Curvas y Superficies

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Ejercicio 1

i)

$$X_u \times X_v = \begin{vmatrix} i & j & k \\ \cos(v) & \operatorname{sen}(v) & 0 \\ -u\operatorname{sen}(v) & u\cos(v) & 1 \end{vmatrix} = (\operatorname{sen}(v), -\cos(v), u)$$

$$\Rightarrow X_3 = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{((\operatorname{sen}(v), -\cos(v), u))}{\sqrt{u^2+1}}$$

$$L_{11} = X_3 \cdot X_{uu} = 0$$

$$L_{12} = X_3 \cdot X_{uv} = \frac{1}{\sqrt{u^2+1}}(\operatorname{sen}(v), -\cos(v), u)(-\operatorname{sen}(v), \cos(v), 0) = \frac{-1}{\sqrt{u^2+1}}$$

$$L_{22} = X_3 \cdot X_{vv} = \frac{1}{\sqrt{u^2+1}}(\operatorname{sen}(v), -\cos(v), u) \cdot (-u\cos(v), -u\operatorname{sen}(v), 0) = 0$$

$$E = X_u \cdot X_u = 1$$

$$F = X_u \cdot X_v = 0$$

$$G = X_v \cdot X_v = u^2 + 1$$

Sea $\alpha(t) = X(u(t), v(t))$

$$II_{\alpha(t)}(\alpha'(t)) = (u'(t))^2 L_{11}(u(t), v(t)) + 2u'(t)v'(t)L_{12}(u(t), v(t)) + (v'(t))^2 L_{22}(u(t), v(t)) = -\frac{2u'(t)v'(t)}{(u(t)^2+1)^{\frac{1}{2}}}$$

$$K(X(u,v)) = \frac{L_{11}(u,v)L_{22}(u,v) - (L_{12}(u,v))^2}{E(u,v)G(u,v) - (F(u,v)^2)} = \frac{-1}{(u^2+1)^2}$$

$$H(X(u,v)) = \frac{1}{2}\frac{-2FL_{12} + GL_{11} + EL_{22}}{EG - F^2} = 0$$

$$\begin{cases} K = \lambda_1\lambda_2 \\ H = \frac{1}{2}(\lambda_1 + \lambda_2) \end{cases} \Rightarrow \begin{cases} K = -\lambda_1^2 \\ \lambda_2 = -\lambda_1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = \frac{1}{u^2+1} \\ \lambda_2 = \frac{-1}{u^2+1} \end{cases} \quad \text{ó viceversa}$$

ii)

$$\|\alpha'(t)\|^2 = I_{\alpha(t)}(\alpha'(t)) = (u'(t))^2 E(u(t), v(t)) + 2u'(t)v'(t)F(u(t), v(t)) + (v'(t))^2 G(u(t), v(t))$$

$$= cosh^2(t) + 1^2(senh^2(t) + 1) = 2cosh^2(t)$$

$$k_n(\alpha'(t)) = \frac{1}{\|\alpha'(t)\|} II_{\alpha(t)}(\alpha'(t)) = \frac{1}{cosh^2(t)}$$

iii)

β línea asintótica $\quad\Leftrightarrow\quad (u'(t))^2 L_{11}(u(t), v(t)) + 2u'(t)v'(t)L_{12}(u(t), v(t)) + (v'(t))^2 L_{22}(u(t), v(t)) = 0.$

$$(u'(t))^2 L_{11} + 2u'(t)v'(t)L_{12} + (v'(t))^2 L_{22} = -\frac{2}{\sqrt{t^2+1}} \cdot 1 \cdot 0 = 0 \Rightarrow \beta \text{ línea asintótica}$$

β línea de curvatura $\Leftrightarrow$

$$\begin{vmatrix} (u'(t))^2 & -u'(t)v'(t) & (v'(t))^2 \\ E & F & G \\ L_{11} & L_{12} & L_{22} \end{vmatrix} = 0$$

$$\begin{vmatrix} (u'(t))^2 & -u'(t)v'(t) & (v'(t))^2 \\ E & F & G \\ L_{11} & L_{12} & L_{22} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & t^2+1 \\ 0 & \frac{-1}{\sqrt{t^2+1}} & 0 \end{vmatrix} = \frac{t^2+1}{\sqrt{t^2+1}} \neq 0 \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \quad \beta \text{ no es línea de curvatura}$$

