1 Exercises

1.1 Logic06

Theorem (Double Negation). $A \dashv\vdash \neg\neg A$

Proof. In the forwards direction:

(1)	$\neg\neg A \vdash \neg\neg A$	(Ref)
(2)	$\neg\neg A, \neg A \vdash \neg\neg A$	(+, 1)
(3)	$\neg A \vdash \neg A$	(+)
(4)	$\neg\neg A, \neg A \vdash \neg A$	(+, 3)
(5)	$\neg\neg A \vdash \neg A$	($\neg$ −, 1, 3)

In the backwards direction:

(1)	$A, \neg A \vdash A$	($\in$)
(2)	$A, \neg A \vdash \neg A$	($\in$)
(3)	$A \vdash \neg\neg A$	($\neg$ +, 1, 2)

□

Theorem (Inconsistency Rule). $\neg A, A \vdash B$

Proof.

(1)	$A, \neg A, \neg B \vdash A$	($\in$)
(2)	$A, \neg A, \neg B \vdash \neg A$	($\in$)
(3)	$A, \neg A, \vdash B$	($\neg$ −, 1, 2)

□

Theorem (Disjunctive Syllogism). $A \vee B, \neg A \vdash B$

Proof.

(1)	$\neg A, B \vdash B$	($\in$)
(2)	$\neg A, A \vdash B$	(Inconsistency Rule)
(3)	$\neg A, A \vee B \vdash B$	($\vee$ −, 1, 2)

□

Theorem (Modus Tollens). $\neg B, A \to B \vdash \neg A$

Proof.

(1)	$\neg B, \neg A \vee B \vdash \neg A$	(Disjunctive Syllogism)
(2)	$\neg A \vee B \vdash A \to B$	(Implication)
(3)	$\neg B, A \to B \vdash \neg A$	(Repl., 1, 2)

□

Theorem (Contrapositive). $A \to B \vdash \neg A \to \neg B$

Proof. In the forwards direction:

(1)	$A \to B, \neg B \vdash \neg A$	(Modus Tollens)
(2)	$A \to B \vdash \neg B \to \neg A$	$(\to +, 1)$

In the backwards direction:

(1)	$B \vee \neg A, \neg\neg A \vdash B$	(Disjunctive Syllogism)
(2)	$B \vee \neg A, A \vdash B$	(Repl., Double Negation, 1)
(3)	$\neg\neg B \vee \neg A, A \vdash B$	(Repl., Double Negation, 2)
(4)	$\neg B \to \neg A, A \vdash B$	(Repl., Implication, 3)
(5)	$\neg B \to \neg A \vdash A \to B$	$(\to +, 4)$

□

Theorem (Affirmation)**.** *$A \vdash B$ if and only if $\emptyset \vdash A \to B$*

Proof. In the forwards direction:

(1)	$\emptyset \vdash A \to B$	(Premise)
(2)	$A \vdash A \to B$	$(+, 1)$
(3)	$A \vdash A$	(Ref)
(4)	$A \vdash B$	$(\to -, 3, 2)$

The backwards direction is just $(\to +)$ on the premise. □

Theorem (Flip-Flop)**.** *If $A \vdash B$, then $\neg B \vdash \neg A$.*

Proof. Suppose $A \vdash B$. Then,

(1)	$A \vdash B$	(Premise)
(2)	$\emptyset \vdash A \to B$	$(\to +, 1)$
(3)	$\emptyset \vdash \neg B \to \neg A$	(Repl., Contrapositive, 2)
(4)	$\neg B \vdash \neg A$	(Affirmation, 3)

as desired. □

Theorem (De Morgan 1)**.** $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$

Proof. In the forwards direction:

(1)	$\neg(A \vee B), A \vdash A$	$(\in)$
(2)	$\neg(A \vee B), A \vdash A \vee B$	$(\vee +, 1)$
(3)	$\neg(A \vee B), A \vdash \neg(A \vee B)$	$(\in)$
(4)	$\neg(A \vee B) \vdash \neg A$	$(\neg +, 2, 3)$
(5)	$\neg(A \vee B), B \vdash B$	$(\in)$
(6)	$\neg(A \vee B), B \vdash A \vee B$	$(\vee +, 5)$
(7)	$\neg(A \vee B), B \vdash \neg(A \vee B)$	$(\in)$
(8)	$\neg(A \vee B) \vdash \neg B$	$(\neg +, 6, 7)$
(9)	$\neg(A \vee B) \vdash \neg A \wedge \neg B$	$(\wedge +, 5, 8)$

Backwards:

(1) $\neg A \wedge \neg B, A \vee B \vdash A \vee B$ $(\in)$
(2) $\neg A \wedge \neg B, A \vee B \vdash \neg A \wedge \neg B$ $(\in)$
(3) $\neg A \wedge \neg B, A \vee B \vdash \neg A$ $(\wedge -, 2)$
(4) $\neg A \wedge \neg B, A \vee B \vdash \neg B$ $(\wedge -, 2)$
(5) $\neg A \wedge \neg B, A \vee B \vdash A$ (Disjunctive Syllogism, 1, 4)
(6) $\neg A \wedge \neg B \vdash \neg(A \vee B)$ $(\neg +, 3, 5)$

□

Theorem (Implication)**.** $A \to B \dashv\vdash \neg A \vee B$

Proof. Backwards:

(1) $\neg A \vee B, A \vdash A$ $(\in)$
(2) $\neg A \vee B, A \vdash \neg\neg A$ (Double Negation, 1)
(3) $\neg A \vee B, A \vdash \neg A \vee B$ $(\in)$
(4) $\neg A \vee B, A \vdash B$ (Disjunctive Syllogism, 3, 2)
(5) $\neg A \vee B \vdash A \to B$ $(\to +, 4)$

□