Essential Laws of Propositional Logic

$$\text{Double Negation}\begin{cases}\neg(\neg p) \mathrel{\models\!\!\!\dashv} p\end{cases}$$

$$\text{Excluded Middle}\begin{cases}p \vee \neg p \mathrel{\models\!\!\!\dashv} 1\end{cases}$$

$$\text{Contradiction}\begin{cases}p \wedge \neg p \mathrel{\models\!\!\!\dashv} 0\end{cases}$$

$$\text{Idempotence}\begin{cases}p \wedge p \mathrel{\models\!\!\!\dashv} p\\ p \vee p \mathrel{\models\!\!\!\dashv} p\end{cases}$$

$$\text{Identity}\begin{cases}p \wedge 1 \mathrel{\models\!\!\!\dashv} p\\ p \vee 0 \mathrel{\models\!\!\!\dashv} p\end{cases}$$

$$\text{Domination}\begin{cases}p \wedge 0 \mathrel{\models\!\!\!\dashv} 0\\ p \vee 1 \mathrel{\models\!\!\!\dashv} 1\end{cases}$$

$$\text{Commutativity}\begin{cases}p \wedge q \mathrel{\models\!\!\!\dashv} q \wedge p\\ p \vee q \mathrel{\models\!\!\!\dashv} q \vee p\\ p \leftrightarrow q \mathrel{\models\!\!\!\dashv} q \leftrightarrow p\end{cases}$$

$$\text{Associativity}\begin{cases}p \wedge (q \wedge r) \mathrel{\models\!\!\!\dashv} (p \wedge q) \wedge r\\ p \vee (q \vee r) \mathrel{\models\!\!\!\dashv} (p \vee q) \vee r\end{cases}$$

$$\text{Distributivity}\begin{cases}p \wedge (q \vee r) \mathrel{\models\!\!\!\dashv} (p \wedge q) \vee (q \wedge r)\\ p \vee (q \wedge r) \mathrel{\models\!\!\!\dashv} (p \vee q) \wedge (q \vee r)\end{cases}$$

$$\text{Implication}\begin{cases}p \to q \mathrel{\models\!\!\!\dashv} \neg p \vee q\end{cases}$$

$$\text{Contrapositive}\begin{cases}p \to q \mathrel{\models\!\!\!\dashv} \neg q \to \neg p\end{cases}$$

$$\text{Equivalence}\begin{cases}p \leftrightarrow q \mathrel{\models\!\!\!\dashv} (p \to q) \wedge (q \to p)\end{cases}$$

$$\text{De Morgan}\begin{cases}\neg(p \wedge q) \mathrel{\models\!\!\!\dashv} \neg p \vee \neg q\\ \neg(p \vee q) \mathrel{\models\!\!\!\dashv} \neg p \wedge \neg q\end{cases}$$

$$\text{Absorption I}\begin{cases}p \wedge (p \vee q) \mathrel{\models\!\!\!\dashv} p\\ p \vee (p \wedge q) \mathrel{\models\!\!\!\dashv} p\end{cases}$$

$$\text{Absorption II}\begin{cases}(p \vee q) \wedge (\neg p \vee q) \mathrel{\models\!\!\!\dashv} q\\ (p \wedge q) \vee (\neg p \wedge q) \mathrel{\models\!\!\!\dashv} q\end{cases}$$

Eleven Rules of Formal Deduction (+ Six of First Order Logic)

