MATH 135 Fall 2020: Extra Practice 3

Warm-Up Exercises

WE01. Prove the following two quantified statements.

(a) $\forall n \in \mathbb{N}, n+1 \geq 2$

Proof. Let $n \in \mathbb{N}$. Recall that 1 is the smallest natural. $n \geq 1 \iff n+1 \geq 2$. $\square$

(b) $\exists n \in \mathbb{Z}, \frac{5n-6}{3} \in \mathbb{Z}$

Proof. Select $n = 3$. Then, $\frac{5n-6}{3} = \frac{15-6}{3} = \frac{9}{3} = 3 \in \mathbb{Z}$. $\square$

WE02. Prove that for all $k \in \mathbb{Z}$, if k is odd, then $4k+7$ is odd.

Proof. Let k be an odd integer. Then, it can be written as $2n+1$ for some integer n.

Substituting, $4k+7 = 4(2n+1)+7 = 8n+11 = 2(4n+5)+1$. By definition, since $4k+7$ can be written as $2m+1$ where $m = 4n+5$ is an integer, it is odd. $\square$

WE03. Consider the following proposition

For all $a, b \in \mathbb{Z}$, if $a^3 \mid b^3$, then $a \mid b$.

We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

(a) *Consider $a = 2$, $b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.*

This proof is erroneous as it only considers one specific case of a and b and not the general case of integer a and b.

(b) *Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3a^3$. Since $k^3 \in \mathbb{Z}$, then $a^3 \mid b^3$.*

This proof supposes the conclusion instead of the hypothesis.

(c) *Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.*

The proof does not guarantee that $\frac{ka^2}{b^2}$ is an integer.

WE04. Let x be a real number. Prove that if $x^3 - 5x^2 + 3x \neq 15$, then $x \neq 5$.

Proof. Suppose for the contrapositive that $x = 5$. Then, $x^3 - 5x^2 + 3x = (5)^3 - 5(5)^2 + 3(5) = 15$, as required. Since the contrapositive is true, the original implication must be true. $\square$

WE05. Prove that there do not exist integers x and y such that $2x + 4y = 3$.

Proof. For the sake of contradiction, suppose the negation is true.

Consider the negation of the statement: there exist integers x and y such that $2x+4y=3$. Let x and y be such integers. Then, $x+2y$ is an integer. Therefore, $2x+4y=2(x+2y)$ is even. However, 3 is odd. An integer cannot be both even and odd, therefore, the negation is false, and the original statement is true. □

WE06. Prove that an integer is even if and only if its square is an even integer.

Proof. ($\Rightarrow$) Let n be an even integer. Then, $n=2k$ for some integer k. $n^2=(2k)^2=4k^2=2(2k^2)$. Since $2k^2$ is an integer, n^2 is even.

($\Leftarrow$) Let n be an even square integer. Then, $n=2k$ for some integer k and $n=x\cdot x$ for some integer x. Since $2k=x\cdot x$, and 2 is prime, 2 must divide x. Therefore, $x=2y$ for some integer y, which is the definition of being even.

Since the implication is true in both directions, the biconditional is true. □

Recommended Problems

RP01. Prove that $x^2+9\geq 6x$ for all real numbers x.

Proof. Let x be a real number. $x^2+9\geq 6x \iff x^2-6x+9\geq 0 \iff (x-3)^2\geq 0$. Since the square of a real is always non-negative, the statements are true. □

