1 Exercises to Prepare for Test 3

Q01. Let $C \in M_{n\times n}(\mathbb{F})$ be invertible, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{F}^n$. Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, then so is $\{C\mathbf{v}_1, \ldots, C\mathbf{v}_k\}$.

Proof (from Rajvi). Suppose that $\sum a_i C\mathbf{v}_i = \mathbf{0}$. We must show that all $a_i = 0$.

By the linearity of matrix multiplication, $\sum a_i C\mathbf{v}_i = C\sum a_i\mathbf{v}_i$. However, since C is invertible, we have $\sum a_i\mathbf{v}_i = \mathbf{0}$. Since $\{\mathbf{v}_i\}$ is linearly independent, this only occurs if all $a_i = 0$. $\square$

Proof (more complicated). Proceed by the contrapositive.

Suppose that $\{C\mathbf{v}_i\}$ is linearly dependent. Then, $\sum a_i C\mathbf{v}_i = \mathbf{0}$ for some non-zero a_i. By linearity, $C\sum a_i\mathbf{v}_i = \mathbf{0}$. Since C is invertible, $\sum a_i\mathbf{v}_i = \mathbf{0}$. This is exactly what it means for $\{\mathbf{v}_i\}$ to be linearly dependent. $\square$

Q02. Let $L : \mathbb{F}^n \to \mathbb{F}^m$ be a linear mapping, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{F}^n$.

(a) Prove or disprove: if L is one-to-one and $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$ is linearly independent, then so is $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

Proof. Proceed by the contrapositive. Suppose that $\{\mathbf{v}_i\}$ is linearly dependent, so $\sum c_i\mathbf{v}_i = \mathbf{0}$ for non-zero c_i. Now, if we apply L to both sides, $\sum c_i L(\mathbf{v}_i) = L(\mathbf{0})$ by linearity. But $L(\mathbf{0}) = \mathbf{0}$, so we are done. $\square$

(b) Prove or disprove: if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, then so is $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$.

Solution. For a counterexample, define L by the mapping $\mathbf{x} \mapsto \mathbf{0}$.

Then, $L(\mathbf{v}_1) = \mathbf{0}$ so any set containing it is linearly dependent. $\square$

Q03. Let $A \in M_{n\times n}(\mathbb{F})$. We say that A is nilpotent if there exists a positive integer n such that $A^n = \mathcal{O}_{n\times n}$. Prove that $\lambda = 0$ is the only eigenvalue of A.

Proof. Start by taking the determinant on both sides. Then, $\det(A^n) = \det(A)^n = \det(\mathcal{O}) = 0$. Therefore, $\det(A) = 0$.

Then, we have a non-trivial solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$. But this is just $A\mathbf{x} = \lambda\mathbf{x}$ for $\lambda = 0$. Therefore, 0 is an eigenvalue of A.

Now, we prove uniqueness. Suppose that $A\mathbf{x} = \lambda\mathbf{x}$ for arbitrary λ and non-zero $\mathbf{x}$. Multiply on the left by A^{n-1}. Then, $A^n\mathbf{x} = A^{n-1}\lambda\mathbf{x}$. But this expands as $A^n\mathbf{x} = \lambda^n\mathbf{x}$. Since $A^n = \mathcal{O}$, we have $\mathbf{0} = \lambda^n\mathbf{x}$. But $\mathbf{x}$ is non-zero, so $\lambda^n = 0$ and $\lambda = 0$.

Therefore, the only eigenvalue of A is 0. $\square$

Q04. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{F}^n$, and let $c_1, \ldots, c_n \in \mathbb{F}$ be non-zero scalars. Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{F}^n$, then so is $\{c_1\mathbf{v}_1, \ldots, c_n\mathbf{v}_n\}$.

Proof. We must show that $\{c_i\mathbf{v}_i\}$ is both spanning and linearly independent.

For spanning, notice that it follows trivially from the definition that multiplying a term of a linear combination by a non-zero scalar does not change the span.

Let $B = \{\mathbf{v}_i\}$ and let $[C]_B = diag(c_i)$. Then, $C\mathbf{v}_i = c_i\mathbf{v}_i$ and C is invertible since it is diagonal. But by Q01, $\{C\mathbf{v}_i\}$ is linearly independent. Therefore, since $\{c_i\mathbf{v}_i\}$ is spanning and linearly independent, it is a basis. $\square$

Q05. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{F}^n$, and let B be a basis of $\mathbb{F}^n$. Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{F}^n$ if and only if $\{[\mathbf{v}_1]_B, \ldots, [\mathbf{v}_n]_B\}$ is a basis of $\mathbb{F}^n$.

