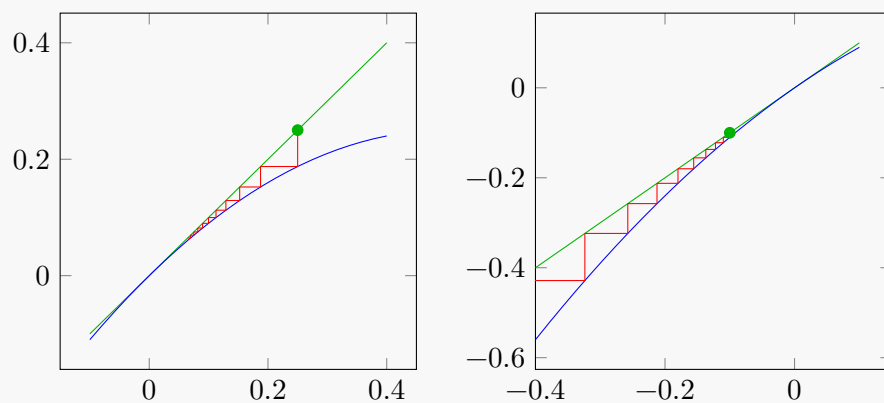
Neutral fixed points can exhibit a lot of different behaviours.

*Lecture 6  
Jan 19*

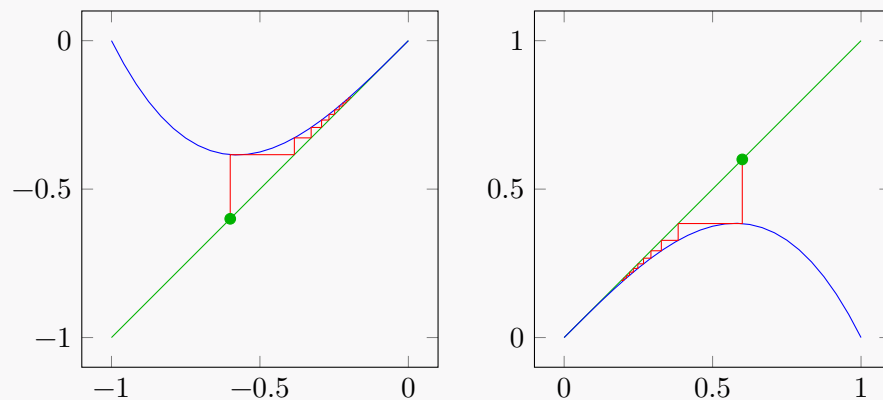
**Example 3.2.1.** For  $f(x) = -x$ ,  $0$  is a fixed point with  $|f'(0)| = 1$ . The orbit bounces:



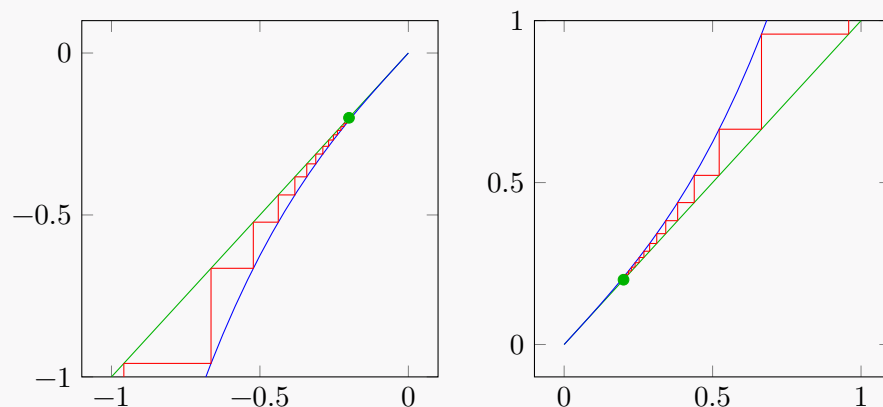
**Example 3.2.2.** For  $f(x) = x - x^2$ ,  $|f'(1)| = 1$  is a neutral fixed point. It is attracting from the right and repelling from the left:



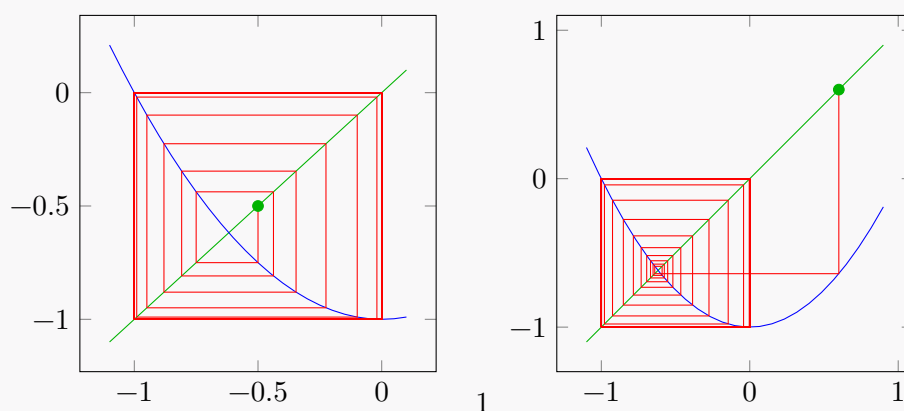
**Example 3.2.3.** For  $f(x) = x - x^3$ ,  $|f'(0)| = 1$  is a neutral fixed point. It is weakly attracting, attracting but too slowly.



**Example 3.2.4.** For  $f(x) = x + x^3$ ,  $|f'(0)| = 1$  is a neutral fixed point. It is weakly repelling, repelling but too slowly:



**Example 3.2.5.** Consider  $f(x) = x^2 - 1$ . The orbit at  $a = 0$  is periodic  $(0, -1, 0, -1, \dots)$  with period 2. Near 0, the orbit tends to the  $(0, -1)$ -cycle:



We will call 0 an attracting periodic point because 0 is an attracting point of  $f^2(x)$ .

↑ Lectures 5 and 6 adapted from *Rosie* ↑

### Definition 3.2.6

Let  $a$  be a periodic point for  $f(x)$  with period  $n$ .

We say  $a$  is an attracting/repelling/neutral periodic point if  $a$  is an attracting/repelling/neutral fixed point of  $f^n$

Finding a closed form expression for something like  $f^{10}(x)$  is a nightmare, so we need a better way.

Lecture 7  
Jan 22

**Proposition 3.2.7**

Let  $f(x)$  be a differentiable function. Then,  $(f^n)'(x) = f'(x) \cdot f'(f(x)) \cdots f'(f^{n-1}(x))$ .

*Proof.* Proceed by induction on  $n$ .

If  $n = 1$ , we have  $f'(x) = f'(x)$  and we are done.

Suppose  $(f^n)'(x) = \prod_{k=0}^{n-1} f'(f^k(x))$  for some  $n \geq 1$ . Consider  $f^{n+1}$ :

$$\frac{d}{dx} f^{n+1}(x) = \frac{d}{dx} f(f^n(x)) = f'(f^n(x)) \cdot (f^n)'(x)$$

by the chain rule. Then,

$$\begin{aligned} (f^{n+1})'(x) &= f'(f^n(x)) \cdot (f^n)'(x) \\ &= f'(f^n(x)) \cdot \prod_{k=0}^{n-1} f'(f^k(x)) \\ &= \prod_{k=0}^n f'(f^k(x)) \end{aligned}$$

completing the proof. □

**Example 3.2.8.** Analyze the periodic point  $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$ ,  $a = 0$

*Solution.* The orbit is  $(0, 1, 2, 0, 1, 2, \dots)$  with period 3.

We have  $f'(x) = -3x + \frac{5}{2}$ . Then,  $(f^3)'(0) = f'(0)f'(1)f'(2) = (-\frac{7}{2})(-\frac{1}{2})(\frac{5}{2}) = \frac{35}{8} > 1$ .

Therefore, the point is repelling. □

## Chapter 4

# Bifurcations

In general, bifurcation theory is the study of how a family of curves can change when a defining parameter is changed.

Consider the quadratic family:

$$Q_C(x) = x^2 + C$$

defined by the parameter  $C \in \mathbb{R}$ .

### Problem 4.0.1

How does the behaviour (fixed points, orbits, etc.) of  $Q_C$  change based on  $C$ ?

First, we can find the fixed points (if they exist) by solving

$$Q_C(x) = x \iff x^2 - x + C = 0 \iff x = \frac{1 \pm \sqrt{1-4C}}{2}$$

and note that  $Q_C(x)$  has 2 fixed points when  $C < \frac{1}{4}$ , 1 fixed point when  $C = \frac{1}{4}$ , and no fixed points when  $C > \frac{1}{4}$ .

Suppose  $C > \frac{1}{4}$ . Then, we must have  $Q_C^n(x) \rightarrow \infty$  for all  $x$ .

Instead, if  $C = \frac{1}{4}$ ,  $Q_C(x)$  has the unique fixed point  $p = \frac{1}{2}$ . Since  $Q'_C(x) = 2x$  and  $Q'_C(p) = 1$ , this is a neutral fixed point. In fact, it attracts to one side and repels from the other.

Finally, if  $C < \frac{1}{4}$ ,  $Q_C(x)$  has two fixed points  $p_+ = \frac{1+\sqrt{1-4C}}{2}$  and  $p_- = \frac{1-\sqrt{1-4C}}{2}$ . Then,  $Q'_C(p_+) =$

$1 + \sqrt{1 - 4C} > 1$  is repelling. Next,

$$\begin{aligned}
 & -1 < Q'_C(p_-) < 1 \\
 \Leftrightarrow & -1 < 1 - \sqrt{1 - 4C} < 1 \\
 \Leftrightarrow & -2 < -\sqrt{1 - 4C} < 0 \\
 \Leftrightarrow & 0 < \sqrt{1 - 4C} < 2 \\
 \Leftrightarrow & -\frac{3}{4} < C < \frac{1}{4}
 \end{aligned}$$

and in fact if  $C < -\frac{3}{4}$ ,  $Q'_C(p_-) < -1$  and if  $C = -\frac{3}{4}$ ,  $Q'_C(p_-) = -1$ .

### Theorem 4.0.2

For the family

$$Q_C(x) = x^2 + C,$$

depending on  $C$ :

1. All orbits tend to  $\infty$  if  $C > \frac{1}{4}$ .
2. When  $C = \frac{1}{4}$ ,  $Q_C(x)$  has a unique fixed point  $\frac{1}{2}$  and it is neutral.
3. If  $C < \frac{1}{4}$ ,  $Q_C(x)$  has two fixed points  $p_+$  and  $p_-$ . The point  $p_+$  is repelling. Moreover,
  - (a) if  $-\frac{3}{4} < C < \frac{1}{4}$ ,  $p_-$  is attracting;
  - (b) if  $C = -\frac{3}{4}$ ,  $p_-$  is neutral; and
  - (c) if  $C < -\frac{3}{4}$ ,  $p_-$  is repelling.

### Definition 4.0.3 (bifurcation)

We say a family of functions  $F_\lambda(x)$  undergoes a bifurcation at  $\lambda_0$  if there is a change in fixed point structure at  $\lambda_0$ .