RP02. Prove that for all $r\in\mathbb{R}$ where $r\neq -1$ and $r\neq -2$,

$$\frac{2^{r+1}}{r+2}-\frac{2^r}{r+1}=\frac{r(2^r)}{(r+1)(r+2)}$$

Proof. Let r be a real number that is neither -1 nor -2. Then,

$$\begin{aligned}
LHS &= \frac{2^{r+1}}{r+2}-\frac{2^r}{r+1}\\
&= \frac{2^{r+1}(r+1)-2^r(r+2)}{(r+1)(r+2)}\\
&= \frac{r2^{r+1}+2^{r+1}-r2^r-2\cdot 2^r}{(r+1)(r+2)}\\
&= \frac{r2^{r+1}+\cancel{2^{r+1}}-r2^r-\cancel{2^{r+1}}}{(r+1)(r+2)}\\
&= \frac{r(2^{r+1}-2^r)}{(r+1)(r+2)}\\
&= \frac{r(2^r\cdot 2-2^r)}{(r+1)(r+2)}\\
&= \frac{r(2^r+\cancel{2^r}-\cancel{2^r})}{(r+1)(r+2)}\\
&= \frac{r(2^r)}{(r+1)(r+2)}\\
&= RHS
\end{aligned}$$

Since the left side equals the right side, the equality is true. □

RP03. Prove that there exists a real number x such that $x^2 - 6x + 11 \leq 2$.

Proof. Let $x = 3$. $x^2 - 6x + 11 = (3)^2 - 6(3) + 11 = 9 - 18 + 11 = 2 \leq 2$, as required. Since 3 is a real number, the statement is true. □

RP04. Prove or disprove each of the following statements.

(a) $\forall n \in \mathbb{Z}$, $\frac{5n-6}{3}$ is an integer.

Proof. Let $n = 1$ as a counter-example. Then, $\frac{5n-6}{3} = \frac{5-6}{3} = -\frac{1}{3}$, which is not an integer. Therefore, the statement is false. □

(b) $\forall a \in \mathbb{Z}$, $a^3 + a + 2$ is even.

Proof. Let a be an integer. Then, a is either even or odd. Suppose that a is even and can be written as $a = 2k$ for an integer k. Then, $a^3 + a + 2 = (2k)^3 + 2k + 2 = 8k^3 + 2k + 2 = 2(4k^3 + k + 1)$, an even number.

Suppose a is odd and can be written as $a = 2k+1$ for an integer k. Then, $a^3+a+2 = (2k + 1)^3 + (2k + 1) + 2 = 8k^3 + 12k^2 + 8k + 4 = 2(4k^3 + 6k^2 + 4k + 2)$, an even number.

Therefore, the statement is true. □

(c) For every prime number p, $p + 7$ is composite.

Proof. Let p be a prime number.

If p is even, then $p = 2$, and $p + 7 = 9$ which is composite.

If p is odd, $p = 2k + 1$ for some integer $k \geq 0$ (as there are no negative primes). Then, $p + 7 = 2k + 8 = 2(k + 4)$, which is even. The only even prime is 2, but $2k + 8 \geq 8$, so $p + 7$ is composite.

Therefore, since all primes are either even or odd, $p+7$ is composite for all primes. □

(d) For all $x \in \mathbb{R}$, $|x - 3| + |x - 7| \geq 10$.

Proof. Let $x = 3$ as a counter-example. Then, $|x-3|+|x-7| = |(3)-3|+|(3)-7| = 0 + 4 = 4 \ngeq 10$. Therefore, the statement is false. □

(e) There exists a natural number $m < 123456$ such that $123456m$ is a perfect square.

Proof. Let $m = 1929$, which is a natural number less than 123456. Then, $123456m = 238146624 = 15432^2$. Since $123456m$ can be written as n^2 where $n = 15432 \in \mathbb{Z}$, it is a perfect square, and the statement is true. □

Note: To find $m = 1929$, notice that if $123456m = n^2$, then $\sqrt{123456m} = 8\sqrt{1929m}$ (after simplifying by prime factorization) must be an integer.

(f) $\exists k \in \mathbb{Z}, 8 \nmid (4k^2 + 12k + 8)$.

Proof. Consider the negation, $\forall k \in \mathbb{Z}, 8 \mid (4k^2 + 12k + 8)$. Notice that the open sentence is logically equivalent to $8 \mid (4k^2 + 12k)$. Let k be a natural number. Then, k is either even or odd.

