MATH 239 Fall 2022:

Exercises/Problem Sets

Part I

Enumeration

1 Basic Principles

Exercise 1.0.1. Fix integers $n \geq 0$ and $t \geq 1$. Consider a randomly chosen multiset of size n with elements of t types. For each part below, calculate the probability that the multiset has the stated property, and give a brief explanation.

(a) Every type of element occurs at most once.

Solution. Every element either appears or not. That is, we have $2^t / \binom{n+t-1}{t-1}$ $\square$

(b) Every type of element occurs at least once.

Solution. This is equivalent to including every element, then creating a multiset of the remaining $n-t$ spots. That gives $\binom{n-1}{t-1} / \binom{n+t-1}{t-1}$. $\square$

(c) Every type of element occurs an even number of times.

Solution. Make a multiset of size $\frac{n}{2}$ and then double every item: $\binom{\frac{n}{2}+t-1}{t-1} / \binom{n+t-1}{t-1}$. Note that 0 is even, so we don't require all types appearing. $\square$

(d) Every type of element occurs an odd number of times.

Solution. Include every element, then create an even multiset with the remaining $n-t$ spots: $\binom{\frac{n-t}{2}+t-1}{t-1} / \binom{n+t-1}{t-1}$. $\square$

(e) For $k \in \mathbb{N}$, exactly k types of element occur with multiplicity at least one.

Solution. Pick k types, then a multiset of size $n-k$: $\binom{t}{k}\binom{n-k+t-1}{t-1} / \binom{n+t-1}{t-1}$. $\square$

(f) For $k \in \mathbb{N}$, exactly k types of element occur with multiplicity at least two.

Solution. Pick k types, then a multiset of size $n-2k$: $\binom{t}{k}\binom{n-2k+t-1}{t-1} / \binom{n+t-1}{t-1}$. $\square$

Exercise 1.0.2. Consider rolling six fair 6-sided dice, which are distinguishable, so that there are $6^6 = 46656$ equally likely outcomes. Count how many outcomes are of each of the following types:

(a) Six-of-a-kind. $\boxed{6}$

(b) Five-of-a-kind and a single. $\boxed{(6 \cdot 5) \cdot \binom{6}{1} = 180}$

(c) Four-of-a-kind and a pair. $\boxed{(6 \cdot 5) \cdot \binom{6}{4} = 450}$

(d) Four-of-a-kind and two singles. $\boxed{(6 \cdot 5 \cdot 4) \cdot \frac{\binom{6}{4}\binom{2}{1}}{2} = 1800}$

(e) Two triples. $\boxed{(6 \cdot 5) \cdot \frac{\binom{6}{3}}{2} = 300}$

(f) A triple, a pair, and a single. $\boxed{(6 \cdot 5 \cdot 4) \cdot \binom{6}{3}\binom{3}{2} = 7200}$

(g) A triple and three singles. $\boxed{(6 \cdot 5 \cdot 4 \cdot 3) \cdot \frac{\binom{6}{3}\binom{3}{1}\binom{2}{1}}{3 \cdot 2} = 7200}$

(h) Three pairs. $\boxed{(6 \cdot 5 \cdot 4) \cdot \frac{\binom{6}{2}\binom{4}{2}}{3 \cdot 2} = 1800}$

(i) Two pairs and two singles. $\boxed{(6 \cdot 5 \cdot 4 \cdot 3) \cdot \frac{\binom{6}{2}\binom{4}{2}}{2 \cdot 2} = 16200}$

(j) One pair and four singles. $\boxed{(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) \cdot \binom{6}{2} = 10800}$

(k) Six singles. $\boxed{6! = 720}$

Exercise 1.0.3. Let $m \geq 1$, $d \geq 2$, and $k \geq 0$ be integers. When rolling m fair dice, each of which has d sides, what is the probability of rolling exactly k pairs and $m - 2k$ singles

Solution. There are $k + (m - 2k) = m - k$ distinct sides in the roll. There are $\frac{d!}{(d-(m-k))!}$ ways to pick those sides.

Then, there are $\binom{m}{2}\binom{m-2}{2} \cdots \binom{m-2(k-1)}{2} = \prod_{i=0}^{k-1} \binom{m-2i}{2}$ ways to choose the locations of the pairs. Since some of these are duplicates, we divide out $k!$ permutations of pairs.

Finally, this gives us $\frac{d!}{(d-m+k)!k!} \prod_{i=0}^{k-1} \binom{m-2i}{2}$. ☐

Exercise 1.0.4.

(a) Prove that $\rightleftharpoons$ is an equivalence relation.

Proof. We must show identity, reflexivity, and transitivity. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be sets.

Notice that $\text{id} : \mathcal{A} \to \mathcal{A} : a \mapsto a$ is both surjective (there always exists a such that $\text{id}(a) = a$, namely a) and injective ($\text{id}(a) = \text{id}(b) \implies a = b$). Therefore, $\mathcal{A} \rightleftharpoons \mathcal{A}$.

Suppose $\mathcal{A} \rightleftharpoons \mathcal{B}$. Then, there exists a bijection $f : \mathcal{A} \to \mathcal{B}$. Because f is surjective, a preimage under f exists for all $a \in \mathcal{A}$. Since f is a injective, the preimage of a under f is a single element b.

Define $g : \mathcal{B} \to \mathcal{A}$ by that preimage. This is surjective (for all a, there exists $b = f(a)$ such that $g(b) = a$) and injective ($g(b) = g(b') \implies f(g(b)) = f(g(b')) \implies b = b'$). Then, g is a bijection and $\mathcal{B} \rightleftharpoons \mathcal{A}$.

Suppose $\mathcal{A} \rightleftharpoons \mathcal{B} \rightleftharpoons \mathcal{C}$. Then, there exist bijections $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$. Define $h = g \circ f : \mathcal{A} \to \mathcal{C}$. Then, for all $c \in \mathcal{C}$, there exists $a \in \mathcal{A}$ such that $h(a) = g(f(a)) = c$ because of the surjectivity of g and f. Also, $h(a) = h(a') \implies g(f(a)) = g(f(a')) \implies f(a) = f(a') \implies a = a'$ by the injectivity of g and f, so h is injective. Therefore, h is bijective and $\mathcal{A} \rightleftharpoons \mathcal{C}$.

It follows that $\rightleftharpoons$ is an equivalence relation. ☐

(b) Prove Proposition 1.11.

Proof. Suppose $g(f(a)) = a$ and $f(g(b)) = b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

WLOG, consider $f : \mathcal{A} \to \mathcal{B}$.

Surjectivity: Let $b \in \mathcal{B}$. Since $f(g(b)) = b$ and $g(b)$ exists, f is surjective.

Injectivity: Let $a, a' \in \mathcal{A}$ and suppose $f(a) = f(a')$. Then, $g(f(a)) = g(f(a'))$ which means $a = a'$ by supposition.

Therefore, f is bijective and likewise for g.

Now, suppose $f(a) = b$. Then, $g(f(a)) = a = g(b)$. Likewise if $g(b) = a$, then $f(g(b)) = b = f(a)$. Therefore, $f(a) = b \iff g(b) = a$, as desired. $\square$

Exercise 1.0.5. Define $f : \mathbb{Z} \to \mathbb{N}$ as follows: for $a \in \mathbb{Z}$, $f(a) = \begin{cases} 2a & a \geq 0 \\ -1-2a & a < 0 \end{cases}$

Show that f is a bijection by Proposition 1.11.

Proof. We define the function $g : \mathbb{N} \to \mathbb{Z} : b \mapsto \begin{cases} \frac{b}{2} & b \bmod 2 = 0 \\ -\frac{b+1}{2} & b \bmod 2 = 1 \end{cases}$

Then, consider $g(f(a))$. If $a \geq 0$, then $g(f(a)) = g(2a) = \frac{2a}{2} = a$ since $2a$ is even. Otherwise, $g(f(a)) = g(-(1+2a)) = -\frac{-(2a+1)+1}{2} = a$ since $-(2a+1)$ is odd.

Now, consider $f(g(b))$. If $b = 2k$ is even, then $f(g(2k)) = f(k) = 2k = b$ since $k \geq 0$ (because $b \in \mathbb{N}$). Likewise, if $b = 2k+1$ is odd, then $f(g(2k+1)) = f(-(k+1)) = -1-2(-(k+1)) = 2k+2-1 = 2k+1 = b$.

Therefore, by Proposition 1.11, f is a bijection. $\square$

Exercise 1.0.6. Complete Example 1.13.

Proof. We must show that f is a bijection from the set of subsets of $[n+t-1]$ of size $t-1$ to set of all multisets of size n with t types $\mathcal{M}(n,t)$:

> Given a subset $S \subseteq [n+t-1]$, sort it in increasing order $s_1 < \cdots < s_{t-1}$. Then, define $m_i = s_i - s_{i-1} - 1$. For convenience, define $s_0 = 0$ and $s_t = n + t$.
>
> Define $f(S) = (m_1, \dots, m_t)$.

The inverse is also given:

> Given a multiset $\mu = (m_1, \dots, m_t)$ of size n, let $s_i = m_1 + \cdots + m_i + i$.
>
> Define $f^{-1}(\mu) = \{s_1, \dots, s_{t-1}\}$.

Consider $f(f^{-1}(\mu)) = f(f^{-1}(m_1, \dots, m_t)) = f(\{s_1, \dots, s_{t-1}\})$. Notice that the s_i are already in increasing order because we add on the m_i terms. Then,

$$\begin{aligned} m_i' &= s_i - s_{i-1} - 1 \\ &= \left(\sum_{j=1}^{i} m_j + i\right) - \left(\sum_{j=1}^{i-1} m_j + i - 1\right) - 1 \\ &= m_i + \sum_{j=1}^{i-1} (m_j - m_j) + (i - i + 1 - 1) \\ &= m_i \end{aligned}$$

and we have that $f(f^{-1}(\mu)) = (m_1', \dots, m_t') = (m_1, \dots, m_t) = \mu$.

Conversely, consider $f^{-1}(f(S)) = f^{-1}(f(\{s_1, \dots, s_{t-1}\})) = f^{-1}(m_1, \dots, m_t)$. WLOG, suppose that the s_i are already in increasing order. Then,

$$\begin{aligned} s_i' &= i + \sum_{j=0}^{i} m_j \\ &= \sum_{j=0}^{i} (m_j + 1) \\ &= \sum_{j=0}^{i} ((s_j - s_{j-1} - 1) + 1) \\ &= (s_1 + \cdots + s_i) - (s_1 + \cdots + s_{i-1}) \\ &= s_i \end{aligned}$$

and it follows that $f^{-1}(f(S)) = \{s_1', \dots, s_{t-1}'\} = \{s_1, \dots, s_{t-1}\} = S$.

Therefore, by Proposition 1.11, f is a bijection, completing Example 1.13. □

Exercise 1.0.7. Give bijective proofs of the following identities:

(a) For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}$

Proof. Consider the set $\mathcal{S}$ of subsets of $[n]$ with one "highlighted" element. For example, $\{1, 2, 3, \underline{4}, 10, 12\} \subseteq [12]$.

We can construct $\mathcal{S} = \bigcup \mathcal{S}_k$ where $\mathcal{S}_k := \{S \in \mathcal{S} : |S| = k\}$. To construct an element S of $\mathcal{S}_k$, create a subset of size k, of which there are $\binom{n}{k}$, then select one of those k elements to highlight. This gives $|\mathcal{S}_k| = \binom{n}{k}k$. As a disjoint union, $|\mathcal{S}| = \sum \binom{n}{k}k$.

Alternatively, construct $S \in \mathcal{S}$ directly. Pick a single element from $[n]$ to highlight, of which there are n. Then, fill out the rest of the subset using the remaining $n-1$ items, of which there are 2^{n-1}. That is, $|\mathcal{S}| = n2^{n-1}$.

Therefore, under the identity bijection, $\sum \binom{n}{k}k = |\mathcal{S}| = n2^{n-1}$, as desired. □

(b) For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{n}{k} k(k-1) = n(n-1)2^{n-2}$

Proof. Proceed analogously to part (a), but with two highlighted entries in the subset, e.g., $\{1, \underline{2}, \overline{4}, 6\} \subseteq [10]$. Let this set of subsets be $\mathcal{T}$.

As in (a), consider elements of $\mathcal{T}_k$. We select *two* elements to underline from the k elements in the subset, giving us $\binom{n}{k}k(k-1)$. Then, as a disjoint union, $|\mathcal{T}| = \sum |\mathcal{T}_k| = \sum \binom{n}{k}k(k-1)$.

Again, considering an element of $\mathcal{T}$ directly, pick two elements to highlight from n and the $n-1$ elements remaining, then of the remaining $n-2$ elements construct a subset. This gives $|\mathcal{T}| = n(n-1)2^{n-2}$.

Therefore, $\sum \binom{n}{k}k(k-1) = n(n-1)2^{n-2}$, as desired. □

Exercise 1.0.8. For an integer $n \geq 1$, give a bijective proof that $\sum_{2|n} \binom{n}{k} = \sum_{2\nmid n} \binom{n}{k}$.

Proof. We must establish a bijection between the set of even subsets $\mathcal{E} = \{S \subseteq [n] : 2 \mid |S|\}$ and the set of odd subsets $\mathcal{O} = \{S \subseteq [n] : 2 \nmid |S|\}$.

Define $f : S_n \to S_n : f(S) = \begin{cases} S \cup \{1\} & 1 \notin S \\ S \setminus \{1\} & 1 \in S \end{cases}$

Let $f_{\mathcal{E}}$ and $f_{\mathcal{O}}$ be f restricted to the respective set.

Notice that $f(S)$ always either increases or decreases the size of a set by 1, meaning that it will send sets in $\mathcal{E}$ to $\mathcal{O}$ and vice versa.

Also, it is obvious that $f(f(S)) = S \cup \{1\} \setminus \{1\}$ or $S \setminus \{1\} \cup \{1\} = S$, so f is its own inverse.

It follows by Proposition 1.11 that $\mathcal{E} \rightleftharpoons \mathcal{O}$, as desired. $\square$

Exercise 1.0.9. Let n be a positive integer. Let $\mathcal{S}_n$ be the set of ordered pairs of subsets (A, B) in which $A \subseteq B \subseteq [n]$. Let $\mathcal{T}_n$ be the set of all functions $f : [n] \to [3]$.

(a) What is $|\mathcal{T}_n|$?

Solution. Set-theoretically, a function $f : [n] \to [3]$ is a set of ordered pairs for each value $\{(1, f(1)), (2, f(2)), \dots, (n, f(n))\}$. We pick n values here for $f(1), \dots, f(n) \in [3]$. That is, 3^n choices. Therefore, $|\mathcal{T}_n| = 3^n$. $\square$

(b) Define a bijection $g : \mathcal{S}_n \to \mathcal{T}_n$. Explain why $g((A, B)) \in \mathcal{T}_n$ for any $(A, B) \in \mathcal{S}_n$.

Solution. Given $(A, B) \in \mathcal{S}_n$, every element $i \in [n]$ is either: (1) not in A or B, (2) in B but not in A, or (3) in A (and B since $A \subseteq B$).

Let $f(i)$ be the number of the case listed above. This is a function $[n] \to [3]$, so $f \in \mathcal{T}_n$. $\square$

(c) Define the inverse function $g^{-1} : \mathcal{T}_n \to \mathcal{S}_n$ of the bijection g from part (b).

Solution. Construct A and B from $f \in \mathcal{T}_n$:

Read the case list in (b) in reverse. For all $i \in [n]$: if $f(i) = 2$, place $i \in B$; if $f(i) = 3$, place $i \in A$.

Finally, place all elements of A in B. Then, we have $(A, B) \in \mathcal{S}_n$. $\square$

Exercise 1.0.10. Fix integers $n \geq 0$ and $k \geq 1$. Let $\mathcal{A}(n, k)$ be the set of sequences $(a_i) \in \mathbb{N}^k$ such that $\sum a_i = n$ and $j \mid a_j$ for all j.

Let $\mathcal{B}(n, k)$ be the set of sequences $(b_i) \in \mathbb{N}^k$ such that $\sum b_i = n$ and $b_1 \geq b_2 \geq \cdots \geq b_k$.

Construct a pair of mutually inverse bijections between the sets $\mathcal{A}(n, k)$ and $\mathcal{B}(n, k)$.

Solution. Fix n and k and imply the parameters on $\mathcal{A}$ and $\mathcal{B}$. We will treat the sequences as vectors, i.e., $\mathbf{a} = (a_1, \dots, a_k)$.

Let $f : \mathcal{A} \to \mathcal{B}$. We will take the sum of vectors of ones. Let $\mathbb{1}_i = (\underbrace{1, \dots, 1}_{i \text{ ones}}, \underbrace{0, \dots, 0}_{k-i \text{ zeroes}})$.

Then, define $f(\mathbf{a}) = \sum_{i=1}^{k} \frac{a_i}{i} \mathbb{1}_i$. For example, if $n = 7$ and $k = 3$,

$$f((2, 2, 3)) = 2(1, 0, 0) + \frac{2}{2}(1, 1, 0) + \frac{3}{3}(1, 1, 1) = (4, 2, 1)$$

Notice that since the $\mathbb{1}_i$ are non-increasing for all i, their linear combination with positive coefficients is also non-increasing. Also, we are "distributing" the multiples of i into i ones, meaning that the sum $\sum a_i = n$ does not change.

That is, for all $\mathbf{a} \in \mathcal{A}$, $f(\mathbf{a}) = \mathbf{b}$ for some $\mathbf{b} \in \mathcal{B}$.

We can define an inverse $f^{-1}(\mathbf{b})$ by starting at $i = k$ and recursively taking out the largest muliple of i from all k entries.

For the above example, start with $(4, 2, 1)$ and take out 1 from all 3 entries. This sets $a_3 = 1$ and gives $(3, 1, 0)$. Then, take out 1 from the first 2 entries, setting $a_2 = 1$ and giving $(2, 0, 0)$. Finally, take out 2 from the first 1 entry, setting $a_1 = 2$. This gives $\mathbf{a} = (2, 1, 1)$, as expected.

Formally, we define $\mathbf{a} = f^{-1}(\mathbf{b})$ as follows:

$$\begin{aligned} a'_i &= \min\{b_1 - \sum_{j>i} a'_i, \dots, b_i - \sum_{j>i} a'_i\} \\ \mathbf{a} &= (a'_1, 2a'_2, \dots, ka'_k) \end{aligned}$$

which follows the process described above.

Then, since the processes are inverses, f is a bijection and $\mathcal{A} \rightleftharpoons \mathcal{B}$. $\square$

Exercise 1.0.11. For $n \geq 0$ and $t \geq 2$, prove bijectively that $\binom{n+t-1}{t-1} = \sum_{k=0}^{n} \binom{n-k+t-2}{t-2}$.

Proof. The left-hand side counts the set $\mathcal{S}$ of multisets of size n and t types.

Since there are at least 2 types, partition $\mathcal{S}$ according to the number of times that 1 appears in the multiset. Let $\mathcal{S}_k = \{S \in \mathcal{S} : |\{1 \in S\}| = k\}$.

Then, we can ignore the 1's. This means to create an element of $\mathcal{S}_k$, we must create a multiset of size $n-k$ with $t-1$ types and then add k 1's. This gives us $|\mathcal{S}_k| = \binom{(n-k)+(t-1)-1}{(t-1)-1} = \binom{n-k+t-2}{t-2}$.

Finally, since the number of 1's in a multiset is unique, this is a disjoint union and $|\mathcal{S}| = \sum |\mathcal{S}_k| = \sum \binom{n-k+t-2}{t-2}$.

Therefore, $\binom{n+t-1}{t-1} = |\mathcal{S}| = \sum \binom{n-k+t-2}{t-2}$, as desired. $\square$

Exercise 1.0.12. For $n \geq 1$ and $t \geq 1$, prove bijectively that $\binom{n+t-1}{t-1} = \sum_{k=0}^{t} \binom{t}{k}\binom{n-1}{k-1}$.

