Entrega 5

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Problema:

Probar que, si $\mathbb{X} : \mathcal{U} \in \mathbb{R}^2 \longrightarrow \mathbb{X}(\mathcal{U}) \subset S$, tal que $F = 0$, entonces

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

Solución:

Por definición de los símbolos de Christoffel

$$\begin{cases} \mathbb{X}_{uu} = \Gamma_{11}^1\mathbb{X}_u + \Gamma_{11}^2\mathbb{X}_v + eN \\ \mathbb{X}_{uv} = \Gamma_{12}^1\mathbb{X}_u + \Gamma_{12}^2\mathbb{X}_v + fN \\ \mathbb{X}_{vv} = \Gamma_{22}^1\mathbb{X}_u + \Gamma_{22}^2\mathbb{X}_v + gN \end{cases}$$

Como $F = 0$, $\mathbb{X}_u\mathbb{X}_v = 0$. Despejamos los 6 símbolos de Christoffel.

$$\mathbb{X}_u\mathbb{X}_{uu} = \Gamma_{11}^1\mathbb{X}_u\mathbb{X}_u + \Gamma_{11}^2\mathbb{X}_u\mathbb{X}_v + e\mathbb{X}_uN$$
$$\mathbb{X}_u\mathbb{X}_{uu} = \Gamma_{11}^1E$$

$$\mathbb{X}_u\mathbb{X}_{uv} = \Gamma_{12}^1\mathbb{X}_u\mathbb{X}_u + \Gamma_{12}^2\mathbb{X}_u\mathbb{X}_v + f\mathbb{X}_uN$$
$$\mathbb{X}_u\mathbb{X}_{uv} = \Gamma_{12}^1E$$

$$\mathbb{X}_u\mathbb{X}_{vv} = \Gamma_{22}^1\mathbb{X}_u\mathbb{X}_u + \Gamma_{22}^2\mathbb{X}_u\mathbb{X}_v + g\mathbb{X}_uN$$
$$\mathbb{X}_u\mathbb{X}_{vv} = \Gamma_{22}^1E$$

$$\mathbb{X}_v\mathbb{X}_{uu} = \Gamma_{11}^1\mathbb{X}_v\mathbb{X}_u + \Gamma_{11}^2\mathbb{X}_v\mathbb{X}_v + e\mathbb{X}_uN$$
$$\mathbb{X}_v\mathbb{X}_{uu} = \Gamma_{11}^2G$$

$$\mathbb{X}_v\mathbb{X}_{uv} = \Gamma_{12}^1\mathbb{X}_v\mathbb{X}_u + \Gamma_{12}^2\mathbb{X}_v\mathbb{X}_v + f\mathbb{X}_vN$$
$$\mathbb{X}_v\mathbb{X}_{uv} = \Gamma_{12}^2G$$

$$\mathbb{X}_v\mathbb{X}_{vv} = \Gamma_{22}^1\mathbb{X}_v\mathbb{X}_u + \Gamma_{22}^2\mathbb{X}_v\mathbb{X}_v + g\mathbb{X}_vN$$
$$\mathbb{X}_v\mathbb{X}_{vv} = \Gamma_{22}^2G$$

Ahora observamos que

$$(\mathbb{X}_u\mathbb{X}_u)_u = \mathbb{X}_u\mathbb{X}_{uu} + \mathbb{X}_{uu}\mathbb{X}_u = 2\mathbb{X}_{uu}\mathbb{X}_u$$
$$E_u = 2\mathbb{X}_{uu}\mathbb{X}_u$$

$$(\mathbb{X}_u\mathbb{X}_u)_v = \mathbb{X}_u\mathbb{X}_{uv} + \mathbb{X}_{uv}\mathbb{X}_u = 2\mathbb{X}_{uv}\mathbb{X}_u$$
$$E_v = 2\mathbb{X}_{uv}\mathbb{X}_u$$

$$(\mathbb{X}_v\mathbb{X}_v)_u = \mathbb{X}_v\mathbb{X}_{vu} + \mathbb{X}_{vu}\mathbb{X}_v = 2\mathbb{X}_{vu}\mathbb{X}_v$$
$$G_u = 2\mathbb{X}_{vu}\mathbb{X}_v$$

$$(\mathbb{X}_v\mathbb{X}_v)_v = \mathbb{X}_v\mathbb{X}_{vv} + \mathbb{X}_{vv}\mathbb{X}_v = 2\mathbb{X}_{vv}\mathbb{X}_v$$
$$G_v = 2\mathbb{X}_{vv}\mathbb{X}_v$$

