Entrega 7

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Problema:
Ver si las curvas coordenadas son geodésicas para una superficie de revolución.
Solución:
Si la superficie viene parametrizada por $\mathbb{X}(u,v) = (f(u)\cos v, f(u)\,\mathrm{sen}\, v, g(u))$, las curvas coordenadas son

$$\alpha_u : \mathbb{X}(u_0,t) = (f(u_0)\cos t, f(u_0)\,\mathrm{sen}\, t, g(u_0))$$

$$\alpha_v : \mathbb{X}(t,v_0) = (f(t)\cos v_0, f(t)\,\mathrm{sen}\, v_0, g(t))$$

Para que las curvas sean geodésicas necesitamos que la componente tangencial de sus segundas derivadas sea nula.

$$\mathbb{X}_u = (f'(u)\cos v, f'(u)\,\mathrm{sen}\, v, g'(u))$$

$$\mathbb{X}_v = (-f(u)\,\mathrm{sen}\, v, f(u)\cos v, 0)$$

$$N = \mathbb{X}_u \wedge \mathbb{X}_v = \begin{vmatrix} i & j & k \\ f'(u)\cos v & f'(u)\,\mathrm{sen}\, v & g'(u) \\ -f(u)\,\mathrm{sen}\, v & f(u)\cos v & 0 \end{vmatrix} =$$

$$= i\begin{vmatrix} f'(u)\,\mathrm{sen}\, v & g'(u) \\ f(u)\cos v & 0 \end{vmatrix} - j\begin{vmatrix} f'(u)\cos v & g'(u) \\ -f(u)\,\mathrm{sen}\, v & 0 \end{vmatrix} + k\begin{vmatrix} f'(u)\cos v & f'(u)\,\mathrm{sen}\, v \\ -f(u)\,\mathrm{sen}\, v & f(u)\cos v \end{vmatrix} =$$

$$= ig'(u)f(u)\cos v + jg'(u)f(u)\,\mathrm{sen}\, v + k(f'(u)f(u)) =$$

$$= (g'(u)f(u)\cos v, g'(u)f(u)\,\mathrm{sen}\, v, f'(u)f(u))$$

Empecemos por α_u. Veamos si está parametrizada por longitud de arco:

$$\alpha_u{}' = (-f(u_0)\,\mathrm{sen}\, t, f(u_o)\cos t, 0)$$

$$\|\alpha_u'\| = |f(u_o)|$$

Tenemos que,

$$\beta_u' = \frac{\alpha_u{}'}{f(u_o)}$$

Donde β_u es α_u parametrizada por longitud de arco.

$$\beta_u' = (-\,\mathrm{sen}\, t, \cos t, 0)$$

$$\beta_u'' = (-\cos t, -\,\mathrm{sen}\, t, 0)$$

Como $\beta_u'\beta_u'' = 0$, bastaría con ver que $\beta_u'' \cdot (N \wedge \beta_u') = 0$.

$$N \wedge \beta_u' = \begin{vmatrix} i & j & k \\ g'(u_0)f(u_0)\cos t & g'(u_0)f(u_0)\,\mathrm{sen}\, t & f'(u_0)f(u_0) \\ -\,\mathrm{sen}\, t & \cos t & 0 \end{vmatrix} =$$

$$= (f'(u_0)f(u_0)\cos t, -f'(u_0)f(u_0)\,\mathrm{sen}\, t, g'(u_0)f(u_0)\cos^2 t + g'(u_0)f(u_0)\,\mathrm{sen}^2\, t) =$$

$$= (f'(u_0)f(u_0)\cos t, -f'(u_0)f(u_0)\,\mathrm{sen}\, t, g'(u_0)f(u_0))$$

$$\beta_u'' \cdot (N \wedge \beta_u') = (-\cos t, -\,\mathrm{sen}\, t, 0)\cdot(f'(u_0)f(u_0)\cos t, -f'(u_0)f(u_0)\,\mathrm{sen}\, t, g'(u_0)f(u_0)) =$$

$$= -f'(u_0)f(u_0)\cos^2 t + f'(u_0)f(u_0)\,\mathrm{sen}^2\, t = (f'(u_0)f(u_0))(1 - 2\cos^2 t)$$

Si $f(u_0) = 0$, $\|\alpha_u{}'\| = 0$ y la curva no es regular. Cuando $f'(u_0) = 0$, α_u es geodésica.
Veamos que ocurre con α_v.