Proof. Notice that the proof goes in both directions if we consider bases generally. Again, we must show spanning and linear independence.

Since $\{\mathbf{v}_i\}$ is a basis, the matrix $(\mathbf{v}_i)$ is invertible. Then, $([\mathbf{v}_i]_B) = [(\mathbf{v}_i)]_B = {}_B[I]_S(\mathbf{v}_i)_S[I]_B$ must also be invertible as the product of invertible matrices. Therefore, $\{[\mathbf{v}_i]_B\}$ is invertible and therefore spanning.

Proceed as in Q04 to show linear independence with $C = {}_B[I]_S$.

Conversely, consider when $B = S$ and $S = B$. $\square$

Q06. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{F}^n$. Prove that if for every vector $\mathbf{x} \in \mathbb{F}^n$, there exist unique scalars $c_1, \ldots, c_n \in \mathbb{F}$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{F}^n$.

Proof. We must prove spanning and linear independence. Spanning follows immediately from the hypothesis by definition.

By Lemma 17C.11, $\{\mathbf{v}_i\}$ is linearly independent since there are n vectors.

Therefore, it is a basis. $\square$

Q07. Find all real numbers a and b such that $\operatorname{Span}\left(\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\a\\2\end{pmatrix}, \begin{pmatrix}0\\1\\b\end{pmatrix}\right\}\right) \neq \mathbb{R}^3$.

Solution. Consider $A = \begin{pmatrix}1 & 1 & 0\\ 2 & a & 1\\ 1 & 2 & b\end{pmatrix}$. We consider when $\operatorname{Col}(A) \neq \mathbb{R}^3$.

By the Rank-Nullity Theorem, we must find when $N(A) \neq \{\mathbf{0}\}$. This occurs only when $\det(A) = 0$. Expanding the determinant, $ab - 2b - 1 = 0$, so $b = \frac{1}{a-2}$.

Therefore, for all $(a, b) \in \{(k, \frac{1}{k-2}) : k \in \mathbb{R} \setminus \{2\}\}$, $\operatorname{Col}(A) \neq \mathbb{R}^3$. $\square$

Q08. Let $V = \left\{\begin{pmatrix}a^2\\b\\b\end{pmatrix} : a, b \in \mathbb{R}\right\}$ be a subset of $\mathbb{F}^3$. Prove or disprove:

(a) If $\mathbb{F} = \mathbb{R}$, then V is a subspace of $\mathbb{F}^3$.

(b) If $\mathbb{F} = \mathbb{C}$, then V is a subspace of $\mathbb{F}^3$.

Solution. Notice that V is defined as a subset of $\mathbb{R}^3$ since the parameters are in $\mathbb{R}$. Then, we know $\mathbf{x} = (1, 0, 0)^T \in V$ with $a = 1$ and $b = 0$.

However, $-2\mathbf{x} = (-2, 0, 0)^T \notin V$ because there exists no $a \in \mathbb{R}$ such that $a^2 = -2$. Therefore, V is not closed under scalar multiplication.

Since $-2 \in \mathbb{R}$ and $-2 \in \mathbb{C}$, V is neither a subspace of $\mathbb{R}^3$ nor $\mathbb{C}^3$. $\square$

Q09. We call a square matrix A idempotent if $A^2 = A$. Prove that if A is idempotent, then so is $I - A$. Is the converse of this statement true? Explain why or why not.

Proof. Suppose that $A^2 = A$. Then, $(I - A)^2 = (I - A)(I - A) = I^2 - AI - IA + A^2 = I - 2A + A^2 = I - A$ by properties of the identity matrix and the distributivity of matrix multiplication. Therefore, $I - A$ is idempotent.

Suppose conversely that $(I - A)^2 = I - A$. Then, $I - 2A + A^2 = I - A$ as above, but then $-A + A^2 = \mathbb{O}$. It follows $A = A^2$ and A is idempotent. $\square$

Q10. Let $A, B \in M_{n\times n}(\mathbb{F})$.

(a) Prove or disprove: if $\mathbf{v}$ is an eigenvector of both A and B, then it is an eigenvector of both AB and BA.

Proof. Suppose $A\mathbf{v} = \lambda_A \mathbf{v}$ and $B\mathbf{v} = \lambda_B \mathbf{v}$.

If we multiply the first equation by B, we have $BA\mathbf{v} = B\lambda_A\mathbf{v} = \lambda_A B\mathbf{v} = \lambda_A\lambda_B\mathbf{v}$.