Proof. Again, the LHS counts the set $\mathcal{S}$ of multisets of size n and t types.

Notice that a multiset need not use all t types. Consider the set $\mathcal{S}_k$ of multisets of size n which use $k \leq t$ types. This set will have at least one of each of the k types and the remainder is a multiset of k types and size $n-k$. That is, there are $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$ of these. We also had to pick the $\binom{t}{k}$ types. Therefore, $|\mathcal{S}_k| = \binom{t}{k}\binom{n-1}{k-1}$.

Since the number of types used by a multiset is unique, this is a disjoint union and $|\mathcal{S}| = \sum |\mathcal{S}_k| = \sum \binom{t}{k}\binom{n-1}{k-1}$.

Therefore, $\binom{n+t-1}{t-1} = |\mathcal{S}| = \sum \binom{t}{k}\binom{n-1}{k-1}$. $\square$

Exercise 1.0.13. Choose a permutation σ of $\{1, 2, \dots, 7\}$ at random, so that each of the $7! = 5040$ permutations are equally likely. What are the probabilities of the following events?

1. Numbers 1 and 2 are consecutive

   Solution. Let σ_i be the index of i in σ. That is, $\sigma_1 = 2$ means 1 is in position 2.

   Consider when 12 appears in the permutation. There are 6 choices to place $\sigma_1 = 1, \dots, 6$ so that $\sigma_2 = 2, \dots, 7$. There are also 6 choices to place 21. Fill the remaining spots with 5!. Therefore, the probability is $\frac{(6+6)5!}{5040} = \frac{2}{7}$. $\square$

2. Number 1 is to the left of 2

   Solution. There are 7 choices for σ_2. Then, there are $\sigma_2 - 1$ choices for σ_1. That is, there are $\sum_{\sigma_2=1}^{7}(\sigma_2 - 1) = \frac{7 \cdot 8}{2} - 7 = 21$ of these permutations. Fill the remaining 5 spots with 5!. Therefore, the probability is $\frac{21 \cdot 5!}{5040} = \frac{1}{2}$. $\square$

3. No two odd numbers are consecutive

Solution. There are four odd numbers and three evens. This means the only way to separate them is to write OEOEOEO. We can permute the odd numbers in 4! ways and evens in 3! ways. This gives a probability $\frac{4!\cdot 3!}{5040} = \frac{2}{70}$. □

Exercise 1.0.14. Let $r \geq 2$ and $s \geq 2$ be integers. Consider a (non-standard) deck of rs cards, divided into s suits each with cards of r different values. The cards in each suit are numbered $A, 2, 3, \dots, r$, and A can be either below 2 or above r. Choose five cards from such a deck in one of $\binom{rs}{5}$ ways. How many ways are there to produce each kind of hand for this "poker in an alternate universe"?

(a) Count "quints" (five-of-a-kinds). $\boxed{r}$

(b) Count straight flushes. $\boxed{(r-4)s}$

(c) Count quads. $\boxed{r\binom{s}{4} \cdot \binom{(r-1)s}{1}}$

(d) Count full houses. $\boxed{r\binom{s}{3} \cdot (r-1)\binom{s}{2}}$

(e) Count flushes. $\boxed{s(r! - (r-4))}$

(f) Count straights. $\boxed{(r-4)(5^s - 5)}$

(g) Count trips. $\boxed{r\binom{s}{3} \cdot \left(s^2\binom{r-1}{2}\right)}$

(h) Count two-pairs. $\boxed{\binom{r}{2}\binom{s}{2}\binom{s}{2} \cdot (s(r-2))}$

(i) Count one-pairs. $\boxed{r\binom{s}{2} \cdot \binom{(r-1)s}{3}}$

(j) Count busted hands. $\boxed{\left(\binom{r}{5} - (r-4)\right) \cdot (s^5 - s)}$

Exercise 1.0.15. The game called "Crowns and Anchors" or "Birdcage" was popular on circus midways early in the 20th century. It is a game between a Player and the House, played as follows. First, the Player wagers w dollars on an integer p from one to six. Next, the House rolls three six-sided dice. For every die that shows p dots on top, the House pays the Player w dollars, but if no dice show p dots on top then the Player's wager is forfeited, and goes to the House. (Assume that the dice are fair, so that every outcome is equally likely.)

For example, if I wager two dollars on the number five, and the dice show five, five, and three dots, respectively, then the House pays me four dollars for a total of six (a profit of four dollars). However, if in this case the dice show four, three, and two dots, respectively, then the House takes my wager for a total of zero (a loss of two dollars).

(a) For every dollar that the Player wagers, how much money should the Player expect to win back in the long run? Would you play this game?

Solution. Consider the expected value for each dollar the Player wagers on k:

There are $6^3 = 216$ total outcomes. Of these, there is 1 where k appears 3 times, 5 where k appears twice, and $5^2 = 25$ where k appears once.

This gives an expected value of $\frac{1}{216}(3k + 5(2k) + 25k) = \frac{38}{216}k$.

The Player will want to maximize payout and always pick $k = 6$.

This gives an expected payout of $\frac{33 \cdot 6}{216} \approx \$1.06$. This is more than the \$1 wager, so the game is worth playing. $\square$

(b) In a parallel universe there is a game of Crowns and Anchors being played with $m \geq 1$ dice, each of which has $d \geq 2$ sides. (Assume that the dice are fair, so that every outcome is equally likely.) In which universes does the Player win in the long run? In which universes does the House win in the long run? In which universes is the game completely fair?

Solution. The Player, as above, will always place a dollar on the highest number d.

Then, there are d^m total outcomes. For each possible payout $1 \leq i \cdot d \leq m \cdot d$, there are i occurrences of d and there are $(d-1)^{m-i}$ ways to pick the remaining $m - i$ dice. As above, we sum to calculate the expected value.

This gives $\frac{1}{d^m} \sum_{i=1}^{m} i(d-1)^{m-i}$. $\square$

2 Generating Series

Exercise 2.0.1. Calculate the following coefficients

(a) $[x^8](1-x)^{-7}$

Solution. Apply the Negative Binomial Series to get $\binom{8+7-1}{7-1} = \binom{14}{6} = 3003$ $\square$

(b) $[x^{10}]x^6(1-2x)^{-5}$

Solution. $[x^{10}]x^6(1-2x)^{-5} = [x^4](1-2x)^{-5}$ where $(1-2x)^{-5} = \sum\binom{n+5-1}{5-1}2^n x^n$ by NBS so we have $\binom{4+5-1}{4}2^4 = 16\binom{8}{4} = 1120$ $\square$

(c) $[x^8](x^3+5x^4)(1+3x)^6$

Solution. Expand: $[x^8](x^3+5x^4)(1+3x)^6 = [x^8](x^3(1+3x)^6) + [x^8](5x^4(1+3x^6)) = [x^5](1+3x)^6 + 5[x^4](1+3x)^6$.

Apply the Binomial Theorem: $(1+3x)^6 = \sum\binom{6}{n}3^n x^n$ giving $[x^n](1+3x)^6 = 3^n\binom{6}{n}$.

Finally, $[x^8](x^3+5x^4)(1+3x)^6 = 3^6\binom{6}{5} + 5(3^4\binom{6}{4}) = 10449$ $\square$

(d) $[x^9]((1-4x)^5 + (1-3x)^{-2})$

Solution. Expand: $[x^9]((1-4x)^5 + (1-3x)^{-2}) = [x^9](1-4x)^5 + [x^9](1-3x)^{-2}$.

Apply BT: $(1+(-4x))^5 = \sum_{n=1}^{5}\binom{5}{n}(-4)^n x^n$ so $[x^9](1-4x)^5 = 0$.

Apply NBS: $(1-3x)^{-2} = \sum\binom{n+2-1}{2-1}3^n x^n$ so $[x^9](1-3x)^{-2} = 3^9\binom{10}{1} = 196830$ $\square$

(e) $[x^n](1-2tx)^{-k}$

Solution. Apply NBS: $(1-2tx)^{-k} = \sum\binom{n+k-1}{k-1}(2t)^n x^n$, so we have $\binom{n+k-1}{k-1}(2t)^n$. $\square$

(f) $[x^{n+1}]x^k(1-4x)^{-2k}$

Solution. Expand: $[x^{n+1}]x^k(1-4x)^{-2k} = [x^{n-k+1}](1-4x)^{-2k}$.

Apply NBS: $(1-4x)^{-2k} = \sum\binom{n+2k-1}{2k-1}4^n x^n$.

This gives $4^{n-k+1}\binom{n-k+1+2k-1}{2k-1} = 4^{n-k+1}\binom{n+k}{2k-1}$. $\square$

(g) $[x^n]x^k(1-x^2)^{-m}$

Solution. Expand: $[x^n]x^k(1-x^2)^{-m} = [x^{n-k}](1-x^2)^{-m}$.

Apply NBS: $(1-x^2)^{-m} = \sum\binom{n+m-1}{m-1}x^{2n}$.

This gives $[x^n]x^k(1-x^2)^{-m} = \begin{cases}\binom{\frac{n}{2}+m-1}{m-1} & n \text{ even}\\ 0 & n \text{ odd}\end{cases}$ $\square$

(h) $[x^n]((1-x^2)^{-k} + (1-7x^3)^{-k})$

Solution. Expand and apply NBS:

$$\begin{aligned}
&[x^n]((1-x^2)^{-k}+(1-7x^3)^{-k})\\
&=[x^n](1-x^2)^{-k}+[x^n](1-7x^3)^{-k}\\
&=[x^n]\sum_{m=0}^{\infty}\binom{m+k-1}{k-1}x^{2m}+[x^n]\sum_{m=0}^{\infty}\binom{m+k-1}{k-1}7^m x^{3m}\\
&=\begin{cases}\binom{\frac{n}{2}+k-1}{k-1} & 2\mid n\\ 0 & 2\nmid n\end{cases}+\begin{cases}7^n\binom{\frac{n}{3}+k-1}{k-1} & 3\mid n\\ 0 & 3\nmid n\end{cases}\\
&=\begin{cases}\binom{\frac{n}{2}+k-1}{k-1} & 2\mid n, 3\nmid n\\ 7^n\binom{\frac{n}{3}+k-1}{k-1} & 2\nmid n, 3\mid n\\ \binom{\frac{n}{2}+k-1}{k-1}+7^n\binom{\frac{n}{3}+k-1}{k-1} & 6\mid n\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

which is a mess. □

Exercise 2.0.2. In each case, find an instance of a Binomial Series that begins as shown.

(a) $1-2x+3x^2-4x^3+5x^4-6x^5+\cdots$

Solution. This has coefficients $(-1)^n(n+1)=(-1)^n\binom{n+2-1}{2-1}$ which comes from the series $(1+x)^{-2}$. □

(b) $1+3x+6x^2+10x^3+15x^4+21x^5+\cdots$

Solution. Coefficients are $\binom{2}{2},\binom{3}{2},\binom{4}{2},\ldots$ which match $\binom{n+3-1}{3-1}$, which comes from the series $(1-x)^{-3}$. □

(c) $1-x^3+x^6-x^9+x^{12}-x^{15}+\cdots$

Solution. Consider x^{3n}. Coefficients $(-1)^n=(-1)^n\binom{n+1-1}{1-1}$ match $(1+x^3)^{-1}$. □

(d) $1+2x^2+4x^4+8x^6+16x^8+32x^{10}+\cdots$

Solution. Consider x^{2n}. Coefficients 2^n match $(1-2x^2)^{-1}$. □

(e) $1-4x^2+12x^4-32x^6+80x^8-192x^{10}+\cdots$

Solution. Consider x^{2n}. Divide through 2^n to get $(1,-2,3,-4,5,-6)$. Coefficients $(-1)^n 2^n(n+1)=(-2)^n\binom{n+2-1}{2-1}$ match $(1+2x^2)^{-2}$ □

(f) $1+6x+24x^2+80x^3+240x^4+672x^5+\cdots$

Solution. Again, divide through by 2^n to get $(3,6,10,\ldots)$ which we recognize from (b). Coefficients $2^n\binom{n+3-1}{3-1}$ come from $(1-2x)^{-3}$. □

Exercise 2.0.3. Give algebraic proofs of these identities from Exercise 1.0.7.

(a) For all $n\in\mathbb{N}$, $\sum_{k=0}^{n}\binom{n}{k}k=n2^{n-1}$

Proof. Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Then:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} k &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} k \\
&= \sum_{k=0}^{n} \frac{n!}{(k-1)!(n-k)!} \\
&= n \sum_{k=0}^{n} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\
&= n \sum_{k=-1}^{n-1} \binom{n-1}{k} \\
&= n2^{n-1}
\end{aligned}$$

as desired. ☐

(b) For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{n}{k} k(k-1) = n(n-1)2^{n-2}$

Proof. As in (a) above:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} k(k-1) &= \sum_{k=0}^{n} \frac{n!}{(k-2)!(n-k)!} \\
&= n(n-1) \sum_{k=0}^{n} \frac{(n-2)!}{(k-2)!((n-2)-(k-2))!} \\
&= n(n-1) \sum_{k=-2}^{n-2} \binom{n-2}{k} \\
&= n(n-1)2^{n-2}
\end{aligned}$$

as desired. ☐

Exercise 2.0.4. Calculate $[x^n](1+x)^{-2}(1-2x)^{-2}$. Give the simplest expression you can find.

Solution. First, find $A(x) = (1+x)^{-2} = \sum a_n x^n$ and $B(x) = (1-2x)^{-2} = \sum b_n x^n$:

$$\begin{aligned}
(1+x)^{-2} &= \sum \binom{n+1}{1} (-1)^n x^n = \sum (-1)^n (n+1) x^n \\
(1-2x)^{-2} &= \sum \binom{n+1}{1} 2^n x^n = \sum 2^n (n+1) x^n
\end{aligned}$$

Then, by definition of multiplication of power series,

$$\begin{aligned}
[x^n](1+x)^{-2}(1-2x)^{-2} &= [x^n]A(x)B(x) \\
&= [x^n] \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n \\
&= \sum_{k=0}^{n} (-1)^k (k+1) 2^{n-k} (n-k+1) \\
&= 2^n \sum_{k=0}^{n} \frac{(k+1)(n-k+1)}{(-2)^k}
\end{aligned}$$

which does not look like it's getting simpler. ☐

Exercise 2.0.5.

(a) Let $a \geq 1$ be an integer. For each $n \in \mathbb{N}$, extract the coefficient x^n from both sides of this power series identity:

$$\frac{(1+x)^a}{(1-x^2)^a} = \frac{1}{(1-x)^a}$$

to show that $\binom{n+a-1}{a-1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{a}{n-2k}\binom{k+a-1}{a-1}$

Proof. Apply the Binomial Series to get $[x^n]\frac{1}{(1-x)^a} = \binom{n+a-1}{a-1}$.

For the left-hand side, consider a product $A(x) \cdot B(x) = (1+x)^a \cdot (1-x^2)^{-a}$.

Then, the Binomial Theorem gives $A(x) = \sum_{n=0}^{a} \binom{a}{n} x^n$ and the Binomial Series gives $B(x) = \sum_{n=0}^{\infty} \binom{n+a-1}{a-1} x^{2n}$.

After multiplying, we can consider contributions from $B(x)$. We can get x^{2k} for $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$ leaving x^{n-2k} from $A(x)$.

This gives a coefficient $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{a}{n-2k}\binom{k+a-1}{a-1}$.

Therefore, by definition of equality of power series, $\binom{n+a-1}{a-1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{a}{n-2k}\binom{k+a-1}{a-1}$, as desired. □

(b) Can you think of a combinatorial proof?

Proof. We are considering the set of multisets of size n and a types.

In a multiset, every type t can appear either an even $m_t = 2k_t$ or odd $m_t = 2k_t + 1$ number of times. Let $k = \sum k_t$ and partition the set of multisets on k. Notice that $\sum m_t = (\sum 2k_t) + |\{t : m_t \text{ is odd}\}| = n$, which means that there are $n - 2k$ types appearing an odd number of times.

Now, we can instead pick a multiset of size k with a types in $\binom{k+a-1}{a-1}$ ways, double every entry, and then add the remainders for the $\binom{a}{n-2k}$ odd entries. Since k runs from 0 to $\lfloor n/2 \rfloor$ (above gives $\binom{a}{n-2k} = 0$), $\binom{n+a-1}{a-1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{a}{n-2k}\binom{k+a-1}{a-1}$, as desired. □

Exercise 2.0.6. Prove the Infinite Sum Lemma.

Proof. Suppose $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ are pairwise disjoint sets and let $\mathcal{B} = \bigcup_{j=0}^{\infty} \mathcal{A}_j$. Also, let $\omega : \mathcal{B} \to \mathbb{N}$ be a weight function. We want to show that $\Phi_{\mathcal{B}}^{\omega}(x) = \sum_{j=0}^{\infty} \Phi_{\mathcal{A}_j}^{\omega}(x)$.

Proceed by the definition of equality and addition for power series.

Let $\Phi_{\mathcal{A}_j}(x) = \sum a_{j,n} x^n$ and $\Phi_{\mathcal{B}}(x) = \sum b_n x^n$.

We must show that $b_n = \sum_{j=0}^{\infty} a_{j,n}$ for all $n \geq 0$.

Since ω is a weight function, $\omega^{-1}(n)$ is a finite set.[1]

Let $m = \max\{j \in \mathbb{N} : \exists k \in \omega^{-1}(n), k \in \mathcal{A}_j\}$, the minimum m where $\omega^{-1}(n) \subseteq \bigcup_{j=0}^{m} \mathcal{A}_j$.

Proceed by induction on m to show that $b_n = \sum_{j=0}^{m} a_{j,n}$.

If $m = 0$, then $\omega^{-1}(n) \subseteq \mathcal{A}_0$ and we just let $b_n = a_{0,n}$. If $m = 1$, then $\omega^{-1}(n) \subseteq \mathcal{A}_0 \cup \mathcal{A}_1$ and by the ordinary Sum Lemma, $b_n = [x^n]\Phi_{\mathcal{A}_0}(x) + [x^n]\Phi_{\mathcal{A}_1}(x) = a_{0,n} + a_{1,n}$.

Otherwise, $m \geq 1$ and we apply the Sum Lemma to $\mathcal{A}_m$ and $\bigcup_{j=0}^{m-1} \mathcal{A}_{j-1}$ to get that $b_n = a_{j,n} + \sum_{j=0}^{m-1} a_{j,n} = \sum_{j=0}^{m} a_{j,n}$, as desired.

[1] Let $\omega^{-1}(n)$ be the preimage of n under ω.

Then, since $\omega^{-1}(n)$ contains no elements in $\mathcal{A}_{m+1}, \dots$ we know that $a_{m+1,n} = a_{m+2,n} = \dots = 0$ and we have that $b_n = \sum_{j=0}^{m} a_{j,n} = \sum_{j=0}^{\infty} a_{j,n}$.

Finally, $\Phi_{\mathcal{B}}(x) = \sum \Phi_{\mathcal{A}_j}(x)$, as desired. □

Exercise 2.0.7. Extend the Product Lemma to the product of finitely many sets with weight functions.

Proof. Consider finitely many sets $\mathcal{A}_1, \dots, \mathcal{A}_n$ for $n \geq 2$. Let $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}$. Define a weight function $\omega : \mathcal{A} \to \mathbb{N}$ a weight function which we can restrict to each set $\mathcal{A}_i$. On the partial unions $\mathcal{B}_j = \bigcup_{i=1}^{j} \mathcal{A}_i$, define $\sigma_j : \mathcal{B}_j \to \mathbb{N} : (a_1, \dots, a_j) \mapsto \sum_{i=1}^{j} \omega(a_i)$. Note that $\mathcal{B}_n = \mathcal{A}$ and write $\sigma = \sigma_n$.

Proceed to show that $\Phi_{\mathcal{A}}^{\omega}(x) = \prod \Phi_{\mathcal{A}_i}^{\omega}(x)$ by induction.