Lecture 8  
Jan 24

**Example 4.0.4.** The quadratic family  $Q_C(x) = x^2 + C$  undergoes a bifurcation at  $\lambda_0 = \frac{1}{4}$ .

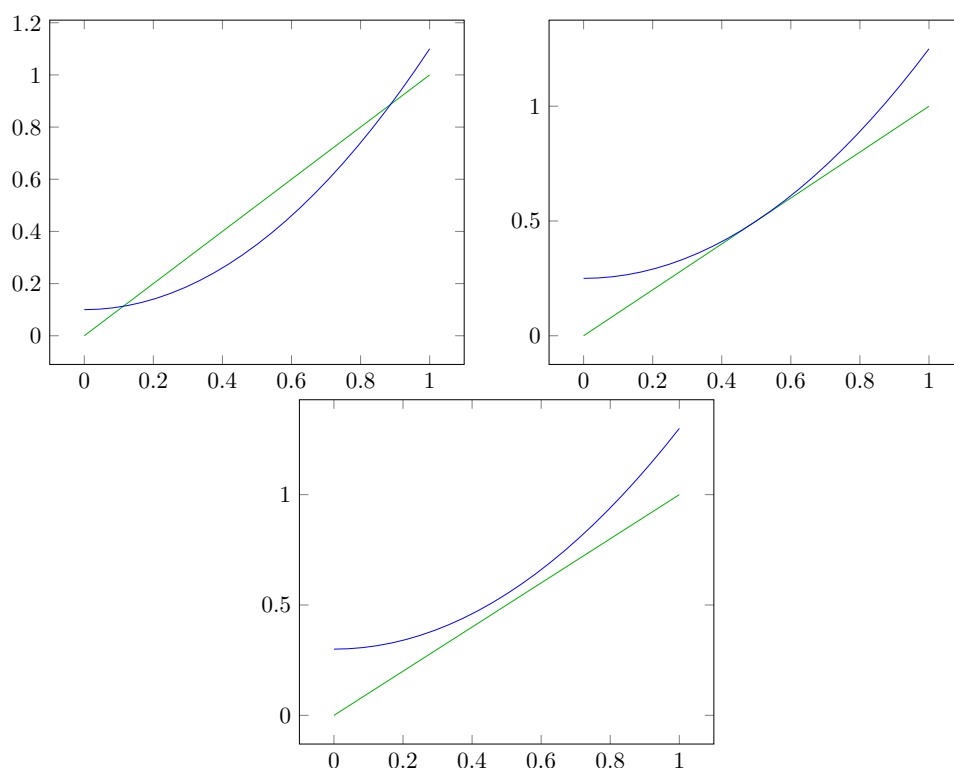
**Definition 4.0.5** (tangent bifurcation)

A family  $F_\lambda(x)$  undergoes a tangent bifurcation at  $\lambda_0$  if there is an open interval  $I$  and an  $\varepsilon > 0$  such that:

1. for  $\lambda_0 - \varepsilon < \lambda < \lambda_0$ ,  $F_\lambda(x)$  has no fixed points on  $I$ ;
2. for  $\lambda = \lambda_0$ ,  $F_\lambda(x)$  has one fixed point and it is neutral; and
3. for  $\lambda_0 < \lambda < \lambda_0 + \varepsilon$ ,  $F_\lambda(x)$  has two fixed points in  $I$ , one of which is attracting and the other repelling.

(or with all inequalities flipped)

Visually, you have situations like



for  $\lambda < \lambda_0$ ,  $\lambda = \lambda_0$ , and  $\lambda > \lambda_0$ .

**Example 4.0.6.** Consider the exponential family  $E_\lambda(x) = e^x + \lambda$  at  $\lambda_0 = -1$ .

This is a tangent bifurcation.

**Example 4.0.7.**  $F_\lambda(x) = \lambda x(1-x)$ ,  $\lambda_0 = 1$

Here, we have two fixed points on one side of  $\lambda_0$  and one fixed point on the other. So this is a bifurcation but not a tangent bifurcation.

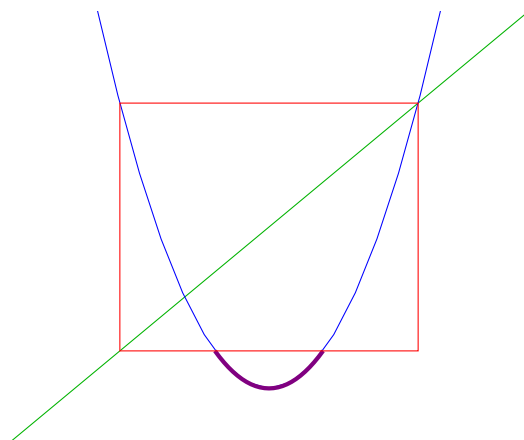


# Chapter 5

## Cantor set

Recall the quadratic family  $Q_C(x) = x^2 + C$  for  $C < -2$ . Then,  $p_+ = \frac{1+\sqrt{1-4C}}{2} > 2$  and  $-p_+ < -2$ . Consider the interval/region  $I = [-p_+, p_+]$  and  $I \times I$ .

Draw the picture of  $y = x$ ,  $y = Q_C(x)$ , and the box  $I \times I$ :



Let  $J_1 \subseteq I$  be the interval such that  $Q_C(x) \notin I$  for all  $x \in J_1$ .

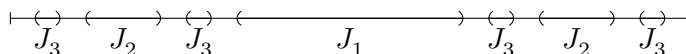
For  $x \in J_1$ ,  $Q_C^n(x) \rightarrow \infty$ . Moreover, if there exists  $n$  such that  $Q_C^n(x) \in J_1$ , then  $Q_C^n(x) \rightarrow \infty$ .

Consider the set of points  $\Lambda = \{x \in I : \forall n, Q_C^n(x) \in I\}$  with “interesting” orbits staying inside  $I$ .

Now, let  $J_2 = \{x \in I : Q_C(x) \in J_1\} = \{x \in I : Q_C^2(x) \notin I\}$  and define higher  $J_n$  likewise.

Then,  $\Lambda = I \setminus (J_1 \cup J_2 \cup \dots)$  is a Cantor set, that is, a fractal. (roll credits!)

Drawing  $\Lambda$  on the  $x$ -axis, we get something that looks like



↓ Lecture 9 adapted from Imaad ↓

Lecture 9  
Jan 26

**Definition 5.0.1** (Cantor middle thirds set)

Let  $C_0 = [0, 1]$ . Remove the open middle third interval each time.

That is,  $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ ,  $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ , and so on.

The set  $K = \bigcap_{n=1}^{\infty} C_n$  is the Cantor (middle thirds) set.

**Proposition 5.0.2**

Suppose a bunch of sets  $A_n \subseteq \mathbb{R}$  are closed. Then,  $\bigcap A_n$  is also closed.

*Proof.* Let  $(a_k) \subseteq \bigcap A_n$  where  $(a_k) \rightarrow a$ .

Note that for all  $n$ ,  $(a_k) \subseteq A_n \implies a \in A_n \implies a \in \bigcap A_n$  □

**Proposition 5.0.3**

Let  $A, B \subseteq \mathbb{R}$  be closed. Then,  $A \cup B$  is closed.

*Proof.* Let  $(a_n) \subseteq A \cup B$  where  $a_n \rightarrow a$ .

WLOG,  $\{n : a_n \in A\}$  is infinite. This allows us to construct  $(b_n) \subseteq A$  such that  $b_n \rightarrow a$ .

Since  $A$  is closed,  $a \in A \subseteq A \cup B$ . □

**Theorem 5.0.4** (Cantor sets are closed)

Any Cantor set, in particular  $K$ , is closed.

**Theorem 5.0.5**

$K$  contains no non-empty open intervals.

*Proof.* Consider  $I \subseteq K$ . Then  $\forall n, I \subseteq C_n$ .

Then  $\ell(I) \leq \frac{1}{3^n} \implies \ell(I) = 0 \implies I = \emptyset$ , contradiction. □

Now, let's consider the base-3 expansion of  $x \in [0, 1]$ .  $x = 0.s_1s_2s_3, \dots, s_i \in \{0, 1, 2\}$

Consider  $\underbrace{[0, 1/3]}_{s_1=0}$  and  $\underbrace{[2/3, 1]}_{s_1=2}$  and  $\underbrace{[0, 1/9]}_{s_1=0, s_2=0}$   $[2/9, 1/3]$   $[2/3, 7/9]$   $[8/9, 1]$ .

**Remark 5.0.6.**  $x \in K$  if and only if  $x$  can be written in base 3 using only 0s and 2s

**Example 5.0.7.**  $\frac{1}{3} \in K$ .  $\frac{1}{3} = 0.1_3 = 0.02222\dots_3$

**Theorem 5.0.8**

$K$  is uncountable and  $|K| = |\mathbb{R}|$ .

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↑ *Lecture 9 adapted from Imaad* ↑

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# Chapter 6

## Symbolic dynamics

Lecture 10  
Jan 29

Recall the construction of the Cantor set from the quadratic family:

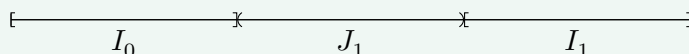
Fix  $C < -2$  and consider  $Q_C(x) = x^2 + C$ . Define an interval  $I = [-p_+, p_+]$  for a fixed point  $p_+ = \frac{1+\sqrt{1-4C}}{2}$ . Then, let

$$\begin{aligned} J_1 &= \{x \in I : Q_C(x) \notin I\} \\ J_2 &= \{x \in I : Q_C(x) \in J_1\} \\ J_3 &= \{x \in I : Q_C(x) \in J_2\} \\ &\vdots \end{aligned}$$

and define  $\Lambda = I \setminus (\bigcup J_i) = \{x \in I : \forall n, Q_C^n(x) \in I\}$ .

We proceed to do some analysis of  $\Lambda$  by translating into some sort of sequence space, doing analysis, and then going back to the Cantor set.

**Notation.** Define closed intervals  $I_0 \cup I_1 := I \setminus J_1$  on the left/right of  $J_1$ :



### Definition 6.0.1

For  $x \in \Lambda$ , the itinerary of  $x$  is the sequence  $S(x) = (x_0 x_1 x_2 x_3 \dots)$  with  $x_i \in \{0, 1\}$  where  $x_i = 0 \iff Q_C^i(x) \in I_0$  and  $x_i = 1 \iff Q_C^i(x) \in I_1$ .

Our goal is to understand  $S(x)$  better so that we can glean information about  $\Lambda$ .

**Notation.** Let  $\Sigma = \{(x_0 x_1 x_2 \dots) : x_i \in \{0, 1\}\}$  be the sequence space. Write elements of  $\Sigma$  as binary strings. Then,  $S : \Lambda \rightarrow \Sigma$  is a function.

It would be helpful to define some PMATH 351/topology shit.

## 6.1 Intro to topology

### Definition 6.1.1 (metric space)

Let  $X$  be a set. A function  $d : X \times X \rightarrow [0, \infty)$  is a metric if

1.  $d(x, y) = 0 \iff x = y$  (positive definiteness),
2.  $d(x, y) = d(y, x)$  (symmetry), and
3.  $d(x, y) \leq d(x, z) + d(z, y)$  (triangle inequality).

The pair  $(X, d)$  is a metric space.

Once we have a metric space with a notion  $d$  of distance, we can adapt all our definitions from real analysis to an abstract space.

**Example 6.1.2.** The following are all metrics:

- $X = \mathbb{R}$ ,  $d(x, y) = |x - y|$
- $X = \mathbb{R}^n$ ,  $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$
- For any set  $X$ , the discrete metric  $d(x, y) = [x \neq y]$  (but is not particularly useful).
- For a subset  $A \subseteq \mathbb{R}$ ,  $d(x, y) = |x - y|$  is a metric.

Extremely helpfully, we can define a metric on the sequence space.

### Definition 6.1.3 (Cantor space)

Let  $X = \Sigma$ . Define  $d(x, y) = \sum_{i=0}^{\infty} 2^{-i} |x_i - y_i|$ .

This is well-defined (converges) since  $|x_i - y_i| \leq 1$  and  $\sum 2^{-i}$  converges.

**Example 6.1.4.** Let  $x = (1111\cdots)$  and  $y = (1010\cdots)$ . Calculate  $d(x, y)$ .

*Solution.* By definition,

$$\begin{aligned}
 d(x, y) &= \sum_{i=0}^{\infty} \frac{x_i - y_i}{2^i} \\
 &= \sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} && \text{(even indices cancel)} \\
 &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^i} \\
 &= \frac{1}{2} \left( \frac{1}{1 - \frac{1}{4}} \right) = \frac{1}{2} \left( \frac{4}{3} \right) = \frac{4}{6} = \frac{2}{3}
 \end{aligned}$$

□

We don't want to do this manual calculation every time.