Suppose that k is even and can be written as $k = 2n$. Then, $4k^2 = 16n^2 = 8(2n^2)$, so $8 \mid 4k^2$. Likewise, $12k = 24n = 8(3n)$, so $8 \mid 12k$. By DIC, $8 \mid (4k^2 + 12k)$.

Now, suppose that k is odd and can be written as $k = 2n + 1$. Then, $4k^2 + 12k = 4(4n^2 + 2n + 1) + 12(2n + 1) = 16n^2 + 40n + 16 = 8(2n^2 + 5n + 1)$, so $8 \mid (4k^2 + 12k)$.

Therefore, the negation is true, so the original statement is false. □

RP05. Prove or disprove each of the following statements involving nested quantifiers.

(a) For all $n \in \mathbb{Z}$, there exists an integer $k > 2$ such that $k \mid (n^3 - n)$.

Proof. Let n be an integer. If $n = 0$ or $n = \pm 1$, $n^3 - n = 0$ and all integers (including any k) divide zero.

If $n > 1$, we select $k = n + 1 > 2$. Factor: $n^3 - n = n(n - 1)(n + 1)$. Then, $n^3 - n = [n(n - 1)](n + 1)$, so $k \mid (n^3 - n)$.

If $n < 1$, first let $m = -n$ so $n^3 - n = (-m)^3 + m = -(m^3 - m)$. Now, select $k = m + 1 > 2$. Then, $n^3 - n = -m(m - 1)(m + 1)$, so $k \mid (n^3 - n)$.

Therefore, the statement is true. □

(b) For every positive integer a, there exists an integer b with $|b| < a$ such that b divides a.

Proof. We disprove by counter-example. Let $a = 1$. Then, $|b| < 1$, and the only such integer is 0. However, $0 \nmid 1$ since there is no integer k where $k \cdot 0 = 1$. Therefore, the statement is false. □

(c) There exists an integer n such that $m(n - 3) < 1$ for every integer m.

Proof. Choose $n = 3$ and let m be an integer. Then, $m(n - 3) = m(3 - 3) = 0 < 1$, as desired. Therefore, the statement is true. □

(d) $\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, -nm < 0$

Proof. Consider the negation $\forall n \in \mathbb{N}, \exists m \in \mathbb{Z}, -nm \geq 0$. Let n be a natural number.

We can choose an integer m, namely $m = -1$. Notice that because n is a natural number, $n > 0 \iff n(-1)(-1) > 0 \iff -nm > 0 \iff -nm \geq 0$.

Because the negation is true, the original statement is false. □

RP06. Prove that for all integers a and b, if $a \mid (2b + 3)$ and $a \mid (3b + 5)$, then $a \mid 13$.

Proof. Let a and b be arbitrary integers, and assume that $a \mid (2b + 3)$ and $a \mid (3b + 5)$.

Recall the divisibility of integer combinations: since $2b + 3$ and $3b + 5$ are integers, a must divide $n(2b + 3) + m(3b + 5)$ for all integers n and m. Specifically, let $n = -39$ and $m = 26$. Then, $n(2b + 3) + m(3b + 5) = -78b - 117 + 78b + 130 = 13$. Therefore, $a \mid 13$. □

RP07. Let a, b, c and d be positive integers. Prove that if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof. Let a, b, c and d all be positive integers. Suppose $\frac{a}{b} < \frac{c}{d}$, which means $ad < bc$, because b and d are positive. Now, adding ab and cd to both sides, respectively:

$$\begin{aligned} ad &< bc & \qquad ad &< bc \\ ad + ab &< bc + ab & ad + cd &< bc + cd \\ a(b+d) &< b(c+a) & d(a+c) &< c(b+d) \\ \frac{a}{b} &< \frac{a+c}{b+d} & \frac{a+c}{b+d} &< \frac{c}{d} \end{aligned}$$

Therefore, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. □

RP08. Prove that for all integers n, if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

Proof. Let n be an integer where $1 - n^2 > 0$. Since squares of integers are positive, $1 > n^2$. This is only true when $|n| < 1$, but the only such integer is 0. $3(0) - 2 = -2$, which is even. □

RP09. Let a and b be integers. Prove each of the following implications.

