Q01. Evaluate the following integrals

(a) $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-16}}$

(b) $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}\,\mathrm{d}x$ using a trigonometric substitution

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{x^5}{\sqrt{x^2+2}}\,\mathrm{d}x$

(e) $\displaystyle\int_1^3 x^5 \ln x^2\,\mathrm{d}x$

(f) $\displaystyle\int e^{2x}\cos x\,\mathrm{d}x$

(g) $\displaystyle\int_0^2 e^{2x}\cos e^x\,\mathrm{d}x$

(h) $\displaystyle\int \arcsin x\,\mathrm{d}x$

(i) $\displaystyle\int \frac{x^2-x+6}{x^3+3x}\,\mathrm{d}x$

(j) $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,\mathrm{d}x$

(k) $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x$

(l) $\displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,\mathrm{d}x$

(m) $\displaystyle\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,\mathrm{d}x$

Q02. An integrand with trigonometric functions in the numerator and denominator can often be converted to a rational integrand using the substitution $u = \tan(x/2)$ or $x = 2\tan^{-1} u = 2\arctan u$.

(a) With this substitution, prove that $\cos x = \frac{1-u^2}{1+u^2}$ and $\sin x = \frac{2u}{1+u^2}$.

(b) Using this substitution and part (a), evaluate the following integrals:

i. $\displaystyle\int \frac{1}{1+\cos x}\,\mathrm{d}x$

ii. $\displaystyle\int \frac{\mathrm{d}x}{1-\cos x+\sin x}$

Q03. It has been shown that $\int e^{x^2}\,\mathrm{d}x$ and $\int x^2e^{x^2}\,\mathrm{d}x$ do not have elementary antiderivatives. However, $\int(2x^2+1)e^{x^2}\,\mathrm{d}x$ does. Evaluate

$$\int (2x^2+1)e^{x^2}\,\mathrm{d}x$$

[Hint: integration by parts]

Q04. (a) Evaluate $\displaystyle\int_0^1 \frac{x^4(1-x^4)}{1+x^2}\,\mathrm{d}x$.

(b) Prove, using part (a), that $\frac{22}{7} > \pi$.

Q05. Use integration by parts to prove each of the following *reduction formulas*, for integers $n \geq 2$:

(a) $\displaystyle\int (\ln x)^n\,\mathrm{d}x = x(\ln x)^n - n\int (\ln x)^{n-1}\,\mathrm{d}x$

(b) $\displaystyle\int x^n(\ln x)^n\,\mathrm{d}x = \frac{x^{n+1}(\ln x)^n}{n+1} - \frac{n}{n+1}\int x^n(\ln x)^{n-1}\,\mathrm{d}x$