Entrega 6

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Problema:
Probar que el elipsoide es un ovaloide.
Solución:
El elipsoide, $\mathcal{E}$, es compacto y conexo por ser homeomorfo a $\mathbb{S}^2$.

$$\begin{array}{rrcl} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x,y,z) & \longmapsto & (ax,by,cz) \\ & \mathbb{S}^2 & \longmapsto & \mathcal{E} \end{array}$$

f es continua por ser lineal. Calculemos la curvatura de Gauss. Tomamos la siguiente parametrización,

$$\mathbb{X}(u,v) = (a \operatorname{sen} v \cos u, b \operatorname{sen} v \operatorname{sen} u, c \cos v) \quad v \in (0,\pi) \quad u \in (0,2\pi)$$

$$\begin{array}{cc} \mathbb{X}_u = (-a \operatorname{sen} v \operatorname{sen} u, b \operatorname{sen} v \cos u, 0) & \mathbb{X}_v = (a \cos v \cos u, b \cos v \operatorname{sen} u, -c \operatorname{sen} v) \\ \mathbb{X}_{uu} = (-a \operatorname{sen} v \cos u, -b \operatorname{sen} v \operatorname{sen} u, 0) & \mathbb{X}_{vu} = (-a \cos v \operatorname{sen} u, b \cos v \cos u, 0) \\ \mathbb{X}_{uv} = (-a \cos v \operatorname{sen} u, b \cos v \cos u, 0) & \mathbb{X}_{vv} = (-a \operatorname{sen} v \cos u, -b \operatorname{sen} v \operatorname{sen} u, -c \cos v) \end{array}$$

$$E = \mathbb{X}_u \mathbb{X}_u = a^2 \operatorname{sen}^2 v \operatorname{sen}^2 u + b^2 \operatorname{sen}^2 v \cos^2 u = \operatorname{sen}^2 v (a^2 \operatorname{sen}^2 u + b^2 \cos^2 u) = \operatorname{sen}^2 v ((a^2 - b^2) \operatorname{sen}^2 u + b^2)$$

$$F = \mathbb{X}_u \mathbb{X}_v = -a^2 \operatorname{sen} v \operatorname{sen} u \cos v \cos u + b^2 \operatorname{sen} v \cos u \cos v \operatorname{sen} u = (b^2 - a^2) \operatorname{sen} v \operatorname{sen} u \cos v \cos u$$

$$G = \mathbb{X}_v \mathbb{X}_v = a^2 \cos^2 v \cos^2 u + b^2 \cos^2 v \operatorname{sen}^2 u + c^2 \operatorname{sen}^2 v = \cos^2 v ((a^2 - b^2) \cos^2 u + b^2 - c^2) + c^2$$

$$\mathbb{X}_u \wedge \mathbb{X}_v = \begin{vmatrix} i & j & k \\ -a \operatorname{sen} v \operatorname{sen} u & b \operatorname{sen} v \cos u & 0 \\ a \cos v \cos u & b \cos v \operatorname{sen} u & -c \operatorname{sen} v \end{vmatrix} = \begin{pmatrix} -bc \operatorname{sen}^2 v \cos u \\ -ac \operatorname{sen}^2 v \operatorname{sen} u \\ -ab \operatorname{sen} v \operatorname{sen}^2 u \cos v - ab \operatorname{sen} v \cos^2 u \cos v \end{pmatrix}$$

$$\mathbb{X}_u \wedge \mathbb{X}_v = \begin{pmatrix} -bc \operatorname{sen}^2 v \cos u \\ -ac \operatorname{sen}^2 v \operatorname{sen} u \\ -ab \operatorname{sen} v \cos v \end{pmatrix} \quad \|\mathbb{X}_u \wedge \mathbb{X}_v\| = \frac{1}{n} \geq 0$$

$$N = (-nbc \operatorname{sen}^2 v \cos u, -nac \operatorname{sen}^2 v \operatorname{sen} u, -nab \operatorname{sen} v \cos v)$$

$$e = N\mathbb{X}_{uu} = nabc \operatorname{sen}^3 v \cos^2 u + nabc \operatorname{sen}^3 v \operatorname{sen}^2 u = nabc \operatorname{sen}^3 v$$

$$f = N\mathbb{X}_{uv} = nabc \operatorname{sen}^2 v \cos u \cos v \operatorname{sen} u - nabc \operatorname{sen}^2 v \operatorname{sen} u \cos u \cos v = 0$$

$$g = N\mathbb{X}_{vv} = nabc \operatorname{sen}^3 v \cos^2 u + nabc \operatorname{sen}^3 v \operatorname{sen}^2 u + nabc \operatorname{sen} v \cos^2 v = nabc(\operatorname{sen}^3 v + \operatorname{sen} v \cos^2 v) = nabc \operatorname{sen} v$$

$$K = \frac{eg - f^2}{EG - F^2} = \frac{nabc \operatorname{sen}^3 v nabc \operatorname{sen} v}{EG - F^2} = \frac{(nabc)^2 \operatorname{sen}^4 v}{EG - F^2}$$