(a) If $ab = 4$, then $(a - b)^3 - 9(a - b) = 0$

Proof. Let a and b be integers with product 4.

Consider the possible values for a and b. 4's divisor pairs are $(\pm 1, \pm 4)$ and $(\pm 2, \pm 2)$. For all of these pairs, either $a = b$ or $a = b \pm 3$. Specifically:

- If $b = \pm 2$, then $a = b$
- If $b = 1$, then $a = 4 = b + 3$ (for $b = -1$, $a = -4 = b - 3$)
- If $b = 4$, then $a = 1 = b - 3$ (for $b = -4$, $a = -1 = b + 3$)

Notice that the conclusion factors to $(a - b)(a - b - 3)(a - b + 3) = 0$. This is true when $a = b$ or $a = b \pm 3$, which we just showed. □

(b) If a and b are positive, then $a^2(b + 1) + b^2(a + 1) \geq 4ab$

Proof. Let a and b be positive integers, i.e., at least 1.

If a and b are both at least 1, then $a + b \geq 2$, or $a + b - 2 \geq 0$. Likewise, ab is a positive integer, so $ab(a + b - 2) \geq 0$.

$$\begin{aligned} ab(a+b-2) &\geq 0 \\ a^2b + b^2a - 2ab &\geq 0 \end{aligned}$$

Recall that squares are non-negative:

$$\begin{aligned} (a-b)^2 + a^2b + b^2a - 2ab &\geq 0 \\ a^2 - 2ab + b^2 + a^2b + b^2a - 2ab &\geq 0 \\ a^2 + a^2b + b^2 + b^2a &\geq 4ab \\ a^2(b+1) + b^2(a+1) &\geq 4ab \end{aligned}$$

□

RP10. Let a, b, c and d be integers. Prove that if $a \mid b$ and $b \mid c$ and $c \mid d$, then $a \mid d$.

Proof. Let a, b, c, and d be integers where $a \mid b$, $b \mid c$, and $c \mid d$.

Recall the transitivity of divisibility: for integers x, y, and z, if $x \mid y$ and $y \mid z$, then $x \mid z$.

Then, $a \mid b$ and $b \mid c$ implies $a \mid c$. Likewise, $a \mid c$ and $c \mid d$ implies $a \mid d$. $\square$

RP11. Prove that the product of any four consecutive integers is one less than a perfect square.

Proof. The statement is equivalently expressed that for any integer k, $k(k+1)(k+2)(k+3) = r^2 - 1$ for some positive integer r.

Let k be an integer. The product $k(k+1)(k+2)(k+3)$ expands to $k^4 + 6k^3 + 11k^2 + 6k$. As a fourth-degree polynomial, its square root would be a quadratic.

Expanding algebraically, the square of a quadratic in x, $ax^2 + bx + c$, is $a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2$.

Notice that when $a = c = 1$ and $b = 3$, this formula becomes $x^4 + 6x^3 + 11x^2 + 6x + 1$. The coefficients on x are precisely our original product (with a constant $+1$). Therefore, $x^4 + 6x^3 + 11x^2 + 6x = (x^2 + 3x + 1)^2 - 1$ for all real x.

We can now let $r = k^2 + 3k + 1$, which is a positive integer such that

$$\begin{aligned} r^2 - 1 &= (k^2 + 3k + 1)^2 - 1 \\ &= k^4 + 6k^3 + 11k^2 + 6k + 1 - 1 \\ &= k(k+1)(k+2)(k+3) \end{aligned}$$

and conclude that the statement is true. $\square$

RP12. Let $x, y \in \mathbb{R}$. Prove that if $xy + 2x - 3y - 6 < 0$, then $x < 3$ or $y < -2$.