Ejercicio 2

i)

Sea $f(x,y,z) = x^2 + y^2 - z$. $S = f^{-1}(0)$ regular $\Leftrightarrow \nabla f(x,y,z) \neq (0,0,0) \quad \forall (x,y,z) \in S$.

$$Df(x,y,z) = (2x, 2y, -1) \neq (0,0,0) \forall (x,y,,z) \in S \quad \Rightarrow \quad S \text{ resgular}$$

ii)

$$X(u,v) = (u, v, u^2 + v^2)$$

$$X_u(u,v) = (1,0,2u), X_v(u,v) = (0,1,2v)$$

$$\Rightarrow X_u \times X_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1) \quad \Rightarrow \quad X_3 = \frac{(-2u,-2v,1)}{\sqrt{4u^2 + 4v^2 + 1}}$$

$$L_{11} = \frac{2}{\sqrt{4u^2 + 4v^2 + 1}} \quad , \quad L_{12} = 0 \quad , \quad L_{22} = \frac{2}{\sqrt{4u^2 + 4v^2 + 1}}$$

$$E = 1 + 4u^2 \quad , \quad F = 4uv \quad , \quad G = 1 + av^2$$

$$K = \frac{4}{4u^2 + 4v^2 + 1} \frac{1}{4u^2 + 4v^2 + 1} = \frac{4}{(4u^2 + 4v^2 + 1)^2}$$

$$H = \frac{1}{2} \frac{4}{4u^2 + 4v^2 + 1} \left(-8uv0 + (1 + 4v^2) \frac{2}{\sqrt{4u^2 + 4v^2 + 1}} + (1 + 4u^2) \frac{2}{\sqrt{4u^2 + 4v^2 + 1}} \right) = \frac{2 + 4v^2 + 4u^2}{(4u^2 + 4v^2 + 1)^{\frac{3}{2}}}$$

$$\left\{ \begin{array}{l} K = \lambda_1 \lambda_2 \\ H = \frac{1}{2}(\lambda_1 + \lambda_2) \end{array} \right. \Rightarrow \left\{ \begin{array}{l} K = \lambda_1 \lambda_2 \\ \lambda_1 = 2H - \lambda_2 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} K = 2H\lambda_2 - \lambda_2^2 \\ \lambda_1 = 2H - \lambda_2 \end{array} \right. \Rightarrow -\lambda^2 + 2H\lambda - K = 0 \Rightarrow \lambda = H \pm \sqrt{H^2 - K}$$

$$H^2 - K = \frac{(2 + 4v^2 + 4u^2)^2}{(4u^2 + 4v^2 + 1)^3} - \frac{4}{(4u^2 + 4v^2 + 1)^2} = \frac{16(u^2 + v^2)^2}{(4u^2 + 4v^2 + 1)^3}$$

$$\Rightarrow \left\{ \begin{array}{l} \lambda_1 = H + \sqrt{H^-K} = 2\frac{1 + 4u^2 + 4v^2}{(4u^2 + 4v^2 + 1)^{\frac{3}{2}}} \\ \lambda_2 = H - \sqrt{H^-K} = \frac{2}{(4u^2 + 4v^2 + 1)^{\frac{3}{2}}} \end{array} \right.$$

iii)

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_(t)) = (t, t, 2t^2)$$

$$\alpha_1^2(t) + \alpha_2^2 = t^2 + t^2 = \alpha_3(t) \Rightarrow \alpha \text{ esta sobre } S.$$

$$\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t)) = (senh(t), 1, senh^2(t))$$

$$\beta_1^2(t) + \beta_2^2(t) = senh^2(t) + 1 \neq senh^2(t) = \beta_2 \Rightarrow \beta \text{ no esta sobre } S$$

$$\alpha(t) = X(u,v)$$

$$(u'(t))^2 L_{11}(u(t), v(t)) + 2u'(t)v'(t)L_{12}(u(t), v(t)) + (v'(t))^2 L_{22}(u(t), v(t))$$