If $n = 2$, this is the standard Product Lemma and we get that σ_2 is a weight function and $\Phi_{\mathcal{B}_2}^{\omega_2}(x) = \prod_{i=1}^{2} \Phi_{\mathcal{A}_i}^{\omega}(x)$.

If $n \geq 3$, suppose that $\Phi_{\mathcal{B}_{n-1}}^{\omega}(x) = \prod_{i=1}^{n-1} \Phi_{\mathcal{A}_i}^{\omega}(x)$.

Then, by the Product Lemma, since $\sigma_n(a_1, \dots, a_n) = \sigma_{n-1}(a_1, \dots, a_{n-1}) + \omega(a_n)$, we can say that σ_n is a weight function and $\Phi_{\mathcal{B}_{n-1} \times \mathcal{A}_n}^{\sigma_n}(x) = \Phi_{\mathcal{A}}^{\sigma}(x) = \Phi_{\mathcal{A}_n}^{\omega}(x) \cdot \prod_{i=1}^{n-1} \Phi_{\mathcal{A}_i}^{\omega}(x) = \prod_{i=1}^{n} \Phi_{\mathcal{A}_i}^{\omega}(x)$ as desired.

Therefore, by induction, the Generalized Product Lemma holds. □

Exercise 2.0.8. Show that for $m, n, k \in \mathbb{N}$, $\sum_{j=0}^{k} (-1)^j \binom{n+j-1}{j} \binom{m}{k-j} = \binom{m-n}{k}$.

Exercise 2.0.9.

(a) Make a list of all the four-letter "words" that can be formed from the "alphabet" $\{a, b\}$. Define the weight of a word to be the number of occurrences of ab in it. Determine how many words there are of weight 0, 1 and 2. Determine the generating series.

Solution. There are $2^4 = 16$ possible words.

There is one word of weight 2, namely $abab$.

A single occurrence of ab can be placed at the start, middle, or end. If at the start/end, there are 3 ways to fill the other 2 letters (aa, bb, and ba). Otherwise, there are 4 (since we do not worry about accidentally making another ab). This gives $3 + 4 + 3 = 10$ words of weight 1.

The remaining 4 words have weight 0.

This gives a generating series $4 + 10x + 2x^2$. □

(b) Do the same for five-letter words over the same alphabet, but preferably, without listing the words separately.

Solution. There are $2^5 = 32$ possible words.

Two occurrences can be placed at the start ($ababX$), end ($Xabab$), or split ($abXab$). The X can be either a or b, giving $3 \times 2 = 6$ words of weight 2.

A word with no ab in it can only go from a run of b's to a's, so if we consider $b^* a^*$ we have 6 words with weight 0.

This means there are $32 - 6 - 6 = 20$ words with weight 1.

This gives a genearting series $6 + 20x + 6x^2$. □

(c) Do the same for six-letter words.

Solution. There are $2^6 = 64$ possible words.

Three occurrences fit six letters, so there is 1 word of weight 3.

Two occurrences can be placed in 4 positions: $ababXX$, $abXabX$, $abXXab$, $XababX$, $XabXab$, $XXabab$. Like in (a), there are 3, 4, 3, 4, 4, and 3 ways to fill the X's. This gives $3+4+3+4+4+3 = 21$ words of weight 2.

As in (b), there are 7 words with weight 0.

There remains $64 - 1 - 21 - 7 = 35$ words with weight 1.

The generating series is therefore $7 + 35x + 21x^2 + x^3$. □

Exercise 2.0.10.

(a) Consider throwing two six-sided dice, one red and one green. The weight of a throw is the total number of pips showing on the top faces of both dice (that is, the usual score). Make a table showing the number of throws of each weight, and write down the generating series.

Solution. Construct the table:

Sum	Outcomes	Count
0	∅	0
1	∅	0
2	11	1
3	12 21	2
4	13 22 31	3
5	14 23 32 41	4
6	15 24 33 42 51	5
7	16 25 34 43 52 61	6
8	26 35 44 53 62	5
9	36 45 54 63	4
10	46 55 64	3
11	56 65	2
12	66	1

with generating series $x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}$. □

(b) Do the same as for part (a), but throwing three dice: one red, one green, and one white

Solution. Construct the table:

Sum	Outcomes	Count
0	∅	0
1	∅	0
2	∅	0
3	111	1
4	112 121 211	3
5	113 122 131 212 221 311	6
6	114 123 132 141 213 222 231 312 321 411	10
7	115 124 133 142 151 214 223 232 241 313 322 331 412 421 511	15
8	116 125 134 143 152 161 215 224 233 242 251 314 323 332 341 413 422 431 512 521 611	21
9	126 135 144 153 162 216 225 234 243 252 261 315 324 333 342 351 414 423 432 441 513 522 531 612 621	25
10	136 145 154 163 226 235 244 253 262 316 325 334 343 352 361 415 424 433 442 451 514 523 532 541 613 622 631	27
11	146 155 164 236 245 254 263 326 335 344 353 362 416 425 434 443 452 461 515 524 533 542 551 614 623 632 641	27
12	156 165 246 255 264 336 345 354 363 426 435 444 453 462 516 525 534 543 552 561 615 624 633 642 651	25
13	166 256 265 346 355 364 436 445 454 463 526 535 544 553 562 616 625 634 643 652 661	21
14	266 356 365 446 455 464 536 545 554 563 626 635 644 653 662	15
15	366 456 465 546 555 564 636 645 654 663	10
16	466 556 565 646 655 664	6
17	566 656 665	3
18	666	1

which gives generating series $x^3 + 3x^4 + 6x^5 + 10x^6 + 15x^7 + 21x^8 + 25x^9 + 27x^{10} + 27x^{11} + 25x^{12} + 21x^{13} + 15x^{14} + 10x^{15} + 6x^{16} + 3x^{17} + x^{18}$. □

Exercise 2.0.11. Construct a table, as in Exercise 2.10(a), if the weight of a throw is defined to be the absolute value of the difference between the number of pips showing on the two dice. Also, write down the generating series.

Solution. Construct the table:

Sum	Outcomes	Count
0	11 22 33 44 55 66	6
1	12 21 23 32 34 43 45 54 56 65	10
2	13 24 31 35 42 46 53 64	8
3	14 25 36 41 52 63	6
4	15 26 51 62	4
5	16 61	2

giving generating series $6 + 10x + 8x^2 + 6x^3 + 4x^4 + 2x^5$. □

Exercise 2.0.12. Let $\mathcal{S}$ be the set of ordered pairs (a, b) of integers with $0 \leq |b| \leq a$. Each part gives a function ω defined on the set $\mathcal{S}$. Determine whether or not ω is a weight function on the set $\mathcal{S}$. If it is not, then explain why not. If it is a weight function, then determine the generating series $\Phi_{\mathcal{S}}(x)$ of $\mathcal{S}$ with respect to ω, and write it as a polynomial or a quotient of polynomials.

(a) For $(a, b) \in \mathcal{S}$, let $\omega((a, b)) = a$.

Solution. This is a weight function. First, $a \geq 0$ for all (a, b).

Given finite $a \in \mathbb{Z}$ with $a \geq |b| \geq 0$, there are finitely many b to choose from, namely, $-a \leq b \leq a$. That is, there are $2a + 1$ options for (a, b) given $\omega((a, b)) = a$.

Therefore, $\Phi_{\mathcal{S}}(x) = \sum(2n+1)x^n = 2x\sum nx^{n-1} + \sum x^n = \frac{2x}{(1-x)^2} + \frac{1}{1-x} = \frac{1+x}{(1-x)^2}$. □

(b) For $(a, b) \in \mathcal{S}$, let $\omega((a, b)) = a + b$.

Solution. This is not a weight function. Notice that $\omega^{-1}(0)$ has infinite size, namely, given any $a \geq 0$, set $b = a$ so that $\omega((a, b)) = 0$. □

(c) For $(a, b) \in \mathcal{S}$, let $\omega((a, b)) = 2a + b$.

Solution. This is a weight function. First, $a \geq |b| \geq 0$ means $2a + b \geq 0$.

Given $\omega((a, b)) = n$, we can construct n as $2a+b$ for $a = \lceil \frac{n}{3} \rceil, \dots, n$ and corresponding $b = n - 2a = n - 2\lceil \frac{n}{3} \rceil, \dots, -n$. That is, $\omega^{-1}(n)$ has size $n - \lceil \frac{n}{3} \rceil + 1$.

This means $\Phi_{\mathcal{S}}(x) = \sum(n - \lceil \frac{n}{3} \rceil + 1)x^n = \sum nx^n - \sum \lceil \frac{n}{3} \rceil x^n + \sum x^n$. Write

$$\begin{aligned}
\sum \left\lceil \frac{n}{3} \right\rceil x^n &= x + x^2 + x^3 + 2x^4 + 2x^5 + 2x^6 + \cdots \\
&= (x + 2x^4 + \cdots) + (x^2 + 2x^5 + \cdots) + (x^3 + 2x^6 + \cdots) \\
&= x(1 + 2x^3 + \cdots) + x^2(1 + 2x^3 + \cdots) + x^3(1 + 2x^3 + \cdots) \\
&= (x + x^2 + x^3)\sum(n+1)x^{3n} \\
&= \frac{x + x^2 + x^3}{(1 - x^3)^2}
\end{aligned}$$

By part (a), we know the other series give $\frac{x}{(1-x)^2} - \frac{x+x^2+x^3}{(1-x^3)^2} + \frac{1}{1-x}$. □

Exercise 2.0.13. Let $\mathcal{S} = [6]^4$ be the set of outcomes when rolling four six-sided dice. For $(a, b, c, d) \in S$, define its weight to be $\omega(a, b, c, d) = a + b + c + d$. Consider the generating series $\Phi_{\mathcal{S}}(x)$ of $\mathcal{S}$ with respect to ω.

(a) Explain why $\Phi_{\mathcal{S}}(x) = \left(\frac{x-x^7}{1-x}\right)^4$.

Solution. The inner term gives the finite geometric series for $1 + x + x^2 + x^3 + x^4 + x^5 + x^6$. This is exactly $\Phi_{[6]}(x)$. By the Generalized Product Lemma (Exercise 2.0.7), the generating series for $\mathcal{S} = [6]^4$ is $(\Phi_{[6]}(x))^4$. □

(b) How many outcomes in $\mathcal{S}$ have weight 19?

(c) Let m, d, k be positive integers. When rolling m dice, each of which has exactly d sides, how many different ways are there to roll a total of k pips on the top faces of the dice?

Solution. The general generating series is $\Phi(x) = \left(\frac{x-x^{d+1}}{1-x}\right)^m$. We want $[x^k]\Phi(x)$. □

Exercise 2.0.14. Let $\mathcal{A}$ be a set with weight function $\omega : \mathcal{A} \to \mathbb{N}$. Show that for any $n \in \mathbb{N}$, the number of $\alpha \in \mathcal{A}$ with $\omega(\alpha) \le n$ is $[x^n]\frac{1}{1-x}\Phi_{\mathcal{A}}(x)$.

Proof. By the Binomial Series, $B(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots$, i.e., $b_n = 1$ for all n.

When we multiply $\Phi_{\mathcal{A}}(x) = \sum a_n x^n$ by $\frac{1}{1-x}$, the product coefficient is by definition $\sum_{i=0}^n a_i b_{n-i} = \sum_{i=0}^n a_i$. That is, the sum of the a_i for $i \le n$.

But since a_i is the number of elements of $\alpha \in \mathcal{A}$ with $\omega(\alpha) = i$, we have counted the number of elements $\omega(\alpha) \le n$. □

Exercise 2.0.15. For each of the following sets of compositions, obtain a rational function formula for the generating series of that set with respect to size.

(a) Let $\mathcal{A}$ be the set of compositions of length congruent to 1 (modulo 3).

Solution. The allowed sizes for a part are $P = \{1, \ldots,\}$ so $\Phi_P(x) = \frac{x}{1-x}$.

We can generate compositions of length $3k + 1$ as $P(P^3)^k$. In general, $\mathcal{A} = P(P^3)^*$.

Then, by the Product Lemma, the generating series for P^3 is $(\frac{x}{1-x})^3$ and by the String Lemma, the generating series for $(P^3)^*$ is $\sum(\frac{x}{1-x})^{3j} = \frac{1}{1-x^3/(1-x)^3} = \frac{(1-x)^3}{(1-x)^3-x^3}$.

Finally, $\Phi_{\mathcal{A}}(x) = \frac{x(1-x)^3}{(1-x)(1-3x+3x^2-2x^3)} = \frac{x-2x^2+x^3}{1-3x+3x^2-2x^3}$. □

(b) Let $\mathcal{B}$ be the set of compositions of length congruent to 2 (modulo 3).

Solution. Proceed as in (a) up to the last multiplication, then square Φ_P to get $\Phi_{\mathcal{B}}(x) = \frac{x^2(1-x)^3}{(1-x)^2(1-3x+3x^2-2x^3)} = \frac{x^2-x^3}{1-3x+3x^2-2x^3}$. □

(c) Let $\mathcal{C}$ be the set of compositions of even length, with each part being at most 3.

Solution. Here, $P = \{1, 2, 3\}$, so $\Phi_P(x) = x + x^2 + x^3$.

Then, to get even length, $\mathcal{C} = (P^2)^*$.

By the Product Lemma, $\Phi_{P^2}(x) = (x+x^2+x^3)^2$ and by the String Lemma, we have $\Phi_{\mathcal{C}}(x) = \frac{1}{1-(x+x^2+x^3)^2} = \frac{1}{1-x^2-2x^3-3x^4-2x^5-x^6}$. □

(d) Let $\mathcal{D}$ be the set of compositions of odd length, with each part being at least 2.

Solution. Let $P = \{2, 3, \dots\}$ so $\Phi_P(x) = x^2 + x^3 + \cdots = x^2(1 + x + \cdots) = \frac{x^2}{1-x}$.

Then, to get odd length, $\mathcal{D} = P(P^2)^*$. By the Product Lemma, $\Phi_{P^2} = \frac{x^4}{(1-x)^2}$ and by the String Lemma $\Phi_{(P^2)^*} = \frac{1}{1-x^4(1-x)^{-2}} = \frac{(1-x)^2}{(1-x)^2-x^4} = \frac{1-2x+x^2}{1-2x+x^2-x^4}$.

Finally, by the Product Lemma, $\Phi_{\mathcal{D}} = \frac{x^2-2x^3+x^4}{(1-x)(1-2x+x^2-x^4)} = \frac{x^2-2x^3+x^4}{1-3x+3x^2-x^3-x^4+x^5}$ □

(e) Let $\mathcal{E}$ be the set of compositions $\gamma = (c_i)$ of any length, in which each part c_i is congruent to i (modulo 2). So c_1 is odd, c_2 is even, c_3 is odd, and so on.

Solution. If $\mathcal{O} = \{1, 3, 5, \dots\}$ and $\mathcal{P} = \{2, 4, 6, \dots\}$, we can write $\mathcal{E} = (\mathcal{O}\mathcal{P})^* \cup (\mathcal{O}\mathcal{P})^*\mathcal{O} = (\mathcal{O}\mathcal{P})^*(\{\varepsilon\} \cup \mathcal{O})$ depending on if the length is even or odd.

Write $\Phi_{\mathcal{O}} = x + x^3 + \cdots = x(1 + x^2 + \cdots) = \frac{x}{1-x^2}$ and $\Phi_{\mathcal{P}} = x^2(1 + x^2 + \cdots) = \frac{x^2}{1-x^2}$.

Then, by the Product Lemma, $\Phi_{\mathcal{O}\mathcal{P}} = \frac{x^3}{(1-x^2)^2}$ and by the String Lemma, $\Phi_{(\mathcal{O}\mathcal{P})^*} = \frac{1}{1-x^3(1-x^2)^{-2}} = \frac{(1-x^2)^2}{(1-x^2)^2-x^3}$.

Finally, by the Product and Sum Lemmas, $\Phi_{\mathcal{E}} = \frac{(1-x^2)^2}{(1-x^2)^2-x^3}\left(1 + \frac{x}{1-x^2}\right) = \frac{(1-x^2)^2}{(1-x^2)^2-x^3} \cdot \frac{1+x-x^2}{1-x^2} = \frac{(1-x^2)(1+x-x^2)}{(1-x^2)^2-x^3} = \frac{1+x-2x^2-x^3+x^4}{1-2x^2-x^3+x^4}$. □

3 Binary Strings

Exercise 3.0.1. Prove Lemma 3.9 (Unambiguous Expressions)

Proof. Let R and S be regular expressions producing $\mathcal{R}$ and $\mathcal{S}$. Proceed by cases.

If $\mathsf{R} = \varepsilon$, 0, or 1, then $\mathcal{R} = \{\varepsilon\}$, $\{0\}$, or $\{1\}$. Notice each is produced exactly once, so R is unambiguous.

Now, suppose R and S are unambiguous. We proceed by contrapositives.

Suppose $\mathsf{R} \smile \mathsf{S}$ which produces $\mathcal{R} \cup \mathcal{S}$ is ambiguous. Then, since R and S are unambiguous, the union operation must produce a duplicate. That is, $\mathcal{R} \cap \mathcal{S} \neq \varnothing$. Conversely, if $\mathcal{R} \cap \mathcal{S}$ is non-empty, then whatever expressions are in the intersection are produced twice by $\mathsf{R} \smile \mathsf{S}$. Therefore, $\mathsf{R} \smile \mathsf{S}$ is ambiguous if and only if $\mathcal{R} \cap \mathcal{S} \neq \varnothing$.

Suppose RS is ambiguous. Then, $\mathcal{RS} = \{\rho\sigma : \rho \in \mathcal{R}, \sigma \in \mathcal{S}\}$ produces some element $\rho\sigma = \rho'\sigma'$ twice. Notice that for all $(\rho,\sigma) \in \mathcal{R} \times \mathcal{S}$, $f(\rho,\sigma) = \rho\sigma \in \mathcal{RS}$. That is, $|\mathcal{R} \times \mathcal{S}| \geq |\mathcal{RS}|$. However, since $\rho\sigma = \rho'\sigma'$, f is not injective. Therefore, $|\mathcal{R} \times \mathcal{S}| \neq |\mathcal{RS}|$. This means $\mathcal{R} \times \mathcal{S} \not\Leftrightarrow \mathcal{RS}$.

Conversely, suppose $\mathcal{R} \times \mathcal{S} \not\Leftrightarrow \mathcal{RS}$. Then, $|\mathcal{R} \times \mathcal{S}| > |\mathcal{RS}|$. This means that under f, multiple pairs must be sent to one string, which is exactly what it means for $\mathcal{RS}$ to be ambiguous.

Finally, consider R^* is ambiguous. Then, since R is unambiguous, the ambiguity must be introduced by either a single R^k being ambiguous or the union. By induction on the second point, the union is unambiguous if and only if $\bigcup_{k=0}^{\infty} \mathcal{R}^k$ is a disjoint union. Therefore, R^* is ambiguous if and only if all the R^k are unambiguous and the union of the $\mathcal{R}^k$ is disjoint. □

Exercise 3.0.2. Let $\mathsf{A} = (10 \smile 101)$ and $\mathsf{B} = (001 \smile 100 \smile 1001)$. For each of AB and BA, is the expression unambiguous? What is the generating series (by length) of the set it produces?

Solution. Write out $\mathcal{AB} = \{10001, 10100, 101001, 101001, 101100, 1011001\}$. Notice 101001 appears twice, so AB is ambiguous. The (meaningless) generating series is $2x^5 + 2x^6 + x^7$.

Write out $\mathcal{BA} = \{00110, 001101, 10010, 100101, 100110, 1001101\}$. No element appeared twice, so BA is unambiguous. The generating series is $2x^5 + 3x^6 + x^7$. □

Exercise 3.0.3. Let $\mathsf{A} = (00 \smile 101 \smile 11)$ and $\mathsf{B} = (00 \smile 001 \smile 10 \smile 110)$. Prove that A^* is unambiguous and B^* is ambiguous. Find the generating series (by length) for the set $\mathcal{A}^*$ produced by A^*.