(Abbr.)	From	Conclude	Rule
(Ref)	$\emptyset$	$A \vdash A$	Reflexivity
$(+)$	$\Sigma \vdash A$	$\Sigma, \Sigma' \vdash A$	Addition of premises
$(\neg\, -)$	$\Sigma, \neg A \vdash B$ $\Sigma, \neg A \vdash \neg B$	$\Sigma \vdash A$	$\neg$ elimination
$(\to\, -)$	$\Sigma \vdash A \to B$ $\Sigma \vdash A$	$\Sigma \vdash B$	$\to$ elimination (modus ponens)
$(\to\, +)$	$\Sigma, A \vdash B$	$\Sigma \vdash A \to B$	$\to$ introduction
$(\wedge\, -)$	$\Sigma \vdash A \wedge B$	$\Sigma \vdash A$ $\Sigma \vdash B$	$\wedge$ elimination
$(\wedge\, +)$	$\Sigma \vdash A$ $\Sigma \vdash B$	$\Sigma \vdash A \wedge B$	$\wedge$ introduction
$(\vee\, -)$	$\Sigma, A \vdash C$ $\Sigma, B \vdash C$	$\Sigma, A \vee B \vdash C$	$\vee$ elimination
$(\vee\, +)$	$\Sigma \vdash A$	$\Sigma \vdash A \vee B$ $\Sigma \vdash B \vee A$	$\vee$ introduction
$(\leftrightarrow\, -)$	$\Sigma \vdash A \leftrightarrow B$ $\Sigma \vdash A$	$\Sigma \vdash B$	$\leftrightarrow$ elimination
	$\Sigma \vdash A \leftrightarrow B$ $\Sigma \vdash B$	$\Sigma \vdash A$	
$(\leftrightarrow\, +)$	$\Sigma, A \vdash B$ $\Sigma, B \vdash A$	$\Sigma \vdash A \leftrightarrow B$	$\leftrightarrow$ introduction
$(\forall\, -)$	$\Sigma \vdash \forall x\, A(x)$	$\Sigma \vdash A(t)$	$\forall$ elimination
$(\forall\, +)$	$\Sigma \vdash A(u)$ u not in Σ	$\Sigma \vdash \forall x\, A(x)$	$\forall$ introduction
$(\exists\, -)$	$\Sigma, A(u) \vdash B$ u not in Σ, B	$\Sigma, \exists x\, A(x) \vdash B$	$\exists$ elimination
$(\exists\, +)$	$\Sigma \vdash A(u)$	$\Sigma \vdash \exists x\, A(x)$	$\exists$ introduction
$(\approx\, -)$	$\Sigma \vdash A(t_1)$ $\Sigma \vdash t_1 \approx t_2$	$\Sigma \vdash A(t_2)$	$\approx$ elimination
$(\approx\, +)$	$\emptyset$	$\emptyset \vdash u \approx u$	$\approx$ introduction

N.B.: In $(\forall\, +)$ and $(\exists\, +)$, $A(x)$ is $A(u)$ with some but not necessarily all occurences of u replaced by x.

Proved Theorems

(Abbr.)	From	Conclude	Theorem
$(\in)$	$A \in \Sigma$	$\Sigma \vdash A$	Membership
(Tr.)	$\Sigma \vdash \Sigma'$ $\Sigma' \vdash A$	$\Sigma \vdash A$	Transitivity
$(\neg +)$	$\Sigma, A \vdash B$ $\Sigma, A \vdash \neg B$	$\Sigma \vdash \neg A$	Reductio ad absurdum
(Repl.)	$A \dashv\vdash A'$ $C = B$ with A' for A	$B \dashv\vdash C$	Replaceability
	$A \vdash B$	$\neg B \vdash \neg A$	Flip-Flop

$\neg\neg A \dashv\vdash A$ (Double Negation)

$\emptyset \vdash A \vee \neg A$ (Excluded Middle)

$\emptyset \vdash \neg(A \wedge \neg A)$ (Non-Contradiction)

$A, \neg A \vdash B$ (Inconsistency Rule)

$A \to B, B \to C \vdash A \to C$ (Hypothetical Syllogism)

$A \vee B, \neg B \vdash A$ (Disjunctive Syllogism)

$A \to B \dashv\vdash \neg A \vee B$ (Implication)

$A \to B \dashv\vdash \neg B \to \neg A$ (Contrapositive)

$\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B$
$\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$ (De Morgan)

Definitions

Alphabet (§16a, 13). Finite set of symbols Σ. Σ^* contains all possible strings, including empty string λ. Subset of Σ^* is language.

Propositional language (§02, 3). $\mathcal{L}^p$ has $\Sigma = \{p, q, r, \dots, \neg, \wedge, \vee, \to, \leftrightarrow, (,)\}$.
First-order language (§11, 4). $\mathcal{L}$ extends above with quantifier $(\forall, \exists)$, free (u, v, w) and bound variable (x, y, z), individual $(a, b, c, \dots)$, relation (F, G, H), and function (f, g, h) symbols.

Expression (§02, 4). Element of language, including empty expression ϵ.

Segment (§02, 5). V segment of U if $U = W_1 V W_2$. Initial if $W_1 = \epsilon$, final if $W_2 = \epsilon$. Proper if $V \neq U$.

Term (§11, 6). Term($\mathcal{L}$) is smallest set that contains all individual symbols, free variable symbols, and functions of terms.

Atom (§02, 6). Atom($\mathcal{L}^p$) has expressions that are one proposition symbol.
Atom (§11, 8). Atom($\mathcal{L}$) has $F(t_1, t_2, \dots, t_n)$ and $t_1 \approx t_2$ for $t_i \in$ Term($\mathcal{L}$).

