1 Exercises to Prepare for Test 3

Q01. Let $C \in M_{n\times n}(\mathbb{F})$ be invertible, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{F}^n$. Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, then so is $\{C\mathbf{v}_1, \ldots, C\mathbf{v}_k\}$.

Proof (from Rajvi). Suppose that $\sum a_i C\mathbf{v}_i = \mathbf{0}$. We must show that all $a_i = 0$.

By the linearity of matrix multiplication, $\sum a_i C\mathbf{v}_i = C \sum a_i \mathbf{v}_i$. However, since C is invertible, we have $\sum a_i \mathbf{v}_i = \mathbf{0}$. Since $\{\mathbf{v}_i\}$ is linearly independent, this only occurs if all $a_i = 0$. □

Proof (more complicated). Proceed by the contrapositive.

Suppose that $\{C\mathbf{v}_i\}$ is linearly dependent. Then, $\sum a_i C\mathbf{v}_i = \mathbf{0}$ for some non-zero a_i. By linearity, $C\sum a_i\mathbf{v}_i = \mathbf{0}$. Since C is invertible, $\sum a_i \mathbf{v}_i = \mathbf{0}$. This is exactly what it means for $\{\mathbf{v}_i\}$ to be linearly dependent. □

Q02. Let $L : \mathbb{F}^n \to \mathbb{F}^m$ be a linear mapping, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{F}^n$.

(a) Prove or disprove: if L is one-to-one and $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$ is linearly independent, then so is $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

Proof. Proceed by the contrapositive. Suppose that $\{\mathbf{v}_i\}$ is linearly dependent, so $\sum c_i\mathbf{v}_i = \mathbf{0}$ for non-zero c_i. Now, if we apply L to both sides, $\sum c_i L(\mathbf{v}_i) = L(\mathbf{0})$ by linearity. But $L(\mathbf{0}) = \mathbf{0}$, so we are done. □

(b) Prove or disprove: if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, then so is $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$.

Solution. For a counterexample, define L by the mapping $\mathbf{x} \mapsto \mathbf{0}$.

Then, $L(\mathbf{v}_1) = \mathbf{0}$ so any set containing it is linearly dependent. □

Q03. Let $A \in M_{n\times n}(\mathbb{F})$. We say that A is nilpotent if there exists a positive integer n such that $A^n = \mathcal{O}_{n\times n}$. Prove that $\lambda = 0$ is the only eigenvalue of A.

Proof. Start by taking the determinant on both sides. Then, $\det(A^n) = \det(A)^n = \det(\mathcal{O}) = 0$. Therefore, $\det(A) = 0$.

Then, we have a non-trivial solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$. But this is just $A\mathbf{x} = \lambda\mathbf{x}$ for $\lambda = 0$. Therefore, 0 is an eigenvalue of A.

Now, we prove uniqueness. Suppose that $A\mathbf{x} = \lambda\mathbf{x}$ for arbitrary λ and non-zero $\mathbf{x}$. Multiply on the left by A^{n-1}. Then, $A^n\mathbf{x} = A^{n-1}\lambda\mathbf{x}$. But this expands as $A^n\mathbf{x} = \lambda^n\mathbf{x}$. Since $A^n = \mathcal{O}$, we have $\mathbf{0} = \lambda^n\mathbf{x}$. But $\mathbf{x}$ is non-zero, so $\lambda^n = 0$ and $\lambda = 0$.

Therefore, the only eigenvalue of A is 0. □

Q04. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{F}^n$, and let $c_1, \ldots, c_n \in \mathbb{F}$ be non-zero scalars. Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{F}^n$, then so is $\{c_1\mathbf{v}_1, \ldots, c_n\mathbf{v}_n\}$.

Proof. We must show that $\{c_i\mathbf{v}_i\}$ is both spanning and linearly independent.

For spanning, notice that it follows trivially from the definition that multiplying a term of a linear combination by a non-zero scalar does not change the span.

Let $B = \{\mathbf{v}_i\}$ and let $[C]_B = diag(c_i)$. Then, $C\mathbf{v}_i = c_i\mathbf{v}_i$ and C is invertible since it is diagonal. But by Q01, $\{C\mathbf{v}_i\}$ is linearly independent. Therefore, since $\{c_i\mathbf{v}_i\}$ is spanning and linearly independent, it is a basis. □

Q05. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{F}^n$, and let B be a basis of $\mathbb{F}^n$. Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{F}^n$ if and only if $\{[\mathbf{v}_1]_B, \ldots, [\mathbf{v}_n]_B\}$ is a basis of $\mathbb{F}^n$.

Proof. Notice that the proof goes in both directions if we consider bases generally. Again, we must show spanning and linear independence.