Proof. Let x and y be real solutions to $xy + 2x - 3y - 6 < 0$.

Notice that the inequality factors to $(x - 3)(y + 2) < 0$. This is true when x and y are non-zero and have opposite signs: either $x < 3$ and $y > -2$, or $x > 3$ and $y < -2$. Therefore, either $x < 3$ or $y < -2$. $\square$

RP13. Is the following implication true for all integers a, b and c? Prove that your answer is correct.

$$a \mid b \textit{ if and only if } ac \mid bc$$

Solution. The statement is false. Consider the counterexample $a = 2$, $b = 3$, and $c = 0$. Then, the backwards implication's hypothesis is true $(0 \mid 0)$ but the conclusion is false $(2 \nmid 3)$. $\square$

RP14. Let n be an integer. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

Proof. Consider the two implications of the biconditional statement:

($\Rightarrow$) Let n be an integer where 2 divides $n^4 - 3$. This means there is an integer k where $n^4 - 3 = 2k$. Notice that this means $n^4 - 3$ is even, so $n^4 = 2(k+1) + 1$ is odd. Even numbers raised to the fourth power remain even, so n must be odd. Therefore, $n = 2m+1$ for some integer m.

Now, expand $n^2 + 3$:

$$\begin{aligned} n^2 + 3 &= (2m+1)^2 + 3 \\ &= 4m^2 + 4m + 1 + 3 \\ &= 4(m^2 + m + 1) \end{aligned}$$

Because $m^2 + m + 1$ is an integer, $4 \mid (n^2 + 3)$.

($\Leftarrow$) Let n be an integer where 4 divides $n^2 + 3$. This means there is an integer k where $n^2 + 3 = 4k$ or $n^2 = 4k - 3$, and

$$\begin{aligned} n^2 &= 4k - 3 \\ n^4 &= (4k-3)^2 \\ n^4 &= 16k^2 - 24k + 9 \\ n^4 - 3 &= 16k^2 - 24k + 6 \\ &= 2(8k^2 - 12k + 3) \end{aligned}$$

Because $8k^2 - 12k + 3$ is an integer, $2 \mid (n^4 - 3)$.

Therefore, since both expressions imply the other, $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$. $\square$

RP15. Let x and y be integers. Prove that if $xy = 0$ then $x = 0$ or $y = 0$.

Proof. Consider the contrapositive, $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$.

Let x and y be non-zero integers. WLOG, take $x \leq y$.

Now, take cases of the signs of x and y:

- If $0 < x \leq y$, then $xy > 0$, since two positive numbers' product is a positive number.
- xy is also positive when $x \leq y < 0$, with two negative numbers.
- When $x < 0 < y$, i.e. the signs are opposite, $xy < 0$.

Since xy can never be 0 for any combination of non-zero integers, the contrapositive, and by extension, the original implication, is true. $\square$

RP16. Prove that $\forall a, b \in \mathbb{Z}, [(a \mid b \wedge b \mid a) \iff a = \pm b]$.

Proof. Let a and b be integers. Suppose a divides b and vice versa. Equivalently, integers p and q exist such that $a = pb$ and $b = qa$. Substituting, $a = pb = p(qa) \iff 1 = pq \iff p = \frac{1}{q}$.

The only integers of the form $\frac{1}{k}$ with integer k are 1 and -1. Therefore, $p = \frac{1}{q}$ if and only if $p = \pm 1$, i.e., $a = \pm b$. $\square$

RP17. Let a be an integer. Prove that $a^2 + 2a - 3$ is even if and only if a is odd.