If we instead multiply the second by A, we have $AB\mathbf{v} = A\lambda_B\mathbf{v} = \lambda_B A\mathbf{v} = \lambda_B\lambda_A\mathbf{v}$.

Therefore, $\mathbf{v}$ is an eigenvector of AB and BA. $\square$

(b) Prove or disprove: if λ is an eigenvalue of both A and B, then it is an eigenvalue of both AB and BA

Solution. We consider for a counterexample $A = B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $\lambda = 2$ is an eigenvalue of A and B.

However, $AB = BA = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ and $\lambda = 2$ is not an eigenvalue. $\square$

2 Exercises to Prepare for the Exam

Sourced from Piazza @4051.

Q01. We say that a subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of $\mathbb{C}^n$ is orthogonal if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for all $i \neq j$. Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is orthogonal if and only if $\{\overline{\mathbf{v}_1}, \ldots, \overline{\mathbf{v}_k}\}$ is orthogonal.

Proof. Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is orthogonal. Then, $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for all $i \neq j$.

But $\langle \overline{\mathbf{v}_i}, \overline{\mathbf{v}_j} \rangle = \sum \overline{v_{ii}} v_{ji} = \overline{\sum v_{ii} \overline{v_{ji}}} = \overline{\langle \mathbf{v}_i, \mathbf{v}_j \rangle} = \overline{0} = 0$. Thus, $\{\overline{\mathbf{v}_1}, \ldots, \overline{\mathbf{v}_k}\}$ is orthogonal.

Conversely, notice that $\overline{\overline{\mathbf{v}_i}} = \mathbf{v}_i$ for any vector. □

Q02. Let $A \in M_{n\times n}$ be diagonalizable with not necessarily distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $B = A - \lambda_1 I$ is diagonalizable. What are the eigenvalues of B?

Proof. Let $\mathcal{B} = \{\mathbf{v}_i\}$ be the eigenvectors of A with eigenvalues λ_i.

Then, $B\mathbf{v}_i = (A - \lambda_1 I)\mathbf{v}_i = A\mathbf{v}_i - \lambda_1 \mathbf{v}_i = (\lambda_i - \lambda_1)\mathbf{v}_i$ for all i.

Therefore, $[B]_{\mathcal{B}} = \operatorname{diag}(0, \lambda_2 - \lambda_1, \ldots, \lambda_n - \lambda_1)$ so B is diagonalizable. Notice that because $|\mathcal{B}| = n$, these are the only eigenvalues of B. □

Q03. Let $B \in M_{n\times n}$. Prove or disprove: if $B^2\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$, then B is not invertible.

Proof. Recall that by the Invertible Matrix Theorem, $B^2\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$ iff B^2 is not invertible. Proceed by the contrapositive. Suppose that B is invertible. Then, $B^2 = BB$ must be invertible as the product of invertible matrices. □

Q04. Let $A \in M_{n\times n}$ be diagonal with not necessarily distinct non-zero eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that there exist eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of A such that the set $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis of $\operatorname{Col}(A)$.

Proof. Notice that the diagonal entries of A are either the k non-zero eigenvalues or zero. Therefore, $\operatorname{rank}(A) = \dim(\operatorname{Col}(A)) = k$, since all other columns are zero.

Without loss of generality, suppose that λ_i is in the ith column of A. Then, $A\mathbf{e}_i = \lambda_i \mathbf{e}_i$ for each standard basis vector with $i \leq k$. Notice also that $\mathbf{e}_i$ is an eigenvector of A. Also, since λ_i is non-zero, $A(\lambda_i^{-1}\mathbf{e}_i) = \mathbf{e}_i$ so we have $\mathbf{e}_i \in \operatorname{Col}(A)$.

Now, let $\mathbf{v}_i = \mathbf{e}_i$ for each $i \leq k$. The set B is linearly independent as a subset of S with $|B| = k$ and $B \subset \operatorname{Col}(A)$. By Lemma 17C.11, it is a basis for $\operatorname{Col}(A)$. □

Q05. Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear transformation. Prove or disprove: if T is not invertible, then there exists a basis B for $\mathbb{F}^n$ for which the matrix $[T]_B$ has a column of zeroes.

Proof. Suppose T is not invertible. Recall that by the Invertible Matrix Theorem, the nullity of T is at least 1. Therefore, there exists a basis B_1 of $N(T)$ with $|B_1| \geq 1$.