**Proposition 6.1.5**

Let  $x, y \in \Sigma$ .

1. If  $x_i = y_i$  for  $i \leq n$ , then  $d(x, y) \leq \frac{1}{2^n}$ .
2. If  $d(x, y) < \frac{1}{2^n}$ , then  $x_i = y_i$  for  $i \leq n$ .

*Proof.* Suppose  $x_i = y_i$  for  $i \leq n$ . Then,  $d(x, y) \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k}$  since the first  $n$  terms will be 0 and  $|x_i - y_i| \leq 1$ . That is,  $d(x, y) \leq \frac{1/2^{n+1}}{1 - \frac{1}{2}} = \frac{1}{2^n}$ .

Conversely, suppose  $d(x, y) < \frac{1}{2^n}$  and for a contradiction that there exists  $k \leq n$  where  $x_k \neq y_k$ . Then, there will be a  $\frac{1}{2^k}$  term in the sum, so  $d(x, y) \geq \frac{1}{2^k} \geq \frac{1}{2^n}$ . Contradiction.  $\square$

**Example 6.1.6.** Let  $x = (0000\cdots)$  and  $y = (1000\cdots)$ . Then, the distance is  $\frac{1}{2^0} = 1$ . However,  $x_0 \neq y_0$ .

**Definition 6.1.7** (shift map)

The map  $\sigma : \Sigma \rightarrow \Sigma : (x_0x_1x_2\cdots) \mapsto (x_1x_2x_3\cdots)$  that shifts a bitstring one bit to the left.

**Remark 6.1.8.**  $\sigma^k(x_0x_1x_2\cdots) = x_kx_{k+1}x_{k+2}\cdots$

**Definition 6.1.9** (continuity in metric spaces)

Suppose  $(X, d)$  and  $(Y, d')$  are (possibly distinct) metric spaces.

A function  $f : X \rightarrow Y$  is continuous at  $y \in X$  if for all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in X$ ,

$$d(x, y) < \delta \implies d'(f(x), f(y)) < \varepsilon$$

We say  $f$  is continuous if it is continuous at every  $y \in X$

**Proposition 6.1.10**

The shift map  $\sigma : \Sigma \rightarrow \Sigma$  is continuous.

*Proof.* Fix  $y = (y_0y_1y_2\cdots) \in \Sigma$  and let  $\varepsilon > 0$ . Take  $n \in \mathbb{N}$  such that  $\frac{1}{2^n} < \varepsilon$ .

Consider  $\delta = \frac{1}{2^{n+1}}$ . Let  $x = (x_0x_1x_2\cdots) \in \Sigma$  such that  $d(x, y) < \delta$ .

Therefore, by prop. 6.1.5,  $x_i = y_i$  for  $i = 0, 1, \dots, n+1$ . Then,  $\sigma(x) = (x_1x_2x_3\cdots)$  and  $\sigma(y) = (y_1y_2y_3\cdots)$  agree for the first  $n$  terms.

Again by prop. 6.1.5,  $d(\sigma(x), \sigma(y)) \leq \frac{1}{2^n} < \varepsilon$ .  $\square$

Lecture 11  
Jan 31

**Definition 6.1.11** (convergence in metric spaces)

Let  $(X, d)$  be a metric space,  $(x_n) \subseteq X$ , and  $x \in X$ .

We say  $(x_n)$  converges to  $x$  ( $x_n \rightarrow x$ ) if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$n \geq N \implies d(x_n, x) < \varepsilon.$$

**Proposition 6.1.12** (sequential characterization of continuity in metric spaces)

Let  $(X, d)$  and  $(Y, d')$  be metric spaces and  $f : X \rightarrow Y$ . Then,  $f$  is continuous if and only if  $f(x_n) \rightarrow f(x)$  whenever  $x_n \rightarrow x$ .

**Definition 6.1.13** (homeomorphism)

Let  $(X, d)$  and  $(Y, d')$  be metric spaces. A function  $f : X \rightarrow Y$  is a homeomorphism if

1.  $f$  is injective,
2.  $f$  is surjective,
3.  $f$  is continuous, and
4.  $f^{-1}$  is continuous.

Suppose  $f : X \rightarrow Y$  is a homeomorphism. Then, if  $(x_n) \subseteq X$  with  $x_n \rightarrow x$ , then  $f(x_n) \rightarrow f(x)$ .

Likewise, suppose  $(y_n) \subseteq Y$  with  $y_n \rightarrow y$ . Say  $y_n = f(x_n)$  and  $y = f(x)$ . Then,  $f(x_n) \rightarrow f(x)$ , so  $f^{-1}(f(x_n)) \rightarrow f^{-1}(f(x))$  and  $x_n \rightarrow x$ .

That is,  $f$  is a *relabelling* of  $X$  to  $Y$ . We think of  $X$  and  $Y$  as the “same metric space”.

## 6.2 Revisiting the itinerary

**Remark 6.2.1.** Suppose we have  $x \in \Lambda$  with  $S(x) = (x_0 x_1 \cdots)$ . Then, by definition,  $x \in I_{x_0}$ ,  $Q_c(x) \in I_{x_1}$ ,  $Q_c^2(x) \in I_{x_2}$ , etc. Therefore,  $S(Q_c(x)) = (x_1 x_2 \cdots) = \sigma(S(x))$ .

Iterating,  $S(Q_c^n(x)) = \sigma^n(x)$ .

**Theorem 6.2.2**

$S : \Lambda \rightarrow \Sigma$  is a homeomorphism.

We will prove this with some more tools. Recall from MATH 137:

**Theorem 6.2.3** (monotone convergence theorem)

If  $(a_n) \subseteq \mathbb{R}$  is increasing/decreasing and bounded, then  $(a_n)$  converges.

Instead of using this directly, we use a lemma:

**Lemma 6.2.4** (nested intervals lemma)

If  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$  are closed intervals, then  $\bigcap_{i=1}^{\infty} I_i \neq \emptyset$ .

*Proof.* Let  $I_k = [a_k, b_k]$ .

That is,  $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \dots$ .

Then,  $(a_n)$  is increasing and  $(a_n) \subseteq [a_1, b_1]$ . Likewise,  $(b_n)$  is decreasing and  $(b_n) \subseteq [a_1, b_1]$ . By the [monotone convergence theorem](#),  $a_n \rightarrow a$  and  $b_n \rightarrow b$  for some limit points  $a$  and  $b$ .

Therefore (handwavey),  $\emptyset \neq [a, b] \subseteq \bigcap_{n=1}^{\infty} I_n$ . □

We will now prove thm. 6.2.2. It is true for  $c < -2$ , but we will show it for  $c < -\frac{5+2\sqrt{5}}{4}$ .

Lecture 12  
Feb 2

*Proof.* (injective) Suppose  $x, y \in \Lambda$  with  $S(x) = S(y)$  but  $x \neq y$ . Then, for all  $n$ ,  $Q_c^n(x)$  and  $Q_c^n(y)$  live in the same  $I_0$  or  $I_1$ . Recall from Assignment 2 that for all  $x \in I \setminus J_1 = I_0 \cup I_1$ , we have  $|Q'_c(x)| \geq \mu > 1$ . By the mean value theorem,

$$|Q_c(x) - Q_c(y)| \geq \mu|x - y|.$$

Since  $Q_c$  is injective on  $I_0$  and  $I_1$ , we have that  $Q_c(x) \neq Q_c(y)$ . Thus,

$$\begin{aligned} |Q_c^2(x) - Q_c^2(y)| &\geq \mu^2|x - y| \\ &\vdots \\ |Q_c^n(x) - Q_c^n(y)| &\geq \mu^n|x - y| \end{aligned}$$

Since  $\mu > 1$ , we have  $\mu^n|x - y| \rightarrow \infty$ . However,  $|Q_c^n(x) - Q_c^n(y)| \leq \max\{\ell(I_0), \ell(I_1)\}$ , so it cannot blow up to infinity. Contradiction, so we have injectivity.

(surjective) Let  $y = (y_0 y_1 \dots) \in \Sigma$ . For  $n \in \mathbb{N}$ , define

$$I_{y_0 y_1 \dots y_n} := \{x \in I : x \in I_{y_0}, Q_c(x) \in I_{y_1}, \dots, Q_c^n(x) \in I_{y_n}\}.$$

It is enough to show there exists

$$x \in \bigcap_{n=1}^{\infty} I_{y_0 y_1 \dots y_n}$$

which would imply  $S(x) = y$ . Clearly, by definition,  $I_{y_0} \supseteq I_{y_0 y_1} \supseteq I_{y_0 y_1 y_2} \supseteq \dots$

We claim that each  $I_{y_0 y_1 \dots y_n}$  is a closed interval. Proceed by induction.

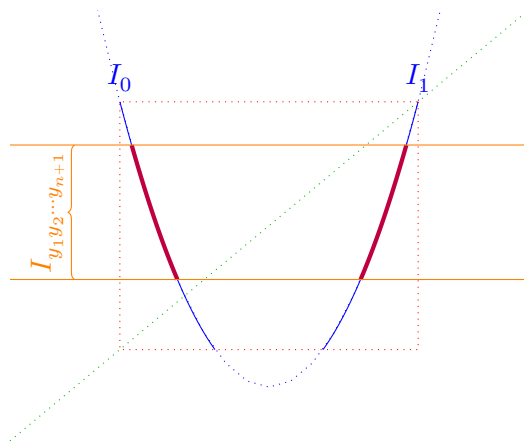
First,  $I_{y_0} \in \{I_0, I_1\}$  so it is closed. Assume  $I_{y_0 y_1 \dots y_n}$  is closed for some  $n \geq 0$ . Note:

$$\begin{aligned} x &\in I_{y_0 y_1 \dots y_{n+1}} \\ \iff x &\in I_{y_0}, Q_c(x) \in I_{y_1}, Q_c(Q_c(x)) \in I_{y_2}, Q_c(Q_c^2(x)) \in I_{y_3}, \dots, Q_c(Q_c^n(x)) \in I_{y_{n+1}} \\ \iff x &\in I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \dots y_{n+1}}) \end{aligned} \tag{*}$$

By the inductive hypothesis,  $I_{y_1 y_2 \dots y_{n+1}}$  is a closed interval (the subscript has length  $n$ ).

We have





That is,  $Q_c^{-1}(I_{y_1 y_2 \dots y_{n+1}})$  is a union of **two disjoint closed intervals**, one in  $I_0$  and one in  $I_1$ .

In particular, returning to  $(\star)$ ,  $I_{y_0 y_1 \dots y_{n+1}} = I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \dots y_{n+1}})$  is one of these closed intervals.

By the **nested intervals lemma**, there must exist  $x \in \bigcap_{n=1}^{\infty} I_{y_0 y_1 \dots y_n}$ . Hence,  $S(x) = y$  and we have surjectivity.

(continuous) Fix  $y \in \Lambda$  and say  $S(y) = (y_0 y_1 y_2 \dots)$ . Let  $\varepsilon > 0$  and choose  $n$  such that  $\frac{1}{2^n} < \varepsilon$ .

Consider the  $2^{n+1}$  disjoint, closed intervals  $I_{t_0 t_1 \dots t_n}$ .

Pick  $\delta > 0$  such that  $(y - \delta, y + \delta)$  only overlaps with  $I_{y_0 y_1 \dots y_n}$ . We know  $\delta$  exists since we have a finite set of disjoint closed intervals.

For  $x \in \Lambda$  with  $|x - y| < \delta$ ,  $x \in I_{y_0 y_1 \dots y_n}$  and so  $d(S(x), S(y)) \leq \frac{1}{2^n} < \varepsilon$ .

(continuous inverse) Similar. □

# Chapter 7

## Chaos

### 7.1 Prerequisites to chaos

Lecture 13  
Feb 5

#### Definition 7.1.1 (density)

Let  $(X, d)$  be a metric space. We say  $A \subseteq X$  is dense in  $X$  if for all  $x \in X$  and  $\varepsilon > 0$ , there exists  $a \in A$  such that  $d(a, x) < \varepsilon$ .