Veamos que $EG - F^2$ es positivo.

$$EG = \left(\operatorname{sen}^2 v((a^2 - b^2) \operatorname{sen}^2 u + b^2)\right) \left(\cos^2 v((a^2 - b^2) \cos^2 u + b^2 - c^2) + c^2\right) =$$

$$= \operatorname{sen}^2 v \left(((a^2 - b^2) \operatorname{sen}^2 u + b^2)(\cos^2 v((a^2 - b^2) \cos^2 u + b^2 - c^2) + c^2)\right) =$$

$$\begin{array}{ll} = \operatorname{sen}^2 v(& = \operatorname{sen}^2 v(\\ (a^2 - b^2) \operatorname{sen}^2 u \cos^2 v((a^2 - b^2) \cos^2 u + b^2 - c^2) + & (a^2 - b^2) \operatorname{sen}^2 u \cos^2 v((a^2 - b^2) \cos^2 u + b^2 - c^2) + \\ (a^2 - b^2) \operatorname{sen}^2 u c^2 + & c^2 (a^2 - b^2) \operatorname{sen}^2 u + \\ b^2 \cos^2 v((a^2 - b^2) \cos^2 u + b^2 - c^2) + & (a^2 - b^2) b^2 \cos^2 v \cos^2 u + b^4 \cos^2 v - b^2 c^2 \cos^2 v + \\ b^2 c^2) = & b^2 c^2) = \end{array}$$

$$\begin{array}{l} = \operatorname{sen}^2 v(\\ (a^2 - b^2) \operatorname{sen}^2 u \cos^2 v((a^2 - b^2) \cos^2 u + b^2 - c^2) + \\ c^2 (a^2 - b^2) \operatorname{sen}^2 u + \\ (a^2 - b^2) b^2 \cos^2 v \cos^2 u + b^4 \cos^2 v - b^2 c^2 \cos^2 v + \\ b^2 c^2) \end{array}$$

$$F^2 = (b^2 - a^2)^2 \operatorname{sen}^2 v \operatorname{sen}^2 u \cos^2 v \cos^2 u$$

$$
\begin{aligned}
&EG-F^2=\operatorname{sen}^2 v(\\
&+(a^2-b^2)\operatorname{sen}^2 u\cos^2 v((a^2-b^2)\cos^2 u+b^2-c^2)\\
&+c^2(a^2-b^2)\operatorname{sen}^2 u\\
&+(a^2-b^2)b^2\cos^2 v\cos^2 u+b^4\cos^2 v-b^2c^2\cos^2 v\\
&+b^2c^2\\
&-(b^2-a^2)^2\operatorname{sen}^2 u\cos^2 v\cos^2 u)=
\end{aligned}
$$

Es suficiente probar que $\frac{EG-F^2}{\operatorname{sen}^2 v}$ es positivo.

$$
\begin{aligned}
&(a^2-b^2)\operatorname{sen}^2 u\cos^2 v((a^2-b^2)\cos^2 u+b^2-c^2)\\
&+c^2(a^2-b^2)\operatorname{sen}^2 u\\
&+(a^2-b^2)b^2\cos^2 v\cos^2 u+b^4\cos^2 v-b^2c^2\cos^2 v\\
&+b^2c^2\\
&-(b^2-a^2)^2\operatorname{sen}^2 u\cos^2 v\cos^2 u=
\end{aligned}
\quad
\begin{aligned}
&=(a^2-b^2)\operatorname{sen}^2 u\cos^2 v(a^2-b^2)\cos^2 u\\
&+(a^2-b^2)\operatorname{sen}^2 u\cos^2 v(b^2-c^2)\\
&+c^2(a^2-b^2)\operatorname{sen}^2 u\\
&+(a^2-b^2)b^2\cos^2 v\cos^2 u+b^4\cos^2 v-b^2c^2\cos^2 v\\
&+b^2c^2\\
&-(b^2-a^2)^2\operatorname{sen}^2 u\cos^2 v\cos^2 u=
\end{aligned}
$$

$$
\begin{aligned}
&=(a^2-b^2)^2\operatorname{sen}^2 u\cos^2 v\cos^2 u\\
&+(a^2-b^2)(b^2-c^2)\operatorname{sen}^2 u\cos^2 v\\
&+c^2(a^2-b^2)\operatorname{sen}^2 u\\
&+(a^2-b^2)b^2\cos^2 v\cos^2 u\\
&+b^4\cos^2 v\\
&+b^2c^2\\
&-(b^2-a^2)^2\operatorname{sen}^2 u\cos^2 v\cos^2 u\\
&-b^2c^2\cos^2 v=
\end{aligned}
\quad
\begin{aligned}
&=(a^2-b^2)(b^2-c^2)\operatorname{sen}^2 u\cos^2 v\\
&+c^2(a^2-b^2)\operatorname{sen}^2 u\\
&+(a^2-b^2)b^2\cos^2 v\cos^2 u\\
&+b^2c^2\\
&+b^4\cos^2 v\\
&-b^2c^2\cos^2 v=
\end{aligned}
$$