$$= \frac{2}{\sqrt{8t^2 + 1}} + \frac{2}{\sqrt{8t^2 + 1}} = \frac{4}{\sqrt{8t^2 + 1}} \neq 0 \quad \forall t \in \mathbb{R} \quad \Rightarrow \quad \alpha \text{ no es línea asintótica}$$

$$\begin{vmatrix} (u'(t))^2 & -u'(t)v'(t) & (v'(t))^2 \\ E & F & G \\ L_{11} & L_{12} & L_{22} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 1 + 4t^2 & 4t^2 & 1 + 4t^2 \\ \frac{2}{\sqrt{8t^2+1}} & 0 & \frac{2}{\sqrt{8t^2+1}} \end{vmatrix} = 0$$

$$\Rightarrow \quad \alpha \text{ es línea de curvatura}$$

Ejercicio 3

i)

$$X_u = (1, 0, 2u), X_v = (0, 1, -2v)$$

$$X_u \times X_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix} = (-2u, 2v, 1) \Rightarrow X_3 = \frac{(-2u, 2v, 1)}{\sqrt{4u^2+4v^2+1}}$$

$$L_{11} = \frac{2}{\sqrt{4u^2+4v^2+1}} \quad , \quad L_{12} = 0 \quad , \quad L_{22} = \frac{-2}{\sqrt{4u^2+4v^2+1}}$$

$$E = 1+4u^2 \quad , \quad F = -4uv \quad , \quad G = 1+4v^2$$

Sea $\alpha(t) = X(u(t), v(t))$

$$I_{\alpha(t)}(\alpha'(t)) = (u'(t))^2 E(u(t),v(t)) + 2u'(t)v'(t)F(u(t),v(t)) + (v'(t))^2 G(u(t),v(t))$$

$$= (u'(t))^2(1+4u^2) + 8u'(t)v'(t)u(t)v(t) + (v'(t))^2(1+4v^2)$$

$$II_{\alpha(t)}(\alpha'(t)) = \frac{2}{\sqrt{4u^2+4v^2+1}}((u'(t))^2 - (v'(t))^2)$$

$$K = \frac{-4}{4u^2+4v^2+1}\frac{1}{4u^2+4v^2+1} = \frac{-4}{(4u^2+4v^2+1)^2}$$

ii)

$$\alpha \text{ línea asintótica} \quad \Leftrightarrow \quad II_{\alpha(t)}(\alpha'(t)) = 0 \quad \Leftrightarrow \quad \frac{2}{\sqrt{4u^2+4v^2+1}}((u'(t))^2 - (v'(t))^2) = 0$$

$$\Leftrightarrow \quad (u'(t))^2 = (v'(t))^2 \quad \Leftrightarrow \quad u'(t) = \pm v'(t) \quad \Leftrightarrow \quad u(t) = \pm v(t) + \text{cte.}$$

$$\alpha(t) = X(t,0) = (t, 0, t^2) \Rightarrow \alpha'(t) = (1, 0, 2t) =$$

$$k_n(\alpha'(t)) = \frac{1}{\|\alpha'(t)\|^2} II_{\alpha(t)}(\alpha'(t)) = \frac{2}{\sqrt{4t^2+1}}\frac{1}{\sqrt{4t^2+1}} = \frac{2}{4t^2+1}$$

$$k_n(\alpha'(t)) \cdot X_3(\alpha(t)) = \frac{2}{4t^2+1}\frac{(-2t,0,1)}{\sqrt{4t^2+1}} = \frac{2}{(4t^2+1)^{\frac{3}{2}}}(-2t, 0, 1)$$

$$t \neq \text{cte.} \quad \Rightarrow \quad \alpha \text{ no es línea asintótica.}$$

$$\begin{vmatrix} (u'(t))^2 & -u'(t)v'(t) & (v'(t))^2 \\ E & F & G \\ L_{11} & L_{12} & L_{22} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1+4t^2 & 0 & 1 \\ \frac{2}{\sqrt{4t^2+1}} & 0 & \frac{-2}{\sqrt{4t^2+1}} \end{vmatrix} = 0$$

$$\Rightarrow \quad \alpha \text{ línea de curvatura}$$