Informally, there is always something “that close” to any point.

**Example 7.1.2.**  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . Given a real number, there is always a decimal approximation with arbitrary accuracy.

$\mathbb{Z}$  is not dense in  $\mathbb{R}$ . Given  $x = \frac{1}{2} \in \mathbb{R}$ , there are no integers within  $\varepsilon = \frac{1}{4}$ .

**Example 7.1.3.** Let  $A = \{x \in \Sigma : \exists N, \forall i > N, x_i = 0\}$ , i.e., the sequences which are eventually constant 0s. This is dense in  $\Sigma$ .

*Proof.* Let  $x = (x_0 x_1 x_2 \dots) \in \Sigma$  and let  $\varepsilon > 0$ . As usual, take  $n \in \mathbb{N}$  such that  $\frac{1}{2^n} < \varepsilon$ .

Consider  $y = (x_0 x_1 x_2 \dots x_n 0000 \dots) \in A$ . Then, by prop. 6.1.5,  $d(x, y) \leq \frac{1}{2^n} < \varepsilon$ .  $\square$

**Exercise 7.1.4.** Let  $A = \{x \in \Sigma : x \text{ is periodic}\}$ . Show that this is dense in  $\Sigma$ .

**Remark 7.1.5.**  $A$  in exercise 7.1.4 is exactly the set of periodic points for the shift map  $\sigma : \Sigma \rightarrow \Sigma$ .

#### Proposition 7.1.6

There exists  $z \in \Sigma$  such that  $\{\sigma^k(z) : k \in \mathbb{N} \cup \{0\}\}$  is dense in  $\Sigma$ .

*Proof.* Take  $z = (0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ \dots)$  to contain all possible blocks of increasing sizes.

Let  $x \in \Sigma$  and  $\varepsilon > 0$ . Again, let  $\frac{1}{2^n} < \varepsilon$ .

For some  $k$ ,  $\sigma^k(z)$  and  $x$  agree on the first  $n$  terms. This must exist because  $z$  has *every possible* sequence of  $n$  terms. That is, by prop. 6.1.5,  $d(\sigma^k(z), x) \leq \frac{1}{2^n} < \varepsilon$ .  $\square$

Warning: def. 7.1.7 is not the normal definition from applied math textbooks, but it is what we will use in the course.

**Definition 7.1.7** (dynamical system)

A metric space  $(X, d)$  together with a continuous function  $f : X \rightarrow X$ .

This is an abstract space in which we can do orbit analysis and all our fun stuff.

**Example 7.1.8.**  $\sigma : \Sigma \rightarrow \Sigma$  is a dynamical system (see thm. 6.2.2).

**Definition 7.1.9** (transitivity)

We say a dynamical system  $f : X \rightarrow X$  is transitive if for all  $x, y \in X$  and  $\varepsilon > 0$ , there exists  $z \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$  such that  $d(x, f^n(z)) < \varepsilon$  and  $d(y, f^m(z)) < \varepsilon$ .

Informally, given any two points, there is a special point whose orbit gets arbitrarily close to both points.

**Proposition 7.1.10**

$\sigma : \Sigma \rightarrow \Sigma$  is transitive.

*Proof.* Take  $z$  from prop. 7.1.6 such that the orbit is dense in  $\Sigma$ .

Then, for all  $\varepsilon > 0$  and  $x, y \in \Sigma$ , there must exist by the definition of density  $n$  and  $m$  such that  $d(x, \sigma^n(z)) < \varepsilon$  and  $d(y, \sigma^m(z)) < \varepsilon$ .  $\square$

**Definition 7.1.11** (sensitive dependence on initial conditions)

Let  $f : X \rightarrow X$  be a dynamical system.

We say  $f$  is sensitively dependent on initial conditions (or just sensitive) if

$$\exists \beta > 0, \forall \varepsilon > 0, \forall x \in X, \exists y \in X, \exists k \in \mathbb{N}$$

such that  $d(x, y) < \varepsilon$  and  $d(f^k(x), f^k(y)) \geq \beta$ .

Informally, there exists a “wrongness”  $\beta$  that can always be achieved in the orbit no matter how close two starting points are.

**Proposition 7.1.12**

$\sigma : \Sigma \rightarrow \Sigma$  is sensitive.

*Proof.* Take  $\beta = 1$ .

Let  $\varepsilon > 0$  and let  $x \in \Sigma$ . Say  $\frac{1}{2^n} < \varepsilon$  and pick  $y \in \Sigma$  such that  $0 < d(x, y) < \frac{1}{2^n}$ . That is,  $x$  and  $y$  must agree on the first  $n$  terms by prop. 6.1.5, but they are not equal.

Therefore, there exists  $k \geq n$  such that  $x_k \neq y_k$ .

In the distance  $d(\sigma^k(x), \sigma^k(y)) \geq \frac{|x_k - y_k|}{2^0} \geq 1 = \beta$ . □

## 7.2 Defining chaos

↓ Lectures 14 and 15 adapted from Imaad ↓

Lecture 14  
Feb 7

**Definition 7.2.1** (chaos)

A dynamical system  $f : X \rightarrow X$  is chaotic if

1. the periodic points for  $f$  are dense in  $X$ ,
2.  $f$  is transitive, and
3.  $f$  is sensitive.

**Theorem 7.2.2**

$\sigma : \Sigma \rightarrow \Sigma$  is chaotic.

*Proof.* By props. 7.1.6, 7.1.10 and 7.1.12. □

**Proposition 7.2.3**

Let  $(X, d), (Y, d')$  be metric spaces.

Suppose  $f : X \rightarrow Y$  is continuous and surjective. If  $A \subseteq X$  is dense in  $X$ , then  $f(A)$  is dense in  $Y$ .

*Proof.* Let  $y \in Y$  and say  $y = f(x)$ .

Let  $\epsilon > 0$ . Since  $f$  is continuous at  $x$ , there exists  $\delta > 0$  such that

$$d(z, x) < \delta \implies d'(f(z), f(x)) < \epsilon$$

for any  $z$ . In particular, since  $A$  is dense in  $X$ , we may find  $a \in A$  such that

$$d(a, x) < \delta \implies d'(f(a), f(x)) = d'(f(a), y) < \epsilon$$

as desired. □

**Theorem 7.2.4**

Let  $c < \frac{-(5+2\sqrt{5})}{4}$ . Then,  $Q_c : \Lambda \rightarrow \Lambda$  is chaotic.

*Proof.* (periodic point density) Note that  $Q_c^n(x) = x \iff S(Q_c^n(x)) = S(x) \iff \sigma^n(S(x)) = S(x)$ .

By prop. 7.2.3 applied to  $S^{-1} : \Sigma \rightarrow \Lambda$ , the periodic points for  $Q_c$  are dense in  $\Lambda$ .

(transitivity) Take  $z \in \Sigma$  from prop. 7.1.6 such that  $\{\sigma^K(z) : K \in \mathbb{N} \cup \{0\}\}$  is dense in  $\Sigma$ . Again by prop. 7.2.3,  $\{S^{-1}(\sigma^K(z)) : K \in \mathbb{N} \cup \{0\}\}$  is dense in  $\Lambda$ .

Note: Say  $S(x) = z$ , we know  $(S(Q_c^K(x))) = \sigma^K(S(x)) \iff Q_c^K(x) = S^{-1}(\sigma^K(S(x)))$

This,  $\{Q_c^K(x) : K \in \mathbb{N} \cup \{0\}\}$  is dense in  $\Lambda$ . As in prop. 7.1.10, we have that  $Q_c$  is transitive.

(sensitivity) Recall that  $\Lambda \subseteq I \setminus J_1 = I_0 \cup I_1$ . Let  $\beta > 0$  be less than the gap between  $I_0$  and  $I_1$ .

For  $x, y \in \Lambda$  with  $x \neq y$ , suppose  $S(x) \neq S(y)$ . Then, there must exist a  $k$  where  $k^{\text{th}}$  term of  $S(x)$  does not equal the  $k^{\text{th}}$  term of  $S(y)$ .

Hence,  $|Q_c^k(x) - Q_c^k(y)| > \beta$  and  $Q_c$  is sensitive. □

## Chapter 8

# Sarkovskii's Theorem

### Theorem 8.0.1 (period 3)

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. If  $f$  has a point with period 3, then  $f$  has a point with period  $n$  for all  $n \in \mathbb{N}$ .

Lecture 15  
Feb 9

### Proposition 8.0.2

Let  $I \subseteq J$  be closed intervals and suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous. If  $f(I) \supseteq J$ , then  $f(x)$  has a fixed point in  $I$ .

### Proposition 8.0.3

Let  $I, J$  be closed intervals,  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous, and  $f(I) \supseteq J$ . Then, there exists a closed interval  $I' \subseteq I$  such that  $f(I') = J$ .

We can now prove thm. 8.0.1.

*Proof.* Let  $a \in \mathbb{R}$  be a period 3 point for  $f(x)$ . Say  $f(a) = b, f(b) = c, f(c) = a$ . WLOG, suppose  $a < b$  and  $a < c$ .

Suppose  $a < b < c$ . The case where  $a < c < b$  is left as an exercise.

Let  $I = [a, b]$  and  $J = [b, c]$ . Then,  $f(a) = b$  and  $f(b) = c$  imply by IVT that  $[b, c] = J \subseteq f(I)$ . Likewise,  $f(b) = c$  and  $f(c) = a$  imply by IVT that  $[a, c] = I \cup J \subseteq f(J)$ .

Since  $J \subseteq f(J)$ , there exists a closed interval  $A_1 \subseteq J$  such that  $f(A_1) = J$  by prop. 8.0.3. Again,  $A_1 \subseteq J = f(A_1)$ , so there exists a closed interval  $A_2 \subseteq A_1$  such that  $f(A_2) = A_1$ .

Now, fix  $n > 3$ . Repeating the above process, we can find  $A_{n-2} \subseteq A_{n-3} \subseteq \dots \subseteq A_2 \subseteq A_1 \subseteq J$  such that  $f(A_i) = A_{i-1}$ . Now,  $f(I) \supseteq J \supseteq A_{n-2}$  means there exists a closed interval  $A_{n-1} \subseteq I$  such that  $f(A_{n-1}) = A_{n-2}$ .

Moreover,  $f(J) \supseteq I \supseteq A_{n-1}$  which means there exists a closed interval  $A_n \subseteq J$  such that  $f(A_n) = A_{n-1}$ .

We have  $f^n(A_n) = J$  and  $A_n \subseteq J$ . By prop. 8.0.2, there exists  $x_0 \in A_n$  such that  $f^n(x_0) = x_0$ .

Note: for  $x_0 \in A_n$ ,  $f(x_0) \in A_{n-1} \subseteq I$ ,  $f^i(x_0) \in J$  for  $i = 2, 3, \dots, n$ .

For contradiction, suppose  $f^i(x_0) = x_0$  for  $i < n$ .

Then,  $\overbrace{f(x_0)}^{\in I} = \overbrace{f^{i+1}(x_0)}^{\in J} = b$  so  $f(x_0) = b$ ,  $f^2(x_0) = c$ , and  $f^3(x_0) = a$ , which is a contradiction because  $f^3(x_0) \in J$  but  $a \notin J$ . Hence,  $x_0$  has period  $n$ .

That is,  $f$  has a periodic point with period  $n$  for all  $n > 3$ .

Further,  $f(J) \supseteq J$  and so by prop. 8.0.2,  $f$  has a fixed point (aka period 1) in  $J$ .

Finally,  $f(I) \supseteq J$  means  $J = f(I')$  and  $f(J) \supseteq I'$  means  $f(J') = I'$ . This implies  $f^2(j') = f(I') = J \supset J'$ . Therefore, we know there exists  $x \in J'$  such that  $f^2(x) = x$ .

If  $f(x) = x$ , then  $x \in J'$  and  $f(x) \in I'$ , meaning  $x = b$ . But,  $f(b) \neq b = c$ , contradiction.