Solution. Notice that A is unambiguous. There is no way to combine any two of the three strings to create the other one: creating 00 from 101 and 11 is clearly impossible; 11 from 00 and 101 can only be made by 101101 but 10 and 01 cannot be made by 00; 101 cannot be made from 00 and 11 since there is no single 0. Therefore, A^* is unambiguous.

For B, notice that $00110 = (001)(10) = (00)(110)$, so it is ambiguous.

By Theorem 3.13, $\Phi_{\mathcal{A}^*} = (A^*)(x) = \frac{1}{1-A(x)} = \frac{1}{1-2x^2-x^3}$. □

Exercise 3.0.4. For each of the following sets of binary strings, write an unambiguous expression which produces that set.

(a) Binary strings that have no block of 0's of size 3, and no block of 1's of size 2.

Solution. A valid block of 0's is matched by $0 \smile 00 \smile 00000^*$. Likewise, a valid block of 1's is $1 \smile 1111^*$. Then, the block decomposition is

$$\mathsf{R} = (\varepsilon \smile 0 \smile 0^2 \smile 0^4 0^*)((1 \smile 1^3 1^*)(0 \smile 0^2 \smile 0^4 0^*))^*(\varepsilon \smile 1 \smile 1^3 1^*)$$

which is, as a block decomposition, unambiguous. $\square$

(b) Binary strings that have no substring of 0's of length 3, and no substring of 1's of length 2.

Solution. This means blocks of 0's have length 1 or 2, i.e., $(0 \smile 00)$ and blocks of 1's have length 1, i.e., (1). Then, the block decomposition is

$$\mathsf{R} = (\varepsilon \smile 0 \smile 00)(1(0 \smile 00))^*(\varepsilon \smile 1)$$

which an unambiguous block decomposition. $\square$

(c) Binary strings in which the substring 011 does not occur.

Solution. There are no ways that 011 overlaps itself. Therefore, we need only force blocks of 1's after a 0 to have length 1. Using a block decomposition depending if we start with 1 or 0,

$$\mathsf{R} = \varepsilon \smile (11^*)(00^*1)^*0^* \smile (00^*)(100^*)^*(\varepsilon \smile 1)$$

which is unambiguous. $\square$

(d) Binary strings in which the blocks of 0's have even length and the blocks of 1's have odd length.

Solution. Blocks of 0's are matched by $00(00)^*$ and 1's by $1(11)^*$. Then,

$$(00)^*(1(11)^*00(00)^*)^*(\varepsilon \smile 1(11)^*)$$

is an unambiguous block decomposition. $\square$

Exercise 3.0.5. Let $\mathsf{G} = 0^*((11)^*1(00)^*00 \smile (11)^*11(00)^*0)^*$, and let $\mathcal{G}$ be the set of binary strings produced by G.

(a) Give a verbal description of the strings in the set $\mathcal{G}$.

Solution. The set of binary strings where blocks of 0's have the opposite parity of the preceding block of 1's. $\square$

(b) Find the generating series (by length) of $\mathcal{G}$.

Solution. We know by Theorem 3.13 $\Phi_{\mathcal{G}}(x) = G(x)$, so

$$\begin{aligned}
\Phi_{\mathcal{G}}(x) &= \frac{1}{1-x} \cdot \frac{1}{1 - \left(\frac{x}{1-x^2} \cdot \frac{x^2}{1-x^2} + \frac{x^2}{1-x^2} \cdot \frac{x}{1-x^2}\right)} \\
&= \frac{1}{1-x} \cdot \frac{1}{1 - \frac{2x^3}{(1-x^2)^2}} \\
&= \frac{1}{1-x} \cdot \frac{(1-x^2)^2}{(1-x^2)^2 - 2x^3} \\
&= \frac{(1+x)(1-x^2)}{1-2x^2-2x^3+x^4} \\
&= \frac{1+x-x^2-x^3}{1-2x^2-2x^3+x^4}
\end{aligned}$$

as desired. $\square$

(c) For $n \in \mathbb{N}$, let g_n be the number of strings in $\mathcal{G}$ of length n. Give a recurrence relation and enough initial conditions to uniquely determine g_n for all $n \in \mathbb{N}$.

Solution. Apply Theorem 4.8 and read off the linear recurrence relation:

$$g_n - 2g_{n-2} - 2g_{n-3} + g_{n-4} = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ -1 & n = 2 \\ -1 & n = 3 \\ 0 & n \geq 4 \end{cases}$$

from which we calculate initial conditions $g_0 = 1$, $g_1 = 1$, $g_2 = 2g_0 - 1 = 1$, $g_3 = 2g_1 + 2g_0 - 1 = 3$. □

Exercise 3.0.6.

(a) Show that the generating series (by length) for binary strings in which every block of 0's has length at least 2 and every block of 1's has length at least 3 is $\frac{(1-x+x^3)(1-x+x^2)}{1-2x+x^2-x^5}$.

Solution. This set of strings is produced by $\mathsf{R} = (\varepsilon \smile 000^*)(1111^*000^*)^*(\varepsilon \smile 1111^*)$. This leads to the rational function

$$\begin{aligned} R(x) &= \left(1 + \frac{x^2}{1-x}\right) \frac{1}{1 - (\frac{x^3}{1-x} \cdot \frac{x^2}{1-x})} \left(1 + \frac{x^3}{1-x}\right) \\ &= \frac{1-x+x^2}{1-x} \cdot \frac{1}{1 - \frac{x^5}{(1-x)^2}} \cdot \frac{1-x+x^3}{1-x} \\ &= \frac{(1-x+x^2)(1-x+x^3)}{(1-x)^2} \cdot \frac{(1-x)^2}{(1-x)^2 - x^5} \\ &= \frac{(1-x+x^2)(1-x+x^3)}{1-2x+x^2-x^5} \end{aligned}$$

which is equal to the generating series by Theorem 3.13. □

(b) Give a recurrence relation and enough initial conditions to determine the coefficients of this power series.

Solution. Expand the numerator to get $\Phi_{\mathcal{R}}(x) = \frac{1-2x+2x^2-x^4+x^5}{1-2x+x^2-x^5}$. Then, apply Theorem 4.8 to read off the linear recurrence relation:

$$r_n - 2r_{n-1} + r_{n-2} - r_{n-5} = \begin{cases} 1 & n = 0 \\ -2 & n = 1 \\ 2 & n = 2 \\ 0 & n = 3 \\ -1 & n = 4 \\ 1 & n = 5 \\ 0 & n \geq 6 \end{cases}$$

and calculate $r_0 = 1$, $r_1 = -2 + 2r_0 = 0$, $r_2 = 2 + 2r_1 - r_0 = 1$, $r_3 = 0 + 2r_2 - r_1 = 2$, $r_4 = -1 + 2r_3 - r_2 = 2$, and $r_5 = 1 + 2r_4 - r_3 + r_0 = 4$. □

Exercise 3.0.7.

(a) For $n \in \mathbb{N}$, let h_n be the number of binary strings of length n such that each even-length block of 0's is followed by a block of exactly one 1 and each odd-length block of 0's is followed by a block of exactly two 1's. Show that $h_n = [x^n]\frac{1+x}{1-x^2-2x^3}$.

Solution. Let $\mathcal{H}$ be the relevant set and notice it is produced by the block decomposition $\mathsf{H} = 1^*(00(00)^*1 \smile 0(00)^*11)$. Note that since a block of 0's is followed by a block of 1's, we must end on a block of 1's. By Theorem 3.11, $\Phi_{\mathcal{H}}(x) = H(x)$ which is

$$\begin{aligned} H(x) &= \frac{1}{1-x} \cdot \frac{1}{1-(\frac{x^3}{1-x^2}+\frac{x^3}{1-x^2})} \\ &= \frac{1}{1-x} \cdot \frac{1-x^2}{1-x^2-2x^3} \\ &= \frac{1+x}{1-x^2-2x^3} \end{aligned}$$

and $h_n = [x^n]H(x)$ by Proposition 2.7, as desired. □

(b) Give a recurrence relation and enough initial conditions to determine h_n for all $n \in \mathbb{N}$.

Solution. Again, read off a recurrence relation from $H(x)$ by Theorem 4.8:

$$h_n - h_{n-2} - 2h_{n-3} = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

and calculate $h_0 = 1$, $h_1 = 1$, $h_2 = 0 + h_0 = 1$, and $h_3 = 0 + h_1 + 2h_0 = 3$. □

Exercise 3.0.8. Let $\mathcal{K}$ be the set of binary strings in which any block of 1's which immediately follows a block of 0's must have length at least as great as the length of that block of 0's.

(a) Derive a formula for $K(x) = \sum_{\kappa\in\mathcal{K}} x^{\ell(\kappa)}$.

Solution. First, we recursively define $\mathsf{L} = 11^* \smile 0\mathsf{L}1$ so that $\mathcal{L} = 0^i1^j$ where $j \geq i \geq 1$ which is unambiguous. Now, define $\mathsf{K} = 1^*\mathsf{L}^*0^*$ as a block decomposition.

We can now calculate $L(x) = \frac{x}{1-x} + x^2L(x)$ so $L(x) = \frac{x}{(1-x)(1-x^2)}$. Then,

$$\begin{aligned} K(x) &= \frac{1}{1-x} \cdot \frac{1}{1-\frac{x}{(1-x)(1-x^2)}} \cdot \frac{1}{1-x} \\ &= \frac{1}{(1-x)^2} \cdot \frac{(1-x)(1-x^2)}{(1-x)(1-x^2)-x} \\ &= \frac{1+x}{1-2x-x^2+x^3} \end{aligned}$$

which is the generating series $\Phi_{\mathcal{K}}(x)$ by Theorem 3.11. □

(b) Give a recurrence relation and enough initial conditions to determine the coefficients $[x^n]K(x)$ for all $n \in \mathbb{N}$.

Solution. Let $k_n = [x^n]K(x)$. By Theorem 4.8, we have the recurrence relation

$$k_n - 2k_{n-1} - k_{n-2} + k_{n-3} = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

and calculate $k_0 = 1$, $k_1 = 1 + 2k_0 = 3$, and $k_2 = 0 + 2k_1 + k_0 = 7$. □

Exercise 3.0.9.

(a) Fix an integer $m \geq 1$. Find the generating series (by length) of the set of binary strings in which no block has length greater than m.

Solution. Blocks of 0's and 1's become $\smile_{i=1}^{m} 0^i$ and $\smile_{i=1}^{m} 1^i$. Then, the block decomposition

$$\mathsf{R} = (\underset{i=0}{\overset{m}{\smile}} 0^i)(\underset{i=1}{\overset{m}{\smile}} 1^i \underset{i=1}{\overset{m}{\smile}} 0^i)^*(\underset{i=0}{\overset{m}{\smile}} 1^i)$$

produces our desired set. By Theorem 3.11, the generating series $\Phi_{\mathcal{R}}(x) = R(x)$ which is

$$\begin{aligned}
R(x) &= \left(\sum_{i=0}^{m} x^i\right)\left(\frac{1}{1-(\sum_{i=1}^{m} x^i)^2}\right)\left(\sum_{i=0}^{m} x^i\right) \\
&= \left(\frac{1-x^{m+1}}{1-x}\right)^2\left(\frac{1}{1-(\frac{x-x^{m+1}}{1-x})^2}\right) \\
&= \frac{(1-x^{m+1})^2}{(1-x)^2}\cdot\frac{(1-x)^2}{(1-x)^2-(x-x^{m+1})^2} \\
&= \frac{(1-x^{m+1})^2}{(1-x)^2-(x-x^{m+1})^2}
\end{aligned}$$

which I'm sure simplifies further. □

(b) Fix integers $m, k \geq 1$. Find the generating series (by length) of the set of binary strings in which no block of 0's has length greater than m and no block of 1's has length greater than k.

Solution. Proceed as above: we instead get the block decomposition

$$\mathsf{R} = (\underset{i=0}{\overset{m}{\smile}} 0^i)(\underset{i=1}{\overset{k}{\smile}} 1^i \underset{i=1}{\overset{m}{\smile}} 0^i)^*(\underset{i=0}{\overset{k}{\smile}} 1^i)$$

which produces the set we want. By Theorem 3.11,

$$\begin{aligned}
\Phi_{\mathcal{R}}(x) &= R(x) \\
&= \left(\sum_{i=0}^{m} x^i\right)\left(\frac{1}{1-(\sum_{i=1}^{k} x^i)(\sum_{i=1}^{m} x^i)}\right)\left(\sum_{i=0}^{k} x^i\right) \\
&= \left(\frac{1-x^{m+1}}{1-x}\right)\left(\frac{1}{1-(\frac{x-x^{k+1}}{1-x})(\frac{x-x^{m+1}}{1-x})}\right)\left(\frac{1-x^{k+1}}{1-x}\right) \\
&= \frac{(1-x^{m+1})(1-x^{k+1})}{(1-x)^2}\cdot\frac{(1-x)^2}{(1-x)^2-(x-x^{k+1})(x-x^{m+1})} \\
&= \frac{(1-x^{m+1})(1-x^{k+1})}{(1-x)^2-(x-x^{k+1})(x-x^{m+1})}
\end{aligned}$$

as desired. □

Exercise 3.0.10. Let $\mathcal{L}$ be the set of binary strings in which each block of 1's has odd length, and which do not contain the substring 0001. Let $\mathcal{L}_n$ be the set of strings in $\mathcal{L}$ of length n and let $L(x) = \sum_{n=0}^{\infty} |\mathcal{L}_n| x^n$.

(a) Give an expression that produces the set $\mathcal{L}$ unambiguously, and explain briefly why it is unambiguous and produces $\mathcal{L}$.

Solution. Notice that 0001 does not overlap itself. Then, we need only prevent a non-terminal block of 3 or more 0's. Also, blocks of 1's must be odd. Write $\mathsf{L} = (\varepsilon \smile 1(11)^*)((0 \smile 00)1(11)^*)^*0^*$ and notice it is unambiguous as a block decomposition. □

(b) Use your expression from part (a) to show that $L(x) = \frac{1+x-x^2}{1-x-x^2+x^3+x^4}$.

Solution. By Theorem 3.11, the generating series $L(x)$ is

$$\begin{aligned} L(x) &= (1+\frac{x}{1-x^2})\frac{1}{1-\frac{(x+x^2)x}{1-x^2}}\frac{1}{1-x} \\ &= \frac{1+x-x^2}{(1-x^2)(1-x)}\cdot\frac{1-x^2}{1-x^2-(x+x^2)x} \\ &= \frac{1+x-x^2}{(1-x)(1-2x^2-x^3)} \\ &= \frac{1+x-x^2}{1-x-x^2+x^3+x^4} \end{aligned}$$

as desired. □

Exercise 3.0.11. Let $\mathcal{M}$ be the set of binary strings in which each block of 0's has length at most two and which do not contain 00111 as a substring. Let $\mathcal{M}_n$ be the set of strings in $\mathcal{M}$ of length n and let $M(x) = \sum_{n=0}^{\infty} |\mathcal{M}_n| x^n$.

(a) Give an expression that produces the set $\mathcal{M}$ unambiguously, and explain briefly why it is unambiguous and produces $\mathcal{M}$.

Solution. Write $\mathsf{M} = 1^*(011^* \smile (001 \smile 0011))^*(\varepsilon \smile 0 \smile 00)$. We split the middle blocks by whether there are one or two zeroes. If there are two zeroes, then we can only have two ones to avoid 00111. This is unambiguous as it is an (albeit weird-looking) block decomposition. □

(b) Use your expression from part (a) to show that $M(x) = \frac{1+x+x^2}{1-x-x^2-x^3+x^5}$.

Solution. Write by Theorem 3.11 that

$$\begin{aligned} M(x) &= \frac{1}{1-x}\cdot\frac{1}{1-(\frac{x^2}{1-x}+x^3+x^4)}\cdot(1+x+x^2) \\ &= \frac{1+x+x^2}{1-x}\cdot\frac{1-x}{(1-x^3-x^4)(1-x)-x^2} \\ &= \frac{1+x+x^2}{1-x-x^2-x^3+x^5} \end{aligned}$$

as desired. □

Exercise 3.0.12. Let $\mathcal{N}$ be the set of binary strings in which each block of 0's has odd length and each block of 1's has length 1 or 2. Let $\mathcal{N}_n$ be the set of strings in $\mathcal{N}$ of length n and let $N(x) = \sum_{n=0}^{\infty} |\mathcal{N}_n| x^n$.

(a) Show that $N(x) = \frac{1+2x+x^2-x^4}{1-2x^2-x^3} = -2+x+\frac{3+x-3x^2}{1-2x^2-x^3}$.

Solution. Write $\mathsf{N} = (\varepsilon \smile 0(00)^*)((1 \smile 11)0(00)^*)^*(\varepsilon \smile 1 \smile 11)$. By Theorem 3.11,

$$\begin{aligned} N(x) &= \left(1 + \frac{x}{1-x^2}\right) \cdot \frac{1}{1 - \frac{(x+x^2)x}{1-x^2}} \cdot (1 + x + x^2) \\ &= \frac{(1+x-x^2)(1+x+x^2)}{1-x^2} \cdot \frac{1-x^2}{(1-x^2)-(x^2+x^3)} \\ &= \frac{(1+x-x^2)(1+x+x^2)}{1-2x^2-x^3} \\ &= \frac{1+2x+x^2-x^4}{1-2x^2-x^3} \end{aligned}$$

as desired. □

(b) Derive an exact formula for $|\mathcal{N}_n|$ as a function of n.

Solution. Apply partial fractions on $\frac{3+x-3x^2}{1-2x^2-x^3} = \frac{A}{1+x} + \frac{B+Cx}{1-x-x^2}$. Equate numerators to get $3+x-3x^2 = A(1-x-x^2)+(B+Cx)(1+x) = (A+B)+(-A+B+C)x+(-C)x^2$. This gives the system

$$\begin{aligned} A + B \quad\quad &= 3 \\ -A + B + C &= 1 \\ -A \quad\quad + C &= -3 \end{aligned}$$

which solves to $A = -1$, $B = -4$, $C = 4$. Finally, we have that

$$\begin{aligned} [x^n]\frac{3+x-3x^2}{1-2x^2-x^3} &= -[x^n]\frac{1}{1+x} - 4[x^n]\frac{1}{1-x-x^2} + 4[x^n]\frac{x}{1-x-x^2} \\ &= -(-1)^n - 4f_n + 4f_{n-1} \\ &= (-1)^{n+1} - 4(f_n - f_{n-1}) \\ &= 4f_{n-2} - (-1)^n \end{aligned}$$

where f_n is the nth Fibonacci number. □

Exercise 3.0.13. For $n \in N$, let p_n be the number of binary strings of length n in which every block of 0's is followed by a block of 1's with the same parity of length.

(a) Determine the generating series $P(x) = \sum_{n=0}^{\infty} p_n x^n$

Solution. Write $\mathsf{P} = 1^*(00(00)^*11(11)^* \smile 0(00)^*1(11)^*)0^*$. Then, by Theorem 3.11,

$$\begin{aligned} P(x) &= \frac{1}{1-x} \cdot \frac{1}{1 - \left(\frac{x^4}{(1-x^2)^2} + \frac{x^2}{(1-x^2)^2}\right)} \cdot \frac{1}{1-x} \\ &= \frac{1}{(1-x)^2} \cdot \frac{1}{1 - \frac{x^4+x^2}{(1-x^2)^2}} \\ &= \frac{1}{(1-x)^2} \cdot \frac{(1-x^2)^2}{(1-x^2)^2 - x^2 - x^4} \\ &= \frac{1+2x+x^2}{1-3x^2} \end{aligned}$$

as desired. □

(b) Show that if $n \geq 2$, then $p_n = 2 \cdot 3^{\lfloor n/2 \rfloor - 1}$.