Formula (§02, 6). Form($\mathcal{L}^p$) defined by: Atom($\mathcal{L}^p$) $\subset$ Form($\mathcal{L}^p$).
$A, B \in$ Form($\mathcal{L}^p$) $\Rightarrow (\neg A), (A \wedge B), (A \vee B), (A \to B), (A \leftrightarrow B) \in$ Form($\mathcal{L}^p$).
No other expressions are in Form($\mathcal{L}^p$).
Formula (§11, 9). Form($\mathcal{L}$) has same formation rules but also includes $A(u) \in$ Form($\mathcal{L}$), $x \notin A(u) \Rightarrow \forall x\, A(x), \exists x\, A(x) \in$ Form($\mathcal{L}$)

Sentence (§11, 17). Sent($\mathcal{L}$) $\subset$ Form($\mathcal{L}$) with no free variables.

Scope (§02, 45). In $(\neg A)$, A is the scope of $\neg$. In $(A \star B)$, A is the left scope and B is the right scope of $\star$.

Truth valuation (§03, 6). A function $t :$ Atom($\mathcal{L}^p$) $\to \{0, 1\}$.
Valuation (§12, 7). A domain D and function v such that
$a^v, u^v \in D$ for all individual symbols a and free variables u
$w^{v(u/d)} = d$ if $w = u$ and w^v otherwise for free variables u, w, d
$F^v \subseteq D^n$ for all n-ary relations F with $\approx^v = \{(x, x), x \in D\}$
$f^v : D^n \to D$ for all n-ary functions f

Satisfiablity (§03, 9). t satisfies A if $A^t = 1$. Set satisfied if members satisfied.

Tautology/Universally valid (§03, 14, §12, 23). For all t, $A^t = 1$.
Contradiction/Unsatisfiability (§03, 14). For all t, $A^t = 0$.
Contingent (§03, 14). A is neither a tautology nor contradiction.

Tautological consequence (§03, 21). $\Sigma \models A$ if for all t, $\Sigma^t = 1$ gives $A^t = 1$.
Tautological equivalence (§03, 25). $A \models\!\!\mid B$ if $A \models B$ and $B \models A$.

Literal (§04, 10). A formula of the form p or $\neg p$.

Disjunctive clause (§04, 12). Disjunction with literal disjuncts.
Conjunctive clause (§04, 12). Conjunction with literal conjuncts.

DNF (§04, 13). Disjunction with conjunctive clause disjuncts.
CNF (§04, 13). Conjunction with disjunctive clause conjuncts.

Definability/Reducibility (§05, 2). Connective $\star$ reducible to set $\mathcal{S}$ if $A \star B \models\!\!\mid C$ where C uses only A, B, and connectives in $\mathcal{S}$.

Adequate (§05, 7). Connectives which express any truth table/connective.

Formal deducibility (§06, 15). $\Sigma \vdash A$ generated by finite deduction rules.
Syntactic equivalence (§06, 15). $A \dashv\vdash B$ if $A \vdash B$ and $B \vdash A$.

Consistency (§06, 67). There is no F such that $\Sigma \vdash F$ and $\Sigma \vdash \neg F$.

Resolution (§07, 8). $C \vee p, D \vee \neg p \vdash_r C \vee D$ if C and D disjunctive clauses. Resolve parent clauses over p to resolvent $C \vee D$. Commutativity and idempotence allowed. Note $p, \neg p \vdash_r \{\}$ (empty clause, representing contradiction).

Set-of-Support Strategy (§07, 19). Let $\Sigma' = \neg C$. Resolve all premises against Σ' and add resolvents to Σ'. Repeat until $\{\}$.
Davis-Putnam Procedure (§07, 29). For each p: discard tautolgies from Σ, resolve all pairs over p, discard clauses with p. Always outputs $\emptyset$ or $\{\}$.

Prenex Normal Form (§15, 4). All quantifiers at start. Prefix (quantifiers) + matrix (rest). If no $\exists$, it is $\exists$-free PNF.

Skolem function (§15, 14). Given $\forall x_i \, \exists y$, a function $f(x_i) = y$. Note that Skolemized sentence is not equivalent to original.

Unification (§15, 26). A unifies with B if exists unifiers $x_i := t_i$ (replacing variable x_i with term t_i) that make A equal B.