$$\alpha_v{}' = (f'(t)\cos v_0, f'(t)\,\mathrm{sen}\, v_0, g'(t))$$

$$\|\alpha_v{}'\| = \sqrt{f'(t)^2 + g'(t)^2}$$

$$\beta_v' = \frac{\alpha_v{}'}{\sqrt{f'(t)^2 + g'(t)^2}}$$

$$\beta_v'' = (\cos v_0 h(t), \operatorname{sen} v_0 h(t), l(t))$$

Con

$$h(t) = \frac{f''(t)\sqrt{f'(t)^2+g'(t)^2} - f'(t)\frac{f'(t)f''(t)+g'(t)g''(t)}{\sqrt{f'(t)^2+g'(t)^2}}}{f'(t)^2+g'(t)^2}$$

$$l(t) = \frac{g''(t)\sqrt{f'(t)^2+g'(t)^2} - g'(t)\frac{f'(t)f''(t)+g'(t)g''(t)}{\sqrt{f'(t)^2+g'(t)^2}}}{f'(t)^2+g'(t)^2}$$

$$\beta_v'\beta_v'' = \frac{{\alpha_v}'}{\|{\alpha_v}'\|}\beta_v'' = \frac{1}{\|{\alpha_v}'\|}(f'(t)\cos v_0, f'(t)\operatorname{sen} v_0, g'(t)) \cdot (\cos v_0 h(t), \operatorname{sen} v_0 h(t), l(t))$$

$$= \frac{1}{\|{\alpha_v}'\|}(f'(t)h(t)\cos^2 v_0 + f'(t)h(t)\operatorname{sen}^2 v_0 + g'(t)l(t)) = \frac{1}{\|{\alpha_v}'\|}(f'(t)h(t) + g'(t)l(t))$$

Nos centramos en $f'(t)h(t)+g'(t)l(t)$.

$$f'(t)h(t)+g'(t)l(t) =$$

$$= \frac{f'(t)f''(t)\sqrt{f'(t)^2+g'(t)^2} - f'(t)^2\frac{f'(t)f''(t)+g'(t)g''(t)}{\sqrt{f'(t)^2+g'(t)^2}} + g'(t)g''(t)\sqrt{f'(t)^2+g'(t)^2} - g'(t)^2\frac{f'(t)f''(t)+g'(t)g''(t)}{\sqrt{f'(t)^2+g'(t)^2}}}{f'(t)^2+g'(t)^2} =$$

$$= \frac{(f'(t)f''(t)+g'(t)g''(t))\sqrt{f'(t)^2+g'(t)^2} - (f'(t)^2+g'(t)^2)\frac{f'(t)f''(t)+g'(t)g''(t)}{\sqrt{f'(t)^2+g'(t)^2}}}{f'(t)^2+g'(t)^2}$$

Entonces,

$$\beta_v'\beta_v'' = \frac{(f'(t)f''(t)+g'(t)g''(t)) - (f'(t)^2+g'(t)^2)\frac{f'(t)f''(t)+g'(t)g''(t)}{f'(t)^2+g'(t)^2}}{f'(t)^2+g'(t)^2} = 0$$

Veamos que ocurre con $\beta_v'' \cdot (N \wedge {\alpha_v}')$.

$$N \wedge {\alpha_v}' = \begin{vmatrix} i & j & k \\ g'(t)f(t)\cos v_0 & g'(t)f(t)\operatorname{sen} v_0 & f'(t)f(t) \\ f'(t)\cos v_0 & f'(t)\operatorname{sen} v_0 & g'(t) \end{vmatrix} =$$

$$= (g'(t)^2 f(t)\operatorname{sen} v_0 - f'(t)^2 f(t)\operatorname{sen} v_o, -g'(t)^2 f(t)\cos v_0 + f'(t)^2 f(t)\cos v_0, 0)$$

$$= (\operatorname{sen} v_0(g'(t)^2 f(t) - f'(t)^2 f(t)), -\cos v_0(g'(t)^2 f(t) - f'(t)^2 f(t)), 0)$$

$$= (g'(t)^2 f(t) - f'(t)^2 f(t))(\operatorname{sen} v_0, -\cos v_0, 0)$$

$$\frac{\beta_v'' \cdot (N \wedge {\alpha_v}')}{(g'(t)^2 f(t) - f'(t)^2 f(t))} = (\operatorname{sen} v_0, -\cos v_0, 0) \cdot (\cos v_0 h(t), \operatorname{sen} v_0 h(t), l(t)) = 0$$

Y se tiene que α_v es geodésica.