PMATH 370 Winter 2024:
Lecture Notes

Lecture notes taken, unless otherwise specified, by myself during the Winter 2024 offering of PMATH 370, taught by Blake Madill.

Lectures

Chapter 1

Iteration and Orbits

1.1 Orbits

Lecture 1
Jan 8

Definition 1.1.1 (iteration)
Let $f : A \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$. For $a \in A$ we may iterate the function at a:

$$x_1 = a, x_2 = f(a), x_3 = \underbrace{f(f(a))}_{f^2(a)}, \dots, x_i = f^{i-1}(a), \dots .$$

The sequence $(x_n)_{n=1}^{\infty}$ is the orbit of a under f (abbreviated (x_n) without limits).

Example 1.1.2. Let $f(x) = x^4 + 2x^2 - 2$, $a = -1$. What is the orbit of a under f?

Solution. $a = -1$, $f(a) = 1$, $f(f(a)) = f(1) = 1$, so we have $-1, 1, 1, 1, \dots$. We call this eventually constant. ☐

Example 1.1.3. Let $f(x) = -x^2 - x + 1$, $a = 0$. What is the orbit of a under f?

Solution. Calculate: $0, 1, -1, 1, -1, 1, \dots$. We call this eventually periodic (with period 2). ☐

Example 1.1.4. Let $f(x) = x^3 - 3x + 1$, $a = 1$. What is the orbit of a under f?

Solution. Calculate the first few terms: $1, -1, 3, 19, \dots$ (too big). This is a divergence to infinity. ☐

Example 1.1.5. Let $f(x) = x^2 + 2x$, $a = -0.5$. What is the orbit of a under f?

Solution. Calculate: $-0.5, -0.75, -0.9375, -0.9961 \dots$ and we make an educated guess that this converges to -1 since $f(-1) = -1$, a fixed point. ☐

Example 1.1.6. Let $f(x) = x^3 - 3x$, $a = 0.75$. What is the orbit of a under f?

Solution. Calculate: $0.75, -1.828, -0.625, 1.631, -0.552, \ldots$. There is no clear pattern, so we call this chaotic. In fact, the orbit is dense in a neighbourhood of 0. □

We can start to formalize the examples.

Definition 1.1.7 (fixed point)

Let $f : A \to \mathbb{R}$ such that $f(A) \subseteq A$. A point $a \in A$ is fixed if $f(a) = a$.

Then, the orbit of a under f is $(a, a, a, \ldots)$ which is constant.

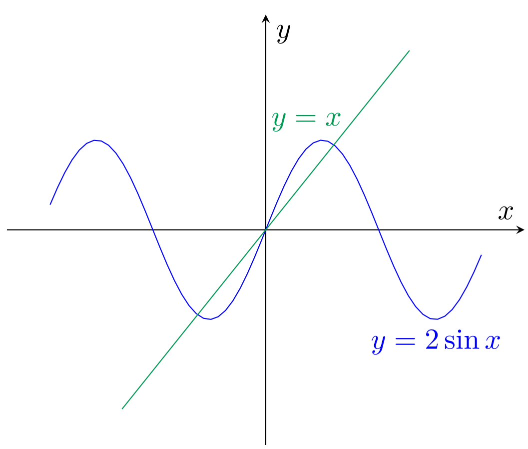
Example 1.1.8. Find all fixed points of $f(x) = x^2 + x - 4$.

Solution. We find points where $f(x) = x$, i.e., $x^2 + x - 4 = x$.

$$x^2 + x - 4 = x \iff x^2 = 4 \iff x = \pm 2$$ □

Example 1.1.9. How many fixed points does $f(x) = 2\sin x$ have?

Solution. Consider where the curve $y = 2\sin x$ meets $y = x$:



We can see there are three fixed points. □

Example 1.1.10. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Proof. We must show there is a solution to $x^4 - 3x + 1 \iff x^4 - 4x + 1 = 0$. Let $g(x) = x^4 - 4x + 1$. Since $g(x)$ is continuous, $g(0) = 1 > 0$, and $g(1) = -2 < 0$, by the Intermediate Value Theorem, there must exist a root of g on the interval $(0, 1)$. That is, a fixed point of f. □

Definition 1.1.11 (periodicity)

Let $f : A \to \mathbb{R}, f(A) \subseteq A$.

(a) A point $a \in A$ is periodic for f if its orbit is periodic. An orbit is periodic if for some $n \in \mathbb{N}$, $f^n(a) = a$. The smallest n is the period of (the orbit of) a.

(b) An orbit (of a point) is eventually periodic if there exists $n < m$ such that $f^n(a) = f^m(a)$. The smallest difference $m - n$ is the period of the orbit.

List of Named Results

Index of Defined Terms