De donde obtenemos las siguientes expresiones para los símbolos de Christoffel

$$\Gamma_{11}^1 = \frac{E_u}{2E}$$

$$\Gamma^1_{12} = \frac{E_v}{2E}$$
$$\Gamma^1_{22} = \frac{-G_u}{2E}$$
$$\Gamma^2_{11} = \frac{-E_v}{2G}$$
$$\Gamma^2_{12} = \frac{G_u}{2G}$$
$$\Gamma^2_{22} = \frac{G_v}{2G}$$

Sustituimos en la ecuación de la curvatura en términos de los símbolos de Christoffel.

$$K = -\frac{1}{E}\left(\left(\Gamma^2_{12}\right)_u - \left(\Gamma^2_{11}\right)_v + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \left(\Gamma^2_{12}\right)^2 - \Gamma^2_{11}\Gamma^2_{22}\right) =$$

$$= -\frac{1}{E}\left(\left(\frac{G_u}{2G}\right)_u - \left(\frac{-E_v}{2G}\right)_v + \frac{E_v}{2E}\frac{-E_v}{2G} - \frac{E_u}{2E}\frac{G_u}{2G} + \left(\frac{G_u}{2G}\right)^2 - \frac{-E_v}{2G}\frac{G_v}{2G}\right) =$$

$$= -\frac{1}{2E}\left(\left(\frac{G_u}{G}\right)_u + \left(\frac{E_v}{G}\right)_v - \frac{E_v^2}{2EG} - \frac{E_uG_u}{2EG} + \frac{G_u^2}{2G^2} + \frac{E_vG_v}{2G^2}\right) =$$

$$= -\frac{1}{2E}\left(\frac{G_{uu}G - G_uG_u}{G^2} + \frac{E_{vv}G - E_vG_v}{G^2} - \frac{E_v^2}{2EG} - \frac{E_uG_u}{2EG} + \frac{G_u^2}{2G^2} + \frac{E_vG_v}{2G^2}\right) =$$

$$= -\frac{1}{2EG}\left(G_{uu} - \frac{G_u^2}{G} + E_{vv} - \frac{E_vG_v}{G} - \frac{E_v^2}{2E} - \frac{E_uG_u}{2E} + \frac{G_u^2}{2G} + \frac{E_vG_v}{2G}\right)$$

$$= -\frac{1}{2EG}\left(G_{uu} + E_{vv} - \frac{E_v^2}{2E} - \frac{E_uG_u}{2E} - \frac{G_u^2}{2G} - \frac{E_vG_v}{2G}\right)$$

$$= -\frac{1}{2EG}\left(G_{uu} + E_{vv} - E_v\frac{E_vG}{2EG} - G_u\frac{E_uG}{2EG} - G_u\frac{EG_u}{2EG} - E_v\frac{EG_v}{2EG}\right)$$

$$= -\frac{1}{2EG}\left(E_{vv} - E_v\frac{E_vG + EG_v}{2EG} + G_{uu} - G_u\frac{E_uG + EG_u}{2EG}\right)$$

$$= -\frac{1}{2}\left(\frac{E_{vv} - E_v\frac{E_vG+EG_v}{2EG}}{EG} + \frac{G_{uu} - G_u\frac{E_uG+EG_u}{2EG}}{EG}\right)$$

$$= -\frac{1}{2\sqrt{EG}}\left(\frac{E_{vv}\sqrt{EG} - E_v\frac{E_vG+EG_v}{2\sqrt{EG}}}{EG} + \frac{G_{uu}\sqrt{EG} - G_u\frac{E_uG+EG_u}{2\sqrt{EG}}}{EG}\right)$$

$$= -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$