Solution. By part (a), we have that

$$\begin{aligned} p_n &= [x^n]\frac{1+2x+x^2}{1-3x^2} \\ &= [x^n]\frac{1}{1-3x^2} + 2[x^n]\frac{x}{1-3x^2} + [x^n]\frac{x^2}{1-3x^2} \\ &= a_n + 2a_{n-1} + a_{n-2} \end{aligned}$$

where $\sum a_n x^n = \frac{1}{1-3x^2}$ which is valid because $n \geq 2$. Notice that $\frac{1}{1-3x^2} = \sum 3^n x^{2n}$. If n is even, then we have $3^{n/2} + 2(0) + 3^{n/2-1} = 4 \cdot 3^{\lfloor n/2 \rfloor - 1}$. If n is odd, then we have $0 + 2 \cdot 3^{(n-1)/2} + 0 = 2 \cdot 3^{\lfloor n/2 \rfloor - 1}$.

Which isn't right??? □

Exercise 3.0.14.

(a) Let $\mathcal{Q}$ be the set of binary strings that do not contain 11000 as a substring. For $n \in \mathbb{N}$, let $\mathcal{Q}_n$ be the set of strings in $\mathcal{Q}$ with length n. Obtain a formula for the generating series $Q(x) = \sum |\mathcal{Q}_n| x^n$.

Solution. Notice that 11000 cannot overlap itself. Therefore, the generating series $Q(x) = \frac{1+0}{(1-2x)(1+0)+x^5} = \frac{1}{1-2x+x^5}$ by Theorem 3.26. □

(b) Let $\mathcal{R}$ be the set of compositions, of any length, in which each part is at most 4. For $n \in \mathbb{N}$, let $\mathcal{R}_n$ be the set of compositions in $\mathcal{R}$ of size n. Obtain a formula for the generating series $R(x) = \sum |\mathcal{R}_n| x^n$.

Solution. The parts are $\mathcal{P} = \{1, 2, 3, 4\}$ with generating series $P(x) = x + x^2 + x^3 + x^4 = \frac{x - x^5}{1-x}$. Then, by the String Lemma, we have that the generating series for $\mathcal{R} = \mathcal{P}^*$ is $R(x) = \frac{1}{1 - \frac{x-x^5}{1-x}} = \frac{1-x}{1-2x+x^5}$ □

(c) Deduce that for all integers $n \geq 1$, $|\mathcal{R}_n| = |\mathcal{Q}_n| - |\mathcal{Q}_{n-1}|$.

Proof. By (a) and (b), we have that:

$$\begin{aligned} |\mathcal{Q}_n| - |\mathcal{Q}_{n-1}| &= [x^n]\frac{1}{1-2x+x^5} - [x^{n-1}]\frac{1}{1-2x+x^5} \\ &= [x^n]\frac{1}{1-2x+x^5} - [x^n]\frac{x}{1-2x+x^5} \\ &= [x^n]\frac{1-x}{1-2x+x^5} \\ &= |\mathcal{R}_n| \end{aligned}$$

as desired. □

(d) Part (c) implies that for every integer $n \geq 1$, there exists a bijection $\mathcal{R}_n \rightleftharpoons \mathcal{Q}_n \cup \mathcal{Q}_{n-1}$. Can you determine such a bijection precisely?

Exercise 3.0.15. Let $\mathcal{V}$ be the set of binary strings that do not contain 0110 as a substring. Find the generating series for $\mathcal{V}$.

Solution. Notice that 0110 can only overlap itself as (0110)(110), so we consider the set of suffixes {110} Therefore, if we let $C(x) = x^3$, we have by Theorem 3.26 that $\Phi_{\mathcal{V}}(x) = \frac{1+x^3}{(1-2x)(1+x^3)+x^4} = \frac{1+x^3}{1-2x+x^3-x^4}$. □

Exercise 3.0.16.

(a) Let $\mathcal{W}$ be the set of binary strings that do not contain 0101 as a substring. Obtain a formula for the generating series (by length) of $\mathcal{W}$.

Solution. Observe that 0101 only overlaps itself as (0101)(01). The set of suffixes to consider is $\{01\}$ with generating series $C(x) = x^2$. Therefore, by Theorem 3.26, $\Phi_{\mathcal{W}}(x) = \frac{1+x^2}{(1-2x)(1+x^2)+x^4} = \frac{1+x^2}{1-2x+x^2-2x^3+x^4}$. □

(b) Fix a positive integer $r \geq 1$ and consider the binary string $(01)^r$. Obtain a formula for the generating series of the set of binary strings that do not contain $(01)^r$.

Solution. The set of suffixes to consider will be $\{(01)^1, (01)^2, \dots, (01)^{r-1}\}$. This has generating series $C(x) = x^2 + x^4 + \cdots + x^{2(r-1)} = x^2(1 + x^2 + \cdots + x^{2(r-2)}) = \frac{x^2 - x^{2r}}{1-x^2}$. Notice that $1 + C(x) = \frac{1-x^{2r}}{1-x^2}$. Now, by Theorem 3.26,

$$\begin{aligned}\Phi(x) &= \frac{\frac{1-x^{2r}}{1-x^2}}{(1-2x)(\frac{1-x^{2r}}{1-x^2}) + x^{2r}} \\ &= \frac{1-x^{2r}}{(1-2x)(1-x^{2r}) + x^{2r}(1-x^2)} \\ &= \frac{1-x^{2r}}{1-2x+2x^{2r+1}-x^{2r+2}}\end{aligned}$$

as desired. □

Exercise 3.0.17. Let $\mathsf{S} = \mathsf{A}^*\mathsf{B}$ be an unambiguous prefix decomposition producing some set of strings $\mathcal{S} \subseteq \{0,1\}^*$. Show that the recursion $\mathsf{R} = \mathsf{B} \smile \mathsf{AR}$ defines an expression R that produces the same set of strings $\mathcal{S} \subseteq \{0,1\}^*$. Also check that both S and R lead to the rational function $B(x)/(1-A(x))$

Proof. Let $\mathcal{S}_n = \mathcal{A}^n\mathcal{B}$. By definition, $\mathcal{S} = \bigcup_{n\geq 0} \mathcal{S}_n$ and because of unambiguity, this is a disjoint union.

Now, consider the set produced by R. Produce a disjoint union based on the number of times m we choose to recurse (i.e., choose AR) before terminating (i.e., choosing B). Then, $\mathcal{R}_m = \mathcal{A}^m\mathcal{B} = \mathcal{S}_m$. Therefore, the set produced is $\bigcup_{n\geq 0} \mathcal{R}_n = \bigcup_{n\geq 0} \mathcal{S}_n = \mathcal{S}$.

For the generating series, notice that we can write $S(x) = \frac{1}{1-A(x)} \cdot B(x) = \frac{B(x)}{1-A(x)}$ and $R(x) = B(x) + A(x)R(x) \implies R(x)(1-A(x)) = B(x) \implies R(x) = \frac{B(x)}{1-A(x)}$ as desired. □

4 Recurrence Relations

Exercise 4.0.1. For each of the sets of compositions from Exercise 2.0.15, do the following:

- Derive a recurrence relation and initial conditions for the coefficients of the corresponding generating series.
- Calculate the coefficients for $n = 0$ up to 9.

(a) $A(x) = \frac{x-2x^2+x^3}{1-3x+3x^2-2x^3}$

Solution. Apply Theorem 4.8 to read:

$$a_n - 3a_{n-1} + 3a_{n-2} - 2a_{n-3} = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ -2 & n = 2 \\ 1 & n = 3 \\ 0 & n \geq 4 \end{cases}$$

and calculate the initial conditions

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 + 3a_0 = 1 \\ a_2 &= -2 + 3a_1 - 3a_0 = 1 \\ a_3 &= 1 + 3a_2 - 3a_1 + 2a_0 = 1 \end{aligned}$$

as desired. □

(b) $B(x) = \frac{x^2-x^3}{1-3x+3x^2-2x^3}$

Solution. Apply Theorem 4.8 to read:

$$b_n - 3b_{n-1} + 3b_{n-2} - 2b_{n-3} = \begin{cases} 0 & n = 0 \\ 0 & n = 1 \\ 1 & n = 2 \\ -1 & n = 3 \\ 0 & n \geq 4 \end{cases}$$

and calculate the initial conditions

$$\begin{aligned} b_0 &= 0 \\ b_1 &= 0 + 3b_0 = 0 \\ b_2 &= 1 + 3b_1 - 3b_0 = 1 \\ b_3 &= -1 + 3b_2 - 3b_1 + 2b_0 = 2 \end{aligned}$$

as desired. □

(c) $C(x) = \frac{1}{1-x^2-2x^3-3x^4-2x^5-x^6}$

Solution. Apply Theorem 4.8 to read:

$$c_n - c_{n-2} - 2c_{n-3} - 3c_{n-4} - 2c_{n-5} - c_{n-6} = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases}$$

and calculate the initial conditions

$$\begin{aligned}
c_0 &= 1 \\
c_1 &= 0 \\
c_2 = c_0 &= 1 \\
c_3 = c_1 + 2c_0 &= 2 \\
c_4 = c_2 + 2c_1 + 3c_0 &= 4 \\
c_5 = c_3 + 2c_2 + 3c_1 + 2c_0 &= 6 \\
c_6 = c_4 + 2c_3 + 3c_2 + 2c_1 + c_0 &= 12
\end{aligned}$$

as desired. □

(d) $D(x) = \frac{x^2-2x^3+x^4}{1-3x+3x^2-x^3-x^4+x^5}$

Solution. Apply Theorem 4.8 to read:

$$d_n - 3d_{n-1} + 3d_{n-2} - d_{n-3} - d_{n-4} + d_{n-5} = \begin{cases} 0 & n = 0 \\ 0 & n = 1 \\ 1 & n = 2 \\ -2 & n = 3 \\ 1 & n = 4 \\ 0 & n \geq 5 \end{cases}$$

and calculate the initial conditions

$$\begin{aligned}
d_0 &= 0 \\
d_1 = 3d_0 &= 0 \\
d_2 = 1 + 3d_1 - 3d_0 &= 1 \\
d_3 = -2 + 3d_2 - 3d_1 + d_0 &= 1 \\
d_4 = 1 + 3d_3 - 3d_2 + d_1 + d_0 &= 1 \\
d_5 = 3d_4 - 3d_3 + d_2 + d_1 - d_0 &= 1
\end{aligned}$$

as desired. □

(e) $E(x) = \frac{1+x-2x^2-x^3+x^4}{1-2x^2-x^3+x^4}$

Solution. Apply Theorem 4.8 to read:

$$e_n - 2e_{n-2} - e_{n-3} + e_{n-4} = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ -2 & n = 2 \\ -2 & n = 3 \\ 1 & n = 4 \\ 0 & n \geq 5 \end{cases}$$

and calculate the initial conditions

$$\begin{aligned} e_0 &= 1 \\ e_1 &= 1 \\ e_2 &= -2 + 2e_0 = 0 \\ e_3 &= -2 + 2e_1 + e_0 = 1 \\ e_4 &= 1 + 2e_2 + e_1 - e_0 = 1 \end{aligned}$$

as desired. □

Exercise 4.0.2. Let $\mathcal{K}$ be the set of compositions $\gamma = (c_i)$ with at least one part such that the first part is odd. Let $K(x)$ be the generating series for $\mathcal{K}$ with respect to size.

(a) Show that $K(x) = \frac{x}{(1+x)(1-x)}$.

Solution. Write $\mathcal{K} = \{1, 3, 5, \dots\} \times \{1, 2, 3, \dots\}^*$.

The generating series for the odd parts is $\frac{x}{1-x^2}$ and for the normal parts $\frac{x}{1-x}$.

By the String Lemma, we get $\frac{1}{1-\frac{x}{1-x}} = \frac{1-x}{1-2x}$.

Finally, by the Product Lemma, we have $K(x) = \frac{x(1-x)}{(1-x^2)(1-2x)} = \frac{x-x^2}{1-2x-x^2+2x^3}$. □

(b) Use part (a) to show that among all 2^{n-1} compositions of size $n \geq 1$, the fraction of these compositions in the set $\mathcal{K}$ is $\frac{2}{3} + \frac{1}{3}(\frac{-1}{2})^{n-1}$.

Solution. Use partial fractions to decompose $\frac{x-x^2}{(1-x)(1+x)(1-2x)} = \frac{1/3}{1-2x} - \frac{1/3}{1+x}$.

Then, $k_n = \frac{1}{3}[x^n]\frac{1}{1-2x} - \frac{1}{3}[x^n]\frac{1}{1+x} = \frac{1}{3} \cdot 2^n - \frac{1}{3}(-1)^n$ so that $\frac{k_n}{2^{n-1}} = \frac{2}{3} + \frac{1}{3}(\frac{-1}{2})^{n-1}$ as desired. □

Exercise 4.0.3. Consider the power series $\sum_{n=0}^{\infty} c_n x^n = \frac{1-2x^2}{1-5x+8x^2-4x^3} = 1 + 5x + 15x^2 + 39x^3 + \cdots$

(a) Give a linear recurrence relation that (together with the initial conditions above) determines the sequence of coefficients $(c_n : n \geq 0)$ uniquely.

Solution. Apply Theorem 4.8:

$$c_n - 5c_{n-1} + 8c_{n-2} - 4c_{n-3} = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \\ -2 & n = 2 \\ 0 & n \geq 3 \end{cases}$$

and we are given initial conditions 1, 5, 15, 39. □

(b) Derive a formula for c_n as a function of $n \geq 0$.

Solution. Write $\frac{1-2x^2}{1-5x+8x^2-4x^3} = \frac{1-2x^2}{(1-x)(1-2x)^2}$.

Then, partial fractions gives $\frac{-1}{1-x} + \frac{1}{1-2x} + \frac{1}{(1-2x)^2}$.

Extracting coefficients gives $c_n = -1 + 2^n + (n+1)2^n = (n+2)2^n - 1$. □

Exercise 4.0.4. Consider the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{x+7x^2}{1-3x^2-2x^3}$

(a) Write down a linear recurrence relation and enough initial conditions to determine the sequence $(a_n : n \in \mathbb{N})$ uniquely.

Solution. Apply Theorem 4.8 to get $a_n - 3a_{n-2} - 2a_{n-3} = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ 7 & n = 2 \\ 0 & n \geq 3 \end{cases}$ □

(b) Given that $1 - 3x^2 - 2x^3 = (1 - 2x)(1 + x)^2$, obtain a formula for a_n as a function of $n \in \mathbb{N}$.

Solution. The inverse roots are 2 (multiplicity 1) and -1 (multiplicity 2). By the Main Theorem, $a_n = A2^n + (Bn + C)(-1)^n$ for some coefficients A, B, and C.

The initial conditions are $a_0 = 0$, $a_1 = 1$, and $a_2 = 3a_0 + 7 = 7$.

Using the initial conditions, we have the system

$$\begin{aligned} A \phantom{{}-2B} + C &= 0 \\ 2A - B - C &= 1 \\ 4A + 2B + C &= 7 \end{aligned}$$

which solves to $A = 1$, $B = 2$, $C = -1$.

Therefore, $a_n = 2^n + (2n - 1)(-1)^n$ for all n. □

Exercise 4.0.5. Consider the power series $\sum_{n=0}^{\infty} c_n x^n = \frac{3-11x+11x^2}{1-4x+5x^2-2x^3} = 3 + x + x^3 + 6x^4 + 19x^5 + \cdots$

(a) Give a linear recurrence relation that (together with the initial conditions above) determines the sequence of coefficients $(c_n : n \geq 0)$ uniquely.

Solution. Apply Theorem 4.8: $c_n - 4c_{n-1} + 5c_{n-2} - 2c_{n-3} = \begin{cases} 3 & n = 0 \\ -11 & n = 1 \\ 11 & n = 2 \\ 0 & n \geq 3 \end{cases}$ □

(b) Derive a formula c_n as a function of $n \geq 0$.

Solution. Observe that the denominator factors as $-(x - 1)^2(2x - 1)$. The inverse roots are 2 (multiplicity 1) and 1 (multiplicity 2). By the Main Theorem, we have $c_n = A(2)^n + (Bn + C)1^n = A(2)^n + Bn + C$.

The initial conditions give us:

$$\begin{aligned} A \phantom{{}+2B} + C &= c_0 = 3 \\ 2A + B + C &= c_1 = 1 \\ 4A + 2B + C &= c_2 = 0 \end{aligned}$$

which solves to $A = 1$, $B = -3$, and $C = 2$.

Therefore, $c_n = 2^n - 3n + 2$ for all $n \geq 0$. □

Exercise 4.0.6. A sequence of integers is determined by the initial conditions $g_0 = 1$, $g_1 = 2$, $g_2 = 3$, and $g_n = 2g_{n-1} - g_{n-2} + 2g_{n-3}$ for $n \geq 3$.

(a) Obtain a rational function formula for the generating series $G(x) = \sum_{n=0}^{\infty} g_n x^n = 1 + 2x + 3x^2 + 6x^3 + 13x^4 + 26x^5 + 51x^6 + \cdots$

Solution. The denominator will be $1-2x+x^2-2x^3$ (based on the recurrence). Using the initial conditions, we can find the numerator coefficients:

$$\begin{aligned} g_0 - 2g_{-1} + g_{-2} - 2g_{-3} &= g_0 && = 1 \\ g_1 - 2g_0 + g_{-1} - 2g_{-2} &= g_1 - 2g_0 && = 0 \\ g_2 - 2g_1 + g_0 - 2g_{-1} &= g_2 - 2g_1 + g_0 && = 0 \end{aligned}$$

and notice the numerator is just 1.

Therefore, $G(x) = \frac{1}{1-2x+x^2-2x^3}$. □

(b) Obtain a formula for the coefficient g_n as a function of $n \in \mathbb{N}$.

Solution. Notice that the denominator factors as $-(2x-1)(x^2+1)$. The inverse roots are 2, $i^{-1} = -i$, and $(-i)^{-1} = i$ (all multiplicity 1). By the Main Theorem, we can write $g_n = \alpha(2)^n + \beta(-i)^n + \gamma(i)^n$ for complex coefficients α, β, and γ.

The initial conditions give us:

$$\begin{aligned} 1\alpha + \beta + \gamma &= 1 \\ 2\alpha - i\beta + i\gamma &= 2 \\ 4\alpha - \beta - \gamma &= 3 \end{aligned}$$

which solves to $\alpha = \frac{4}{5}$, $\beta = \frac{1}{10} + \frac{1}{5}i$, and $\gamma = \bar{\beta}$.

Therefore, $g_n = \frac{1}{10}[8(2)^n + (1+2i)(-i)^n + (1-2i)(i)^n]$. □

Exercise 4.0.7. Define a sequence of numbers $(c_n : n \in \mathbb{N})$ by $c_0 = 1$, $c_1 = 2$, $c_2 = 3$, and the recurrence relation $c_n = -c_{n-1} + 2c_{n-2} + 2c_{n-3}$ for all $n \geq 3$.

(a) Obtain an algebraic formula for the rational function $C(x) = \sum_{n=0}^{\infty} c_n x^n = 1 + 2x + 3x^2 + 3x^3 + 7x^4 + 5x^5 + \cdots$

Solution. Based on the recurrence, the denominator will be $1 + x - 2x^2 - 2x^3$. The numerator will have coefficients:

$$\begin{aligned} c_0 + c_{-1} - 2c_{-2} - 2c_{-3} &= c_0 && = 1 \\ c_1 + c_0 - 2c_{-1} - 2c_{-2} &= c_1 + c_0 && = 3 \\ c_2 + c_1 - 2c_0 - 2c_{-1} &= c_2 + c_1 - 2c_0 && = 3 \end{aligned}$$

That is, it will be $1 + 3x + 3x^2$.

Therefore, $C(x) = \frac{1+3x+3x^2}{1+x-2x^2-2x^3}$. □

(b) Obtain a formula for c_n as a function of $n \in \mathbb{N}$.

Solution. Factor the denominator as $-(x+1)(2x^2-1) = -(x+1)(\sqrt{2}x-1)(\sqrt{2}x+1)$. This has inverse roots -1, $\sqrt{2}$ and $-\sqrt{2}$ (all multiplicity 1). Therefore, by the Main Theorem, $c_n = A(-1)^n + B(\sqrt{2})^n + C(-\sqrt{2})^n$.

