Seminario 3

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1.

a) Sea $c = \text{mcm}(a,b)$.

$\boxed{(a) \cap (b) \supseteq (c)}$

$$\left.\begin{array}{ccc} a \mid c & \Rightarrow & (c) \subseteq (a) \\ b \mid c & \Rightarrow & (c) \subseteq (b) \end{array}\right\} \Rightarrow (c) \subseteq (a) \cap (b)$$

$\boxed{(a) \cap (b) \subseteq (c)}$

$$x \in (a) \cap (b) \Rightarrow \left\{\begin{array}{ccc} (x) \subseteq (a) & \Rightarrow & a \mid x \\ (x) \subseteq (b) & \Rightarrow & b \mid x \end{array}\right.$$

$$\Rightarrow c \mid x \Rightarrow x \in (c) \Rightarrow (a) \cap (b) \subseteq (c)$$

c) En el DFU $\mathbb{C}[X,Y]$, $(X) + (Y) \neq (\text{MCM}(X,Y))$. Si $C = \text{MCM}(X,Y)$ tenemos que $C = 1$, es decir, $(C) = \mathbb{C}[X,Y]$ pero $1 \notin (X)+(Y)$

2.

Sabemos que $r(a) = \{b \in A : \exists n \in \mathbb{N} : b^n \in (a)\}$. Sea $b \in a$ y su producto en irreducibles $b = q_1^{e_1} \cdots q_r^{e_s}$. Ahora, $b^n = q_1^{ne_1} \cdots q_r^{ne_s}$. Para que $b^n \in (a)$ se necesita que $a \mid b^n$, es decir,

$$\forall i \in \{1, \cdots, m\} \exists j \in \{1, \cdots, r\} :$$

$$\exists n_i \in \mathbb{N}_{>0}\, p_i^{n_i} \mid q_j^{ne_j} \Leftrightarrow p_i \sim q_j$$

Entonces b es de la forma $cp_1 \cdots p_m$ con $c \in A$.

$$r(a) = (p_1 \cdots p_m)$$

7.

a)

$\boxed{\text{ker } \varphi \supseteq (X^2 - Y^3)}$

$$\varphi(X^2 - Y^2) = \varphi(X)^2 + \varphi(Y)^3 = (T^3)^2 - (T^2)^3$$

$$= 0_{K[X,Y]} \Rightarrow \varphi(f(X^2 - Y^3)$$

$$= \varphi(f) 0_{K[X,Y]} \forall f \in K[X,Y]$$

$$\Rightarrow \text{ker } \varphi \supseteq (X^2 - Y^3)$$

$\boxed{\text{ker } \varphi \subseteq (X^2 - Y^3)}$

Sea $f(X,Y) \in \text{ker}\varphi$. Si dividimos $f(X,Y)$ entre $X^2 - Y^3$ respecto de X.

$$f(X,Y) = q(X,Y)(X^2 - Y^3) + r(X,Y)$$

$$\text{con } 2 > \deg_X(r)$$

Podemos escribir $r(X,Y) = a(Y)X + b(Y)$.

$$\varphi(f) = \varphi(q)\varphi(X^2 - Y^3) + \varphi(r)$$

$$0 = \varphi(a(Y))\varphi(X) + \varphi(b(Y))$$

$$0 = \varphi(\sum_{i \geq 0} a_i Y^i) T^3 + \varphi(\sum_{i \geq 0} b_i Y^i)$$

$$0 = \sum_{i \geq 0} \varphi(a_i) T^{2i+3} + \sum_{i \geq 0} \varphi(b_i) T^{2i}$$

$$\Rightarrow \varphi(a_i) = \varphi(b_i) = 0 \quad \forall i \geq 0$$

Como K es cuerpo y $1 \neq 0$ en $K[T]$.

$$a_i = b_i = 0 \quad \forall i \geq 0$$

$$\Rightarrow r(X,Y) = 0 \Rightarrow f(X,Y) \in (X^2 - Y^3)$$

b)

$\boxed{K[T^2, T^3] \supseteq \{\sum_{i \geq 0} c_i T^i \in K[T] \mid c_1 = 0\}}$

$$\sum_{i \geq 0} c_i T^i = \underbrace{c_0}_{\in K[T^2,T^3]} + \underbrace{c_2 T^2}_{\in K[T^2,T^3]} + \underbrace{c_3 T^3}_{\in K[T^2,T^3]}$$

$$+ \underbrace{c_4 T^4}_{= c_4 (T^2)^2 \in K[T^2,T^3]} + \underbrace{c_5 T^5}_{= c_5 T^2 T^3 \in K[T^2,T^3]} \cdots \in K[T^2, T^3]$$

En general si i es par $c_i T^i = c_i (T^2)^{i/2} \in K[T^2, T^3]$, cuando $i \geq 3$ es impar $c_i T^i = c_i T^3 (T^2)^{\frac{i-3}{2}} \in K[T^2, T^3]$.

$\boxed{K[T^2, T^3] \subseteq \{\sum_{i \geq 0} c_i T^i \in K[T] \mid c_1 = 0\}}$

Como $\{\sum_{i \geq 0} c_i T^i \in K[T] \mid c_1 = 0\}$ es subanillo de $K[T]$ y $T^2, T^3 \in \{\sum_{i \geq 0} c_i T^i \in K[T] \mid c_1 = 0\}$. Se tiene que,

$$K[T^2, T^3] \subseteq \{\sum_{i \geq 0} c_i T^i \in K[T] \mid c_1 = 0\}$$