$$
\begin{aligned}
&=(a^2b^2-a^2c^2-b^4+b^2c^2)\operatorname{sen}^2 u\cos^2 v\\
&+(a^2c^2-b^2c^2)\operatorname{sen}^2 u\\
&+(a^2b^2-b^4)\cos^2 v\cos^2 u\\
&+b^2c^2\\
&+b^4\cos^2 v\\
&-b^2c^2\cos^2 v=
\end{aligned}
\quad
\begin{aligned}
&=a^2b^2\operatorname{sen}^2 u\cos^2 v\\
&-a^2c^2\operatorname{sen}^2 u\cos^2 v\\
&-b^4\operatorname{sen}^2 u\cos^2 v\\
&+b^2c^2\operatorname{sen}^2 u\cos^2 v\\
&+a^2c^2\operatorname{sen}^2 u\\
&-b^2c^2\operatorname{sen}^2 u\\
&+a^2b^2\cos^2 v\cos^2 u\\
&-b^4\cos^2 v\cos^2 u\\
&+b^2c^2\\
&+b^4\cos^2 v\\
&-b^2c^2\cos^2 v=
\end{aligned}
$$

$$
\begin{aligned}
&=a^2b^2(\operatorname{sen}^2 u\cos^2 v+\cos^2 v\cos^2 u)\\
&+a^2c^2(\operatorname{sen}^2 u-\operatorname{sen}^2 u\cos^2 v)\\
&+b^2c^2(1+\operatorname{sen}^2 u\cos^2 v-\operatorname{sen}^2 u-\cos^2 v)\\
&+b^4(\cos^2 v-\operatorname{sen}^2 u\cos^2 v-\cos^2 v\cos^2 u)=
\end{aligned}
\quad
\begin{aligned}
&=a^2b^2(\operatorname{sen}^2 u\cos^2 v+\cos^2 v\cos^2 u)\\
&+a^2c^2\operatorname{sen}^2 u(1-\cos^2 v)\\
&+b^2c^2(1+\operatorname{sen}^2 u\cos^2 v-\operatorname{sen}^2 u-(1-\operatorname{sen}^2 v))\\
&+b^4\cos^2 v(1-\operatorname{sen}^2 u-\cos^2 u)=
\end{aligned}
$$

$$
\begin{aligned}
&=a^2b^2(\operatorname{sen}^2 u\cos^2 v+\cos^2 v\cos^2 u)\\
&+a^2c^2\operatorname{sen}^2 u\operatorname{sen}^2 v\\
&+b^2c^2(1+\operatorname{sen}^2 u\cos^2 v-\operatorname{sen}^2 u-1+\operatorname{sen}^2 v)=
\end{aligned}
\quad
\begin{aligned}
&=a^2b^2(\operatorname{sen}^2 u\cos^2 v+\cos^2 v\cos^2 u)\\
&+a^2c^2\operatorname{sen}^2 u\operatorname{sen}^2 v\\
&+b^2c^2(\operatorname{sen}^2 u\cos^2 v-\operatorname{sen}^2 u+\operatorname{sen}^2 v)=
\end{aligned}
$$

$$
\begin{aligned}
&=a^2b^2(\operatorname{sen}^2 u\cos^2 v+\cos^2 v\cos^2 u)\\
&+a^2c^2\operatorname{sen}^2 u\operatorname{sen}^2 v\\
&+b^2c^2(\operatorname{sen}^2 u(\cos^2 v-1)+\operatorname{sen}^2 v)=
\end{aligned}
\quad
\begin{aligned}
&=a^2b^2(\operatorname{sen}^2 u\cos^2 v+\cos^2 v\cos^2 u)\\
&+a^2c^2\operatorname{sen}^2 u\operatorname{sen}^2 v\\
&+b^2c^2\operatorname{sen}^2 v(1-\operatorname{sen}^2 u)=
\end{aligned}
$$

$$
\begin{aligned}
&=a^2b^2(\operatorname{sen}^2 u\cos^2 v+\cos^2 v\cos^2 u)\\
&+a^2c^2\operatorname{sen}^2 u\operatorname{sen}^2 v\\
&+b^2c^2\operatorname{sen}^2 v\cos^2 u>0
\end{aligned}
$$

Tenemos que $K>0$. Los cálculos funcionan cuando $\operatorname{sen} v\neq 0$ que nunca ocurre con $v\in(0,\pi)$. El resto de cartas necesarias para cubrir $\mathcal{E}$ son similares a $\mathbb{X}$ y los cálculos son casi idénticos.