The initial conditions give us:

$$\begin{aligned} A + B + C &= 1 \\ -A + \sqrt{2}B - \sqrt{2}C &= 2 \\ A + 2B + 2C &= 3 \end{aligned}$$

which solves to $A = -1$, $B = 1 + \frac{\sqrt{2}}{4}$, and $C = 1 - \frac{\sqrt{2}}{4}$.

Therefore, $c_n = (-1)^{n+1} + (1 + \frac{\sqrt{2}}{4})(\sqrt{2})^n + (1 - \frac{\sqrt{2}}{4})(-\sqrt{2})^n$. □

Exercise 4.0.8.

(a) Obtain a formula for the coefficients of the rational function $B(x) = \sum_{n=0}^{\infty} b_n x^n = \frac{1+3x-x^2}{1-3x^2-2x^3}$.

Solution. Since the denominator factors as $-(2x-1)(x+1)^2$, we can write as a partial fraction $B(x) = -\frac{A}{2x-1} - \frac{B}{(x+1)^2} - \frac{C}{(x+1)}$. Then, multiplying by the denominator, we have $1+3x-x^2 = A(x+1)^2 + B(2x-1) + C(x+1)(2x-1)$.

When $x = -1$, we have $-3 = -3B$, i.e., $B = 1$.

When $x = \frac{1}{2}$, we have $\frac{9}{4} = \frac{9}{4}A$, i.e., $A = 1$.

When $x = 0$, we have $1 = A - B - C$, i.e., $C = -1$.

Therefore, we can write

$$\begin{aligned} B(x) &= \frac{1}{2x-1} + \frac{1}{(x+1)^2} - \frac{1}{x+1} \\ &= \sum_{n=0}^{\infty}(2x)^n + \sum_{n=0}^{\infty}\binom{n+1}{1}(-x)^n - \sum_{n=0}^{\infty}(-x)^n \\ &= \sum_{n=0}^{\infty}(2^n - (n+1)(-1)^n + (-1)^n)x^n \\ &= \sum_{n=0}^{\infty}(2^n - n(-1)^n)x^n \end{aligned}$$

and conclude that $b_n = 2^n - n(-1)^n$. □

(b) Derive a recurrence relation and use it to check your answer.

Solution. Read off the recurrence relation $b_n - 3b_{n-2} - 2b_{n-3} = \begin{cases} 1 & n=0 \\ 3 & n=1 \\ -1 & n=2 \end{cases}$

We get initial conditions $b_0 = 1$, $b_1 = 3$, $b_2 = -1 + 3(1) = 2$.

Generating some coefficients by both methods yields 1, 3, 2, 11, 12, 37, 58, 135. □

Exercise 4.0.9. Define a sequence $(h_n : n \in \mathbb{N})$ by $h_0 = 1$, $h_1 = 2$, $h_2 = 0$, $h_3 = 5$, and the recurrence relation $h_n = -2h_{n-1} + h_{n-2} + 4h_{n-3} + 2h_{n-4}$ for all $n \geq 4$.

(a) Obtain an algebraic formula for the rational function $H(x) = \sum_{n=0}^{\infty} h_n x^n = 1 + 2x + 0x^2 + 5x^3 + 0x^4 + 9x^5 + \cdots$

Solution. Reading the recurrence relation, by the Main Theorem, the denominator will be $1 + 2x - x^2 - 4x^3 - 2x^4$. The coefficients of the numerator will be:

$$\begin{aligned} h_0 + 2h_{-1} - h_{-2} - 4h_{-3} - 2h_{-4} &= 1 \\ h_1 + 2h_0 - h_{-1} - 4h_{-2} - 2h_{-3} &= 4 \\ h_2 + 2h_1 - h_0 - 4h_{-1} - 2h_{-2} &= 3 \\ h_3 + 2h_2 - h_1 - 4h_0 - 2h_{-1} &= -1 \end{aligned}$$

so we have $H(x) = \frac{1+4x+3x^2-x^3}{1+x-x^2-4x^3-2x^4}$. □

(b) Obtain a formula for h_n as a function of $n \in \mathbb{N}$.

Solution. The denominator factors as $(1+x)^2(1-2x^2)$. This has inverse roots -1 (multiplicity 2) and $\pm\sqrt{2}$ (multiplicity 1). Therefore, by the Main Theorem, $h_n = (An+B)(-1)^n + C(\sqrt{2})^n + D(-\sqrt{2})^n$.

Construct the linear system for $n = 0, 1, 2, 3$:

$$\begin{aligned} B + \quad C + \quad D &= 1 \\ -A - B + \sqrt{2}C - \sqrt{2}D &= 2 \\ 2A + B + \quad 2C + \quad 2D &= 0 \\ -3A - B + 2\sqrt{2}C - 2\sqrt{2}D &= 5 \end{aligned}$$

and solve with a computer to get $A = -1$, $B = 0$, $C = \frac{1}{2} + \frac{\sqrt{2}}{4}$, $D = \frac{1}{2} - \frac{\sqrt{2}}{4}$.

Therefore, $h_n = (-1)^{n+1} + (\frac{1}{2} + \frac{\sqrt{2}}{4})(\sqrt{2})^n + (\frac{1}{2} - \frac{\sqrt{2}}{4})(-\sqrt{2})^n$. □

Part II

Graph Theory

4 Inroduction to Graph Theory

4.4 Definitions, Isomorphism, Degree, Bipartite Graphs

Exercise 4.4.1. For the graphs G_1, G_2, G_3, and H in Figure 4.13, prove that no two of G_1, G_2, or G_3 are isomorphic. Prove that one of them (which?) is isomorphic to H by giving a suitable bijection.

Solution. Isomorphism preserves adjacency, so if a k-cycle exists in a graph, it must also appear in an isomorphic graph.

Notice that in G_1, to move from one adjacent "outer" vertex to another, either take the edge connecting them or move down a 4-cycle.

There is also a 4-cycle in G_3.

In G_2, there are no 4-cycles, so $G_1 \not\simeq G_2 \not\simeq G_3$.

Consider G_1 and G_3. There are exactly two 5-cycles in G_1 (outer and inner loops) but at least three in G_2 (outer, inner, triangle at the bottom). We cannot map vertices not in a cycle to a cycle, so $G_1 \not\simeq G_3$.

Finally, consider H. Notice that travelling between adjacent outer edges can be accomplished by a 4-cycle. This indicates that $H \simeq G_1$ and indeed if we map the inner vertices to their vertical reflections, we recover G_1. □

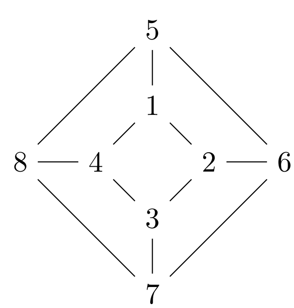
Exercise 4.4.2. A cubic graph is one in which every vertex has degree three. Find all the nonisomorphic cubic graphs with 4, 6 and 8 vertices.

Solution. The only cubic graph with 4 vertices is K_4 (every vertex must connect to the other 3, so there is no room to modify).

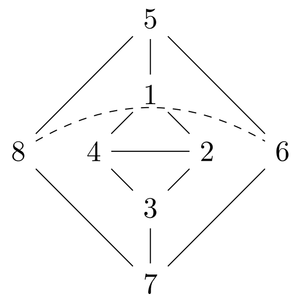
In 6 vertices, notice that $K_{3,3}$ is cubic (each vertex connecting to the 3 opposite). If we construct a triangular prism, this is also cubic.

In 8 vertices:

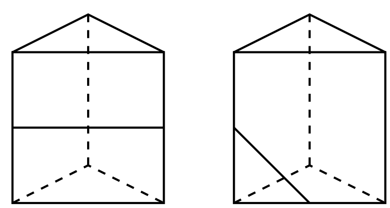
- The 3-cube is cubic (for obvious reasons).



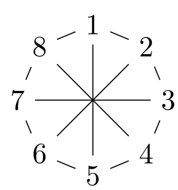
- From the 3-cube, we can remove two "vertical" edges and add diagonals to the top/bottom faces.



- If we remove the other two "verticals", we get two copies of K_4.
- From the triangular prism, we can add vertices into two "vertical" edges or one "horizontal" and one "vertical edge" (adding to "horizontal" edges gives a cube).



- We can take an octagon and pair up vertices:



which, according to Wikipedia, is exhaustive. □

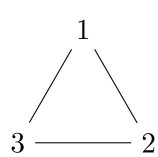
Exercise 4.4.3. For the subset graph $S_{n,k}$ defined in Example 4.1.5, find the number of vertices and the number of edges.

Solution. There are $\binom{n}{k}$ vertices. Each edge takes one of the k elements and replaces it with a different one from the $n-k$ others. That is, each node has $k(n-k)$ edges. By Corollary 4.3.3, this means there are $\frac{1}{2}k(n-k)\binom{n}{k}$ edges. □

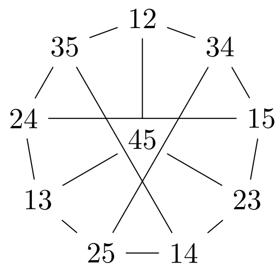
Exercise 4.4.4. The odd graph O_n is the graph whose vertices are the n-subsets of a $(2n+1)$-set, two such subsets being adjacent if and only if they are disjoint.

(a) Draw O_1 and O_2.

Solution. The 1-subsets of [3] are $\{1\}$, $\{2\}$, and $\{3\}$ where they are all disjoint. Then, we can draw O_1 (omitting curly braces) as



This is isomorphic to K_3. The 2-subsets of [5] are 12, 13, 14, 15, 23, 24, 25, 34, 35, and 45. We can draw O_2 as



as desired. □

(b) Prove that O_2 is isomorphic to the Peterson graph.

Solution. Observe that the layout of O_2 is identical to that of H in Figure 4.8. □

(c) How many vertices and edges does O_n have?

Solution. There are $\binom{2n+1}{n}$ vertices.

Each each vertex has $\binom{(2n+1)-2}{2} = \binom{2n-1}{2}$ adjacent vertices, giving by Corollary 4.3.3 a total of $\frac{1}{2}\binom{2n+1}{n}\binom{2n-1}{2}$ edges. □

Exercise 4.4.5. The line-graph $L(G)$ of a graph G is the graph whose vertex set is $E(G)$ and in which two vertices are adjacent if and only if the corresponding edges of G are incident with a common vertex.

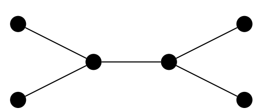
(a) Find a graph G such that $L(G)$ is isomorphic to G.

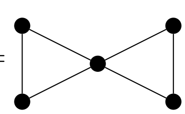
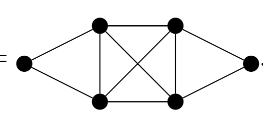
Solution. Let $G = K_3 =$
```
  1
 / \
3 — 2
```
. Then, $L(G) =$
```
  12
 /  \
13 – 23
```
isomorphic to G. □

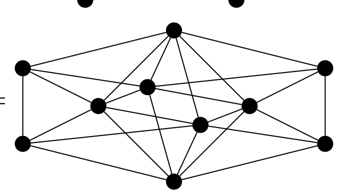
(b) Find nonisomorphic graphs G, G' such that $L(G)$ is isomorphic to $L(G')$.

Solution. Let $G = K_3$ as above and $G' =$
```
  1
 /4\
3 — 2
```
.

Since 4 has no edges in G', it does not affect $L(G')$. Then, $L(G) = L(G')$ (the same graph as in part (a)). □

(c) If G is the graph  find $L(G)$, $L(L(G))$ and $L(L(L(G)))$.

Solution. Calculate $L(G) =$  and $L(L(G)) =$ .

Finally, $L(L(L(G))) =$  □

Exercise 4.4.6. For integer $n \geq 0$, define the graph G_n as follows: $V(G_n)$ is the set of all binary strings of length n having at most one block of 1's. Two vertices are adjacent if they differ in exactly one position.

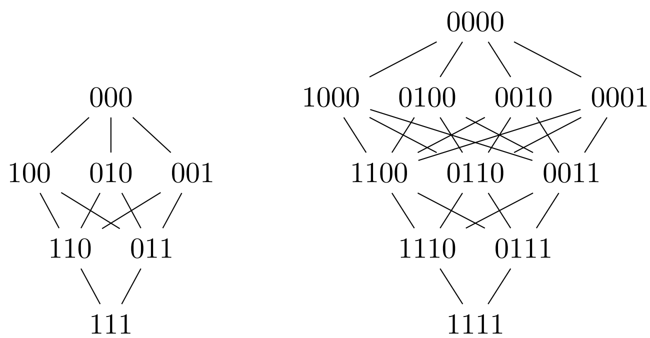
(a) Find $|V(G_n)|$.

Solution. We can map $V(G_n) \rightleftharpoons S_{n+1,2} \cup \{\varnothing\}$ as follows: given $\alpha = 0^i 1^j 0^k \in V(G_n)$, let $f(\alpha) = \{i, j\}$. Then, $f^{-1}(\{\ell, m\}) = 0^\ell 1^m 0^{n-\ell-m}$. Map $\varnothing$ to the string 0^n with no 1's. Notice that for this to work, we define $[n+1] = \{0, \dots, n\}$.

Therefore, $|V(G_n)| = \binom{n+1}{2} + 1 = \frac{1}{2}n(n+1) + 1 = \frac{1}{2}n^2 + \frac{1}{2}n + 1$. □

(b) Make drawings of G_3 and G_4.

Solution. Construct $V(G_3) = \{000, 100, 010, 001, 110, 011, 111\}$. Likewise, construct $V(G_4) = \{0000, 1000, 0100, 0010, 0001, 1100, 0110, 0011, 1110, 0111, 1111\}$. Then, notice that differing by exactly one position can either add a 1 (swap $0 \to 1$) or remove a 1 ($1 \to 0$). Then, we can draw



as desired. □

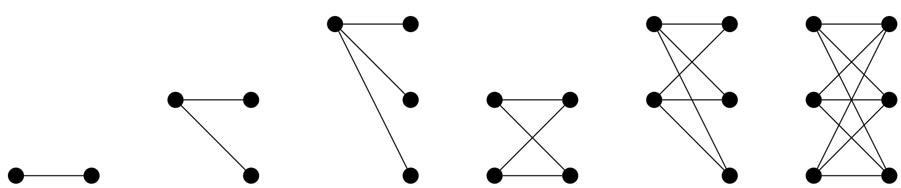
(c) Find $|E(G_n)|$.

Solution. Consider the triangle and the 0^n separately. Between each layer with $i-1$ and i 0's, there are $i(i-1)$ edges. Then, $\sum_{i=2}^{n} i(i-1) = 2\sum_{i=1}^{n}\binom{i}{2} = 2\binom{n+1}{3}$. Finally, there are n edges incident to 0^n, so we have $|E(G_n)| = 2\binom{n+1}{3} + n$. □

Exercise 4.4.7.

(a) Draw $K_{m,n}$ for all m, n such that $1 \le m \le n \le 3$.

Solution. Draw:



as desired. □

(b) How many vertices does $K_{m,n}$ have?

Solution. There are $\left|V(K_{m,n})\right| = m + n$ vertices. □

(c) Let K be a complete bipartite graph on p vertices. Prove that K has at most $\lfloor p^2/4 \rfloor$ edges.

Solution. Suppose $K \simeq K_{m,n}$. Then, there are $|E(K)| = \left|E(K_{m,n})\right| = m \times n$ edges to connect each of the m left vertices to n right vertices.

WLOG, suppose $m \le n$. By high school algebra, $|E(K)|$ is maximized when $m = n$.

If p is even, let $m = n = \frac{p}{2}$. Then, $|E(K)| \leq \frac{p^2}{4}$.

If $p = 2k+1$ is odd, let $m = k = \lfloor \frac{p}{2} \rfloor$ and $n = k+1 = \lfloor \frac{p}{2} \rfloor + 1$. Notice that $\lfloor \frac{p^2}{4} \rfloor = \frac{4k^2+4k+1}{4} = k^2 + k$. Then, $|E(K)| \leq mn = k^2 + k = \lfloor \frac{p^2}{4} \rfloor$,

Therefore, K has at most $\lfloor \frac{p^2}{4} \rfloor$ edges. □

(d) Let G be a bipartite graph on p vertices. Prove that G has at most $\lfloor p^2/4 \rfloor$ edges.

Solution. Since G is bipartite, all edges cross the bipartition (A, B) where $|A| = m$ and $|B| = n$. That is, all edges exist in $K_{m,n}$.

Thus, G is a subgraph of $K_{m,n}$ and by (c), $|E(G)| \leq \left|E(K_{m,n})\right| \leq \lfloor \frac{p^2}{4} \rfloor$, as desired. □

(e) Let G be a k-regular bipartite graph with bipartition (X, Y). Prove that $|X| = |Y|$ if $k > 0$. Is this still valid when $k = 0$?

Proof. Let $k \geq 1$ and suppose $|X| \neq |Y|$.

Then, notice that since G is bipartite, by definition, the edges incident to X are the edges incident to Y, i.e., $|\delta(X)| = |\delta(Y)|$. But each of X (resp. Y) consist of $|X|$ vertices of k degree going to vertices in Y, so $|\delta(X)| = k|X| = k|Y| = |\delta(Y)|$.

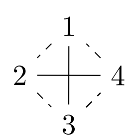
But $|X| \neq |Y|$, implying $k = 0$. Contradiction.

This is not valid when $k = 0$. Notice that if $V(G) = \{1\}$, $E(G) = \varnothing$, $X = \{1\}$, and $Y = \varnothing$, we have a 0-regular bipartite graph with bipartition (X, Y) and $|X| \neq |Y|$. □

Exercise 4.4.8. The complement of a graph G, denoted $\bar{G}$, is the graph with $V(\bar{G}) = V(G)$ and the edge $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$.

(a) Let G have vertices 1, 2, 3, 4 and edges 12, 23, 34, 14. Draw $\bar{G}$.

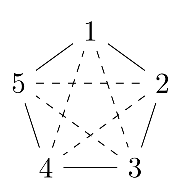
Solution. Note that $E(\bar{G}) = \{13, 24\}$ (with $E(G)$ in dashes):



as desired. □

(b) Find a 5-vertex graph that is isomorphic to its complement.

Solution. Consider the 5-cycle C_5 and its complement (dashed) $\bar{C_5}$:



and notice that the complement is the cycle $\{1, 3, 5, 2, 4\}$. □

(c) Prove that no 6-vertex graph is isomorphic to its complement.

Proof. Notice that for a graph G, we have $E(G) \cup E(\bar{G})$ is a disjoint union of the set of subsets of $V(G)$ with size 2.

Notice that for a 6-vertex graph, there are $\binom{6}{2} = 15$ subsets of $V(G)$ with size 2. These cannot be partitioned into $|E(G)| = \left|E(\bar{G})\right|$ because 15 is odd. ☐

(d) Let G_1 and G_2 be two graphs. Prove that G_1 is isomorphic to G_2 if and only if $\bar{G_1}$ is isomorphic to $\bar{G_2}$.

Proof. Suppose $G_1 \simeq G_2$. Then, there exists an isomorphism $\phi : V(G_1) \to V(G_2)$ such that $uv \in E(G_1) \iff \phi(u)\phi(v) \in E(G_2)$. It follows that

$$uv \in E(\bar{G_1}) \iff uv \notin E(G_1) \iff \phi(u)\phi(v) \notin E(G_2) \iff \phi(u)\phi(v) \in E(\bar{G_2})$$

so ϕ is also an isomorphism between $\bar{G_1}$ and $\bar{G_2}$.

Finally, since $\bar{\bar{G}} = G$ (because $E(G)$ and $E(\bar{G})$ are set complements), this goes in both directions. ☐

(e) Find all 2-regular non-isomorphic graphs on 6 vertices (prove that these are the only ones).