Hence,  $x$  has period 2.

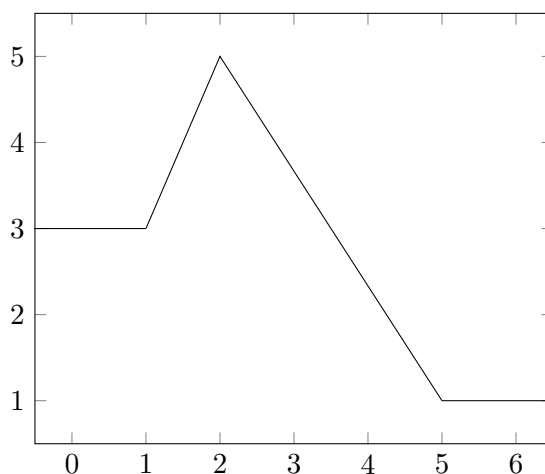
Therefore, since we already supposed  $f$  has a period 3 point,  $f$  has a period  $n$  point for all  $n$ .  $\square$

**Exercise 8.0.4.** Complete the proof for the case where  $a < c < b$ .

↑ Lectures 14 and 15 adapted from Imaad ↑

Draw the continuous function

Lecture 16  
Feb 12



Then, the orbit of 1 is  $1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5 \mapsto 1$  and 1 has period 5.

**Claim 8.0.5.**  $f$  has no point with period 3.

*Proof.* Suppose that  $f$  has a point  $x$  with period 3. Then,  $1 \leq x \leq 5$ .

Suppose  $x \in [1, 2]$ . Then,  $x \in [1, 2] \cap f^3([1, 2])$  since  $x = f^3(x)$ . But  $f^3([1, 2]) = [2, 5]$ , so  $x = 2$ . However, 2 has period 5 since it is on the same 5-cycle given above.

Suppose instead that  $x \in [2, 3]$ . Then,  $x \in [2, 3] \cap f^3([2, 3]) = [2, 3] \cap [3, 5] = \{3\}$  which is also on the 5-cycle.

If  $x \in [4, 5]$ , then  $x \in [4, 5] \cap f^3([4, 5]) = [4, 5] \cap [1, 4] = \{4\}$  which is, again, on the 5-cycle.

Finally, suppose that  $x \in [3, 4]$ . Then,  $f([3, 4]) = [2, 4]$  and it is strictly decreasing. Further,  $f([2, 4]) = [2, 5]$  and it is also strictly decreasing. Once more,  $f([2, 5]) = [1, 5]$  and it is again strictly decreasing. Since  $f^3$  is strictly decreasing, it has a unique fixed point in  $[3, 4]$ , but it is just the fixed point of  $f$ .

Since we have covered the entire interval  $[1, 5]$ ,  $x$  must not exist.  $\square$

**Example 8.0.6.** The function  $f(x) = \begin{cases} 1 & x < -1 \\ -x & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$  has a period 1 point at  $x = 0$ , period 2 points  $[-1, 1] \setminus \{0\}$ , and no other periodic points.

**Definition 8.0.7** (Sarkovskii ordering)

Start by ordering the odd numbers  $3 \prec 5 \prec 7 \prec 9 \prec \dots$

Then, all those are  $\dots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec \dots$

All those are  $\dots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec \dots$

Complete the ordering as  $\dots \prec 2^n \prec 2^{n-1} \prec \dots \prec 2^2 \prec 2 \prec 1$ .

This is a total order on the natural numbers.

**Example 8.0.8.**

- $26 = 2 \cdot 13 \prec 2^2 \cdot 5 = 40$  because the exponent of 2 is smaller.
- $3072 = 2^{10} \cdot 3 \prec 2^5 = 32$  because powers of 2 are big.
- $n \prec 1$  for all  $n$ .
- $2^{15} \prec 2^3$  since the powers of 2 are ordered backwards.

**Theorem 8.0.9** (Sarkovskii's theorem)

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. Suppose  $n \prec m$  in the Sarkovskii ordering. Then, if  $f$  has a point with period  $n$ , then it has a point with period  $m$ .



# Chapter 9

## Fractals

### 9.1 Definitions and dimensions

#### Definition 9.1.1

Define a few things from topology.

- For  $\mathbf{x} \in \mathbb{R}^n$ , the norm  $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$
- $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$  is our default metric on  $\mathbb{R}^n$
- For  $\mathbf{x} \in \mathbb{R}^n$ ,  $\varepsilon > 0$ , the open ball of radius  $\varepsilon$  centered at  $x$  is  $B_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| < \varepsilon\}$
- We say  $U \subseteq \mathbb{R}^n$  is open if for all  $\mathbf{x} \in U$ , there exists  $\varepsilon > 0$  such that  $B_\varepsilon(\mathbf{x}) \subseteq U$ .
- The boundary  $\delta(A)$  of a set  $A \subseteq \mathbb{R}^n$  is the closure of  $A$  without the interior of  $A$ .

Lecture 17  
Feb 14

#### Definition 9.1.2 (topological dimension (zero case))


We say  $S \subseteq \mathbb{R}^n$  has topological dimension  $\dim_t S = 0$  if for all  $\mathbf{x} \in S$ , there exists arbitrarily small open sets  $U \ni \mathbf{x}$  such that  $\delta(U) \cap S = \emptyset$ .

**Example 9.1.3.** Let  $X = \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{0\}$ . Then, since we can draw balls  $B_\varepsilon(\frac{1}{n})$  separating each point,  $\dim_t X = 0$ .

**Example 9.1.4.**  $X = \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{0\}$  has topological dimension 0.

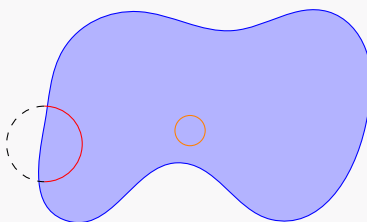
#### Definition 9.1.5 (topological dimension (non-zero case))

A set  $S \subseteq \mathbb{R}^n$  has topological dimension  $k \in \mathbb{N}$  if for all  $\mathbf{x} \in S$ , there exists arbitrarily small  $U \ni \mathbf{x}$  such that  $\delta(U) \cap S$  has topological dimension  $k-1$ , where  $k$  is minimal with this property.

**Example 9.1.6.** Consider a line  $X = \text{---}$ . Then, since any ball's boundary  creates an intersection made of two distinct points (i.e., a set with topological dimension 0), we know that  $\dim_t X = 1$ .

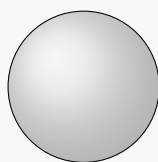
**Example 9.1.7.** Let  $X$  be a circle . Again, any ball's boundary  still only has two intersecting points, so  $\dim_t X = 1$ .

**Example 9.1.8.** Let  $X$  be a filled 2D region.



Then, the intersection of a ball's boundary will give either a **circle** or an **arc**, which have topological dimension 1, so the region has topological dimension 2.

**Example 9.1.9.** Let  $X$  be a non-filled sphere.



Then, the intersection of a 3D ball's boundary will give a circle, which has topological dimension 1, so  $\dim_t X = 2$ .

**Example 9.1.10.** Let  $X$  be a filled sphere.

Then, a 3D ball's boundary's intersection is either a hollow sphere or a spherical cap, which each have topological dimension 2, so  $\dim_t X = 3$ .

**Definition 9.1.11** (fractal dimension)

We say  $S \subseteq \mathbb{R}^n$  is self-similar if  $S$  may be divided into  $K$  congruent subsets, each of which may be magnified by a fixed  $M$  to yield  $S$  itself.

The fractal dimension of  $S$  is given by  $\dim_f S = \frac{\ln K}{\ln M}$ .

**Definition 9.1.12** (fractal)

A fractal is a self-similar  $S \subseteq \mathbb{R}^n$  such that  $\dim_f S > \dim_t S$ .

## 9.2 Fractal gallery

**Example 9.2.1.** Let  $X = \text{•} \text{---} \text{•}$  be a line. Then, since we can divide it into  $n$  smaller lines each of size  $\frac{1}{n}$ , it has fractal dimension  $\dim_f X = \frac{\ln n}{\ln n} = 1$ . The topological dimension is  $\dim_t X = 1$ .

So this is not a fractal, and is indeed just boring (not a fractal).

**Example 9.2.2** (Sierpinski triangle). Let  $X$  be the Sierpinski triangle, i.e., the limiting point of the process:



Then, the topological dimension is  $\dim_t X = 1$  because, in the limit, any ball will touch only single points. In particular, we can imagine balls touching the three points of a triangle.

However, the fractal dimension is  $\dim_f X = \frac{\ln 3}{\ln 2} \approx 1.58 > 1$  because each step is consisted of 3 previous steps scaled by  $\frac{1}{2}$ . so  $X$  is a fractal!

**Example 9.2.3** (Cantor set). Let  $K$  be a middle-thirds Cantor set, i.e., the limiting point of the process:



For any point in the Cantor set, we can find a small empty region around it since we keep

Lecture 18  
Feb 16

cutting away from the sides. That is,  $\dim_t K = 0$ . However,  $\dim_f K = \frac{\ln 2}{\ln 3} > 0$ .

**Example 9.2.4** (Koch curve). Let  $X$  be the Koch curve, where each line segment is replaced by a bump:



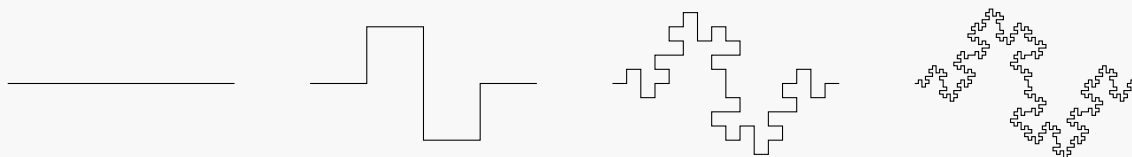
As a continuous line, intersection with a ball boundary gives points, so  $\dim_t X = 1$ . We have four copies scaled by  $\frac{1}{3}$ , so  $\dim_f X = \frac{\ln 4}{\ln 3} > 1$ .

**Example 9.2.5** (box fractal). Let  $X$  be a box fractal, where we delete edge pieces of a  $3 \times 3$  grid:



Then, since the squares are solid, we have topological dimension 1 but fractal dimension  $\dim_f X = \frac{\ln 5}{\ln 3} > 1$ .

**Example 9.2.6** (Minkowski sausage). Let  $X$  be the Minkowski sausage, where each line segment is replaced by a square wave:



Then, as a continuous line,  $\dim_t X = 1$ , but we have  $\dim_f X = \frac{\ln 8}{\ln 4} = \frac{3}{2} > 1$ .

There is a hidden connection between iterated systems and fractals! For example, playing around with the website <http://www.shodor.org/interactivate/activities/TheChaosGame/> has a process where each iteration moves a point halfway to one of the vertices.

...one reading week later...

Lecture 19  
Feb 26

Recall the chaos game:

1. Start with the vertices  $(v_1, v_2, v_3)$  of an equilateral triangle.
2. Pick  $p \in_{\mathbb{R}} \mathbb{R}^2$ .
3. Pick  $v_i \in_{\mathbb{R}} \{v_1, v_2, v_3\}$ .

4. Replace  $p$  with the midpoint of  $p$  and  $v_i$ .
5. Iterate.

Where does the orbit of  $p$  end up? Somehow, exactly in the Sierpinski triangle. Our goal is to formalize this.