Proof. Consider the two implications of the biconditional statement:

($\Rightarrow$) Let a be an odd integer, or, $a = 2k + 1$ for some integer k. Then,

$$\begin{aligned}
&2a^2 + 2a - 3 \\
&\qquad = (2k+1)^2 + 2(2k+1) - 3 \\
&\qquad = 4k^2 + 4k + 1 + 4k + 2 - 3 \\
&\qquad = 4k^2 + 8k - 2 \\
&\qquad = 2(2k^2 + 4k - 1)
\end{aligned}$$

which is even, because $2k^2 + 4k - 1$ is an integer.

($\Leftarrow$) Consider the contrapositive, where even a implies odd $a^2 + 2a - 3$. Let a be an even integer, i.e., $a = 2k$ for some integer k. Then,

$$\begin{aligned}
a^2 + 2a - 3 &= (2k)^2 + 2(2k) - 3 \\
&= 4k^2 - 4k - 3 \\
&= 2(2k^2 - 2k - 2) + 1
\end{aligned}$$

which is odd, because $2k^2 - 2k - 2$ is an integer. Since the contrapositive is true, the original implication is also true.

Therefore, since both implications hold, the statement is true. $\square$

RP18. Prove or disprove each of the following for any integers x and y.

(a) If $2 \nmid xy$ then $2 \nmid x$ and $2 \nmid y$.

(b) If $2 \nmid y$ and $2 \nmid x$ then $2 \nmid xy$.

Proof. First, notice that if $2 \mid n$ for an integer n, then $n = 2k$ for some integer k. This is the definition of saying n is even. Therefore, $2 \nmid n$ is the same as saying n is not even, i.e., n is odd.

Let x and y be odd integers. Equivalently, $x = 2p+1$ and $y = 2q+1$ for some integers p and q. Substituting into xy, $(2p+1)(2q+1) = 2pq+2p+2q+1 = 2(pq+p+q)+1$. By definition, since $pq + p + q$ is an integer, xy is odd.

Therefore, x and y are odd if and only if xy is odd, so (a) and (b) are both true. $\square$

(c) If $10 \nmid xy$ then $10 \nmid x$ and $10 \nmid y$.

Proof. Consider the contrapositive, "if $10 \mid x$ and $10 \mid y$ then $10 \mid xy$". Let x and y be integers where 10 divides both.

This means $x = 10n$ and $y = 10m$ for some integers n and m. Then, $xy = (10n)(10m) = 10(10nm)$, and since $10nm$ is an integer, $10 \mid xy$.

Since the contrapositive, the original implication is true. $\square$

(d) If $10 \nmid x$ and $10 \nmid y$ then $10 \nmid xy$.

Proof. For a counterexample, let $x = 5$ and $y = 2$. $10 \nmid x$ and $10 \nmid y$ since $2 < 5 < 10$. However, $xy = 10$ and $10 \mid 10$, so the statement is false. $\square$

RP19. For every odd integer n, prove that there exists a unique integer m such that $n^2 = 8m + 1$.

Proof. Let n be an odd integer, i.e., $n = 2k + 1$ for some other integer k. Then, $n^2 = (2k+1)^2 = 4k^2+4k+1$. We must show that $8m = 4k^2+4k \iff 2m = k^2+k$, or, k^2+k is even. Now, consider k's parity:

Suppose k is even. Then, $k = 2p$ for an integer p and $k^2 + k = 4p^2 + 2p = 2(2p^2 + p)$, which means that $k^2 + k$ is even.

Now, suppose k is odd. Then, $k = 2p + 1$ for an integer p and $k^2 + k = 4p^2 + 6p + 2 = 2(2p^2 + 3p + 1)$, which means that $k^2 + k$ is even.

Since k is either even or odd, $k^2 + k$ is even for all k.