Then, by the Replacement Theorem, there exists a basis B for $\mathbb{F}^n$ consisting of B_1 and some of the standard basis vectors. Now, let $\mathbf{v} \in B_1$. Then, $T\mathbf{v} = \mathbf{0}$, so the column of $[T]_B$ corresponding to $\mathbf{v}$ has all zero entries. □

Q06. Let T be an invertible linear operator on $\mathbb{F}^n$.

(a) Show that if λ is an eigenvalue of T then λ^{-1} is an eigenvalue of T^{-1}.

Proof. Let $T(\mathbf{v}) = \lambda\mathbf{v}$. Then, $\mathbf{v} = T^{-1}(T(\mathbf{v})) = \lambda T^{-1}(\mathbf{v})$ and so $T^{-1}(\mathbf{v}) = \lambda^{-1}\mathbf{v}$. □

(b) Show that the eigenspace of T corresponding to λ is equal to the eigenspace of T^{-1} corresponding to λ^{-1}.

Proof. Notice that in (a) the same vector $\mathbf{v}$ was the eigenvector of T corresponding to λ and of T^{-1} corresponding to λ^{-1}.

Therefore, $E_\lambda(T) = \text{Span}(\{\mathbf{v}, \dots\}) = E_{\lambda^{-1}}(T^{-1})$. □

(c) Prove or disprove: T is diagonalizable iff T^{-1} is diagonalizable.

Proof. Recall that T is diagonalizable iff T has n linearly independent eigenvectors. However, the eigenvectors of T and T^{-1} are the same. Therefore, T^{-1} is diagonalizable iff T has n linearly independent eigenvectors iff T is diagonalizable. □

Q07. Suppose that $A \in M_{n\times n}$ has two distinct eigenvalues λ_1 and λ_2. Show that if the geometric multiplicity of one of the eigenvalues is $n-1$, then A must be diagonalizable. Is the converse true?

Proof. Suppose $g_{\lambda_1} = n-1$. Then, by definition, a basis B_1 for E_{λ_1} has $n-1$ vectors.

Now, let B_2 be a basis for E_{λ_2}. By Lemma 19B.6, $E_{\lambda_1} \cup E_{\lambda_2}$ is linearly independent. But if $\dim(B_2) > 1$, then $\dim(E_{\lambda_1} \cup E_{\lambda_2}) > n$. Therefore, $\dim(B_2) = g_{\lambda_2} = 1$.

We now have n linearly independent eigenvectors, so A is diagonalizable.

The converse is not true. Consider $A = \text{diag}(1, 2, 3)$. No eigenvalue has multiplicity 2. □

Q08. Suppose $A \in M_{n\times n}$ is similar to the upper triangular matrix $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 2 & 3 & \cdots & n \\ 0 & 0 & 3 & \cdots & n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & n \end{pmatrix}$.

Show that A is diagonalizable.

Solution. Let B be the given upper triangular matrix. Then, we can evaluate along the diagonal $\Delta_B(t) = (1-t)(2-t)(3-t)\cdots(n-t)$.

Since A is similar to B, it has the same characteristic polynomial. That is, A also has eigenvalues $\lambda = 1, \dots, n$. Since there are n distinct eigenvalues, A is diagonalizable. □

Q09. Let $X = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ and $Y = (\mathbf{u}_1, \dots, \mathbf{u}_k)$ be in $M_{n\times k}$.

Prove that if $\text{Col}(X) \subseteq \text{Col}(Y)$, then there exists $A \in M_{k\times k}$ such that $X = YA$.

Proof. □

Q10. Suppose that $L : \mathbb{F}^4 \to \mathbb{F}^7$ is linear. Prove that $R(L) \neq \mathbb{F}^7$.

Proof. Suppose $R(L) = \mathbb{F}^7$. Then, $\dim(\text{Col}(L)) = 7$. However, there are only four columns of $[L]_S$. Therefore, $\dim(\text{Col}(L)) \leq 4$, so by contradiction, $R(L) \neq \mathbb{F}^7$. □

Q11. Let $A \in M_{3\times 3}(\mathbb{R})$ with three distinct, real, non-negative eigenvalues.

Prove that there exists $B \in M_{3\times 3}(\mathbb{R})$ such that $B^2 = A$.

Proof. As the eigenvalues are distinct, we may diagonalize A. Let $\mathcal{B}$ be the eigenbasis for A such that $[A]_{\mathcal{B}} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Now, since all eigenvalues are non-negative, let $[B]_{\mathcal{B}} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})$.