Turing machine (§16a, 13). Finite set of states S, alphabet I containing blank B, start state $s_0 \in S$, transition function $f : S \times I \to S \times I \times \{L, R\}$ or transition rules (s, x, s', x', d). State s final if no rules for s. Halts if no rule for (s, x), accepts if s final (reject otherwise). TM total if always halts.

Decidability (§16a, 22). Decision problem can be solved by TM that accepts "yes" answers. Function that can be computed by TM is computable.

P and NP (§16b, 42). Problem in P if DTM can solve in polynomial time. In NP if NDTM can solve (or DTM can verify) in polynomial time. Problem NP-complete if any NP can be reduced to it in polynomial time.

Theorems (...the general kind)

Lemma (§02, 29). Every formula has equal number of left/right parentheses.

Unique Readability Theorem (§02, 32). Every formula in Form($\mathcal{L}^p$) is exactly one form of exactly one of $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \leftrightarrow B)$.
Theorem (§11, 15). The same applies to Form($\mathcal{L}$) plus $(\forall x \, A(x))$, $(\exists x \, A(x))$.
Theorem (§11, 15). Also to Term($\mathcal{L}$) with free/individual/function symbols.

Lemma (§03, 23). Equivalent statements of $\{A_i\} \models C$:
Argument with premises A_i and conclusion C is valid.
$(\bigwedge A_i) \to C$ is a tautology; $(\neg C \wedge \bigwedge A_i)$ is a contradiction.
Formula $(\neg C \wedge \bigwedge A_i)$ or set $\{\neg C, A_i\}$ is not satisfiable.

Replaceability of tautologically equivalent formulas (§03, 44, §13, 16). If $A \models\!\!\!\dashv A'$ and A is a subformula of B, then $B \models\!\!\!\dashv B'$ where B' is B with some of the As replaced by A'.

Duality Theorem (§03, 44). If A has only $\neg$, $\wedge$, $\vee$ and $\Delta(A)$ replaces atoms with negations and swaps $\wedge$ with $\vee$, then $\neg A \models\!\!\!\dashv \Delta(A)$.
Duality in FoL (§13, 16). If A is as above plus $\forall$, $\exists$, and $\Delta(A)$ replaces atoms with negations, swaps $\wedge$ with $\vee$, and swaps $\forall$ with $\exists$, then $\neg A \models\!\!\!\dashv \Delta(A)$.

Theorem (§04, 22). All formulas have $F \models\!\!\!\dashv \mathrm{DNF}(F)$ based on truth table.
Theorem (§04, 24). All formulas have $F \models\!\!\!\dashv \mathrm{CNF}(F) = \Delta(\mathrm{DNF}(\neg F))$.

Theorem (§05, 8). The set $S_0 = \{\neg, \wedge, \vee\}$ is adequate.

Finiteness of Premise Set (§06, 31). If $\Sigma \vdash A$, $\Sigma^0 \vdash A$ with finite $\Sigma^0 \subseteq \Sigma$.

Soundness Theorem (§06, 45). If $\Sigma \vdash A$ then $\Sigma \models A$.
Completeness Theorem (§06, 49). If $\Sigma \models A$ then $\Sigma \vdash A$.

Lemma (§06, 67). Σ is satisfiable if and only if Σ is consistent.

Theorem (§07, 20). $\vdash_r$ is complete with the set of support strategy.
Theorem (§07, 35). $\vdash_r$ is sound and complete with DPP.
Theorem (§15, 31). $\vdash_r$ in FoL is sound and complete with unification.

Theorem (§12, 17). All terms in Term($\mathcal{L}$) have $t^v \in D$.
Theorem (§12, 20). All formulas in Form($\mathcal{L}$) have $A^v \in \{0, 1\}$.

Theorem (§12, 28). There is no algorithm to decide validity or satisfiability in first-order logic.

Theorem (§14, 33). First order formal deduction is sound and complete.

Theorem (§15, 19). All sentences F in Sent($\mathcal{L}$) have a corresponding $\exists$-free PNF F' such that F satisfiable iff F' satisfiable.

Theorem (§15, 20). All $\exists$-free PNFs F can construct a finite set of disjunctive clauses C_F such that F satisfiable iff C_F satisfiable.

Theorem (§15, 21). Argument $\Sigma \models A$ valid iff $C_{\neg A} \cup \bigcup_{F \in \Sigma} C_F$ unsatisfiable.

Theorem (§16a, 7). The halting problem is undecidable.
Lemma (§16a, 28). Satisfiablity and validity problems for FoL undecidable.

Lemma (§16b, 50). Satisfiablity for propositional logic NP-complete.