Proof. Notice that a 2-regular component is a cycle. The smallest cycle is the 3-cycle, so we can either have two 3-cycles or one 6-cycle. ☐

(f) Prove that there are only two 3-regular non-isomorphic graphs on 6 vertices.

Proof. Let G be such a graph. For each vertex, there are three incident edges. The other two potential edges lie in the complement. Therefore, $\bar{G}$ is a 2-regular graph on 6 vertices. By (d), the complement preserves isomorphism. By (e), there are only two classes of 2-regular isomorphic graphs on 6 vertices. Therefore, there must be only two 3-regular non-isomorphic graphs on 6 vertices. ☐

Exercise 4.4.9. Make drawings of the 15 non-isomorphic graphs having six vertices and six edges, such that every vertex has degree at least one.

Exercise 4.4.10. Are the graphs in Figure 4.14 isomorphic? Justify your answer.

Exercise 4.4.11. For n a positive integer, define the prime graph B_n to be the graph with vertex set $\{1, 2, \ldots, n\}$, where $\{u, v\}$ is an edge if and only if $u+v$ is a prime number. Prove that B_n is bipartite.

Proof. Notice that for $u+v > 2$ to be prime, it must be odd (since we cannot have a $\{1, 1\}$ loop to make 2). Therefore, exactly one of u and v must be odd. It follows that if we partition $V(B_n) = [n]$ into $X = \{i \in [n] : i \bmod 2 = 0\}$ and $Y = \{i \in [n] : i \bmod 2 = 1\}$, we have a bipartition of B_n. ☐

4.5 How to Specify a Graph

Not covered in Fall 2022 offering.

4.6 Paths and Cycles

Exercise 4.6.1. Let G be a graph with minimum degree k, where $k \geq 2$. Prove that:

(a) G contains a path of length at least k

Proof. Proceed by induction on k. Let $v \in V(G)$ be a vertex.

If $k = 0$, then notice that v is a path of length 0.

If $k = 1$, then v must have a neighbour u. Then, v, u is a path of length 1.

For $k \geq 2$, since $k > k-1$, suppose there exists a path $P = v_0, \dots, v_{k-1}$ of length $k-1$. Since v_{k-1} has degree at least k, $|N_G(v_{k-1})| \geq k$, so $|N_G(v_{k-1}) \setminus \{v_0, \dots, v_{k-2}\}| \geq 1$. Let $v_k \in N_G(v_{k-1}) \setminus \{v_0, \dots, v_{k-2}\}$ and notice that $P + v_k$ is a path of length k. □

(b) G contains a cycle of length at least $k + 1$

Proof. By part (a), a path of length at least k exists in G. Let $P = v_0, \dots, v_k, \dots, v_\ell$ be the longest such path. Then, as above, notice that $|N_G(v_\ell) \setminus \{v_{\ell-k}, \dots, v_{\ell-1}\}| \geq 1$. Select any $x \in N_G(v_\ell) \setminus \{v_{\ell-k+1}, \dots, v_{\ell-1}\}$. If $x \notin P$, then $P' = P + x$ is a longer path. Otherwise, $x = v_i$ for $0 \leq i \leq \ell - k$.

The cycle $v_i, \dots, v_\ell$ has length at least $k + 1$. □

Exercise 4.6.4. Let G be the graph whose set of vertices is the set of all “lower 48” states of the United States, plus Washington, DC, with two vertices being adjacent if they share a boundary. Let H be the subgraph of G whose vertices are those of G whose first letter is one of W, O, M, A, N, and whose edges are the edges of G whose ends have this property. Find a path in H from Washington to Washington, DC.

Solution. Consult a map and get WA, OR, NV, AZ, NM, OK, MO, NB, WY, MT, ND, MN, WI, MI, OH, WV, MD, DC. □

Exercise 4.6.5. Consider the word graph W_n defined in Example 4.1.2.[2]

(a) Find a cycle through *math* in W_4.

Solution. (*math*, *meth*, *mete*, *mate*, *math*) is a cycle in W_4. □

(b) Find a path from *pink* to *blue* in W_4.

Solution. (*pink*, *tink*, *tank*, *tans*, *tens*, *fens*, *feus*, *flus*, *flue*, *blue*) using words in Merriam–Webster. □

Exercise 4.6.6. For $n \geq 2$, prove that the n-cube contains a Hamilton cycle.

Proof. Let G_n denote the n-cube. Proceed by induction on n.

Notice that for $n = 2$, $(00, 01, 11, 10, 00)$ is a Hamilton cycle.

Let $n \geq 3$. Partition $V(n) = V_0 \cup V_1$ based on the first digit in the binary string. Now, the subgraphs induced by V_0 and V_1 are isomorphic to G_{n-1} under the isomorphism $f_b : G_{n-1} \to V_b : \alpha \mapsto b\alpha$ and inverse $f_b^{-1} : V_b \to G_{n-1} : b_1 \cdots b_n \to b_2 \cdots b_n$.

Assume G_{n-1} has a Hamilton cycle. WLOG, suppose that it begins at 0^{n-1} and ends at $0^{n-2}1$, so that V_0 has a Hamilton cycle $(00^{n-2}0, \dots, 00^{n-2}1)$ and V_1 has a Hamilton cycle $(10^{n-2}0, \dots, 10^{n-2}1)$.

Then, $(00^{n-2}0, \dots, 00^{n-2}0, 10^{n-2}0, \dots, 10^{n-2}1, 00^{n-2}1)$ is a Hamilton cycle for G_n. □

Exercise 4.6.7. Prove that the complete bipartite graph $K_{m,n}$ has a Hamilton cycle if and only if $m = n$ and $m > 1$.

Proof. Suppose $m = n$ and $m > 1$. Then, $(s_1, t_1, s_2, t_2, \dots, s_m, t_n)$ is a Hamilton cycle.

Conversely, assume WLOG that $m \leq n$ and that there is a Hamilton cycle.

[2] $V(W_n)$ is the set of English words of length n with adjacency if they differ by exactly one letter.

When $m = 0$, there are no edges and no Hamilton cycle. For $m = 1$, trace the Hamilton cycle starting at s_1. First $N(s_1) = \{t_i\}$, so the next vertex must be t_k for some k. However, $N(t_k) = \{s_1\}$ and we cannot continue because $s_1 t_k$ is already in the cycle.

Otherwise, suppose $m \neq n$ and $1 < m < n$. Every edge is $s_i t_j$, so the cycle goes $s_{a_1}, t_{b_1}, s_{a_2}, t_{b_2}, \dots$. At t_{b_m}, we have visited m vertices s_i, i.e., all of them. Therefore, the cycle must end here and return to s_{a_1}. However, there are vertices $t_{b_{m+1}}, \dots, t_{b_n}$ which are not in the cycle. That is, the cycle is not a Hamilton cycle.

Therefore, there is a Hamilton cycle if and only if $m = n$ and $m > 1$. $\square$

Exercise 4.6.8. Show that if there is a closed walk of odd length in the graph G, then G contains an odd cycle (that is, G has a subgraph which is a cycle on an odd number of vertices).

Proof. Suppose that $W = v_1, \dots, v_n, v_1$ is the shortest closed walk of odd length that exists in G (i.e., n is odd).

Induct on the number of such vertices in W that appear multiple times.

If W is a path, i.e., there are no such vertices, W is a cycle and we are done.

Otherwise, there exists i and j such that $v_i = v_j$. Then, the edges in W can be partitioned into two closed walks $W_1 = v_1, \dots, v_{i-1}, v_i, v_{j+1}, \dots, v_1$ and $W_2 = v_i, v_{i+1}, \dots, v_{j-1}, v_i$. Since W has an odd number of edges and $E(W_1) \cup E(W_2)$ partition $E(W)$, exactly one of W_1 and W_2 has an odd number of edges. By the inductive hypothesis, since that walk necessarily has fewer duplicates, there exists an odd cycle in that walk.

Therefore, there exists an odd cycle in G. $\square$

Exercise 4.6.9. A diagonal of a cycle in a graph is an edge that joins vertices that are not consecutive in the cycle.

(a) Prove that a shortest cycle has no diagonal.

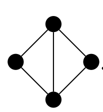
Proof. Let $C = v_1, \dots, v_k, v_1$ be a shortest cycle. Suppose a diagonal $v_i v_j$, $i < j$, $j \neq i + 1$ exists. Then, $v_1, \dots, v_i, v_j, \dots, v_k, v_1$ is a shorter cycle. Contradiction. $\square$

(b) Prove that a shortest odd cycle has no diagonal.

Proof. Let $C = v_1, \dots, v_k, v_1$ be a shortest odd cycle, i.e., k is odd. Suppose a diagonal $v_i v_j$, $i < j$, $j \neq i + 1$ exists. Then, as in exercise 4.6.8, we can create two cycles $C_1 = v_1, \dots, v_i, v_j, \dots, v_k, v_1$ and $C_2 = v_i, \dots, v_j, v_i$.

Since $V(C_1) \cap V(C_2) = \{v_i, v_j\}$, we have $|V(C)| = |V(C_1)| + |V(C_2)| - 2$. That is, $1 \equiv |V(C_1)| + |V(C_2)| \pmod 2$. Then, either C_1 or C_2 must be a shorter odd cycle. Contradiction. $\square$

(c) Give an example of a graph in which a shortest even cycle has a diagonal.

Solution. The minimal example . $\square$

Exercise 4.6.10.

(a) Prove that a k-regular graph of girth 4 has at least $2k$ vertices ($k \geq 2$).

(b) For $k = 2, 3$, find a k-regular graph of girth 4 with precisely $2k$ vertices. Generalize these examples to find one for each $k \geq 2$.

(c) Prove that a k-regular graph of girth 5 has at least k^2+1 vertices ($k \geq 2$).

(d) Prove that a k-regular graph of girth $2t$, where $t \geq 2$, has at least $\frac{2(k-1)^t-2}{k-2}$ vertices.

(e) Prove that a k-regular graph of girth $2t+1$, where $t \geq 2$, has at least $\frac{k(k-1)^t-2}{k-2}$ vertices.

(f) For $k = 2, 3$, give an example of a k-regular graph of girth 5 with exactly k^2+1 vertices.

4.10 Equivalence Relations, Connectedness, Eulerian Circuits, Bridges

Exercise 4.10.1. Prove that the prime graph B_n defined in Exercise 4.4.11 is connected for every n. You may use without proof the following fact: For every integer $k \geq 2$, there is a prime number r such that $k < r < 2k$.

Proof. Proceed by induction on n. Clearly, B_1 is connected with a single vertex.

Let $n \geq 2$. We may assume that $B_{n-1} = B_n - n$ is connected.

There exists a prime number r such that $n < r < 2n$. That is, $r = n + m$ for $1 \leq m \leq n-1$. Since $m \in \{1, \dots, n-1\} \subseteq [n] = V(B_n)$, the edge mn exists in B_n. For any $k \in V(B_n) \setminus \{n\} = V(B_{n-1})$, since B_{n-1} is connected, there exists a k, m-path. Adjoining mn, there exists a k, n-path in B_n.

Therefore, by Theorem 4.8.2, B_n is connected. □

Exercise 4.10.2. Prove that if G is connected, any two longest paths have a vertex in common.

Proof. Let P_1 be a s, t-path and P_2 be a u, v-path in G such that both P_1 and P_2 are longest paths in G with length k.

Then, suppose that $V(P_1) \cap V(P_2) = \varnothing$.

Since G is connected, there exists a t, u-path P_3 in G. Consider the walk $W = P_1P_3P_2$. By the process in the proof in Theorem 4.6.2, there exists a s, v-path P using all the non-overlapping vertices in P_1 and P_2 (but only some of P_3). That is, P has length at least $2k$, meaning that P_1 and P_2 are not longest paths.

Therefore, by contradiction, P_1 and P_2 must share a vertex. □

Exercise 4.10.3. Which graphs, with at least one edge, have the property that every edge is a bridge?

Solution. By Theorem 4.10.3, an edge is a bridge if and only if it is not contained in a cycle. Therefore, every edge is a bridge if and only if the graph contains no cycles. That is, the graph is a forest. □

Exercise 4.10.4. If every vertex of a graph H with p vertices has degree at least $\frac{p}{5}$, prove that H cannot have more than 4 components.

Proof. Let $v \in V(H)$. Since $\deg v \geq \frac{p}{5}$, its component has size at least $\frac{p}{5}+1$.

This applies to all components, so if there are k components, $p \geq k(\frac{p}{5}+1)$, that is, $k \leq \frac{p}{\frac{p}{5}+1} < \frac{p}{p/5} = 5$. Therefore, there are at most 4 components. □

Exercise 4.10.5. If an edge e is not a bridge of a connected graph G, prove that e is an edge of some cycle.

Proof. If $e = uv \in E(G)$ is not a bridge, then $G - e$ remains connected. But that means there is a u, v-path P in $G - e$. Then, $P + e$ is a cycle in G containing e. ☐

Exercise 4.10.6. Prove that a 4-regular graph has no bridge.

Proof. Consider an edge $e = uv$ in a 4-regular component G. Then, by the proof of Exercise 4.6.1, e lies in a 5-cycle. Since e lies in a cycle, it is not a bridge by Theorem 4.10.3. ☐

Exercise 4.10.7. Let A_n be the graph whose vertices are the $\{0, 1\}$-strings of length $n \geq 2$, and edges are between strings that differ in exactly two positions.

(a) How many edges does A_n have?

Solution. Since each of the 2^n strings have $\binom{n}{2}$ neighbours (must pick two positions to change), we have $|E(A_n)| = \binom{n}{2}2^{n-1}$ by the Handshaking Lemma. ☐

(b) Is A_n bipartite for any $n \geq 2$?

Solution. When $n = 2$, $V(A_2) = \{00, 01, 10, 11\}$ and $E(A_2) = \{\{00, 11\}, \{01, 10\}\}$ which is bipartite with bipartition $(X, Y) = (\{00, 11\}, \{01, 10\})$.

For $n \geq 3$, there exists an odd cycle $(0000^{n-3}, 0110^{n-3}, 1100^{n-3}, 0000^{n-3})$. Therefore, by Theorem 5.3.2, A_n is not bipartite. ☐

(c) How many components does A_n have?

Solution. Claim: there are two components.

Divide the elements of $V(A_n)$ by parity. Notice that changing exactly two positions either sends two 0's to two 1's, two 1's to two 0's, or one of each (cancelling). These do not change the parity. Therefore, no path exists between any string of even and odd parity.

Now, consider two strings with the same parity. Then, they differ by an even number of positions. By changing two at a time, there exists a path between them.

Therefore, there are exactly two components of A_n. ☐

Exercise 4.10.8. Let B_n be the graph whose vertices are the $\{0, 1\}$-strings of length $n \geq 2$, and edges are between strings that differ in exactly two consecutive positions.

(a) How many edges does B_n have?

Solution. The neighbours of a string can differ in the positions starting at $1, \ldots, n-1$. That is, the degree of every vertex is $n - 1$. By the Handshaking Lemma, since $|V(B_n)| = 2^n$, $|E(B_n)| = (n - 1)2^{n-1}$. ☐

(b) How many components does B_n have?

Solution. There are at least two components by Exercise 4.10.7.

Consider two strings with even parity. Follow a path sending each instance of 01 to 10. This will give a string of the form $1^{2k}0^{n-2k}$. Then, there is clearly a path to 0^n. Therefore, all the even parity strings form a single component by Theorem 4.8.2.

Likewise, for odd parity strings, there is always a path to 10^{n-1} and they form a single component by Theorem 4.8.2.

Therefore, there are exactly two components of B_n. ☐

Exercise 4.10.9. Let G be a graph in which exactly two of the vertices u and v have odd degree. Prove that G contains a path from u to v.

Proof. Suppose there is not a u, v-path. That is, u and v lie in different components. Then, every $w \in [u]$, $w \neq u$ has even degree. By the Handshaking Lemma,

$$0 \equiv 2|E(G)| \equiv \sum_{w \in [u]} \deg(w) \equiv \deg(v) + \sum_{\substack{w \in [u] \\ w \neq v}} \deg(w) \equiv 1 + \sum_{\substack{w \in [u] \\ w \neq v}} 0 \equiv 1 \pmod 2$$

which is a contradiction. Therefore, $v \in [u]$ and there must exist a u, v-path. $\square$

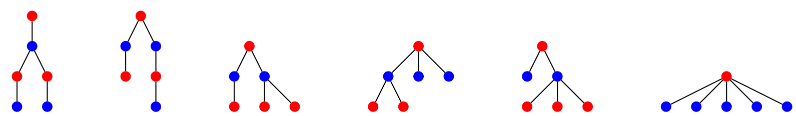
5 Trees

5.1 Trees

Exercise 5.1.1. (a) Draw one tree from each isomorphism class of trees on six or fewer vertices.

(b) For each tree in (a), exhibit a bipartition (X, Y) by coloring the vertices in X with one colour and the vertices in Y with another.

Solution. Draw the trees:



where we colour the vertices an even distance away from the root in red and those an odd distance away in blue. □

Exercise 5.1.2. Prove that every tree is bipartite.

Proof. Let T be a non-empty tree and $r \in V(T)$.

Then, by Lemma 5.1.3, for all vertices $u \in V(T)$, there exists a unique path P_u from u to r. Define $P_r = \varnothing$. Claim that $X = \{u \in V(T) : \text{length of } P_u \text{ even}\}$ and $Y = \{u \in V(T) : \text{length of } P_u \text{ odd}\}$ are a bipartition of T.

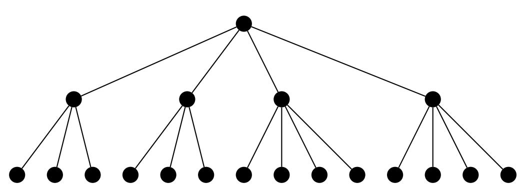
Let $uv \in E(T)$. Then, either $P_v = P_u + uv$ or $P_u = P_v + uv$ because if uv is in both paths, then there exists a cycle based on the walk (r, P_u, u, v, P_v, r). WLOG, suppose $P_v = P_u + uv$. Then, the length of P_v is exactly one more than the length of P_u. That is, the parity is different. Therefore, u and v are partitioned by (X, Y).

It follows that T is bipartite. □

Exercise 5.1.3. What is the smallest number of vertices of degree 1 in a tree with 3 vertices of degree 4 and 2 vertices of degree 5? Justify your answer by proving that this is the smallest possible number and by giving a tree with this many vertices of degree 1.

Solution. Let n_r be the number of vertices with degree r in such a tree. By the alternate proof of Theorem 5.1.8, $n_1 = 2 + \sum_{r\geq 3}(n-2)n_r = 2 + n_3 + 2n_4 + 3n_5 + \sum_{r\geq 6}(n-2)n_r$. If $n_4 = 3$ and $n_5 = 2$, then $n_1 \geq 2 + 6 + 6 = 14$.

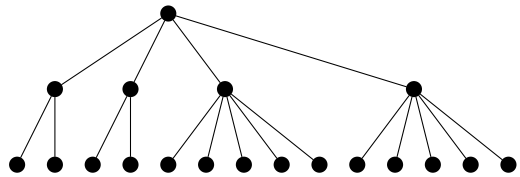
Construct the tree:



which has 14 leaves. □

Exercise 5.1.4. Find the smallest number r of vertices in a tree having two vertices of degree 3, one vertex of degree 4, and two vertices of degree 6. Justify your answer by proving that any such tree has at least r vertices, and by giving an example of such a tree with exactly r vertices.

Solution. As in exercise 5.1.3, write $n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \sum_{r\geq 7}(n-2)n_r$ and notice that if $n_3 = 2$, $n_4 = 1$, and $n_6 = 2$, then $n_1 \geq 2 + 2 + 2 + 8 = 14$. Draw the tree:



which has 14 leaves. □

Exercise 5.1.5. A cubic tree is a tree whose vertices have degree either 3 or 1. Prove that a cubic tree with exactly k vertices of degree 1 has $2(k-1)$ vertices.