Therefore, $B = {}_S[I]_{\mathcal{B}} \,\text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})_{\mathcal{B}}[I]_S \in M_{3\times 3}(\mathbb{R})$ such that $B^2 = A$. □

Q12. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{F}^n$.

Prove that for any basis B for $\mathbb{F}^n$, $\{[\mathbf{v}_1]_B, \ldots, [\mathbf{v}_n]_B\}$ is a basis for $\mathbb{F}^n$.

Proof. This is a specific case of Question 1.05 above. $\square$

Q13. Let $A, B \in M_{n\times n}$, with $AB = BA$. Suppose that every eigenvalue of A has algebraic multiplicity 1. Prove that every eigenvector of A is an eigenvector of B.

Proof. Let $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Now, because $AB = BA$, we can write $AB\mathbf{v}_i = BA\mathbf{v}_i = \lambda_i(B\mathbf{v}_i)$. But this means $B\mathbf{v}_i$ is an eigenvector for A with eigenvalue λ_i, that is, $B\mathbf{v}_i \in E_{\lambda_i}$.

By Lemma 19B.5, if $a_{\lambda_i} = 1$, then $g_{\lambda_i} = 1$. Then, $\dim(E_{\lambda_i}) = 1$, so $E_{\lambda_i} = \operatorname{Span}(\{\mathbf{v}_i\})$. Therefore, $B\mathbf{v}_i = k\mathbf{v}_i$ by definition of the span. But this is exactly what it means for $\mathbf{v}_i$ to be an eigenvector of B. $\square$

Q14. Let $A, B \in M_{4\times 4}$ with $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ and $B = (\mathbf{a}_4, \mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_1)$.

Prove that $A - B$ is not invertible.

Proof. Notice that we have $A - B = (\mathbf{a}_1 - \mathbf{a}_4, \mathbf{a}_2 - \mathbf{a}_3, \mathbf{a}_3 - \mathbf{a}_2, \mathbf{a}_4 - \mathbf{a}_1)$. But if we perform $C_4 + C_1$ and $C_3 + C_2$ we have $A - B = (\mathbf{a}_1 - \mathbf{a}_4, \mathbf{a}_2 - \mathbf{a}_3, 0, 0)$. Therefore, $\operatorname{rank}(A - B) \leq 2$ since $\dim(\operatorname{Col}(A - B)) = \dim(\operatorname{Row}(A - B))$. It follows $A - B$ is not invertible. $\square$

Q15. Let $\mathbf{n} = (i, 1, 0, 1 + i)^T$. Compute a basis of a subspace $S = \{\mathbf{x} \in C^4 : \langle \mathbf{x}, \mathbf{n}\rangle = 0\}$. What is $\dim(S)$?

Solution. Let $\mathbf{x} = (a, b, c, d)^T \in S$. Then, $\langle \mathbf{x}, \mathbf{n}\rangle = a(-i) + b + d(1 - i) = 0$. Equivalently, $b = ia - (1 - i)d$ for arbitrary a and d.

Let $a = s$, $c = t$, and $d = u$. Then, $S = \{(s, is + (-1 + i)u, t, u)^T : s, t, u \in \mathbb{C}\}$.

But this is equivalently $S = \operatorname{Span}\left(\left\{\begin{pmatrix}1\\i\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1+i\\0\\1\end{pmatrix}\right\}\right)$.

The set is linearly independent, so it forms a basis for S. Therefore, $\dim(S) = 3$. $\square$

Q16. Let $A \in M_{n\times n}$ be non-invertible. Prove that the eigenspace of A corresponding to the eigenvalue $\lambda = 0$ is equal to the nullspace of A.

Proof. By the Invertible Matrix Theorem, since A is singular, E_0 exists. Now, notice that E_0 is defined by $N(A - 0I) = N(A)$. $\square$

Q17. Find the projection of $\mathbf{v} = \begin{pmatrix}-2\\0\\3\end{pmatrix}$ onto the plane $S = \operatorname{Span}\left(\left\{\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}\right)$.

Solution. Who needs fancy matrices? We must take $\mathbf{v} - \operatorname{Proj}_{\mathbf{n}}(\mathbf{v})$ for a normal vector $\mathbf{n}$. Note that we have a trivial $\mathbf{n} = (2, 1, 1)^T \times (1, 1, -1)^T = (-2, 3, 1)^T$. Then, we can calculate that $\operatorname{Proj}_{\mathbf{n}}(\mathbf{v}) = \frac{1}{2}(-2, 3, 1)^T$.

Therefore, the projection is $\frac{1}{2}(-2, -3, 5)^T$. $\square$