### 9.3 Iterated function systems

Fix some  $\mathbf{p}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$  and contraction factor  $0 < \beta < 1$ . Consider

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \beta \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

i.e.,  $F(\mathbf{p}) = \beta(\mathbf{p} - \mathbf{p}_0) + \mathbf{p}_0$ . Then,

1.  $F(\mathbf{p}_0) = \mathbf{p}_0$
2.  $\|F(\mathbf{p}) - F(\mathbf{p}_0)\| = \|\beta(\mathbf{p} - \mathbf{p}_0)\| = \beta\|\mathbf{p} - \mathbf{p}_0\|$
3.  $\|F^n(\mathbf{p}) - \mathbf{p}_0\| = \beta^n\|\mathbf{p} - \mathbf{p}_0\| \rightarrow \mathbf{0}$  so  $F^n(\mathbf{p}) \rightarrow \mathbf{p}_0$

#### Definition 9.3.1

Let  $0 < \beta < 1$  and  $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{R}^2$ . For each  $i = 1, \dots, n$ , let

$$F_i(\mathbf{p}) = \beta(\mathbf{p} - \mathbf{p}_i) + \mathbf{p}_i$$

Then,  $\{F_1, \dots, F_n\}$  is an iterated function system (IFS).

Fix  $\mathbf{q}_0 \in \mathbb{R}^2$ . Randomly select an  $F_i$ . Let  $\mathbf{q}_1 = F_i(\mathbf{q}_0)$ . Repeat. The set of points in which the orbit  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \dots$  lives is the attractor for the IFS.

**Example 9.3.2.** Formalize the chaos game. Let  $\mathbf{p}_1 = v_1$ ,  $\mathbf{p}_2 = v_2$ ,  $\mathbf{p}_3 = v_3$ , and  $\beta = \frac{1}{2}$ . Then,  $F_i(\mathbf{p}) = \frac{1}{2}(\mathbf{p} - \mathbf{p}_i) + \mathbf{p}_i = \frac{1}{2}(\mathbf{p} + \mathbf{p}_i)$  is the midpoint.

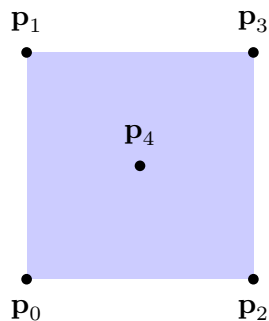
The set  $\{F_1, F_2, F_3\}$  is an iterated function system whose attractor is the Sierpinski triangle.

Note that we can construct pathologically unlucky sequences of  $F_i$ 's that give us point sequences that never reach the attractor. However, we ignore those :)

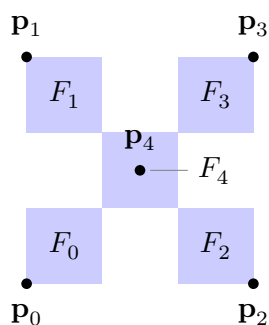
**Example 9.3.3.** Let  $\mathbf{p}_0 = (0, 0)^\top$ ,  $\mathbf{p}_1 = (1, 0)^\top$ ,  $\mathbf{p}_2 = (0, 1)^\top$ ,  $\mathbf{p}_3 = (1, 1)^\top$ ,  $\mathbf{p}_4 = (\frac{1}{2}, \frac{1}{2})^\top$ , and  $\beta = \frac{1}{3}$ .

What fractal does this produce?

*Solution.* Draw the points:



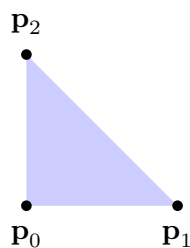
Divide the square into thirds (since we are using  $\beta = \frac{1}{3}$ ). Then, colour in the images of the square under each  $F_i$ :



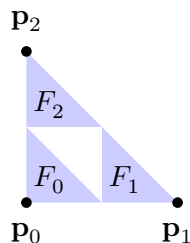
This is going to produce the [box fractal](#). □

**Example 9.3.4.** Repeat with  $p_0 = (0, 0)^\top$ ,  $p_1 = (1, 0)^\top$ ,  $p_2 = (0, 1)^\top$ , and  $\beta = \frac{1}{2}$ .

*Solution.* Again, draw the points:



Shrink the right triangle by a factor of  $\beta = \frac{1}{2}$  around each point:



This will generate a Sierpinski-like triangle.  $\square$

**Example 9.3.5.** Let  $\mathbf{p}_0 = (0, 0)^\top$ ,  $\mathbf{p}_1 = (1, 0)^\top$ ,  $\beta = \frac{1}{3}$ . Repeat.

Lecture 20  
Feb 28

*Solution.* Write the functions explicitly

$$F_0(\mathbf{x}) = \frac{1}{3} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x \\ \frac{1}{3}y \end{bmatrix} \quad \text{and} \quad F_1(\mathbf{x}) = \frac{1}{3} \begin{bmatrix} x-1 \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x + \frac{2}{3} \\ \frac{1}{3}y \end{bmatrix}$$

and pick a point  $\mathbf{q}_0 = (x_0, y_0)^\top \in \mathbb{R}^2$ . We say that the orbit of  $q_0$  under  $\{F_0, F_1\}$  is  $q_0, q_1, q_2, \dots$  with random selections  $s_1, s_2, s_3, \dots \in \{0, 1\}$  where  $q_i = F_{s_i}(q_{i-1})$ .

First, notice that no matter which one we choose,  $y_i = \frac{1}{3}y_{i-1}$ . Therefore,  $y_n = \frac{1}{3^n}y_0 \rightarrow 0$ .

For the  $x$ -coordinate, we can write it out explicitly to find the pattern:

$$\begin{aligned} x_1 &= \frac{1}{3}x_0 + \frac{2s_1}{3} \\ x_2 &= \frac{1}{3^2}x_0 + \frac{2s_1}{3^2} + \frac{2s_2}{3} \\ &\vdots \\ x_n &= \frac{1}{3^n}x_0 + \frac{2s_1}{3^n} + \frac{2s_2}{3^{n-1}} + \dots + \frac{2s_n}{3} \end{aligned}$$

As  $n \rightarrow \infty$ , the first term disappears. The remaining term looks like a funny ternary expansion. Therefore,  $x_n$  gets arbitrarily close to points of the form  $\sum_{i=1}^{\infty} \frac{t_i}{3^i}$  where  $t_i \in \{0, 2\}$ .

However, the set of points whose ternary expansion uses only 0s and 2s is exactly the Cantor set from def. 5.0.1 (see rem. 5.0.6).

Therefore, the attractor of the IFS  $\{F_0, F_1\}$  is  $\{(x, 0)^\top : x \in \text{Cantor set}\}$ .  $\square$

## 9.4 Generated iterated function systems

We want to generalize our definition of IFSs and fractals so that we can play with things that look exactly like fractals (for example, where the scaling factor differs for each piece).

### Definition 9.4.1 (affine transformation)

A function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by  $F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$  where  $A \in M_n(\mathbb{R})$  and  $\mathbf{b} \in \mathbb{R}^n$ . If  $\mathbf{b} = \mathbf{0}$ , we recover the linear transformations.

We call  $F$  a linear contraction if there exists  $0 < \lambda < 1$  such that  $\|F(\mathbf{x}) - F(\mathbf{y})\| < \lambda\|\mathbf{x} - \mathbf{y}\|$ .

In general, “affine” just means linear but shifted.

**Example 9.4.2.** Let  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  and  $0 < \beta < 1$ .

Then,  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $F(\mathbf{x}) = (\beta A\mathbf{x} + \mathbf{b})$  is a linear contraction.

This linear contraction (1) scales by  $\beta$ , (2) rotates counter-clockwise by  $\theta$ , and (3) translates by  $\mathbf{b}$ .

**Definition 9.4.3** (compactness)

A subset  $A \subseteq \mathbb{R}^n$  is compact if  $A$  is closed and bounded.

Write  $\mathcal{K}_n$  for the set of all non-empty compact subsets of  $\mathbb{R}^n$ .

**Definition 9.4.4** (generalized iterated function system)

Let  $F_1, \dots, F_k : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be linear contractions. We call  $F : \mathcal{K}_n \rightarrow \mathcal{K}_n$  given by

$$F(A) = F_1(A) \cup F_2(A) \cup F_3(A) \cup \dots \cup F_k(A)$$

a (generalized) iterated function system.

This is well-defined since finite unions and the  $F_i$ 's continuity preserve closure and compactness.

We will now:

1. Equip  $\mathcal{K}_n$  with a metric.
2. Show  $F$  has a unique fixed point  $A^*$  and for all  $A \in \mathcal{K}_n$ ,  $F^n(A) \rightarrow A^*$ . The point  $A^*$  is the attractor of  $F$  (and is a fractal!).

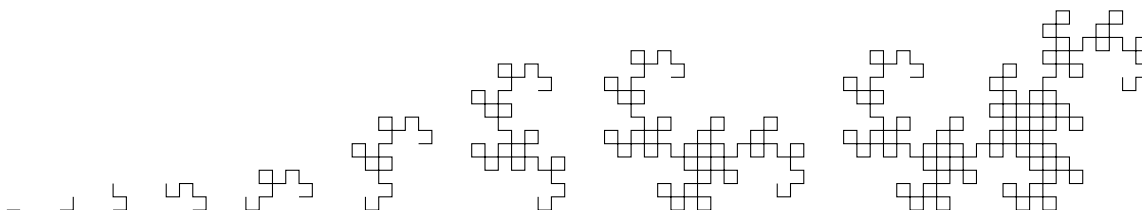
**Example 9.4.5.** Let  $F_1(\mathbf{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{x}$  and  $F_2(\mathbf{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

Find the attractor.

*Solution.* Notice that  $F_1$  will (1) scale by  $\frac{1}{\sqrt{2}}$  and (2) rotate by  $\frac{\pi}{4}$ . Then,  $F_2$  will (1) scale by  $\frac{1}{\sqrt{2}}$ , (2) rotate by  $\frac{3\pi}{4}$ , and (3) shift one unit left.

Consider the line  $L$  from  $(0,0)$  to  $(1,0)$ .

Then, we can draw:



This fractal, the dragon fractal tiles the space. □



**Remark 9.4.6.** For all  $A \in \mathcal{K}_n$ ,  $F_i(A) \in \mathcal{K}_n$ . This is because the continuous image of a compact set is compact (beyond the scope of this course).

We can now equip  $\mathcal{K}_n$  with a metric. We will consider  $F : \mathcal{K}_n \rightarrow \mathcal{K}_n : A \mapsto F_1(A) \cup \dots \cup F_k(A)$ . This is well-defined since we already showed that the finite union of closed sets are closed, and it is trivial to show that the finite union of bounded sets is bounded.

We will then show that  $F$  has a unique fixed point  $A^* \in \mathcal{K}_n$  and that for all  $A \in \mathcal{K}_n$ ,  $F^n(A) \rightarrow A^*$ .

**Example 9.4.7.** Let  $F_1(\mathbf{x}) = \frac{1}{2}\mathbf{x}$ ,  $F_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ , and  $F_3(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{4} \end{bmatrix}$ .

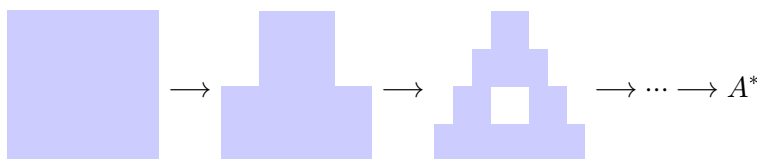
Find the attractor.

*Solution.* Let  $A$  be the filled triangle with vertices  $(0,0)^\top$ ,  $(1,0)^\top$ ,  $(\frac{1}{2}, \frac{\sqrt{3}}{2})^\top$ :



This is the Sierpinski triangle.

Alternatively, we could have started with a square:



or with a goose:

TODO

but these all converge to the same attractor. □

**Example 9.4.8.** Repeat with  $F_1(\mathbf{x}) = \frac{1}{3}\mathbf{x}$ ,  $F_2(\mathbf{x}) = \frac{1}{3} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$ ,  $F_3(\mathbf{x}) = \frac{1}{3} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$ , and  $F_4(\mathbf{x}) = \frac{1}{3}\mathbf{x} + \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix}$ .