Therefore, $m = \frac{k^2+k}{2}$ is an integer, but recall $k = \frac{n-1}{2}$, so:

$$m = \frac{k^2+k}{2} = \frac{\left(\frac{n-1}{2}\right)^2 + \frac{n-1}{2}}{2} = \frac{\frac{(n-1)^2}{4} + \frac{n-1}{2}}{2} = \frac{(n-1)^2 + 2(n-1)}{8} = \frac{n^2-1}{8}$$

is the same integer, but $m = \frac{n^2-1}{8}$ if and only if $n^2 = 8m+1$, so the statement is true. $\square$

RP20. Prove the following statements.

(a) There is no smallest positive real number.

Proof. Suppose, for a contradiction, that there is a smallest positive real number n. Now, consider $\frac{n}{2}$.

This number is still real ($\mathbb{R}$ is closed under division). $\frac{n}{2}$ is positive because n and 2 are positive. Therefore, $\frac{n}{2}$ is a positive real number.

Clearly $\frac{n}{2} < n$, so the supposition must be false. Therefore, there is no smallest positive real number. $\square$

(b) For every even integer n, n cannot be expressed as the sum of three odd integers.

Proof. We will prove by the contrapositive. Let r, s, and t be arbitrary integers so $2r + 1$, $2s + 1$, and $2t + 1$ are odd.

Then, $r + s + t = 2r + 2s + 2t + 3 = 2(r + s + t + 1) + 1$, so this sum is odd.

Therefore, the sum of three odd integers is always odd, and no even integer may be expressed as such a sum. $\square$

(c) Let $a, b \in \mathbb{Z}$. If a is an even integer and b is an odd integer, then $4 \nmid (a^2 + 2b^2)$.

Proof. Let a and b be integers and suppose, for a contradiction, that the negation is true. Then, a is even, b is odd, and $4 \mid (a^2 + 2b^2)$.

Rewrite $a = 2n$ and $b = 2m+1$ with some integers n and m. Now, expand $a^2+2b^2 = (2n)^2 + 2(2m + 1)^2 = 4n^2 + 8m^2 + 8m + 2$.

We can extract a factor of four, and get $4 \mid (4(n^2 + 2m^2 + 2m) + 2)$. Then, 4 must divide 2, which is a contradiction.

Therefore, the negation is false, so the original statement is true. $\square$

(d) For every integer m with $2 \mid m$ and $4 \nmid m$, there are no integers x and y that satisfy $x^2 + 3y^2 = m$.

Proof. Those negations are ugly so we can consider the contrapositive:

If $x^2 + 3y^2 = m$ has integer solutions in x and y, m is odd or $4 \mid m$.

Notice that $2 \nmid m \vee 4 \mid m \equiv 4 \mid m$.

Suppose integers x and y so $x^2 + 3y^2 = m$ exist. Break into cases for x and y's parities.

- If x and y are odd, they can respectively be expressed as $2p + 1$ and $2q + 1$ for integers p and q. Then, $m = (2p+1)^2+3(2q+1)^2 = 4p^2+4p+1+3(4q^2+4q+1)$. This simplifies to $4(p^2 + 3q^2 + p + 3q + 1)$, so $m \mid 4$.
- If x and y are even, let $x = 2p$ and $y = 2q$. Then, $m = (2p)^2 + 3(2q)^2 = 4p^2 + 12q^2 = 4(p^2 + 3q^2)$, so $m \mid 4$.
- If x is odd and y is even, let $x = 2p + 1$ and $y = 2q$. Then, $m = (2p + 1)^2 + 3(2q)^2 = 4p^2 + 4p + 1 + 3(4q^2) = 2(2p^2 + 2p + 6q^2) + 1$, so m is odd.
- If x is even and y is odd, let $x = 2p$ and $y = 2q + 1$. Then, $m = (2p)^2 + 3(2q + q)^2 = 4p^2 + 3(4q^2 + 4q^2 + 1) = 2(2p^2 + 6q^2 + 6q + 1) + 1$, so m is odd.