Proof. As in exercise 5.1.3, $k =: n_1 = 2 + n_3$ so $n_3 = k - 2$. Then, there are $n_1 + n_3 = k + k - 2 = 2(k-1)$ vertices in total. □

Exercise 5.1.6. Prove that a forest with p vertices and q edges has $p - q$ components.

Proof. Let G be such a forest and $T_1, \ldots, T_k$ be its k components, which are all trees because they are connected and have no cycles. By Theorem 5.1.5, $|E(T_i)| = |V(T_i)| - 1$, so

$$q = E(G) = \sum_{i=1}^{k} |E(T_i)| = \sum_{i=1}^{k}(|V(T_i) - 1|) = \left(\sum_{i=1}^{k} |V(T_i)|\right) - k = p - k$$

so $k = p - q$, as desired. □

Exercise 5.1.7. Let $p \geq 2$. Show that a sequence of $(d_1, \ldots, d_p)$ of positive integers is the degree sequence of a tree on p vertices if and only if $\sum_{i=1}^{p} d_i = 2p - 2$.

Proof. Suppose that T is a tree with $V(T) = \{v_1, \ldots, v_p\}$. Let $d_i = \deg(v_i)$. Then, apply Theorem 5.1.5 ($|E(T)| = p - 1$) and the Handshaking Lemma ($\sum d_i = 2|E(T)| = 2p - 2$), as desired.

Suppose that $\delta = (d_1, \ldots, d_p)$ is a sequence with sum $2p - 2$. We induct on p.

If $p = 2$, then $2p - 2 = 2$. There is only one composition of 2, $\delta = (1, 1)$. Draw the tree  with two vertices of degree 1.

Let $p \geq 3$. Since all the p parts of the sequence are positive, i.e., $d_i \geq 1$, we have $\sum d_i \geq p$. Then, by the pigeonhole principle, since there remains $p - 2$ weight to distribute into p parts of the sequence, there exists $d_i = 1$ and $d_j > 1$ for distinct i, j. Let δ' be δ with $d_j \mapsto d_j - 1$ and skipping d_i. This is a sequence of $p - 1$ elements that sums to $(2p - 2) - 2 = 2(p - 1) - 2$.

Inductively, a tree T' of $p - 1$ vertices exists with degree sequence δ'. Then, add a leaf to $v_j \in V(T)$, i.e., a vertex of degree 1, and increase the degree of v_j by 1. This is a tree of p vertices with degree sequence δ.

Therefore, by induction, δ is the degree sequence of a tree of p vertices. □

5.3 Spanning Trees, Characterizing Bipartite Graphs

Exercise 5.3.1. Let r be a fixed vertex of a tree T. For each vertex v of T, let $d(v)$ be the length of the path from v to r in T. Prove that

(a) for each edge uv of T, $d(u) \neq d(v)$, and

Proof. Let P_u and P_v be the paths from u and v to r, respectively.

Suppose $uv \notin P_u$ and $uv \notin P_v$. Then, (u, P_u, r, P_v, v) is a (u, v)-walk in $T - uv$, meaning a (u, v)-path exists in $T - uv$. But then T has two (u, v)-paths because (u, uv, v) is a path, contradicting Lemma 5.1.3. Therefore, $uv \in P_u$ or $uv \in P_v$.

WLOG, suppose $uv \in P_u$. That is, $P_u = (u, uv, v, \dots, r) = u + P_v$ because P_v is the unique path from v to r in T. Therefore, $d(u) = |E(P_u)| \neq |E(P_v)| + 1 = d(v)$. ☐

(b) for each vertex x of T other than r, there exists a unique vertex y such that y is adjacent to x and $d(y) < d(x)$.

Proof. Let $P_x = (x, y, \dots, r)$ be the unique (x, r)-path (note that y could be r). Clearly, $\delta(y) < \delta(x)$ since the unique (y, r)-path is contained inside P_x.

Let $z \in N_T(x)$, $z \neq y$. Suppose P_z does not contain P_x. Then, $(z, P_z, r, P_x, x, xz, z)$ is a cycle in T. By contradiction, P_z contains P_x and $d(z) > d(x)$.

Therefore, y is unique. ☐

5.4 Breadth-First Search

Not covered in Fall 2022 offering.

5.5 Applications of Breadth-First Search

Not covered in Fall 2022 offering.

6 Codes

Not covered in Fall 2022 offering.

7 Planar Graphs

7.3 Planarity, Euler's Formula, Stereographic Projection

Exercise 7.3.1. Prove that every planar embedding has either a vertex of degree at most 3 or a face of degree 3.

Proof. Let G be a component of a planar graph.

If G has no faces (besides the outer face), then G is a tree with at least two leaves (by Theorem 5.1.8) and those leaves have degree $1 \leq 3$.

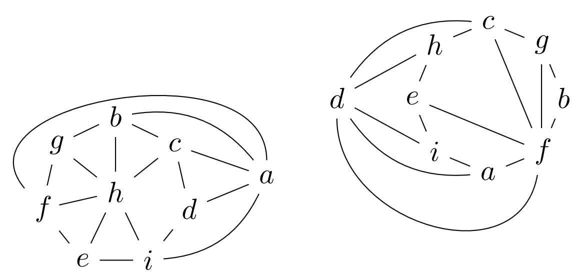
Otherwise, let $V = |V(G)|$, $E = |E(G)|$, and $F = |F(G)|$. Then, by Euler's Formula, $V - E + F = 2$.

Suppose that all vertices have degrees at least 4 and all faces have degrees at least 4. Then, by the Handshaking Lemma, $2E = \sum \deg(v) \geq 4V$ so $E \geq 2V$. Likewise, by the Faceshaking Lemma, $2E = \sum \deg(f) \geq 4F$ so $E \geq 2F$. That is, $2E \geq 2(V + F)$, i.e., $E \geq V + F$. But then $2 = (V + F) - E < 0$, a contradiction.

Therefore, there either exists a vertex with degree at most 3 or a face of degree 3.[3] $\square$

Exercise 7.3.2. Prove that each of the graphs shown in Figure 7.6 is planar, by exhibiting a planar embedding.

Solution. Draw planar embeddings:



$\square$

Exercise 7.3.3. Let $n \geq 3$ be an integer. Suppose that a convex n-gon is drawn in the plane and then each pair of nonadjacent corner points is joined by a straight line through the interior. Suppose that no three of these lines through the interior meet at a common point in the interior.

Let f_n be the number of regions into which the interior of the n-gon is divided by this process. Use Euler's Formula to find f_n.

Solution. Let v_n, e_n, and f_n be the number of vertices, edges, and faces as described. Picking four corners of the n-gon uniquely determines a pair of intersecting lines (i.e., an added vertex), so we have in total $v_n = n + \binom{n}{4}$ vertices.

Each vertex from the n-gon has degree $n - 1$ and each new vertex has degree 4 (as the intersection of two lines), so by the Handshaking Lemma, $2e = n(n-1) + 4\binom{n}{4}$ or equivalently $e = \binom{n}{2} + 2\binom{n}{4}$.

Therefore, by Euler's Formula, discounting the outer face because we are counting only the interior of the n-gon, $f_n = 1 - v_n + e_n = 1 - n + \binom{n}{2} + \binom{n}{4}$.[4] $\square$

[3]Note that inner faces have degree at least 3 and smaller degree implies a tree.
[4]This is a 3Blue1Brown video.

7.4 Platonic Solids

Not covered in Fall 2022 offering.

7.6 Nonplanar Graphs, Kuratowski's Theorem

Exercise 7.6.1. For each of the graphs in Figure 7.13, determine if it is planar. Prove your conclusion in each case.

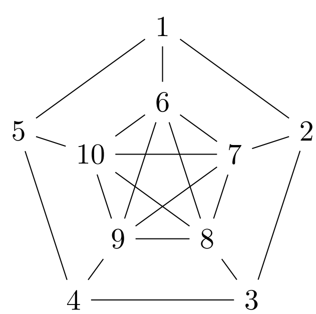
Exercise 7.6.2. Let G be a connected planar graph with p vertices and q edges and girth k. Show that $q \leq \frac{k(p-2)}{k-2}$ and if equality holds, all faces of G have degree k.

Proof (sooshi). Assume that $k < \infty$. By Lemma 7.5.1, the boundary walk of every face in G contains a cycle. Therefore, the degree of each face is at least k. It follows that by Lemma 7.5.2, $q \leq \frac{k}{k-2}(p-2)$.

Suppose $q = \frac{k(p-2)}{k-2}$. Then $kq - 2q = kp - k2$ and we have $2 = p - q + \frac{2}{k}q$. But by Euler's Formula, since G is connected, $2 = p - q + |F(G)|$, so we must have $2q = k|F(G)|$. By the Faceshaking Lemma, $2q = \sum \deg(f) \geq k|F(G)|$ with equality if and only if $\deg(f) = k$ for all $f \in F(G)$. Therefore, if equality holds on the original inequality, every face has degree k. □

Exercise 7.6.3. Prove that the Peterson graph is nonplanar, without using any form of Kuratowski's Theorem.

Proof. Recall the Peterson graph P:

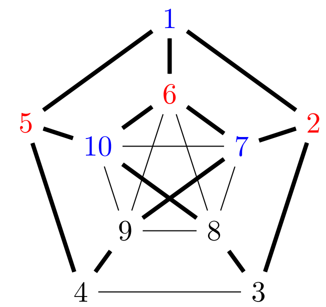


Suppose it is planar. Notice that $|E(P)| = 25$ and $|V(P)| = 10$. By Theorem 7.5.3, since $|V(P)| \geq 3$, we have $25 = |E(P)| \leq 3|V(P)| - 6 = 24$, a contradiction. Therefore, P is nonplanar. □

Exercise 7.6.4.

(a) Prove that the Peterson graph is nonplanar by Kuratowski's Theorem, finding a subgraph that is an edge subdivision of $K_{3,3}$.

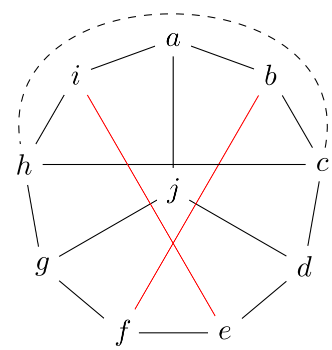
Proof. Highlight the edge subdivision of $K_{3,3} = (A, B)$.



Therefore, since such an edge subdivision exists, by Kuratowski's Theorem, the Peterson graph is nonplanar. □

(b) Show that there exist two edges of the Peterson graph whose deletion leaves a planar graph.

Proof. Recall from Figure 4.8 that this is an isomorphic representation of the Peterson graph:



If we delete the edges in red, then we have a planar graph (bend the edge ch above the rest of the graph). □

Exercise 7.6.5. Prove that the n-cube is not planar when $n \geq 4$, without using any form of Kuratowski's Theorem.

Proof. Recall from Problem 4.4.2 that there are 2^n vertices and $n2^{n-1}$ edges in the n-cube for $n \geq 0$. Also recall from Problem 4.4.3 that the n-cube is bipartite.

Suppose that the n-cube is planar for some $n \geq 4$. Then, by Theorem 7.5.6, since $2^n \geq 3$, we have $n2^{n-1} \leq 2(2^n) - 4 = 4(2^{n-1}) - 4$ which implies $4 \leq 2^{n-1}(4-n)$. But $4 - n \leq 0$ and $2^{n-1} > 0$, so we have $4 \leq 0$, a contradiction.

Therefore, the n-cube is nonplanar for $n \geq 4$. □

Exercise 7.6.6.

(a) Prove that the 4-cube is nonplanar by Kuratowski's Theorem, finding a subgraph that is an edge subdivision of $K_{3,3}$.

Proof. Consider the sets of vertices $(\{0000, 0101, 0110\}, \{0001, 0010, 0100\})$. Namely:

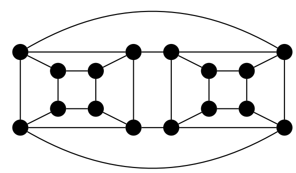
- 0000 is adjacent to 0001, 0010, and 0100
- 0101 is adjacent to 0001 and 0100. Take $0101 \to 0111 \to 0011 \to 0010$.

- 0110 is adjacent to 0010 and 0100. Take 0110 → 1110 → 1010 → 1011 → 1001 → 0001.

since no vertices appear twice in any of these paths, this is an edge subdivision of $K_{3,3}$. Therefore, the 4-cube is nonplanar by Kuratowski's Theorem. □

(b) Show that there exist four edges of the 4-cube whose deletion leaves a planar graph.

Proof. Consider the standard planar embedding of the 3-cube. Connect two to create a 4-cube. Delete the four edges between the two "inner" 2-cubes. Then, we can draw the planar embedding:



as desired. □

(c) Show that no matter which 3 edges are deleted from a 4-cube, the resulting graph is not planar.

Proof. Such a subgraph would have 16 vertices and 29 edges. As a subgraph of a bipartite graph, it is bipartite. Suppose that it is planar.

Then, by Theorem 7.5.6, since $16 > 3$, we have $29 \leq 2(16) - 4 = 28$, a contradiction. Therefore, this graph is nonplanar. □

7.8 Colouring and Planar Graphs, Dual Planar Maps

8 Matchings

8.2 Matchings, Covers

Exercise 8.2.1. Show that a tree has at most one perfect matching.

Proof. Consider a tree T with two perfect matchings M_1 and M_2 and the subgraph D induced by the symmetric difference $E(D) = M_1 \triangle M_2$.

Let $v \in V(T)$. Since both matchings are perfect, there exists $xv \in M_1$ and $yv \in M_2$. If $xv = yv$, then $xv \notin M_1 \triangle M_2$ and $\deg_D(v) = 0$. Otherwise, $xv, yv \in M_1 \triangle M_2$ and $\deg_D(v) = 2$.

Therefore, D consists only of isolated vertices with degree 0 and cycles with vertices of degree 2. Since T contains no cycles, D contains only isolated vertices. That is, $xv = yv$ for all v. Therefore, $M_1 = M_2$. $\square$

Exercise 8.2.2. How many perfect matchings are there in K_n? How many in $K_{m,n}$?

Solution. By Corollary 8.6.1, there are 0 perfect matchings in $K_{m,n}$ with $m \neq n$.

For $K_{n,n} = (A, B)$, we can consider "mapping" each of the n vertices in A to exactly one of the n vertices in B. These are the permutations of $[n]$, of which there are $n!$.

For K_n, notice that there cannot be a perfect matching if $2 \nmid n$.

Othewrise, if $n = 2k$ is even, pick one of the $2k$ vertices v. Then, for each of the other $2k-1$ vertices u, recursively count the perfect matchings in the graph $K_{2k} - v - u$ which is always isomorphic to $K_{2(k-1)}$.

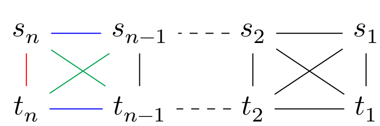
That is, we have $m_k = (2k-1)m_{k-1}$ perfect matchings, which we can unwrap to get $m_k = (2k-1)(2k-3)\cdots 3 \cdot 1 = (2k-1)!!$. $\square$

Exercise 8.2.3. How many perfect matchings does the graph L_n of Figure 8.3 have?

Solution. Let a_n denote the number of perfect matchings in L_n.

Clearly, $a_1 = 1$. Also, $a_2 = 3$ by Exercise 8.2.2 since $L_2 \cong K_4$.

Consider the end of the graph L_n:



To include s_n and t_n in the matching, we can pick one of $\{s_n t_n\}$, $\{s_n s_{n-1}, t_n t_{n-1}\}$, or $\{s_n t_{n-1}, t_n s_{n-1}\}$.

For the former, we pick a perfect matching in the remaining graph $L_n - \{s_n, t_n\} = L_{n-1}$. Otherwise, we find one in the graph $L_n - \{s_n, t_n, s_{n-1}, t_{n-1}\} = L_{n-2}$.

That is, we have $a_n = a_{n-1} + 2a_{n-2}$.

We apply the Main Theorem. Factor $2x^2 + x - 1 = (2x-1)(x+1)$ to get inverse roots 2 and -1. Then, $a_n = \lambda_1 2^n + \lambda_2 (-1)^n$. With the initial conditions $a_1 = 1 = 2\lambda_1 - \lambda_2$ and $a_2 = 3 = 4\lambda_1 + \lambda_2$ which solve to get $\lambda_1 = \frac{2}{3}$ and $\lambda_2 = \frac{1}{3}$.

Therefore, $a_n = \frac{2}{3}(2)^n + \frac{1}{3}(-1)^n$. $\square$

Exercise 8.2.4. Show that for $n \geq 1$, the n-cube has a perfect matching.

Solution. We can construct a matching $M = \{\{\alpha 0, \alpha 1\} : \alpha \in \{0,1\}^{n-1}\}$.

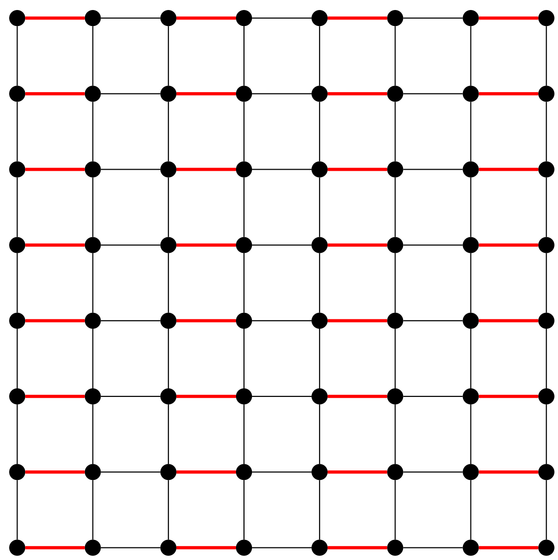
Notice that $\alpha 0$ and $\alpha 1$ differ by one digit, so they are adjacent in the n-cube.

Also, $\bigcup_{\alpha \in \{0,1\}^{n-1}} \{\alpha 0, \alpha 1\} = \{0,1\}^n$ so we saturate all vertices.

Therefore, M is a perfect matching for the n-cube. □

Exercise 8.2.5. Show that the 64 squares of a chessboard can be covered with 32 dominoes, each of which covers two adjacent squares.

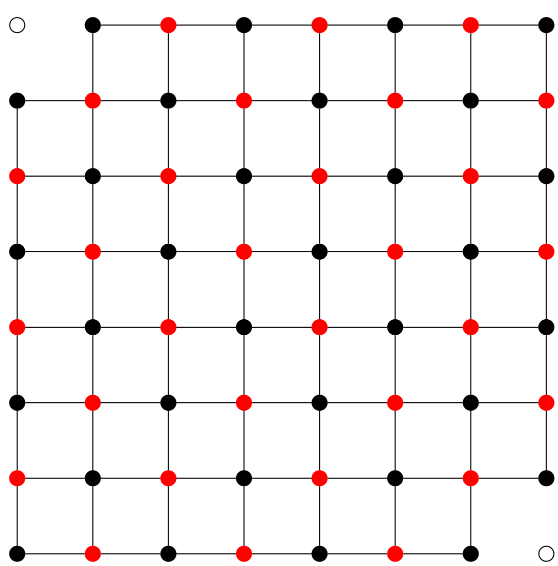
Proof. Notice that this is equivalent to finding a perfect matching for the 8×8 grid. We can simply match to adjacent squares:



□

Exercise 8.2.6. Show that if two opposite corner squares of a chessboard are removed, then the resulting board cannot be covered with 31 dominoes.

Solution (sooshi). Consider the cover created by every second diagonal.



This cover has size $2+4+6+6+6+4+2=30$. Therefore, by Lemma 8.2.1, any matching has size at most 30 and no matching has size 31. $\square$

8.3 König's Theorem

8.6 Applications of König's Theorem, Systems of Distinct Representatives, Perfect Matchings in Bipartite Graphs

8.8 Edge-Colouring, Application to Timetabling

Not covered in Fall 2022 offering.