*Solution.* Let  $L$  be the line segment from  $(0,0)^\top$  to  $(1,0)^\top$ . Then:



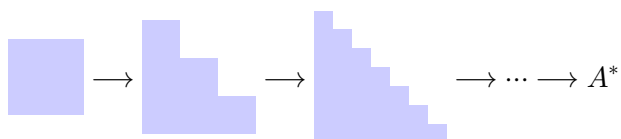
The attractor converges to the Koch curve. □

**Example 9.4.9.** Let  $A = [0, 1] \times [0, 1]$  (i.e., the filled square).

Repeat with  $F_1(\mathbf{x}) = \mathbf{x}$ ,  $F_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $F_3(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

**Remark 9.4.10.** Since  $F_1$  is not a linear contraction,  $\lim_{n \rightarrow \infty} F^n(A)$  will depend on  $A$ .

*Solution.* Draw the  $[0, 1] \times [0, 1]$  square and iterate:



This is not a fractal by our strict definition (it is not even self-similar), but in our eyes and our hearts it's a fractal.  $\square$

**Definition 9.4.11** (Hausdorff metric)

Let  $\mathbf{v} \in \mathbb{R}^n$ ,  $A, B \in \mathcal{K}_n$ . First, define

$$d(\mathbf{v}, B) := \min\{\|\mathbf{v} - \mathbf{b}\| : \mathbf{b} \in B\}$$

(this should be an  $\inf\{\dots\}$  but since  $B$  is compact, the extreme value theorem gives us  $\min\{\dots\}$  instead)

Then, define

$$d(A, B) := \max\{d(\mathbf{a}, B) : \mathbf{a} \in A\}$$

i.e., the length of the longest direct path between points in  $A$  and  $B$ .

Finally, define

$$D(A, B) := \max\{d(A, B), d(B, A)\}$$

to fix the fact that  $d$  is not symmetric.

**Fact 9.4.12.**  $D$  is a metric on  $\mathcal{K}_n$

Lecture 22  
Mar 6

We take this fact without proof.

**Example 9.4.13.**  $A = \{(1, 1)\}$ , let  $B = \{(x, 0) : 0 \leq x \leq 1\}$

[figure]

Then,  $d(A, B) = 1$ ,  $d(B, A) = \sqrt{2}$ , and  $D(A, B) = \max\{1, \sqrt{2}\} = \sqrt{2}$ .

**Lemma 9.4.14**

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear contraction such that  $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq \lambda \|\mathbf{x} - \mathbf{y}\|$  for some  $\lambda \in (0, 1)$ .

Then, for  $A, B \in \mathcal{K}_n$ ,  $D(f(A), f(B)) \leq \lambda D(A, B)$ .

*Proof.* First, we have

$$d(f(a), f(B)) = \min_{b \in B} \|f(a) - f(b)\| \leq \min_{b \in B} \lambda \|a - b\| = \lambda \min_{b \in B} \|a - b\| = \lambda d(a, B)$$

and so

$$d(f(A), f(B)) = \max_{a \in A} d(f(a), f(B)) \leq \lambda \max_{a \in A} d(a, B) = \lambda d(A, B) \leq \lambda D(A, B)$$

Therefore,  $d(f(A), f(B)) \leq \lambda D(A, B)$ . Similarly,  $d(f(B), f(A)) \leq \lambda D(A, B)$ .

Hence,  $D(f(A), f(B)) \leq \lambda D(A, B)$ . □

**Lemma 9.4.15**

For  $A_1, A_2, B_1, B_2 \in \mathcal{K}_n$ ,

$$D(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$$

*Proof.* First,

$$\begin{aligned} d(A_1 \cup A_2, B_1 \cup B_2) &= \max_{a \in A_1 \cup A_2} d(a, B_1 \cup B_2) \\ &= \max \left\{ \max_{a \in A_1} d(a, B_1 \cup B_2), \max_{a \in A_2} d(a, B_1 \cup B_2) \right\} \\ &\leq \max \left\{ \max_{a \in A_1} d(a, B_1), \max_{a \in A_2} d(a, B_2) \right\} \end{aligned} \quad (\star)$$

by the min in the definition.

$$= \max\{d(A_1, B_1), d(A_2, B_2)\} \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$$

$$\text{Hence, } d(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$$

$$\text{Similarly, } d(B_1 \cup B_2, A_1 \cup A_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$$

$$\text{Therefore } D(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\} \quad \square$$

**Lemma 9.4.16**

Let  $F_1, \dots, F_k$  be linear contractions with contraction factor  $\lambda \in (0, 1)$ .

Consider  $F : \mathcal{K}_n \rightarrow \mathcal{K}_n$ ,  $F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$ . Then,  $D(F(A), F(B)) \leq \lambda D(A, B)$ .

*Proof.* We have,  $D(F(A), F(B)) \leq \max_{i=1, \dots, k} D(F_i(A), F_i(B))$  by lem. 9.4.15. By lem. 9.4.14,  $\leq \max_{i=1, \dots, k} \lambda D(A, B) = \lambda D(A, B)$ .  $\square$

### Definition 9.4.17

Let  $(X, d)$  be metric space.

1.  $(x_n) \subseteq X$  is Cauchy if  $\forall \epsilon > 0, \exists n \in \mathbb{N}$ , such that  $n, m \geq N \implies d(x_n, x_m) < \epsilon$ .
2.  $X$  is complete if every Cauchy sequence  $(x_n) \subseteq X$  converges to some  $x \in X$ .

**Fact 9.4.18.**  $(K_n, D)$  is complete.

We do not prove this.

### Theorem 9.4.19

Let  $F_1, \dots, F_k$  be linear contractions with contraction factor  $\lambda \in (0, 1)$ .

Let  $F : \mathcal{K}_n \rightarrow \mathcal{K}_n$  be the corresponding IFS. Then,

1.  $F$  has a unique fixed point  $A^*$ , which we call the attractor.
2. For all  $A \in \mathcal{K}_n$ ,  $F^m(A) \rightarrow A^*$ .

*Proof.* Fix  $A \in \mathcal{K}_n$ . Consider its orbit  $F^m(A)$ . Look at the distance

$$D(F^{m+1}(A), F^m(A)) = D(F^m(F(A)), F^m(A)) \leq \lambda^m D(F(A), A)$$

by lem. 9.4.16. Let  $\epsilon_m = \lambda^m D(F(A), A)$ . Then,  $\sum \epsilon_m$  converges, since  $|\lambda| < 1$ . Therefore, the sequence  $(F^m(A)) \subseteq \mathcal{K}_n$  is strongly Cauchy. In particular,  $F^m(A)$  is Cauchy, so there exists some  $F^m(A) \rightarrow A^* \in \mathcal{K}_n$  because  $\mathcal{K}_n$  is complete.

Since  $F$  is continuous,  $F^{m+1}(A) \rightarrow F(A^*)$ . Hence,  $F(A^*) = A^*$ .

Now, consider uniqueness. Suppose  $A^*$  and  $B^*$  are fixed points for  $F$ . Then,

$$D(A^*, B^*) = D(F(A^*), F(B^*)) \leq \lambda D(A^*, B^*)$$

but  $\lambda \in (0, 1)$ . This forces  $D(A^*, B^*) = 0$ , so  $A^* = B^*$ .  $\square$

Lecture 23  
Mar 8

## Chapter 10

# Complex Functions

### Definition 10.1.1 (complex derivative)

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$ . Then,

1. For  $z_0 \in \mathbb{C}$ , we say that

$$\lim_{z \rightarrow z_0} f(z) = L \in \mathbb{C}$$

if for all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$0 < |z - z_0| < \delta \implies |f(z) - L| < \varepsilon$$

2. The derivative of  $f(z)$  at  $z_0$  is

$$f'(z) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided the limit exists.

In general, we will write  $f(x)$  for a real-valued function and  $f(z)$  for a complex-valued function. Then, analogous to real-valued functions, we can consider complex fixed points.

### Definition 10.1.2 (complex fixed points)

Let  $a \in \mathbb{C}$  be a fixed point of  $f(z)$ . Then,

1.  $a$  is attracting if  $|f'(a)| < 1$ ,
2.  $a$  is repelling if  $|f'(a)| > 1$ , and
3.  $a$  is neutral if  $|f'(a)| = 1$ .

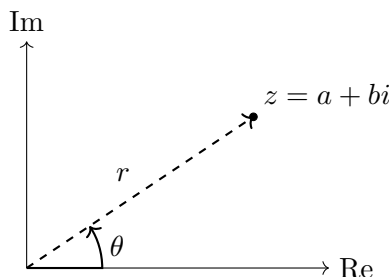
**Remark 10.1.3** (attracting/repelling complex fixed point theorems). We can obtain complex analogues of the proofs of the real-valued attracting/repelling fixed point theorems by replacing intervals around fixed points with open discs.

**Example 10.1.4.** Analyze the fixed points of  $f(z) = z^2 + z + 1$ .

*Solution.* The fixed points are  $z^2 + z + 1 = z \iff z^2 + 1 = 0 \iff z = \pm i$ .

Then,  $f'(z) = 2z + 1$ , so  $|f'(i)| = |2i + 1| = \sqrt{5} > 1$  and  $|f'(-i)| = |-2i + 1| = \sqrt{5} > 1$ , so both are repelling.  $\square$

Recall polar form. For some complex number  $z = a + ib$ , we can plot it as  $(a, b)$ :



Then, we can recall from MATH 135 that we can write  $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$  and we have really nice multiplication.

**Fact 10.1.5 (PMC, MATH 135).**  $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$  and  $(re^{i\theta})^n = r^n e^{in\theta}$ , which is just so much prettier than Cartesian multiplication.

In particular, for complex numbers of the form  $e^{2\pi i/n}$ , we have  $(e^{2\pi i/n})^n = e^{2\pi i} = 1$ , which is a nice way to generate periodic points.

**Example 10.1.6.** Let  $z = e^{2\pi i/3}$  and  $f(w) = w^2$ .

Lecture 24  
Mar 11

*Solution.* Write  $z = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ .

Then,  $f(z) = e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$  and  $f^2(z) = e^{8\pi i/3} = e^{2\pi i/3} = z$ .

That is,  $z$  is periodic with period 2.

We can then find  $|(f^2)'(z)| = |f'(z)f'(f(z))| = |-1 + i\sqrt{3}| \cdot |-1 - i\sqrt{3}| = 4 > 1$ , so  $z$  is attracting.  $\square$

# Chapter 11

## Julia Sets

### 11.1 Construction

**Notation** (quadratic family). For  $c \in \mathbb{C}$ , write  $Q_c(z) = z^2 + c$  just like the real one.

#### Definition 11.1.1

The filled Julia set for  $c$  is  $K_c = \{z \in \mathbb{C} : (Q_c^n(z)) \text{ is bounded}\}$ .

Equivalently,  $\{z \in \mathbb{C} : \exists M > 0, \forall n \in \mathbb{N}, |Q_c^n(z)| \leq M\}$ .

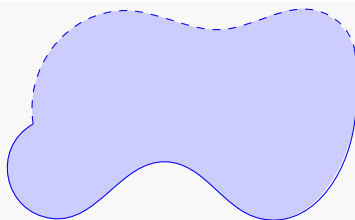
**Remark 11.1.2.** This is the complex analogue of  $\Lambda$  for  $Q_c(x) = x^2 + c$  where  $c \in \mathbb{R}$  and  $c < -2$ .

#### Definition 11.1.3

Let  $(X, d)$  be a metric space and  $A \subseteq X$ .

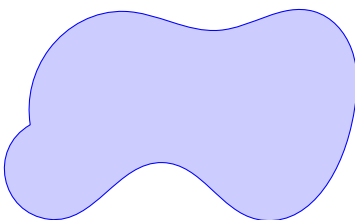
1. The closure of  $A$  is  $\overline{A} = \{x \in X : \exists (a_n) \subseteq A, a_n \rightarrow x\}$ .
2. The interior of  $A$  is  $\text{Int}(A) = \{x \in X : \exists \varepsilon > 0, B_\varepsilon(x) \subseteq A\}$ .
3. The boundary of  $A$  is  $\partial(A) = \overline{A} \setminus \text{Int}(A)$ .