Therefore, either 4 divides m or m is odd, so the contrapositive, and by extension the original statement, is true. □

(e) The sum of a rational number and an irrational number is irrational.

Proof. First, recall that rational numbers are those which can be expressed by $\frac{p}{q}$ for integers p and q.

Let x be a rational number and suppose for a contradiction that y is irrational such that $x + y$ is rational.

Then, $x = \frac{p}{q}$ for integers p and q. Also, $x+y = \frac{n}{m}$ for integers n and m. Substituting, $\frac{p}{q} + y = \frac{n}{m}$. Rearranging,

$$\frac{p}{q} + y = \frac{n}{m} \iff p + yq = \frac{qn}{m} \iff y = \frac{qn - mp}{qm}$$

but if y equals the ratio of two integers ($qn-mp$ and qm), by definition, y is rational.

Therefore, by contradiction, the sum of a rational number and an irrational number is irrational. □

(f) Let x be a non-zero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.

Proof. Let x be a non-zero real such that $x + \frac{1}{x} < 2$. Then,

$$\begin{aligned} x + \frac{1}{x} &< 2 \\ x + \frac{1}{x} - 2 &< 0 \\ \frac{x^2 + 1 - 2x}{x} &< 0 \\ \frac{(x-1)^2}{x} &< 0 \end{aligned}$$

Because $(x - 1)^2$ is a square, so is always non-negative, $\frac{1}{x} < 0$, which is true if and only if $x < 0$. □

Challenges

C01. Let n be an integer. Prove that if $2 \mid n$ and $3 \mid n$, then $6 \mid n$.

Proof. Let n be an integer such that $2 \mid n$ and $3 \mid n$. Then, there exist integers $2p = n$ and $3q = n$. Equivalently, $6p = 3n$ and $6q = 2n$. Subtracting, $n = 6(p - q)$, and since $p - q$ is an integer, $6 \mid n$. □

C02. Let $a, b, c \in \mathbb{R}$. Prove that if $a^2 + b^2 + c^2 = 1$, then $-1/2 \leq ab + bc + ca \leq 1$.

Proof. Let a, b, and c be real numbers. Recall that squares of reals are non-negative. Then, notice that we can create $2ab$-type terms in squares of binomials:

$$\begin{aligned}
0 &\leq (a-b)^2 + (b-c)^2 + (c-a)^2 \\
0 &\leq 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \\
0 &\leq (a^2 + b^2 + c^2) - (ab + bc + ca) \\
ab + bc + ca &\leq 1
\end{aligned}$$

Likewise, we can create these terms in the square of a trinomial:

$$\begin{aligned}
0 &\leq (a+b+c)^2 \\
0 &\leq a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \\
-(a^2 + b^2 + c^2) &\leq 2(ab + bc + ca) \\
-\frac{1}{2} &\leq ab + bc + ca
\end{aligned}$$

Therefore, combining these inequalities, $-\frac{1}{2} \leq ab + bc + ca \leq 1$, as desired. □

C03. Show that if p and $p^2 + 2$ are prime, then $p^3 + 2$ is also prime.

C04. Express the following statement in symbolic form and prove that it is true.

> There exists a real number L such that for every positive real number ϵ, there exists a positive real number δ such that for all real numbers x, if $|x| < \delta$, then $|3x - L| < \epsilon$.

Proof. In symbolic form, with $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$:

$$\exists L \in \mathbb{R}, \forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x| < \delta \implies |3x - L| < \epsilon$$

We propose $L = 0$. Let $\epsilon > 0$. Select $\delta = \frac{\epsilon}{3}$. Now, suppose that $|x| < \delta$. Then, $|3x - L| = |3x| = 3|x| < 3\delta = 3\frac{\epsilon}{3} = \epsilon$, as desired. □

C05. Prove that there are no positive integers a and b such that $b^4 + b + 1 = a^4$.

C06. Prove that the length of at least one side of a right-angled triangle with integer side lengths must be divisible by 3.