Since $\{\mathbf{v}_i\}$ is a basis, the matrix $(\mathbf{v}_i)$ is invertible. Then, $([\mathbf{v}_i]_B) = [(\mathbf{v}_i)]_B = {}_B[I]_S(\mathbf{v}_i)_S[I]_B$ must also be invertible as the product of invertible matrices. Therefore, $\{[\mathbf{v}_i]_B\}$ is invertible and therefore spanning.

Proceed as in Q04 to show linear independence with $C = {}_B[I]_S$.

Conversely, consider when $B = S$ and $S = B$. $\square$

Q06. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{F}^n$. Prove that if for every vector $\mathbf{x} \in \mathbb{F}^n$, there exist unique scalars $c_1, \ldots, c_n \in \mathbb{F}$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{F}^n$.

Proof. We must prove spanning and linear independence. Spanning follows immediately from the hypothesis by definition.

By Lemma 17C.11, $\{\mathbf{v}_i\}$ is linearly independent since there are n vectors.

Therefore, it is a basis. $\square$

Q07. Find all real numbers a and b such that $\operatorname{Span}\left(\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\a\\2\end{pmatrix}, \begin{pmatrix}0\\1\\b\end{pmatrix}\right\}\right) \neq \mathbb{R}^3$.

Solution. Consider $A = \begin{pmatrix}1 & 1 & 0\\2 & a & 1\\1 & 2 & b\end{pmatrix}$. We consider when $\operatorname{Col}(A) \neq \mathbb{R}^3$.

By the Rank-Nullity Theorem, we must find when $N(A) \neq \{\mathbf{0}\}$. This occurs only when $\det(A) = 0$. Expanding the determinant, $ab - 2b - 1 = 0$, so $b = \frac{1}{a-2}$.

Therefore, for all $(a, b) \in \{(k, \frac{1}{k-2}) : k \in \mathbb{R} \setminus \{2\}\}$, $\operatorname{Col}(A) \neq \mathbb{R}^3$. $\square$

Q08. Let $V = \left\{\begin{pmatrix}a^2\\b\\b\end{pmatrix} : a, b \in \mathbb{R}\right\}$ be a subset of $\mathbb{F}^3$. Prove or disprove:

(a) If $\mathbb{F} = \mathbb{R}$, then V is a subspace of $\mathbb{F}^3$.

(b) If $\mathbb{F} = \mathbb{C}$, then V is a subspace of $\mathbb{F}^3$.

Solution. Notice that V is defined as a subset of $\mathbb{R}^3$ since the parameters are in $\mathbb{R}$. Then, we know $\mathbf{x} = (1, 0, 0)^T \in V$ with $a = 1$ and $b = 0$.

However, $-2\mathbf{x} = (-2, 0, 0)^T \notin V$ because there exists no $a \in \mathbb{R}$ such that $a^2 = -2$. Therefore, V is not closed under scalar multiplication.

Since $-2 \in \mathbb{R}$ and $-2 \in \mathbb{C}$, V is neither a subspace of $\mathbb{R}^3$ nor $\mathbb{C}^3$. $\square$

Q09. We call a square matrix A idempotent if $A^2 = A$. Prove that if A is idempotent, then so is $I - A$. Is the converse of this statement true? Explain why or why not.

Proof. Suppose that $A^2 = A$. Then, $(I - A)^2 = (I - A)(I - A) = I^2 - AI - IA + A^2 = I - 2A + A^2 = I - A$ by properties of the identity matrix and the distributivity of matrix multiplication. Therefore, $I - A$ is idempotent.

Suppose conversely that $(I - A)^2 = I - A$. Then, $I - 2A + A^2 = I - A$ as above, but then $-A + A^2 = \mathbb{O}$. It follows $A = A^2$ and A is idempotent. $\square$

Q10. Let $A, B \in M_{n\times n}(\mathbb{F})$.

(a) Prove or disprove: if $\mathbf{v}$ is an eigenvector of both A and B, then it is an eigenvector of both AB and BA.

Proof. Suppose $A\mathbf{v} = \lambda_A \mathbf{v}$ and $B\mathbf{v} = \lambda_B \mathbf{v}$.

If we multiply the first equation by B, we have $BA\mathbf{v} = B\lambda_A\mathbf{v} = \lambda_A B\mathbf{v} = \lambda_A\lambda_B\mathbf{v}$.

If we instead multiply the second by A, we have $AB\mathbf{v} = A\lambda_B\mathbf{v} = \lambda_B A\mathbf{v} = \lambda_B\lambda_A\mathbf{v}$.

Therefore, $\mathbf{v}$ is an eigenvector of AB and BA. $\square$

(b) Prove or disprove: if λ is an eigenvalue of both A and B, then it is an eigenvalue of both AB and BA

Solution. We consider for a counterexample $A = B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $\lambda = 2$ is an eigenvalue of A and B.

However, $AB = BA = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ and $\lambda = 2$ is not an eigenvalue. $\square$