**Example 11.1.4.** Let  $A$  be the blob

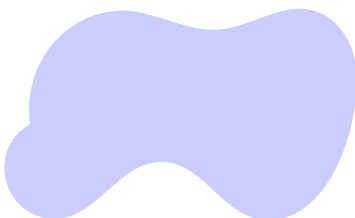


Find the closure, interior, and boundary.

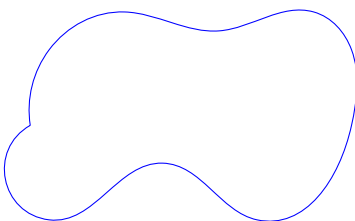
*Solution.* Since we can make a sequence of points that reaches the dashed open parts, the closure  $\overline{A}$  will simply be



Then, since we can draw a ball on the shaded inside but not on the edge, the interior  $\text{Int}(A)$  is



Finally, the boundary  $\partial(A)$  is



□

**Remark 11.1.5.**  $A$  is closed if and only if  $A = \overline{A}$ .

**Lemma 11.1.6** (Assignment 4)

$K_c$  is closed.



**Definition 11.1.7**

The Julia set for  $c$  is  $J_c = \partial(K_c)$ .

**Remark 11.1.8.** Since  $K_c$  is closed,  $J_c = \partial(K_c) = \overline{K_c} \setminus \text{Int}(K_c) = K_c \setminus \text{Int}(K_c)$ .

**Example 11.1.9.** Let  $c = 0$ , so  $Q_0(z) = z^2$ . What do  $K_0$  and  $J_0$  look like?

*Solution.* Let  $z = re^{i\theta}$ . Then,  $|Q_0(z)| = |r^2 e^{2i\theta}| = r^2$ . Likewise,  $|Q_0^2(z)| = |r^4 e^{2i\theta}| = r^4$ . Clearly,  $|Q_0^n(z)| = r^{2^n}$ . Therefore,  $K_0 = \{z \in \mathbb{C} : |z| \leq 1\}$  since that is when  $|z|^{2^n}$  is bounded.

This is the unit disc in the complex plane. Therefore,  $J_0 = \{z \in \mathbb{C} : |z| = 1\}$ , the unit circle.  $\square$

**Example 11.1.10.** Repeat with  $c = -2$ .

*Solution.* First, let  $R = \{z \in \mathbb{C} : |z| > 1\}$  and define a function  $H : R \rightarrow \mathbb{C} : z \mapsto z + \frac{1}{z}$ .

Then, we claim that  $H$  is injective. Suppose  $H(z) = H(w)$ . Then,

$$\begin{aligned} z + \frac{1}{z} &= w + \frac{1}{w} \\ zw &= z^2 + 1 - \frac{z}{w} \\ &= w^2 + 1 - \frac{w}{z} \\ w^2 - z^2 &= \frac{w}{z} - \frac{z}{w} = \frac{w^2 - z^2}{zw} \end{aligned}$$

This means that either  $zw = 1$  or  $w^2 - z^2 = 0$ . However,  $|zw| = |z| \cdot |w| > 1$ , so  $w = \pm z$ . Since  $H(w) = H(z)$ , we must pick  $w = +z$ , and we are done.

Now, claim that  $H : R \rightarrow \mathbb{C} \setminus [-2, 2]$  is surjective. Suppose that  $H(z) = w$ . Then,

$$\begin{aligned} z + \frac{1}{z} &= w \\ z^2 - wz + 1 &= 0 \\ z &= \frac{1}{2}(w \pm \sqrt{w^2 - 4}) \end{aligned}$$

and write  $z_+$  or  $z_-$  for the two possible  $z$ 's. Since these are roots of a polynomial with constant 1, we must have  $z_+ z_- = 1$ .

That is, either (1)  $|z_+| > 1$  and  $|z_-| < 1$ , (2)  $|z_+| < 1$  and  $|z_-| > 1$ , or (3)  $|z_+| = |z_-| = 1$ .

If either root is in  $R$ , then either  $H(z_+) = w$  or  $H(z_-) = w$ .

Otherwise,  $|z_+| = |z_-| = 1$ . Then,  $H(z) = H(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2 \cos \theta \in [-2, 2]$ .

Therefore,  $H$  is well-behaved (i.e., invertible) on  $R \rightarrow \mathbb{C} \setminus [-2, 2]$ .

Consider now  $H(Q_0(z)) = H(z^2) = z^2 + \frac{1}{z^2}$ . Note that  $Q_{-2}(H(z)) = (z + \frac{1}{z})^2 - 2 = z^2 + \frac{1}{z^2}$ . Hence,  $H(Q_0^n(z)) = Q_{-2}^n(H(z))$ .

This looks quite similar to  $S(Q_c^n(x)) = \sigma^n(S(x))$  in  $\mathbb{R}$ . We can say that  $H$  plays a similar role as  $S$ . In fact, (not course content),  $Q_0$  and  $Q_{-2}$  are conjugate because  $H$  is a homeomorphism between them.

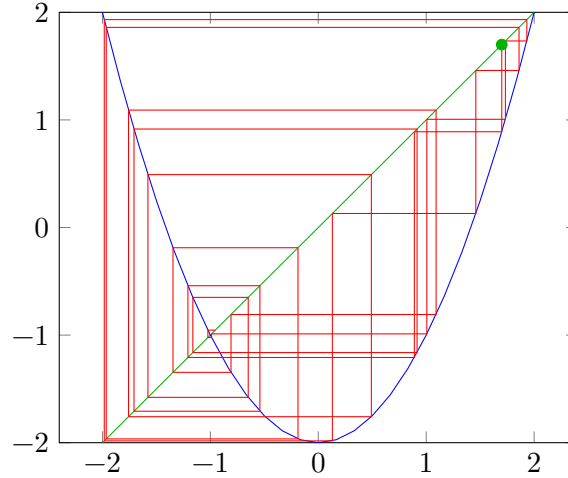
Let  $z_n$  be a diverging sequence  $|z_n| \rightarrow \infty$ . Note that  $|H(z_n)| = \left| z_n + \frac{1}{z_n} \right| \geq |z_n| - \frac{1}{|z_n|} \rightarrow \infty$ . Therefore, the image of the sequence  $|H(z_n)| \rightarrow \infty$  also diverges.

Let  $z \in \mathbb{C} \setminus [-2, 2]$ . Since  $H$  is surjective, we know there exists a  $w \in R$  such that  $z = H(w)$ , and see that

$$|Q_{-2}^n(z)| = |Q_{-2}^n(H(w))| = |H(\underbrace{Q_0^n(w)}_{\rightarrow \infty})| \rightarrow \infty$$

by the previous claim. Hence,  $z \notin K_{-2}$  and we have that  $K_{-2} \subseteq [-2, 2]$ .

Finally, let  $z \in [-2, 2]$ . By graphical analysis,



there is no way to escape the box. That is,  $z \in K_{-2}$ , i.e.,  $[-2, 2] \subseteq K_{-2}$ .

Therefore,  $K_{-2} = [-2, 2]$ , and we have that  $J_{-2} = [-2, 2]$ . □

### Proposition 11.1.11 (Escape Criterion)

If  $|z| \geq |c| > 2$ , then  $|Q_c^n(z)| \rightarrow \infty$ . In particular,  $z \notin K_c$ .

*Proof.* We can write

$$|Q_c(z)| = |z^2 + c| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1)$$

Suppose  $|z| > 2 + \lambda$  for some  $\lambda > 0$ . Then, we have that  $|z| - 1 > 1 + \lambda$ . Therefore,  $|Q_c(z)| \geq |z|(1 + \lambda)$ .

Iterating, we see that  $|Q_c^n(z)| \geq |z|(1 + \lambda)^n \rightarrow \infty$ . □

**Corollary 11.1.12.** Suppose  $|c| > 2$ . Then,  $|Q_c^n(0)| \rightarrow \infty$  and  $0 \notin K_c$ .

*Proof.* Let  $z = Q_c(0) = c$  and  $|z| = |c| > 2$ . By the [Escape Criterion](#),  $|Q_c^n(0)| \rightarrow \infty$ .  $\square$

**Corollary 11.1.13.** Let  $M = \max\{|c|, 2\}$ . If  $|z| > M$ , then  $|Q_c^n(z)| \rightarrow \infty$ . That is, we have that  $K_c \subseteq \{z : |z| \leq M\}$ .

*Proof.* We have  $|Q_c^n(z)| \geq (1 + \lambda)^n |z| \rightarrow \infty$  by the proof of the Escape Criterion (not the Escape Criterion itself because we don't know if  $|z| < 2$ ).  $\square$

**Remark 11.1.14** (assignment hint!). The fact that  $K_c$  is inside this bounded disc will help with the proof of its closedness.

**Corollary 11.1.15.** If there exists a  $k$  such that  $|Q_c^k(z)| > \max\{|c|, 2\}$ , then  $|Q_c^n(z)| \rightarrow \infty$ . That is,  $z \notin K_c$ .

Based on these results, we can develop the

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**Algorithm 1** Filled Julia set algorithm

---

```

1: Choose a large  $N \in \mathbb{N}$ .
2: for points  $z$  do
3:   if  $|Q_c^i(z)| > \max\{|c|, 2\}$  for any  $i \leq N$  then
4:     Colour  $z$  white
5:   else if  $|Q_c^i(z)| \leq \max\{|c|, 2\}$  for all  $i \leq N$  then
6:     Colour  $z$  black
```

---

whose black-shaded region approximates  $K_c$ .

# List of Named Results

1.2.5	Proposition (convergence implies boundedness)	6
1.2.7	Proposition (limit laws)	7
1.2.10	Theorem (completeness of $\mathbb{R}$ )	8
1.3.6	Theorem (Banach contraction mapping theorem)	9
3.1.5	Theorem (attracting fixed point theorem)	17
3.1.7	Theorem (repelling fixed point theorem)	18
5.0.4	Theorem (Cantor sets are closed)	26
6.1.12	Proposition (sequential characterization of continuity in metric spaces)	31
6.2.3	Theorem (monotone convergence theorem)	31
6.2.4	Lemma (nested intervals lemma)	32
8.0.1	Theorem (period 3)	38
8.0.9	Theorem (Sarkovskii's theorem)	40
11.1.6	Lemma (Assignment 4)	56
11.1.11	Proposition (Escape Criterion)	58

# Index of Defined Terms

- affine transformation, 47
- attractor, 45, 52
- bifurcation, 23
- boundary, 41, 55
- Cantor set, 26
- Cantor space, 29
- Cauchy, 52
- chaos, 36
- closure, 55
- cobweb plot, 12
- compactness, 48
- complete, 52
- contraction, 9
- density, 34
- doubling function, 5
- dragon fractal, 48
- dynamical system, 35
- filled Julia set, 55
- fixed point, 4
  - attracting, 17
  - weakly, 19
  - neutral, 17
  - repelling, 17
  - weakly, 20
- fractal, 43
- fractal dimension, 43
- function
  - continuity, 8
- generalized iterated
  - function system, 48
- Hausdorff metric, 50
- homeomorphism, 31
- interior, 55
- interval
  - closure, 9
- iterated function system, 45
- iteration, 3
- itinerary, 28
- Julia set, 57
- linear contraction, 47
- metric, 29
- metric space, 29
  - continuity, 30
  - convergence, 31
- non-isolated, 17
- norm, 41
- open, 41
- open ball, 41
- orbit, 3
  - constant, 4
  - eventually periodic, 5
  - periodic, 5
- period, 5
- periodic point, 5
  - attracting, 20
  - neutral, 20
  - repelling, 20
- periodicity, 5
- quadratic family, 22
- Sarkovskii ordering, 40
- self-similar, 43
- sensitive, 35
- sequence
  - bounded, 6
  - Cauchy, 7
  - convergence, 6
- shift map, 30
- strongly-Cauchy, 10
- tangent bifurcation, 24
- topological dimension, 41
- transitivity, 35